FIFE, EARL DENNETT. Non-Continuous Set-Valued Functions. (1973)
Directed by: Dr. Hughes B. Hoyle, III. Pp. 24.

The author has defined the concept of a somewhat continuous set-valued function. With this definition the author has proven some theorems analogous to those for somewhat continuous functions.

NON-CONTINUOUS SET-VALUED FUNCTIONS

by

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A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
1973

Approved by

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ACKNOWLEDGMENT

The author wishes to express sincere appreciation to Dr. Hughes B. Hoyle, III and Dr. Karl R. Gentry for their assistance in the preparation of this thesis.

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INTRODUCTION

In [1] the concept of continuous set-valued functions is defined and developed. Somewhat continuous functions are defined in [2]. The purpose of this thesis is to extend the concept of somewhat continuity to set-valued functions and prove or disprove the natural extensions of theorems in [1] and [2].

In Chapter I, certain properties of functions which do not extend to set-valued functions are demonstrated, and their weakened extensions are proved.

In Chapter II, the concept of a somewhat continuous set-valued function is defined in terms of upper semi-somewhat continuity and lower semi-somewhat continuity. The relationship between these types of set-valued functions and those which are upper semi-continuous and lower semi-continuous is investigated.

In Chapter III, extensions and restrictions of upper semi-somewhat continuous and lower semi-somewhat continuous set-valued functions are investigated.

In Chapter IV, the operation of composition on set-valued functions is defined. Conditions on functions to guarantee their composition is upper semi-comewhat continuous or lower semi-somewhat continuous are studied.

The reader is presumed to have a basic knowldege of elementary point-set topology. For definitions and results not developed in the text of this thesis, the reader can find the appropriate information in [3].

CHAPTER I

Definition 1: [1] For any sets X and Y, $F: X \to 2^Y$ is a set-valued function provided that for each $x \in X$, $F(x)$ is a non-empty subset of Y.

Definition 2: [1] Let $F: X \to 2^Y$ be a set-valued function. Then

(i) For each $A \subset X$, $F(A) = \cup \{F(x) : x \in A\}$.

(ii) For each $B \subset Y$, $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \phi\}$.

Example 1: Let $X = \{a, b, c, d\}$, $Y = \{a, b, c, d\}$, and $U \subset Y$ such that $U = \{a, b, c\}$. Define $F: X \to 2^Y$ by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, $F(c) = \{c, d\}$, and $F(d) = \{a, d\}$. Then $F(F^{-1}(U)) \neq U \cap F(X)$.

Proof: Because $F(X) = \{a, b, c, d\}$, $U \cap F(X) = \{a, b, c\}$. Now $F^{-1}(U) = \{a, b, c, d\}$, so $F(F^{-1}(U)) = \{a, b, c, d\}$. Therefore $F(F^{-1}(U)) \neq U \cap F(X)$.

Theorem 1: Let X and Y be sets, and let $F: X \to 2^Y$ be a set-valued function. Then if $U \subset Y$, $U \cap F(X) \subset F(F^{-1}(U))$.

Proof: Let $U \subset Y$. Clearly if $U \cap F(X) = \phi$, then $U \cap F(X) \subset F(F^{-1}(U))$. So suppose $U \cap F(X) \neq \phi$. Let $x \in U \cap F(X)$. Then there exists a $p \in X$ such that $x \in F(p)$. So $F(p) \cap U \neq \phi$. Hence $p \in F^{-1}(U)$. This implies that $F(p) \subset F(F^{-1}(U))$. Therefore $x \in F(F^{-1}(U))$. Hence $U \cap F(X) \subset F(F^{-1}(U))$.

<u>Example 2</u>: Let $X = \{a, b, c, d\}$, $Y = \{a, b, c, d\}$, and let $U \subset Y$ such that $U = \{a, b, c\}$. Define $F: X \to 2^Y$ by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, $F(c) = \{c, d\}$, and $F(d) = \{a, d\}$. Then $F^{-1}(Y - U) \neq X - F^{-1}(U)$.

Proof: Since $Y - U = \{d\}$, $F^{-1}(Y - U) = \{c, d\}$. But $F^{-1}(U) = \{a, b, c, d\}$ so $Y - F^{-1}(U) = \phi$. Therefore $F^{-1}(Y - U) \neq X - F^{-1}(U)$.

<u>Theorem 2</u>: Let X and Y be sets, and let $F: X \to 2^Y$ be a set-valued function. Then if $U \subset Y$, $X - F^{-1}(U) \subset F^{-1}(Y - U)$.

Proof: Let $U \subset Y$. Clearly if $X - F^{-1}(U) = \phi$, then $X - F^{-1}(U) \subset F^{-1}(Y - U)$. So suppose $X - F^{-1}(U) \neq \phi$. Let $x \in X - F^{-1}(U)$. Then $F(x) \cap U = \phi$. So $F(x) \subset Y - U$. Therefore $x \in F^{-1}(Y - U)$, and hence $X - F^{-1}(U) \subset F^{-1}(Y - U)$.

CHAPTER II

Definition 3: [1] Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function. Then

(i) F is an upper semi-continuous function provided that for each closed $B \subset Y$, $F^{-1}(B)$ is a closed subset of X,

(ii) F is a lower semi-continuous function provided that for each open $U \subset Y$, $F^{-1}(U)$ is an open subset of X,

(iii) F is a continuous function provided that F is both upper semi-continuous and lower semi-continuous.

