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In this thesis the author discusses iterative processes as a method for finding zeros of various polynomials. The material is divided into four sections: deriving the iterative function, an illustrative computer program, attaining an initial value, and another illustrative computer program.

ASPECTS OF NUMERICAL ANALYSIS

RELATIVE TO COMPUTING

by

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TABLE OF CONTENTS

INTRODUCTION

The purpose of this thesis is to investigate the problem of finding zeros of various functions through iterative methods. Chapter One is concerned with deriving an iterative function in such a manner as to insure that its fixed point is a zero of the function with which we started.

The relevance of the Lipschitz Condition is unveiled as criteria are hypothesized and proven to insure the existence of the fixed point property. After certain conditions have been formulated, Chapter One then deals with the actual derivation of an iterative function. It also specifies under what conditions convergence to a fixed point can be accelerated.

Chapter Two then takes the concepts of Chapter One, incorporates them into a computer program and illustrates what has been previously proven. Two different iterative methods were applied to the same function and then accelerated processes were used on them.

While Chapter One deals with the derivation of an iterative function and guaranteeing its convergence once the iteration process has started, Chapter Three theroizes a method to give an initial starting value for the process. Thus, Chapter Three deals with Bernoulli's Method and the conditions which must be imposed to insure the proper starting value.

Chapter Four is also a computer program with its data which is used to illustrate the theory of Chapter Three. Chapter Four also gives a process of accelerating Bernoulli's Method.

The author will assume the following well-known properties of real-valued continuous functions.

1) The Identity function I defined by $I(x) = x$ for all real numbers x, is continuous for any real number x.

2) Let m and b be real numbers and define f by $f(x) = (mx + b)$ for all real numbers x, then for any real number a, $\lim_{x \to a} f(x) = ma + b$.

3) Let c be a real number and define f by $f(x) = c$ for all real numbers x, then for any real number a, $\lim_{x \to a} f(x) = c$.

4) For any real number a, $\lim_{x \to a} I(x) = a$.

5) Let a, L, M be real numbers, let f and g be functions and suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f + g)(x) = L + M$.

6) Let a, L, M be real numbers, let f and g be functions and suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f \cdot g)(x) = L \cdot M$.

7) Let a, L, M be real numbers with $M \neq 0$, let f and g be functions and suppose $\lim_{x \to a} g(x) = M$ and $\lim_{x \to a} f(x) = L$.

Define $F(x) = f(x) / g(x)$ for all x in both the domain of f and g such that $g(x) \neq 0$. Then, $\lim_{x \to a} F(x) = L/M$.

8) All Log functions are increasing on the positive real number line.

If any difficulty is encountered in understanding the concepts and material of this thesis, please refer to [4, pp 61-96, pp 146-161].

CHAPTER I

ITERATIVE FUNCTIONS

Definition 1: Let X be a set and let f be a function from X into X. Then, α is said to be a fixed point of f provided $f(\alpha) = \alpha$.

Definition 2: A function f from the interval $[a,b]$ into the real numbers is said to satisfy the Lipschitz condition provided there is an $L \geq 0$, such that if $x_1, x_2 \in [a,b]$ then $| f(x_1) - f(x_2) | \leq L \cdot | x_1 - x_2 |$. The number L is referred to as a Lipschitz constant for f on $[a,b]$.

Definition 3: A sequence is a function whose domain is the set of non-negative integers.

Theorem 1: Let $f : [a,b] \to$ Reals satisfy the Lipschitz condition, then f is continuous.

Proof: Since f satisfies the Lipschitz condition, there is an $L \geq 0$ such that if $x_1, x_2 \in [a,b]$ then $| f(x_1) - f(x_2) | \leq L \cdot | x_1 - x_2 |$.

Let $P \in [a,b]$, and let $\epsilon > 0$. Let $\delta = \epsilon / (L + 1)$ and let $x \in [a,b]$ such that $| x - P | < \delta$. Then

$| f(x) - f(P) | \leq L \cdot | x - P | \leq$

$L \cdot \delta = L \cdot [\epsilon / (L + 1)] = \epsilon \cdot (L / L + 1) < \epsilon$.

Therefore f is continuous.

<u>Theorem 2</u>: If $f : [a,b] \to [a,b]$ and f satisfies the Lipschitz condition with a Lipschitz constant L for f on $[a,b]$ less than 1, then f has exactly one fixed point.

Proof: Let I be the identity function on $[a,b]$ and let $f : [a,b] \to [a,b]$ be a function satisfying the Lipschitz condition with a Lipschitz constant L for f on $[a,b]$ less than 1. Define a function g on $[a,b]$ by $g(x) = I(x) - f(x)$ for all $x \in [a,b]$.

By theorem 1, f is continuous on $[a,b]$. Since the identity function I is continuous on $[a,b]$, then g is continuous on $[a,b]$, [1, Th. 1, p 94].

Since $f : [a,b] \to [a,b]$ then $f(a) \geq a$ and $f(b) \leq b$. Thus, $g(a) = a - f(a) \leq a - a = 0$ and $g(b) = b - f(b) \geq b - b = 0$.

Since g is continuous, then by the Intermediate Value Theorem, [1, TH. 1, p 100], there exists an $s \in [a,b]$ such that $g(s) = 0$. But $s - f(s) = g(s) = 0$ and therefore $f(s) = s$. Hence f has a fixed point, s.

Suppose there is a number $s_1 \in [a,b]$ such that $f(s_1) = s_1$. Then $f(s_1) - f(s) = s_1 - s$ and $| f(s_1) - f(s) | = | s_1 - s |$. Since f satisfies the Lipschitz condition with Lipschitz constant L for f on $[a,b]$ less than 1, then $| f(s_1) - f(s) | \leq L \cdot | s_1 - s |$ thus $| s_1 - s | \leq L \cdot | s_1 - s |$.

Since $L < 1$ then $s_1 = s$. Hence, f has exactly one fixed point.

<u>Example 1</u>: Let f be the function on $[0,1]$ defined by $f(x) = (x + 4) / 2$ for all $x \in [0,1]$. Then f satisfies the

Lipschitz condition with Lipschitz constant L for f on [0,1] less than 1, f does not map [0,1] into [0,1] and f has no fixed points in [0,1].

Proof: Let $x_1, x_2 \in [0,1]$, then

$| f(x_1) - f(x_2) | = | [(x_1 + 4) / 2] - [(x_2 + 4) / 2]| = | (x_1 - x_2)/ 2|$.

Thus f satisfies the Lipschitz condition with Lipschitz constant L for f on [0,1] less than 1.

Since $f(0) = 2$, f does not map [0,1] into [0,1]. Also by [1, THS. 1,3,5,6, pp 144-146] $f'(x) = 1/2$ for all $x \in [0,1]$. Therefore by [3, TH. 6.7, p 133] f is increasing on [0,1].

Since $f(0) = 2$ and f is increasing then f has no fixed points in [0,1].

<u>Example 2</u>: Let f be a function on [1/2,4] defined by $f(x) = (x^2 + 4) / 5$ for all $x \in [1/2,4]$. Then f satisfies the Lipschitz condition on [1/2,4], f does not satisfy the Lipschitz condition on [1/2,4] with Lipschitz constant L for f on [1/2,4] less than 1, and f has 2 fixed points.

Proof: Let $L = 2$ and let $x_1, x_2 \in [1/2,4]$. By the Mean Value Theorem [3, TH. 6.6 p 131], there exists a number c in (1/2,4) such that $f(x_1) - f(x_2) = f'(c) \cdot (x_1 - x_2)$. But by [1, THS. 1,3,5,6, pp 144-146], $f'(x) = (2x) / 5$ for all $x \in [1/2,4]$, and $1/5 < f'(c) < 8/5$, hence $| f'(c) | < 2$.

Thus

$| f(x_1) - f(x_2) | = | f'(c) \cdot (x_1 - x_2) | =$

$| f'(c) | \cdot | x_1 - x_2 | < 2 \cdot | x_1 - x_2 |$.

Therefore f satisfies the Lipschitz condition on $[1/2,4]$.

Since $f'(x) = (2x) / 5$ for all $x \in [1/2,4]$, f' is positive on $[1/2,4]$. Thus by [3, TH. 6.7, p 133] f is increasing on $[1/2,4]$. Now, $f(1/2) = 17/20 > 1/2$ and $f(4) = 4$, Therefore, $f : [1/2,4] \to [1/2,4]$.

But, f has two fixed points $1, 4 \in [1/2,4]$ so by Theorem 2, f does not satisfy the Lipschitz condition on $[1/2,4]$ with Lipschitz constant L for f on $[1/2,4]$ less than 1.

<u>Theorem 3</u>: Let $f : [a,b] \to [a,b]$ satisfy the Lipschitz condition with Lipschitz constant L for f on $[a,b]$ less than 1. Let $x_0 \in [a,b]$ and for eacn non-negative integer n define $x_{n+1} = f(x_n)$. Then, if s is the fixed point of f, x has limit s.

Proof: Since f satisfies the Lipschitz condition,

$| x_1 - s | = | f(x_0) - f(s) | \leq L \cdot | x_0 - s |$.

Assume $| x_{k-1} - s | \leq L^{k-1} \cdot | x_0 - s |$, then

$$| x_k - s | = | f(x_{k-1}) - f(s) | \leq L \cdot | x_{k-1} - s | \leq$$
$$L \cdot L^{k-1} \cdot | x_0 - s | = L^k \cdot | x_0 - s |.$$

Thus, by mathematical induction, if n is a non-negative integer, then $| x_n - s | \leq L^n \cdot | x_0 - s |$.

Let $\epsilon > 0$. There is a positive integer $N > (\log [\epsilon / (| x_0 - s | + 1)]) / (\log L)$. Let $n > N$. Since $L < 1$ then $\log L < 0$. Therefore, $n \cdot \log L < \log [\epsilon / (| x_0 - s | + 1)]$ or $\log L^n < \log [\epsilon / (| x_0 - s | + 1)]$. By [3, TH. 9.22, p 264] the log function is increasing, so $L^n < \epsilon / (| x_0 - s | + 1)$.

Thus,

$$| x_n - s | \le L^n \cdot | x_0 - s | \le$$

$$[\epsilon / (| \ x_0 - s | + 1)] \cdot | x_0 - s | \le \epsilon.$$

Therefore $\lim_{n\to\infty} x_n = s$

Theorem 4: Let f, f' be continuous functions on $[a,b]$ and suppose there is an $s \in (a,b)$ such that $f(s) = s$ and $f'(s) = 0$. Then there is a $d > 0$ such that if $x_0 \in [a,b]$ such that $| x_0 - s | < d$ and for each non-negative integer n, $x_{n+1} = f(x_n)$, then x has limit s.

Proof: Let $d_0 = (1/2)$ [minimum $\{s - a, b - s\}$]. Since $a, b > 0$ and $a < s < b$ then $d_0 > 0$. Let $x \in [s - d_0, s + d_0]$ then, $a \le s - d_0 \le x$ and $x \le d_0 + s \le b$ thus $x \in [a,b]$. Hence $[s - d_0, s + d_0] \subset [a,b]$.

If $0 \le d \le d_0$ then $[s - d, s + d] \subset [a,b]$. Let L be a number such that $0 < L < 1$. Since f' is continuous then there exists a $d_1 > 0$ such that if $| s - x | < d_1$ then $| f'(x) - f'(s) | < L$. Let d = minimum $\{d_0, d_1\}$. Let $x \in [s - d, s + d]$. By the Mean Value Theorem, [3, TH. 6.6, p 131], there exists a number c between x and s such that $f(x) - f(s) = f'(c) \cdot (x - s)$. Since $| s - c | < d_1$, $| f'(c) | = | f'(c) - f'(s) | < L$.

Therefore

$$| f(x) - s | \le | f(x) - f(s) | = | f'(c) \cdot (x - s) | \le$$

$$L \cdot | x - s | < L \cdot d < d .$$

Thus, $f(x) \in [s - d, s + d]$ and $f : [s - d, s + d] \to [s - d, s + d]$.

Now, for $x_1, x_2 \in [s - d, s - d]$ by the Mean Value Theorem, [3, TH. 6.6, p 131], there exists a number k such that $f(x_1) - f(x_2) = f'(k) \cdot (x_1 - x_2)$. But $| f(x_1) - f(x_2) | = | f'(k) | \cdot | x_1 - x_2 | < L \cdot | x_1 - x_2 |$. Thus, $f : [s - d, s + d] \to [s - d, s + d]$ satisfies the Lipschitz condition with Lipschitz constant L for f on $[s - d, s + d]$ less than 1. So by Theorem 3, x has limit s.

<u>Theorem 5</u>: Let f, f', f'' be continuous functions on $[a,b]$ and suppose there is an $s \in [a,b]$ such that $f(s) = 0$ and f' is never zero. Then there is a number $d > 0$ such that if $x_0 \in [a,b]$ such that $| x_0 - s | < d$ and for each non-negative integer n, $x_{n+1} = x_n - [f(x_n) / f'(x_n)]$ then x has limit s.

Proof: Define G by $G(x) = x - [f(x) / f'(x)]$ for every $x \in [a,b]$. By [1, TH. 1, p 94], G is continuous.
Now, $G(s) = s - [f(s) / f'(s)] = s$.

By [1, THS. 2,3,4, pp 144-145],
$G'(x) = 1 - [(f'(x) \cdot f'(x) - f(x) \cdot f''(x)) / (f'(x))^2]$.
By [1, TH. 1, p 94] G' is continuous. Also,
$G'(s) = 1 - [(f'(s) \cdot f'(s) - f(s) \cdot f''(s)) / (f'(s))^2] = 1 - 1 = 0$.

Now since G is continuous on $[a,b]$ $G(s) = s$, $G'(s) = 0$ then by Theorem 4, there exists a $d > 0$ such that if $x_0 \in [a,b]$ such that $| x_0 - s | < d$ and for each non-negative integer n, $x_{n+1} = G(x_n)$ then x has limit s.

It is clear that this number d will work and the theorem is proved.

For the remainder of this chapter, the symbols once defined will not change.

Let $f : [a,b] \to [a,b]$ satisfy the Lipschitz condition with Lipschitz constant L for f on $[a,b]$ less than 1, and let f' be continuous and never zero on (a,b). Suppose $x_0 \in [a,b]$ and $f(x_0) \neq x_0$. Define a sequence x by if n is a non-negative integer then $x_{n+1} = f(x_n)$.

Theorem 6: For any positive integer n, $f(x_n) \neq x_n$.

Proof: Suppose there is a positive integer j such that $f(x_j) = x_j$. Let k be the smallest positive integer such that $f(x_k) = x_k$. Then $f(x_k) = x_k = f(x_{k-1})$ and $x_{k-1} \neq x_k$ since k is the smallest positive integer such that $f(x_k) = x_k$ and $f(x_{k-1}) = x_k$.

By the Mean Value Theorem, [3, TH. 6.6, p 131], there exists a number c between x_{k-1} and x_k such that $f(x_{k-1}) - f(x_k) = f'(c) \cdot (x_{k-1} - x_k)$. But since $f(x_{k-1}) - f(x_k) = 0$ and $x_{k-1} - x_k \neq 0$, $f'(c) = 0$ which contradicts the fact that f' is never zero on $[a,b]$.

Therefore, for any positive integer n, $f(x_n) \neq x_n$.

Let $s \in [a,b]$ such that $f(s) = s$. For each non-negative integer n define $d_n = x_n - s$. By Theorem 6, if n is a non-negative integer then $d_n \neq 0$.

