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In view of the fact that most treatments of the Riemann-Stieltjes integral usually consider only real-valued functions, and whereas, complex-valued functions of a certain type are closely related to real-valued functions, it is the purpose of this thesis to demonstrate explicitly how the theory of Riemann-Stieltjes integration for complex-valued functions is developed.

The development of the complex theory follows the same general pattern as the real case. In fact, the natural extensions of the latter are obvious after some preliminary results are shown.

One result demonstrates clearly how the integral for complex-valued functions may be computed by expressing the integral in terms of real-valued functions. The final theorem is a special case in which the integration technique takes on the exact form of the real case.

A CONSIDERATION OF THE RIEMANN-STIELTJES INTEGRAL FOR COMPLEX-VALUED FUNCTIONS DEFINED ON A REAL CLOSED INTERVAL

by

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INTRODUCTION

The theory and development of the Riemann-Stieltjes integral for real-valued functions f and g, defined and bounded on the closed interval [a,b] can be extended in a direct way to complex-valued functions f and g, also defined and bounded on [a,b]. Actually, the sum of products of the form $f(s_k)[g(t_k)-g(t_{k-1})]$ might very well be interpreted as complex-valued and the results that are established in the case of real-valued functions compared to the results in the case of complex-valued functions.

Furthermore, there is a result which enables us to extend most important properties of the real case to the complex case by simply defining such concepts as continuity, differentiation, and bounded variation in terms of the components of the complex-valued function. In particular, the so-called fundamental theorem of integral calculus can be shown to remain valid in the complex case.

CHAPTER I

DEFINITIONS AND PRELIMINARY REMARKS

DEFINITION 1.1: By a partition P of the closed, real interval $[a,b]$ we will mean a finite, increasing sequence $\{t_0, t_1, t_2, \cdot \cdot \cdot, t_n\}$ in which $a = t_0 < t_1 < t_2 < \cdot \cdot \cdot < t_n = b$. Usually we will denote P by $\{t_k\}_{k=0}^{n}$. By a refinement of P on $[a,b]$ we will mean any partition P' such that $P \subset P'$. The norm of a partition P, denoted by $||P||$, will be the greatest of the numbers $\Delta_k t = t_k - t_{k-1}$. For the function g defined on $[a,b]$ we will use the notation $\Delta_k g = g(t_k) - g(t_{k-1})$.

DEFINITION 1.2: By the statement z is a complex number we will mean that z may be represented in the form $z = a + bi$ where a and b are real numbers $(a,b \in R)$, and i is defined as the unique solution to $x^2 = -1$. We may consider z as the ordered pair (a,b) when appropriate. By the modulus of z, denoted by $|z|$, we mean $|z| = (a^2 + b^2)^{1/2}$. Two operations, addition and multiplication, are defined in the usual way as follows: For any two complex numbers $z = a + bi$ and $w = c + di$

(1) $z + w = (a + c) + (b + d)i = (a + c, b + d)$.

(2) $z \cdot w = (ac - bd) + (ad + bc)i = (ac - bd, ad + bc)$.

If $z = a + bi$ then we call "a" the real part of z and denote it by $Re(z)$. The imaginary part of z is "b", denoted by Im(z). Moreover, a and b are called the <u>components</u> of z. Let C represent the set of all complex numbers as R represents the set of all real numbers. Now for a fixed $z_0 = (a,b)$ we can indicate a circle in the real plane. With center (a,b) and radius r the interior of a circle may be indicated by the set $B(z_0;r) = \{z \mid |z - z_0| < r\}$. The set $B(z_0;r)$ is often called <u>an open ball</u> about z_0 with radius r.

<u>DEFINITION 1.3</u>: By a complex-valued function we will mean a function whose domain is a subset of the real line and whose range is a subset of C.

<u>DEFINITION 1.4</u>: A complex-valued function f is said to be <u>continuous at t_0</u> provided that for each real $\epsilon > 0$ there exists some $\delta > 0$ such that $|f(t) - f(t_0)| < \epsilon$ whenever $|t - t_0| < \delta$. Geometrically this definition requires that for each open ball $B(f(t_0);\epsilon)$ we can find some interval about t_0, with radius δ, such that $f(t) \in B(f(t_0);\epsilon)$ whenever $|t-t_0| < \delta$. We will say that a complex-valued function f is <u>bounded on a set S</u> provided that there is some real number $M > 0$ such that $|f(t)| < M$ for all $t \in S$.