Definition 4: Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function. Then

(i) F is a lower semi-somewhat continuous function provided that for each $U \in T$ such that $F^{-1}(U) \neq \phi$, there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset F^{-1}(U)$,

(ii) F is an upper semi-somewhat continuous function provided that for each set A closed in (Y, T) such that $F^{-1}(A) \neq X$, there exists a set B closed in (X, S) such that $B \neq X$ and $F^{-1}(A) \subset B$.

Theorem 3: Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function such that F is lower semi-continuous. Then F is lower semi-somewhat continuous.

Proof: Let $U \in T$ such that $F^{-1}(U) \neq \phi$. Since F is lower semi-continuous, then $F^{-1}(U) \in S$. Therefore F is lower semi-somewhat continuous.

Example 3: Let $X = \{a, b, c\}$, $Y = \{a, b, c\}$, $S = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$, and let $T = \{\phi, Y, \{b\}\}$. Define $F: (X, S) \to 2^Y$ by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, and $F(c) = \{a, c\}$. Then F is lower semi-somewhat continuous, but not lower semi-continuous.

Proof: The open sets of (Y, T) are Y, ϕ, and $\{b\}$. So $F^{-1}(Y) = X$, $F^{-1}(\phi) = \phi$, and $F^{-1}(\{b\}) = \{a, b\}$. Since $\{a\} \in S$, and $\{a\} \subset \{a, b\}$ and $\{a\} \subset X$, F is lower semi-somewhat continuous. But $\{a, b\} \notin S$, so F is not lower semi-continuous.

Theorem 4: Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function such that F is upper semi-continuous. Then F is upper semi-somewhat continuous.

Proof: Let A be a closed subset of Y such that $F^{-1}(A) \neq X$. Since F is upper semi-continuous, $F^{-1}(A)$ is a closed subset of X. Therefore F is upper semi-somewhat continuous.

Example 4: Let $X = \{a, b, c, d\}$, $Y = \{a, b, c, d\}$, $S = \{\phi, X, \{d\}\}$, and $T = \{\phi, Y, \{a, c, d\}\}$. Define $F: (X, S) \to 2^Y$ by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, $F(c) = \{c, d\}$, and $F(d) = \{a, d\}$. Then F is upper semi-somewhat continuous, but not upper semi-continuous.

Proof: The closed sets of (X, S) are ϕ, X, and $\{a, b, c\}$. The closed sets of (Y, T) are ϕ, Y, and $\{b\}$. It follows then that $F^{-1}(\phi) = \phi$, $F^{-1}(Y) = X$, and $F^{-1}(\{b\}) = \{a, b\}$. Since ϕ is closed in (X, S) and $\{a, b\} \subset \{a, b, c\}$ which is closed in (X, S), F is upper semi-somewhat continuous. But $F^{-1}(\{b\}) = \{a, b\}$ which is not a closed subset of X, so F is not upper semi-continuous.

<u>Example 5</u>: Let $X = \{a, b, c\}$, $Y = \{a, b, c\}$, $S = \{\phi, X, \{a, c\}\}$, and $T = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Let $F: (X, S) \to 2^Y$ be defined by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, and $F(c) = \{a, c\}$. Then F is lower semi-somewhat continuous, but not upper semi-somewhat continuous.

Proof: The open sets of (Y, T) are ϕ, Y, $\{a\}$, $\{a, b\}$, and $\{a, c\}$. It follows then that $F^{-1}(\phi) = \phi$, $F^{-1}(Y) = X$, $F^{-1}(\{a\}) = \{a, c\}$, $F^{-1}(\{a, b\}) = X$, and $F^{-1}(\{a, c\}) = X$. Since $\{a, c\} \in S$, and $\{a, c\} \subset X$ and $\{a, c\} \subset \{a, c\}$, F is lower semi-somewhat continuous.

The closed sets of (X, S) are ϕ, X, and $\{b\}$. The set $\{b\}$ is closed in (Y, T) and $F^{-1}(\{b\}) = \{a, b\}$. The only closed subset of X containing $\{a, b\}$ is X itself. Therefore F is not upper semi-somewhat continuous.

<u>Example 6</u>: Let $X = \{a, b, c\}$, $Y = \{a, b, c\}$, $S = \{\phi, X, \{b\}\}$, and $T = \{\phi, Y, \{a\}\}$. Let $F: (X, S) \to 2^Y$ be defined by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, and $F(c) = \{a, c\}$. Then F is upper semi-somewhat continuous, but not lower semi-somewhat continuous.

Proof: The closed sets of (Y, T) are ϕ, Y, and $\{b, c\}$. It follows that $F^{-1}(\phi) = \phi$, $F^{-1}(Y) = X$, and $F^{-1}(\{b, c\}) = X$. Since $\{a, c\}$ is closed in (X, S) and $F^{-1}(\phi) \subset \{a, c\}$, F is upper semi-somewhat continuous.

The open sets of (X, S) are ϕ, X, and $\{b\}$. Since $\{a\} \in T$ and $F^{-1}(\{a\}) = \{a, c\}$, the only open subset of $F^{-1}(\{a\})$ in (X, S) is ϕ. Therefore F is not lower semi-somewhat continuous.

Definition 5: Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function. Then F is somewhat continuous provided that F is both upper semi-somewhat continuous and lower semi-somewhat continuous.