Let n be a non-negative integer. Since $s \in [a,b]$ and $x_n \in [a,b]$, then $d_n + s$ is in $[a,b]$. Since f is continuous on

$[a,b]$ and differentiable on (a,b), then f is continuous on $[s, d_n + s]$ and differentiable on $(s, d_n + s)$ if $d_n > 0$; and f is continuous on $[d_n + s, s]$ and differentiable on $(d_n + s, s)$ if $d_n < 0$.

Therefore, by the Mean Value Theorem, [3, TH. 6.6, p 131], there exists a number c between s and $d_n + s$ such that $f(d_n + s) - f(s) = f'(c) \cdot (d_n + s - s)$.

Since c is between s and $d_n + s$ there is a θ_n where $0 \le \theta_n < 1$ such that $c = s + \theta_n \cdot d_n$. Thus, $f(d_n + s) - f(s) = f'(s + \theta_n \cdot d_n) \cdot d_n$. Therefore, if m is a non-negative integer, there is a number θ_m such that $0 \le \theta_m < 1$ and $f(d_m + s) - f(s) = f'(s + \theta_m \cdot d_m) \cdot d_m$.

If m is a non-negative integer, define ϵ_m to be $f'(s + \theta_m \cdot d_m) - f'(s)$.

<u>Theorem 7</u>: Then $\lim_{m\to\infty} \epsilon_m = 0$.

Proof: By Theorem 4, $\lim_{m\to\infty} x_m = s$. Since for each non-negative integer m, $d_m = x_m - s$, then $\lim_{m\to\infty} d_m = \lim_{m\to\infty} x_m - \lim_{m\to\infty} s = s \cdot s = 0$. Since for each non-negative integer m, θ_m is greater than or equal to 0 and less than 1, $| \theta_m \cdot d_m | < | d_m |$. Since $\lim_{m\to\infty} d_m = 0$, then $\lim_{\to\infty} (\theta_m \cdot d_m) = 0$. By the continuity of f', $\lim_{m\to\infty} f'(s + \theta_m \cdot d_m) = f'(s)$. Thus, $\lim_{m\to\infty} \epsilon_m = \lim_{m\to\infty} f'(s + \theta_m \cdot d_m) - \lim_{m\to\infty} f'(s) = f'(s) - f'(s) = 0$.

<u>Theorem 8</u>: If u is a non-negative integer then $d_{u+1} = f'(s + \theta_u \cdot d_u) \cdot d_u$ and $\lim_{m\to\infty} [(d_{m+1}) / (d_m)] = f'(s)$.

Proof: Let u be a non-negative integer. Then

$d_{u+1} = x_{u+1} - s = f(x_u - s) = f(d_u + s) - f(s) =$

$f'(s + \theta_u \cdot d_u) \cdot d_u$.

If m is a non-negative integer, since

$\epsilon_m = f'(s + \theta_m \cdot d_m) - f'(s)$, then $f'(s + \theta_m \cdot d_m) = \epsilon_m + f'(s)$ and $d_{m+1} = [\epsilon_m + f'(s)] \cdot d_m$, and $[(d_{m+1}) / d_m] = [\epsilon_m + f'(s)]$. Thus $\lim_{m\to\infty} [(d_{m+1}) / (d_m)] = \lim_{m\to\infty} [\epsilon_m + f'(s)] = \lim_{m\to\infty} \epsilon_m + \lim_{m\to\infty} f'(s)$.

But by Theorem 7, $\lim_{m\to\infty} \;_m = 0$. Therefore,

$\lim_{m\to\infty} [(d_{m+1}) / (d_m)] = 0 + \lim_{m\to\infty} f'(s) = 0 + f'(s) = f'(s)$.

Let $A = f'(s)$. Now, $f(s) = \lim_{x\to s} [(f(x) - f(s)) / (x - s)]$.
For each $x \in (a,b)$, since f satisfies the Lipschitz condition with Lipschitz constant L for f on $[a,b]$ less than 1 then

$| f(x) - f(s) | \le L \cdot | x - s |$; and therefore

$| (f(x) - f(s)) / (x - s) | \le L$ and

$- L \le (f(x) - f(s)) / (x - s) \le L$. Thus $- L \le f'(s) \le L$; and therefore, $A = f'(s) \le L < 1$. If m is a non-negative integer, define ϵ_m' to be $A \cdot (\epsilon_m + \epsilon_{m+1}) - 2\, \epsilon_m + \epsilon_m \cdot \epsilon_{m+1}$.

<u>Theorem 9</u>: Then, $\lim_{m\to\infty} \epsilon_m' = 0$.

Proof: We have

$\lim_{m\to\infty} \epsilon_m' = \lim_{m\to\infty} (A \cdot \epsilon_m) + \lim_{m\to\infty} (A \cdot \epsilon_{m+1}) - \lim_{m\to\infty} (2 \cdot \epsilon_m) + \lim_{m\to\infty} (\epsilon_m \cdot \epsilon_{m+1})$.
By Theorem 7, $\lim_{m\to\infty} \epsilon_m = 0$, thus $\lim_{m\to\infty} \epsilon_m' = 0 + 0 + 0 + 0 = 0$.

For each non-negative integer m, define

$\Delta x_m = x_{m+1} - x_m$ and $\Delta^2 x_m = x_{m+2} - 2\, x_{m+1} + x_m$. Define a sequence x' by if m is a non-negative integer, then

$x_m' = x_m - [(\Delta x_m)^2 / (\Delta^2 x_m)]$.

Theorem 10: The sequence x' converges to s and x' converges to s faster than x in the sense that $\lim_{m\to\infty} [(x_m' - s) / (x_m - s)] = 0$.

Proof: Let w be a non-negative integer. By Theorem 8, $d_{w+1} / d_w = f'(s) + \epsilon_w$. Since f is Lipschitz,

$d_{w+1} = (A + \epsilon_w) \cdot d_w$, and

$d_{w+2} = (A + \epsilon_{w+1}) \cdot d_{w+1} = (A + \epsilon_{w+1}) \cdot (A + \epsilon_w) \cdot d_w$.

Since $x_w = d_w + s$ then

$$\Delta^2 x_w = d_{w+2} + s - 2\, d_{w+1} - 2\, s + d_w + s = d_{w+2} - 2\, d_{w+1} + d_w =$$
$$(A + \epsilon_{w+1}) \cdot (A + \epsilon_w)\, d_w - 2 \cdot (A + \epsilon_w) \cdot d_w + d_w =$$
$$d_w \cdot [(A - 1)^2 + \epsilon_w'].$$

Thus, if m is a non-negative integer, then $\Delta^2 x_m = d_m \cdot [(A - 1)^2 + \epsilon_m']$. By Theorem 9, $\lim_{m\to\infty} \epsilon_m' = 0$. Since $A = f'(s) \le L < 1$, then $A - 1 \neq 0$. Since if p is a non-negative integer, then $d_p \neq 0$, and $(A - 1)^2 > 0$, and $\lim_{m\to\infty} \epsilon_m' = 0$, there is a non-negative integer z such that if $m \ge z$, then $\Delta^2 x_m \neq 0$.

Let q be an integer such that $q \ge z$. Then

$$\Delta x_q = (x_{q+1} - x_q) = (d_{q+1} + s - d_q - s) = (d_{q+1} - d_q) =$$
$$(A + \epsilon_q) \cdot d_q - d_q = (A + \epsilon_q - 1) \cdot d_q\ .$$

Also,

$$x_q' - s = x_q - s - ((\Delta x_q)^2 / (\Delta^2 x_q)) =$$
$$d_q - [((A + \epsilon_q - 1)^2 \cdot d_q^2) / (((A - 1)^2 + \epsilon_q') \cdot d_q)] =$$
$$d_q \cdot [(1 - (A + \epsilon_q - 1)^2) / ((A - 1)^2 + \epsilon_q')] =$$
$$d_q \cdot [(\epsilon_q' - 2 \epsilon_q (A - 1) - \epsilon_q^2) / ((A - 1)^2 + \epsilon_q')].$$

Thus, if m is an integer such that $m \geq z$, then

$$x_m' - s = d_m \cdot [(\epsilon_m' - 2 \epsilon_m (A - 1) - \epsilon_m^2) / ((A - 1)^2 + \epsilon_m')].$$

By Theorems 7 and 9 $\lim_{m \to \infty} \epsilon_m = 0$ and $\lim_{m \to \infty} \epsilon_m' = 0$.

Thus $\lim_{m \to \infty} x_m' - s = 0$ and $\lim_{m \to \infty} x_m' = s$. So, x' converges to s.

Let t be an integer such that $t \quad z$. Then

$$(x_t' - s) / (x_t - s) = (x_t' - s) / d_t =$$
$$(\epsilon_t' - 2 \epsilon_t (A - 1) - \epsilon_t^2) / ((A - 1)^2 + \epsilon_t').$$

Therefore $\lim_{m \to \infty} [(x_m' - s) / (x_m - s)] = 0$.

CHAPTER II

PROGRAM ONE

In this chapter we use the techniques discussed in Chapter One and apply them to the function:

$$f(x) = x^3 - x^2 - x - 1.$$

The following program uses two different methods to derive the iterative schemes and then uses some accelerated methods on their iterates. The first iterative scheme is determined by setting $f(x) = 0$ and solving for an x. Thus, the iterative scheme is defined by

$$x_{n+1} = 1 + (1 / x_n) + (1 / x_n^2)$$

The second method used to derive an iterative scheme is called Newton's Method. It is defined as follows:

$$x_{n+1} = x_n - [f(x_n) / f'(x_n)].$$

Note how much faster this scheme converges than the first one.

With the sequence of numbers obtained from the two previous iterative methods, an accelerated process, Steffenson's Method, (referred to in the program data as little Aitken's Method), is applied. It is easily seen how much faster the iterates converge to their fixed point than either of the previous methods. Then for the final process, Aitken's δ^2 Method, (referred to as big Aitken's Method), was incorporated, using as its initial values, the first three iterates obtained from the first and second iterative schemes.

Aitken's δ^2 method can be seen to converge quicker than any of the previously mentioned schemes.