<u>DEFINITION 1.5</u>: Consider two complex-valued functions f and g, each defined and bounded on the interval $[a,b]$. Let P be a partition of $[a,b]$, $P = \{t_k\}_{k=0}^{n}$, and let $s_k \in [t_{k-1},t_k]$. We form what is generally called a <u>Riemann-Stieltjes sum of f with respect to g and the partition P</u> by $\sum_{k=1}^{n} f(S_k)[g(t_k) - g(t_{k-1})]$ and usually denote the sum

by $S(P,f,g)$. Now, we will say that <u>f is Riemann-Stieltjes integrable with respect to g on $[a,b]$</u>, denoted by $f \in I(g)$, provided that there is a complex number A with the following property: For each positive ϵ we can find a partition P_0 of $[a,b]$ such that for each refinement P of P_0 and for any $s_k \in [t_{k-1},t_k]$ we are assured of the inequality $|S(P,f,g)-A| < \epsilon$. The complex number A is unique, when it exists, since if we assume that there are two such numbers A and A', the triangle inequality which is applicable for complex numbers shows that $|A - A'| < \epsilon$ for all positive ϵ. To apply the triangle inequality we need to choose a common refinement P of the partitions $P(A)$ and $P(A')$ in order to guarantee that $|A - A'| < \epsilon$ from the inequality $|(S(P,f,g)-A)+(A'-S(P,f,g))| < \epsilon/2 + \epsilon/2 = \epsilon$. This unique complex number A will be denoted by $\int_a^b f dg$, or by $\int_a^b f(t)dg(t)$.

Throughout this paper, unless specified otherwise, the symbols f, g, and h are assumed to represent <u>bounded</u>, <u>complex-valued</u> functions.

CHAPTER II

ELEMENTARY PROPERTIES OF THE RIEMANN-STIELTJES INTEGRAL

We can show that the integral behaves in a reasonably linear fashion by showing that integrability is preserved under addition and that the product of $f \in I(g)$ on $[a,b]$ and $c \in C$ is also integrable on $[a,b]$.

PROPOSITION 2.1: If $f \in I(g)$ on $[a,b]$ and $c \in C$ then $\int_a^b c \cdot f dg = c \int_a^b f dg$.

Proof: For $\epsilon > 0$ there exists a partition P_0 of $[a,b]$ such that if P refines P_0 then for $c \neq (0,0)$ $\left| S(P,f,g) - \int_a^b f dg \right| < \frac{\epsilon}{|c|}$. Using a rule for the product of two moduli we have $\left| c \sum_{k=1}^{n} f(s_k)\Delta_k g - c\int_a^b f dg \right| =$

$$\left| \sum_{k=1}^{n} cf(s_k)\Delta_k g - c\int_a^b f dg \right| < \epsilon \text{, for } s_k \in [t_{k-1},t_k].$$

The inequality follows as a result of the distributive law for complex numbers. The last conclusion implies that $c\int_a^b f dg = \int_a^b c f dg$, for $c \in C$.

PROPOSITION 2.2: If $f \in I(g)$ on $[a,b]$ and $h \in I(g)$ on $[a,b]$ then $f + h \in I(g)$ on $[a,b]$, and furthermore,

$$\int_a^b (f+h)dg = \int_a^b f dg + \int_a^b h dg.$$

Proof: For each $\epsilon > 0$ we know that there exist two partitions P_0 and P_1 of $[a,b]$ such that if P is a common refinement of P_0 and P_1 then

$\left|S(P,f,g) - \int_a^b fdg + S(P,h,g) - \int_a^b hdg\right| < \epsilon/2 + \epsilon/2 = \epsilon$ which may be written as

$\left|\sum_{k=1}^{n} [f(s_k)+h(s_k)] \Delta_k g - (\int_a^b fdg + \int_a^b hdg)\right| < \epsilon$. The first inequality is a result of the triangle inequality. The second is a combination of the commutative, associative and distributive properties which hold for complex numbers. Furthermore, the above result implies that $\int_a^b (f+h)dg = \int_a^b fdg + \int_a^b hdg$.