Definition 6: Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function. Then F is single-valued provided that for every $x \in X$, $F(x)$ is a singleton set.

Theorem 5: Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function such that F is single-valued. Then F is lower semi-somewhat continuous if and only if F is upper semi-somewhat continuous.

Proof: Suppose F is lower semi-somewhat continuous. Let A be a closed subset of Y such that $F^{-1}(A) \neq X$. Then $Y - A$ is an open subset of Y. Since $F^{-1}(A) \neq X$, there exists a $y \in Y - A$ such that $F^{-1}(y) \subset X - F^{-1}(A)$. Thus $F^{-1}(Y - A) \neq \phi$. So there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset F^{-1}(Y - A)$. Since F is single-valued, $F^{-1}(Y - A) = X - F^{-1}(A)$. So $V \subset X - F^{-1}(A)$. By De Morgan's law, $F^{-1}(A) \subset X - V$. Since $V \in S$, $X - V$ is a closed subset of X, and since $V \neq \phi$, $X - V \neq X$. Therefore F is upper semi-somewhat continuous.

Suppose F is upper semi-somewhat continuous. Let $U \in T$ such that $F^{-1}(U) \neq \phi$. Then $Y - U$ is a closed subset of Y. Since F is single-valued, $F^{-1}(Y - U) = X - F^{-1}(U)$. So $F^{-1}(Y - U) \neq X$. Hence there exists a set D closed in (Y, T) such that $D \neq X$ and $F^{-1}(Y - U) \subset D$. Since D is a closed subset of X, then $X - D \in S$. And since $D \neq X$, $X - D \neq \phi$. By De Morgan's law $X - D \subset F^{-1}(U)$. Therefore F is lower semi-somewhat continuous.

<u>Example 7</u>: Let $X = \{a, b, c\}$, $Y = \{a, b, c\}$, $S = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$, and $T = \{\phi, Y, \{a, b\}\}$. Define $F: (X, S) \to 2^Y$ by $F(a) = \{a\}$, $F(b) = \{b\}$, and $F(c) = \{c\}$. Then F is somewhat continuous, but not continuous.

Proof: The open sets of (Y, T) are ϕ, Y, and $\{a, b\}$. Thus $F^{-1}(\phi) = \phi$, $F^{-1}(Y) = X$, and $F^{-1}(\{a, b\}) = \{a, b\}$. Since $\{a\} \in S$, and $\{a\} \subset X$ and $\{a\} \subset \{a, b\}$, F is lower semi-somewhat continuous. Because F is single-valued, F is also upper semi-somewhat continuous. Hence F is somewhat continuous.

Since $F^{-1}(\{a, b\}) = \{a, b\}$, and $\{a, b\}$ is not an open set in (X, S), F is not lower semi-continuous. Therefore F is not continuous.

<u>Definition 7</u>: [2] Let (X, S) and (Y, T) be topological spaces. Then a function $f: (X, S) \to (Y, T)$ is said to be <u>somewhat continuous</u> provided that if $U \in T$ such that $f^{-1}(U) \neq \phi$, then there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset f^{-1}(U)$.

<u>Theorem 6</u>: Let (X, S) and (Y, T) be topological spaces. Let $F: (X, S) \to 2^Y$ be a set-valued function defined by

$F(x) = \{f(x)\}$ for some function $f: (X, S) \to (Y, T)$. Then F is somewhat continuous if and only if f is somewhat continuous.

Proof: Suppose F is somewhat continuous. Then F is lower semi-somewhat continuous. Let $U \in T$ such that $f^{-1}(U) \neq \phi$. Then $F^{-1}(U) \neq \phi$ so there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset F^{-1}(U)$. Then $V \subset f^{-1}(U)$. Therefore f is somewhat continuous.

Suppose f is somewhat continuous. Let $U \in T$ such that $F^{-1}(U) \neq \phi$. Then $f^{-1}(U) \neq \phi$, so there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset f^{-1}(U)$. Then $V \subset F^{-1}(U)$. Therefore F is lower semi-somewhat continuous. Since F is single-valued, F is also upper semi-somewhat continuous. Hence F is somewhat continuous.

Theorem 7: Let (X, S) and (Y, T) be topological spaces, and let $F: (X, S) \to 2^Y$ be a set-valued function. Then F is lower semi-somewhat continuous if and only if F has the property that if M is a dense subset of X, then $F(M)$ is a dense subset of $F(X)$.

Proof: Suppose F has the property that if M is a dense subset of X, then $F(M)$ is a dense subset of $F(X)$. Let $U \in T$ such that $F^{-1}(U) \neq \phi$. Assume $F^{-1}(U)$ does not contain a non-empty open subset of X. Then every non-empty open subset of X intersects $X - F^{-1}(U)$. So $X - F^{-1}(U)$ is a dense subset of X. By the supposition, $F(X - F^{-1}(U))$ is a dense subset of $F(X)$. If $F(X - F^{-1}(U)) \cap U \neq \phi$, there exists a $y \in Y$ such that $y \in F(X - F^{-1}(U))$ and $y \in U$. So there exists an $x \in X - F^{-1}(U)$ such that $y \in F(x)$. Hence $x \notin F^{-1}(U)$, so $F(x) \cap U = \phi$. But if

$y \in U$ and $y \in F(x)$, then $F(x) \cap U \neq \phi$. Therefore $F(X - F^{-1}(U)) \cap U = \phi$. This implies that $F(X - F^{-1}(U)) \subset F(X) - U$. So $F(X) - U$ is a dense subset of $F(X)$. Since $U \in T$, $U \cap F(X) \in T_{F(X)}$ so $U \cap F(X) = \phi$. This says $F^{-1}(U) = \phi$. But this is a contradiction of $F^{-1}(U) \neq \phi$. Therefore the assumption was incorrect, and $F^{-1}(U)$ contains a non-empty open subset of X. Therefore F is lower semi-somewhat continuous.