The program and data are as follows:

```
*PL/C  ('TIME=(,003),PAGES=010,NOBOUNDARY')

*OPTIONS IN EFFECT*     PAGES=010,TIME=(000,003),ERRORS=(050,050),SORMGIN=(002,072,001),LINECNT=060,NOBOUNDARY,
*OPTIONS IN EFFECT*     FLAGW,XREF,ATR,NOLIST,UDEF,SOURCE,DUMP,NODUMPARRAY,M91,NOCOMMENTS,CHECK

   ROUGH: PROCEDURE OPTIONS (MAIN);                                           PL/C-R6.5000  03/05/73 23:53 PAGE   1

  STMT LEVEL NEST BLOCK      SOURCE STATEMENT                                                              ID FIELD

    1                        ROUGH: PROCEDURE OPTIONS (MAIN);
    2     1          1       DECLARE A(100),B(100),C(100),D(100),E(100),F(100),G(100);
    3     1          1       DO I=1 TO 100;
    4     1     1    1       A(I)=I;
    5     1     1    1       END;
    6     1          1       X=2.8;
    7     1          1       XT=100;
    8     1          1       B(1)=X;
    9     1          1       DO I=2 TO 100;
   10     1     1    1       X=1+1/X+1/X**2;
   11     1     1    1       IF ABS(ABS(X)-ABS(XT))<.0000001 THEN DO;
   13     1     2    1       N1=I;
   14     1     2    1       B(I)=X;
   15     1     2    1       GO TO NEWT;
   16     1     2    1       END;
   17     1     1    1       XT=X;
   18     1     1    1       B(I)=X;
   19     1     1    1       N1=I;
   20     1     1    1       END;
   21     1          1       NEWT:
                             X=2.8;
   22     1          1       XT=100;
   23     1          1       C(1)=2.8;
   24     1          1       DO I=2 TO 100;
   25     1     1    1       W=X**3-X**2-X-1;
   26     1     1    1       Z=3*X**2-2*X-1;
   27     1     1    1       X=X-(W/Z);
   28     1     1    1       IF ABS(ABS(X)-ABS(XT))<.0000001 THEN DO;
   30     1     2    1       N2=I;
   31     1     2    1       C(I)=X;
   32     1     2    1       GO TO ALGACC;
   33     1     2    1       END;
   34     1     1    1       C(I)=X;
   35     1     1    1       XT=X;
   36     1     1    1       N2=I;
   37     1     1    1       END;
   38     1          1       ALGACC: D(1)=2.8;
   39     1          1       XT=100;
   40     1          1       DO I=2 TO N1-2;
   41     1     1    1       W=(B(I+1)-B(I))**2;
   42     1     1    1       Z=B(I+2)-2*B(I+1)+B(I);
   43     1     1    1       X=B(I)-(W/Z);
   44     1     1    1       IF ABS(ABS(X)-ABS(XT))<.0000001 THEN DO;
   46     1     2    1       N3=I;
   47     1     2    1       D(I)=X;
   48     1     2    1       GO TO NEWACC;
```

```
STMT LEVEL NEST BLOCK      SOURCE STATEMENT                                        ID FIELD

  49     1    2     1      END;
  50     1    1     1      D(I)=X;
  51     1    1     1      XT=X;
  52     1    1     1      N3=I;
  53     1    1     1      END;
  54     1          1      NEWACC:E(1)=2.8;
  55     1          1      XT=100;
  56     1          1      DO I=2 TO N2-2;
  57     1    1     1      W=(C(I+1)-C(I))**2;
  58     1    1     1      Z=C(I+2)-2*C(I+1)+C(I);
  59     1    1     1      X=C(I)-(W/Z);
  60     1    1     1      IF ABS(ABS(X)-ABS(XT))<.0000001 THEN DO;
  62     1    2     1      N4=I;
  63     1    2     1      E(I)=X ;
  64     1    2     1      GO TO ALGBAC;
  65     1    2     1      END;
  66     1    1     1      E(I)=X ;
  67     1    1     1      XT=X;
  68     1    1     1      N4=I;
  69     1    1     1      END;
  70     1          1      ALGBAC: F(1)=2.8;
  71     1          1      XT=100;
  72     1          1      X=2.8;
  73     1          1      DO I=2 TO 100;
  74     1    1     1      X1=1+1/X+1/X**2;
  75     1    1     1      X2=1+1/X1+1/X1**2;
  76     1    1     1      X=X-((X1-X)**2)/(X2-2*X1+X);
  77     1    1     1      IF ABS(ABS(X)-ABS(XT))<.0000001 THEN DO;
  79     1    2     1      N5=I;
  80     1    2     1      F(I)=X;
  81     1    2     1      GO TO NEWBAC;
  82     1    2     1      END;
  83     1    1     1      F(I)=X;
  84     1    1     1      XT=X;
  85     1    1     1      N5=I;
  86     1    1     1      END;
  87     1          1      NEWBAC: G(1)=2.8;
  88     1          1      XT=100;
  89     1          1      X=2.8;
  90     1          1      DO I=2 TO 100;
  91     1    1     1      W=X**3-X**2-X-1;
  92     1    1     1      Z=3*X**2-2*X-1;
  93     1    1     1      X1=X-(W/Z);
  94     1    1     1      W=X1**3-X1**2-X1-1;
  95     1    1     1      Z=3*X1**2-2*X1-1;
  96     1    1     1      X2=X1-(W/Z);
  97     1    1     1      X=X-((X1-X)**2)/(X2-2*X1+X);
  98     1    1     1      IF ABS(ABS(X)-ABS(XT))<.0000001 THEN DO;
 100     1    2     1      N6=I;
 101     1    2     1      G(I)=X;
 102     1    2     1      GO TO FINISH;
 103     1    2     1      END;
 104     1    1     1      G(I)=X;
 105     1    1     1      XT=X;
 106     1    1     1      N6=I;
```

```
STMT LEVEL NEST BLOCK      SOURCE STATEMENT                                                          ID FIELD

 107     1     1     1     END;
 108     1           1     FINISH:
                           PUT PAGE;
 109     1           1     PUT EDIT('THE FOLLOWING COLUMNS LIST THE NUMBER OF ITERATIONS NEEDED FO
                           R 6 DIFFERENT ITERATION METHODS TO CONVERGE TO A',     'SPECIFIED ACCUR
                           ACY.',                                                                 '
                           EACH COLUMN REPRESENTS THE FOLLOWING:',     'COLUMN 1 - ITERATIONS BY F
                           (X)=1+1/X+1/X**2',  'COLUMN 2 - ITERATIONS USING LITTLE AITKENS METHOD
                           AND THE VALUES FROM F(X)=1+1/X+1/X**2',    'COLUMN 3 - ITERATIONS USING
                           NEWTONS METHOD',     'COLUMN 4 - ITERATIONS USING LITTLE AITKENS METHOD
                            AND THE VALUES FROM NEWTONS METHOD',      'COLUMN 5 - ITERATIONS USING
                            BIG AITKENS METHOD AND F(X)=1+1/X+1/X**2',    'COLUMN 6 - ITERATIONS U
                           SING BIG AITKENS METHOD AND F(X) DERIVED FROM NEWTON"S METHOD')
                           (X(3),A,SKIP,X(3),A,SKIP,X(5),A,SKIP(2),6(X(10),A,SKIP(2)));
 110     1           1     PUT SKIP(4);
 111     1           1     PUT EDIT('(1)','(2)','(3)','(4)','(5)','(6)')(X(19),A,5(X(13),A));
 112     1           1     PUT SKIP;
 113     1           1            J=MAX(N1,N2,N3,N4,N5,N6);
 114     1           1     DO I=1 TO J;
 115     1     1     1     PUT EDIT(A(I),'.')(X(5),F(4),A);
 116     1     1     1      IF N1>=I THEN PUT EDIT(B(I))(X(4),F(12,8));
 118     1     1     1     ELSE PUT EDIT('                ')(A);
 119     1     1     1      IF N3>=I THEN PUT EDIT(D(I))(X(4),F(12,8));
 121     1     1     1     ELSE PUT EDIT('                ')(A);
 122     1     1     1      IF N2>=I THEN PUT EDIT(C(I))(X(4),F(12,8));
 124     1     1     1     ELSE PUT EDIT('                ')(A);
 125     1     1     1      IF N4>=I THEN PUT EDIT(E(I))(X(4),F(12,8));
 127     1     1     1     ELSE PUT EDIT('                ')(A);
 128     1     1     1      IF N5>=I THEN PUT EDIT(F(I))(X(4),F(12,8));
 130     1     1     1     ELSE PUT EDIT('                ')(A);
 131     1     1     1      IF N6>=I THEN PUT EDIT(G(I))(X(4),F(12,8));
 133     1     1     1     PUT SKIP(2);
 134     1     1     1     END;
 135     1           1     END ROUGH;
```

```
DCL NO.       IDENTIFIER                         ATTRIBUTES AND REFERENCES

   2          A                                  (*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                                 2,4,115

              ABS                                GENERIC,BUILT-IN FUNCTION
                                                 11,11,11,28,28,28,44,44,44,60,60,60,77,77,77,98,98,98

  38          ALGACC                             STATEMENT LABEL CONSTANT
                                                 32,38

  70          ALGBAC                             STATEMENT LABEL CONSTANT
                                                 64,70

   2          B                                  (*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                                 2,8,14,18,41,41,42,42,42,43,117
```

```
DCL NO.   IDENTIFIER        ATTRIBUTES AND REFERENCES


   2      C                 (*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                            2,23,31,34,57,57,58,58,58,59,123

   2      D                 (*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                            2,38,47,50,120

   2      E                 (*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                            2,54,63,66,126

   2      F                 (*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                            2,70,80,83,129

 108      FINISH            STATEMENT LABEL CONSTANT
                            102,108

   2      G                 (*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                            2,87,101,104,132

   3      I                 AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
                            3,4,4,9,13,14,18,19,24,30,31,34,36,40,41,41,42,42,42,43,46,47,50,52,56,5
                            58,58,59,62,63,66,68,73,79,80,83,85,90,100,101,104,106,114,115,116,117,1
                            122,123,125,126,128,129,131,132

 113      J                 AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
                            113,114

          MAX               GENERIC,BUILT-IN FUNCTION
                            113

  54      NEWACC            STATEMENT LABEL CONSTANT
                            48,54

  87      NEWBAC            STATEMENT LABEL CONSTANT
                            81,87

  21      NEWT              STATEMENT LABEL CONSTANT
                            15,21

  13      N1                AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
                            13,19,40,113,116

  30      N2                AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
                            30,36,56,113,122

  46      N3                AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
                            46,52,113,119

  62      N4                AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
                            62,68,113,125

  79      N5                AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
                            79,85,113,128

 100      N6                AUTOMATIC,ALIGNED,BINARY,FIXED(15,0)
```

```
DCL NO.        IDENTIFIER                  ATTRIBUTES AND REFERENCES

                                           100,106,113,131

   1           ROUGH                       ENTRY,DECIMAL,FLOAT(6)
                                           1

  25           W                           AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                           25,27,41,43,57,59,91,93,94,96

   6           X                           AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                           6,8,10,10,10,11,14,17,18,21,25,25,25,26,26,27,27,28,31,34,35,43,44,47,50,
                                           60,63,66,67,72,74,74,76,76,76,76,77,80,83,84,89,91,91,91,92,92,93,97,97,9
                                           98,101,104,105

   7           XT                          AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                           7,11,17,22,28,35,39,44,51,55,60,67,71,77,84,88,98,105

  74           X1                          AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                           74,75,75,76,76,93,94,94,94,95,95,96,97,97

  75           X2                          AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                           75,76,96,97

  26           Z                           AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6)
                                           26,27,42,43,58,59,92,93,95,96

WARNING  CG0C NO FILE SPECIFIED. SYSIN/SYSPRINT ASSUMED.
```

The following results are the data from the previous computer program illustrating the various methods on the function

$$f(x) = x^3 - x^2 - x - 1.$$

Column 1 is iterations derived from the iterative sequence:

$$x_{n+1} = 1 + (1 / x_n) + (1 / x_n^2)$$

Column 2 is iterations derived from using Steffenson's accelerated method on the iterations derived from Column 1. Note that the convergence occurs in 1/2 the time.

Column 3 is iterations obtained from using Newton's Method:

$$x_{n+1} = x_n - (f(x_n) / f'(x_n))$$

Column 4 is iterations using Steffenson's accelerated method on the iterations from Newton's Method. Note accelerance.

Column 5 is iterations using Aitken's δ^2 method of accelerated convergence using the first three values of Column 1. Note that it is faster than Steffenson's Method.

Column 6 is iterations using Aitken's δ^2 accelerated method using as its values the first three iterates derived from Newton's Method.

THE FOLLOWING COLUMNS LIST THE NUMBER OF ITERATIONS NEEDED FOR 6 DIFFERENT ITERATION METHODS TO CONVERGE TO A SPECIFIED ACCURACY.
EACH COLUMN REPRESENTS THE FOLLOWING:

COLUMN 1 - ITERATIONS BY F(X)=1+1/X+1/X**2

COLUMN 2 - ITERATIONS USING LITTLE AITKENS METHOD AND THE VALUES FROM F(X)=1+1/X+1/X**2

COLUMN 3 - ITERATIONS USINGNEWTONS METHOD

COLUMN 4 - ITERATIONS USING LITTLE AITKENS METHOD AND THE VALUES FROM NEWTONS METHOD

COLUMN 5 - ITERATIONS USING BIG AITKENS METHOD AND F(X)=1+1/X+1/X**2

COLUMN 6 - ITERATIONS USING BIG AITKENS METHOD AND F(X) DERIVED FROM NEWTON"S METHOD

	(1)	(2)	(3)	(4)	(5)	(6)
1.	2.80000000	2.80000000	2.80000000	2.80000000	2.80000000	2.80000000
2.	1.48469387	1.86742042	2.19054373	1.82165411	1.91634294	1.67324419
3.	2.12719500	1.84996563	1.91073914	1.83906420	1.84018608	1.84341690
4.	1.69109914	1.84331702	1.84316771	1.83928671	1.83928688	1.83928670
5.	1.94100419	1.84082320	1.83929913	1.83928675	1.83928675	1.83928675
6.	1.78062543	1.83987033	1.83928675		1.83928675	
7.	1.87699546	1.83950912	1.83928675			
8.	1.81660631	1.83937135				
9.	1.85350193	1.83931897				
10.	1.83060029	1.83929901				
11.	1.84467859	1.83929142				
12.	1.83597210	1.83928853				
13.	1.84133662	1.83928743				
14.	1.83802370	1.83928701				
15.	1.84006676	1.83928685				
16.	1.83880572	1.83928679				
17.	1.83958365					
18.	1.83910359					
19.	1.83939978					
20.	1.83921701					
21.	1.83932978					

```
22.        1.83926020

23.        1.83930313

24.        1.83927664

25.        1.83929299

26.        1.83928290

27.        1.83928912

28.        1.83928529

29.        1.83928765

30.        1.83928619

31.        1.83928709

32.        1.83928654

33.        1.83928688

34.        1.83928667

35.        1.83928680

36.        1.83928672


IN STMT  135  PROGRAM RETURNS FROM MAIN PROCEDURE


ALL ACTIVE BLOCKS AND SCALAR AUTOMATIC VARIABLES
 BLOCK #    1 (MAIN PROCEDURE)

                                               N5=       6               X2= 1.83928E+00           X1= 1.83928E+00
J=    36              N6=     5                N2=       7               Z= 5.47035E+00            W= 9.99200E-15
N4=     5             N3=     16               X= 1.83928E+00            I=      37
N1=   36              XT= 1.83928E+00


                                                                                                   LABEL/ENTRY STMT COUNT
                                               LABEL/ENTRY STMT COUNT    LABEL/ENTRY STMT COUNT    NEWACC      0054 00001
LABEL/ENTRY STMT COUNT  LABEL/ENTRY STMT COUNT NEWBAC      0087 00001    ALGBAC      0070 00001
ROUGH       0001 00001  FINISH      0108 00001
ALGACC      0038 00001  NEWT        0021 00001

*STMT* DYNAMIC FLOW TRACE                                                              0122->0124      0125->0127
*0135*        0131->0133     0134->0114     0116->0117     0119->0121                  0119->0121      0122->0124
*0135*        0128->0130     0131->0133     0134->0114     0116->0117                  0114->0135      0135->0000
*0135*        0125->0127     0128->0130     0131->0133     0134->0114

 CORE USAGE(BYTES): SYMBOL TABLE 3896, OBJECT CODE 8300, STATIC AND EXTERNAL STORAGE 0000, AUTOMATIC STORAGE 6082, UNUSED 677

COMPILE TIME    1.53 SECONDS.

EXECUTION TIME    1.18 SECONDS.
```

CHAPTER III

BERROULLI'S METHOD

Definition: If $P(x) = a_0 z^N + a_1 z^{N-1} + \cdot \cdot \cdot \cdot + a_N$ is a polynomial then its <u>associated</u> <u>difference</u> <u>equation</u> is

$a_0 x_n + a_1 x_{n-1} + \cdot \cdot \cdot + a_N x_{n-N} = 0.$

Definition: A <u>dominant</u> <u>zero</u> of a polynomial is a zero whose modulus is not exceeded by the modulus of any other zero.

For the remainder of this chapter, the symbols once defined will not change.

Let $P(z) = a_0 z^N + a_1 z^{N-1} + \cdot \cdot \cdot \cdot + a_N$ be a polynomial with an associated difference equation

$a_0 x_n + a_1 x_{n-1} + \cdot \cdot \cdot + a_N x_{n-N}$ and let $z_1, z_2 \cdot \cdot \cdot z_N$ be the roots of P. Let X be a solution of

$a_0 x_n + \cdot \cdot \cdot a_n x_{n-N} = 0$. Then by [4, TH. 6.3, p 136] there are numbers $c_1, c_2, c_3, \cdot \cdot \cdot c_N$ such that

$X_n = c_1 z_1^n + \cdot \cdot \cdot \cdot c_n z_N^n$. Suppose z_1 is a single dominant zero of P. Suppose also that $c_1 \neq 0$. Define a sequence q by, for each non-negative integer n, $q_n = (X_{n+1}) / (X_n)$.

<u>Theorem 9</u>: Then, q converges to z_1.

Proof: If n is a non-negative integer, then

$$\frac{x_{n+1}}{x_n} = \frac{c_1(z_1)^{n+1}+c_2(z_1)^{n+1}+\cdots+c_N(z_n)^{n+1}}{c_1(z_1)^n+c_2(z_2)^n+\cdots+c_N(z_n)^n} =$$

$$z_1 \cdot \left[\frac{(1+(c_2/c_1)(z_2^{\,n+1}/z_1^{\,n+1})+\cdots+(c_N/c_1)(z_N^{\,n+1}/z_1^{\,n+1}))}{(1+(c_2/c_1)(z_2^{\,n}/z_1^{\,n})+\cdots+(c_N/c_1)(z_N^{\,n}/z_1^{\,n}))}\right]$$

$$z_1 \cdot \left[\frac{(1+(c_2/c_1)(z_2/z_1)^{n+1}+\cdots+(c_N/c_1)(z_N/z_1)^{n+1})}{(1+(c_2/c_1)(z_2/z_1)^n+\cdots+(c_N/c_1)(z_N/z_1)^n)}\right].$$

Since $|z_1| > |z_i|$ for $i = 2,3,\cdots N$ then $\dfrac{z_i}{z_1} < 1.$

Now,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left\{z_1 \cdot \left[\frac{(1+(c_2/c_1)(z_2/z_1)^{n+1}+\cdots+(c_N/c_1)(z_N/z_1)^{n+1})}{(1+(c_2/c_1)(z_2/z_1)^n+\cdots+(c_N/c_1)(z_N/z_1)^n)}\right]\right.$$

$$\lim_{n\to\infty} z_1 \cdot \lim_{n\to\infty} \left[\frac{1+(c_2/c_1)(z_2/z_1)+\cdots+(c_N/c_1)(z_n/z_1)^{n+1}}{1+(c_2/c_1)(z_2/z_1)^n+\cdots+(c_N/c_1)(z_n/z_1)^n}\right]$$

Since $(z_i / z_1) < 1$ then by [5, TH. 320 (e), p 50],

$\lim_{n\to\infty} [(z_i / z_1)^n] = 0$ for $i = 2,3,\cdots N$. Thus

$$\lim_{n\to\infty} \left[\frac{(1+(c_2/c_1)(z_2/z_1)+\cdots+(c_N/c_1)(z_N/z_1)^{n+1}}{(1+(c_2/c_1)(z_2/z_1)+\cdots+(c\ /c_1)(z_N/z_1)^n}\right] = 1$$

and since z_1 is a constant $\operatorname{limit}_{n\to\infty} z_1 = z_1$ therefore

$\lim_{n\to\infty} (x_{n+1} / x_n) = z_1$ and q converges to z_1.

CHAPTER IV

PROGRAM TWO

In this chapter, the function $P(x) = z^3 - 3z^2 + 4$ is used to illustrate Bernoulli's method. By [4, Alg 7.1, p 147] the sequence determined by the recurrence relation is $x_n = 3x_{n-1} - 4x_{n-3}$. This relation is used to obtain the x_n and x_{n+1} values. From these values, the ratio x_{n+1} / x_n is formed. With the condition imposed as previously stated in Chapter 3, the convergence to a zero of $P(x)$ is guaranteed. Or, if the convergence is slow then the ratio can be used as an initial value for one of the previously mentioned methods.