<u>PROPOSITION 2.3</u>: If $f \varepsilon I(g)$ on $[a,b]$, and $h \varepsilon I(g)$ on $[a,b]$, and if c_1 and c_2 are complex numbers then

$$\int_a^b (c_1 f + c_2 h)dg = c_1 \int_a^b fdg + c_2 \int_a^b fdg.$$

<u>Proof</u>: Proposition 2.1 gives us

$c_1 \int_a^b fdg = \int_a^b c_1 fdg$ and $c_2 \int_a^b hdg = \int_a^b c_2 hdg$; and this result with Proposition 2.2 above yields

$$c_1 \int_a^b fdg + c_2 \int_a^b hdg = \int_a^b c_1 fdg + \int_a^b c_2 hdg = \int_a^b (c_1 f + c_2 h)dg.$$

In the expression $\int_a^b fdg$ we refer to f as the <u>integrand</u> and to g as the <u>integrator</u>. What we have demonstrated is the linearity of the integral with respect to the integrand and constant multiples of an integrable function on $[a,b]$. Next, we will show that the Riemann-Stieltjes integral is linear with respect to the integrator as well as the integrand.

<u>PROPOSITION 2.4</u>: Suppose that $f \varepsilon I(g)$ on $[a,b]$ and $c \varepsilon C$. Then $f \varepsilon I(cg)$ on $[a,b]$, and $\int_a^b fd(cg) = c \int_a^b fdg$.

<u>Proof</u>: Let $\epsilon > 0$. There exists a partition P_0 of $[a,b]$ such that if P is a refinement of P_0 then

$\left|cS(P,f,g) - c\int_a^b fdg\right| < \epsilon$. Moreover, we can write this inequality as $\left| \sum_{k=1}^{n} f(s_k)[cg(t_k)-cg(t_{k-1})] - c\int_a^b fdg \right| < \epsilon$, $s_k \in [t_{k-1},t_k]$, by the commutative and distributive properties of the complex numbers. The latter inequality implies that $\int_a^b fd(cg) = c\int_a^b fdg$, and moreover, that $f \in I(cg)$ on $[a,b]$.

PROPOSITION 2.5: If $f \in I(g)$ and $f \in I(h)$ on $[a,b]$ then $f \in I(g+h)$ on $[a,b]$ and $\int_a^b fd(g+h) = \int_a^b fdg + \int_a^b fdh$.

Proof: Let $\epsilon > 0$. Then there exist two partitions P_0 and P_1 of $[a,b]$ such that if P is a partition of $[a,b]$ in which $P_0 \cup P_1 \subseteq P$ (P a common refinement of P_0 and P_1) then

$$\left| [S(P,f,g)-\int_a^b fdg] + [S(P,f,h)-\int_a^b fdh] \right| \leq$$
$$\left| S(P,f,g)-\int_a^b fdg \right| + \left| S(P,f,h)-\int_a^b fdh \right| < \epsilon/2 + \epsilon/2 = \epsilon .$$

We may write this inequality as

$$\left| \sum_{k=1}^{n} f(s_k)[(g(t_k)+h(t_k)) - (g(t_{k-1})+h(t_{k-1}))]-(\int_a^b fdg+\int_a^b fdh) \right| < \epsilon$$

for all $s_k \in [t_{k-1},t_k]$ and over all partitions $P = \{t_k\}_{k=0}^{n}$ of $[a,b]$ such that $P_0 \cup P_1 \subseteq P$. The latter inequality verifies the proposition, $f \in I(g+h)$ on $[a,b]$ and that $\int_a^b fd(g+h) = \int_a^b fdg + \int_a^b fdh$.

PROPOSITION 2.6: If $f \in I(g)$ and $f \in I(h)$ on $[a,b]$ and if $c_1, c_2 \in C$ then $f \in I(c_1g+c_2h)$ on $[a,b]$ and $\int_a^b fd(c_1g+c_2h) = c_1\int_a^b fdg + c_2\int_a^b fdh$.

Proof: Proposition 2.5 yields $\int_a^b fd(c_1g+c_2h) = \int_a^b fd(c_1g) + \int_a^b fd(c_2h)$. Proposition 2.4 and this equation verify that $\int_a^b fd(c_1g+c_2h) = c_1\int_a^b fdg + c_2\int_a^b fdh$.