Suppose F is lower semi-somewhat continuous. Let M be a dense subset of X. Assume $F(X)$ is not a dense subset of $F(X)$. Then there exists a set $U \in T_{F(X)}$ such that $U \neq \phi$ and $U \cap F(M) = \phi$. Then $F^{-1}(U) \neq \phi$, so there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset F^{-1}(U)$. Since M is a dense subset of X, $V \cap M \neq \phi$. So there exists an $m \in V \cap M$. Hence $F(m) \cap U \neq \phi$. But $F(m) \subset F(M)$, so $U \cap F(M) \neq \phi$. This is a contradiction of $U \cap F(M) = \phi$. Therefore the assumption was erroneous, and hence $F(M)$ must be a dense subset of $F(X)$.

<u>Example 8</u>: Let $X = \{a, b, c, d\}$, $Y = \{a, b, c, d\}$, $S = \{\phi, X, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$, and $T = \{\phi, Y, \{c\}, \{a, b, d\}\}$, and let $A \subset X$ such that $A = \{c, d\}$. Define $F: (X, S) \to 2^Y$ by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, $F(c) = \{c, d\}$, and $F(d) = \{a, d\}$. Then $X - A$ is a dense subset of X, F is upper semi-somewhat continuous, but $Y - F(A)$ is not a dense subset of $F(X)$.

Proof: The closed sets of (X, S) are ϕ, X, $\{b\}$, $\{c\}$, and $\{b, c\}$. Since $X - A = \{a, b\}$ and the only closed set containing

$\{a, b\}$ is X, $X - A$ is a dense subset of X.

The closed sets of (Y, T) are ϕ, Y, $\{c\}$, and $\{a, b, d\}$. Hence $F^{-1}(\phi) = \phi$, $F^{-1}(Y) = X$, $F^{-1}(\{a, b, d\}) = X$, and $F^{-1}(\{c\}) = \{b, c\}$. It follows that both ϕ and $\{b, c\}$ are subsets of $\{b, c\}$, and $\{b, c\}$ is a closed subset of X. Therefore F is upper semi-somewhat continuous.

Since $F(A) = \{a, c, d\}$, $Y - F(A) = \{b\}$. Now $\{b\} \subset \{a, b, d\}$ which is a closed subset of Y, so $Y - F(A)$ is not a dense subset of $Y = F(X)$.

<u>Example 9</u>: Let $X = \{a, b, c, d\}$, $Y = \{a, b, c, d\}$, $S = \{\phi, X, \{a\}\}$, and $T = \{\phi, Y, \{a, b\}, \{a, b, c\}\}$. Define $F: (X, S) \to 2^Y$ by $F(a) = \{a, b, c, d\}$, $F(b) = \{b, c, d\}$, $F(c) = \{c, d\}$, and $F(d) = \{b\}$. Then F has the property that if $E \subset X$ such that $X - E$ is a dense subset of X, then $Y - F(E)$ is a dense subset of $F(X)$. But F is not upper semi-somewhat continuous.

Proof: The closed sets in (X, S) are ϕ, X, and $\{b, c, d\}$. The closed sets in (Y, T) are ϕ, Y, $\{c, d\}$, and $\{d\}$. Let $E \subset X$ such that $X - E$ is a dense subset of X. Obviously $a \notin E$, for if $a \in E$ then $X - E \subset \{b, c, d\}$ which is a closed subset of X, and hence $X - E$ would not be a dense subset of X. So $E \subset \{b, c, d\}$, and $X - \{b, c, d\} = \{a\}$. Since the only closed subset of X containing $\{a\}$ is X, $X - \{b, c, d\}$ is a dense subset of X. So $F(E) \subset F(\{b, c, d\}) = \{b, c, d\}$, and $Y - \{b, c, d\} = \{a\}$. Since the only closed subset of Y containing $\{a\}$ is Y itself, $Y - \{b, c, d\}$ is a dense subset of

$Y = F(X)$. Therefore $Y - F(E)$ is a dense subset of $Y = F(X)$.

Now $\{c, d\}$ is a closed subset of Y and $F^{-1}(\{c, d\}) \neq X$. However $F^{-1}(\{c, d\}) = \{a, b, c\}$, and the only closed subset of X containing $\{a, b, c\}$ is X itself. Therefore F is not upper semi-somewhat continuous.

CHAPTER III

Theorem 8: Let (X, S) and (Y, T) be topological spaces, let A be a dense subset of X, and let $F: (X, S) \to 2^Y$ be a set-valued function such that F is lower semi-somewhat continuous. Then $F|_A$ is lower semi-somewhat continuous.