Once again we are confronted by what values with which to start. By [4, STMT. 7-15, p 153] if the following starting values are chosen

$$x_{-n+t} = x_{-n+2} = \cdot \cdot \cdot = x_{-1} = 0, \; x_0 = 1$$

then it is assured that one of our condition for convergence is met - c_1 0. However by [4, TH. 7.4, p 153] if we choose the starting values in the following manner

$$x_k = (-1/a_0) \cdot [(k+1)\, a_{k+1} + a_k x_0 + a_{k-1} x_1 + \cdot \cdot \cdot + a_1 x_{k-1}]$$

where $k = 1, 2, \cdot \cdot \cdot N - 1$ then faster convergence is assured. The following program illustrates Bernoulli's method using both sets of starting values. The X_n values are derived from the reccurrance relation. The Q_n values are the ratio of each two consecutive X_n's. The Q_n" values are derived from using

Steffenson's method to accelerate convergence of the Q_n's.

A number of the form

$$n.nnnnE - 17 \quad \text{means} \quad n.nnnn \times 10^{17}$$

where n is an integer.

Note the difference that the starting values made and also Steffenson's method on the second set of numbers. The program and data are as follows.

*PL/C

```
*OPTIONS IN EFFECT*     PAGES=030,TIME=(000,015),ERRORS=(050,050),SORMGIN=(002,072,001),LINECNT=060,BOUNDARY,
*OPTIONS IN EFFECT*     FLAGW,XREF,ATR,NOLIST,UDEF,SOURCE,DUMP,NODUMPARRAY,M91,NOCOMMENTS,CHECK

   TEST: PROCEDURE OPTIONS (MAIN);                                          PL/C-R6.5000  03/06/73   0:19 PAGE    1

  STMT LEVEL NEST BLOCK      SOURCE STATEMENT                                                                 ID FIELD

    1                        TEST: PROCEDURE OPTIONS (MAIN);
    2      1          1      DECLARE(QF,RF) FLOAT(16);
    3      1          1      DECLARE(J,JJ,X3,X2,X1,Y1,Y2,Y3,QI,RI) FLOAT(16);
    4      1          1      DECLARE(X(999),Y(999),Q(999),R(999),QP(999),RP(999),Z(999),
                             ZZ(999)) FLOAT(16);
    5      1          1      QI=10; RI=10; A=-3;
    8      1          1      RF=10; QF=10;
   10      1          1      DO K=1 TO 3;
   11      1     1    1      Z(K)=0; ZZ(K)=0;
   13      1     1    1      END;
   14      1          1      ZZ(3)=1;
   15      1          1      GET LIST (X(3),X(2),X(1));
   16      1          1      X3=X(3); X2=X(2); X1=X(1);
   19      1          1      DO I=4 TO 999;
   20      1     1    1      X(I)=3*X3-4*X1;
   21      1     1    1      X1=X2; X2=X3; X3=X(I);
   24      1     1    1      Z(I)=Z(I-1);
   25      1     1    1      IF X(I)>= 10 THEN DO;X1=X1/10; X2=X2/10; X3=X3/10; X(I)=X(I)/10;
   31      1     2    1      Z(I)=Z(I)+1;
   32      1     2    1      END;
   33      1     1    1      IF X(I-1)>X(I) THEN J=X(I-1)/10;
   35      1     1    1      ELSE J=X(I-1);
   36      1     1    1      Q(I-3)=X(I)/J;
   37      1     1    1      IF I<8 THEN GO TO BYPASS;
   39      1     1    1      IF ABS(Q(I-3)-QI)<.0000001 THEN DO; COUNT1 =I;
   42      1     2    1      GO TO QPRIME;
   43      1     2    1      END;
   44      1     1    1      BYPASS: QI=Q(I-3);
   45      1     1    1      END;
   46      1          1      COUNT1=I-1;
   47      1          1      QPRIME: COUNT11=COUNT1-5;
   48      1          1      DO P=1 TO COUNT11;
   49      1     1    1      T=(Q(P+2)-2*Q(P+1)+Q(P));
   50      1     1    1      S=((Q(P+1)-Q(P))**2)     ;
   51      1     1    1      QP(P)=Q(P)-(S/T);
   52      1     1    1      IF ABS(QP(P)-QF)<.0000001 THEN DO; COUNT11=P+1;       GO TO LAST1;
   56      1     2    1      END;
   57      1     1    1      QF=QP(P);
   58      1     1    1      END;
   59      1          1      COUNT11=P;
   60      1          1      LAST1:
                             GET LIST (Y(3),Y(2),Y(1));
   61      1          1      Y3=Y(3); Y2=Y(2); Y1=Y(1);
   64      1          1      DO I=4 TO 999;
   65      1     1    1      Y(I)=3*Y3-4*Y1;
   66      1     1    1      Y1=Y2; Y2=Y3; Y3=Y(I);
```

```
STMT LEVEL NEST BLOCK       SOURCE STATEMENT                                                                    ID FIELD

  69     1     1     1      ZZ(I)=ZZ(I-1);
  70     1     1     1      IF Y(I)>= 10 THEN DO;Y1=Y1/10; Y2=Y2/10; Y3=Y3/10; Y(I)=Y(I)/10;
  76     1     2     1      ZZ(I)=ZZ(I)+1;
  77     1     2     1      END;
  78     1     1     1      IF Y(I-1)>Y(I) THEN JJ=Y(I-1)/10;
  80     1     1     1      ELSE JJ=Y(I-1);
  81     1     1     1      R(I-3)=Y(I)/JJ;
  82     1     1     1      IF ABS(R(I-3)-RI)<.0000001 THEN DO; COUNT2 =I;
  85     1     2     1      GO TO QQPRIME;  END;
  87     1     1     1      RI=R(I-3);
  88     1     1     1      END;
  89     1           1      COUNT2=I-1;
  90     1           1      QQPRIME: COUNT22=COUNT2-5;
  91     1           1      Y(3)=Y(3)/10;
  92     1           1      DO P=1 TO COUNT22;
  93     1     1     1      T=(R(P+2)-2*R(P+1)+R(P));
  94     1     1     1      S=((R(P+1)-R(P))**2)   ;
  95     1     1     1      RP(P)=R(P)-(S/T);
  96     1     1     1      IF ABS(RP(P)-RF)<.0000001 THEN DO; COUNT22=P+1;       GO TO LAST2;
 100     1     2     1      END;
 101     1     1     1      RF=RP(P);
 102     1     1     1      END;
 103     1           1      COUNT22=P;
 104     1           1      LAST2:
                            NUM=MAX(COUNT1,COUNT2);
 105     1           1      PUT EDIT ('**********************************************************'
                            ,'*********************************************************')(A,A);
 106     1           1      PUT SKIP;
 107     1           1      PUT EDIT('N','XN','QN',    'QN"','N','XN','QN','QN"')(X(2),A,X(12),A,
                            X(18),A,X(12),A,X(9),A,X(11),A,X(18),A,X(13),A);
 108     1           1      PUT SKIP;
 109     1           1      PUT EDIT ('**********************************************************'
                            ,'*********************************************************')(A,A);
 110     1           1      PUT SKIP;
 111     1           1      DO L=1 TO 3;
 112     1     1     1      A=A+1;
 113     1     1     1      PUT EDIT(A,X(L),'E+',    Z(L),'*',A,Y(L),'E+',ZZ(L))(F(3),X(2),F(14,8),
                                    A,F(3), COLUMN(59),A,     F(3),X(2), F(14,8), A,F(3));
 114     1     1     1      PUT SKIP;
 115     1     1     1      END;
 116     1           1      DO M=L TO NUM;
 117     1     1     1      IF COUNT1>=M THEN DO;
 119     1     2     1      PUT EDIT(M-3,X(M),'E+',Z(M),Q(M-3))(F(3),X(2),F(14,8),A,F(3),X(2),
                            F(14,8));
 120     1     2     1      IF M-3<COUNT11 THEN PUT EDIT(QP(M-3))(X(2),F(14,8));
 122     1     2     1      END;
 123     1     1     1      PUT EDIT ('*')(COLUMN(59),A);
 124     1     1     1      IF COUNT2>=M THEN DO;
 126     1     2     1      PUT EDIT(M-3,Y(M),'E+',ZZ(M),R(M-3))(F(3),X(2),     F(14,8),A,F(3),
                            X(2),F(14,8));
 127     1     2     1      IF M-3<COUNT22 THEN PUT EDIT(RP(M-3))(X(2),F(14,8));
 129     1     2     1      END;
 130     1     1     1      PUT SKIP;
 131     1     1     1      END;
 132     1           1      END TEST;
```

STMT LEVEL NEST BLOCK SOURCE STATEMENT ID FIELD

DCL NO.	IDENTIFIER	ATTRIBUTES AND REFERENCES
7	A	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 7,112,112,113,113
	ABS	GENERIC,BUILT-IN FUNCTION 39,52,82,96
44	BYPASS	STATEMENT LABEL CONSTANT 38,44
41	COUNT1	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 41,46,47,104,117
47	COUNT11	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 47,48,54,59,120
84	COUNT2	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 84,89,90,104,124
90	COUNT22	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 90,92,98,103,127
19	I	AUTOMATIC,ALIGNED,BINARY,FIXED(15,0) 19,20,23,24,24,25,30,30,31,31,33,33,34,35,36,36,37,39,41,44,46,64,65,68,6 70,75,75,76,76,78,78,79,80,81,81,82,84,87,89
3	J	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,34,35,36
3	JJ	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,79,80,81
10	K	AUTOMATIC,ALIGNED,BINARY,FIXED(15,0) 10,11,12
111	L	AUTOMATIC,ALIGNED,BINARY,FIXED(15,0) 111,113,113,113,113,116
60	LAST1	STATEMENT LABEL CONSTANT 55,60
104	LAST2	STATEMENT LABEL CONSTANT 99,104
116	M	AUTOMATIC,ALIGNED,BINARY,FIXED(15,0) 116,117,119,119,119,119,120,121,124,126,126,126,126,127,128
	MAX	GENERIC,BUILT-IN FUNCTION 104

DCL NO.	IDENTIFIER	ATTRIBUTES AND REFERENCES
104	NUM	AUTOMATIC,ALIGNED,BINARY,FIXED(15,0) 104,116
48	P	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 48,49,49,49,50,50,51,51,52,54,57,59,92,93,93,93,94,94,95,95,96,98,101,10
4	Q	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,36,39,44,49,49,49,50,50,51,119
2	QF	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 2,9,52,57
3	QI	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,5,39,44
4	QP	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,51,52,57,121
47	QPRIME	STATEMENT LABEL CONSTANT 42,47
90	QQPRIME	STATEMENT LABEL CONSTANT 85,90
4	R	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,81,82,87,93,93,93,94,94,95,126
2	RF	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 2,8,96,101
3	RI	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,6,82,87
4	RP	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,95,96,101,128
50	S	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 50,51,94,95
49	T	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(6) 49,51,93,95
1	TEST	ENTRY,DECIMAL,FLOAT(6) 1
4	X	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,15,15,15,16,17,18,20,23,25,30,30,33,33,34,35,36,113,119
3	X1	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,18,20,21,27,27
3	X2	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,17,21,22,28,28

DCL NO.	IDENTIFIER	ATTRIBUTES AND REFERENCES
3	X3	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,16,20,22,23,29,29
4	Y	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,60,60,60,61,62,63,65,68,70,75,75,78,78,79,80,81,91,91,113,126
3	Y1	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,63,65,66,72,72
3	Y2	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,62,66,67,73,73
3	Y3	AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 3,61,65,67,68,74,74
4	Z	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,11,24,24,31,31,113,119
4	ZZ	(*)AUTOMATIC,ALIGNED,DECIMAL,FLOAT(16) 4,12,14,69,69,76,76,113,126

WARNING   CGOC NO FILE SPECIFIED. SYSIN/SYSPRINT ASSUMED.

The following pages are the data from the previous mention program Two different sets of data were used. The first set of data is described by the first four columns of the output, the second set of data is described by the second four columns of output

The N columns are the subscripts for our XN column. The first XN value being given the notation x_{n-2}, the second x_{n-1} and so forth.

The XN columns are the successive iterants defined by our recursive relationship, they get large as N gets large.

The QN columns are the ratio of each set of consecutive XN's. these values eventually converge to Z_1.

The QN" columns are an accelerated method used on the QN's.

Note how much faster the data using the specially selected starting points attain these approprate values to six decimal place accuracy.

N	XN	QN	QN"
-2	0.00000000E+ 0		
-1	0.00000000E+ 0		
0	1.00000000E+ 0		
1	3.00000000E+ 0	3.00000000	3.00000000
2	9.00000000E+ 0	3.00000000	2.46198830
3	2.30000000E+ 1	2.55555555	2.73913043
4	5.70000000E+ 1	2.47826086	2.27697070
5	1.35000000E+ 2	2.36842105	1.52412412
6	3.12999999E+ 2	2.31851851	2.17981169
7	7.10999999E+ 2	2.27156549	2.06250668
8	1.59299999E+ 3	2.24050632	2.12457545
9	3.52699999E+ 3	2.21406151	2.08338226
10	7.73699999E+ 3	2.19364899	2.09416573
11	1.68389999E+ 4	2.17642497	2.07744841
12	3.64089999E+ 4	2.16218302	2.07682414
13	7.82789999E+ 4	2.14999038	2.06874719
14	1.67480999E+ 5	2.13953934	2.06574964
15	3.56806999E+ 5	2.13043270	2.06093747
16	7.57304999E+ 5	2.12244995	2.05784330
17	1.60199099E+ 6	2.11538415	2.05446533
18	3.37874499E+ 6	2.10909112	2.05176625
19	7.10701499E+ 6	2.10344817	2.04915816
20	1.49130809E+ 7	2.09836070	2.04688660
21	3.12242629E+ 7	2.09374997	2.04477004
22	6.52447289E+ 7	2.08955224	2.04286031
23	1.36081862E+ 8	2.08571428	2.04109423
24	2.83348536E+ 8	2.08219178	2.03947454
25	5.89066694E+ 8	2.07894736	2.03797423
26	1.22287263E+ 9	2.07594936	2.03658559
27	2.53522375E+ 9	2.07317073	2.03529399
28	5.24940447E+ 9	2.07058823	2.03409097
29	1.08567228E+ 10	2.06818181	2.03296700
30	2.24292736E+ 10	2.06593406	2.03191491
31	4.62902030E+ 10	2.06382978	2.03092782
32	9.54437176E+ 10	2.06185567	2.03000000
33	1.96614058E+ 11	2.05999999	2.02912621
34	4.04681363E+ 11	2.05825242	2.02830188
35	8.32269218E+ 11	2.05660377	2.02752293
36	1.71035142E+ 12	2.05504587	2.02678571
37	3.51232881E+ 12	2.05357142	2.02608695
38	7.20790955E+ 12	2.05217391	2.02542372
39	1.47823229E+ 13	2.05084745	2.02479338
40	3.02976537E+ 13	2.04958677	2.02419354
41	6.20613229E+ 13	2.04838709	2.02362204
42	1.27054676E+ 14	2.04724409	2.02307692
43	2.59973415E+ 14	2.04615384	2.02255639
44	5.31674956E+ 14	2.04511278	2.02205882
45	1.08680616E+ 15	2.04411764	2.02158273
46	2.22052481E+ 15	2.04316546	2.02112676
47	4.53487462E+ 15	2.04225352	2.02068965
48	9.25739923E+ 15	2.04137931	2.02027027
49	1.88900984E+ 16	2.04054054	2.01986754
50	3.85307968E+ 16	2.03973509	2.01948051
51	7.85627934E+ 16	2.03896103	2.01910828
52	1.60127986E+ 17	2.03821656	2.01874999
53	3.26260773E+ 17	2.03749999	2.01840490
54	6.64531145E+ 17	2.03680981	2.01807228

