GOOLSBY, RONNIE CHRISTIAN. Near-Continuous Functions and Near-Continuous Homotopy. (1972)
Directed by: Dr. Hughes B. Hoyle, III. Pp. 42.

A function $f:(X,S) \to (Y,T)$ is said to be near-continuous provided that if $\mathcal{U}$ is an open cover of Y then there is an open cover $\mathcal{V}$ of X such that if $V \in \mathcal{V}$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Theorem. If $f:(X,S) \to (Y,T)$ is a continuous function then f is near-continuous, and if f is a near-continuous function and (Y,T) is a T_1 - space then f is continuous. Theorem. If (Y,T) is a T_0 - space with the property that if (X,S) is a topological space and f is a near-continuous function from (X,S) into (Y,T) then f is continuous, then (Y,T) is a T_1 - space. In this paper, near-continuous homotopy is defined and theorems which parallel results for usual homotopy are proved. Examples are proved which show that the near-continuous fundamental group is non-trival and different from fundamental group.

NEAR-CONTINUOUS FUNCTIONS AND

NEAR-CONTINUOUS HOMOTOPY

by

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A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
July, 1972

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ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. Hughes B. Hoyle and to Dr. Karl R. Gentry for their aid and guidance during the preparation of this thesis.

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INTRODUCTION

The purpose of this thesis is to investigate and study the ideas of near-continuous functions and near-continuous homotopy. The ideas of near-continuous functions and near-continuous homotopy originated in [3] and [4]. The reader is expected to have a knowledge of point set topology, and is referred to [1] for definitions and results not given in this thesis.

In Chapter 1, near-continuous is defined and relationships between near-continuous and continuous functions are proved. Surprisingly, a characterization of T_1 - spaces also arises.

In Chapter 2, certain topological properties which are preserved by continuous functions are seen to be preserved by near-continuous functions. It is also shown that, although separable is preserved under continuous functions, separable is not preserved under near-continuous functions.

In Chapter 3, near-continuous retracts and near-continuous fixed points are defined and studied.

In Chapter 4, the near-continuous fundamental group is defined, and although the proofs of theorems turn out by necessity to be different, the usual theorems about fundamental groups are proved.

In Chapter 5, examples are proved to show that the near-continuous fundamental group is non-trivial and different from the fundamental group.

CHAPTER I

Definition 1: Let (X,S) and (Y,T) be topological spaces. A function $f:(X,S) \to (Y,T)$ is near-continuous provided if $\mathcal{U}$ is an open cover of Y, then there exists an open cover $\mathcal{V}$ of X such that if $V \in \mathcal{V}$, then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$.

Theorem 1: If (X,S) and (Y,T) are topological spaces and $f:(X,S) \to (Y,T)$ is continuous, then f is near-continuous.

Proof: Let $\mathcal{U}$ be an open cover of Y. Let $x \in X$. Then there is a $U_x \in \mathcal{U}$ such that $f(x) \in U_x$. Since f is continuous at x, there is a $V_x \in S$ such that $x \in V_x$ and $f(V_x) \subset U_x$. Let $\mathcal{V} = \{ V_x \mid x \in X \}$. Let $x \in X$. Then f is continuous at x and thus $x \in V_x$. Therefore $x \in \cup\{ V_x \mid x \in X \}$ and thus $\mathcal{V}$ is an open cover of X. If $V_x \in \mathcal{V}$ then there is a $U_x \in \mathcal{U}$ such that $f(V_x) \subset U_x$. Hence, f is near-continuous.

Theorem 2: If (X,S) and (Y,T) are topological spaces such that (Y,T) is T_1 and $f:(X,S) \to (Y,T)$ is near-continuous then f is continuous.

Proof: Let $x \in X$ and let U be an open set containing $f(x)$. Since (Y,T) is T_1, $Y - \{f(x)\}$ is open. Let $D = \{ U, Y - \{f(x)\} \}$. Let $y \in Y$. Then either $y = f(x)$ or $y \neq f(x)$. If $y = f(x)$ then $y \in U$. If $y \neq f(x)$ then $y \in Y - \{f(x)\}$. Therefore D is an open cover of (Y,T). Since f is near-continuous there is an open cover C of (X,S) such that if $V \in C$ then there is a $W \in D$ such that $f(V) \subset W$. Since C is an open cover of X, there is a $V \in C$ such that $x \in V$.

Now since U is the only element of D containing f(x), f(V) ⊂ U. Hence, f is continuous.

<u>Theorem 3</u>: If (Y,T) is a T_0 - space with the property that if (X,S) is a topological space and f is a near-continuous function from (X,S) into (Y,T) then f is continuous, then (Y,T) is a T_1 - space.

<u>Proof</u>: Suppose (Y,T) is not a T_1 - space. Then there are points p,q ∈ Y such that every open set which contains p contains q. Let X = {p,q} and let S = { ϕ, X }, and define f:(X,S) → (Y,T) by f(p) = p and f(q) = q. Let $\mathcal{U}$ be an open cover of (Y,T), and let D = { X }. Then D is an open cover of X. Let U ∈ $\mathcal{U}$ such that p ∈ U. Then q ∈ U. Thus f(X) ⊂ U. Therefore f is near-continuous. But since (Y,T) is a T_0 - space there is an element W ∈ T such that q ∈ W and p ∉ W. Now $f^{-1}(W)$ = {q} which is not contained in S. Therefore f is not continuous. But this is a contradiction to the supposition that each near-continuous function is continuous. Hence, (Y,T) is a T_1 - space.

<u>Definition 2</u>: Let (X,S) and (Y,T) be topological spaces and let f:(X,S) → (Y,T) be a function. Then f is <u>somewhat continuous</u> provided if U ≠ ϕ and U ∈ T then there is a V ∈ S such that V ≠ ϕ and V ⊂ $f^{-1}(U)$.

<u>Example 1</u>: Let X = Y = reals. Let S be the usual topology for X and let T = { U | U ⊂ Y, 0 ∉ U } ∪ { Y }. Define f:(X,S) → (Y,T) by f(x) = x for all x ∈ X. Then f is near-continuous, but f is not somewhat continuous.

<u>Proof</u>: Let $\mathcal{U}$ be an open cover of Y. Let D = { X }. Then D is

an open cover of X. Since Y is the only element of T which contains 0, $f(X) \subset Y$ and f is near-continuous. Now $f^{-1}(\{1\}) = \{1\}$ and $\{1\}$ does not contain a non-empty open set from the usual topology. Hence, f is not somewhat continuous.

Example 2: Let $Y = \{a,b\}$ and let $T = \{\phi, Y\}$. Then every near-continuous function from any space, (X,S), into (Y,T) is continuous and yet (Y,T) is not a T_1 - space.

Proof: Let (X,S) be a topological space, and let $f:(X,S) \rightarrow (Y,T)$ be near-continuous. Let $U \in T$. Then $U = \phi$ or $U = Y$. If $U = \phi$ then $f^{-1}(\phi) = \phi$ which is contained in S. If $U = Y$ then $f^{-1}(Y) = X$ which is contained in S. Hence, f is continuous. Now (Y,T) is not a T_1 - space since any open set containing a contains b.

Theorem 4: If (X,R), (Y,S), and (Z,T) are topological spaces and $f:(X,R) \rightarrow (Y,S)$ and $g:(Y,S) \rightarrow (Z,T)$ are near-continuous, then $gf:(X,R) \rightarrow (Z,T)$ is near-continuous.

Proof: Let $\mathcal{W}$ be an open cover of Z. Then since g is near-continuous there is an open cover $\mathcal{U}$ of Y such that if $U \in \mathcal{U}$ then there is a $W \in \mathcal{W}$ such that $g(U) \subset W$. Since f is near-continuous there is an open cover $\mathcal{V}$ of X such that if $V \in \mathcal{V}$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Let $V \in \mathcal{V}$. Then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$, and there is a $W \in \mathcal{W}$ such that $g(U) \subset W$. Now, $gf(V) = g(f(V)) \subset g(U) \subset W$, and hence, gf is near-continuous.

Definition 3: A topological space, (X,S), is a T_{-1} - space provided there is an open cover of X which does not have X as an element.

Definition 4: A topological space, (X,S), is a strongly T_0 - space if and only if (X,S) is a T_{-1} - space and a T_0 - space.

Example 3: Let $X = Y = \{a,b,c\}$, let $S = \{ \phi, X, \{a,c\}, \{c,b\}, \{c\} \}$ and $T = \{ \phi, Y, \{a,b\}, \{b,c\}, \{b\} \}$. Define $f:(X,S) \rightarrow (Y,T)$ by $f(a) = b$, $f(b) = b$, and $f(c) = c$. Then (Y,T) is a strongly T_0 - space, f is near-continuous, but f is not continuous.

Proof: Since $\{b\}$ is an open set containing b and not containing a or c, and $\{a,b\}$ is an open set containing a but not c, (Y,T) is a T_0 - space. Let $D = \{ \{a,b\}, \{b,c\} \}$. Then D is an open cover of Y that does not contain Y as an element. Hence, (Y,T) is a T_{-1} - space. Now, let $\mathcal{U}$ be an open cover of Y. Then either $Y \in \mathcal{U}$ or $\{a,b\}$ and $\{b,c\}$ are elements of $\mathcal{U}$. Let $D = \{ X \}$. Then D is an open cover of X. If $Y \in \mathcal{U}$ then $f(X) \subset Y$, and if $\{a,b\}$ and $\{b,c\}$ are elements of $\mathcal{U}$ then $f(X) \subset \{b,c\}$. Hence, f is near-continuous. The function f is not continuous since $f^{-1}(\{b\}) = \{a,b\}$ which is not an element of S.

Example 4: Let $X = Y =$ reals. Let S be the usual topology for X, and let $T = \{ U \mid U \in S, 0 \notin U \} \cup \{ Y \}$. Define $f:(X,S) \rightarrow (Y,T)$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases} .$$

Then f is near-continuous and not continuous anywhere.

Proof: Let $\mathcal{U}$ be an open cover of Y. Then $Y \in \mathcal{U}$ since Y is the only set in T which contains zero. Let D = { X }. Now D is an open cover of X and $f(X) \subset Y$. Therefore, f is near-continuous. Let $x \in X$. Then either x is rational or x is irrational. If x is rational then f(x) = 1. Let $U = (1/2,3/4) \in T$. Now $f^{-1}(U)$ is the set of all rational numbers, which is not an open set of S. If x is irrational then f(x) = -1. Let $U = (-3/4,-1/2) \in T$. Now $f^{-1}(U)$ is the set of all irrational numbers which again is not an open set of S. Hence, f is not continuous anywhere.

Example 5: Let X = Y = reals. Let S be the usual topology for X and let $T = \{ (a,\infty) \mid a \in Y \} \cup \{ [a,\infty) \mid a \in Y \} \cup \{ \phi, Y \}$. Define $f:(X,S) \to (Y,T)$ by f(x) = x for all $x \in X$. Then f is one-to-one, onto, a near-continuous function which is not continuous anywhere, and (Y,T) is strongly - T_0.

