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The ways teachers understand mathematics that is useful for teaching, known as mathematical knowledge for teaching (MKT), has been a center of attention for the research and policy communities. A teacher's MKT has been shown to be a significant factor in predicting student outcomes and positively related to the quality of mathematics instructional quality. Yet researchers have almost exclusively focused on elementary teachers' MKT, leaving very little understanding of MKT for secondary teachers and whether MKT relates to student outcomes and instructional quality in the same ways. Given the gatekeeper functionality of algebra and the seemingly intractable opportunity gap, it is imperative that we build knowledge about MKT at the secondary level.

This exploratory, multi-case design study investigated the realms of knowledge used by expert mathematics educators when engaging with student quadratic function work. The experts participated in a series of interviews revealing their use of mathematical knowledge while unpacking student strategies, mathematical understandings, and needs for further instruction. Ball and colleagues' Mathematical Knowledge for Teaching (MKT) Framework serves as a lens to identify and categorize the realms of knowledge made explicit during the experts' engagement with the student work.

The use of multiple cases provides convergent and complementary evidence of the knowledge used when engaging with student quadratic function work, therefore supporting the evolution of an innovative conception of secondary MKT. While findings

from the study specifically address the nature of secondary MKT for teaching quadratics, study approaches also address engagement with student written work and the needed assimilation of MKT research to advance secondary MKT understanding. Three manuscripts collectively convey these results.

The first manuscript explores the nature of mathematical knowledge for teaching quadratic functions by using Ball and colleague's (2008) Mathematical Knowledge for Teaching Framework as a guide and student written work as a source. Through a series of task-based interviews with the six experts, findings indicate that the nature of mathematical knowledge for teaching quadratic functions can be characterized as six interrelated entities: content knowledge, connections, interpretations, anticipations, instructional moves, and resources. The second manuscript details a set of six questions that emerged from my analysis. These questions, which direct a strengths-based engagement with student work and the exploration of one's own MKT, can provide meaningful learning experiences for individuals, professional learning communities, and large group professional development activities. The third manuscript addresses the advances that have been made in understanding secondary MKT and the barriers that could be hindering progress. Ideas that help to reevaluate differences in the literature are presented to motivate the mathematics education community to continue efforts to develop a unified vision of secondary MKT.

EXPLORING MATHEMATICAL KNOWLEDGE FOR TEACHING QUADRATIC FUNCTIONS THROUGH STUDENT WORK

by

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Approved by

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Committee Chair

To my husband, Edgar Sr. - One who embodies completely giving of self to change the lives of others through education.

APPROVAL PAGE

This dissertation written by STACEY CHANELLE ZIMMERMAN has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I

INTRODUCTION

Mathematical Knowledge for Teaching or *MKT* is the phrase that is commonly used to reference the knowledge used and needed by those providing mathematics instruction. Over the last few decades, researchers have diligently worked to identify, describe, categorize, and connect MKT to student learning. Notably, Ball and colleagues (Ball & Bass, 2002; Ball et al., 2004, 2005; Hill et al., 2004), often credited for the MKT phrase, developed a practice-based theory that represents "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill et al., 2005, p. 373). This practice-based theory, grounded in the work of elementary teachers, has been widely used to study and create measures of teacher knowledge. Through this work, researchers have linked MKT to the quality of instruction (Hill, Ball, et al., 2008) and student achievement (Hill et al., 2005).

While Ball and colleagues' theory of MKT has been widely accepted and recognized in the elementary mathematics community, research explicitly regarding secondary MKT lags that of the elementary field. The secondary mathematics education community is yet to reach a common conceptualization of MKT. Though gains in secondary MKT research have been made, more work is needed to produce a widely accepted theory. Perhaps the diversity of secondary MKT research has not availed itself to a unified vision of the knowledge needed to teach secondary mathematics. For example, several researchers have built upon the work of Ball and colleagues to generate topic specific conceptualizations of secondary MKT (e.g., Hatisaru & Erbas, 2017; Steele et al., 2013), while others have taken Ball and colleagues' framework as defined and applied it to secondary studies (e.g., Campbell & Lee, 2017; Khakasa & Berger, 2016). Still others, question the application of Ball and colleagues' theory of teacher knowledge at the secondary level (Silverman & Thompson, 2008; Speer et al., 2015).

Although mostly diverse, I have found one commonality across the secondary MKT research efforts. Whether investigative, evaluative, or developmental in nature, the studies tend to enter the work with a predefined conception of secondary MKT. For example, Khakasa and Berger (2016) consider identifying patterns in student errors and assessing the usability of nonstandard approaches as evidence of specialized content knowledge. Whereas, Hatisaru and Erbas (2017) deem "working with the definition of function, having a neat repertoire of examples, conceiving the function as an object, knowing multiple representations of functions, knowing essential features of functions, knowing the importance of the function concept, and communicating the importance with students" evidence of teachers' specialized content knowledge (p. 707). Whether it is the use of Ball and colleagues' descriptions (i.e., Khakasa and Berger) or using teaching and learning research to expand upon Ball and colleagues' descriptions to reflect a specific secondary mathematics topic (i.e., Hatisaru and Erbas), researchers establish how MKT should be evident in the data. The researchers have established what MKT is for their study.

With a predefined conception of MKT, such as the before mentioned, research can take an evaluative perspective; one that illustrates the presence or absence of MKT. Hence, the researchers' question, whether explicit or subliminal, is not *what is secondary MKT,* but rather, *does this teacher possess what we define as secondary MKT?*

Considering the current state of secondary MKT research, I enter my study with two motivating thoughts. First, there is an ongoing need to work toward developing a unified conception of secondary mathematics. Even if research desires to determine the existence or nonexistence of MKT, a unified vision will bridge studies and present more compelling evidence of MKT across our secondary mathematics community. Further, if indeed "<u>all</u> teachers need continuing opportunities to deepen and strengthen their *mathematical knowledge for teaching*" (Conference Board of the Mathematical Sciences, 2012, p. 68), we need to know what MKT is for <u>all</u> teachers. Examinations of specific conceptions of MKT will not contribute to efforts to articulate and accumulate this knowledge, but a unified vision will. A common conception of MKT will be a step towards ensuring the equitable preparation and development of <u>all</u> secondary mathematics teachers.

My second motivation comes from existing MKT efforts. Several secondary mathematics researchers have put forth significant efforts to understand this complex construct. Although diverse, I consider existing teacher knowledge research valuable. The methods, outcomes, and recommendations for MKT studies can and should inform ongoing efforts. While one study alone cannot conquer the challenge of producing a vision of secondary MKT that is acceptable by all, a combination of efforts may. I am motivated to build upon the work of others with the intention of bringing more clarity, and possibly agreement, to the ways that we conceptualize secondary MKT. However, to do this, I feel we must first answer a question that has yet to be directly asked - what is secondary MKT?

Statement of Research Problem

Questions concerning teacher knowledge continue to drive research studies as efforts are made to improve the mathematics achievement of all students. While we know teachers must comprehend the subject that they teach (Ball et al., 2008), we are still uncertain about the specifics of the knowledge needed to teach well (Wilson et al., 2017). The specifics of the knowledge needed to teach well is possibly embedded in our conceptualization of secondary MKT. Therefore, in this dissertation I address the question, what is secondary MKT, by exploring the mathematical knowledge needed to teach a secondary mathematics concept, quadratic functions.

Significance of Study

Considering the state of secondary mathematical knowledge for teaching research and the ongoing desire to improve secondary mathematics instruction, there is a need to continue the work towards developing a unified understanding and conception of the knowledge that teachers need, possess, and use to teach secondary mathematics. My study contributes to this need in several ways. First, my approach to secondary MKT research differs from existing studies. As previously stated, existing secondary MKT studies start with a predefined conception of MKT, which can lead to a deficit perspective of teachers' MKT. Instead, I seek to understand what secondary MKT is by exploring the knowledge that is demonstrated by secondary mathematics educators. The methods I use are replicable and can be used to explore other areas of secondary mathematics content, contributing to the understanding of secondary MKT, possibly across the secondary mathematics curriculum.

Second, by using quadratic functions as the mathematical content for the study, I am also contributing to the field as little research exists that addresses teachers' knowledge of teaching quadratic functions. Most of the research on quadratic functions is based on student learning of the subject. Research does not explicitly address the ways that teachers know quadratic functions that is useful and meaningful for student learning. This study provides insight into the ways that experienced mathematics educators know quadratic functions.

Finally, the conception of secondary MKT that is a result of my empirical exploration can be influential in both research and practice. This new conception provides secondary scholars an innovative vision and possible means to studying secondary teacher knowledge. In addition, considering the preparation and ongoing development of mathematics teachers, the conception clearly identifies aspects of teacher knowledge that are useful for teaching quadratic functions.

Overview of the Dissertation

This dissertation is divided into seven chapters. Chapter I introduces the research problem and highlights its significance within the mathematics education community. Chapter II provides a review of the literature, as well as the researcher stance and research question. Chapter III delves into the research methods. To fully covey the results of my study, the findings are presented in three manuscripts, which account for Chapters IV, V, and VI. The first manuscript (Chapter IV), titled "Exploring Mathematical Knowledge for Teaching Quadratic Functions When Analyzing Student Written Work" explores the nature of mathematical knowledge for teaching quadratic functions by using Ball and colleagues' (2008) Mathematical Knowledge for Teaching framework as a guide and student written work as a source. Through a series of task-based interviews with six experienced mathematics educators, findings indicate that the nature of mathematical knowledge for teaching quadratic functions can be characterized as six interrelated entities: content knowledge, connections, interpretations, anticipations, instructional moves, and resources.

The second manuscript (Chapter V), titled "Beyond Right or Wrong: A Strengths-Based Approach to Examining Student Work" stemmed from ways in which the experienced mathematics educators engaged with student written work. While the purpose of the mathematics educators' engagement with student written work was for me to gain insight into their MKT, the ways in which the participants responded to student work was insightful. This manuscript details a set of questions that can be used to engage with student written work in a strengths-based way. In addition, the six questions can also provide opportunities to explore one's own MKT. These questions, which can provide meaningful learning experiences for individuals, professional learning communities, and large group professional development activities, quite possibly address the Conference Board of the Mathematical Sciences' (2012) call for all teachers to have opportunities to deepened and strengthen their MKT.

The third and final manuscript (Chapter VI), titled "Research Commentary: Moving Secondary Mathematical Knowledge for Teaching (MKT) Forward" encourages scholars to continue to work towards a unified conception of secondary MKT. In this manuscript, through the literature, I discuss the progress that has been made in understanding secondary MKT and the barriers that could be impeding advancement. In motivating scholars to continue MKT efforts, I present ideas that can help reevaluate the differences in the literature and move the mathematics education community toward a more unified conception of secondary MKT.

In the closing (Chapter VII), after a brief review of the manuscripts, I discuss implications for researchers, mathematics teacher educators working with prospective and practicing teachers, teachers, and policy makers.

CHAPTER II

LITERATURE REVIEW

In this section, I provide a brief review of the literature that lays the foundation for exploring mathematical knowledge for teaching on the secondary level. I begin by discussing the idea of teacher knowledge and the development of mathematical knowledge for teaching at the elementary level. From there, I present the work that provides insight and direction for the continued pursuit of understanding secondary mathematical knowledge for teaching. Based on these insights, I discuss the need and reason to be content specific in my exploration, focusing on an area of need, quadratic functions. I conclude by clarifying my researcher stance and research question.

Of note to the reader, though this dissertation contains three manuscripts that collectively present a comprehensive reflection of my study, each manuscript is written to stand alone. Hence, each manuscript shares a common review of the literature that is presented here.

Teacher Knowledge

For decades, scholars have recognized that teacher knowledge is necessary for meaningful student learning. Although views of which type or types of teacher knowledge is most vital for educational improvement has swayed between a focus on content and a focus on pedagogy, content knowledge is still recognized as a primary requirement for teaching certification (National Research Council, 2001). However, with the work of Begle (1972, 1979) and Monk (1994) revealing that advanced content knowledge does not guarantee meaningful instruction and Shulman's push for teacher knowledge to be conceptualized as a type of professional knowledge that encompasses the "ability to transform one's knowledge into teaching" (Shulman, 1986, p. 14), scholars have been inclined to take a more holistic approach to understanding the domains of knowledge that contribute to teaching.

Shulman (1987) could be considered the leader in the endeavor to broaden the conceptualization of teacher knowledge as he suggested several domains that could comprise a teacher's knowledge base: content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends, purposes and values. One of Shulman's domains, pedagogical content knowledge, has been of special interest to scholars as it is a departure from considering content knowledge and pedagogical knowledge separately. Pedagogical content knowledge, the domain described as a "particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9), blends subject matter understanding with the methods and practice of teaching. This domain provides a theoretical lens for exploring how teachers' subject matter understanding, their content knowledge, plays out in their instruction.

Shulman's ideas concerning the categorization of teacher knowledge drew great interest and continues to do so since his work was first published in 1986. According to Google Scholar, his *Harvard Education Review* article has been cited over 20,000 times.

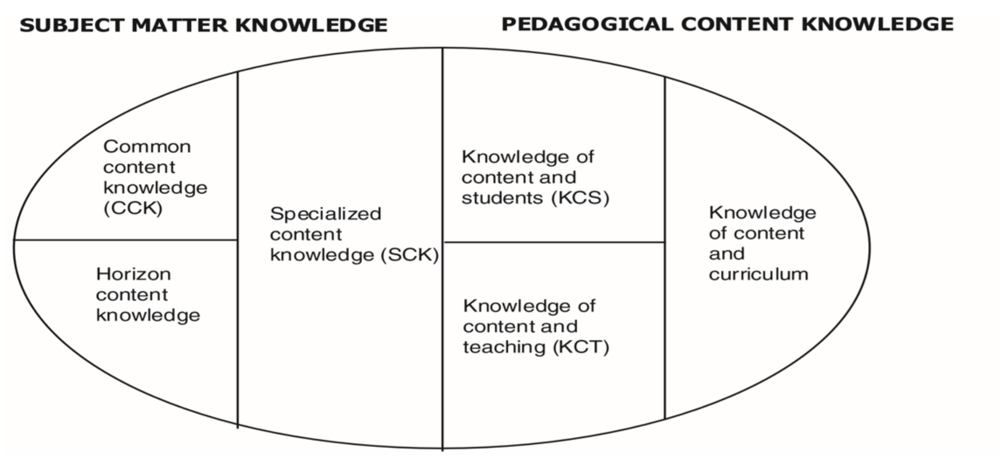
As his work spans subject areas, from science to music, the interest in pedagogical content knowledge has been widespread and has been very influential in the mathematics education field.

Elementary Mathematical Knowledge for Teaching

Reconfiguring and expanding upon the earlier pedagogical content knowledge work of Shulman, Ball and colleagues (2008), through the work of the Learning Mathematics for Teaching (LMT) research group, developed a framework for conceptualizing the knowledge needed specifically for teaching elementary mathematics. The LMT research group took a practice-based approach, examining the work of teaching, to gain an understanding of the mathematical knowledge used by elementary teachers. The Mathematical Knowledge for Teaching Framework (Figure 1.1) divides the domain of mathematical knowledge for teaching into two overarching domains, pedagogical content knowledge, PCK, and subject matter knowledge, SMK. The PCK domain, which goes beyond the basic understanding of mathematical content, focuses on what is needed to successfully instruct. For example, from a discursive perspective, this would include a teacher's ability to understand student statements and respond in a way that promotes learning. The domain of PCK is composed of three subdomains, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. Knowledge of content and students, KCS, is a combination of teachers knowing their students and knowing about mathematics (Ball et al., 2008). For example, a teacher that has knowledge of their students and mathematics will know that a common misconception among students is to incorrectly calculate 3^2 as 6 because the student interprets the expression to be 3 x 2 and not 3 x 3 x 3. Knowing which strategies to use or knowing how to design instruction to facilitate understanding of mathematical concepts, such as exponentiation, for example, requires that teachers possess knowledge of content and knowledge of teaching, KCT. Lastly, in the pedagogical domain, knowledge of content and curriculum, KCC, provides the teacher with the awareness of the programs and instructional materials available to teach certain topics and the value or effectiveness of the programs and materials (Ball et al., 2008).