To round out the linearity we verify another result that is also found in the real case. An "additive" quality of the integral with respect to $[a,b]$, the interval of integration, exists.

PROPOSITION 2.7: If $f \in I(g)$ on $[a,c]$ and $f \in I(g)$ on $[c,b]$ for $c \in (a,b)$ then $\int_a^b f dg$ exists and $\int_a^b f dg = \int_a^c f dg + \int_c^b f dg$.

Proof: We know that for each $\epsilon > 0$ there are two partitions P_0' of $[a,c]$ and P_0'' of $[c,b]$ such that whenever P' is a refinement of P_0' and P'' is a refinement of P_0'' we can write the two inequalities

$$\left| S(P',f,g) - \int_a^c f dg \right| < \epsilon/2 \text{ and}$$

$$\left| S(P'',f,g) - \int_c^b f dg \right| < \epsilon/2.$$

Now, if we let $P = P' \cup P''$ then we have that

$$\left| S(P,f,g) - \left(\int_a^c f dg + \int_c^b f dg \right) \right| =$$

$$\left| S(P',f,g) - \int_a^c f dg + S(P'',f,g) - \int_c^b f dg \right| < \epsilon.$$

This inequality is true for all refinements of the partition P and, thus, verifies that $\int_a^b f dg = \int_a^c f dg + \int_c^b f dg$.

As in the real case we can make the following definitions:

(1) For $a < b$, we define $\int_b^a f dg = - \int_a^b f dg$ when $\int_a^b f dg$ exists.

(2) We define $\int_a^a f dg = 0$.

CHAPTER III

MAJOR EXTENSIONS OF THE RIEMANN-STIELTJES INTEGRAL

A Reciprocity Relation

A connection of some interest between the integrand and the integrator of the Riemann-Stieltjes integral is

PROPOSITION 3.1: For complex-valued f and g, if $f \in I(g)$ on $[a,b]$, then $g \in I(f)$ on $[a,b]$.

Proof: Given any $\epsilon > 0$ we can find a partition P_0 of $[a,b]$ such that if P' is any refinement of P_0 on $[a,b]$ then $\left|\sum_{k=1}^{n} f(s_k)[g(t_k)-g(t_{k-1})] - \int_a^b f dg\right| < \epsilon$, for $P' = \{t_k\}_{k=0}^{n}$ and for each $s_k \in [t_{k-1},t_k]$. Now let $Q = \{x_k\}_{k=0}^{m}$ be any refinement of P' and form the Riemann-Stieltjes sum $S(Q,g,f)$. $S(Q,g,f) = \sum_{k=1}^{m} g(r_k)[f(x_k) - f(x_{k-1})]$, $r_k \in [x_{k-1},x_k]$. If we subtract from the left and right side of this equation, respectively, the left and right hand expressions in the identity

$$f(b)g(b) - f(a)g(a) = \sum_{k=1}^{m} [f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})]$$

then we obtain

$$\left|f(b)g(b) - f(a)g(a) - S(Q,g,f)\right| =$$

$$\left|\sum_{k=1}^{m} g(r_k)[f(x_k)-f(x_{k-1})] - [f(x_k)g(x_k)-f(x_{k-1})g(x_{k-1})]\right| =$$

$$\left|\sum_{k=1}^{m} \left\{f(x_k)[g(x_k)-g(r_k)] + f(x_{k-1})[g(r_k)-g(x_{k-1})]\right\}\right| .$$

Now we let $P = \{p_k\}_{k=0}^{2m}$ be the partition of $[a,b]$ formed by the sequence $\left\{r_k\right\}_{k=1}^{m}$ and the partition $Q = \left\{x_k\right\}_{k=0}^{m}$ described above. Since P is a refinement of Q, then consequently, P is a refinement of P_0. We recall that $x_{k-1} \leqslant r_k \leqslant x_k$, for $k = 1,2,\cdots m$. Hence we can rewrite the right hand side of the equation above in the form $\sum_{k=1}^{m} f(q_k)[g(\overline{p}_k)-g(\overline{p}_{k-1})]$ where $q_k \varepsilon [\overline{p}_{k-1},\overline{p}_k]$, $q_k = x_k$, $\overline{p}_k = p_{2k+1} = r_{k+1}$ for $k = 0,1,2,\cdots,m-1$. (Note: $\overline{p}_{k-1} = x_0$ for $k = 0$ and $\overline{p}_k = x_m$ for $k = m$). Thus, the fact that P is a refinement of P_0 on $[a,b]$ yields the inequality $\left|\sum_{k=1}^{m} f(q_k)[g(\overline{p}_k)-g(\overline{p}_{k-1})]-\int_a^b fdg\right| < \epsilon$. Furthermore, this result means that