Proof: Since f is the identity function, f is one-to-one and onto. Let $\mathcal{U}$ be an open cover of Y. Let $p \in X$. Then there is a $U_p \in \mathcal{U}$ such that $p - 1 \in U_p$. Then either $U_p = (a,\infty)$ or $U_p = [a,\infty)$ for some $a \in$ reals, $a \leq p - 1$. Let $V_p = (p - 1/2,\infty)$. Then $p \in V_p \subset U_p$ and V_p is open. Let $\mathcal{V} = \{ V_p \mid p \in X \}$. Since each V_p is open and if $p \in X$ then $p \in V_p$, $\mathcal{V}$ is an open cover of X. Let $V_p \in \mathcal{V}$. Then since f is the identity function, $f(V_p) = V_p \subset U_p$. Hence, f is near-continuous. Let $x,y \in Y$ and $x < y$. Then $[y,\infty)$ is an open set containing y and not containing x. Thus, Y is a T_0 - space. Let $\mathcal{W} = \{ W \mid W = [-N,\infty)$, N is a positive integer $\}$. If $W \in \mathcal{W}$ then W is open. Now if $z \in Y$ then there is a positive integer, N, such that $-N < z$. Thus $z \in [-N,\infty)$ and therefore $\mathcal{W}$ is an open cover of Y and Y is not an element of $\mathcal{W}$. Therefore, (Y,T) is a T_0 - space and a T_{-1} - space and hence, (Y,T) is a strongly T_0 - space. Let $p \in X$ and let $U = [p,\infty)$. Then $U \in T$ and $f(p) = p \in U$, and $f^{-1}(U) = [p,\infty)$ which is not an element of S. Hence, f is not continuous anywhere.

<u>Theorem 5</u>: If (X,S) and (Y,T) are topological spaces and $f:(X,S) \to (Y,T)$ is near-continuous and $A \subset X$, then $f|_A:(A,S_A) \to (Y,T)$ is near-continuous (where S_A is the relative topology on A induced by S).

<u>Proof</u>: Let $\mathcal{U}$ be an open cover of Y. Then there is an open cover $\mathcal{V}$ of X such that if $V \in \mathcal{V}$, then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Let $\mathcal{W}$ { $V \cap A \mid V \in \mathcal{V}$ }. Now if $W \in \mathcal{W}$ then $W = V \cap A$ for some $V \in \mathcal{V}$. There is a $U \in \mathcal{U}$ such that $f(V) \subset U$. But $f(W) = f(V \cap A) \subset f(V) \subset U$. Hence, $f|_A$ is near-continuous.

<u>Theorem 6</u>: If (X,S) and (Y,T) are topological spaces and $f:(X,S) \to (Y,T)$ is a function, and A and B are open subsets of X such that $X = A \cup B$, and $f|_A$ and $f|_B$ are near-continuous, then f is near-continuous.

<u>Proof</u>: Let $\mathcal{U}$ be an open cover of Y. Then there is an open cover $\mathcal{V}_1$ of A by elements in S_A such that if $V \in \mathcal{V}_1$ then there is a $U \in \mathcal{U}$ such that $f|_A(V) \subset U$. There is an open cover $\mathcal{V}_2$ of B by elements in S_B such that if $V \in \mathcal{V}_2$ then there is a $U \in \mathcal{U}$ such that $f|_B(V) \subset U$. Now if $V \in \mathcal{V}_1$, then $V = A \cap W$ where $W \in S$. Since A is open, $V = A \cap W$ is open. If $V \in \mathcal{V}_2$ then $V = B \cap W$ where $W \in S$. Since B is open, $V = B \cap W$ is open. Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. Since, $A \cup B = X$ and $\mathcal{V}_1$ covers

A and $\mathcal{V}_2$ covers B then $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ covers X. Let $V \in \mathcal{V}$. Then either $V \in \mathcal{V}_1$ or $V \in \mathcal{V}_2$, say $V \in \mathcal{V}_1$. Then there is a $U \in \mathcal{U}$ such that $f|_A(V) \subset U$. But since $V \in \mathcal{V}_1$, $V \subset A$ and $f(A) = f(V \cap A) = f|_A(V) \subset U$. Hence, f is near-continuous.

Example 6: Let X be the reals and let S be the usual topology on X. Let $Y = \{a,b,c\}$ and let $T = \{ \phi, Y, \{b\}, \{a,b\}, \{b,c\} \}$. Let $A = (-\infty,0]$, $B = [0,\infty)$ and define $f:(X,S) \to (Y,T)$ by

$$f(x) = \begin{cases} a & \text{if } x < 0 \\ b & \text{if } x = 0 \\ c & \text{if } x > 0 \end{cases} .$$

Then A and B are closed subsets of X such that $X = A \cup B$ and $f|_A$ and $f|_B$ are near-continuous, and yet f is not near-continuous.

Proof: The interval $(0,\infty)$ and $(-\infty,0)$ are open in X. Thus $X - (0,\infty) = A$ and $X - (-\infty,0) = B$ are closed. Now $A \cup B = (-\infty,0] \cup [0,\infty) = X$. Let $\mathcal{U}$ be an open cover of Y. Then either $Y \in \mathcal{U}$ or $\{a,b\}$ and $\{b,c\}$ are elements of $\mathcal{U}$. If $Y \in \mathcal{U}$ then $\{ A \}$ and $\{ B \}$ are open covers of A and B respectively such that $f|_A(A) \subset Y$ and $f|_B(B) \subset Y$. If $\mathcal{U}$ contains the sets $\{a,b\}$ and $\{b,c\}$ then again $\{ A \}$ and $\{ B \}$ are open covers of A and B respectively and $f|_A(A) \subset \{a,b\}$ and $f_A(B) \subset \{b.c\}$. Hence, $f|_A$ and $f|_B$ are near-continuous. Suppose f is near-continuous, and let $\mathcal{U} = \{ \{a,b\}, \{b,c\} \}$. Since $\{a,b\}$ and $\{b,c\}$ are elements of T, $\mathcal{U}$ is an open cover of Y. Since f is near-continuous, there is an open cover $\mathcal{V}$ of X such that if $V \in \mathcal{V}$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Let V be the open set of $\mathcal{V}$ containing zero. Since V is a usual open subset of the reals, there is an open interval, (r,s), such that $0 \in (r,s)$ and $(r,s) \subset V$. Then $f((r,s)) = f(V) = \{a,b,c\}$ which is not contained in $\{a,b\}$ or $\{b,c\}$. This is a contradiction and hence, f is not near-continuous.

<u>Theorem 7</u>: If (X,S) and (Y,T) are topological spaces and $f:(X,S) \to (Y,T)$ is a function, and A and B are closed subsets of X such that $X = A \cup B$ and f_A is near-continuous and $f|_B$ is continuous, then f is near-continuous.

<u>Proof</u>: Let $\mathcal{U}$ be an open cover of Y. Since $f|_A$ is near-continuous, there is an open cover $\mathcal{V}_1$ of A by elements in S_A such that if $V \in \mathcal{V}_1$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Let $\alpha = \{ V \cap (X - B) \mid V \in \mathcal{V}_1 \}$. Let $V \in \mathcal{V}_1$. Then there is $\mathcal{O} \in S$ such that $V = A \cap \mathcal{O}$. Since $X - A \subset A$, then $V \cap (X - B) = (A \cap \mathcal{O}) \cap (X - B) = \mathcal{O} \cap (X - B)$. Since B is closed, $X - B$ is open and thus $\mathcal{O} \cap (X - B) = V \cap (X - B)$ is open. Since $X - B \subset A$ and $\mathcal{V}_1$ covers A, α is an open cover of $X - B$. Now if $W \in \alpha$ then $W = V \cap (X - B)$ for some $V \in \mathcal{V}_1$. There is a $U \in \mathcal{U}$ such that $f(V) \subset U$. But $W \subset V$ and hence $f(W) \subset f(V) \subset U$.

Let $b \in X - A$. Then there is an element $U \in \mathcal{U}$ such that $f(b) \subset U$. Since $X - A \subset B$ and $f|_B$ is continuous, there is an element $N_b \in S_b$ containing b such that $f(N_b) \subset U$. There is an open set $\mathcal{O} \in S$ such that $N_b = \mathcal{O} \cap B$. Let $M_b = \mathcal{O} \cap (X - A)$. Now $b \in N_b = \mathcal{O} \cap B$. Thus $b \in \mathcal{O}$ and therefore $b \in \mathcal{O} \cap (X - A) = M_b$. Since $X - A \subset B$, $M_b \subset N_b$. Let $\beta = \{ M_b \mid b \in X - A \}$. If $M_b \in \beta$ then $M_b = \mathcal{O} \cap (X - A)$ for some $\mathcal{O} \in S$. Since A is closed, $X - A$ is open and therefore

$\mathcal{O} \cap (X - A) = M_b$ is open. If $b \in X - A$ then $b \in M_b$. Therefore β is an open cover of $X - A$. Now let $M_b \in \beta$. Then $M_b \subset N_b$ and there is a $U \in \mathcal{U}$ such that $f(N_b) \subset U$. Hence $f(M_b) \subset f(N_b) \subset U$. Let $p \in A \cap B$. Then since $\mathcal{V}_1$ covers A there is a $V \in \mathcal{V}_1$ such that $p \in V$. Since $f|_A$ is near-continuous, there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Since $f|_B$ is continuous at p, there is a $W \in S_B$ such that $p \in W$ and $f(W) \subset U$. Since $V \in S_A$, there is an $\mathcal{O}_1 \in S$ such that $V = \mathcal{O}_1 \cap A$. There is $\mathcal{O}_2 \in S$ such that $W = \mathcal{O}_2 \cap B$. Let $\mathcal{O}_p = \mathcal{O}_1 \cap \mathcal{O}_2$. Let $x \in \mathcal{O}_p$. If $x \in A$, then $x \in V$ and thus $f(x) \in U$. If $x \in B$, then $x \in W$ and thus $f(x) \in U$. Therefore, $f(\mathcal{O}_p) \subset U$. Since $p \in V$ and $p \in W$ then $p \in \mathcal{O}_1 \cap A$ and $p \in \mathcal{O}_2 \cap B$. Therefore $p \in \mathcal{O}_1$ and $p \in \mathcal{O}_2$ and thus $p \in \mathcal{O}_1 \cap \mathcal{O}_2 = \mathcal{O}_p$. Let $\gamma = \{ \mathcal{O}_p \mid p \in A \cap B \}$. Now if $\mathcal{O}_p \in \gamma$ then $\mathcal{O}_p$ is open, and if $x \in A \cap B$ then $x \in \mathcal{O}_x$. Thus γ is an open cover of $A \cap B$. Let $\mathcal{O}_p \in \gamma$ then there is a $U \in \mathcal{U}$ such that $f(\mathcal{O}_p) \subset U$. Now $\alpha \cup \beta \cup \gamma$ is an open cover of X and if $V \in \alpha \cup \beta \cup \gamma$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Hence, f is near-continuous.