Figure 2.1

Mathematical Knowledge for Teaching Framework (Adapted from Ball et al., 2008)



The domain of subject matter knowledge, SMK, includes three subdomains, common content knowledge, specialized content knowledge, and horizon content knowledge. Common content knowledge, CCK, is the knowledge and skills of mathematics that can be used in settings outside of teaching (Ball et al., 2008). For example, having the skill or knowing the algorithm to multiply two-digit numbers is content knowledge that is not limited to instruction. Knowledge that allows the teacher to connect the multiplication algorithm to other concepts in mathematics, for example place value or division, is specialized content knowledge, SCK. SCK is a knowledge and skill that is unique to teaching (Ball et al., 2008). Horizon content knowledge, HCK, is knowledge of how mathematical concepts span the mathematical careers of students. For example, knowing that the algorithm for multiplication is related to binomial multiplication is an aspect of HCK.

The Mathematical Knowledge for Teaching framework has been instrumental at the elementary level. Once researchers established a better understanding of the knowledge domains used to teach elementary mathematics, researchers were able to link mathematical knowledge for teaching to student achievement (Hill et al., 2005) and to the quality of instruction (Hill, Ball, et al., 2008). Although the Mathematical Knowledge for Teaching Framework has drawn criticisms for its loosely defined terminology (Howell, 2012) and the inability to distinguish between domains (Hill, Ball, et al., 2008), it has become one of the mostly widely used frameworks for exploring teacher knowledge in the mathematics education field. While deemed an elementary framework, the Mathematical Knowledge for Teaching framework has been used in several efforts to study the mathematical knowledge of secondary mathematics teachers.

Secondary Mathematical Knowledge for Teaching

Although the LMT research group never recommended the generalization of the Mathematical Knowledge for Teaching Framework to other content areas or grade levels, a few secondary mathematics scholars have successfully utilized the Mathematical

Knowledge for Teaching Framework in the exploration of secondary mathematics knowledge. Of note is the work of Michael D. Steele. Steele has used the Mathematical Knowledge for Teaching Framework to explore the relationship between mathematical knowledge for teaching and practice, to understand how mathematical knowledge for teaching develops in a methods course, and to design assessment tasks to measure secondary mathematical knowledge for teaching.

Using Ball and colleagues' Mathematical Knowledge for Teaching Framework as a guide, Steele and Rogers (2012) developed a framework to explicitly study the relationship between teacher knowledge and the practice of teaching proof in high school. Their framework for teaching proof, the MKT-P framework, focused primarily on Ball and colleagues' CCK and SCK subdomains. Built from literature on student learning of proof and teacher's knowledge of proof, the MKT-P framework outlines components of proof knowledge, such as defining proof, identifying proofs and non-proofs, creating proofs, and understanding the roles of proof in mathematics. Steele and Rogers acknowledge that the MKT-P framework is not fully representative of the mathematical knowledge for teaching proof, but state that it "provides a fruitful starting point grounded in previous research for investigating teachers' mathematical knowledge" (2012, p. 161). Using the MKT-P framework to identify evidence of teacher knowledge, researchers analyzed interviews, written assessments, and teaching observations of two contrasting cases, a novice teacher, and an expert teacher. Results indicated that mathematical knowledge for teaching evident in clinical settings (i.e., written assessment and interviews) played out differently in the classroom. While both the novice and expert demonstrated a wealth of knowledge in clinical settings, the expert's classroom observation revealed greater utilization of this knowledge in practice. Such results may provide insights into how to facilitate the growth of mathematical knowledge for teaching in novice teachers to that of an expert level.

In a subsequent study, Steele and colleagues (2013) focused on how mathematical knowledge for teaching, specifically that related to teaching functions in a secondary curriculum, evolved through the matriculation in a graduate level methods course designed for prospective and practicing teachers. Using a teaching experiment methodology (Steffe & Thompson, 2000) the researchers intentionally designed to provide opportunities for students to develop and connect CCK and SCK. Twenty-one teachers with various backgrounds in teaching licensure, subject-matter preparation, and teaching experience, were given pre- and post-course written assessments and interviewed to track their growth. In addition, video records of class meetings, field notes, and instructional artifacts were collected as data sources. Analysis focused on changes in written work and interviews related to aspects of CCK and SCK over time. Here, both CCK and SCK followed the general definitions provided through Ball and colleagues' Mathematical Knowledge for Teaching Framework, but specifically related to functions. For example, knowing the univalence property of functions (i.e., each element of the domain maps to exactly one element of the range) was considered an aspect of CCK, whereas connecting different representations of functions was an aspect of SCK. Findings indicated that all students enrolled in the graduate level course demonstrated some level of growth in their mathematical knowledge for teaching as there was a significant increase in the number of students providing accurate definitions of functions, connecting generalizations to visual patterns, and generating and connecting function representations. Studies of this kind indicate that courses can be designed to impact one's mathematical knowledge for teaching and that prospective and practicing teachers can both benefit from opportunities to develop common and specialized content knowledge.

Tackling another area of secondary mathematics content, Steele (2013) turned his focus to geometry and measurement, an area known for little gain by American students on national and international assessments (National Center for Educational Statistics, 2012; National Mathematics Advisory Panel, 2008). Here, Steele was more focused on the ability to develop meaningful tasks and less focused on the evaluation or categorization of teachers' knowledge. Presumably, "the construction of such tasks could aid the work of assessing the impact of mathematics teacher education and professional development efforts" (Steele, 2013, p. 265). With the obvious need to improve learning in the area of geometry and measurement, Steele sought to design and implement tasks that would illicit mathematical knowledge for teaching. Again, using the definitions of CCK and SCK from Ball and colleagues' framework, Steele designed open-ended tasks that were grounded in the context of teaching, focused on measuring aspects of and relationships between CCK and SCK related to geometry, and capturing the nuances of teacher knowledge beyond correct and incorrect answers (2013, p. 248). Tasks were administered to 25 teachers enrolled in a 6-week course focused teaching geometry and measurement. Based on task assessment results, Steele concluded that it is possible to design tasks that elicit aspects of mathematical knowledge for teaching, differentiate teacher performance, and show connections between CCK and SCK. Steele (2013) concludes, “Understanding these interactions through teacher knowledge assessments could significantly advance the field’s understanding of the nature and use of mathematical knowledge for teaching” (p. 265).

Like the task design work of Steele, several other secondary mathematics scholars used Ball and colleagues’ framework as a conceptual guide for designing tools or assessment items to measure secondary teachers’ mathematical knowledge for teaching. Khakasa and Berger (2016) developed the MKT proficiency status tool to measure and describe secondary teachers’ mathematical knowledge for teaching across the Kenyan secondary mathematics curriculum. Khakasa and Berger administered the proficiency status tool, a questionnaire aimed at eliciting teachers’ interpretations of students’ problem-solving strategies, to practicing teachers with varying levels of teaching experience. They analyzed one hundred seventeen responses using a rubric developed by the team. In the rubric, a score of 0 (zero) represented a non-response or incorrect response, and a score of 4, represented a response that demonstrated all MKT subdomains. Teacher’s proficiency levels were categorized based on their scoring. Only those teachers that demonstrated synergy across all six of the Mathematical Knowledge for Teaching framework’s subdomains through their written responses to the MKT proficiency tool were considered proficient in their mathematical knowledge for teaching. Like other scholars, the authors suggest that knowledge of this type is needed as it can be used to inform teacher education and professional development programs.

Similarly, in the assessment development arena, Herbst and Kosko (2014) reported the successful pilot of 34 tasked aimed at measuring secondary teachers' mathematical knowledge for teaching geometry. Targeting four of the six Mathematical Knowledge for Teaching Framework subdomains, KCT, KCS, CCK, and SCK, findings from the pilot indicate a correlation between the specified MKT subdomains and years of teaching experiences. Herbst and Kosko conclude, while noting that the initial findings are promising, that more testing is necessary to gain greater understanding into the differences among teachers and how teachers possibly struggle with varying domains or aspects of mathematical knowledge for teaching.

Although a few scholars had already applied Ball and colleagues' Mathematical Knowledge for Teaching Framework on the secondary level, Howell and colleagues (2016) conducted a study solely focused on the utility of the Mathematical Knowledge for Teaching Framework on the secondary level. To determine if the Mathematical Knowledge for Teaching Framework would indeed extend to the secondary level, Howell et al. (2016) designed assessment items aimed at obtaining evidence of mathematical knowledge for teaching at the secondary level. Think-aloud cognitive interviews with 23 prospective and practicing teachers revealed that the items measured aspects of MKT as designed and demonstrated the ability to measure knowledge, such as SCK, that extends beyond conventional mathematics knowledge. This validation study was a step in proving that the design principles used in Ball and colleagues' exploration of elementary MKT can be applied on the secondary level.

Still other researchers have used Ball and colleagues' Mathematical Knowledge for Teaching Framework as a theoretical and conceptual guide to develop Mathematical Knowledge for Teaching Frameworks directly related to secondary mathematics (see Hatisaru & Erbas, 2017; Taşdan & Koyunkaya, 2017). These frameworks provide an opportunity to provide specific and detailed descriptions of the six subdomains that compose the MKT-framework. Using the elementary Mathematical Knowledge for Teaching Framework as a shell, one can fill in the details that are precise to teaching a specific content area in secondary mathematics. The work to validate and expand upon these proposed secondary mathematics frameworks may be a very important step in the work of exploring and understanding secondary mathematics knowledge for teaching and moving towards a unified vision of secondary MKT. To categorize or measure secondary mathematical knowledge for teaching, one must first know exactly what it is. The Mathematical Knowledge for Teaching Framework may be a tool that can aid in this work.

A Content Specific Study

Although sparse, studies of secondary mathematics knowledge for teaching have typically followed two paths: a broad study across secondary curriculum or a focused content-specific study. While both have been fruitful in gaining a greater understanding into secondary mathematical knowledge for teaching, a content specific study will potentially yield results that can inform both teacher education and professional development efforts. For example, in Campbell and Lee's (2017) examination of the opportunities to develop secondary mathematical knowledge for teaching through professional learning communities (PLCs), the researchers took a non-content specific approach to exploring mathematical knowledge for teaching by analyzing the conversations and interactions of teachers in learning communities. The discussion in the PLCs often covered numerous topics, from student assessment data to student behavior issues, but were not necessarily related to a particular mathematics course or content area. Discussions that did focus on mathematics content, such as teachers discussing methods to multiply binomials, were considered opportunities to develop mathematical knowledge for teaching by the researchers. While opportunities to expand mathematical knowledge for teaching were identified, one is left to infer what parts of the discussion can grow what aspects of the Mathematical Knowledge for Teaching Framework domains. Whereas Steele and colleagues' (2013) content-specific approach to identifying aspects of mathematical knowledge of teaching during the classroom discussions of 21 teachers resulted in very detailed examples of conversations that relate to specific domains of the Mathematical Knowledge for Teaching Framework. For example, one's ability to correctly define a function was identified as a part of the CCK subdomain and the ability to vocally and/or visually make connections between various function representations was an aspect of the SCK. Specifics, such as these, can be used to guide the design and implementation of educational opportunities for prospective and practicing teachers to further develop their mathematical knowledge for teaching.

Following the idea that using a subject-specific framework for investigating secondary mathematical knowledge for teaching will "give researchers and teachers a window into the ways in which teacher knowledge influences the work that they do with students" (Steele & Rogers, 2012, p. 178), I examine the secondary mathematical knowledge for teaching quadratic functions. Focusing on a specific content area such as quadratic functions allows in-depth examination of individual teacher knowledge, simplify the development of instruments used to explore MKT, and possibly limit the cognitive demand placed on teachers during the research study.

Why Quadratics Functions?

Although functions are a "unifying theme in United States mathematics curricula" (Steele et al., 2013, p. 454), little research exists on teacher knowledge of quadratic functions. Quadratics functions are frequently viewed as one of the most conceptually challenging areas in the secondary mathematics curriculum (Lobato et al., 2012; Zaslavsky, 1997). Instruction on functions can begin as early as elementary school (Blanton & Kaput, 2011) and continue to grow in complexity throughout middle school, high school, and college, where the understanding of quadratic functions is vital. For students to be successful in mathematics, they must develop a rich conceptual understanding of *all* functions (Cooney et al., 2010; Thompson & Carlson, 2017). I aid in this process by exploring the mathematical knowledge needed to successful provide quadratic function instruction.

Typically, students' first encounter with nonconstant rates of change is through quadratic functions. Quadratic functions are often used to model real life events and model numerous physical phenomena, such as the speed of a falling object. Quadratic functions, according to definition can be written in the form $f(x) = ax^2 + bx + c$, for some real numbers *a*, *b*, and *c*, where *a* is not zero. According to the National Council of Teachers of Mathematics (NCTM),

> Quadratic functions are characterized by a linear rate of change, so the rate of change of the rate of change (the second derivative), of a quadratic function is constant. Reasoning about the vertex form of a quadratic allows deducing that a quadratic has a maximum or minimum value and that if the zeros of the quadratic are real, they are symmetric about the x-coordinate of the maximum or minimum point. (Cooney et al., 2010, p. 9)

This statement represents what NCTM considers an "essential understanding" of quadratic functions. This very detailed description of quadratic functions supports NCTM's big idea (or essential knowledge for teachers), that functions can be classified into different function families, all of which have unique characteristics and can be used to model varying real-world events.

Unfortunately, empirical studies regarding the teacher knowledge of quadratic functions are nonexistent, and there is a dearth of research on student learning of quadratic functions. Research on student learning, for the most part, focuses on the difficulties that students encounter when learning quadratic functions. For example, Zaslavsky (1997), after analyzing data from over 800 students enrolled in 10th and 11th grade high school in Israel, identified five obstacles encountered by student when learning: (1) the interpretation of graphical information, (2) the relation between a quadratic function and a quadratic equation, (3) the analogy between a quadratic function and a linear function, (4) the seeming change in form of a quadratic function whose parameter (either *b* or c in $ax^2 + bx + c$) is zero, and (5) the over-emphasis on only one coordinate of special points (e.g., the x-coordinate of the vertex). Celik and Guzel (2017), along with Eraslan and colleagues (2007), further this line of research by presenting the results of clinical interviews which reveal student misconceptions and errors when working with quadratic functions.

A few scholars have made recommendations to overcome such learning obstacles. After collecting and analyzing the data of 60 students enrolled in an undergraduate mathematics course focused on modeling, researchers concluded that is it vital to explore student's personal meanings for concepts of quadratics, such as the vertex (Childers & Vidakovic, 2014). Childers and Vidakovic found students' personal meanings that represented a misconception of the vertex performed lower than fellow students and struggled to complete algebraic and real-world quadratic problems. Lobato and colleagues (2012) suggested that conceptual learning goals should be explicitly defined prior to quadratic function instruction. Based on the exploration and analysis of student reasoning related to quadratic functions, which led to the identification of pivotal student conceptions, the team of scholars proposed five conceptual learning goals for quadratic functions:

> *Goal 1*: comprehend a quadratic function situation as containing a set of changes in the dependent variable, as quantities, meaning that these are entities that can be mentally operated on (e.g., through comparison), interpreted in terms of their meaning in particular contexts, and assigned correct units of measure.
>
> *Goal 2*: comprehend a quadratic function situation as containing a set of changes in the independent variable and to grant them the same stature as the corresponding changes in the dependent variable.
>
> *Goal 3*: conceive of a quadratic function situation as necessitating the construction of a sequence of ratios (as composed units) of the changes in the dependent variable to the corresponding changes in the independent variable.