$\left|f(b)g(b) - f(a)g(a) - S(P,g,f) - \int_a^b fdg\right| < \epsilon$ for all refinements P of P_0 on $[a,b]$. Our assertion is verified and we conclude that $\int_a^b gdf$ exists and that

$$\int_a^b gdf = f(b)g(b) - f(a)g(a) - \int_a^b fdg.$$

We often refer to this equation as the "partial integration formula" and it is useful when for convenience we need to express $\int_a^b fdg$ in terms of $\int_a^b gdf$.

A Reduction of the Riemann-Stieltjes Integral

By using our definition of the Riemann-Stieltjes integral for complex-valued functions f and g defined on $[a,b]$ we can reduce $\int_a^b fdg$ to the real case.

<u>PROPOSITION 3.2</u>: If $f(t) = u(t) + iv(t)$ and $g(t) = r(s) + is(t)$ where u, v, r, and s are each defined and bounded on $[a,b]$ and real-valued, then

$\int_a^b f\,dg = \left(\int_a^b u\,dr - \int_a^b v\,ds\right) + i\left(\int_a^b v\,dr + \int_a^b u\,ds\right)$ provided that each of the four integrals on the right exists.

Proof: For $f = u + iv$ and $g = r + is$ and for each partition P of $[a,b]$ we can form the Riemann-Stieltjes sum

$$S(P,f,g) = \sum_{k=1}^{n} f(s_k)[g(t_k)-g(t_{k-1})]$$

$$= \sum_{k=1}^{n} [u(s_k)-iv(s_k)][r(t_k)+is(t_k)-r(t_{k-1})-is(t_{k-1})]$$

$$= (S(P,u,r)-S(P,v,s))+i(S(P,v,r)+S(P,u,s)),$$

for all $s_k \in [t_{k-1},t_k]$. Now, for each $\epsilon > 0$ there exist partitions P_1,P_2,P_3, and P_4 of $[a,b]$ such that if P refines each of P_1, P_2,P_3, and P_4, then

$$\left|(S(P,u,r)-S(P,v,s))+i(S(P,v,r)+S(P,u,s))-(A_1+A_2+A_3+A_4)\right| < \epsilon,$$

where $A_1 = \int_a^b u\,dr$, $A_2 = \int_a^b v\,ds$, $A_3 = \int_a^b v\,dr$, $A_4 = \int_a^b u\,ds$. The inequality above verifies the proposition and expresses the integral in terms of integrals of real-valued functions; namely, the components of the complex-valued f and g.

The Derivative and Uniform Continuity

We will say that the complex-valued function g has a derivative at t provided, for $g(t) = x(t) + iy(t)$, that $x'(t)$ and $y'(t)$ exist, where x' and y' are the derivatives of the two real-valued component functions of g. We define the <u>derivative</u> of g by $g'(t) = x'(t) + iy'(t)$.

We will say that the complex-valued function g is <u>uniformly continuous</u> on a set S provided, for $g(t) = x(t) + iy(t)$, that each of x and y are uniformly continuous on S. Recall that the real-valued function x is uniformly continuous on S provided that for each $\epsilon > 0$ there is some $\delta > 0$ such that whenever t_1 and t_2 are in S and $|t_1 - t_2| < \delta$, then $|x(t_1)-x(t_2)| < \epsilon$. We also recall that if the real-valued function x is uniformly continuous on a set S then x is continuous at each point t of S. Furthermore, with respect to the continuity of a function, we have the following

PROPOSITION 3.3: The complex-valued function g is continuous at t if and only if each of the real-valued components of g are continuous at t.

Proof: If $g(t) = x(t) + iy(t)$ and g is continuous at t, with each of x and y real-valued, then for each $\epsilon > 0$ there exists $\delta > 0$ such that $|g(t) - g(h)| < \epsilon$ whenever $|t - h| < \delta$. Since $|g(t) - g(h)| < \epsilon$ implies that $|x(t) - x(h)| < \epsilon$ and $|y(t) - y(h)| < \epsilon$ whenever $|t - h| < \delta$ we see that x and y are each continuous at t.