<u>Theorem 8</u>: If (X,S) and (Y,T) are topological spaces, and $f:(X,S) \to (Y,T)$ is a function. If A, B, and C are closed subsets of X with $A \cap C = \phi$, and $X = A \cup B \cup C$, and $f|_A$ and $f|_C$ are near-continuous and f_B is continuous, then f is near-continuous.

<u>Proof</u>: By Theorem 7, $f|_{A \cup B}$ and $f|_{B \cup C}$ are near-continuous. Let $\mathcal{U}$ be an open cover of Y. Then there are open covers $\mathcal{V}_1$ and $\mathcal{V}_2$ of $A \cup B$ and $B \cup C$ respectively by elements of $S_{A \cup B}$ and $S_{B \cup C}$ respectively such that if $V \in \mathcal{V}_1$ or $V \in \mathcal{V}_2$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Let $\alpha = \{ V \cap (X - C) \mid V \in \mathcal{V}_1 \}$ and let

$\beta = \{ V \cap (X - A) \mid V \in \mathcal{V}_2 \}$. If $W \in \alpha$ then $W = V \cap (X - C)$ for $V \in \mathcal{V}_1$. Since $V \in \mathcal{V}_1$, there is an open set $\mathcal{O}$ such that $V = \mathcal{O} \cap (A \cup B)$. Thus $W = (\mathcal{O} \cap (A \cup B)) \cap (X - C)$. Since $(X - C) \subset A \cup B$ then $W = \mathcal{O} \cap (X - C)$. But $(X - C)$ is open since C is closed and thus W is open. If $x \in X - C$ then $x \in A \cup B$ since $X - C \subset A \cup B$. Thus $x \in V \cap (X - C)$ for some $V \in \mathcal{V}_1$. Hence α is an open cover of $X - C$. Similarly β is an open cover of $X - A$. Let $x \in X$. Then $x \in A$ or $x \notin A$. If $x \in A$ then $x \in X - C$ since $A \cap C = \phi$. Thus there is a $W \in \alpha$ such that $x \in W$. If $x \notin A$ then $x \in X - A$ since $A \cap C = \phi$, and thus there is a $W \in \beta$ such that $x \in W$. Therefore $\alpha \cup \beta$ is an open cover of X. Let $W \in \alpha \cup \beta$. Then either $W \in \alpha$ or $W \in \beta$ or both. If $W \in \alpha$ then $W = V \cap (X - C)$ for some $V \in \mathcal{V}_1$. Now there is a $U \in \mathcal{U}$ such that $f(V) \subset U$ since $f|_{A \cup B}$ is near-continuous. Thus since $V \cap (X - C) \subset V$, $f(W) = f(V \cap (X - C)) \subset U$. Similarly if $W \in \beta$ there is a $U \in \mathcal{U}$ such that $f(W) \subset U$. Hence, f is near-continuous.

<u>Theorem 9</u>: If (X,S) and (Y,T) are topological spaces, and $f:(X,S) \to (Y,T)$ is a function. If A, B, and C are closed subsets of X with $A \cap C = \phi$, and $X = A \cup B \cup C$, and $f|_A$ and $f|_C$ are continuous and $f|_B$ is near-continuous then f is near-continuous.

<u>Proof</u>: By Theorem 7, $f|_{A \cup B}$ and $f|_{B \cup C}$ are near-continuous. Let $\mathcal{U}$ be an open cover of X. Then there are open covers $\mathcal{V}_1$ and $\mathcal{V}_2$ of $A \cup B$ and $B \cup C$ respectively by elements of $S_{A \cup B}$ and $S_{B \cup C}$ respectively such that if $V \in \mathcal{V}_1$ or $V \in \mathcal{V}_2$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Let $\alpha = \{ V \cap (X - C) \mid V \in \mathcal{V}_1 \}$ and let

$\beta = \{ V \cap (X - A) \mid V \in \mathcal{V}_2 \}$. If $W \in \alpha$ then $W = V \cap (X - C)$ for some $V \in \mathcal{V}_1$. Since $V \in \mathcal{V}_1$, there is an open set $\mathcal{O}$ in S such that $V = \mathcal{O} \cap (A \cup B)$. Thus $W = (\mathcal{O} \cap (A \cup B)) \cap (X - C)$. Since $X - C \subset A \cup B$ then $W = \mathcal{O} \cap (X - C)$. But $X - C$ is open since C is closed, and thus W is open. If $x \in X - C$ then $x \in A \cup B$ since $X - C \subset (A \cup B)$. Therefore $x \in V$ for some $V \in \mathcal{V}_1$. Thus $x \in V \cap (X - C)$ for some $V \in \mathcal{V}_1$. Hence α is an open cover of $X - C$. Similarly β is an open cover of $X - A$. Let $x \in X$. Then either $x \in A$ or $x \notin A$. If $x \in A$ then $x \in X - C$ since $A \cap C = \phi$. Thus there is a $W \in \alpha$ such that $x \in W$. If $x \notin A$ then $x \in X - A$, and thus there is a $W \in \beta$ such that $x \in W$. Therefore $\alpha \cup \beta$ is an open cover of X. Let $W \in \alpha \cup \beta$. Then either $W \in \alpha$ or $W \in \beta$ or both. If $W \in \alpha$, then $W = V \cap (X - C)$ for some $V \in \mathcal{V}_1$. Now, there is a $U \subset \mathcal{U}$ such that $f(V) \subset U$, since $f|_{A \cup B}$ is near-continuous. Thus since $V \cap (X - C) \subset V$, $f(W) = f(V \cap (X - C)) \subset U$. Similarly if $W \in \beta$ there is a $U \in \mathcal{U}$ such that $F(V) \subset U$. Hence, f is near-continuous.

CHAPTER II

Theorem 10: If $f:(X,S) \to (Y,T)$ is near-continuous and onto and (X,S) is compact, then (Y,T) is compact.

Proof: Let $\mathcal{U}$ be an open cover of (Y,T). Then there is an open cover $\mathcal{V}$ of (X,S) such that if $V \in \mathcal{V}$, then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Since (X,S) is compact there is a finite subcover $V_1, V_2, V_3, \cdots, V_N$ of $\mathcal{V}$ which covers X. If $1 \leq i \leq N$, then there is an element $U_i \in \mathcal{U}$ such that $f(V_i) \subset U_i$. Now let $y \in Y$. Then since f is onto, there is a $x \in X$ such that $f(x) = y$. Now $x \in V_i$ for some i, $1 \leq i \leq N$. Thus $y \in f(V_i) \subset U_i$. Therefore the subcollection $U_1, U_2, U_3, \cdots, U_N$, of $\mathcal{U}$ covers Y. Hence, (Y,T) is compact.

Definition 5: A topological space (X,T) is said to be connected provided X cannot be represented as the union of two non-empty, open sets.

Theorem 11: If $f:(X,S) \to (Y,T)$ is near-continuous and onto and (X,S) is connected, then (Y,T) is connected.

Proof: Suppose (Y,T) is not connected. Then $Y = A \cup B$ where A and B are disjoint, non-empty, open sets. Thus $\mathcal{U} = \{ A, B \}$ is an open cover of Y. Since f is near-continuous, there is an open cover $\mathcal{V}$ of X such that if $V \in \mathcal{V}$ then there is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Let $M = \cup\{ V \mid V \in \mathcal{V}, f(V) \subset A \}$ and let $N = \cup\{ V \mid V \in \mathcal{V}, f(V) \subset B \}$. Let $y \in A$. Then since f is onto, there is an $x \in X$ such that $f(x) = y$. Since $\mathcal{V}$ is an open cover of X, there is a $V \in \mathcal{V}$ such that $x \in V$. Since

$f(V) \subset A$, $M \neq \phi$. Similarly, $N \neq \phi$. The sets M and N are the union of open sets and thus are open. Suppose $p \in M \cup N$. Then there is a $W \in \mathcal{V}$ such that $p \in W \subset M$ and $p \in V \subset N$. But $f(W) \subset A$ and $f(V) \subset B$. Thus $f(p) \in A \cap B$. But this contradicts $A \cap B = \phi$. Therefore $M \cap N = \phi$. Now M and N are open, non-empty, disjoint sets such that $M \cup N = X$. But this is impossible since X is connected and hence, (Y,T) is connected.

Definition 6: A topological space (X,T) is said to be separable provided there is a subset M of X which is countable and if $U \in T$ then there is a $p \in M$ such that $p \in U$.

Example 7: Let $X = Y = [-1,1)$, let R = reals, let S be the topology on R generated by $\{ (a,b] \mid a < b \}$, and let S_1 be the topology on X induced by S. Let $T = \{ U \mid U \subset Y, 0 \notin U \} \cup \{ (-1,1), Y \}$. Define $f:(X,S_1) \to (Y,T)$ by $f(x) = x$ for all $x \in X$. Then (Y,T) is strongly - T_0, f is one-to-one, onto, and near-continuous, and (X,S_1) is separable and yet (Y,T) is not separable.

Proof: The set $D = \{ (-1,1), \{1\} \}$ is a subset of T and $Y = \{ (-1,1) \} \cup \{-1\}$. Thus there is an open cover of Y which does not contain Y, and hence, (Y,T) is a T_{-1} - space. Let $x,z \in Y$. Then either x or z is not zero, say $x \neq 0$. Then $\{x\} \in T$ and $y \notin \{x\}$. Thus Y is T_0. Hence (Y,T) is strongly - T_0. Since f is the identity function, f is one-to-one and onto. Let $\mathcal{U}$ be an open cover of Y. Then either $\{ Y \} \in \mathcal{U}$ or $\mathcal{U}$ contains the set, $\{ (-1,1) \}$, and a set W which contains -1. If $\{ Y \} \in \mathcal{U}$ then let $D = \{ X \}$. Then D is an open cover of X and $f(X) \subset Y$. If $\mathcal{U}$ contains the set, $\{ (-1,1) \}$, and a set W which contains -1 then $(-2,-1] \in S$ and therefore, $X \cap (-2,-1] \in S_1$. Therefore $\{-1\}$ is open in X. Let $C = \{ \{-1\}, (-1,1) \}$. Then C is an open cover of X and $f(\{-1\}) \subset W$ and $f(\{ (-1,1) \}) \subset (-1,1)$. Therefore, f is near-continuous. Now X is separable since the rational numbers are a countably dense subset in X. Suppose Y is separable. Then there is a countable dense subset H of Y such that if $U \in T$ then there is a $p \in H$ such that $p \in U$. Now if $p \in [-1,1)$ and $p \neq 0$, then $\{p\} \in T$. Therefore, $p \in H$. B ut this is impossible, since by [6, Cor.; p.36], $Y - \{0\}$ is uncountable. Hence, Y is not separable.