> *Goal 4*: conceive of rates of change as quantities that can be mentally compared to yield a new quantity, which is a rate, not a change in the dependent quantity.
>
> *Goal 5*: conceive of a quadratic function situation as necessitating the construction of a rate of change of the rates of change, meaning that students compose the change in the average rates of change (in the dependent variable with respect to the corresponding changes in independent variable) with the associated interval on which the domain of the independent variable is segmented, and understand that this quantity is constant for quadratic functions. (Lobato et al., 2012, p. 112)

These goals provide very specific and detailed information that could play a major role in designing and planning instructional activities for students. They could also serve to identify the knowledge needed by teachers to successfully instruct.

When pursuing a content-specific mathematical knowledge for teaching study, to some “an important first step is using the research literature on student and teacher learning in the development of an Mathematical Knowledge for Teaching Framework for the content in question” (Steele, 2013, p. 265). The MKT research that has followed this recommendation has been able to determine the existence or nonexistence of the predefined MKT. However, the absence of existing quadratic teaching and learning research could lead to an inadequate framework. In addition, my primary goal is to discover and understand the MKT that those doing the work of teaching possess. For that reason, I do not utilize a predefined conception of MKT for quadratics, but rather use the quadratic research to make sense of the MKT that is evident in my study.

Researcher Stance

After reviewing and reflecting on the teacher knowledge literature, four considerations inform my approach to exploring secondary MKT. These considerations, discussed below, also inform my research question, which follows the discussion.

First, I believe work at the elementary level, specifically Ball and colleagues' conceptualization of MKT, which is grounded in practice and proven useful in empirical studies, provides a theoretical guide for teacher knowledge exploration. I consider this conceptualization a fruitful guide or starting point. Bearing in mind the vastness and complexities of secondary mathematics, I consider Ball and colleagues' conception a place to work from; not a comprehensive and complete framework for exploring the particulars of the knowledge needed to teach secondary mathematics. Therefore, Ball and colleagues' conception serves as a theoretical guide.

Second, believing that a comprehensive picture of mathematical knowledge for teaching is difficult to observe, even if both clinical methods (e.g., written assessments, interviews) and classroom observations are utilized, I acknowledge the challenge of designing a study that reflects authentic activities of teaching, yet allows unrestricted demonstration of knowledge. In addition, I note that researchers conducting clinical studies of MKT through written assessments and/or interviews only (e.g., Herbst & Kosko, 2014, Howell et al., 2016, or Steele et al., 2013) have not addressed all six of the Mathematical Knowledge for Teaching Framework subdomains. Since I wish to understand every possible aspect of secondary MKT, through the literature, I am aware of the limitations that could result from certain methods. Though classroom observations can be used to see knowledge in action, the teacher knowledge that is displayed may be influenced or restricted by the needs of the students or may be obscured for reasons unknown to the observer. Of course, this is ideal instruction as the teacher is addressing student needs, but this may prevent or limit a full understanding of the teacher's knowledge. Therefore, I engage my study participants in a series of interviews that allow for unrestricted flow of thought, inquiry, reflection, and member-checking, to gain the most complete picture of teacher knowledge possible.

Third, researchers have shared that experience may impact the level of MKT. In Taşdan and Koyunkaya's (2017) MKT study with preservice teachers, subdomains HCK and KCC were omitted because the researchers felt preservice teachers needed more experience to possess such knowledge. Similarly, in the case study exploration of an experienced teacher and novice teacher, the experienced teacher demonstrated greater levels of MKT (Steele & Rogers, 2012). Therefore, to gain the broadest view of secondary MKT, I engage with those that have significant experience in mathematics education.

Lastly, content specific studies, similar to the work of those studying functions or geometry (Hatisaru & Erbas, 2017; Herbst & Kosko, 2014; Steele, 2013; Steele et al., 2013; Taşdan & Koyunkaya, 2017), provide detailed information regarding the links between knowledge and practice. The Mathematical Knowledge for Teaching Framework, which "links knowledge, teaching practice, and students' learning" (Speer et al., 2015, p. 120), may be better suited at the secondary level as a guide if a specific mathematical topic is used as a catalyst to understand MKT. By focusing on the knowledge needed to teach a specific topic, I capture the nuances needed to describe and categorize secondary MKT.

Research Question

Using the Mathematical Knowledge for Teaching Framework as a theoretical guide while engaging experienced educators in a series of interviews with a content specific focus, I gain a greater understanding of the knowledge needed to teach secondary mathematics content by utilizing a multi-case exploratory designed that answers the question: *What is the nature of the knowledge expert mathematics educators use when engaging with student written work on quadratic function tasks?*

CHAPTER III

METHODS

To gain a greater understanding of the knowledge needed to teach quadratics functions, thus providing insight into secondary MKT, I utilize a multi-case exploratory design. In this section, I review the study design, data collection and analysis. Again, as of note to the reader, though this dissertation contains three manuscripts that collectively present a comprehensive reflection of my study, each manuscript is written to stand alone. Hence, each manuscript shares the methods discussed here.

Design

Exploratory Multi-Case Approach

Exploratory case studies are ideal for investigating phenomena on which little to no research exists and for which existing data is too sparse to build solid hypotheses (Streb, 2010). This research method provides a way to gain insight into something that is not sufficiently understood (Stake, 1995). The exploration of secondary MKT, a meagerly researched area benefits from a case study approach as case studies are ideal for gaining understanding of real-life, uncontrollable, complex phenomena that are dependent upon context (Yin, 2009). The complexities of teacher knowledge may not be sufficiently explored via other research approaches, such as surveys or experiments, but demands the compilation of multiple sources of evidence, which case studies allow (Creswell, 2012; Yin, 2009).

Given the purpose of this study is to gain a greater understanding of secondary MKT and the productive ways of knowing mathematics, I employ elements of appreciative inquiry to inform my exploratory multi-case study. Appreciative inquiry, an ideology rooted in organizational development, utilizes a four-phase approach (discovering, dreaming, designing and delivering) to focus on positive experiences and creatively plan for the future (Cooperrider & Srivastva, 1987). An "appreciative stance broadens our capacity for seeing the good" (Godwin, 2016, p. 27). Guiding this exploration with ideas from the appreciative inquiry phases ensures that the focus remains on positive aspects of secondary MKT. This too, helps me with my positionality throughout the study and makes me keenly aware of my potential researcher bias. Having more than 20+ years of mathematics teaching experience, I have strong views of productive and nonproductive ways of knowing mathematics.

In addition, the Mathematical Knowledge for Teaching Framework (Ball et al., 2008), a theoretical framework derived from the practice of elementary mathematics teachers serves as a conceptual lens in the exploration of the knowledge needed to teach secondary mathematics. The knowledge demonstrated by experts in the field of mathematics education, will converge and be complementary, enabling the building of a conception that is reflective of secondary level mathematical knowledge for teaching. Acknowledging the vastness of secondary mathematics content is beyond the realm of this study, to pursue this endeavor, I use an exploratory multi-case study approach to discover the mathematical knowledge needed for teaching, specifically, teaching quadratic functions. Further, given the expectancy that the knowledge of experts in the field is convergent and complementary, my exploratory case study employs a multi-case design or collective design (Stake, 1995). To maximize what can be learned, with the intent of presenting more compelling and robust evidence, multiple cases, experts in the mathematics education field were selected as participants. This selection is informed by the discovering phase of appreciative inquiry which encourages identifying the "best and most positive experiences" (Shuayb et al., 2009, p. 3).

Selection of the Cases

Matsuura and colleagues (2013), mathematicians engaged in doing mathematics with secondary teachers for more than 20 years, suggest that the ways that teachers know mathematics can be categorized into four overlapping categories: knowing mathematics as a scholar, knowing mathematics as an educator, knowing mathematics as a mathematician, and knowing mathematics as a teacher. To know mathematics as a scholar, one understands the origins of mathematics, the important contributions of the mathematics fields, and how the history of mathematics connects to current mathematics instruction. As an educator, one knows how mathematical thinking evolves in learners and how this thinking supports the various branches of mathematics. As a mathematician, teachers know the work of a mathematician involves struggle, experiment, abstractions, and theory development. Lastly, knowing mathematics as a teacher implies that one understands how mathematics is used to instruct, engage in deep thought, and promote understanding in learners. Assuming scholar, the first way of knowing, permeates the other ways, I selected participants that represent the best and most positive experience by choosing those that epitomize knowing mathematics as a teacher educator, a mathematician, and a secondary teacher.

Selecting participants that know mathematics as a teacher educator, a mathematician, and a secondary teacher constitute stratified purposeful sampling. Purposefully sampling has the potential to maximize the learning potential (Stake, 1995). Participants were purposefully sampled, the deliberately selected based on characteristics highly relevant to the phenomenon under study (Wiersma & Jurs, 2005) while being stratified by current or primary occupation. Description of the selection criteria are below.

> *Expert Mathematics Teacher Educator.* An expert mathematics teacher educator has a doctorate in mathematics or mathematics education and has served in the role of teacher educator for at least ten years through his or her work with preservice and/or in-service secondary mathematics teachers. Ideally, this expert will have also spent considerable time teaching mathematics on the secondary level before serving as a preservice/in-service mathematics educator.
>
> *Expert Mathematician.* An expert mathematician has a doctorate in pure or applied mathematics, worked at least ten years in the mathematics field, and currently engages in mathematics related research. In addition, the expert mathematician demonstrates an interest in mathematics education, possibly through teaching and/or research.
>
> *Expert Secondary Teacher*. An expert secondary teacher has a degree in mathematics or mathematics education and has taught mathematics on the secondary level for at least 10 years. The expert has demonstrated success on the secondary level, possibly through teaching, mentoring teachers, serving in leadership roles, and/or continued engagement with professional learning. His or her excellence in the field has be acknowledged through state and local recognitions and awards.

Base on the selection criteria, six participants[1] were purposefully selected from a statewide network of mathematics education leaders from public school districts and universities to ensure that best practices were represented. The network of mathematics education leaders, from a state located in the southeastern United States, all actively engage in activities designed to improve mathematics education. Below, is a brief description of each participant.

Cameron

Cameron holds both a doctoral degree and a master's degree in mathematical sciences and bachelor's degree in computer science. He has over 30 years of teaching experience at the university level, where he has taught both mathematical content and teacher preparation courses. He currently teaches at the university level and frequently serves as a mathematics education leader throughout his state as he constantly works to improve mathematics education. Cameron, although formally trained as a mathematician, at the time of our interview self-categorized as a mathematics teacher educator. He stated,

> if we were to take a snapshot at this instant, I am a mathematics teacher educator, but it has been a transition. I started in the math field, so I started as a mathematician, and now I have transitioned over to mathematics teacher educator.

Cameron's views of MKT center on knowing the content well first. He believes that after knowing the content well, one can become a good facilitator of learning.

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[1] I used pseudonyms for all participants in the study.

Christian

Christian holds all three degrees (doctoral, master's and bachelor's) in mathematics education. With over 16 years of teaching experience, Christian started as a high school mathematics teacher and has spent the last 8 years at the university level where she provides instruction and supervision to aspiring mathematics teachers, while providing leadership throughout the state. To this self-categorized mathematics teacher educator, MKT means,

> the egg and all the components that go into that. I do not think it is just knowing the mathematics. I think it is knowing about students and how students think about the mathematics. I think it is knowing about the curriculum, and the tools that you are using to interact with students. I think it is the teaching techniques that you use, the pedagogy that you use, to engage students with mathematics. And I think it is a strong conception of, whatever topic I am teaching, understanding what comes before and what comes after.

Jamie

Jamie holds a master's degree in mathematics education, a bachelor's degree in mathematics education, and a bachelor's degree in applied mathematics. Jamie, who has been recognized numerous times for her work in mathematics educations, spent 32 years as a high teacher, teaching across the secondary curriculum, before moving the university level, where she has worked for the past 9 years. Jamie considers herself both a mathematician and mathematics teacher. She describes MKT as

> just the content knowledge that a teacher needs in order to effectively teach mathematics but that's way more than content knowledge that a teacher needs … as a teacher, I have to know a variety of ways of understanding and learning and processing that content.

Jeremy

Jeremy holds a bachelor's degree in mathematics with certificates in teaching and AIG (academically intelligent or gifted education). Jeremy, currently a high school teacher, has taught grades 7th through 12th and spent the past 17 years in the teaching profession. Prior to teaching, Jeremy worked in STEM related areas which he credits with informing his teaching. This National Board Certified teacher has been recognized at the local and state levels for his many contributions to the classroom. Jeremy primarily self-categorizes as a mathematics teacher but acknowledges his experiences in a mathematics teacher educator role. When asked what MKT means to him, Jeremy stated,

> I think you have to not just know what you are teaching to the kids that year, but you have to know where it is going. And you have to know why what you are teaching is relevant to something that they are going to have to learn later on. And you need to know those things that they are going to be doing so that you can have a good idea of how to break down what you are doing so that it makes sense.

Kurin

Kurin holds both a doctoral and master's degree in mathematical sciences and a bachelor's degree, with teaching certification, in mathematics. With over 24 years of teaching experience, Kurin taught grades K-8 for two years prior to moving to the university level, where she primarily teaches mathematics courses. She has over 18 years of experience designing, facilitating, and leading professional development activities for K-12 teachers. Kurin states that her self-categorization of mathematics teacher, mathematician, and then mathematics teacher educator is based on what she "spends the most time doing." When asked what MKT means to her, Kurin stated

> To me, it is a specialized way of thinking about mathematics because you are thinking about the ways in which you are going to engage others in learning. It is a combination of understanding of mathematical content and ideas and connections across mathematics. Then you are thinking about helping students make sense of it. So, what are the tools you are going to use to communicate those ideas? How are the tools going to engage students and develop their understanding? What are the understandings that they, the students, already have that you can connect to as you share new ideas? It is something in between just pedagogy and math.

Rena

Rena holds a doctoral degree in curriculum and instruction, a post-master's certificate in administration and supervision, a master's degree in mathematics education, and a bachelor's in mathematics and library education. Prior to her recent retirement, Rena had served the mathematics education community for 49 years. Most of her experiences (38 years) were spent at the university level where she taught mathematics methods courses, supervised preservice teachers, and designed, facilitated, and led professional development opportunities for teachers in grades K-12. This self-categorized mathematics teacher educator describes MKT as

> a really strong foundational understanding of mathematics, but also it means understanding the research on learning mathematics. Understanding the research on learning mathematics means being able to identify what will be common student responses, common student errors, common student misconceptions and a variety of different strategies. You understand and are expecting a variety of strategies and solutions from students.

In summary, the participants, deemed *experts* in this study in recognition of their accomplishments and experiences in the field, brought over 170 years of experience in secondary mathematics that spans classroom teaching, professional development and facilitation, preservice teacher supervision, and mathematics education research to the study. Along with their practice, as a combined group they have received more than 50 recognitions for their work in mathematics education. By their own self-categorization, they embody the ways of knowing mathematics; knowing as a mathematician, as a mathematics teacher, and as a mathematics teacher educator.

Data Collection and Analysis

Using student written work as a source to generate data, data were collected through a series of interviews and an artifact review. My stages of data collection and analysis where guided by the appreciative phrases of dreaming and designing. In these phases, creativity is key to designing in ways that reflects the experts' best practices and views (Shuayb et al., 2009). I designed the data collection and analysis to ensure the authentic reflection of the experts' knowledge for teaching. This was accomplished through the sharing of ideas and findings with the experts, along with the gathering and incorporating of expert feedback, throughout my states of data collection and analysis.