Conversely, if each of x and y are continuous at t then for $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|t - h| < \delta$ we have $|x(t) - x(h)| < \sqrt{\epsilon/2}$ and $|y(t) - y(h)| < \sqrt{\epsilon/2}$. The latter inequalities imply that $|g(t) - g(h)| < \epsilon$ whenever $|t - h| < \delta$.

Another Form of the Integral for Complex-valued Functions

From the real case we can verify another result which extends to our complex case. The proof of the result depends upon two theorems not yet mentioned in this paper. The first is the Mean Value Theorem for real-valued functions of a real variable. Since we use the result of this theorem we state it as follows: If the real-valued function F is continuous on $[a,b]$ and differentiable in (a,b) then for some $c \in (a,b)$ $F(b) - F(a) = F'(c)(b-a)$.

The second theorem required for the proof is one due to Heine which states that if F is continuous on a compact set S then F is uniformly continuous [4]. We recall that the Heine-Borel theorem states that the closed and bounded $[a,b]$ is compact [3].

<u>PROPOSITION 3.4</u>: Suppose that we have complex-valued functions f and g, each defined and bounded on $[a,b]$, with g' continuous on $[a,b]$ and $f \in I(g)$ on $[a,b]$. Then $\int_a^b f\cdot g'dt$ exists and $\int_a^b f\cdot g'dt = \int_a^b fdg$.

<u>Proof</u>: Consider any partition P of $[a,b]$ and form the two Riemann sums

(1) $S(P,f,g) = \sum_{k=1}^{n} f(s_k)\Delta_k g$ and

(2) $S(P,h) = \sum_{k=1}^{n} h(s_k)\Delta_k t$, where

$\Delta_k t = t_k - t_{k-1}$, $s_k \in (t_{k-1},t_k)$ and $h(t) = f(t)\cdot g'(t)$. Next, let $f(t) = u(t) + iv(t)$ and $g(t) = r(t) + is(t)$, where by definition each of u, v, r, and s are defined and bounded on $[a,b]$ with r and s each continuous on $[a,b]$ and each differentiable in (a,b). The Mean Value Theorem applies to r and s, so there exist b_k and c_k contained in (t_{k-1},t_k) such that $\Delta_k g = [r'(b_k) + is'(c_k)]\Delta_k t$. Recall that

$g'(t) = r'(t) + is'(t)$. We can write

$$\left|S(P,f,g)-S(P,h)\right| = \left|\sum_{k=1}^{n} f(s_k)[r'(b_k)+is'(c_k)]\Delta_k t - \sum_{k=1}^{n} f(s_k)[r'(s_k)+is'(s_k)]\Delta_k t\right|.$$

$$= \left|\sum_{k=1}^{n} f(s_k)[(r'(b_k)-r'(s_k))+i(s'(c_k)-s'(s_k))]\Delta_k t\right|.$$

Now recall that f is bounded on $[a,b]$ so there exists a positive number M such that $\left|f(t)\right| \leq M$ for all $t \in [a,b]$. Then $[u^2(t) + v^2(t)]^{1/2} \leq M$ for all $t \in [a,b]$. Also, note that r' and s' are each uniformly continuous on $[a,b]$. Hence, we know that for each $\epsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $\left|b_k - s_k\right| < \delta_1$ then $\left|r'(b_k) - r'(s_k)\right| < \frac{\epsilon}{8M(b-a)}$, and if $\left|c_k - s_k\right| < \delta_2$ then $\left|s'(c_k) - s'(s_k)\right| < \frac{\epsilon}{8M(b-a)}$. We remark that each of δ_1 and δ_2 depends only upon the choice of ϵ and not upon the points c_k, s_k and b_k which are contained in $[a,b]$. For each choice of ϵ above let $\delta = \min(\delta_1, \delta_2)$ and examine one of the terms in the right hand expression for $\left|S(P,f,g) - S(P,h)\right|$: Expressing $f(s_k)$ as $u(s_k) + iv(s_k)$ we have