CHAPTER III

Definition 7: If (X,T) is a topological space, then a subset of X is said to be a near-continuous retract of X provided there is a near-continuous function, $f:X \to A$ such that f is the identity on A (i.e. $f(x) = x$ for all $x \in A$).

Theorem 12: A near-continuous retract B of a near-continuous retract A of X is a near-continuous retract of X.

Proof: Since A is a near-continuous retract of X, there is a near-continuous function $f:X \to A$ such that f is the identity on A. Since B is a near-continuous retract of A, there is a near-continuous function $g:A \to B$ such that g is the identity on B. Then by Theorem 4, gf is a near-continuous function. Let $b \in B$. Then since $B \subset A$, $gf(b) = g(f(b)) = g(b) = b$. Hence B is a near-continuous retract of X.

Theorem 13: If X is a space and A is a subset of X, then A is a near-continuous retract of X, if and only if for every space Y, each near-continuous function $f:A \to Y$ can be extended to a near-continuous function $F:X \to Y$.

Proof: (1) Suppose A is a near-continuous retract of X. Let $g:X \to A$ be a near-continuous function such that g ia the identity on A. Let Y be a space and let $f:A \to Y$ be a near-continuous function. Then by Theorem 4, $fg:X \to Y$ is near-continuous. Let $a \in A$. Then $fg(a) = f(g(a)) = f(a)$. Therefore, fg is an extension of f.

(2) Suppose that for every Y, each near-continuous function

$f:A \to Y$ can be extended to a near-continuous function, $F:X \to Y$. Define $f:A \to A$ by $f(a) = a$ for all $a \in A$. Then f is continuous since f is the constant function, and by Theorem 1, f is near-continuous. Therefore, there is a near-continuous function $F:X \to A$ such that $F|_A = f$. Then $F(a) = f(a) = a$. Hence, A is a near-continuous retract of X.

Theorem 14: If X is a space, and X has the near-continuous fixed point property (i.e. If $f:X \to X$ is near-continuous, then there is an element $x \in X$ such that $f(x) = x$) and A is a near-continuous retract of X, then A has the near-continuous fixed point property.

Proof: Let $f:A \to A$ be a near-continuous function. Since A is a near-continuous retract of X, by Theorem 12, f can be extended to a near-continuous function $F:X \to A$. Since X has the near-continuous fixed point property, there is a element $x \in X$ such that $F(x) = x$. Since the range of F is a subset of A, $x \in A$. Thus $f(x) = F(x) = x$, and x is a fixed point of f. Therefore, A has the near-continuous fixed point property.

Example 8: Let $X = \{a,b,c\}$ and let $T = \{ \phi, X, \{a,b\}, \{b\}, \{b,c\} \}$. Then (X,T) is strongly - T_0 and (X,T) has the fixed point property, but (X,T) does not have the near-continuous fixed point property.

Proof: By Example 3, (X,T) is strongly - T_0. Now the set of all continuous functions from X into X are given in Table 1. Each function has a fixed point. Thus (X,T) has the fixed point property.

Table 1

X	f(X)										
a	a	a	a	a	b	b	b	c	c	c	c
b	a	b	b	b	b	b	b	c	b	b	b
c	a	a	c	b	b	a	c	c	c	a	b

The first column is the domain of the functions. Each column after the first represents the range of a possible continuous function.

Define f:(X,T) → (X,T) by f(a) = b, f(b) = a, and f(c) = a. Let $\mathcal{U}$ be an open cover of X. Then X is contained in $\mathcal{U}$ or {a,b}, and {b,c} are contained in $\mathcal{U}$. If X ∈ $\mathcal{U}$ then let D = { X }. Now D is an open cover of X such that f(X) ⊂ X. If {a,b} and {b,c} are elements of $\mathcal{U}$, then let D = { X }. Then again D is an open cover of X and f(X) ⊂ {b,c}. Hence, f is near-continuous. But f does not have a fixed point. Hence, f does not have the near-continuous fixed point property.

Example 9: Let X = [0,1] and let T = { U | U ⊂ X, 0 ∈ U } ∪ { ϕ }. Then (X,T) is not a T_1 - space and (X,T) has the near-continuous fixed point property.

Proof: Let x ∈ X, and x ≠ 0. Then since any open set U that contains x contains 0, X is not a T_1 - space. Suppose f:(X,T) → (X,T) is near-continuous, and f does not have a fixed point. Then there is a z ∈ X such that f(0) = z and z ≠ 0. Also there is a y ∈ X such that f(z) = y and y ≠ z. Let $\mathcal{U}$ = { {p,0} | p ∈ X }. Now if x ∈ X then x ∈ {x,0} ∈ $\mathcal{U}$. Since {x,o} ∈ T, $\mathcal{U}$ is an open cover of X. Since f is near-continuous, there is an open cover $\mathcal{V}$ of (X,T) such that if V ∈ $\mathcal{V}$, then there is a U ∈ $\mathcal{U}$ such that f(V) ⊂ U. Then $\mathcal{V}$ cover X, there is a

$V \in \mathcal{V}$ such that $x \in V$. There is a $U \in \mathcal{U}$ such that $f(V) \subset U$. Since $f(z) = y$, $U = \{y,0\}$ where y may be 0. Now, $\{z,0\} \subset V$ since $0 \in V$. Thus $f(\{z,0\}) \subset f(V) \subset U$. Hence, $f(0) \in U$ and therefore, $f(0) = y$ or $f(0) = 0$. If $f(0) = y$ then $z = y$ which is a contradiction. If $f(0) = 0$ then f has a fixed point which is a contradiction. Hence, f has a fixed point, and thus f has the near-continuous fixed point property.

CHAPTER IV

Definition 8: If A and B are sets then the cross-product of A and B, denoted by $A \times B$, is the set of all ordered pairs of the form (a,b) where $a \in A$ and $b \in B$. The symbol I in the following chapters will denote the closed unit interval under the usual topology.

Definition 9: Let (Y,T) be a topological space and let $y_0 \in Y$. Then $C(Y,y_0)$ is the set of all continuous functions $F:I \to Y$ such that $F(0) = y_0 = F(1)$. Let $f,g \in C(Y,y_0)$. We say f is continuous homotopic to g modulo y_0, denoted $f \underset{y_0}{\sim} g$, provided there is a continuous function $F:I \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$. We say f is near-continuous homotopic to g modulo y_0, denoted by $f \underset{y_0}{\overset{n}{\sim}} g$, provided there is a near-continuous function $F:I \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$.

Lemma 9.1: The functions $K:I \times I \to I \times I$ by $K(x,t) = (x,1 - t)$ for all $(x,t) \in I \times I$, $\alpha:I \times [0,1/4] \to I \times I$ by $\alpha(x,t) = (x,4t)$ for all $(x,t) \in I \times [0,1/4]$, and $\beta:I \times [3/4,1] \to I \times I$ by $\beta(x,t) = (x,4t - 3)$ for all $(x,t) \in I \times [3/4,1]$ are homeomorphisms.

Proof: (1) Let $\varepsilon > 0$ and $(x,y) \in I \times I$. Let $\delta = \varepsilon/2$ and $(r,s) \in I \times I$ such that $|x - r| < \delta$ and $|y - r| < \delta$ then

$$|K(r,s) - K(x,y)| = |(r,1 - s) - (x,1 - y)| = |(r - x,y - s)| \le |r - x| + |y - s| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus K is continuous. Let (x,y) and $(r,s) \in I \times I$ such that

$K(r,s) = K(x,y)$. Then $(r,1 - s) = (x,1 - y)$, and thus $r = x$ and $s = y$. Hence, K is one-to-one. Let $(r,s) \in I \times I$. Then $(r,1 - s) \in I \times I$ and $K(r,1 - s) = (r,s)$. Hence, K is onto. Let $(x,y) \in I \times I$. Then $KK(x,y) = K(K(x,y)) = K(x,1 - y) = (x,y)$. Hence, K is the inverse of K, and thus the inverse of K is continuous. Therefore, K is an homeomorphism.

(2) Let $\varepsilon > 0$ and let $(x,y) \in I \times [0,1/4]$. Let $\delta = \varepsilon/8$ and $(r,s) \in I \times [0,1/4]$ such that $|x - r| < \delta$ and $|y - s| < \delta$. Then

$$|\alpha(r,s) - \alpha(x,y)| = |(r,4s) - (x,4y)| = |(r -x,4(s - y))| \leq$$
$$|(r - x)| + 4|s - y| \leq \varepsilon/8 + \varepsilon/2 < \varepsilon.$$

Hence, α is continuous. Let (r,s) and $(x,y) \in I \times [0,1/4]$ such that $\alpha(r,s) = \alpha(x,y)$. Then $(r,4s) = (x,y)$, and thus $x = r$ and $s = y$. Hence, α is one-to-one. Let $(r,s) \in I \times I$ then $(r,s/4) \in I \times [0,1/4]$ and $\alpha(r,s/4) = (r,s)$. Therefore, α is onto. Define $\gamma:I \times I \to I \times [0,1/4]$ by $\gamma(r,s) = (r,s/4)$ for all $(r,s) \in I \times I$. Then $\alpha\gamma(x,y) = \gamma\alpha(x,y) = (x,y)$ for all $(x,y) \in I \times I$. Thus γ is the inverse of α. Let $(x,y) \in I \times I$. Let $\delta = \varepsilon/2$ and let $(r,s) \in I \times I$ such that

$$|\gamma(r,s) - \gamma(x,y)| = |(r,s/4) - (x,y/4)| =$$
$$|((r - x),(1/4)(s - y))| \leq |r - x| + (1/4)|s - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, γ is continuous, and thus α is an homeomorphism.

(3) Let $\varepsilon > 0$ and let $(x,y) \in I \times [3/4,1]$. Let $\delta = \varepsilon/8$ and let $(r,s) \in I \times [3/4,1]$ such that $|x - r| < \delta$ and $|y - s| < \delta$. Then

$$|\beta(r,s) - \beta(x,y)| = |(r,4s - 3) - (x,4y - 3)| =$$
$$|((r - s),4(s - y))| \leq |r - s| + 4|s - y| < \varepsilon/8 + \varepsilon/2 < \varepsilon.$$

Hence, β is continuous. Let (r,s) and $(x,y) \in I \times [3/4,1]$ such that

$\beta(r,s) = \beta(x,y)$. Then $(r,4s - 3) = (x,4y - 3)$, and thus $x = r$ and $y = s$. Hence, β is one-to-one. Let $(x,y) \in I \times I$ then $(x,1/4y + 3/4) \in I \times [3/4,1]$ and $\beta(x,(1/4)y + 3/4) = (x,y)$. Hence, β is onto. Define $\gamma:I \times I \to I \times [3/4,1]$ by $\gamma(x,y) = (x,(1/4)y + 3/4)$ for all $(x,y) \in I \times I$. Then $\beta\gamma(x,y) = \gamma\beta(x,y) = (x,y)$, and thus γ is the inverse of β. Let $(r,s) \in I \times I$ such that $|x - r| < \delta$ and $|y - s| < \delta$. Then

$$|\gamma(r,s) - \gamma(x,y)| = |((r - x),(1/4)(s - y))| \leq$$
$$|r - s| + (1/4)|s - y| < \varepsilon/2 + \varepsilon/8 < \varepsilon.$$

Hence, γ is continuous, and thus β is a homeomorphism.