Student Written Work

While there are many components of teaching, one important part is engaging with student written work. Student written work, which can be considered "performances of understanding" (McDonald, 2002, p. 121) that possibly demonstrates students' comprehension of content and the impact of instruction. For this study, I use student written work as a means of generating data relevant to teacher knowledge. Empirical studies have shown that studying student work can impact teaching and teacher learning (Crespo, 2000; Goldsmith et al., 2014; Kazemi & Franke, 2004; Little et al., 2003).

The strategically selected student written work (see Appendix B), embedded in the content of quadratic functions, presents varying levels of understanding and skills. Student written work, collected from students ranging in grade 8 through year 1 undergraduate, were purposefully selected to represent quadratic functions as a mathematical concept, not to represent understanding of quadratic functions at a certain point in time. Although presenting anonymous student work may appear to pose limitations as the work is not that of the expert's own student, Jessup (2018) noted that "a lack of substantial evidence was found to distinguish the quality of teacher's overall noticing" (p. 88) when working with the written work of their own students and the written work of unknown students. Teachers basically demonstrated the same quality of professional noticing, whether they knew the student or not.

The student written work presented to experts was composed of two to three anonymous student samples from five mathematics tasks. The five mathematics tasks covered various quadratic concepts while the student written work samples displayed varying levels of student understanding and skills. Below, I describe the task and supporting student work samples. The tasks and student work are included in Appendix B.

Table Task

The purpose of the table task was to identify the function type based on tabular data. Students were specifically directed to determine which of the four tables represented a quadratic function and to explain how they knew. Three student samples were provided with this task. Student responses focused on ideas of symmetry, rates of change, and graphical representations.

Review Task

The purpose of the review task was to identify key features of quadratic functions (e.g., vertex, axis of symmetry, minimum or maximum value). Students were specifically asked to both define and determine the function's features. Two student samples were provided with this task. The student samples represented the diverse use of mathematical language or terminology and varying algebraic calculations.

Stretching Task

The purpose of the stretching task was to demonstrate an understanding of the impact of coefficients on quadratic functions. Students, given three algebraic representations of quadratic functions, were asked to determine the widest parabola from group of three. Two student samples were provided with this task. Student responses demonstrated ideas of range and vertical transformations.

Solutions Task

The purpose of the solutions task was to demonstrate an understanding of the relationship between algebraic and graphical solutions. Students were asked to solve the same quadratic problem through both graphical and algebraic means. Two student samples were provided with this task. Student responses demonstrated ideas of substitution, factoring, and intersection.

Lawnmower Task

The purpose of the lawnmower task was to demonstrate representational fluency (contextual, tabular, algebraic, etc.) in describing quadratic relationships. Specifically, students were given a contextual problem and asked to determine the solution and generate an algebraic representation. Three student samples were provided with this task. Student responses demonstrated ideas of repeated addition and multiple representations (tabular, graphical, and algebraic).

Interviews

Using the student written work samples, a series of semi-structured tasked based interviews (Goldin, 1997) were designed to explore the experts' MKT. While each interview had a purpose, the flexible design of the interview allowed for fluid conversations where new ideas were exposed and explored. Interviews, held in various formats (e.g., in-person, virtually), typically lasted one hour. The first three interviews were conducted individually with each expert, while the final interview took on a focus group format with all experts meeting together.

The purpose of Interview #1 was to get to know the expert, their background, experiences, and perception of teacher knowledge. As we discussed teacher knowledge, I let the experts know that the following interviews would be focused on understanding their MKT. In Interview #2, the expert engaged in a think-aloud session as he or she reviewed the student sample package. As this interview captured the expert's initial thoughts and reactions to seeing the student samples for the very first time, the expert was asked to keep the samples and look back over them as preparation for Interview #3. Prior to Interview #3, each expert was provided a written recap of their Interview #2, highlighting what I considered aspects of teacher knowledge. The recap included the student samples and a summary of the expert's responses to each sample. By providing the written recap, Interview #3 served as a member-checking activity and allowed for a time of reflection and elaboration, as the expert and I discussed the recap and revisited each student sample. For example, during Interview #2, Cameron stated "make sure that the student understands what it means for a graph to be symmetric with respect to the y-axis." When we revisited this statement during Interview #3, Cameron clarified this idea by saying he should have said, "symmetry with respect to the y-axis versus symmetry with respect to another vertical axis." Further, as we continued our review, the time of reflection in Interview #3 was evident as Cameron discussed an idea that was not revealed in Interview #2. In his reflection Cameron stated, "I just didn't pick up on that the first time."

Additionally, during both interview #2 and interview #3, experts occasionally wrote or jotted down notes and math examples to demonstrate their understanding as we navigated through the samples. These writings, along with a written brain dump regarding quadratic functions from each expert, were collected and included as artifacts from Interview #2 and Interview #3.

Focus Group Interview

A final focus group (Morgan, 1996) was held with all experts to discuss my preliminary findings. Intentionally gathering experts after individually collecting data through interviews #1 through #3 allowed for group interactions where one's train of thought encouraged that of another. Specifically, during this time we critically reviewed my proposed organization of teacher knowledge and conception of secondary MKT for quadratic functions. Experts provided valuable input and feedback that was incorporated into the study results.

Data Analysis

Transcripts from Interview #2 and Interview #3, along with written notes provided by the experts, were the primary data sources for identifying and categorizing the knowledge used by experts when engaging with student written work. The data were analyzed in four phases (see Table 3.1). The initial coding, Phase 1, that followed Interview #2, utilized a hypothesis coding approach (Miles et al., 2014) to broadly categorize the data into two predetermined categories, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK and PCK are the two overarching domains in the Mathematical Knowledge for Teaching Framework, and I anticipated that the data would fall into one of the two categories. An expert's idea unit, a statement or statements that convey a single thought (J. K. Jacobs et al., 1997), was categorized as SMK if it reflected quadratic content knowledge; categorized as PCK if it demonstrated the knowledge that "links content, students, and pedagogy" (Ball, 2003, p. 17). For example, when reviewing samples from the Review Task, Rena stated,

> Since the leading coefficient is negative, I know the parabola will open downward. However, I would ask the student a clarifying question like, does every parabola have a low point and a high point? Students have a tendency of viewing parabolas with a minimum, opening upward.

In Rena's remarks, stating that the parabola will open upward based on the sign of the leading coefficient was coded as SMK. The last two statements, Rena expressing her desire to ask a clarifying question, and suggesting that students tend to view parabolas with a minimum, were both categorized as PCK.

Table 3.1

Data Analysis Summary

Phase	Data Source	Description of Analysis
1	Interview #2	— Data categorized base on predetermined codes, SCK and PCK
2	Interview #2	— Thematic coding to investigate emergent themes in data categorized as SCK or PCK — Codebook developed
3	Interview #2 & #3	— Data analyzed using codebook — Reliability established
4	Interview #2 & #3	— Idea unit trace

Phase 1 was followed by a second round of coding (Phase 2) to identify the themes within the SMK and PCK categories. Here, the Mathematical Knowledge for Teaching Framework was a vital tool as I looked for ways to detail demonstrated aspects of SMK and PCK. Using the six subdomains of the framework as a lens for a more in-depth content analysis, in Phase 2, I specifically looked for ways to describe and/or to subcategorize data as aspects of CCK, SCK, HCK, KCS, KCC, or KCT. As expected, considering the ways in which the Mathematical Knowledge for Teaching Framework subdomains are defined, emergent themes were frequently linked to multiple subdomains. For example, Jeremey's statement that "transformations of quadratic functions are typically covered in 2nd or higher-level algebra courses," demonstrated his knowledge of the sequencing of content in his curriculum, which was deemed an aspect of KCC. However, considering that SCK is defined as "knowledge not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400), knowing the sequence of content in the curriculum is useful for teaching, but more than likely not relevant in other professions. Therefore, Jeremy's idea unit was also linked to SCK.

Through iterative pattern coding in Phase 2, I was able to identify six emergent themes that were representative of the knowledge demonstrated by the experts in Interview #2. These themes provided greater detail to the data that had been initially given SMK or PCK codes. From these themes, a codebook (Appendix D) was generated.

In Phase 3, data from Interview #3, which provided elaborations and clarification to Interview #2 data, were merged with data from Interview #2 to form well-detailed idea units. This merged data set was then analyzed using the codebook. To establish reliability, a second coder, an established researcher in the mathematics education field, used the codebook to examine a sample of randomly selected data. An agreement rate of 95% was reached on unit ideas and a rate of 92% agreement was achieved on coding. Three weeks after my Phase 3 coding, I repeated my coding on a sample of the data to test my own coding consistency reliability (99%).

The final phase of analysis, Phase 4, was completed on the merged data set to explore the existence of connections and interactions amongst the themes. During this phase, relationships were characterized and categorized. One way this was done was by tracing an expert's initial idea unit throughout the entire interviews. For instance, when

Christian viewed the first student sample, she stated "I know what the shape of a quadratic looks like. It is a parabola, which has a vertical axis of symmetry." The idea unit here is based upon the graph or shape of a quadratic function and the attribute of symmetry. Therefore, I traced instances where ideas of shapes, graphing, and symmetry reappeared. A partial sample of Christian's trace is displayed in Table 3.2.

Table 3.2

Idea Unit Trace

Line #	Sample	Response
3	Student A	I know what the **shape** of a quadratic looks like. It is a parabola, which has a vertical axis of **symmetry**.
12	Student A	If students don't know what it looks like, they will not **graph**
17	Student A	This student is looking for **symmetry**... This student is showing **symmetry** in their table and **graph**
28	Student A	Since they did not see that **symmetry** in any of the tables, they are going to answer that there are no quadratic functions
40	Student B	It is important to know the **shape** of a linear function
53	Student B	I'd probably ask them to **graph** to confirm
59	Student C	This student did what I would have initially done, which is **graph**
66	Student C	I will encourage the student to get more points to **graph** so they can see the **symmetry** in the tables and then **graph**
93	Student D	It looks like they have a line indicating the axis of **symmetry**
235	Student H	I would use XXXX software to **graph** more efficiently

CHAPTER IV

EXPLORING MATHEMATICAL KNOWLEDGE FOR TEACHING QUADRATIC FUNCTIONS WHEN ANALYZING STUDENT WRITTEN WORK

Abstract: In this study, I explore the nature of mathematical knowledge for teaching quadratic functions. Using Ball and colleagues' (2008) Mathematical Knowledge for Teaching Framework as a guide and student written work as a source, I conducted a series of tasked-based interviews with six experienced mathematics educators. Findings indicate that the nature of mathematical knowledge for teaching quadratic functions can be characterized as six interrelated entities: content knowledge, connections, interpretations, anticipations, instructional moves, and resources. This conception of the mathematical knowledge for teaching quadratic functions has implications for secondary mathematics research.

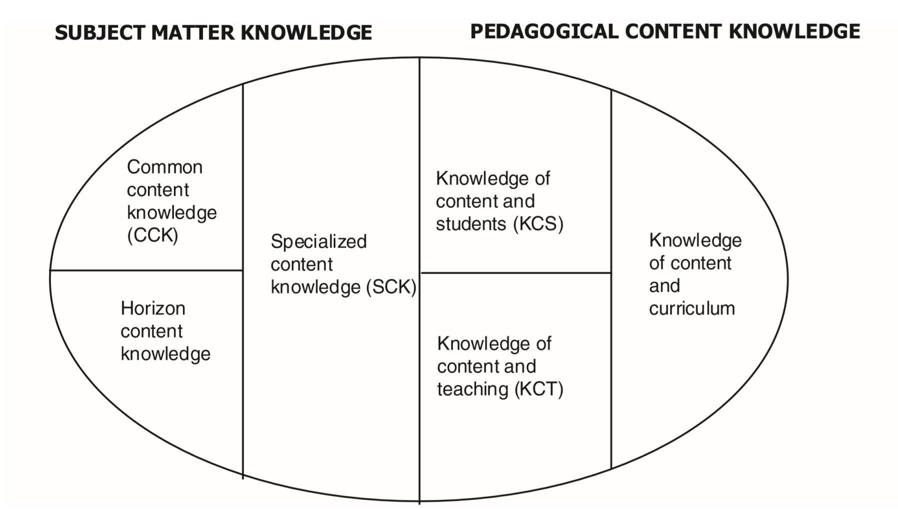
Mathematical Knowledge for Teaching

Teaching mathematics is hard. It requires understanding and knowing mathematics in ways that are meaningful to learners. For years, researchers have worked to understand the knowledge of mathematics teachers. Gaining an understanding of this knowledge, often referred to as mathematical knowledge for teaching or MKT, can lead to improved teacher education and professional development innovations.

The Mathematical Knowledge for Teaching Framework (Figure 4.1), representative of Deborah Ball and colleagues' (2008) practice-based theory of MKT at the elementary level has advanced our understanding of the knowledge needed to carry out the work of teaching mathematics. Influenced by Shulman's (1986) introduction of pedagogical content knowledge, the practice-based theory conceptualizes teacher knowledge into six knowledge subdomains: common content knowledge, specialized content knowledge, horizon content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. This theoretical perspective has been instrumental in detailing ways of knowing elementary grades mathematics useful for teaching (Hill, 2010; Hill, Blunk, et al., 2008). Studies employing this framework have demonstrated a strong association between MKT and student achievement (Hill et al., 2005) and MKT and the quality of instruction (Hill, Ball, et al., 2008).

Figure 4.1

Mathematical Knowledge for Teaching Framework (Adapted from Ball et al., 2008)



The impact of the Mathematical Knowledge for Teaching Framework across the mathematics education community is evident as several secondary mathematics scholars have utilized the framework in the exploration of secondary mathematics (e.g., Campbell & Lee, 2017; Hatisaru & Erbas, 2017; Khakasa & Berger, 2016; Steele, 2013; Steele & Rogers, 2012; Taşdan & Koyunkaya, 2017). Using the Mathematical Knowledge for Teaching Framework has allowed secondary researchers the ability to explore the intersection of theory and practice by gauging proficiency in teaching secondary mathematics, examining the development of secondary mathematical knowledge, and identifying core elements of teaching specific secondary content. Even though progress has been made in understanding the knowledge of secondary mathematics teachers, there is still much to learn. Given that improving student learning of mathematics is dependent on developing teachers' knowledge (Ball & Bass, 2002), the continued exploration of secondary MKT is vital.

I approach this study, first, with the purpose of contributing to the field of secondary mathematics teacher education. The research on secondary mathematics teacher knowledge, though emerging, is sparse. Second, I also consider the narrative, “"little attention has been paid to the ways in which MKT theory is or is not applicable to teachers at secondary and post-secondary levels” (Speer et al., 2015, p. 106). Though scholars have questioned the applicability of the Mathematical Knowledge for Teaching Framework at the secondary level (Howell, 2012; Speer et al., 2015), I feel that the framework can be an influential guide. Hence, while seeking to identify and describe teacher knowledge in the context of quadratic functions, I pay attention to the ways in which MKT theory can be applicable at the secondary level by using the Mathematical Knowledge for Teaching Framework as a tool to explore secondary MKT.

Learning from scholars that have used Ball and colleagues' Mathematical Knowledge for Teaching Framework to study secondary mathematics teacher knowledge, I investigate the nature of knowledge for teaching quadratic functions. In the section that follows, I use the Mathematical Knowledge for Teaching Framework and existing literature on learning quadratic functions to provide a hypothetical literature-based perspective of quadratic function MKT. I proceed in this way because it is common practice among scholars (Hatisaru & Erbas, 2017; Herbst & Kosko, 2014; Steele, 2013; Steele et al., 2013; Steele & Rogers, 2012; Taşdan & Koyunkaya, 2017) to elaborate or redefine the Mathematical Knowledge for Teaching Framework's subdomains to be topic-specific based on what is known about teaching and learning the topic. This provides a literature-based perspective of the Mathematical Knowledge for Teaching Framework in the context of quadratic functions.