Proof: Let $U \in T$ such that $F^{-1}(U) \cap A \neq \phi$. Then $F^{-1}(U) \neq \phi$, so there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset F^{-1}(U)$. It follows that $V \cap A \in S_A$, and since A is a dense subset of X, $V \cap A \neq \phi$. Hence $V \cap A \subset F^{-1}(U) \cap A$, but $F^{-1}(U) \cap A = F|_A{}^{-1}(U)$, so $F|_A$ is lower semi-somewhat continuous.

Theorem 9: Let (X, S) and (Y, T) be topological spaces, and let A be a dense subset of X. Let $F: (X, S) \to 2^Y$ be a set-valued function such that F is upper semi-somewhat continuous. Then $F|_A$ is upper semi-somewhat continuous.

Proof: Let B be a closed set in (Y, T) such that $F^{-1}(B) \cap A \neq A$. Then $F^{-1}(B) \neq X$, so there exists a set D which is closed in (X, S) such that $D \neq X$ and $F^{-1}(B) \subset D$. Because A is a dense subset of X and D is a proper closed subset of X, $A \cap D \neq A$. Hence $A \cap D$ is a set which is closed in (A, T_A) and $F^{-1}(B) \cap A \subset D \cap A$. Therefore $F|_A$ is upper semi-somewhat continuous.

Theorem 10: Let (X, S) and (Y, T) be topological spaces, and let $A \in S$. Let $F: (A, S_A) \to 2^Y$ be a set-valued function such that F is lower semi-somewhat continuous, and $F(A)$ is a dense subset of

Y. Let $G: (X, S) \to 2^Y$ be an extension of F. Then G is lower semi-somewhat continuous.

Proof: Let $U \in T$ such that $G^{-1}(U) \neq \phi$. Trivially $U \neq \phi$, so since F(A) is a dense subset of Y, $U \cap F(A) \neq \phi$. Because F is lower semi-somewhat continuous, there exists a set $V \in S_A$ such that $V \neq \phi$ and $V \subset F^{-1}(U)$. Since A is open, $V \in S$; and since $F^{-1}(U) \subset G^{-1}(U)$, $V \subset G^{-1}(U)$. Therefore G is lower semi-somewhat continuous.

<u>Example 10</u>: Let $X = \{a, b, c, d\}$, $Y = \{a, b, c, d\}$, $S = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, and $T = \{\phi, Y, \{a, c, d\}, \{b, c, d\}, \{c, d\}\}$. Let A be a subset of X such that $A = \{a, b\}$. Define $F: (X, S) \to 2^Y$ by $F(a) = \{a, b\}$, $F(b) = \{b, c\}$, $F(c) = \{c, d\}$, and $F(d) = \{a, d\}$. Then $A \in S$, F(A) is a dense subset of Y, $F|_A$ is upper semi-somewhat continuous, but F is not upper semi-somewhat continuous.

Proof: Clearly $A \in S$. The closed sets of (X, S) are ϕ, X, $\{a, c, d\}$, $\{b, c, d\}$, and $\{c, d\}$. So the closed sets of (A, S_A) are ϕ, A, $\{a\}$, and $\{b\}$. The closed sets of T are ϕ, Y, $\{a\}$, $\{b\}$, and $\{a, b\}$. The only closed set in (Y, T) containing $F(A) = \{a, b, c\}$ is Y, so F(A) is a dense subset of Y.

Since $F^{-1}(\{a\}) = \{a, d\}$, $F^{-1}(\{b\}) = \{a, b\}$, and $F^{-1}(\{a, b\}) = \{a, b, d\}$, Then $F|_A{}^{-1}(\{a\}) = \{a\}$, $F|_A{}^{-1}(\{b\}) = A$, and $F|_A{}^{-1}(\{a, b\}) = A$. As noted above, $\{a\}$ is a closed set in (A, S_A), therefore $F|_A$ is upper semi-somewhat continuous.

However the only closed set in (X, S) containing $F^{-1}(\{a, b\})$ is X itself. Therefore F is not upper semi-somewhat continuous.

Theorem 11: Let (X, S) and (Y, T) be topological spaces, and let $X = A \cup B$ where $A \in S$ and $B \in S$. Let $F: (X, S) \to 2^Y$ be a set-valued function such that $F|_A$ and $F|_B$ are lower semi-somewhat continuous. Then F is lower semi-somewhat continuous.

Proof: Let $U \in T$ such that $F^{-1}(U) \neq \phi$. Then $F|_A{}^{-1}(U) \neq \phi$ or $F|_A{}^{-1}(U) = \phi$.

Suppose $F|_A{}^{-1}(U) = \phi$. Since $F^{-1}(U) \neq \phi$ and $X = A \cup B$, then $F|_B{}^{-1}(U) \neq \phi$. So there exists a set $V \in S_B$ such that $V \neq \phi$ and $V \subset F|_B{}^{-1}(U)$. Since $F|_B{}^{-1}(U) = F^{-1}(U) \cap B$, $F|_B{}^{-1}(U) \subset F^{-1}(U)$. Because B is open in (X, S), V is open in (X, S), and finally $V \subset F^{-1}(U)$.

Suppose $F|_A{}^{-1}(U) \neq \phi$. Then there exists a set $V \in S_A$ such that $V \neq \phi$ and $V \subset F|_A{}^{-1}(U)$. Since $F|_A{}^{-1}(U) = F^{-1}(U) \cap A$, then $F|_A{}^{-1}(U) \subset F^{-1}(U)$. Because A is an open subset of X, V is an open subset of X. Finally $V \subset F^{-1}(U)$.