Theorem 15: The relation $\overset{n}{\underset{y_0}{\sim}}$ is an equivalence relation on $C(Y,y_0)$.

Proof: Let $f \in C(Y,y_0)$. Since $f \underset{y_0}{\sim} f$ and every continuous function is near-continuous by Theorem 1, $f \overset{n}{\underset{y_0}{\sim}} g$.

Let $f,g \in C(Y,y_0)$ and suppose that $f \overset{n}{\underset{y_0}{\sim}} g$. Then there is a near-continuous function $F:I \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$. Define $G:I \times I \to Y$ by $G(x,t) = F(x,1 - t)$ for all $(x,t) \in I \times I$. Then $G(x,0) = F(x,1) = g(x)$ and $G(x,1) = F(x,0) = f(x)$ for all $x \in I$. Define $K:I \times I \to I \times I$ by $K(x,t) = (x,1 - t)$ for all $(x,t) \in I \times I$. Then by Lemma 9.1, K is an homeomorphism. By Theorem 1, K is near-continuous. But $FK(x,t) = F(K(x,t)) = F(x,1 - t) = G(x,t)$ for all $(x,t) \in I \times I$. Therefore, by Theorem 4, G is a near-continuous. Hence, $g \overset{n}{\underset{y_0}{\sim}} f$.

Let $f,g,h \in C(Y,y_0)$ and suppose that $f \overset{n}{\underset{y_0}{\sim}} g$ and $g \overset{n}{\underset{y_0}{\sim}} h$. Then there are near-continuous functions, $F,G:I \times I \to Y$ such that $F(x,0) = f(x)$,

$F(x,1) = g(x)$, and $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$, and $G(x,0) = g(x)$, $G(x,1) = h(x)$, and $G(0,t) = y_0 = G(1,t)$ for all $x \in I$, $t \in I$. Define $H: I \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x,4t) & \text{if } x \in I, \quad 0 \le t \le 1/4 \\ g(x) & \text{if } x \in I, \ 1/4 \le t \le 3/4 \\ G(x,4t-3) & \text{if } x \in I, \ 3/4 \le t \le 1 \end{cases} .$$

Define $\alpha: I \times [0,1/4] \to I \times I$ by $\alpha(x,t) = (x,4t)$ for all $(x,t) \in I \times [0,1/4]$ and define $\beta: I \times [3/4,1] \to I \times I$ by $\beta(x,t) = (x,4t-3)$ for all $(x,t) \to I \times [3/4,1]$. Now if $(x,t) \in I \times [0,1/4]$ then $F\alpha(x,t) = F(\alpha(x,t)) = F(x,4t) = H|_{I \times [0,1/4]}$. If $(x,t) \in I \times [3/4,1]$ then $G\beta(x,t) = G(\beta(x,t)) = G(x, 4t-3) = H|_{I \times [3/4,1]}$. Since F and G are near-continuous and by Lemma 9.1, α and β are homeomorphisms, then by Theorem 1 and Theorem 4, $H|_{I \times [0,1/4]}$ and $H|_{I \times [3/4,1]}$ are near-continuous. Since $H|_{I \times [1/4,3/4]} = g$ and since g is continuous, $H|_{I \times [1/4,3/4]}$ is continuous. By Theorem 8, H is near-continuous. Now, $H(x,0) = F(x,0) = f(x)$ and $H(x,1) = G(x,1) = h(x)$ for all $x \in I$. Also,

$$H(0,t) = \begin{cases} F(0,4t) & \text{if } \quad 0 \le t \le 1/4 \\ g(0) & \text{if } \ 1/4 \le t \le 3/4 = y_0 \\ G(0,4t-3) & \text{if } \ 3/4 \le t \le 1 \end{cases}$$

for all $x \in I$, and

$$H(1,t) = \begin{cases} F(1,4t) & \text{if } \quad 0 \le t \le 1/4 \\ g(1) & \text{if } \ 1/4 \le t \le 3/4 = y_0 \\ G(1,4t-3) & \text{if } \ 3/4 \le t \le 1 \end{cases}$$

for all $x \in I$. Therefore, $f \overset{n}{\underset{y_0}{\sim}} h$. Hence, $\overset{n}{\underset{y_0}{\sim}}$ is an equivalence relation.

Definition 10: Let $f,g \in C(Y,y_0)$. Then $f *_n g$ is the function in $C(Y,y_0)$ defined by,

$$(f *_n g)(x) = \begin{cases} f(4x) & \text{if } 0 \le x \le 1/4 \\ y_0 & \text{if } 1/4 \le x \le 3/4 \\ g(4x - 3) & \text{if } 3/4 \le x \le 1 \end{cases}.$$

The equivalence relation $\overset{n}{\underset{y_0}{\sim}}$ partitions $C(Y,y_0)$ into disjoint equivalence classes. Let $N(Y,y_0)$ be this set of equivalence classes. Let $[f],[g] \in N(Y,y_0)$. Then we define $[f] \cdot [g]$ to be $[f *_n g]$.

Lemma 10.1: Let $f,g \in C(Y,y_0)$. Define $f * g$ to be the function

$$(f * g)(x) = \begin{cases} f(2x) & \text{if } 0 \le x \le 1/2 \\ g(2x - 1) & \text{if } 1/2 \le x \le 1 \end{cases}.$$

Then $f *_n g \underset{y_0}{\sim} f * g$.

Proof: Define $H: I \times I \to Y$ by

$$H(x,t) = \begin{cases} f(4x/(t + 1)) & \text{if } t \ge 4x - 1 \\ y_0 & \text{otherwise} \\ g((4x + t - 3)/(t + 1)) & \text{if } t \ge -4x + 3 \end{cases}.$$

Now

$$H(x,0) = \begin{cases} f(4x) & \text{if } 0 \le x \le 1/4 \\ y_0 & \text{otherwise} \\ g(4x - 3) & \text{if } 3/4 \le x \le 1 \end{cases} = (f *_n g)(x)$$

for all $x \in I$, and

$$H(x,1) = \begin{cases} f(2x) & \text{if } 0 \le x \le 1/2 \\ g(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases} = (f * g)(x).$$

Now

$$H(0,t) = \begin{cases} f(0) & \text{if } t \in I \\ y_0 & \text{otherwise} \end{cases} = y_0,$$

and

$$H(1,t) = \begin{cases} y_0 & \text{otherwise} \\ g(1) & \text{if } t \in I \end{cases} = y_0$$

Now define $h:J \to R$ by $h(x,t) = 4x/(t + 1)$ where R is the reals and $J = I \times I \cap \{ (x,t) \mid t \ge 4x - 1 \}$. Let $\varepsilon > 0$ and let $(x,t) \in J$. Let $\delta = \min\{ \varepsilon/8(t + 1), \varepsilon/2(4x) \}$ and let $(r,s) \in J$ such that $|r - x| < \delta$ and $|t - s| < \delta$ then

$|h(x,t) - h(r,s)| = |4x/(t - 1) - 4r/(s - 1)| =$

$|(4x(s + 1) - 4r(t + 1))/(t + 1)(s + 1)| \le$

$|4x(s + 1) - 4r(t + 1)| =$

$|4x(s + 1) - 4x(t + 1) + 4x(t + 1) - 4r(t + 1)| \le$

$|4x||s - t| + 4|t + 1||x - r| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$

Hence, h is continuous.

Define $K:L \to R$ by $K(x,t) = (4x + t - 3)/(t + 1)$ where $L = I \times I \cap \{ (x,t) \mid t \ge -4x + 3 \}$. Let $\varepsilon > 0$ and let $(x,t) \in L$. Let $\delta = \min\{ \varepsilon/16(|x| + 1), \varepsilon/16(|t| + 1), \varepsilon/16 \}$ and let $(r,s) \in L$ such that $|x - r| < \delta$ and $|t - s| < \delta$. Then

$|K(x,t) - K(r,s)| = |(4x + t - 3)/(t + 1) - (4r + s - 3)/(s + 1)| =$

$|(4x + t - 3)(s + 1) - (4r + s - 3)/(t + 1)(s + 1)| \le$

$|(4x + t - 3)(s + 1) - (4r + s - 3)(t + 1)| =$

$|4xs - 4rt + 2t - 2s + 4s - 4r| \leq$

$4|xs - rt| + 2|t - s| + 4|x - r| \leq$

$4|xs + xt - xt - rt| + 2|t - s| + 4|x - r| \leq$

$4|x||s - t| + 4|t||x - r| + 2|t - s| + 4|x - r| \leq$

$\varepsilon/4 + \varepsilon/4 + \varepsilon/8 + \varepsilon/4 < \varepsilon.$

Hence, K is continuous L. Now f is continuous, and thus fh is continuous, and g is continuous, and gK is continuous. Since $H(x,t) = y_0$ on $I \times I - (J \cup L)$ and its boundaries, H is continuous on M, where M is $I \times I - (J \cup L)$ union its boundaries. Therefore,

$$H(x,t) = \begin{cases} fh(x,t) & \text{if } (x,t) \in J \\ y_0 & \text{if } (x,t) \in I \times I - (J \cup L). \\ gK(x,t) & \text{if } (x,t) \in L \end{cases}$$

Now $fh(x,t) = y_0$ if $(x,t) \in I \times I \cap \{ (x,t) \mid t = 4x - 1 \}$, and $gK(x,t) = y_0$ if $(x,t) \in I \times I \cap \{ (x,t) \mid t = -4x + 3 \}$. Hence, H is continuous on J, L, and M, and agrees on the boundaries of these sets. Therefore, by [1, Thm. 9.23; p. 59], H is continuous, and thus $f *_n g \underset{y_0}{\sim} f * g$.

__Lemma 10.2__: The functions $h:[0,1/4] \times I \to I \times I$ by $h(x,t) = (4x,t)$ and $k:[3/4,1] \times I \to I \times I$ by $k(x,t) = (4x - 3,t)$ are homeomorphisms.