N	XN	QN	QN"
-2	3.00000000E+ 0		
-1	9.00000000E+ 0		
0	1.50000000E+ 1		
1	3.30000000E+ 2	2.20000000	2.00293255
2	6.30000000E+ 2	1.90909090	2.00073260
3	1.29000000E+ 3	2.04761904	2.00018311
4	2.55000000E+ 3	1.97674418	2.00004577
5	5.12999999E+ 3	2.01176470	2.00001144
6	1.02300000E+ 4	1.99415204	2.00000286
7	2.04899999E+ 4	2.00293255	2.00000071
8	4.09499999E+ 4	1.99853587	2.00000017
9	8.19299999E+ 4	2.00073260	2.00000004
10	1.63829999E+ 5	1.99963383	2.00000001
11	3.27689999E+ 5	2.00018311	
12	6.55349999E+ 5	1.99990845	
13	1.31072999E+ 6	2.00004577	
14	2.62142999E+ 6	1.99997711	
15	5.24288999E+ 6	2.00001144	
16	1.04857499E+ 7	1.99999427	
17	2.09715299E+ 7	2.00000286	
18	4.19430299E+ 7	1.99999856	
19	8.38860899E+ 7	2.00000071	
20	1.67772149E+ 8	1.99999964	
21	3.35544329E+ 8	2.00000017	
22	6.71088629E+ 8	1.99999991	
23	1.34217728E+ 9	2.00000004	
24	2.68435454E+ 9	1.99999997	

55	1,35308148E+ 18	2,03614457	2,01775147	*
56	2,75420137E+ 18	2,03550295	2,01744186	*
57	5,60447953E+ 18	2,03488372	2,01714285	*
58	1,14011126E+ 19	2,03428571	2,01685393	*
59	2,31865324E+ 19	2,03370786	2,01657458	*
60	4,71416792E+ 19	2,03314917	2,01630434	*
61	9,58205872E+ 19	2,03260869	2,01604278	*
62	1,94715631E+ 20	2,03208556	2,01578947	*
63	3,95580178E+ 20	2,03157894	2,01554404	*
64	8,03458186E+ 20	2,03108808	2,01530612	*
65	1,63151203E+ 21	2,03061224	2,01507537	*
66	3,31221538E+ 21	2,03015075	2,01485148	*
67	6,72281339E+ 21	2,02970297	2,01463414	*
68	1,36423920E+ 22	2,02926829	2,01442307	*
69	2,76783146E+ 22	2,02884615	2,01421800	*
70	5,61436904E+ 22	2,02843601	2,01401869	*
71	1,13861502E+ 23	2,02803738	2,01382488	*
72	2,30871250E+ 23	2,02764976	2,01363636	*
73	4,68038989E+ 23	2,02727272	2,01345291	*
74	9,48670955E+ 23	2,02690582	2,01327433	*
75	1,92252786E+ 24	2,02654867	2,01310043	*
76	3,89542764E+ 24	2,02620087	2,01293103	*
77	7,89159910E+ 24	2,02586206	2,01276595	*
78	1,59846858E+ 25	2,02553191	2,01260504	*
79	3,23723469E+ 25	2,02521008	2,01244813	*
80	6,55506444E+ 25	2,02489626	2,01229508	*
81	1,32713189E+ 26	2,02459016	2,01214574	*
82	2,68650182E+ 26	2,02429149	2,01199999	*
83	5,43747968E+ 26	2,02399999	2,01185770	*
84	1,10039114E+ 27	2,02371541	2,01171874	*
85	2,22657270E+ 27	2,02343749	2,01158301	*
86	4,50472625E+ 27	2,02316602	2,01145038	*
87	9,11261417E+ 27	2,02290076	2,01132075	*
88	1,84315516E+ 28	2,02264150	2,01119402	*
89	3,72757500E+ 28	2,02238805	2,01107011	*
90	7,53767935E+ 28	2,02214022	2,01094890	*
91	1,52404173E+ 29	2,02189781	2,01083032	*
92	3,08109520E+ 29	2,02166064	2,01071428	*
93	6,22821388E+ 29	2,02142857	2,01060070	*
94	1,25884747E+ 30	2,02120141	2,01048951	*
95	2,54410432E+ 30	2,02097902	2,01038062	*
96	5,14102743E+ 30	2,02076124	2,01027397	*
97	1,03876924E+ 31	2,02054794	2,01016949	*
98	2,09866599E+ 31	2,02033898	2,01006711	*
99	4,23958700E+ 31	2,02013422	2,00996677	*
100	8,56368405E+ 31	2,01993355	2,00986842	*
101	1,72963881E+ 32	2,01973684	2,00977198	*
102	3,49308165E+ 32	2,01954397	2,00967741	*
103	7,05377133E+ 32	2,01935483	2,00958466	*
104	1,42427587E+ 33	2,01916932	2,00949367	*
105	2,87559496E+ 33	2,01898734	2,00940438	*
106	5,80527634E+ 33	2,01880877	2,00931677	*
107	1,17187255E+ 34	2,01863354	2,00923076	*
108	2,36537967E+ 34	2,01846153	2,00914634	*
109	4,77402850E+ 34	2,01829268	2,00906344	*
110	9,63459528E+ 34	2,01812688	2,00898203	*
111	1,94422671E+ 35	2,01796407	2,00890207	*
112	3,92306873E+ 35	2,01780415	2,00882352	*
113	7,91536809E+ 35	2,01764705	2,00874635	*
114	1,59691974E+ 36	2,01749271	2,00867052	*

115	3.22153173E+ 36	2.01734104	2.00859598	*
116	6.49844797E+ 36	2.01719197	2.00852272	*
117	1.31076649E+ 37	2.01704545	2.00845070	*
118	2.64368679E+ 37	2.01690140	2.00837988	*
119	5.33168118E+ 37	2.01675977	2.00831024	*
120	1.07519775E+ 38	2.01662049	2.00824175	*
121	2.16811855E+ 38	2.01648351	2.00817438	*
122	4.37168318E+ 38	2.01634877	2.00810810	*
123	8.81425853E+ 38	2.01621621	2.00804289	*
124	1.77703013E+ 39	2.01608579	2.00797872	*
125	3.58241714E+ 39	2.01595744	2.00791556	*
126	7.22154800E+ 39	2.01583113	2.00785340	*
127	1.45565234E+ 40	2.01570680	2.00779220	*
128	2.93399018E+ 40	2.01558441	2.00773195	*
129	5.91335135E+ 40	2.01546391	2.00767263	*
130	1.19174446E+ 41	2.01534526	2.00761421	*
131	2.40163732E+ 41	2.01522842	2.00755667	*
132	4.83957144E+ 41	2.01511335	2.00749999	*
133	9.75173645E+ 41	2.01499999	2.00744416	*
134	1.96486600E+ 42	2.01488833	2.00738916	*
135	3.95876943E+ 42	2.01477832	2.00733496	*
136	7.97561373E+ 42	2.01466992	2.00728155	*
137	1.60673771E+ 43	2.01456310	2.00722891	*
138	3.23670537E+ 43	2.01445783	2.00717703	*
139	6.51987064E+ 43	2.01435406	2.00712589	*
140	1.31326610E+ 44	2.01425178	2.00707547	*
141	2.64511616E+ 44	2.01415094	2.00702576	*
142	5.32740024E+ 44	2.01405152	2.00697674	*
143	1.07291363E+ 45	2.01395348	2.00692840	*
144	2.16069442E+ 45	2.01385681	2.00688073	*
145	4.35112317E+ 45	2.01376146	2.00683371	*
146	8.76171500E+ 45	2.01366742	2.00678733	*
147	1.76423673E+ 46	2.01357466	2.00674157	*
148	3.55226092E+ 46	2.01348314	2.00669642	*
149	7.15209677E+ 46	2.01339285	2.00665188	*
150	1.43993433E+ 47	2.01330376	2.00660792	*
151	2.89889864E+ 47	2.01321585	2.00656455	*
152	5.83585723E+ 47	2.01312910	2.00652173	*
153	1.17478343E+ 48	2.01304347	2.00647948	*
154	2.36479084E+ 48	2.01295896	2.00643776	*
155	4.76002963E+ 48	2.01287553	2.00639658	*
156	9.58095517E+ 48	2.01279317	2.00635593	*
157	1.92837021E+ 49	2.01271186	2.00631578	*
158	3.88109879E+ 49	2.01263157	2.00627615	*
159	7.81091430E+ 49	2.01255230	2.00623700	*
160	1.57192620E+ 50	2.01247401	2.00619834	*
161	3.16333909E+ 50	2.01239669	2.00616016	*
162	6.36565157E+ 50	2.01232032	2.00612244	*
163	1.28092499E+ 51	2.01224489	2.00608519	*
164	2.57743933E+ 51	2.01217038	2.00604838	*
165	5.18605736E+ 51	2.01209677	2.00601202	*
166	1.04344721E+ 52	2.01202404	2.00597609	*
167	2.09936590E+ 52	2.01195219	2.00594059	*
168	4.22367477E+ 52	2.01188118	2.00590551	*
169	8.49723547E+ 52	2.01181102	2.00587084	*
170	1.70942427E+ 53	2.01174168	2.00583657	*
171	3.43880292E+ 53	2.01167315	2.00580270	*
172	6.91751459E+ 53	2.01160541	2.00576923	*
173	1.39148466E+ 54	2.01153846	2.00573613	*
174	2.79893282E+ 54	2.01147227	2.00570342	*

175	5,62979264E+ 54	2.01140684	2.00567107	*
176	1,13234392E+ 55	2.01134215	2.00563909	*
177	2,27745865E+ 55	2.01127819	2.00560747	*
178	4,58045889E+ 55	2.01121495	2.00557620	*
179	9,21200097E+ 55	2.01115241	2.00554528	*
180	1,85261683E+ 56	2.01109057	2.00551470	*
181	3,72566693E+ 56	2.01102941	2.00548446	*
182	7,49220042E+ 56	2.01096892	2.00545454	*
183	1,50661339E+ 57	2.01090909	2.00542495	*
184	3,02957340E+ 57	2.01084990	2.00539568	*
185	6,09184005E+ 57	2.01079136	2.00536672	*
186	1,22490665E+ 58	2.01073345	2.00533807	*
187	2,46289061E+ 58	2.01067615	2.00530973	*
188	4,95193581E+ 58	2.01061946	2.00528169	*
189	9,95618080E+ 58	2.01056338	2.00525394	*
190	2,00169799E+ 59	2.01050788	2.00522648	*
191	4,02431966E+ 59	2.01045296	2.00519930	*
192	8,09048668E+ 59	2.01039861	2.00517241	*
193	1,62646680E+ 60	2.01034482	2.00514579	*
194	3,26967254E+ 60	2.01029159	2.00511945	*
195	6,57282297E+ 60	2.01023890	2.00509337	*
196	1,32126016E+ 61	2.01018675	2.00506756	*
197	2,65591148E+ 61	2.01013513	2.00504201	*
198	5,33860528E+ 61	2.01008403	2.00501672	*
199	1,07307751E+ 62	2.01003344	2.00499168	*
200	2,15686795E+ 62	2.00998336	2.00496688	*
201	4,33516174E+ 62	2.00993377	2.00494233	*
202	8,71317517E+ 62	2.00988467	2.00491803	*
203	1,75120537E+ 63	2.00983606	2.00489396	*
204	3,51955141E+ 63	2.00978792	2.00487012	*
205	7,07338417E+ 63	2.00974025	2.00484652	*
206	1,42153310E+ 64	2.00969305	2.00482315	*
207	2,85677874E+ 64	2.00964630	2.00479999	*
208	5,74098256E+ 64	2.00959999	2.00477707	*
209	1,15368152E+ 65	2.00955414	2.00475435	*
210	2,31833308E+ 65	2.00950871	2.00473186	*
211	4,65860623E+ 65	2.00946372	2.00470957	*
212	9,36109259E+ 65	2.00941915	2.00468750	*
213	1,88099454E+ 66	2.00937499	2.00466562	*
214	3,77954113E+ 66	2.00933125	2.00464396	*
215	7,59418636E+ 66	2.00928792	2.00462249	*
216	1,52585809E+ 67	2.00924499	2.00460122	*
217	3,06575782E+ 67	2.00920245	2.00458015	*
218	6,15959892E+ 67	2.00916030	2.00455927	*
219	1,23753644E+ 68	2.00911854	2.00453857	*
220	2,48630619E+ 68	2.00907715	2.00451807	*
221	4,99507900E+ 68	2.00903614	2.00449775	*
222	1,00350912E+ 69	2.00899550	2.00447761	*
223	2,01600490E+ 69	2.00895522	2.00445765	*
224	4,04998309E+ 69	2.00891530	2.00443786	*
225	8,13591279E+ 69	2.00887573	2.00441826	*
226	1,63437187E+ 70	2.00883652	2.00439882	*
227	3,28312239E+ 70	2.00879765	2.00437956	*
228	6,59500206E+ 70	2.00875912	2.00436046	*
229	1,32475186E+ 71	2.00872093	2.00434153	*
230	2,66100664E+ 71	2.00868306	2.00432276	*
231	5,34501911E+ 71	2.00864553	2.00430416	*
232	1,07360498E+ 72	2.00860832	2.00428571	*
233	2,15641230E+ 72	2.00857142	2.00426742	*
234	4,33122926E+ 72	2.00853485	2.00424929	*