Therefore it is always possible to find an appropriate set V, and hence F is lower semi-somewhat continuous.

Theorem 12: Let (X, S) and (Y, T) be topological spaces, and let $X = A \cup B$ where A and B are closed subsets of X. Let $F: (X, S) \to 2^Y$ be a set-valued function such that $F|_A$ and $F|_B$ are upper semi-somewhat continuous. Then F is upper semi-somewhat continuous.

Proof: Let C be a closed set in (X, S) such that $F^{-1}(C) \neq X$. Assume $\overline{F^{-1}(C)} = X$. Then $F^{-1}(C)$ is a dense subset of X. Since $F^{-1}(C) \neq X$, $F|_A{}^{-1}(C) \neq A$ or $F|_B{}^{-1}(C) \neq B$.

Suppose $F|_A{}^{-1}(C) \neq A$. Since $F|_A$ is upper semi-somewhat continuous, there exists a set D which is closed in (A, S_A) such that $D \neq A$ and $F|_A{}^{-1}(C) \subset D$. Then $F^{-1}(C) \cap A \subset D$, so $\overline{F^{-1}(C) \cap A} \neq A$. But since $F^{-1}(C)$ is a dense subset of X and A is closed in (X, S), this cannot occur. So $F|_A{}^{-1}(C) = A$, and hence $F|_B{}^{-1}(C) \neq B$.

Since $F|_B{}^{-1}(C) \neq B$ and $F|_B$ is upper semi-somewhat continuous, there exists a set E which is closed in (B, S_B) such that $E \neq B$ and $F|_B{}^{-1}(C) \subset E$. Then, as before, $F^{-1}(C) \cap B \subset E$ so $\overline{F^{-1}(C) \cap B} \neq B$. Again, this is a contradiction. Hence it must be so that $\overline{F^{-1}(C)} \neq X$. Then there exists a set M which is closed in (X, S) such that $M \neq X$ and $F^{-1}(C) \subset M$. Therefore F is upper semi-somewhat continuous.

<u>Definition 8</u>: [2] Let X and Y be sets, and T and T' be topologies for X. Then T is said to be <u>weakly equivalent</u> to T' provided that if $U \in T$ such that $U \neq \phi$, then there exists a set $V \in T'$ such that $V \neq \phi$ and $V \subset U$, and if $U \in T'$ such that $U \neq \phi$, then there exists a set $V \in T$ such that $V \neq \phi$ and $V \subset U$.

<u>Theorem 13</u>: Let (X, S) and (Y, T) be topological spaces, and let S' be a topology for X which is weakly equivalent to S. Let $F: (X, S) \to 2^Y$ be a set-valued function such that F is lower semi-somewhat continuous. Then $F: (X, S') \to 2^Y$ is lower semi-somewhat continuous.

Proof: Let $U \in T$ such that $F^{-1}(U) \neq \phi$. Then there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset F^{-1}(U)$. Since S is weakly equivalent to S', there exists a set $W \in S'$ such that $W \neq \phi$ and $W \subset V$. Then $W \subset F^{-1}(U)$. Therefore $F: (X, S') \to 2^Y$ is lower semi-somewhat continuous.

Theorem 14: Let (X, S) and (Y, T) be topological spaces, and let S' be a topology for X which is weakly equivalent to S. Let $F: (X, S) \to 2^Y$ be a set-valued function such that F is upper semi-somewhat continuous. Then $F: (X, S') \to 2^Y$ is upper semi-somewhat continuous.

Proof: Let A be a closed subset of Y such that $F^{-1}(A) \neq X$. Then there exists a set B which is closed in (X, S) such that $B \neq X$ and $F^{-1}(A) \subset B$. Since $X - B \in S$ and $X - B \neq \phi$, there exists a set $U \in S'$ such that $U \neq \phi$ and $U \subset X - B$. Then $X - U$ is a closed set in (X, S'), $X - U \neq X$, and $B \subset X - U$. Therefore $F^{-1}(A) \subset X - U$, and hence $F: (X, S') \to 2^Y$ is upper semi-somewhat continuous.

Theorem 15: Let (X, S) and (Y, T) be topological spaces, and let S' and T' be topologies such that S' is weakly equivalent to S and T' is weakly equivalent to T. Let $F: (X, S) \to 2^Y$ be a set-valued function such that F is onto and lower semi-somewhat continuous with respect to (Y, T). Then $F: (X, S') \to 2^Y$ is lower semi-somewhat continuous with respect to (Y, T').

Proof: Let $U \in T'$ such that $F^{-1}(U) \neq \phi$. Then $U \neq \phi$, so there exists a set $V \in T$ such that $V \neq \phi$ and $V \subset U$. Since F

is onto, $F^{-1}(V) \neq \phi$, so there exists a set $W \in S$ such that $W \neq \phi$ and $W \subset F^{-1}(V)$. Then $W \subset F^{-1}(U)$. So $F: (X, S) \to 2^Y$ is lower semi-somewhat continuous with respect to (Y, T'). Then by the previous theorem, $F: (X, S') \to 2^Y$ is lower semi-somewhat continuous with respect to (Y, T').

CHAPTER IV

Definition 9: Let X, Y, and Z be sets, and let $F: X \to 2^Y$ and $G: Y \to 2^Z$ be set-valued functions. Then $(G \circ F): X \to 2^Z$ is the function defined by $(G \circ F)(x) = G(F(x))$ for each $x \in X$.