A Literary Perspective of Quadratic Function MKT

When Steele and Rogers (2012) created a topic-specific framework for investigating secondary teacher knowledge, they demonstrated how focusing on a single topic would allow researchers to gain an in-depth view of secondary teachers' MKT. They, like others (see (Hatisaru & Erbas, 2017; Herbst & Kosko, 2014; Steele, 2013; Steele et al., 2013; Taşdan & Koyunkaya, 2017) use the Mathematical Knowledge for Teaching Framework to examine a specific secondary mathematics topic and provide a window into the ways that teacher knowledge influences and supports the work of teaching. Therefore, to gain an in-dept view into secondary teacher knowledge, I follow the lead of theses scholars and use the Mathematical Knowledge for Teaching Framework to examine the MKT for quadratic functions. Quadratic functions, "a very fundamental and basic function in the high school curriculum" (Even, 1990, p. 553), is frequently viewed as being among the most conceptually challenging areas in the secondary mathematics curriculum. Instruction on functions can begin as early as elementary school (Blanton & Kaput, 2011) and continue to grow in complexity throughout middle school, high school, and college, where the understanding of quadratic functions is vital.

Typically, students' first encounter with nonconstant rates of change, or rates of change that involve neither repeated addition nor repeated multiplication, is through quadratic functions. Quadratic functions are often used to model real life events and mode numerous physical phenomena, such as the speed of a falling object. Quadratic functions can be written in the form $f(x) = ax^2 + bx + c$, for some real numbers a, b, and c, where a is not zero. The essential understandings for quadratic functions, according to the National Council of Teachers of Mathematics (NCTM), reveal that

> quadratic functions are characterized by a linear rate of change, so the rate of change of the rate of change (the second derivative), of a quadratic function is constant. Reasoning about the vertex form of a quadratic allows deducing that a quadratic has a maximum or minimum value and that if the zeros of the quadratic are real, they are symmetric about the x-coordinate of the maximum or minimum point. (Cooney et al., 2010, p. 9)

These essential understandings of quadratic functions could be considered *common content knowledge*, CCK, from a Mathematical Knowledge for Teaching Framework view. According to Ball, CCK, a subdomain of subject matter knowledge, is the knowledge and skills of mathematics that are not restricted to teaching (Ball et al., 2008). Other professionals possess and use CCK.

The essential understandings of quadratic functions detail knowledge that is not necessarily unique to teaching. The features of quadratic functions are aspects of content knowledge that are acquired in secondary and college mathematics courses that serve diverse learners, not just mathematics teachers or those that intend to become mathematics teachers. Since the use of the term *common* in CCK is questionable at the secondary level (see Speer et al., 2015), for this literary perspective, I consider CCK a way of viewing the knowledge that is common or foundational amongst those that have had the opportunity to engage in the secondary mathematics concepts; from high school students to college graduates.

Another subdomain of subject matter knowledge is *specialized content knowledge* or SCK. SCK is considered knowledge that is distinctive or unique to teaching. Accordingly,

> SCK is mathematical knowledge not typically needed for purposes other than teaching. In looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general…teachers have to do a kind of mathematical work that others do not do. (Ball et al., 2008, p. 400)

Assumably, this construct applied at the secondary level would resemble that of elementary mathematics teachers. Considering the work of teaching quadratic functions and the documented difficulties of quadratic function learners (e.g., Childers & Vidakovic, 2014; Ellis & Grinstead, 2008; Lobato et al., 2012), those that teach quadratics are challenged with identifying both errors and understandings demonstrated in student work, a demonstration of SCK. However, with SCK being broadly defined as the mathematical knowledge not typically needed for purposes other than teaching, I expect that secondary definitions of SCK will extend well beyond analyzing student work. For example, Taşdan and Koyunkaya (2017) defined secondary teacher's SCK as one's ability to examine and evaluate the usefulness of varying function definitions, purposefully selecting and moving between function representations, and possessing varying examples that can convey the importance of functions. Their construction of SCK evolved from the work of Nyikahadzoyi (2015) and Steele et al. (2013) who viewed SCK as understanding multiple definitions and/or representations in ways that are useful in teaching.

Building from the work of these secondary scholars, SCK for quadratic functions could start with a similar definition: a teacher should be able to examine and evaluate the usefulness of varying quadratic function definitions, purposefully select and move between quadratic function representations (i.e., algebraic, tabular, graphical, contextual), and possess varying quadratic function examples that can convey the importance of functions (e.g., ability to model real-world events). This, of course, cannot represent a complete conception of SCK as the work of teaching is vast and encompasses more than examining, evaluating, or selecting appropriate definitions and representations. Further, as the name SCK implies, this knowledge is *specialized* to those teaching quadratic functions. Given the lack of clarity in the literature and definitions that are not grade-band independent, I feel, a representative conception of SCK demands empirical efforts.

Rounding out the subject matter subdomains is *horizon content knowledge* (HCK). HCK is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Considering quadratic functions, an example of HCK is knowing that the ability to perform arithmetic operations and write equivalent expressions, topics typically covered in prealgebra courses, are vital to constructing equivalent quadratic forms (e.g., from standard form to vertex form). Similarly, the visual representation of a quadratic can help in conceptually understanding extrema discussed in high-level mathematics.

The construct of pedagogical content knowledge, like subject matter knowledge is elaborated into three subdomains: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. *Knowledge of content and students* (KCS) is a combination of teachers knowing their students and knowing about mathematics (Ball et al., 2008). Fortunately, research exists that can inform our knowing about students and quadratic functions. For example, Zaslavsky (1997) analyzed data from over 800 students enrolled in 10^{th} and 11^{th} grade in Israel and identified five obstacles encountered by students when learning: (1) the interpretation of graphical information, (2) the relation between a quadratic function and a quadratic equation, (3) the analogy between a quadratic function and a linear function, (4) the seeming change in form of a quadratic function whose parameter (either b or c in $ax^2 + bx + c$) is zero, and (5) the over-emphasis on only one coordinate of special points (e.g., the x-coordinate of the vertex). Similarly, Celik and Guzel (2017) and Eraslan et al. (2007), further this line of research by presenting the results of clinical interviews which reveal student misconceptions and errors when working with quadratic functions. For example, both scholars found that students face difficulties when connecting quadratic functions across representations.

Knowing how to design and implement instruction to facilitate understanding of quadratic functions and address the thinking of students, requires *knowledge of content and teaching* (KCT). For example, KCT could be demonstrated through the sequencing of quadratic instruction, selection of appropriate examples, or knowing which quadratic representations to use when. Based on research findings, scholars have made some recommendations that could inform quadratic instruction, hence building KCT. For example, after collecting and analyzing the data of 60 students enrolled in an undergraduate mathematics course focused on modeling, scholars concluded that it is vital that quadratic instruction include the exploration of student's personal meanings for concepts of quadratic functions, such as the vertex (Childers & Vidakovic, 2014). In their study, students' personal meanings of vertex that represented a misconception experienced less success when presented with real-world quadratic tasks. Based on Childers and Vidakovic's findings, teachers should include addressing students' personal meanings of concepts. Another example can be seen in the work of Lobato and colleagues. Lobato and colleagues (2012), through exploration and analysis of student quadratic reasoning, identified five conceptual learning goals that should be the target of quadratic instruction. These learning goals, which focused on students' conception and comprehension of quadratic functions could be used to inform instructional design and activities. Given the nonexistence of quadratic KCT research, KCT could be tentatively conceived from studies addressing student quadratic learning.

The last subdomain that comprises pedagogical content knowledge is *knowledge of content and curriculum* (KCC). Though not explicitly defined in the Mathematical Knowledge for Teaching Framework, KCC can be traced back to Shulman's (1986) "curricula knowledge." Curricula knowledge is described as knowing the value and worth of the programs, materials, and instructional resources defined for the teaching of a particular subject. Using this as a definition for KCC, a teacher's selection of resources, such as lessons from reform-oriented curricula, graphing calculators, or web-based learning tools, to increase student quadratic function learning and understanding could be a demonstration of KCC.

In summary, given the absence of studies focusing specifically on the knowledge used or needed to teach quadratic functions, existing literature has allowed me to speculate how all six subdomains in the Mathematical Knowledge for Teaching Framework could be represented for teaching quadratic functions. Essentially, through the work of secondary scholars using the Mathematical Knowledge for Teaching Framework and studies focusing on learning quadratics, I have formed ideas of what the knowledge needed to teach quadratic functions could be. While I consider these ideas insightful, I believe that true understanding of teacher knowledge comes directly from the holders of the knowledge, the teachers. Since MKT is the knowledge that is needed to carry out the work of teaching, I look to those who do the work.

In the next section, I describe my research approach and provide my research question as I turn to the teachers to learn about the mathematical knowledge needed for teaching quadratic functions.

Research Approach and Question

While the Mathematical Knowledge for Teaching Framework provides an entry point for deciphering what MKT for quadratic functions may be through literature and existing research, a more in-depth look is warranted to increase understanding of teacher knowledge. To accomplish this, using elements of appreciative inquiry (Cooperrider & Srivastva, 1987), I take an exploratory multi-case approach to provide empirical evidence of the teacher knowledge used to teach a secondary mathematics concept, quadratic functions. Purposefully seeking to identify and describe the MKT for quadratic functions, I answer the question: *What is the nature of the knowledge expert mathematics educators use when engaging with student written work on quadratic function tasks?*

Although deficits in teacher knowledge are of concern and have been documented (e.g., Ball 1990), the purpose of this study is to gain a greater understanding of secondary MKT and the productive ways of knowing mathematics. Here is where elements of appreciative inquiry inform this exploratory multi-case study. Appreciative inquiry, an ideology rooted in organizational development, utilizes a four-phase approach (discovering, dreaming, designing and delivering) to focus on positive experiences and creatively plan for the future (Cooperrider & Srivastva, 1987). An “appreciative stance broadens our capacity for seeing the good” (Godwin, 2016, p. 27) and helps us “lift up strengths at all levels” (Godwin, 2016, p. 27). Since exploratory case studies are ideal for investigating phenomena on which little to no research exists (Streb, 2010), guiding this exploration with ideas from the appreciative inquiry phases ensures that the focus remains on positive aspects of secondary MKT. This too, helps me with my positionality throughout the study and makes me keenly aware of my potential researcher bias. Having more than 20+ years of mathematics teaching experience, I have strong views of productive and nonproductive ways of knowing mathematics.

Hence, embracing the aspects of appreciative inquiry, I first focused on the discovering phase which encourages identifying the “best and most positive experiences” (Shuayb et al., 2009, p. 3). I accomplished this through the careful selection of experts as study participants. Then, turning to phases of appreciative dreaming and designing, where creativity is key to designing in ways that reflects the participants’ best practices and views (Shuayb et al., 2009), I designed to ensure the authentic reflection of my study participants’ knowledge for teaching. This was accomplished through the sharing of ideas and findings with participants, along with the gathering and incorporating of participant feedback, throughout my phases of data collection and analysis.

Experts as Study Participants

Matsuura and colleagues (2013) suggest that the ways that teachers *know* mathematics can be categorized into four overlapping categories: knowing mathematics as a scholar, knowing mathematics as an educator, knowing mathematics as a mathematician, and knowing mathematics as a teacher. These categories can be easily related to the overarching knowledge domains in the Mathematical Knowledge for Teaching Framework; subject matter knowledge (knowing as a scholar and mathematician) and pedagogical content knowledge (knowing mathematics as an educator and teacher). Considering this frame of thought, along with evidence that knowing as a teacher, or MKT, increases with experience (see Herbst & Kosko, 2014; Leinhardt & Smith, 1985; Steele & Rogers, 2012; Taşdan & Koyunkaya, 2017) six participants were purposefully selected from a statewide network of mathematics education leaders from public school districts and universities to ensure that best practices were represented. The stratified purposeful sampling, the deliberate selection of participants based on current or primary occupation and characteristics highly relevant to the phenomenon under study (Wiersma & Jurs, 2005), was used in order to maximize the learning potential (Stake, 1995). These mathematics education leaders, from a state located in the southeastern United States, actively engage in activities designed to improve mathematics education.

The participants (see Table 4.1), deemed *experts* in this study in recognition of their accomplishments in the field, brought over 170 years of experience in secondary mathematics that spans classroom teaching, professional development and facilitation, preservice teacher supervision, and mathematics education research to the study. Along with their practice, as a combined group they have received more than 50 recognitions for their work in mathematics education. By their own self-categorization, they embody the ways of knowing mathematics, knowing as a mathematician, as a mathematics teacher, and as a mathematics teacher educator.

Table 4.1

Expert Profile

Expert	Self-Categorization*	Years Experience	Degrees Bachelor	Master	PhD	Awards/ Recognitions
Jeremy	MT	16	x			3
Jamie	MT, M	40	x	x		10
Cameron	M, MTE	29	x	x	x	5
Kurin	MT, M, MTE	23	x	x	x	12
Christian	MTE, M	15	x	x	x	6
Rena	MTE	49	x	x	x	16

** M – Mathematician MT – Mathematics Teacher MTE – Mathematics Teacher Educator*

Data Collection and Analysis

Using student written work as a source to generate data, data were collected through a series of interviews and an artifact review.

Student Quadratic Function Work

While there are many components of teaching, one important part is planning for and engaging with student written work. Student written work, which can be considered "performances of understanding" (McDonald, 2002, p. 121), possibly demonstrates students' comprehension of content and the impact of instruction. In addition, a teacher's instructional moves can be informed by the teacher's analysis of the students' work (Ball & Bass, 2000). Given that reviewing student written work is an authentic practice of teaching and knowing that student written work can be used as a tool for teacher learning (Crespo, 2000; Goldsmith et al., 2014; Kazemi & Franke, 2004; Little et al., 2003), I use student written work as a tool to explore teacher knowledge. Therefore, as a source of data generation, anonymous student written work (see Appendix B) was presented to the experts.

The written work presented to experts was composed of two to three anonymous student samples from five mathematics tasks. The five mathematics tasks covered various quadratic concepts (see Table 4.2) while the student written work samples displayed varying levels of student understanding and skills. The anonymous student samples, gathered from students ranging in grade 8 through year 1 undergraduate, were purposefully selected to represent quadratic functions as a mathematical concept. The samples were not selected to represent the understanding that should be possessed by a student at a certain grade level (i.e., a 10th grader should be able to complete the square in a quadratic function to show zeros, extreme values, and symmetry of the graph). The careful selection of student work samples from various mathematics tasks combined to highlight the foundational elements and characteristics of quadratic functions.

Table 4.2

Student Written Work Structure

Math Task	**Purpose**	**# Student Samples**
Table Task	Identify function type	3
Review Task	Identify key features of quadratic functions (e.g., vertex, axis of symmetry)	2
Stretching Task	Demonstrate an understanding of the impact of coefficients on quadratic functions	2
Solutions Task	Demonstrate a conceptual understanding of the relationship between algebraic and graphical solutions of quadratics	2
Lawnmower Task	Demonstrate representational fluency (contextual, tabular, algebraic, etc.) in describing quadratic relationships	3

Interviews

Using the student written work samples, a series of semi-structured tasked based interviews (Goldin, 1997) were designed to explore the experts' MKT. While each interview had a purpose, the flexible design of the interview allowed for fluid conversations where new ideas were exposed and explored. Interviews, held in various formats (e.g., in-person, virtually), typically lasted one hour. The first three interviews were conducted individually with each expert, while the final interview took on a focus group format with all experts meeting together.