$$\left|\sum_{k=1}^{n} u(s_k)(r'(b_k)-r'(s_k))\Delta_k t\right| < M(b-a)\left[\frac{\epsilon}{8M(b-a)}\right] = \frac{\epsilon}{8},$$

whenever we take a partition P_0 of $[a,b]$ such that $\|P_0\| < \delta$, or any refinement of P_0, say P, on $[a,b]$. Moreover, we claim that the remaining three terms in the equation for $\left|S(P,f,g)-S(P,h)\right|$ can be expressed in exactly the same way, yielding by the triangular-inequality

$$\left|S(P_0,f,g)-S(P_0,h)\right| = \left|\sum_{k=1}^{n}u(s_k)[r'(b_k)-r'(s_k)]\Delta_k t - \sum_{k=1}^{n}v(s_k)[s'(c_k)-s'(s_k)]\Delta_k t + i\sum_{k=1}^{n}u(s_k)[s'(c_k)-s'(s_k)]\Delta_k t + i\sum_{k=1}^{n}v(s_k)[r'(b_k)-r'(s_k)]\Delta_k t\right|$$

$< \frac{\epsilon}{2}$, for all partitions P_0 of $[a,b]$ such that $\|P_0\| < \delta$. Now since $f \in I(g)$ on $[a,b]$ we know that for $\epsilon > 0$ there is a partition P_1 of $[a,b]$ such that if P is a refinement of P_1 then $\left| S(P,f,g) - \int_a^b fdg\right| < \frac{\epsilon}{2}$. If $P = P_0 \cup P_1$ or if P is a refinement of $P_0 \cup P_1$ then by using the triangle-inequality again we obtain $\left| S(P,h) - \int_a^b fdg \right| < \epsilon$, which implies that $\int_a^b fdg = \int_a^b f\cdot g'dt$.

The Algebra of Derivatives

We will now consider the derivatives of two complex-valued functions f and g defined on $[a,b]$ by $f(t) = x(t) + iy(t)$ and $g(t) = u(t) + iv(t)$. It is a straight forward task to show that our definition of f' by $f'(t) = x'(t) + iy'(t)$ satisfies the conventional limit of the difference quotient definition. Furthermore, the following propositions are easily verified by results from calculus:

<u>PROPOSITION 3.5</u>: If $H(t) = f(t) \pm g(t)$ then $H'(t) = f'(t) \pm g'(t)$.

Proof: $H'(t) = [x'(t) + iy'(t)] \pm [u'(t) + iv'(t)]$.

PROPOSITION 3.6: If $H(t) = f(t) \cdot g(t)$ then $H'(t) = f'(t)g(t) + g'(t)f(t)$.

Proof: Differentiation of the real-valued terms of the product function H may be carried out in a straight forward way. Then by using the commutative, associative and distributive properties for complex numbers we arrive at the above result.

PROPOSITION 3.7: If $H(t) = f(t)/g(t)$ then $H'(t) = [f'(t)g(t) - g'(t)f(t)]/[g(t)]^2$.

Proof: The result follows from the same procedure which was suggested in Proposition 3.6.

A Fundamental Theorem

We recall in the study of real-valued functions the concept of bounded variation: If u is defined on [a,b], and $P = \{t_k\}_{k=1}^{n}$ is a partition of [a,b] and there exists a real number $M > 0$ such that $\sum_{k=1}^{n} |\Delta_k u| \leq M$, for all partitions P of [a,b], then we say that u is of bounded variation on [a,b]. Hence we make the

Definition: A complex-valued function g defined on [a,b] by $g(t) = u(t) + iv(t)$ is said to be of bounded variation on [a,b] if and only if each of the real-valued functions u and v are of bounded variation on [a,b].

In the study of real analysis it is shown that if r and s are real-valued functions defined on [a,b], if r is continuous on $[a,b]$ and if s is of bounded variation on $[a,b]$ then $r \in I(s)$ on each subinterval of $[a,b]$ [1]. If, in addition to the hypothesis of this result, we have that s is increasing in $[a,b]$ and that s' exists at each $t \in [a,b]$ then we can state the general theorem known as the fundamental theorem of integral calculus [4]. We recall this result in the following

PROPOSITION 3.8: Define $H : [a,b] \rightarrow R$ by $H(t) = \int_a^t r\,ds$, $t \in [a,b]$. With the above hypothesis on r and s it can be verified that $H'(t) = r(t)s'(t)$, for all $t \in [a,b]$.