Lemma 1: Let X, Y, and Z be sets, and let $F: X \to 2^Y$ and $G: Y \to 2^Z$ be set-valued functions. Then if $U \subset Z$, $(G \circ F)^{-1}(U) = F^{-1}(G^{-1}(U))$.

Proof: Let $U \subset Z$. Suppose $x \notin (G \circ F)^{-1}(U)$, then $(G \circ F)(x) \cap U = \phi$, so $G(F(x)) \cap U = \phi$. Thus for every $y \in F(x)$, $G(y) \cap U = \phi$. Consequently $y \notin G^{-1}(U)$ for any $y \in F(x)$. Therefore $x \notin F^{-1}(G^{-1}(U))$. Hence $F^{-1}(G^{-1}(U)) \subset (G \circ F)^{-1}(U)$.

Suppose $x \in (G \circ F)^{-1}(U)$. Then $(G \circ F)(x) \cap U \neq \phi$, so $G(F(x)) \cap U \neq \phi$. Therefore there exists a $y \in F(x)$ such that $G(y) \cap U \neq \phi$. Then $y \in G^{-1}(U)$, but $x \in F^{-1}(y)$. So $x \in F^{-1}(G^{-1}(U))$. Therefore $(G \circ F)^{-1}(U) \subset F^{-1}(G^{-1}(U))$.

By the definition of set equality $(G \circ F)^{-1}(U) = F^{-1}(G^{-1}(U))$.

Theorem 16: Let (X, S), (Y, T), and (Z, P) be topological spaces. Let $F: (X, S) \to 2^Y$ and $G: (Y, T) \to 2^Z$ be set-valued functions such that F is lower semi-somewhat continuous and G is lower semi-continuous. Then $G \circ F$ is lower semi-somewhat continuous.

Proof: Let $U \in P$ such that $(G \circ F)^{-1}(U) \neq \phi$. Since G is lower semi-continuous, $G^{-1}(U) \in T$. From Lemma 1

$F^{-1}(G^{-1}(U)) = (G \circ F)^{-1}(U)$, so $F^{-1}(G^{-1}(U)) \neq \phi$. Since F is lower semi-somewhat continuous, there exists a set $V \in S$ such that $V \neq \phi$ and $V \subset F^{-1}(G^{-1}(U))$. Therefore $V \subset (G \circ F)^{-1}(U)$, and hence $G \circ F$ is lower semi-somewhat continuous.

Theorem 17: Let (X, S), (Y, T), and (Z, P) be topological spaces. Let $F: (X, S) \rightarrow 2^Y$ and $G: (Y, T) \rightarrow 2^Z$ be set-valued functions such that F is upper semi-somewhat continuous and G is upper semi-continuous. Then $G \circ F$ is upper semi-somewhat continuous.

Proof: Let A be a closed subset of Z such that $(G \circ F)^{-1}(A) \neq X$. Since G is upper semi-continuous, $G^{-1}(A)$ is a closed subset of Y. From Lemma 1 $F^{-1}(G^{-1}(A)) = (G \circ F)^{-1}(A)$. So $F^{-1}(G^{-1}(A)) \neq X$. Because F is upper semi-somewhat continuous, there exists a set B which is a closed subset of X such that $B \neq X$ and $F^{-1}(G^{-1}(A)) \subset B$. Therefore $(G \circ F)^{-1}(A) \subset B$, and hence $G \circ F$ is upper semi-somewhat continuous.

Theorem 18: Let (X, S), (Y, T), and (Z, P) be topological spaces. Let $F: (X, S) \rightarrow 2^Y$ and $G: (Y, T) \rightarrow 2^Z$ be set-valued functions such that F is onto and lower semi-continuous and G is lower semi-somewhat continuous. Then $G \circ F$ is lower semi-somewhat continuous.

Proof: Let $U \in P$ such that $(G \circ F)^{-1}(U) \neq \phi$. From Lemma 1 $F^{-1}(G^{-1}(U)) = (G \circ F)^{-1}(U)$, so $F^{-1}(G^{-1}(U)) \neq \phi$. Hence $G^{-1}(U) \neq \phi$. Since G is lower semi-somewhat continuous, there exists a set $V \in T$ such that $V \neq \phi$ and $V \subset G^{-1}(U)$. Because F is lower semi-continuous, $F^{-1}(V) \in S$. Since F is onto and $V \neq \phi$,

$F^{-1}(V) \neq \phi$. So $F^{-1}(V) \subset F^{-1}(G^{-1}(U)) = (G \circ F)^{-1}(U)$. Therefore $G \circ F$ is lower semi-somewhat continuous.

Example 11: Let $X = \{a, b\}$, $Y = \{a, b, c\}$, $Z = \{a, b, c\}$, $S = \{\phi, X, \{a\}\}$, $T = \{\phi, Y, \{c\}, \{a, c\}\}$, and $P = \{\phi, Z, \{c\}, \{a, c\}, \{b, c\}\}$. Define $F: (X, S) \to 2^Y$ by $F(a) = \{a\}$ and $F(b) = \{b\}$. Define $G: (Y, T) \to 2^Z$ by $G(a) = \{a\}$, $G(b) = \{b\}$, and $G(c) = \{c\}$. Then F is lower semi-continuous and G is lower semi-somewhat continuous. But F is not onto, and hence $G \circ F$ is not lower semi-somewhat continuous.