The purpose of Interview #1 was to get to know the expert, their background, experiences, and perception of teacher knowledge. As we discussed teacher knowledge, I let the experts know that the following interviews would be focused on understanding their MKT. In Interview #2, the expert engaged in a think-aloud session as he or she reviewed the student sample package. As this interview captured the expert's initial thoughts and reactions to seeing the student samples for the very first time, the expert was asked to keep the samples and look back over them as preparation for Interview #3. Prior to Interview #3, each expert was provided a written recap of their Interview #2, highlighting what I considered aspects of teacher knowledge. The recap included the student samples and a summary of the expert's responses to each sample. By providing the written recap, Interview #3 served as a member-checking activity and allowed for a time of reflection and elaboration, as the expert and I discussed the recap and revisited each student sample. For example, during Interview #2, Cameron stated "make sure that the student understands what it means for a graph to be symmetric with respect to the y-axis." When we revisited this statement during Interview #3, Cameron clarified this idea by saying he should have said, "symmetry with respect to the y-axis versus symmetry with respect to another vertical axis." Further, as we continued our review, the time of reflection in Interview #3 was evident as Cameron discussed an idea that was not revealed in Interview #2. In his reflection Cameron stated, "I just didn't pick up on that the first time."

Additionally, during both interview #2 and interview #3, experts occasionally wrote or jotted down notes and math examples to demonstrate their understanding as we navigated through the samples. These writings, along with a written brain dump regarding quadratic functions from each expert, were collected and included as artifacts from Interview #2 and Interview #3.

Focus Group Interview

A final focus group (Morgan, 1996) was held with all experts to discuss my preliminary findings. Intentionally gathering experts after individually collecting data through interviews #1 through #3 allowed for group interactions where one's train of thought encouraged that of another. Specifically, during this time we critically reviewed my proposed organization of teacher knowledge and conception of secondary MKT for quadratic functions. Experts provided valuable input and feedback that was incorporated into the study results.

Data Analysis

Transcripts from Interview #2 and Interview #3, along with written notes provided by the experts, were the primary data sources for identifying and categorizing the knowledge used by experts when engaging with student written work. The data were analyzed in four phases. The initial coding, Phase 1, that followed Interview #2, utilized a hypothesis coding approach (Miles et al., 2014) to broadly categorize the data into two predetermined categories, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK and PCK are the two overarching domains in the Mathematical Knowledge for Teaching Framework and I anticipated that the data would fall into one of the two categories. An expert's idea unit, a statement or statements that convey a single thought(J. K. Jacobs et al., 1997), was categorized as SMK if it reflected quadratic content knowledge; categorized as PCK if it demonstrated the knowledge that "links content, students, and pedagogy" (Ball, 2003, p. 17). For example, when reviewing samples from the Review Task, Rena stated,

> Since the leading coefficient is negative, I know the parabola will open downward. However, I would ask the student a clarifying question like, does every parabola have a low point and a high point? Students have a tendency of viewing parabolas with a minimum, opening upward.

In Rena's remarks, stating that the parabola will open upward based on the sign of the leading coefficient was coded as SMK. The last two statements, Rena expressing her desire to ask a clarifying question, and suggesting that students tend to view parabolas with a minimum, were both categorized as PCK.

Phase 1 was followed by a second round of coding (Phase 2) to identify the themes within the SMK and PCK categories. Here, the Mathematical Knowledge for Teaching Framework was a vital tool as I looked for ways to detail demonstrated aspects of SMK and PCK. Using the six subdomains of the framework as a lens for a more in-depth content analysis, in Phase 2, I specifically looked for ways to describe and/or to subcategorize data as aspects of CCK, SCK, HCK, KCS, KCC, or KCT. As expected, considering the ways in which the Mathematical Knowledge for Teaching Framework subdomains are defined, emergent themes were frequently linked to multiple subdomains. For example, Jeremey's statement that "transformations of quadratic functions are typically covered in 2^{nd} or higher-level algebra courses," demonstrated his knowledge of the sequencing of content in his curriculum, which was deemed an aspect of KCC. However, considering that SCK is defined as "knowledge not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400), knowing the sequence of content in the curriculum is useful for teaching, but more than likely not relevant in other professions. Therefore, Jeremy's idea unit was also linked to SCK.

Through iterative pattern coding in Phase 2, I was able to identify six emergent themes that were representative of the knowledge demonstrated by the experts in Interview #2. These themes provided greater detail to the data that had been initially given SMK or PCK codes. From these themes, a codebook (Appendix D) was generated.

In Phase 3, data from Interview #3, which provided elaborations and clarification to Interview #2 data, were merged with data from Interview #2 to form well-detailed idea units. This merged data set was then analyzed using the codebook. To establish reliability, a second coder, an established researcher in the mathematics education field, used the codebook to examine a sample of randomly selected data. An agreement rate of 95% was reached on unit ideas and a rate of 92% agreement was achieved on coding.

Three weeks after my Phase 3 coding, I repeated my coding on a sample of the data to test my own coding consistency reliability (99%).

The final phase of analysis, Phase 4, was completed on the merged data set to explore the existence of connections and interactions amongst the themes. During this phase, relationships were characterized and categorized. One way this was done was by tracing an expert's initial idea unit throughout the entire interviews. For instance, when Christian viewed the first student sample, she stated "I know what the shape of a quadratic looks like. It is a parabola, which has a vertical axis of symmetry." The idea unit here is based upon the graph or shape of a quadratic function and the attribute of symmetry. Therefore, I traced instances where ideas of shapes, graphing, and symmetry reappeared. A partial sample of Christian's trace is displayed in Table 4.3.

Table 4.3

Idea Unit Trace

Line #	Sample	Response
3	Student A	I know what the **shape** of a quadratic looks like. It is a parabola, which has a vertical axis of **symmetry**.
12	Student A	If students don't know what it looks like, they will not **graph**
17	Student A	This student is looking for **symmetry**... This student is showing **symmetry** in their table and **graph**
28	Student A	Since they did not see that **symmetry** in any of the tables, they are going to answer that there are no quadratic functions
40	Student B	It is important to know the **shape** of a linear function
53	Student B	I'd probably ask them to **graph** to confirm
59	Student C	This student did what I would have initially done, which is **graph**
66	Student C	I will encourage the student to get more points to **graph** so they can see the **symmetry** in the tables and then **graph**
93	Student D	It looks like they have a line indicating the axis of **symmetry**
235	Student H	I would use XXXX software to **graph** more efficiently

In the next section, I present the results of my analysis, with descriptive detail and excerpts from the interviews, to describe the nature of the knowledge expert mathematics educators used when engaging with student written work on quadratic function tasks.

Findings

This section reports on the findings obtained from the analysis of the semi-structured interviews and an artifact review. Borrowing from ontological studies in philosophy, I organize findings according to entities, classes, and relationships to provide details into the nature of the knowledge demonstrated by experts when engaging with the student written work. Here, entities are existing knowledge spheres or categories of teacher knowledge. They are the basic elements of the expert's knowledge system. The entities evolved from the recurrent themes in the data. The focus or emphasis of each entity dictates its class, and the interactions between the entities detail the relationships.

The Entities

The knowledge demonstrated by experts are categorized as six entities: content knowledge, connections, interpretations, anticipations, instructional moves, and resources. Table 4.4 provides an overview of each entity. Following the table, I discuss each entity in detail.

Table 4.4

MKT Quadratic Entities

Mathematical Knowledge for Teaching Quadratics Entities	
Content Knowledge	Knowledge of quadratics (functions and equations); the mathematical ideas, concepts, definitions, and procedures foundational to quadratics
Connections	Knowledge of mathematical concepts that inform quadratic learning and understandings; knowledge of mathematical concepts informed by quadratics
Interpretations	Knowledge of mathematical ideas/concepts represented in student work; interpreting/unpacking/explaining the mathematical ideas present in student work
Anticipations	Knowledge of anticipated student methods, strategies, procedures, misconceptions, etc.; knowledge of student justifications for methods, strategies, procedures, etc.
Instructional Moves	Knowledge of educative reasoning for next teaching move/learning technique to promote learning; reasoning and/or purposes regarding the given tasks, suggested next tasks, and/or next suggested instructional move
Resources	Knowledge of instructional materials and resources, such as technology and curricula, that aid in student understanding; suggested materials and resources to assist in student learning

Content Knowledge

Knowing ideas, concepts, definitions, and procedures directly related to a mathematical topic is knowledge of the content. This entity is most closely aligned with the subdomain of CCK in the Mathematical Knowledge for Teaching Framework. Although demonstrated content knowledge of quadratics was similar, or common, across the experts in my study, including *common* as a descriptor may imply that this knowledge is shared by all those engaged in secondary mathematics, as the use of *common* in the Mathematical Knowledge for Teaching Framework implies common math knowledge amongst most adults. Since this study only presents evidence from those involved and the use of *common* as a descriptor has previously drawn criticism (Speer et al., 2015), I simply refer to this entity as content knowledge.

Content knowledge was made evident when participants talked about the mathematics that they knew. For example, when Cameron viewed the Table Task, he stated, "my definition of a quadratic is $f(x) = ax^2+bx+c$, where a is not equal to zero… for recognizing quadratics from table data, I would look for a constant second difference." Here, Cameron provided a definition of quadratic functions and detailed a procedure that he could use in identifying quadratics.

Throughout the interviews, experts demonstrated a vast array of content knowledge. As a result of the rich and detailed data, the entity of content knowledge could be further delineated or categorized into sub-entities. However, for organization purposes and the use of the findings outside of the study, a more meaningful name for the sub-entities was given, *quadratic characteristics*. Therefore, specific sub-entities of content knowledge that were demonstrated by all experts, whether idea, concept, definition, or procedure, is termed a *quadratic characteristic.* Ideas, concepts, definitions, characteristics, and/or procedures that further support, expand and/or bring additional clarity to the quadratic characteristic, that may or may not have been demonstrated by all experts, are termed *elaborations*. These quadratic characteristics and elaborations are important as they provide more detail of the content knowledge entity.

As a part of the MKT for quadratics, the content knowledge deemed a quadratic characteristic included 2nd degree polynomial, linear rate of change, parabolic shape, symmetry, extreme value, two real roots, and multiple representations. Note, this should not be taken as the only knowledge of quadratics that the experts possessed. Rather, this is the content knowledge that was made evident and captured during my study. Table 4.5 summarizes the content knowledge exhibited by the experts, organized by quadratic characteristic and elaborations. In the table, the quadratic characteristic (in bold print) is followed by the elaborations, with the number of experts that demonstrated each elaboration in parenthesis.

Table 4.5

Quadratic Functions Content Knowledge

Content Knowledge – Quadratic Functions	
2nd Degree Polynomial - a quadratic is a polynomial function of degree two	
Elaboration:	Polynomials are degree 2 can be written in standard form $f(x) = ax^2 + bx + c$, for some real numbers a, b, and c, where $a \neq 0$ (6 experts); vertex form $f(x) = a(x-h)^2 + k$, $a \neq 0$ (6 experts); and factored form $f(x) = a(x + e)(x + d)$, $a \neq 0$ (4 experts).
Linear Rate of Change - a quadratic function has a linear rate of change	
Elaboration:	The rate of change of the rate of change of a quadratic function is constant (6 experts); the second difference is constant (6 experts).
Parabolic Shape - the graphical representation of a quadratic function is a U-shaped curve, the parabola	

Elaborations:	Adding constants to the function or multiplying the function by a constant, changes the graph of the quadratic function in predictable ways (6 experts). For example, the stretch/compression factor is **a** in $a(x\text{-}h)^2 + k$ and $ax^2 + bx + c$ (4 experts); the vertical translation is **k** units in $a(x\text{-}h)^2 + k$ or **c** in $ax^2 + bx + c$ (4 experts); the horizontal translation is **h** units in $a(x\text{-}h)^2 + k$ (4 experts); and reflection over x-axis happens when $\mathbf{a < 0}$ in $a(x\text{-}h)^2 + k$ and $ax^2 + bx + c$ (4 experts).
Symmetry - quadratic functions are symmetrical	
Elaborations:	The axis of symmetry, a vertical line that divides the parabola into mirror images, can be determined visually by examining the graph (3 experts); determined algebraically by using the formula $x = -b/2a$ (3 experts); from vertex form where $x=h$ (3 experts); by taking first derivative of the function, setting it equal to zero, then solving for x will yield the value of the axis of symmetry (3 experts).
Extreme Value - quadratic functions have one extreme value, a minimum or maximum value, located at the vertex	
Elaborations:	The leading coefficient of the quadratic indicates whether the extreme value is a minimum or maximum (6 experts). The extreme value of the quadratic can be determined by graphing (6 experts), from vertex from where the extreme value is k (5 experts), or algebraically, where the extreme value is $f(-b/2a)$ (5 experts).
Two Real Roots - quadratic functions have at most two real roots	
Elaborations:	A parabola that sits below the x-axis will have two distant real zeros (6 experts); on the x-axis, one real root with a multiplicity of 2 (4 experts); above the x-axis no real roots, but complex roots (4 experts). Algebraically roots can be found by factoring (6 experts), completing the square (4 experts), and using the quadratic formula (4 experts). Reversibility allows transitions from roots to algebraic representations (4 experts).
Multiple Representations - quadratic functions can be represented in multiple ways	
Elaborations:	Quadratic functions can be represented algebraically (6 experts), graphically (6 experts), tabular (6 experts) and verbally (6 experts).

Connections

The experts frequently discussed how quadratic concepts were informed by, or informed other mathematical concepts. In my study, identifying links between mathematical concepts that aid in learning or understanding other mathematics concepts, is categorized as connections. This knowledge entity, in addition to being an aspect of SCK, aligns with the HCK subdomain in the Mathematical Knowledge for Teaching Framework. Making connections demonstrates knowledge of how mathematical concepts are related over the span of the mathematics curriculum.

Connections linked prior concepts as well as future concepts. When discussing how to identify functions by their rate of change, Kurin stated, "If students know linear functions, they could recognize a constant rate of change and determine which of the functions are not quadratic," indicating how a prior concept could inform the current concept of quadratic functions. Later, when discussing how to represent the same quadratic function in a table, graphically and algebraically, Kurin eluded to how understanding representations will be used in the future. She stated,

> it is going to be important for a student to be able to work all three of these representations, the table, the graph, and the quadratic equation, because we are going to do that in Calculus. We do a lot of work across the different representations.

In summary (see Table 4.6), experts felt that possessing an understanding of linear functions, equivalent algebraic forms, and algebraic procedures was key to understanding the quadratic characteristic of 2nd degree polynomial. Similarly, understanding linear functions and constant/nonconstant rates of change, along with the ability to recognize patterns in data, informs understanding a quadratics' linear rate of change. Familiarity with graphing and graphical representations of different functions is helpful when identifying the attributes and features of a parabola. Finally, being accustomed to the language of mathematics (i.e., roots, zeros, solutions, etc.) allows one to make sense of the quadratic characteristic *two real roots*.

Looking forward, understanding quadratic characteristics, such as symmetry and the extreme value, will be useful as one begins to study the first derivative, typically covered in Calculus. Along the same line, engaging with the multiple representations of quadratics, as expressed by Kurin, is also useful in higher level mathematics courses.

All connections, whether looking to previous or future mathematical concepts were linked to the quadratic characteristics expressed in the content knowledge entity. Table 4.6, organized by quadratic characteristics, displays a summary of the connections provided by the experts.

Table 4.6

Connections

Quadratic Characteristic		**According to # Experts**
	Informed by	
2nd Degree Polynomial	Linear function, degree 1	3
	Equivalent expressions	3
	Algebraic processes (i.e., multiplying, factoring)	4
Linear Rate of Change	Constant/nonconstant rates of change	4
	Recognizing patterns	3
	Linear functions	3
Parabolic Shape	Graphing	3
	Functions and their graphs	3
Two Real Roots	Language of mathematics (i.e., zeros, roots)	5
	Informs	
Symmetry	First derivative	3
Extreme Value	First derivative	3
Multiple Representations	Calculus and higher-level mathematics	5

Interpretations

Through the student samples, experts attended to and explained their understandings of the students' written work. Experts sought to identify the mathematical meanings and ideas embedded in the work, regardless of the accuracy. The knowledge used to explain and make sense of the student work is described as interpretations. The entity of interpretations is most closely aligned to the SCK subdomain, as interpreting student written work is more than likely unique to teaching.