Now we return to the complex situation to extend the fundamental theorem above in a natural way.

PROPOSITION 3.9: Let f and g be complex-valued functions defined on $[a,b]$ such that f is continuous on $[a,b]$, $f(t) = x(t) + iy(t)$ and g is of bounded variation on $[a,b]$, where $g(t) = u(t) + iv(t)$. Also require that u and v be increasing on $[a,b]$ and that u' and v' exist everywhere on $[a,b]$. Then for H defined by $H(t) = \int_a^t f\,dg$, $t \in [a,b]$, $H'(t) = f(t)g'(t)$.

Proof: Since $H(t) = \int_a^t f\,dg$ we can write $H(t) = U(t) + iV(t)$ for U and V real-valued functions defined on $[a,b]$. By Proposition 3.2 we can express each of U and V as Riemann-Stieltjes integrals:

$$U(t) = \int_a^t x\,du - \int_a^t y\,dv \quad \text{and}$$

$$V(t) = \int_a^t y\,du + \int_a^t x\,dv \; .$$

Then by Proposition 3.8 we have that $U'(t) = x(t)u'(t) - y(t)v'(t)$

and that $V'(t) = y(t)u'(t) + x(t)v'(t)$. Hence, we conclude from our definition of the derivative that $H'(t) = f(t)g'(t)$.

The next result which we consider is a theorem used generally to evaluate an integral whose integrand in real-valued. We extend the theorem for the case of a complex-valued integrand and a real-valued integrator.

<u>PROPOSITION 3.10</u>: If f is a complex-valued function with $f \in I[g]$ on $[a,b]$, $g(t) = t$ on $[a,b]$ and if H is a complex-valued function defined on $[a,b]$ such that $H' = f$, then

$$\int_a^b f\,dt = H(b) - H(a).$$

<u>Proof</u>: We express H in terms of its real-valued components; that is, $H(t) = U(t) + iV(t)$. Now, $H'(t) = U'(t) + iV'(t) = f(t)$, for all $t \in [a,b]$. If $f(t) = u(t) + iv(t)$ then $U'(t) = u(t)$ and $V'(t) = v(t)$, for all $t \in [a,b]$. Hence, $U(b) - U(a) = \int_a^b u\,dt$ and $V(b) - V(a) = \int_a^b v\,dt$. Moreover, since $\int_a^b u(t)dt + i\int_a^b v(t)dt = \int_a^b [u(t) + iv(t)]dt$ we have that $H(b) - H(a) = \int_a^b f\,dt$.

Finally, we state a sufficient condition for $f \in I(g)$ on $[a,b]$.

<u>Corollary to Proposition 3.2</u>: If f is continuous on $[a,b]$ and g is of bounded variation on $[a,b]$ then $f \in I(g)$.

<u>Proof</u>: Let $f(t) = x(t) + iy(t)$ and $g(t) = u(t) + iv(t)$. Since f is continuous on $[a,b]$ each of x and y is continuous on $[a,b]$. Also, since g is of bounded variation on $[a,b]$ each of u and v is of bounded variation on $[a,b]$. Hence $\int_a^b f\,dg = \left(\int_a^b x\,du - \int_a^b y\,dv\right) + i\left(\int_a^b x\,dv + \int_a^b y\,du\right)$, according to Proposition 3.2, since each of the integrals on the right exists.

SUMMARY

If we consider complex-valued functions whose domain is the closed interval [a,b] and the Riemann-Stieltjes integral of these functions we see that the development is a natural extension of the real case.

Once we have represented the complex case in terms of the real situation we can realize some generalizations. In particular, we developed elementary properties of the Riemann-Stieltjes integral of our complex-valued functions. Moreover, we found that in order to extend the theorems in the real case we had to define such concepts as continuity, differentiation, and bounded variation in terms of the components of the complex-valued function. Thus, we were able to express the integral of the complex-valued function in terms of four integrals of real-valued functions.

The final consideration of this paper dealt with the special situation of complex-valued integrand and the real-valued identity function as integrator. The result took the form of the formula generally used to compute real-valued integrals.

We have also included in our final remark a sufficient condition for the existence of integrals of this complex type.

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