Proof: Since $F(X) = \{a, b\}$, F is obviously not onto.

The open sets of (Y, T) are ϕ, Y, $\{c\}$, and $\{a, c\}$. So $F^{-1}(\phi) = \phi$, $F^{-1}(Y) = X$, $F^{-1}(\{c\}) = \phi$, and $F^{-1}(\{a, c\}) = \{a\}$. Since ϕ, X, and $\{a\}$ are open sets of (X, S), F is lower semi-continuous.

The open sets of (Z, P) are ϕ, Z, $\{c\}$, $\{a, c\}$, and $\{b, c\}$. So $G^{-1}(\phi) = \phi$, $G^{-1}(Z) = Y$, $G^{-1}(\{c\}) = \{c\}$, $G^{-1}(\{a, c\}) = \{a, c\}$, and $G^{-1}(\{b, c\}) = \{b, c\}$. Since $\{c\} \in T$ and $\{c\}$ is a subset of $\{a, c\}$, $\{b, c\}$, and Y, then G is lower semi-somewhat continuous.

However $(G \circ F)^{-1}(\{b, c\}) = F^{-1}(\{b, c\}) = \{b\}$, so $(G \circ F)^{-1}(\{b, c\}) \neq \phi$. But $\{b\}$ contains no non-empty open subset of X. Therefore $G \circ F$ is not lower semi-somewhat continuous.

Example 12: Let $X = \{a, b, c, d\}$, $Y = \{a, b, c, d\}$, $Z = \{a, b, c, d\}$, $S = \{\phi, X, \{a, b\}\}$, $T = \{\phi, Y, \{a\}, \{a, b, c\}\}$, and $P = \{\phi, Z, \{a, b, c\}\}$. Define $F: (X, S) \to 2^Y$ by

$F(a) = \{a, b\}$, $F(b) = \{b, c\}$, $F(c) = \{c, d\}$, and $F(d) = \{a, d\}$. Define $G: (Y, T) \to 2^Z$ by $G(a) = \{a, b\}$, $G(b) = \{b, c\}$, $G(c) = \{c, d\}$, and $G(d) = \{a, d\}$. Then F is onto and upper semi-continuous and G is upper semi-somewhat continuous. But $G \circ F$ is not upper semi-somewhat continuous.

Proof: Since $F(X) = \{a, b, c, d\}$, F is clearly onto.

The closed sets of (X, S) are ϕ, X, and $\{c, d\}$. The closed sets of (Y, T) are ϕ, Y, $\{d\}$, and $\{b, c, d\}$. Since $F^{-1}(\phi) = \phi$, $F^{-1}(Y) = X$, $F^{-1}(\{d\}) = \{c, d\}$, and $F^{-1}(\{b, c, d\}) = X$, F is upper semi-continuous.

The closed sets of (Z, P) are ϕ, Z, and $\{d\}$. Now $G^{-1}(\phi) = \phi$, $G^{-1}(Z) = Y$, and $G^{-1}(\{d\}) = \{c, d\}$. Since $\{b, c, d\}$ is a closed subset of Y containing ϕ and $\{c, d\}$, G is upper semi-somewhat continuous.

However $(G \circ F)^{-1}(\{d\}) = F^{-1}(G^{-1}(\{d\})) = \{b, c, d\}$, and $\{b, c, d\}$ is not contained in any proper closed subset of X. Therefore $G \circ F$ is not upper semi-somewhat continuous.

<u>Theorem 19</u>: Let (X, S), (Y, T), and (Z, P) be topological spaces. Let $F: (X, S) \to 2^Y$ and $G: (Y, T) \to 2^Z$ be set-valued functions such that F and G are lower semi-somewhat continuous and $F(X)$ is a dense subset of Y. Then $G \circ F$ is lower semi-somewhat continuous.

Proof: Let $U \in P$ such that $(G \circ F)^{-1}(U) \neq \phi$. Then $F^{-1}(G^{-1}(U)) \neq \phi$, so $G^{-1}(U) \neq \phi$. Since G is lower semi-somewhat continuous, there exists a set $V \in T$ such that $V \neq \phi$ and

$V \subset G^{-1}(U)$. So $F^{-1}(V) \subset F^{-1}(G^{-1}(U))$. Assume $F^{-1}(V) = \phi$. Then $F(X) \cap V = \phi$. But $F(X)$ is a dense subset of Y, so there are no non-empty open subsets of Y which do not intersect $F(X)$. This is a contradiction of $F(X) \cap V = \phi$. Therefore $F^{-1}(V) \neq \phi$, so there exists a set $W \in S$ such that $W \neq \phi$ and $W \subset F^{-1}(V)$. Hence $W \subset F^{-1}(G^{-1}(U))$, so $W \subset (G \circ F)^{-1}(U)$. Therefore $G \circ F$ is lower semi-somewhat continuous.

SUMMARY

The author has shown the relationship between upper semi-somewhat continuous functions and lower semi-somewhat continuous functions. Some theorems regarding the restriction, extension, and composition of these functions have been proven. Many of the properties which held for lower semi-somewhat continuous did not hold for upper semi-somewhat continuous. The problem remaining is to strengthen the definition of upper semi-somewhat continuous so that more theorems may be proven.

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