There were several common interpretations across the experts. For example, when viewing the work of Student B (see Appendix B), Jeremy's interpretations of Student B expressed what all experts interpreted in Student B's work. Jeremy stated,

> This student did the 1st and 2nd differences. The student realizes that Table A is a linear function and calculated the slope. In Table B, the student sees that the rate of change of the rate of change is constant and marks it as quadratic. Table D, the student sees a pattern, first difference as being multiplied by 3. Oh, since it is a common ratio [in Table D], the student thinks it is a quadratic too. This student may be confusing common differences and common ratios.

Like Jeremy, all experts interpreted Student B as using second differences as a procedure to identify the function type from the table data, while demonstrating a possible misconception relating to common differences. Table 4.7 presents the interpretations that were common across the specified number of experts per student sample (see Appendix B for student samples). This table is also organized by quadratic characteristics as all interpretations were linked to the quadratic characteristics detailed in the content knowledge entity.

Table 4.7

Interpretations

Quadratic Characteristic	Interpretation	According to # Experts
2nd Degree Polynomial	Student D uses leading coefficient to determine opening	3
	Student H focuses on leading coefficient to select widest function	5
	Student L rewrites equation as an equivalent expression	4
Linear Rate of Change	Student B is using 2nd difference process to determine function type	6
Parabolic Shape	Student C is graphing to determine function type	6
	Student C knows that a graphical image may not be enough to determine function type	5
	Student F states the range of the function	4

Symmetry	Student D finds the vertex to find axis of symmetry (uses -b/2a)	5
	Student D transforms to vertex form (with errors) to find the axis of symmetry	5
	Student O creates symmetric data in the table	3
Extreme Value	Student D views vertex as a "middle" point	5
	Student D connects vertex to min/max value	5
Multiple Representations	Student K does not use both representations to solve or verify solutions	4
	Student M represents context through a table	6
	Student M does not generate algebraic rule from table	3
	Student O goes from context, to table, to graph	4

Anticipations

Experts frequently postulated a student's method and the student's reasoning for the method on the given mathematical tasks. I consider both these instances *anticipations* as experts hypothesized what they thought students would do in a mathematics task (expectations) and hypothesized why students did what they did in the mathematics task (justifications). Both displays of anticipations, whether before or while viewing student written work, stemmed from the experts' understanding of the quadratics content and understanding of students. The entity of anticipations, an aspect of SCK, also aligns with KCS in the Mathematical Knowledge for Teaching Framework. The combined knowledge of students and content allowed the experts to anticipate how a student would approach a mathematics task and anticipate the student's justification for his or her approach.

While the anticipations were sample specific as experts predicted what students would do on certain tasks and justified why they did what they did, several anticipations were shared by the experts. When viewing the table task Jamie anticipated that "*students* *will graph*," which was the anticipation of all experts. However, when examining the work of Student A, experts realized that the student did indeed graph, but not how they had anticipated. Experts were left to reason why the student did what he did. All experts hypothesized that the student was looking for symmetry in the tables and therefore illustrated that in his written work. This is seen in Christian's response below. Christian first reiterates what the student did (her interpretation) and then reasons why this was the student's solution (her anticipation). She states,

> This student is showing symmetry in their table and graph. So, this student understands that quadratic functions have a line of symmetry. Since they did not see that symmetry in any of the tables, they are going to answer that there are no quadratic functions represented in the table data.

As seen in Christian's response, the quadratic characteristics of symmetry and multiple representations is present. Notably, all anticipations by the experts were linked to at least one of the six quadratic characteristics elaborated in the content knowledge entity.

Instructional Moves

The experts, after anticipating and interpreting the student written work, discussed what they would do next to promote and/or enhance student learning based on the student's work. Their ideas ranged from specific mathematical activities to focused conversations with the student. Planning for instruction and selecting appropriate activities for exploring quadratics is vital and more than likely, unique to teaching, hence SCK. In addition, this entity, deemed instructional moves, aligns with the KCT subdomain. Being knowledgeable of quadratic functions and knowledgeable of effective teaching strategies, allowed experts to suggest instructional moves that would enhance students' quadratic function understandings.

Table 4.8 displays a sample of instructional moves that were suggested by at least three experts for a particular student sample. Again, this table is organized by the quadratic characteristics as all instructional moves were linked to quadratic characteristics identified in the content knowledge entity.

Table 4.8

Instructional Moves

Quadratic Characteristic	**Instructional Moves**	**According to # Experts**
2nd Degree Polynomial	Discuss implications of leading coefficients (Student F)	3
	Help student determine formula (Student M)	6
Linear Rate of Change	Discuss multiplicative versus additive constant (Student B)	4
	Introduce 2nd difference strategy (Student C)	3
	Connect pattern in data to function type (Student M)	6
Parabolic Shape	Ask students to sketch graphs of all function (Student A, Student F)	5
	Assist student in getting more points for graph (Student C)	3
Symmetry	Ask students about the meaning of symmetry (Student A)	3
	Ask student about language used, i.e., "middle" (Student D)	5
Extreme Value	Discuss "vertex" and "value" (Student E)	6
Multiple Representations	Talk about solving across representations with students (Student L, Student K)	6
	Help student move from table to formula (Student M)	6

Resources

When reviewing the table task Jamie stated, "they could put the data in the calculator and see the plot on their calculator, that might be a little more efficient, than having them plot the data by hand." Here, Jamie identifies a resource, the calculator, that can possibly aid in the students' quadratic activity. Frequently, experts suggested technology, curricula, and manipulatives that could be useful in quadratic instruction and activities. This entity, the knowledge of learning resources, deemed resources, is an aspect of SCK and aligns closely with the KCC subdomain in the Mathematical Knowledge for Teaching Framework. Knowing quadratics and being knowledgeable of instructional resources allowed the experts to identify resources that could impact student learning. Some resources mentioned were graphing calculators, computer-based graphing software, and curriculum materials that encourage the use of multiple representations.

The Classes

The entities of teacher knowledge for quadratics, content knowledge, connections, interpretations, anticipations, instructional moves, and resources, can be categorized or grouped into classes. While all entities combine to describe the knowledge needed or used by the experts for teaching quadratics, the underlying focus in each entity suggests its class. Based on the focus of the entities, I identify three classes: math, student, and teacher.

Content knowledge and connections, having a focus on the mathematical content, belong to the math class. Interpretations and anticipations, focusing on what the student did, why the student did it, or what the student may do, are entities of the student class.

Lastly, instructional moves and resources, the actions, and identifications by the mathematics teacher, belong to the teacher class.

Table 4.9

Entity Classes

	Entity Classes	
Math	**Student**	**Teacher**
Content Knowledge	Interpretations	Instructional Move
Connections	Anticipations	Resources

The Relationships

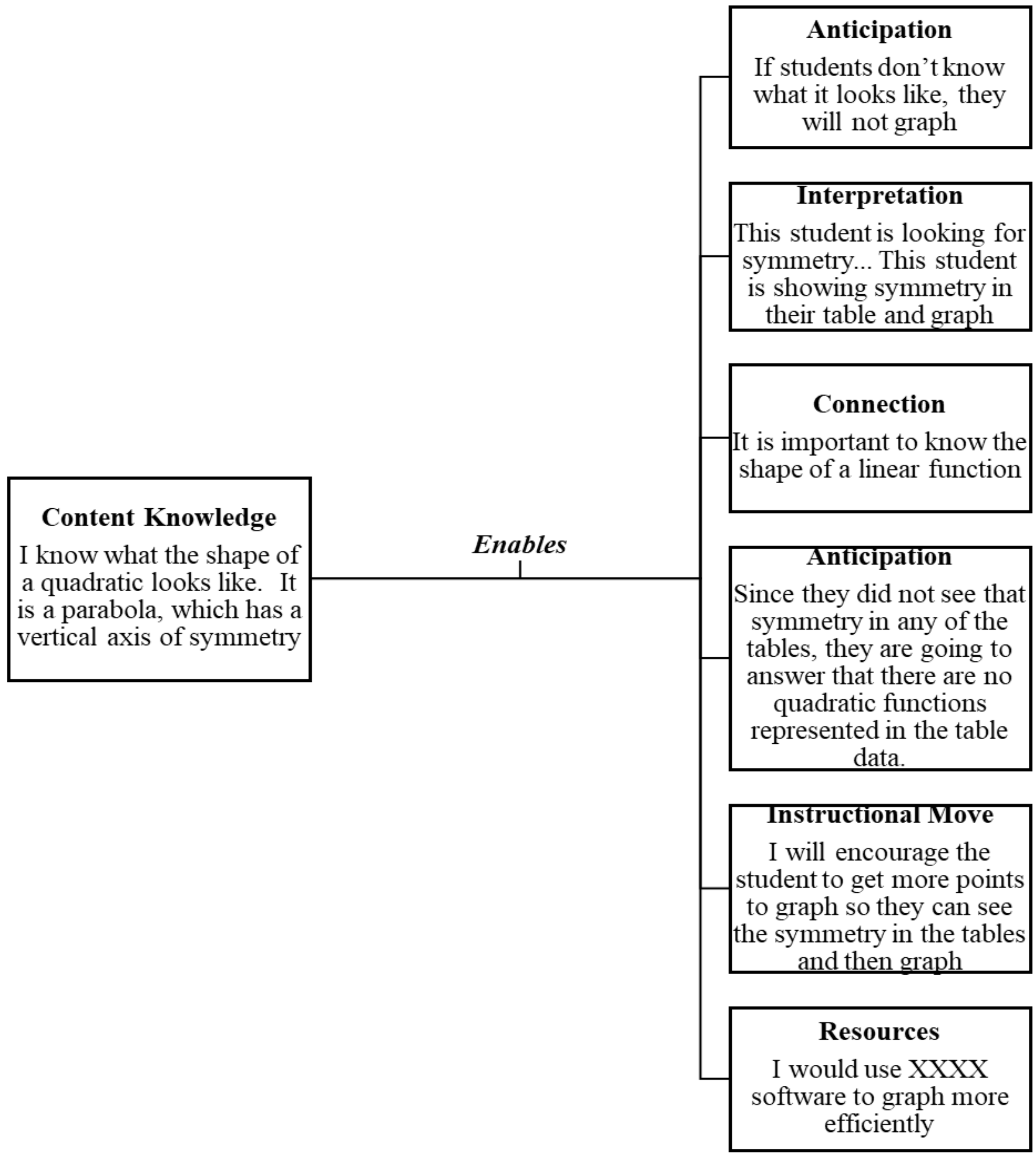
Two significant relationships exist amongst the entities: content knowledge as the enabler and entity synergy. These relationships are detailed in the following sections.

Content Knowledge as the Enabler

Results from the study highlight the vital role that content knowledge plays amongst the entities. I deem this role "the enabler." The retrace of content knowledge throughout the compiled interview data (Interview #2 and Interview #3) provide evidence of content knowledge being the enabler of other knowledge entities. Without content knowledge, it is quite possible that the other entities may not exist for a certain concept. For example, when presented with the Table Task (see Appendix B), Christian quickly stated that she knew the graphical representation of a quadratic, a parabola, which has a vertical axis of symmetry (*content knowledge*). She then *anticipated* that students would not graph if they did not know the shape of a quadratic function. When presented with the written work of Student A, Christian *interpreted* the work as looking for symmetry, which is visible in the graphical representation of a quadratic function and possibly one distinguishing feature between quadratic and linear graphs. Christian further *anticipated* that Student A did not identify any tables as quadratic since symmetry was not visible in the tables. As an *instructional move* to aid in the student's ability to identify quadratic functions through graphical representations, Christian voiced her desire to assist the student in getting more points to graph and possibly using XXXX software as a *resource* to produce more efficient graphs. Here, Christian's anticipation, interpretation, instructional move, and identification of a resource was based upon her foundational understanding of the graphical representation of a quadratic function (see Figure 4.2). Christian's content knowledge - quadratic functions have parabolic shape and symmetry, enabled the other entities. Expressed differently, if Christian did not know the graphical representation of a quadratic function, would any of the other entities, specifically those linked to graphing and symmetry, have existed? In addition to knowing the parabolic shape of a quadratic function, Christian's enacted entities could also be linked to the quadratic characteristics of multiple representations, although parabolic shape was her starting point.

Figure 4.2

Christian's Content Knowledge as the Enabler



Throughout the interviews, a participant's content knowledge proceeded and informed the other entities of connections, interpretations, anticipations, instructional moves, and resources, serving as the enabler. (See Appendix C for additional representations of content knowledge as the enabler). While the *content knowledge as the enabler* relationship is most broadly illuminated in the data, a subtler relationship also exists between the five other entities. There is an interaction, or synergy between the entities of connections, interpretations, anticipations, instructional moves, and resources.

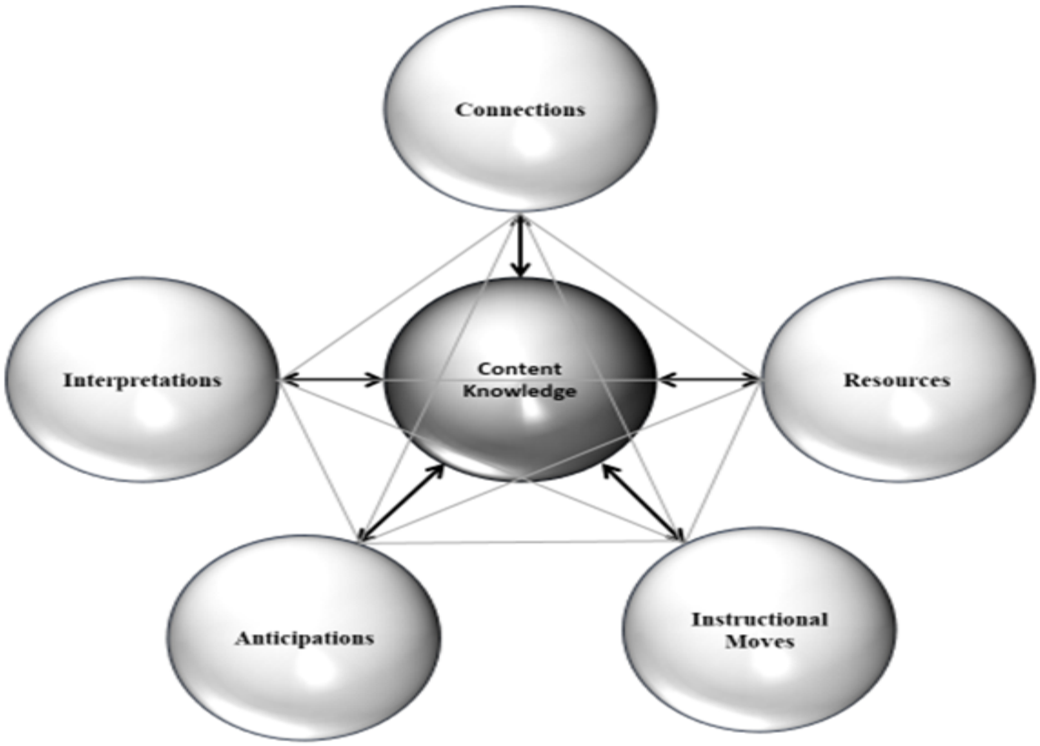
Synergy Between Entities

A synergy or a connection is identifiable between the entities that are enabled by content knowledge. Again using Christian's response to Student A as an example, an interpretation of student written work (*student is looking for symmetry*), can lead to an anticipation of why the student did what they did (*since they did not see that symmetry in any of the tables, they are going to answer that there are no quadratic functions)*. Combined, the interpretation and anticipation impact the instructional move (*I will encourage the student to get more points to graph so they can see the symmetry in the tables and then graph*). Based on the instructional move, useful resources will be utilized (*I would use XXXX software to graph more efficiently)*. Here, the interpretation linked to the anticipation, which informed the instructional move and the identified resource. Synergy between entities, like what is displayed across Christian's responses, is evident in the responses of all participants.