__Proof__: Define $F:I \times I \to I \times I$ by $F(x,t) = (t,x)$ for all $(x,t) \in I \times I$. Let $\varepsilon > 0$ and let $(x,t) \in I \times I$. Let $\delta = \varepsilon/2$ and let $(r,s) \in I \times I$ such that $|x - r| < \delta$ and $|t - s| < \delta$. Then

$|F(x,t) - F(r,s)| = |(t,x) - (s,r)| = |(t - s, x - r)| \leq$

$|t - s| + |x - r| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$

Thus F is continuous. Let $(x,t),(r,s) \in I \times I$ such that

$F(x,t) = F(r,s)$. Then $(t,x) = (s,r)$ and thus $t = s$ and $x = r$. Hence, F is one-to-one. Let $(x,t) \in I \times I$. Then $(t,x) \in I \times I$ and $F(t,x) = (x,t)$. Therefore F is onto. Now if $(x,t) \in I \times I$ then $FF(x,t) = F(F(x,t)) = F(t,x) = (x,t)$. Thus F is its own inverse. Therefore the inverse of F is continuous, and hence, F is an homeomorphism. Now define $\alpha: I \times [0,1/4] \to I \times I$ by $\alpha(x,t) = (x,4t)$ for all $(x,t) \in I \times [0,1/4]$, and define $\beta: I \times [3/4,1] \to I \times I$ by $\beta(x,t) = (x,4t - 3)$ for all $(x,t) \in I \times [3/4,1]$. Since K is an homeomorphism, $K|_{[0,1/4] \times I}$ mapping $[0,1/4] \times I$ into $I \times [0,1/4]$ and $K|_{[3/4,1] \times I}$ mapping $[3/4,1] \times I$ into $I \times [3/4,1]$ are homeomorphisms. By Lemma 9.1, α and β are homeomorphisms. Now if $(x,t) \in I \times [0,1/4]$ then $K\alpha K|_{[0,1/4] \times I}(x,t) = K\alpha(t,x) = K(t,4x) = (4x,t) = h(x,t)$, and if $(x,t) \in [3/4,1] \times I$ then $K\beta K|_{[3/4,1] \times I}(x,t) = K\beta(t,x) = K(t,4x - 3) = (4x - 3,t) = k(x,t)$. Since, K, α, β, $K|_{[0,1/4] \times I}$, and $K|_{[3/4,1] \times I}$ are homeomorphisms, h and k are homeomorphisms.

<u>Lemma 10.3</u>: If $[f],[g] \in N(Y,y_0)$, then $[f] \cdot [g]$ is well-defined.

<u>Proof</u>: Let $f_1,f_2 \in [f]$ and $g_1,g_2 \in [g]$. We want to show that $f_1 *_n g_1 \overset{n}{\underset{y_0}{\simeq}} f_2 *_n g_2$. Since $f_1,f_2 \in [f]$, there is a near-continuous function $F: I \times I \to Y$ such that $F(x,0) = f_1(x)$, $F(x,1) = f_2(x)$, and $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$. Since $g_1,g_2 \in [g]$, there is a near-continuous function $G: I \times I \to Y$ such that $G(x,0) = g_1(x)$, $G(x,1) = g_2(x)$, and $G(0,t) = y_0 = G(1,t)$ for all $x \in I$, $t \in I$. Define a function $H: I \times I \to Y$ by

$$H(x,t) = \begin{cases} F(4x,t) & \text{if } \ 0 \le x \le 1/4, \ t \in I \\ y_0 & \text{if } 1/4 \le x \le 3/4, \ t \in I. \\ G(4x-3,t) & \text{if } 3/4 \le x \le 1, \ \ t \in I \end{cases}$$

Then

$$H(x,0) = \begin{cases} F(4x,0) \\ y_0 \\ G(4x-3,0) \end{cases} = \begin{cases} f_1(4x) \\ y_0 \\ g_1(4x-3) \end{cases} = (f_1 *_n g_1)(x)$$

for all $x \in I$, and

$$H(x,1) = \begin{cases} F(4x,1) \\ y_0 \\ G(4x-3,1) \end{cases} = \begin{cases} f_2(4x) \\ y_0 \\ g_2(4x-3) \end{cases} = (f_2 *_n g_2)(x)$$

for all $x \in I$, and $H(0,t) = F(0,t) = y_0 = G(1,t) = H(1,t)$ for all $t \in I$. Since $F(1,t) = y_0$ and $G(1,t) = y_0$, H is well-defined. Now define $h:[0,1/4] \times I \to I \times I$ by $h(x,t) = (4x,t)$ and define $k:[3/4,1] \times I \to I \times I$ by $k(x,t) = (4x-3,t)$. Then by Lemma 10.2, h and k are homeomorphisms. Since F and G are near-continuous, by Theorem 4, $H|_{[0,1/4] \times I} = Fh$ and $H|_{[3/4,1] \times I} = Gk$ are near-continuous. Since $H|_{[1/4,3/4] \times I} = y_0$, $H|_{[1/4,3/4] \times I}$ is Therefore by Theorem 8, H is near-continuous. Thus $f_1 *_n g_1 \overset{n}{\underset{y_0}{\simeq}} f_2 *_n g_2$. Therefore, $[f_1 *_n g_1] = [f_2 *_n g_2]$ and hence, $[f] \cdot [g]$ is well-defined.

<u>Definition 10</u>: The identity element of $N(Y,y_0)$, denoted by $[e]$, is the equivalence class which contains the function $e:I \to Y$ defined by $e(x) = y_0$ for all $x \in I$.

<u>Lemma 10,4</u>: If $[f] \in N(Y,y_0)$, then $[f] \cdot [e] = [e] \cdot [f] = [f]$.

Proof: By Lemma 10.1,

$[f] \cdot [e] = [f *_n e] = [f * e] = [e * f] = [f]$.

Definition 11: If $[f] \in N(Y,y_0)$, then $[f]^{-1}$ is the element of $N(Y,y_0)$ containing the function $g:I \to Y$ defined by $g(t) = f(1 - t)$ for all $t \in I$.

Lemma 10.5: If $[f] \in N(Y,y_0)$, then $[f] \cdot [f]^{-1} = [e]$.

Proof: If $g(t) = f(1 - t)$ for all $t \in I$. Then

$[f] \cdot [f]^{-1} = [f] \cdot [g] = [f *_n g] = [f * g] = [e]$.

Theorem 16: Let Y be a space and $y_0 \in Y$. Then $(N(Y,y_0), \cdot)$ is a group.

Proof: By Lemmas 10.3, 10.4, and 10.5, it remains only to show that $\cdot$ is associative. Let $[f], [g], [h] \in N(Y,y_0)$. Then

$([f] \cdot [g]) \cdot [h] = [f *_n g] \cdot [h] = [f * g] \cdot [h] =$

$[(f \cdot g) *_n h] = [(f * g) \cdot h] = [f * (g * h)] =$

$[f *_n (g * h)] = [f] \cdot [g * h] = [f] \cdot [g *_n h] =$

$[f] \cdot ([g] \cdot [h])$.

Therefore, $(N(Y,y_0), \cdot)$ is a group.

Theorem 17: Let Y be a space and let $y_0 \in Y$. Then there is an epimorphism $\lambda:\Pi_1(Y,y_0) \to N(Y,y_0)$ where $\Pi_1(Y,y_0)$ denotes the usual fundamental group and $N(Y,y_0)$ denotes the near-continuous fundamental group.

Proof: Let $[f] \in \Pi_1(Y,y_0)$. Define $\lambda([f])$ to be the equivalence class in $N(Y,y_0)$ which contains f. Let $[f] \in \Pi_1(Y,y_0)$ and let $f,g \in [f]$. Then $f \underset{y_0}{\sim} g$ and thus $f \overset{n}{\underset{y_0}{\sim}} g$. Therefore, λ is well-defined. Let $M \in N(Y,y_0)$ and let $f \in M$. Then $[f] \in \Pi_1(Y,y_0)$, and $\lambda([f]) = M$.

Therefore, λ is onto. Let $[f],[g] \in \Pi_1(Y,y_0)$. Then $\lambda([f] \cdot [g]) = \lambda([f * g])$. Thus $\lambda([f] \cdot [g])$ is the equivalence class containing $f * g$ in $N(Y,y_0)$. Now $\lambda([f])$ is the equivalence class containing f in $N(Y,y_0)$ and $\lambda([g])$ is the equivalence class containing g in $N(Y,y_0)$. Hence, $\lambda([f]) \cdot \lambda([g])$ is the equivalence class in $N(Y,y_0)$ containing $f *_n g$. By Lemma 10.1, $f * g \underset{y_0}{\sim} f *_n g$, and therefore, $f * g \overset{n}{\underset{y_0}{\sim}} f *_n g$. Hence, $\lambda([f] \cdot [g]) = \lambda([f]) \cdot \lambda([g])$, and thus λ is an epimorphism.

<u>Theorem 18</u>: If X and Y are spaces, $x_0 \in X$, $y_0 \in Y$, and H is a homeomorphism from X into Y with $H(x_0) = y_0$, then $N(X,x_0)$ is isomorphic to $N(Y,y_0)$.

<u>Proof</u>: Let $f \in C(X,x_0)$. Then $f:I \to X$ and since $H:X \to Y$, $Hf:I \to Y$. Since Hf is continuous and $Hf(0) = H(f(0)) = H(x_0) = y_0 = H(f(1)) = Hf(1)$, $Hf \in C(Y,y_0)$. Define $\lambda:N(X,x_0) \to N(Y,y_0)$ by $\lambda([f]) = [Hf]$.

Let $[f] \in N(X,x_0)$ and let $f,g \in [f]$. Then $f \overset{n}{\underset{x_0}{\sim}} g$. Thus, there is a near-continuous function $F:I \times I \to X$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = x_0 = F(1,t)$ for all $x \in I$, $t \in I$. Then by Theorem 1 and Theorem 4, HF is near-continuous. Now
$HF(x,0) = H(F(x,0)) = H(f(x)) = Hf(x)$ for all $x \in I$,
$HF(x,1) = H(F(x,1)) = H(g(x)) = Hg(x)$ for all $x \in I$, and
$HF(0,t) = H(F(0,t)) = H(x_0) = y_0 = H(x_0) = H(F(1,t)) = HF(1,t)$ for all $t \in I$. Therefore, $Hf \overset{n}{\underset{y_0}{\sim}} Hg$ and $[Hf] = [Hg]$. Hence, λ is well-defined.

Let $[f] \in N(Y,y_0)$. Then $f \in C(Y,y_0)$. Since f is continuous and $(H^{-1}f)(0) = H^{-1}(f(0)) = H^{-1}(y_0) = x_0 = H^{-1}(y_0) = H^{-1}(f(1)) = H^{-1}f(1)$,

$H^{-1}f \in C(X,x_0)$. Therefore $[H^{-1}f]$ is an element of $N(X,x_0)$. Now $\lambda([H^{-1}f]) = [HH^{-1}f] = [f]$. Therefore, λ is onto.