235	8.69926783E+ 72	2.00849858	2.00423131	*
236	1.74721543E+ 73	2.00846262	2.00421348	*
237	3.50915458E+ 73	2.00842696	2.00419580	*
238	7.04775662E+ 73	2.00839160	2.00417827	*
239	1.41544081E+ 74	2.00835654	2.00416088	*
240	2.84266061E+ 74	2.00832177	2.00414364	*
241	5.70887918E+ 74	2.00828729	2.00412654	*
242	1.14648742E+ 75	2.00825309	2.00410958	*
243	2.30239804E+ 75	2.00821917	2.00409276	*
244	4.62364245E+ 75	2.00818553	2.00407608	*
245	9.28497764E+ 75	2.00815217	2.00405953	*
246	1.86453407E+ 76	2.00811907	2.00404312	*
247	3.74414524E+ 76	2.00808625	2.00402684	*
248	7.51844468E+ 76	2.00805369	2.00401069	*
249	1.50971977E+ 77	2.00802139	2.00399467	*
250	3.03150122E+ 77	2.00798934	2.00397877	*
251	6.08712580E+ 77	2.00795755	2.00396301	*
252	1.22224983E+ 78	2.00792602	2.00394736	*
253	2.45414900E+ 78	2.00789473	2.00393184	*
254	4.92759668E+ 78	2.00786369	2.00391644	*
255	9.89379073E+ 78	2.00783289	2.00390117	*
256	1.98647761E+ 79	2.00780234	2.00388601	*
257	3.98839418E+ 79	2.00777202	2.00387096	*
258	8.00766626E+ 79	2.00774193	2.00385604	*
259	1.60770883E+ 80	2.00771208	2.00384122	*
260	3.22776881E+ 80	2.00768245	2.00382653	*
261	6.48023994E+ 80	2.00765306	2.00381194	*
262	1.30098845E+ 81	2.00762388	2.00379746	*
263	2.61185782E+ 81	2.00759493	2.00378310	*
264	5.24347750E+ 81	2.00756620	2.00376884	*
265	1.05264787E+ 82	2.00753768	2.00375469	*
266	2.11320048E+ 82	2.00750938	2.00374064	*
267	4.24221044E+ 82	2.00748129	2.00372670	*
268	8.51603984E+ 82	2.00745341	2.00371287	*
269	1.70953176E+ 83	2.00742574	2.00369913	*
270	3.43171110E+ 83	2.00739827	2.00368550	*
271	6.88871738E+ 83	2.00737100	2.00367197	*
272	1.38280251E+ 84	2.00734394	2.00365853	*
273	2.77572308E+ 84	2.00731707	2.00364520	*
274	5.57168230E+ 84	2.00729040	2.00363196	*
275	1.11838368E+ 85	2.00726392	2.00361881	*
276	2.24486183E+ 85	2.00723763	2.00360576	*
277	4.50591256E+ 85	2.00721153	2.00359281	*
278	9.04420295E+ 85	2.00718562	2.00357995	*
279	1.81531615E+ 86	2.00715990	2.00356718	*
280	3.64358343E+ 86	2.00713436	2.00355450	*
281	7.31306911E+ 86	2.00710900	2.00354191	*
282	1.46779427E+ 87	2.00708382	2.00352941	*
283	2.94594944E+ 87	2.00705882	2.00351699	*
284	5.91262069E+ 87	2.00703399	2.00350467	*
285	1.18666850E+ 88	2.00700934	2.00349243	*
286	2.38162572E+ 88	2.00698486	2.00348027	*
287	4.77982888E+ 88	2.00696055	2.00346820	*
288	9.59281264E+ 88	2.00693641	2.00345622	*
289	1.92519350E+ 89	2.00691244	2.00344431	*
290	3.86364896E+ 89	2.00688863	2.00343249	*
291	7.75382183E+ 89	2.00686498	2.00342075	*
292	1.55606914E+ 90	2.00684150	2.00340909	*
293	3.12274785E+ 90	2.00681818	2.00339750	*
294	6.26671484E+ 90	2.00679501	2.00338600	*

295	1,25758679E+ 91	2.00677200	2.00337457	*
296	2,52366123E+ 91	2.00674915	2.00336322	*
297	5,06429777E+ 91	2.00672645	2.00335195	*
298	1,01625461E+ 92	2.00670391	2.00334075	*
299	2,03929934E+ 92	2.00668151	2.00332963	*
300	4,09217893E+ 92	2.00665926	2.00331858	*
301	8,21151835E+ 92	2.00663716	2.00330760	*
302	1,64773576E+ 93	2.00661521	2.00329670	*
303	3,30633572E+ 93	2.00659340	2.00328587	*
304	6,63439983E+ 93	2.00657174	2.00327510	*
305	1,33122564E+ 94	2.00655021	2.00326441	*
306	2,67114264E+ 94	2.00652883	2.00325379	*
307	5,35966799E+ 94	2.00650759	2.00324324	*
308	1,07541013E+ 95	2.00648648	2.00323275	*
309	2,15777336E+ 95	2.00646551	2.00322234	*
310	4,32945288E+ 95	2.00644468	2.00321199	*
311	8,68671811E+ 95	2.00642398	2.00320170	*
312	1,74290608E+ 96	2.00640341	2.00319148	*
313	3,49693710E+ 96	2.00638297	2.00318133	*
314	7,01612408E+ 96	2.00636267	2.00317124	*
315	1,40767478E+ 97	2.00634249	2.00316122	*
316	2,82424952E+ 97	2.00632244	2.00315126	*
317	5,66629894E+ 97	2.00630252	2.00314136	*
318	1,13681976E+ 98	2.00628272	2.00313152	*
319	2,28075949E+ 98	2.00626304	2.00312174	*
320	4,57575889E+ 98	2.00624349	2.00311203	*
321	9,17999761E+ 98	2.00622406	2.00310237	*
322	1,84169548E+ 99	2.00620475	2.00309278	*
323	3,69478290E+ 99	2.00618556	2.00308324	*
324	7,41234967E+ 99	2.00616649	2.00307377	*
325	1,48702670E+100	2.00614754	2.00306435	*
326	2,98316696E+100	2.00612870	2.00305498	*
327	5,98456101E+100	2.00610997	2.00304568	*
328	1,20055762E+101	2.00609137	2.00303643	*
329	2,40840607E+101	2.00607287	2.00302724	*
330	4,83139382E+101	2.00605449	2.00301810	*
331	9,69195098E+101	2.00603621	2.00300902	*
332	1,94422286E+102	2.00601805	2.00299999	*
333	3,90011107E+102	2.00599999	2.00299102	*
334	7,82355281E+102	2.00598205	2.00298210	*
335	1,56937669E+103	2.00596421	2.00297324	*
336	3,14808566E+103	2.00594648	2.00296442	*
337	6,31483587E+103	2.00592885	2.00295566	*
338	1,26670008E+104	2.00591133	2.00294695	*
339	2,54086597E+104	2.00589390	2.00293829	*
340	5,09666359E+104	2.00587659	2.00292968	*
341	1,02231904E+105	2.00585937	2.00292112	*
342	2,05061074E+105	2.00584225	2.00291262	*
343	4,11316678E+105	2.00582524	2.00290416	*
344	8,25022418E+105	2.00580832	2.00289575	*
345	1,65482295E+106	2.00579150	2.00288739	*
346	3,31920216E+106	2.00577478	2.00287907	*
347	6,65751681E+106	2.00575815	2.00287081	*
348	1,33532586E+107	2.00574162	2.00286259	*
349	2,67829671E+107	2.00572519	2.00285442	*
350	5,37188342E+107	2.00570884	2.00284629	*
351	1,07743468E+108	2.00569259	2.00283822	*
352	2,16098536E+108	2.00567644	2.00283018	*
353	4,33420271E+108	2.00566037	2.00282220	*
354	8,69286942E+108	2.00564440	2.00281425	*

355	1,74346668E+109	2.00562851	2.00280636	*
356	3,49671895E+109	2.00561272	2.00279850	*
357	7,01300910E+109	2.00559701	2.00279069	*
358	1,40651605E+110	2.00558139	2.00278293	*
359	2,82086059E+110	2.00556586	2.00277520	*
360	5,65737813E+110	2.00555041	2.00276752	*
361	1,13460701E+111	2.00553505	2.00275988	*
362	2,27547681E+111	2.00551977	2.00275229	*
363	4,56347918E+111	2.00550458	2.00274473	*
364	9,15200949E+111	2.00548947	2.00273722	*
365	1,83541212E+112	2.00547445	2.00272975	*
366	3,68084469E+112	2.00545950	2.00272232	*
367	7,38173028E+112	2.00544464	2.00271493	*
368	1,48035423E+113	2.00542986	2.00270758	*
369	2,96872483E+113	2.00541516	2.00270027	*
370	5,95348238E+113	2.00540054	2.00269299	*
371	1,19390301E+114	2.00538599	2.00268576	*
372	2,39421912E+114	2.00537153	2.00267857	*
373	4,80126442E+114	2.00535714	2.00267141	*
374	9,62818120E+114	2.00534283	2.00266429	*
375	1,93076670E+115	2.00532859	2.00265721	*
376	3,87179435E+115	2.00531443	2.00265017	*
377	7,76411059E+115	2.00530035	2.00264317	*
378	1,55692649E+116	2.00528634	2.00263620	*
379	3,12206174E+116	2.00527240	2.00262927	*
380	6,26054098E+116	2.00525854	2.00262237	*
381	1,25539169E+117	2.00524475	2.00261551	*
382	2,51735039E+117	2.00523103	2.00260869	*
383	5,04783479E+117	2.00521739	2.00260190	*
384	1,01219375E+118	2.00520381	2.00259515	*
385	2,02964111E+118	2.00519031	2.00258843	*
386	4,06978944E+118	2.00517687	2.00258175	*
387	8,16059328E+118	2.00516351	2.00257510	*
388	1,63632153E+119	2.00515021	2.00256849	*
389	3,28104883E+119	2.00513698	2.00256191	*
390	6,57890919E+119	2.00512382	2.00255536	*
391	1,31914414E+120	2.00511073	2.00254885	*
392	2,64501289E+120	2.00509770	2.00254237	*
393	5,30347501E+120	2.00508474	2.00253592	*
394	1,06338484E+121	2.00507185	2.00252951	*
395	2,13214937E+121	2.00505902	2.00252312	*
396	4,27505813E+121	2.00504625	2.00251677	*
397	8,57163501E+121	2.00503355	2.00251046	*
398	1,71863075E+122	2.00502092	2.00250417	*
399	3,44586900E+122	2.00500834	2.00249791	*
400	6,90895300E+122	2.00499583	2.00249169	*
401	1,38523360E+123	2.00498338	2.00248550	*
402	2,77735320E+123	2.00497100	2.00247933	*
403	5,56847840E+123	2.00495867	2.00247320	*
404	1,11645008E+124	2.00494641	2.00246710	*
405	2,23840896E+124	2.00493421	2.00246103	*
406	4,48783552E+124	2.00492206	2.00245499	*
407	8,99770624E+124	2.00490998	2.00244897	*
408	1,80394828E+125	2.00489795	2.00244299	*
409	3,61671065E+125	2.00488599	2.00243704	*
410	7,25104947E+125	2.00487408	2.00243111	*
411	1,45373552E+126	2.00486223	2.00242522	*
412	2,91452231E+126	2.00485044	2.00241935	*
413	5,84314715E+126	2.00483870	2.00241351	*
414	1,17144993E+127	2.00482703	2.00240770	*

415	2,34854088E+127	2.00481540	2.00240192	*
416	4,70836379E+127	2.00480384	2.00239616	*
417	9,43929162E+127	2.00479233	2.00239043	*
418	1,89237113E+128	2.00478087	2.00238473	*
419	3,79376788E+128	2.00476947	2.00237906	*
420	7,60558700E+128	2.00475812	2.00237341	*
421	1,52472764E+129	2.00474683	2.00236779	*
422	3,05667579E+129	2.00473559	2.00236220	*
423	6,12779257E+129	2.00472440	2.00235663	*
424	1,22844671E+130	2.00471327	2.00235109	*
425	2,46266981E+130	2.00470219	2.00234558	*
426	4,93689242E+130	2.00469116	2.00234009	*
427	9,89689043E+130	2.00468018	2.00233463	*
428	1,98399920E+131	2.00466926	2.00232919	*
429	3,97724063E+131	2.00465838	2.00232377	*
430	7,97296574E+131	2.00464756	2.00231839	*
431	1,59829004E+132	2.00463678	2.00231303	*
432	3,20397386E+132	2.00462606	2.00230769	*
433	6,42273530E+132	2.00461538	2.00230237	*
434	1,28750457E+133	2.00460475	2.00229709	*
435	2,58092418E+133	2.00459418	2.00229182	*
436	5,17367841E+133	2.00458365	2.00228658	*
437	1,03710169E+134	2.00457317	2.00228136	*
438	2,07893541E+134	2.00456273	2.00227617	*
439	4,16733487E+134	2.00455235	2.00227100	*
440	8,35359783E+134	2.00454201	2.00226586	*
441	1,67450518E+135	2.00453172	2.00226073	*
442	3,35658160E+135	2.00452147	2.00225563	*
443	6,72830569E+135	2.00451127	2.00225056	*
444	1,34868963E+136	2.00450112	2.00224550	*
445	2,70343625E+136	2.00449101	2.00224047	*
446	5,41898648E+136	2.00448095	2.00223546	*
447	1,08622009E+137	2.00447093	2.00223048	*
448	2,17728577E+137	2.00446096	2.00222551	*
449	4,36426274E+137	2.00445103	2.00222057	*
450	8,74790784E+137	2.00444115	2.00221565	*
451	1,75345804E+138	2.00443131	2.00221075	*
452	3,51466903E+138	2.00442151	2.00220588	*
453	7,04484395E+138	2.00441176	2.00220102	*
454	1,41206996E+139	2.00440205	2.00219619	*
455	2,83034229E+139	2.00439238	2.00219138	*
456	5,67308930E+139	2.00438276	2.00218658	*
457	1,13709880E+140	2.00437317	2.00218181	*
458	2,27915949E+140	2.00436363	2.00217706	*
459	4,56824275E+140	2.00435413	2.00217233	*
460	9,15633306E+140	2.00434467	2.00216763	*
461	1,83523612E+141	2.00433526	2.00216294	*
462	3,67841125E+141	2.00432588	2.00215827	*
463	7,37270055E+141	2.00431654	2.00215362	*
464	1,47771571E+142	2.00430725	2.00214899	*
465	2,96178264E+142	2.00429799	2.00214438	*
466	5,93626772E+142	2.00428877	2.00213980	*
467	1,18979403E+143	2.00427960	2.00213523	*
468	2,38466903E+143	2.00427046	2.00213068	*
469	4,77950000E+143	2.00426136	2.00212615	*
470	9,57932389E+143	2.00425230	2.00212164	*
471	1,91992955E+144	2.00424328	2.00211714	*
472	3,84798866E+144	2.00423429	2.00211267	*
473	7,71223643E+144	2.00422535	2.00210822	*
474	1,54569910E+145	2.00421644	2.00210378	*

475	3,09790185E+145	2.00420757	2.00209937	*
476	6,20881100E+145	2.00419874	2.00209497	*
477	1,24436365E+146	2.00418994	2.00209059	*
478	2,49393023E+146	2.00418118	2.00208623	*
479	4,99826629E+146	2.00417246	2.00208188	*
480	1,00173442E+147	2.00416377	2.00207756	*
481	2,00763117E+147	2.00415512	2.00207325	*
482	4,02358702E+147	2.00414651	2.00206896	*
483	8,06382336E+147	2.00413793	2.00206469	*
484	1,61609453E+148	2.00412938	2.00206043	*
485	3,23884880E+148	2.00412087	2.00205620	*
486	6,49101707E+148	2.00411240	2.00205198	*
487	1,30086730E+149	2.00410396	2.00204778	*
488	2,60706239E+149	2.00409556	2.00204359	*
489	5,22478036E+149	2.00408719	2.00203942	*
490	1,04708718E+150	2.00407885	2.00203527	*
491	2,09843660E+150	2.00407055	2.00203114	*
492	4,20539765E+150	2.00406228	2.00202702	*
493	8,42784422E+150	2.00405405	2.00202292	*
494	1,68897862E+151	2.00404585	2.00201884	*
495	3,38477681E+151	2.00403768	2.00201477	*
496	6,78319275E+151	2.00402955	2.00201072	*
497	1,35936637E+152	2.00402144	2.00200668	*
498	2,72418840E+152	2.00401337	2.00200267	*
499	5,45928811E+152	2.00400534	2.00199866	*
500	1,09403988E+153	2.00399733	2.00199468	*
501	2,19244428E+153	2.00398936	2.00199071	*
502	4,39361761E+153	2.00398142	2.00198675	*
503	8,80469331E+153	2.00397350	2.00198281	*
504	1,76443027E+154	2.00396563	2.00197889	*
505	3,53584379E+154	2.00395778	2.00197498	*
506	7,08565405E+154	2.00394996	2.00197109	*
507	1,41992410E+155	2.00394218	2.00196721	*
508	2,84543479E+155	2.00393442	2.00196335	*
509	5,70204276E+155	2.00392670	2.00195950	*
510	1,14264318E+156	2.00391900	2.00195567	*
511	2,28975564E+156	2.00391134	2.00195185	*
512	4,58844982E+156	2.00390370	2.00194805	*
513	9,19477672E+156	2.00389610	2.00194426	*
514	1,84253076E+157	2.00388852	2.00194049	*
515	3,69221235E+157	2.00388098	2.00193673	*
516	7,39872636E+157	2.00387346	2.00193298	*
517	1,48260560E+158	2.00386597	2.00192926	*
518	2,97093187E+158	2.00385852	2.00192554	*
519	5,95330508E+158	2.00385109	2.00192184	*
520	1,19294928E+159	2.00384368	2.00191815	*
521	2,39047509E+159	2.00383631	2.00191448	*
522	4,79010325E+159	2.00382897	2.00191082	*
523	9,59851263E+159	2.00382165	2.00190718	*
524	1,92336375E+160	2.00381436	2.00190355	*
525	3,85404995E+160	2.00380710	2.00189993	*
526	7,72274481E+160	2.00379987	2.00189633	*
527	1,54747794E+161	2.00379266	2.00189274	*
528	3,10081384E+161	2.00378548	2.00188916	*
529	6,21334361E+161	2.00377833	2.00188560	*
530	1,24501190E+162	2.00377121	2.00188205	*
531	2,49471018E+162	2.00376411	2.00187852	*
532	4,99879310E+162	2.00375704	2.00187499	*
533	1,00163316E+163	2.00374999	2.00187149	*
534	2,00701543E+163	2.00374298	2.00186799	*

535	4,02152905E+163	2.00373599	2.00186451	*
536	8,05805448E+163	2.00372902	2.00186104	*
537	1,61461017E+164	2.00372208	2.00185758	*
538	3,23521889E+164	2.00371517	2.00185414	*
539	6,48243489E+164	2.00370828	2.00185070	*
540	1,29888640E+165	2.00370141	2.00184729	*
541	2,60257164E+165	2.00369458	2.00184388	*
542	5,21474096E+165	2.00368776	2.00184049	*
543	1,04486772E+166	2.00368098	2.00183710	*
544	2,09357453E+166	2.00367421	2.00183374	*
545	4,19482721E+166	2.00366748	2.00183038	*
546	8,40501071E+166	2.00366076	2.00182704	*
547	1,68407340E+167	2.00365408	2.00182370	*
548	3,37428932E+167	2.00364741	2.00182038	*
549	6,76086367E+167	2.00364077	2.00181708	*
550	1,35462974E+168	2.00363416	2.00181378	*
551	2,71417349E+168	2.00362756	2.00181050	*
552	5,43817502E+168	2.00362100	2.00180722	*
553	1,08960060E+169	2.00361445	2.00180396	*
554	2,18313243E+169	2.00360793	2.00180072	*
555	4,37412728E+169	2.00360144	2.00179748	*
556	8,76397940E+169	2.00359496	2.00179425	*
557	1,75594085E+170	2.00358851	2.00179104	*
558	3,51817163E+170	2.00358208	2.00178784	*
559	7,04892315E+170	2.00357568	2.00178465	*
560	1,41230060E+171	2.00356930	2.00178147	*
561	2,82963316E+171	2.00356294	2.00177830	*
562	5,66933022E+171	2.00355660	2.00177514	*
563	1,13587882E+172	2.00355029	2.00177200	*
564	2,27578320E+172	2.00354400	2.00176886	*
565	4,55961753E+172	2.00353773	2.00176574	*
566	9,13533731E+172	2.00353148	2.00176263	*
567	1,83028791E+173	2.00352526	2.00175953	*
568	3,66701671E+173	2.00351906	2.00175644	*
569	7,34691522E+173	2.00351288	2.00175336	*
570	1,47195940E+174	2.00350672	2.00175029	*
571	2,94907152E+174	2.00350058	2.00174723	*
572	5,90844847E+174	2.00349446	2.00174418	*
573	1,18375078E+175	2.00348837	2.00174114	*
574	2,37162373E+175	2.00348229	2.00173812	*
575	4,75149182E+175	2.00347624	2.00173510	*
576	9,51947234E+175	2.00347021	2.00173210	*
577	1,90719220E+176	2.00346420	2.00172910	*
578	3,82097989E+176	2.00345821	2.00172612	*
579	7,65515073E+176	2.00345224	2.00172314	*
580	1,53366833E+177	2.00344629	2.00172018	*
581	3,07261305E+177	2.00344036	2.00171722	*
582	6,15577888E+177	2.00343445	2.00171428	*
583	1,23326632E+178	2.00342857	2.00171135	*
584	2,47075376E+178	2.00342270	2.00170842	*
585	4,94994973E+178	2.00341685	2.00170551	*
586	9,91678389E+178	2.00341102	2.00170261	*
587	1,98673366E+179	2.00340522	2.00169971	*
588	3,98022109E+179	2.00339943	2.00169683	*
589	7,97394973E+179	2.00339366	2.00169395	*
590	1,59749145E+180	2.00338791	2.00169109	*
591	3,20038592E+180	2.00338218	2.00168823	*
592	6,41157787E+180	2.00337647	2.00168539	*
593	1,28447678E+181	2.00337078	2.00168255	*
594	2,57327597E+181	2.00336511	2.00167973	*

595	5,15519677E+181	2.00335946	2.00167691	*
596	1,03276831E+182	2.00335382	2.00167410	*
597	2,06899456E+182	2.00334821	2.00167130	*
598	4,14490499E+182	2.00334261	2.00166852	*
599	8,30364171E+182	2.00333704	2.00166574	*
600	1,66349468E+183	2.00333148	2.00166297	*
601	3,33252205E+183	2.00332594	2.00166021	*
602	6,67610949E+183	2.00332042	2.00165745	*
603	1,33743497E+184	2.00331491	2.00165471	*
604	2,67929609E+184	2.00330943	2.00165198	*
605	5,36744449E+184	2.00330396	2.00164925	*
606	1,07525935E+185	2.00329851	2.00164654	*
607	2,15405963E+185	2.00329308	2.00164383	*
608	4,31520111E+185	2.00328767	2.00164113	*
609	8,64456590E+185	2.00328227	2.00163844	*
610	1,73174591E+186	2.00327689	2.00163576	*
611	3,46915730E+186	2.00327153	2.00163309	*
612	6,94964555E+186	2.00326619	2.00163043	*
613	1,39219530E+187	2.00326086	2.00162778	*
614	2,78892297E+187	2.00325556	2.00162513	*
615	5,58691071E+187	2.00325027	2.00162249	*
616	1,11919509E+188	2.00324499	2.00161987	*
617	2,24201608E+188	2.00323974	2.00161725	*
618	4,49128398E+188	2.00323450	2.00161463	*
619	8,99707157E+188	2.00322927	2.00161203	*
620	1,80231503E+189	2.00322407	2.00160944	*
621	3,61043151E+189	2.00321888	2.00160685	*
622	7,23246591E+189	2.00321371	2.00160427	*
623	1,44881376E+190	2.00320855	2.00160170	*
624	2,90226867E+190	2.00320341	2.00159914	*
625	5,81381966E+190	2.00319829	2.00159659	*
626	1,16462039E+191	2.00319318	2.00159404	*
627	2,33295371E+191	2.00318809	2.00159151	*
628	4,67333327E+191	2.00318302	2.00158898	*
629	9,36151823E+191	2.00317796	2.00158646	*
630	1,87527398E+192	2.00317292	2.00158394	*
631	3,75648865E+192	2.00316789	2.00158144	*
632	7,52485865E+192	2.00316288	2.00157894	*
633	1,50734800E+193	2.00315789	2.00157645	*
634	3,01944854E+193	2.00315291	2.00157397	*
635	6,04840217E+193	2.00314795	2.00157150	*
636	1,21158145E+194	2.00314300	2.00156903	*
637	2,42696493E+194	2.00313807	2.00156657	*
638	4,86153394E+194	2.00313315	2.00156412	*
639	9,73827602E+194	2.00312825	2.00156168	*
640	1,95069683E+195	2.00312337	2.00155925	*
641	3,90747692E+195	2.00311850	2.00155682	*
642	7,82712034E+195	2.00311364	2.00155440	*
643	1,56785737E+196	2.00310880	2.00155199	*
644	3,14058134E+196	2.00310398	2.00154958	*
645	6,29089589E+196	2.00309917	2.00154718	*
646	1,26012582E+197	2.00309437	2.00154479	*
647	2,52414492E+197	2.00308959	2.00154241	*
648	5,05607641E+197	2.00308483	2.00154004	*
649	1,01277259E+198	2.00308008	2.00153767	*
650	2,02865981E+198	2.00307534	2.00153531	*
651	4,06354888E+198	2.00307062	2.00153295	*
652	8,13955628E+198	2.00306591	2.00153061	*
653	1,63040295E+199	2.00306122	2.00152827	*
654	3,26578931E+199	2.00305654	2.00152594	*

655	6,54154543E+199	2.00305188	2.00152361	*
656	1,31030244E+200	2.00304723	2.00152129	*
657	2,62459161E+200	2.00304259	2.00151898	*
658	5,25715667E+200	2.00303797	2.00151668	*
659	1,05302602E+201	2.00303336	2.00151438	*
660	2,10924142E+201	2.00302877	2.00151209	*
661	4,22486160E+201	2.00302419	2.00150981	*
662	8,46248071E+201	2.00301962	2.00150753	*
663	1,69504764E+202	2.00301507	2.00150526	*
664	3,39519829E+202	2.00301053	2.00150300	*
665	6,80060259E+202	2.00300601	2.00150075	*
666	1,36216172E+203	2.00300150	2.00149850	*
667	2,72840584E+203	2.00299700	2.00149625	*
668	5,46497649E+203	2.00299251	2.00149402	*
669	1,09462825E+204	2.00298804	2.00149179	*
670	2,19252244E+204	2.00298359	2.00148957	*
671	4,39157672E+204	2.00297914	2.00148735	*
672	8,79621714E+204	2.00297471	2.00148514	*
673	1,76185616E+205	2.00297029	2.00148294	*
674	3,52893780E+205	2.00296589	2.00148075	*
675	7,06832656E+205	2.00296150	2.00147856	*
676	1,41575550E+206	2.00295712	2.00147637	*
677	2,83569138E+206	2.00295275	2.00147420	*
678	5,67974353E+206	2.00294840	2.00147203	*
679	1,13762085E+207	2.00294406	2.00146986	*
680	2,27858602E+207	2.00293973	2.00146771	*
681	4,56386065E+207	2.00293542	2.00146555	*
682	9,14109852E+207	2.00293111	2.00146341	*
683	1,83089514E+208	2.00292682	2.00146127	*
684	3,66714118E+208	2.00292255	2.00145914	*
685	7,34498414E+208	2.00291828	2.00145701	*
686	1,47113718E+209	2.00291403	2.00145489	*
687	2,94655507E+209	2.00290979	2.00145278	*
688	5,90167157E+209	2.00290556	2.00145067	*
689	1,18204659E+210	2.00290135	2.00144857	*
690	2,36751776E+210	2.00289715	2.00144648	*
691	4,74188466E+210	2.00289296	2.00144439	*
692	9,49746759E+210	2.00288878	2.00144230	*
693	1,90223317E+211	2.00288461	2.00144023	*
694	3,80994565E+211	2.00288046	2.00143815	*
695	7,63084992E+211	2.00287631	2.00143609	*
696	1,52836170E+212	2.00287218	2.00143403	*
697	3,06110686E+212	2.00286806	2.00143198	*
698	6,13098062E+212	2.00286396	2.00142993	*
699	1,22794950E+213	2.00285986	2.00142789	*
700	2,45940576E+213	2.00285578	2.00142585	*
701	4,92582504E+213	2.00285171	2.00142382	*
702	9,86567711E+213	2.00284765	2.00142180	*
703	1,97594082E+214	2.00284360	2.00141978	*
704	3,95749246E+214	2.00283956	2.00141776	*
705	7,92620656E+214	2.00283553	2.00141576	*
706	1,58748563E+215	2.00283152	2.00141376	*
707	3,17945992E+215	2.00282752	2.00141176	*
708	6,36789714E+215	2.00282352	2.00140977	*
709	1,27537488E+216	2.00281954	2.00140778	*
710	2,55434069E+216	2.00281557	2.00140581	*
711	5,11586323E+216	2.00281162	2.00140383	*
712	1,02460901E+217	2.00280767	2.00140186	*
713	2,05209076E+217	2.00280373	2.00139990	*
714	4,10992699E+217	2.00279981	2.00139794	*

715	8,23134493E+217	2.00279589	2.00139599	*
716	1,64856717E+218	2.00279199	2.00139405	*
717	3,30173072E+218	2.00278810	2.00139211	*
718	6,61265421E+218	2.00278422	2.00139017	*
719	1,32436939E+219	2.00278035	2.00138824	*
720	2,65241588E+219	2.00277649	2.00138632	*
721	5,31218597E+219	2.00277264	2.00138440	*
722	1,06390803E+220	2.00276880	2.00138248	*
723	2,13075775E+220	2.00276497	2.00138057	*
724	4,26739887E+220	2.00276115	2.00137867	*
725	8,54656446E+220	2.00275735	2.00137677	*
726	1,71166623E+221	2.00275355	2.00137488	*
727	3,42803916E+221	2.00274977	2.00137299	*
728	6,86549171E+221	2.00274599	2.00137111	*
729	1,37498101E+222	2.00274223	2.00136923	*
730	2,75372738E+222	2.00273847	2.00136736	*
731	5,51498548E+222	2.00273473	2.00136549	*
732	1,10450323E+223	2.00273099	2.00136363	*
733	2,21201875E+223	2.00272727	2.00136177	*
734	4,43006207E+223	2.00272355	2.00135992	*
735	8,87217327E+223	2.00271985	2.00135808	*
736	1,77684448E+224	2.00271616	2.00135623	*
737	3,55850861E+224	2.00271247	2.00135440	*
738	7,12665652E+224	2.00270880	2.00135257	*
739	1,42725916E+225	2.00270513	2.00135074	*
740	2,85837405E+225	2.00270148	2.00134892	*
741	5,72445954E+225	2.00269784	2.00134710	*
742	1,14643419E+226	2.00269420	2.00134529	*
743	2,29595296E+226	2.00269058	2.00134348	*
744	4,59807509E+226	2.00268696	2.00134168	*
745	9,20848848E+226	2.00268336	2.00133988	*
746	1,84416535E+227	2.00267976	2.00133809	*
747	3,69326603E+227	2.00267618	2.00133630	*
748	7,39640272E+227	2.00267260	2.00133451	*
749	1,48125467E+228	2.00266903	2.00133274	*
750	2,96645760E+228	2.00266548	2.00133096	*
751	5,94081172E+228	2.00266193	2.00132919	*
752	1,18974164E+229	2.00265839	2.00132743	*
753	2,38264190E+229	2.00265486	2.00132567	*
754	4,77160101E+229	2.00265134	2.00132391	*
755	9,55583645E+229	2.00264783	2.00132216	*
756	1,91369417E+230	2.00264433	2.00132042	*
757	3,83244212E+230	2.00264084	2.00131868	*
758	7,67499178E+230	2.00263736	2.00131694	*
759	1,53701986E+231	2.00263388	2.00131521	*
760	3,07808274E+231	2.00263042	2.00131348	*
761	6,16425152E+231	2.00262697	2.00131176	*
762	1,23446751E+232	2.00262352	2.00131004	*
763	2,47216943E+232	2.00262008	2.00130832	*
764	4,95080769E+232	2.00261665	2.00130662	*
765	9,91455303E+232	2.00261324	2.00130491	*
766	1,98549813E+233	2.00260983	2.00130321	*
767	3,97617133E+233	2.00260642	2.00130151	*
768	7,96269279E+233	2.00260303	2.00129982	*
769	1,59460858E+234	2.00259965	2.00129813	*
770	3,19335721E+234	2.00259627	2.00129645	*
771	6,39499452E+234	2.00259291	2.00129477	*
772	1,28065492E+235	2.00258955	2.00129310	*
773	2,56462188E+235	2.00258620	2.00129143	*
774	5,13586785E+235	2.00258286	2.00128976	*