Let $[f],[g] \in N(X,x_0)$ such that $\lambda([f]) = \lambda([g])$. Then $[Hf] = [Hg]$ and therefore, $Hf \overset{n}{\underset{y_0}{\sim}} Hg$. Thus there is a near-continuous function $F:I \times I \to Y$ such that $F(x,0) = Hf(x)$, $F(x,1) = Hg(x)$, and $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$. Then by Theorem 1 and Theorem 4, $H^{-1}F:I \times I \to X$ is a near-continuous function. Now
$H^{-1}F(x,0) = H^{-1}(F(x,0)) = H^{-1}(Hf(x)) = H^{-1}Hf(x) = f(x)$ for all $x \in I$,
$H^{-1}F(x,1) = H^{-1}(F(x,1)) = H^{-1}(Hg(x)) = H^{-1}Hg(x) = g(x)$ for all $x \in I$,
$H^{-1}F(0,t) = H^{-1}(F(0,t)) = H^{-1}(y_0) = x_0$ for all $t \in I$, and
$H^{-1}F(1,t) = H^{-1}(F(1,t)) = H^{-1}(y_0) = x_0$ for all $t \in I$. Therefore, $[f] = [g]$. From the definition of $f *_n g$,
$H(f *_n g) = Hf *_n Hg$. Then

$$\lambda([f] \cdot [g]) = \lambda([f *_n g]) = [H(f *_n g]) = [Hf *_n Hg] =$$
$$[Hf] \cdot [Hg] = \lambda([f]) \cdot \lambda([g]).$$

Hence, λ is an isomorphism.

<u>Theorem 19</u>: If Y is a pathwise connected space and $y_0, y_1 \in Y$, then $N(Y,y_0)$ is isomorphic to $N(Y,y_1)$.

<u>Proof</u>: Since Y is pathwise connected, there is a continuous function $p:I \to Y$ such that $p(0) = y_0$ and $p(1) = y_1$. Define $\bar{p}:I \to Y$ by $\bar{p}(x) = p(1 - x)$ for all $x \in I$. Define $H:I \to I$ by $H(x) = 1 - x$. Let $\varepsilon > 0$ and $x \in I$. Let $\delta = \varepsilon$ and let $r \in I$ such that $|x - r| < \delta$. Then $|H(x) - H(r)| = |(1 - x) - (1 - r)| = |r - x| < \varepsilon$. Thus H is continuous and since $\bar{p} = pH$, $\bar{p}$ is continuous. Let e_0 and e_1 be the functions defined by $e_0(x) = y_0$ and $e_1(x) = y_1$ for all $x \in I$. Now

define $F: I \times I \to Y$ by

$$F(x,t) = \begin{cases} p(4x/(t+1)) & \text{if } t \geq -4x - 1 \\ y_0 & \text{otherwise} \\ \bar{p}((4x + t - 3)/(t+1)) & \text{if } t \geq -4x + 3 \end{cases} .$$

Now $F(0,t) = y_0 = F(1,t)$, and

$$F(x,0) = \begin{cases} p(2x) & \text{if } \quad 0 \leq x \leq 1/2 \\ \bar{p}(2x-1) & \text{if } 1/2 \leq x \leq 1 \end{cases} = (p * \bar{p})(x),$$

and $F(x,1) = e_0$. Now by Lemma 10.1 and the proof of Lemma 10.1, $p * \bar{p} \underset{y_0}{\sim} e_0$ and thus $p * \bar{p} \underset{y_0}{\overset{n}{\sim}} e_0$. Therefore, $[p *_n \bar{p}] = [e_0]$. Similarly, $[\bar{p} *_n p] = [e_1]$. Define $\lambda: N(Y,y_0) \to N(Y,y_1)$ by $\lambda([f]) = [\bar{p} *_n (f *_n p)]$. Since $*_n$ satisfies the associative law up to homotopy, From now on, the parenthesis in $[p *_n (f *_n p)]$ will be omitted.

Let $f,g \in C(Y,y_0)$ such that $f \underset{y_0}{\overset{n}{\sim}} g$. Then there is a near-continuous function $F: I \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$. Now

$$(\bar{p} *_n f *_n p)(x) = \begin{cases} \bar{p}(4x) & \text{if } \quad 0 \leq x \leq 1/4 \\ y_0 & \text{if } \quad 1/4 \leq x \leq 3/4 \\ f(16x - 12) & \text{if } \quad 3/4 \leq x \leq 13/16 \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ p(16x - 15) & \text{if } 15/16 \leq x \leq 1 \end{cases}$$

and

$$(\bar{p} *_n g *_n p)(x) = \begin{cases} \bar{p}(4x) & \text{if } 0 \le x \le 1/4 \\ y_0 & \text{if } 1/4 \le x \le 3/4 \\ g(16x - 12) & \text{if } 3/4 \le x \le 13/16. \\ y_0 & \text{if } 13/16 \le x \le 15/16 \\ p(16x - 15) & \text{if } 15/16 \le x \le 1 \end{cases}$$

Define $G: I \times I \to Y$ by

$$G(x,t) = \begin{cases} \bar{p}(4x) & 0 \le x \le 1/4, \quad t \in I \\ y_0 & 1/4 \le x \le 3/4, \quad t \in I \\ F(16x - 12,t) \text{ if } & 3/4 \le x \le 13/16, \quad t \in I \\ y_0 & \text{if } 13/16 \le x \le 15/16, \quad t \in I. \\ p(16x - 15) & \text{if } 15/16 \le x \le 1, \quad t \in I \end{cases}$$

Now G is well-defined on $I \times I$ except possibly at $x = 1/4$, $x = 3/4$, $x = 13/16$, and $x = 15/16$. But $\bar{p}(1) = y_0$, $F(0,t) = y_0$, $F(1,t) = y_0$ and $p(0) = y_0$. Thus G is well-defined on $I \times I$. Since $G|_{[0,3/4] \times I}$ and $G|_{[13/16,1] \times I}$ are continuous, and $G|_{[3/4,13/16] \times I}$ is near-continuous, by Theorem 9, G is near-continuous. Now

$$G(x,0) = \begin{cases} \bar{p}(4x) \\ y_0 \\ F(16x - 12,0) \\ y_0 \\ p(16x - 15) \end{cases} = \begin{cases} \bar{p}(4x) \\ y_0 \\ f(16x - 12) \\ y_0 \\ p(16x - 15) \end{cases} = (\bar{p} *_n f *_n p)(x)$$

for all $x \in I$, and

$$G(x,1) = \begin{cases} \bar{p}(4x) \\ y_0 \\ F(16x - 12,1) \\ y_0 \\ p(16x - 15) \end{cases} = \begin{cases} \bar{p}(4x) \\ y_0 \\ g(16x - 12) \\ y_0 \\ p(16x - 15) \end{cases} = (\bar{p} *_n g *_n p)(x)$$

for all $x \in I$, and $G(0,t) = \bar{p}(0) = p(1) = y_1 = p(1) = G(1,t)$ for all $t \in I$. Therefore, $\bar{p} *_n f *_n p \overset{n}{\underset{y_1}{\sim}} \bar{p} *_n g *_n p$. Hence, if $f \overset{n}{\underset{y_0}{\sim}} g$, $\bar{p} *_n f *_n p \overset{n}{\underset{y_1}{\sim}} \bar{p} *_n g *_n p$. Similarly, if $h \overset{n}{\underset{y_1}{\sim}} k$, then $p *_n h *_n \bar{p} \overset{n}{\underset{y_0}{\sim}} p *_n k *_n \bar{p}$.

Let $[f] \in N(Y,y_0)$. Then $\bar{p} *_n f *_n p$ is continuous, $(\bar{p} *_n f *_n p)(0) = \bar{p}(0) = p(1) = y_1$, and $(\bar{p} *_n f *_n p)(1) = (f *_n p)(1) = p(4(1) - 3) = p(1) = y_1$. Thus $\bar{p} *_n f *_n p \in C(Y,y_1)$. Therefore, $\lambda([f]) \in N(Y,y_1)$ and hence, λ is onto.

Let $[f[\in N(Y,y_0)$ and $f,g \in [f]$. Then $f \overset{n}{\underset{y_0}{\sim}} g$. Thus $\bar{p} *_n f *_n p \overset{n}{\underset{y_0}{\sim}} \bar{p} *_n g *_n p$. Therefore, $[\bar{p} *_n f *_n p] = [\bar{p} *_n g *_n p]$ and hence, λ is well-defined.

Let $[f],[g] \in N(Y,y_0)$ such that $\lambda([f]) = \lambda([g])$. Then $[\bar{p} *_n f * p] = [\bar{p} *_n g * p]$, and therefore, $\bar{p} *_n f *_n p \overset{n}{\underset{y_1}{\sim}} \bar{p} *_n g *_n p$. Thus, $p *_n \bar{p} *_n f *_n p *_n \bar{p} \overset{n}{\underset{y_0}{\sim}} p *_n \bar{p} *_n g *_n p *_n \bar{p}$ and therefore, $e_0 *_n f *_n e_0 \overset{n}{\underset{y_0}{\sim}} e_0 *_n g *_n e_0$. Hence, $f \overset{n}{\underset{y_0}{\sim}} g$. Therefore, $[f] = [g]$ and λ is one-to-one. Let $[f] \in N(Y,y_1)$. Then $[p *_n f *_n \bar{p}] \in N(Y,y_0)$ $\lambda([p *_n f *_n \bar{p}]) = [\bar{p} *_n p *_n f *_n \bar{p} *_n p] = [e_1 *_n f *_n e_1] = [f]$. Hence, λ is onto. Let $[f],[g]$ $N(Y,y_0)$. Then

$$\lambda([f] \cdot [g]) = \lambda([f *_n g]) = [\bar{p} *_n f *_n g *_n p] =$$

$$[\bar{p} *_n f *_n e_0 *_n g *_n p] = [\bar{p} *_n f *_n p *_n \bar{p} *_n g *_n p] =$$

$$[\bar{p} *_n f *_n p] \cdot [\bar{p} *_n g *_n p] = \lambda([f]) \cdot \lambda([g]).$$

Hence, λ is an isomorphism.

CHAPTER V

Example 10: Let C be the unit circle and let S be the usual topology on C induced by the topology on the plane. Let $p = (2,0)$ and let $q = (3,0)$. Let $X = C \cup \{p,q\}$ and $T = S \cup \{ \{p,q\}, X \}$. Let $y_0 \in C$. Then (X,T) is not T_1 and yet $N(X,y_0)$ is isomorphic to Z where Z is the integers.