775	1,02849838E+236	2.00257953	2.00128810	*
776	2,05964640E+236	2.00257621	2.00128644	*
777	4,12459206E+236	2.00257289	2.00128479	*
778	8,25978265E+236	2.00256959	2.00128314	*
779	1,65407623E+237	2.00256629	2.00128150	*
780	3,31239188E+237	2.00256300	2.00127986	*
781	6,63326258E+237	2.00255972	2.00127822	*
782	1,32834827E+238	2.00255645	2.00127659	*
783	2,66008808E+238	2.00255319	2.00127496	*
784	5,32695922E+238	2.00254993	2.00127334	*
785	1,06674845E+239	2.00254668	2.00127172	*
786	2,13621013E+239	2.00254345	2.00127011	*
787	4,27784671E+239	2.00254022	2.00126849	*
788	8,56654631E+239	2.00253699	2.00126689	*
789	1,71547984E+240	2.00253378	2.00126528	*
790	3,43530083E+240	2.00253057	2.00126368	*
791	6,87928398E+240	2.00252737	2.00126209	*
792	1,37759325E+241	2.00252419	2.00126050	*
793	2,75865944E+241	2.00252100	2.00125891	*
794	5,52426473E+241	2.00251783	2.00125733	*
795	1,10624211E+242	2.00251466	2.00125575	*
796	2,21526257E+242	2.00251151	2.00125418	*
797	4,43608182E+242	2.00250836	2.00125260	*
798	8,88327700E+242	2.00250521	2.00125104	*
799	1,77887807E+243	2.00250208	2.00124947	*
800	3,56220148E+243	2.00249895	2.00124792	*
801	7,13329366E+243	2.00249584	2.00124636	*
802	1,42843686E+244	2.00249272	2.00124481	*
803	2,86043001E+244	2.00248962	2.00124326	*
804	5,72797257E+244	2.00248653	2.00124172	*
805	1,14701702E+245	2.00248344	2.00124018	*
806	2,29687906E+245	2.00248036	2.00123864	*
807	4,59944817E+245	2.00247729	2.00123711	*
808	9,21027643E+245	2.00247422	2.00123558	*
809	1,84433130E+246	2.00247116	2.00123406	*
810	3,69321463E+246	2.00246812	2.00123253	*
811	7,39553333E+246	2.00246507	2.00123102	*
812	1,48092747E+247	2.00246204	2.00122950	*
813	2,96549658E+247	2.00245901	2.00122799	*
814	5,93827641E+247	2.00245599	2.00122649	*
815	1,18911193E+248	2.00245298	2.00122498	*
816	2,38113716E+248	2.00244997	2.00122349	*
817	4,76810093E+248	2.00244698	2.00122199	*
818	9,54785506E+248	2.00244399	2.00122050	*
819	1,91190165E+249	2.00244100	2.00121901	*
820	3,82846458E+249	2.00243803	2.00121753	*
821	7,66625173E+249	2.00243506	2.00121605	*
822	1,53511485E+250	2.00243210	2.00121457	*
823	3,07395874E+250	2.00242914	2.00121310	*
824	6,15537552E+250	2.00242620	2.00121163	*
825	1,23256671E+251	2.00242326	2.00121016	*
826	2,46811665E+251	2.00242033	2.00120870	*
827	4,94219973E+251	2.00241740	2.00120724	*
828	9,89633235E+251	2.00241448	2.00120578	*
829	1,98165304E+252	2.00241157	2.00120433	*
830	3,96807924E+252	2.00240867	2.00120288	*
831	7,94570478E+252	2.00240577	2.00120144	*
832	1,59105021E+253	2.00240288	2.00119999	*
833	3,18591895E+253	2.00239999	2.00119856	*
834	6,37947495E+253	2.00239712	2.00119712	*

835	1,27742239E+254	2.00239425	2.00119569	*
836	2,55789961E+254	2.00239139	2.00119426	*
837	5,12190886E+254	2.00238853	2.00119284	*
838	1,02560369E+255	2.00238568	2.00119142	*
839	2,05365125E+255	2.00238284	2.00119000	*
840	4,11219020E+255	2.00238000	2.00118858	*
841	8,23415582E+255	2.00237717	2.00118717	*
842	1,64878624E+256	2.00237435	2.00118577	*
843	3,30148266E+256	2.00237154	2.00118436	*
844	6,61078565E+256	2.00236873	2.00118296	*
845	1,32372119E+257	2.00236593	2.00118156	*
846	2,65057052E+257	2.00236313	2.00118017	*
847	5,30739731E+257	2.00236034	2.00117878	*
848	1,06273071E+258	2.00235756	2.00117739	*
849	2,12796393E+258	2.00235478	2.00117600	*
850	4,26093288E+258	2.00235201	2.00117462	*
851	8,53187579E+258	2.00234925	2.00117325	*
852	1,70837716E+259	2.00234649	2.00117187	*
853	3,42075833E+259	2.00234374	2.00117050	*
854	6,84952468E+259	2.00234100	2.00116913	*
855	1,37150654E+260	2.00233826	2.00116776	*
856	2,74621628E+260	2.00233553	2.00116640	*
857	5,49883899E+260	2.00233281	2.00116504	*
858	1,10104908E+261	2.00233009	2.00116369	*
859	2,20466072E+261	2.00232738	2.00116234	*
860	4,41444658E+261	2.00232468	2.00116099	*
861	8,83914344E+261	2.00232198	2.00115964	*
862	1,76987874E+262	2.00231928	2.00115830	*
863	3,54385758E+262	2.00231660	2.00115696	*
864	7,09591538E+262	2.00231392	2.00115562	*
865	1,42082311E+263	2.00231124	2.00115428	*
866	2,84492632E+263	2.00230858	2.00115295	*
867	5,69641281E+263	2.00230591	2.00115163	*
868	1,14059459E+264	2.00230326	2.00115030	*
869	2,28381325E+264	2.00230061	2.00114898	*
870	4,57287465E+264	2.00229797	2.00114766	*
871	9,15624557E+264	2.00229533	2.00114635	*
872	1,83334836E+265	2.00229270	2.00114503	*
873	3,67089524E+265	2.00229007	2.00114372	*
874	7,35018750E+265	2.00228745	2.00114242	*
875	1,47171690E+266	2.00228484	2.00114111	*
876	2,94679261E+266	2.00228223	2.00113981	*
877	5,90030284E+266	2.00227963	2.00113852	*
878	1,18140409E+267	2.00227703	2.00113722	*
879	2,36549522E+267	2.00227445	2.00113593	*
880	4,73636454E+267	2.00227186	2.00113464	*
881	9,48347726E+267	2.00226928	2.00113335	*
882	1,89884508E+268	2.00226671	2.00113207	*
883	3,80198945E+268	2.00226415	2.00113079	*
884	7,61257744E+268	2.00226159	2.00112951	*
885	1,52423519E+269	2.00225903	2.00112824	*
886	3,05190981E+269	2.00225648	2.00112697	*
887	6,11069846E+269	2.00225394	2.00112570	*
888	1,22351545E+270	2.00225140	2.00112443	*
889	2,44978245E+270	2.00224887	2.00112317	*
890	4,90506797E+270	2.00224634	2.00112191	*
891	9,82114208E+270	2.00224382	2.00112065	*
892	1,96642964E+271	2.00224131	2.00111940	*
893	3,93726174E+271	2.00223880	2.00111815	*
894	7,88332839E+271	2.00223630	2.00111690	*

895	1,57842666E+272	2.00223380	2.00111565	*
896	3,16037528E+272	2.00223131	2.00111441	*
897	6,32779449E+272	2.00222882	2.00111317	*
898	1,26696768E+273	2.00222634	2.00111193	*
899	2,53675293E+273	2.00222386	2.00111069	*
900	5,07914102E+273	2.00222139	2.00110946	*
901	1,01695523E+274	2.00221893	2.00110823	*
902	2,03616452E+274	2.00221647	2.00110701	*
903	4,07683715E+274	2.00221402	2.00110578	*
904	8,16269054E+274	2.00221157	2.00110456	*
905	1,63434135E+275	2.00220913	2.00110334	*
906	3,27228919E+275	2.00220669	2.00110213	*
907	6,55179137E+275	2.00220426	2.00110091	*
908	1,31180087E+276	2.00220183	2.00109970	*
909	2,62648693E+276	2.00219941	2.00109849	*
910	5,25874425E+276	2.00219699	2.00109729	*
911	1,05290292E+277	2.00219458	2.00109609	*
912	2,10811401E+277	2.00219218	2.00109489	*
913	4,22084432E+277	2.00218978	2.00109369	*
914	8,45092127E+277	2.00218738	2.00109249	*
915	1,69203077E+278	2.00218499	2.00109130	*
916	3,38775460E+278	2.00218261	2.00109011	*
917	6,78289529E+278	2.00218023	2.00108892	*
918	1,35805627E+279	2.00217785	2.00108774	*
919	2,71906699E+279	2.00217548	2.00108656	*
920	5,44404286E+279	2.00217312	2.00108538	*
921	1,08999034E+280	2.00217076	2.00108420	*
922	2,18234424E+280	2.00216841	2.00108303	*
923	4,36941558E+280	2.00216606	2.00108186	*
924	8,74828537E+280	2.00216372	2.00108069	*
925	1,75154791E+281	2.00216138	2.00107952	*
926	3,50687750E+281	2.00215905	2.00107836	*
927	7,02131837E+281	2.00215672	2.00107719	*
928	1,40577634E+282	2.00215439	2.00107604	*
929	2,81457803E+282	2.00215208	2.00107488	*
930	5,63520676E+282	2.00214976	2.00107372	*
931	1,12825149E+283	2.00214745	2.00107257	*
932	2,25892325E+283	2.00214515	2.00107142	*
933	4,52268705E+283	2.00214285	2.00107028	*
934	9,05505521E+283	2.00214056	2.00106913	*
935	1,81294726E+284	2.00213827	2.00106799	*
936	3,62976696E+284	2.00213599	2.00106685	*
937	7,26727881E+284	2.00213371	2.00106571	*
938	1,45500473E+285	2.00213143	2.00106458	*
939	2,91310742E+285	2.00212916	2.00106345	*
940	5,83241076E+285	2.00212690	2.00106232	*
941	1,16772133E+286	2.00212464	2.00106119	*
942	2,33792102E+286	2.00212239	2.00106007	*
943	4,68079877E+286	2.00212014	2.00105894	*
944	9,37151100E+286	2.00211789	2.00105782	*
945	1,87628489E+287	2.00211565	2.00105671	*
946	3,75653515E+287	2.00211342	2.00105559	*
947	7,52100107E+287	2.00211118	2.00105448	*
948	1,50578636E+288	2.00210896	2.00105337	*
949	3,01474503E+288	2.00210674	2.00105226	*
950	6,03583467E+288	2.00210452	2.00105115	*
951	1,20843585E+289	2.00210231	2.00105005	*
952	2,41940955E+289	2.00210010	2.00104895	*
953	4,84389479E+289	2.00209790	2.00104785	*
954	9,69794095E+289	2.00209570	2.00104675	*

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955      1.94161846E+290        2.00209351        2.00104566   *
956      3.88729747E+290        2.00209132        2.00104456   *
957      7.78271605E+290        2.00208913        2.00104347   *
958      1.55816742E+291        2.00208695        2.00104239   *
959      3.11958329E+291        2.00208478        2.00104130   *
960      6.24566346E+291        2.00208261        2.00104022   *
961      1.25043206E+292        2.00208044        2.00103914   *
962      2.50346288E+292        2.00207828        2.00103806   *
963      5.01212327E+292        2.00207612        2.00103698   *
964      1.00346415E+293        2.00207397        2.00103591   *
965      2.00900731E+293        2.00207182        2.00103483   *
966      4.02217262E+293        2.00206967        2.00103376   *
967      8.05266124E+293        2.00206753        2.00103270   *
968      1.61219545E+294        2.00206540        2.00103163   *
969      3.22771730E+294        2.00206327        2.00103057   *
970      6.46208740E+294        2.00206114        2.00102951   *
971      1.29374804E+295        2.00205902        2.00102845   *
972      2.59015720E+295        2.00205690        2.00102739   *
973      5.18563664E+295        2.00205479        2.00102634   *
974      1.03819177E+296        2.00205268        2.00102529   *
975      2.07851245E+296        2.00205058        2.00102424   *
976      4.16128269E+296        2.00204848        2.00102319   *
977      8.33108097E+296        2.00204638        2.00102214   *
978      1.66791931E+297        2.00204429        2.00102110   *
979      3.33924485E+297        2.00204220        2.00102006   *
980      6.68530218E+297        2.00204012        2.00101902   *
981      1.33842293E+298        2.00203804        2.00101798   *
982      2.67957084E+298        2.00203596        2.00101694   *
983      5.36459166E+298        2.00203389        2.00101591   *
984      1.07400832E+299        2.00203183        2.00101488   *
985      2.15019664E+299        2.00202976        2.00101385   *
986      4.30475327E+299        2.00202771        2.00101282   *
987      8.61822650E+299        2.00202565        2.00101180   *
988      1.72538929E+300        2.00202360        2.00101078   *
989      3.45426657E+300        2.00202156        2.00100976   *
990      6.91550910E+300        2.00201952        2.00100874   *
991      1.38449701E+301        2.00201748        2.00100772   *
992      2.77178441E+301        2.00201545        2.00100671   *
993      5.54914960E+301        2.00201342        2.00100569   *
994      1.11094607E+302        2.00201139        2.00100468   *
995      2.22412446E+302        2.00200937                     *
996      4.45271354E+302        2.00200736                     *


IN STMT  132  PROGRAM RETURNS FROM MAIN PROCEDURE


ALL ACTIVE BLOCKS AND SCALAR AUTOMATIC VARIABLES
 BLOCK #    1 (MAIN PROCEDURE)

                                                                    COUNT22= 1.10000E+01    COUNT2= 2.70000E+01
M=   1000                 L=       4           NUM=     999         COUNT11= 9.95000E+02    COUNT1= 9.99000E+02
S= 3.01711E-07            T=-8.23949E-04       P= 1.00000E+01       RI= 2.000000044703483E+00
I=     27                 K=       4           A= 0.00000E+00
QI= 2.002007360321158E+00                      Y3= 2.684354549999983E+00
Y2= 1.342177289999993E+00                      Y1= 6.710886299999968E-01
X1= 1.110946076750548E+00                      X2= 2.224124463099366E+00                             J= 2.224124463099366E+00
X3= 4.452713545395273E+00                      JJ= 1.342177289999993E+00
RF= 2.000000044703484E+00                      QF= 2.001004688500030E+00


LABEL/ENTRY STMT COUNT  LABEL/ENTRY STMT COUNT  LABEL/ENTRY STMT COUNT  LABEL/ENTRY STMT COUNT  LABEL/ENTRY STMT COUNT
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SUMMARY

It has been shown that there are various ways to find the roots of functions. This thesis has shown how to form some of the iterative schemes and under what conditions we can be guaranteed convergence. Programs were written to illustrate and help facililate in the understanding of the methods.

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