Proof: Let R be an open set containing p. Then $q \in R$. Hence, (X,T) is not T_1. Let $F:I \times I \to X$ be a near-continuous function. Let $\mathcal{U} = \{ C, \{p,q\} \}$. Since C is open and $\{p,q\}$ is open, $\mathcal{U}$ is an open cover of X. Since F is near-continuous, there is an open cover $\mathcal{V}$ of $I \times I$ such that if $V \in \mathcal{V}$, then there is a $U \in \mathcal{U}$ such that $F(V) \subset U$. Then if $x \in F^{-1}(\{p,q\})$, then there is an element $V_x \in \mathcal{V}$ such that $x \in V_x$. Since $F(x) \in \{p,q\}$, $F(V_x) \subset \{p,q\}$. Then $F^{-1}(\{p,q\}) = \cup\{ V_x \mid x \in F^{-1}(\{p,q\}) \}$ and thus, $F^{-1}(\{p,q\})$ is open, since each V_x is open. Let U_0 be an open subset of X such that $U_0 \subset C$. Let $x \in F^{-1}(U_0)$. Then $\alpha = \{ U_0, X - \{F(x)\} \}$ is an open cover of X. There is an open cover β of $I \times I$ such that if $V \in \beta$, then there is a $W \in \alpha$ such that $F(V) \subset W$. There is an element $V_x \in \mathcal{V}$ such that $x \in V_x$. There is a $W \in \alpha$ such that $F(V_x) \subset W$. But $F(x) \in W$. Thus $F(V_x) \subset U$. Then $F^{-1}(U) = \cup\{ V_x \mid x \in F^{-1}(U) \}$ and thus $F^{-1}(U)$ is open. Hence F is continuous. Therefore, $\Pi_1(X,y_0) = N(X,y_0)$. Since the continuos image of I is connected, any loop at y_0 is a subset of C. Since the continuous image of $I \times I$ is connected, $\Pi_1(X,y_0) = \Pi_1(C,y_0)$. Hence,

$N(X,y_0) = Z$.

Example 11: Let $X = \{a,b,c,d\}$ and let

$T = \{ \phi, X, \{b\}, \{c\}, \{b,c\}, \{a,b,c\}, \{b,c,d\} \}$. Then $\Pi_1(X,b)$ is not isomorphic to $N(X,b)$.

Proof: Let $f:I \to X$ be the continuous function defined by $f(x) = b$ for all $x \in I$ and let $g:I \to X$ be a continuous function such that $g(0) = b = g(1)$. Then since g is continuous and $\{a,b,c\} \in T$, $g^{-1}(\{a,b,c\})$ is open in I and thus $D = \{ x \mid g(x) = d \}$ is closed in I. Similarly, $A = \{ x \mid g(x) = a \}$ is closed in I. Define $F:I \times I \to X$ by

$$F(x,t) = \begin{cases} d & \text{if } x \in D \text{ and } 0 \le t \le 1/2 \\ a & \text{if } x \in A \text{ and } 0 \le t \le 1/2 \\ g(x) & \text{if } t = 0 \\ b & \text{otherwise} \end{cases}.$$

Then F is well-defined and $F(0,t) = b = F(1,t)$ for all $t \in I$ and $F(x,0) = g(x)$ and $F(x,1) = f(x)$ for all $x \in I$. We wish to show that F is near-continuous. Let $\mathcal{U}$ be an open cover of X. Then either $X \in \mathcal{U}$ or $\{a,b,c\}$ and $\{b,c,d\}$ are in $\mathcal{U}$, then an open cover which will work is $\{ I \times I - D \times [0,1/2], I \times I - A \times [0,1/2] \}$. Hence, F is near-continuous. Thus $N(X,b)$ is trival. We now will show that $\Pi_1(x,b)$ has at least two elements. Let $f:I \to X$ be defined by $f(x) = b$ for all $x \in I$. Define $h:I \to X$ by

$$h(x) = \begin{cases} b & \text{if } 0 \le x < 1/5 \\ a & \text{if } 1/5 \le x \le 2/5 \\ c & \text{if } 2/5 < x < 3/5 \\ d & \text{if } 3/5 \le x \le 4/5 \\ b & \text{if } 4/5 < x \le 1 \end{cases} .$$

Now f and h are loops at b and we wish to show that f and h are not homotopic modulo b. Suppose that there is a continuous function $F: I \times I \to X$ such that $F(x,0) = h(x)$, $F(x,1) = f(x)$, and $F(0,t) = b = F(1,t)$ for all $x \in I$, $t \in I$. Let p and q be the points $p = (2/5,0)$, $q = (3/5,0)$. Let $J = (2/5,3/5) \times \{o\}$. Since $\{c\} \in T$, $F^{-1}(\{c\})$ is an open subset of $I \times I$. Since $F(x,0) = h(x)$ for all $x \in I$, $F^{-1}(\{c\})$ contains J. Let U be the component of $F^{-1}(\{c\})$ which contains J. Then U is open and connected and since F is h on $I \times \{0\}$, f on $I \times \{1\}$, and b on $\{0\} \times I$, the only points on the boundary of $I \times I$ which are in U are in J. Let B be the boundary of U. Let $W = I \times I - \bar{U}$ and let $M = W \cup B \cup J$. Then $W \cup B$ is closed in $I \times I$ and since $p,q \in B$, $W \cup B \cup J$ is closed. Hence, M is closed. Since J is the intersection of the boundary of $I \times I$ and U, the boundary of $I \times I$ is contained in M. Let Q be the component of M which contains the boundary of $I \times I$. Then Q is closed and connected. Since Q is bounded, Q is compact and hence a continuum. Since $I \times I$ is closed in the plane, Q is a continuum in the plane. Since J is a subset of the boundary of $I \times I$ and U is an open, connected subset of $I \times I$ containing J, $U - J$ is connected. Now $U - J$ is a connected subset of the compliment of Q. Let $\mathcal{O}$ be the component of the compliment of Q which contains $U - J$.

We wish to show that the boundary of $\mathcal{O}$ is a subset of J union the boundary of U. Let bd denote the boundary of a set. Let $x \in$ bd $\mathcal{O}$. Then $x \in M$ and thus $x \in W \cup B \cup J$. If $x \in B \cup J$, then clearly $x \in (\text{bd } U) \cup J$. Now suppose $x \in W$. Since W is an open subset of $I \times I$, there is a disc D in the plane such that $x \in D \cap (I \times I) \subset W$. Now $x \in$ bd $\mathcal{O}$ and thus $x \in Q$. But since D is connected and contains x and Q is the component containing x, $D \cap (I \times I) \subset Q$. Now Q contains the boundary of $I \times I$ and $\mathcal{O}$ is a component of the compliment of Q which intersects the interior of $I \times I$. Hence, $\mathcal{O}$ is contained in the interior of $I \times I$ and thus x is neither a point or a limit point of $\mathcal{O}$. Therefore, $x \notin$ bd $\mathcal{O}$. But this is impossible. Hence, $x \notin W$. Thus, bd $\mathcal{O} \subset J \cup (\text{bd } U)$. By [8, Thm. 2.1; p. 105], since $\mathcal{O}$ is a bounded component of the compliment of Q, the bd $\mathcal{O}$ is an continuum. Let K be the boundary of $\mathcal{O}$. Let $L = K - J$. Then $L \subset$ bd U and we now wish to show that L is connected. Since $p,q \in K$ and neither p nor q is in J, $p,q \in L$. Suppose L is not connected. Then L is the union of two non-empty, mutually separated sets A and B with p in one of them. Say $p \in A$. Suppose $q \in A$. Then $K = (A \cup J) \cup B$. Now A and B are mutually separated. Since $\mathcal{O}$ is an open subset of $I \times I$ containing J in its boundary, no point of J is a limit point of $K - J$ and no point of $K - J$ is a limit point of J except p and q. But $p,q \in A$. Hence, J and B are mutually separated. Thus, $A \cup J$ and B are non-empty, mutually separated sets. But this is impossible, since K is connected. Thus, $q \in B$. Now suppose A is not connected. Then $A = \alpha \cup \beta$ where α and β are non-empty, mutually separated sets with $p \in \alpha$. Then $K = \beta \cup (\alpha \cup J \cup B)$ where these two sets once again are mutually separated. Thus, A is connected. Since J is an open subset of K, K - J is closed and thus L is closed. Since A is a component of L, A is closed. Hence, A is a continuum. Similarly, B is a continuum. By [8, Thm. 3.1; p. 108], there is a simple closed curve Γ in the plane such that Γ separates p from q and $\Gamma \cap (A \cup B) = \phi$. Let Z be the boundary of $I \times I$ minus $J \cup \{p,q\}$. Then $J \cup \{p,q\}$ is a connected set containing p and q and since Γ separates p from q, $\Gamma \cap J \neq \phi$. Let $W = \Gamma \cap J$. Similarly, $\Gamma \cap Z \neq \phi$. Let $z \in \Gamma \cap Z$. Since $z \in \Gamma \cap Z$, there is a point k in the unbounded component of the compliment of the boundary of $I \times I$ such that $k \in \Gamma$ and the arc from k to z in Γ not containing W contains no point of J. Since J is in the boundary of $\mathcal{O}$ there is a point $m \in \mathcal{O}$ such that $m \in \Gamma$ and the arc from k to m in Γ containing z, contains no point of J. Let Λ be the arc in Γ from k to m containing z. Then $\Lambda \cap J = \phi$ and since $\Gamma \cap (A \cup B) = \phi$, $\Lambda \cap K = \phi$. But then the component of the compliment of K containing $\mathcal{O}$ is not a subset of the interior of $I \times I$ which is impossible. Hence, L is connected. Since $L = K - J$ and $K \subset (\text{bd } U) \cup J$, $L \subset \text{bd } U$. Hence a connected subset of the boundary of U contains p and q. Let P be the component of the boundary B of U which contains p and q. Then since B is closed, P is closed.

Now U was the component of $F^{-1}(\{c\})$ containing J. Thus, no point of B is in $F^{-1}(\{c\})$, for if $x \in B$ and $F(x) = c$. Then since F is continuous at x, there is a disc E such that $x \in E \cap (I \times I)$ and $F(E \cap (I \times I)) = \{c\}$. But $E \cap U \neq \phi$, since x is in the boundary of U. Hence, $U \cup E$ is connected and U was not maximal since E must also contain a point not in U. Since x is in the boundary of U, no point of B is in $F^{-1}(\{b\})$, for if $x \in B$ and $F(x) = b$. Then since F is continuous at x, there is a disc G such that $x \in G \cap (I \times I)$ and $F(G \cap (I \times I)) = \{b\}$. But G contains no points of $F^{-1}(\{c\})$ and hence, no point of U. Hence, $F(B) \subset \{a,d\}$ But $F(p) = a$ and $F(q) = d$. Hence, $F(B) = \{a,d\}$. Since $\{a,b,c\} \in T$, $F^{-1}(\{d\})$ is closed. Similarly $F^{-1}(\{a\})$ is closed. Since P is closed, $P \cap F^{-1}(\{a\})$ and $P \cap F^{-1}(\{d\})$ are closed. But $P \subset B$ which contains p and q. Thus $P = (P \cap F^{-1}(\{a\})) \cup (P \cap F^{-1}(\{d\}))$ which is a contradiction since P is connected and $P \cap F^{-1}(\{a\})$ and $P \cap F^{-1}(\{d\})$ are non-empty closed sets. Thus, no such continuous function F can exist and f and h are not homotopic modulo b. Hence, $\Pi_1(X,b)$ has at least two elements and $N(X,b)$ can not be isomorphic to $\Pi_1(X,b)$.

SUMMARY

In this thesis the ideas of near-continuous functions and near-continuous homotopy have been investigated. A characterization of T_1 - spaces has been proved, and examples are given where the near-continuous fundamental group is non-trival and different from the fundamental group.

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