A visual representation of the relationships, content knowledge as the enabler and synergy between entities, is represented in Figure 4.3. In this image, secondary MKT for quadratics is centered on content knowledge, the enabling entity. The dark arrows, going from content knowledge to the other entities represents the enabling functionality of content knowledge. The dark arrows going from the entities to content knowledge acknowledge the impact of the entities on content knowledge. Although changes in the experts' content knowledge were not evident in my analysis of the task-based interviews, during the focus group discussion all experts stated that their content knowledge has been enhanced through the work of teaching and recommended the bidirectional arrows from content knowledge to the other entities. I propose that due to the experts' level of experience, changes in content knowledge would not be evident through the engagement with the specific student written work. However, including the bidirectional arrows is representative of the ways that the experts viewed their knowledge. Further, the entities of connections, interpretations, anticipations, instructional moves, and resources are connected by lighter lines, representing the connectedness or synergy between the five entities.

Figure 4.3

MKT Quadratic Functions



Discussion and Implications

The purpose of this study was to explore the knowledge used by experts when engaging with written student quadratic function work. In analyzing the interviews and artifacts, the results indicate that the nature of knowledge displayed by the experts can be detailed by six entities: content knowledge, connections, anticipations, interpretations, instructional moves, and resources. These entities, classified as math, student, or teacher focused, are linked. Namely, the five synergistic entities, connections, anticipations, interpretations, instructional moves, and resources, are enabled by the entity of content knowledge.

The consistency across the experts, the teachers, mathematicians, and mathematics teacher educators, and the ongoing member checking, suggests that I have identified entities that are reflective of their MKT for quadratics. Yet, the absence of literature on the ways that teachers know quadratic functions prevents me from assessing my findings with those of others. However, a wider reflection of literature, looking broadly at the mathematics of quadratic functions and the work of teaching, encourages the credibility of my findings. For instance, the entity of content knowledge, specifically content knowledge of quadratic functions, is representative of what NCTM considers essential for teachers to know about functions; the function concept, covariation and rate of change, families of functions, combining and transforming functions, and multiple representations (Cooney et al., 2010). Likewise, the remaining entities depict what Hill and colleagues specify is meant by MKT, "the mathematical knowledge used to carry out the work of teaching mathematics" (2005, p. 373). Linking to the Mathematical Knowledge for Teaching Framework subdomains of SCK, HCK, KCS, KCT and KCC, the entities that represent connecting concepts, anticipating student strategies and reasoning, interpreting mathematical ideas in student written work, instructing to enhance or promote learning, and identifying resources, are used to carry out the work of teaching quadratics.

Further, the findings from this study contribute to an evidence-based understanding of the knowledge used to teach quadratic functions and emphasize the importance of continued research on secondary MKT, specifically how it is conceived. While the Mathematical Knowledge for Teaching Framework presents a compartmentalized view of the teacher knowledge, the results from my study extend Ball and colleagues' work to represent secondary MKT for quadratics in an innovative way. I propose that my representation of the relationships between entities (Figure 4.3) can represent a conception of secondary MKT for quadratic functions. In this proposed conception, the MKT for quadratics is envisioned as a set of dynamic entities that work to inform and enhance each other.

While my study is limited by its focus on quadratic functions and by its context, six experts engaging with student written work, I feel fairly confident that my conception of secondary MKT for quadratics can be extended in terms of secondary mathematics content and research context. To start, by juxtaposing my conception with the work of scholars that have been guided by the Mathematical Knowledge for Teaching Framework in their topic-specific secondary MKT studies, I see appropriate applicability of my entities to their MKT definitions. For example, Steele and Rogers' (2012) conception of MKT for proof (MKT-P), described as addressing aspects of CCK and SCK only, consisted of four components: defining proof, identify proofs and non-proofs, create mathematical proofs, and understanding the roles of proof in mathematics. These components, though topic-specific, can be aligned to the knowledge entities in my conception: defining proof to content knowledge, identify proofs and non-proofs to interpretations, create mathematical proofs to instructional moves, and understanding the roles of proof in mathematics to connections. Further, Steele and Roger's data were collected via observation of video recorded lessons, written assessment, and interview. With the alignment of my entities to components of their MKT-P conception, it is reasonable to believe that my conception could be used as a framework or guide for studies that employ observations or written assessments.

It is important for a well-formed conception of secondary MKT to be operational in various research contexts and mathematical content. While suggesting the applicability of my conception, determining its true pertinence will require empirical endeavors. Future studies should examine the usefulness of this conception across the secondary mathematics curriculum and across various research context.

CHAPTER V

BEYOND RIGHT OR WRONG: A STRENGTHS-BASED APPROACH TO EXAMINING STUDENT WORK

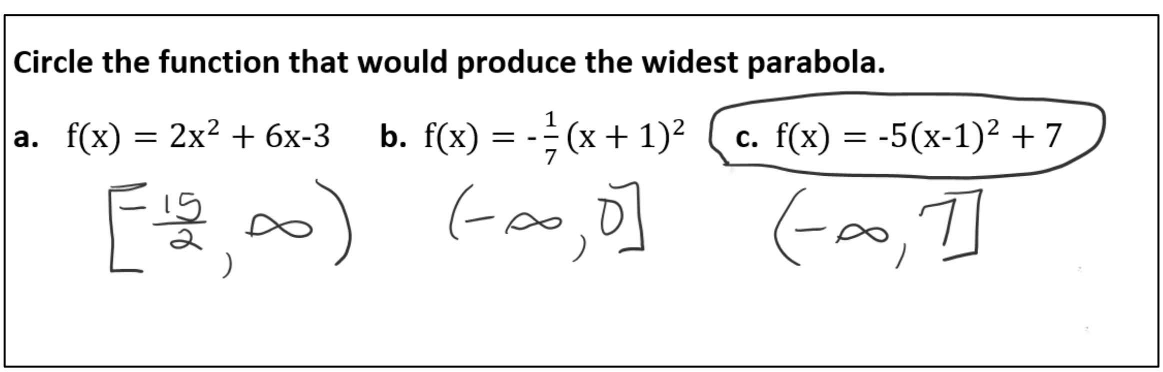
Abstract: Recognizing the strengths of students through their written work takes time, practice, and intentionality. In this article, I detail a set of questions that can be used to intentionally engage with student written work in a strengths-based way. The questions, derived from the exploration of experienced mathematics educators' mathematical knowledge for teaching quadratic functions, place value on student thinking while providing the opportunity for the exploration of one's own mathematical knowledge for teaching. These questions can provide meaningful learning experiences for individuals, professional learning communities, and large group professional development activities.

A Strengths-Based Approach

When you look at the student work presented in Figure 5.1, what do you see?

Figure 5.1

Student Sample 1



Living in a world of high-stakes testing, from end-of-course assessments to college admissions tests, it is not surprising that we tend to focus on correctness. Even with numerous years of teaching experience, sometimes a quick glance at the work in Figure 5.1 may simply reveal an incorrect response. However, what if we were to look a little closer, purposefully seeking to understand the mathematical ideas demonstrated in the student's written work, what would we see?

The purposeful seeking of mathematical ideas embedded in the students' written work is key to taking a strengths-based approach to examining written work. Within the strengths-based educational model, educational principals emphasize the "positive aspects of student effort and achievement" (Lopez & Louis, 2009, p. 1). Here, the work of the student is valued by recognizing what has been done while not ignoring what has not. It requires moving beyond simply identifying what is correct and incorrect. A strengths-based approach allows one to recognize where support is needed and determine ways to build upon the student's understandings (McCarthy et al., 2020). While this may require a degree of intentionality, "we best help a learner by starting where he or she is and building upon his or her current understanding" (Philipp, 2008, p. 23). Hence, paying attention to and building upon the ideas of students can lead to more effective instruction and increased student learning (Bishop et al., 2014).

Interestingly, focusing on the mathematical ideas rooted in student written work not only will benefit the student, but it has the potential to benefit the teacher. As we engage with the written work of students, we have an opportunity to explore and expand our own mathematical knowledge for teaching. Mathematical knowledge for teaching, or

MKT, is the phrase that is commonly referenced to describe the knowledge used and needed by those providing mathematics instruction. It is the knowledge that what we rely upon to convey mathematical concepts to our students in ways that are meaningful and useful. Over the last few decades, researchers have diligently worked to identify, describe, categorize, and connect MKT to student learning. Through this work, researchers have linked MKT to the quality of instruction (Hill, Ball, et al., 2008) and student achievement (Hill et al., 2005). With the importance of MKT recognized through research, it is understandable that “all teachers need continuing opportunities to deepen and strengthen their mathematical knowledge for teaching” (Conference Board of the Mathematical Sciences, 2012, p. 68). Utilizing a strengths-based approach to engage with student written work can provide an opportunity for teachers to deepen and strengthen their MKT.

Student written work can be a powerful tool in mathematics education (Kazemi & Franke, 2004). However, moving from an evaluative, diagnostic, or formulative approach to a strengths-based approach may require some guidance. Fortunately, through my work with six accomplished mathematics educators, who were grounded in the idea that all student written work is valuable and worth careful review, I surmised several questions that can help guide us to a more strengths-based approach when engaging with student written work. After a brief overview of my study, I will detail the questions and provide sample responses from my study participants.

Learning from Experts

Six mathematics educators (two high school teachers, two university teacher educators, and two university mathematicians) participated in my study exploring the MKT for quadratic functions (Zimmerman, 2020). All participants were purposefully selected for the study as they represent expertise in the field. Combined, the participants had over 170 years of teaching experience, amassed over 50 state/local awards and recognition, regularly participated in multi-year professional development activities focused on teacher learning, and actively engaged in mathematics education research. Through a series of semi-structured, think-a-loud interviews, the participants engaged with student written work that represented varying quadratic function concepts. Results from my study indicated that the MKT for quadratic functions demonstrated by my participants could be categorized as content knowledge, anticipations, connections, interpretations, instructional moves, and resources. Further, content knowledge, which deepens as we learn from students, enables effective anticipations, connections, interpretations, instructional moves, and identification of resources.

These entities were made possible because participants engaged with student written work in ways that focused on the students' mathematical ideas. While the participants were keenly aware of the study's purpose to explore MKT for quadratic functions, their engagement with student written work represented attributes of strengths-based education. The participants moved beyond the right or wrong solutions to see the useful and powerful information conveyed by students through writing. This is seen in participant Cameron's and participant Kurin's response to the Student Sample 1 (Figure 5.1). First, Cameron stated

> I would take the absolute value of -1/7. Since that is the smallest number, that parabola, $f(x)=-1/7(x+1)^2$, would be the flattest or the widest. So, this student, I have a sense of what they were trying to do though. Our textbook has certain questions where there are given quadratic functions and they are asked what the intervals for which the function is increasing and decreasing. I think that is what this student is doing when I see their written intervals. Wait, on second thought, that interval notation, they are looking for the range. I might give this student some simpler functions. So, I think this student might just need a simpler set of functions to compare, to get the idea of width across. Once they can see that, then I would introduce more complicated functions.

After determining the solution for the mathematics problem, Cameron discussed the work of the student. Cameron initially connected the student's response to increasing or decreasing intervals, relating it to the problems seen in his textbook. Upon further examination, Cameron decided that the mathematical idea represented in the student's work was the range of the functions, written in interval notation. To aid in the student's understanding, Cameron decided that student engagement with simpler functions was needed. Kurin's response provide additional insight. She remarked,

> I think the width of a parabola is not that well defined without something to reference to. So, I would want to look at all 3 graphs together. Knowing about the different kinds of shifts and changes to functions based upon where you put coefficients, I would say it is the middle one, $f(x)=-1/7(x+1)^2$. So, I think maybe for this question I might say "circle the function that would produce the widest parabola at the same height" or maybe "at the same y-value." Will a student understand what I mean when I add that to it? Now, this student is not connecting what you want the student to connect to in terms of the widest parabola. When they report back the range, you know they are looking vertical instead of looking horizontal. I would ask the student to graph all three functions together and then point out to me in their picture, where they are looking to determine the widest parabola. Then I would just reorient them to the horizontal width instead of

> vertical. I think this is a place where a tool like XXX really comes in handy, where you can graph several of that same function family and change a single coefficient.

Intertwined with the solution to the problem, Kurin expressed her concern regarding the problem itself. Kurin discussed possibly changing the wording of the problem. She too identified range of quadratics as the mathematical idea represented in the student's work. Kurin inferred that the student has a vertical perception of width, therefore using range to identify the widest parabola. Kurin stated that she would use graphing to orient the student to width as a horizontal feature of parabolas.

As demonstrated in the participants' discussions, the think-a-loud responses centered on the written work of the student in a way that was useful. The responses to the student's written work were not evaluative in nature, but rather productive in the sense that a path for teacher action was established based on the student's demonstrated understanding. The participants' strengths-based approach led to identifying the mathematical ideas embedded in the student's written work, recognizing where support may be needed, and devising a plan to expand student understanding.

It should also be noted that both Cameron and Kurin first discussed the mathematics of the problem before engaging with the written work of the student. Then, their final remarks, focused on their role a as teacher. This pattern of focusing on the math, then the student, and finally the teacher, was evident in the interviews of all participants. Participants typically discussed the mathematics of the problem before deeply engaging with the written work of the students. Once the student work was carefully analyzed, participants then turned their focus to their role in furthering or enhancing student understanding.

In my pursuit to understand, describe, and categorize the participants' knowledge for teaching quadratics, I discovered that the participants' responses could be elicited by asking six questions (see Table 5.1). It should be noted that I was not asking these questions in the interviews, rather it was the strengths-based responses from participants that led me to the questions. The six questions connect directly to the knowledge entities revealed in my study and have a math, student, or teacher focus that was evident in the discussion patterns of the experts. While there is not a definitive order to answering the questions, considering the purpose of a strengths-based approach, I propose pursuing a math-student-teacher order that encourages movement away from a teacher-centered approach.

Table 5.1

Quadratic Function MKT Entity and Guiding Question

Knowledge Entity	Entity Description	Strengths-Based Question	Focus
Content Knowledge	Knowledge of quadratics (functions and equations); the mathematical ideas, concepts, definitions, and procedures foundational to quadratics	What is the solution?	Math
Connections	Knowledge of mathematical concepts that inform quadratic learning and understandings; knowledge of mathematical concepts informed by quadratics	What mathematical concepts informed the student work and/or what will the student's work inform?	Math
Interpretations	Knowledge of mathematical ideas/concepts represented in student work; interpreting/unpacking/explaining the mathematical ideas present in student work.	What mathematical understandings or ideas is the student demonstrating?	Student
Anticipations	Knowledge of anticipated student methods, strategies, procedures, misconceptions, etc.; Knowledge of student justifications for methods, strategies, procedures, etc.	How could the student have arrived meaningfully at this solution?	Student
Instructional Moves	Knowledge of educative reasoning for next teaching move/learning technique to promote learning; reasoning and/or purposes regarding the given tasks, suggested next tasks, and/or next suggested instructional move	What could I do?	Teacher
Resources	Knowledge of instructional materials and resources, such as technology and curricula, that aid in student understanding; suggested materials and resources to assist in student learning	What resources are involved?	Teacher

The Guiding Questions

In this section, I will explain each question, discussing the importance of the question and the question's link to the knowledge entities identified in my study.

What is the Solution?

To determine the solution of the mathematics problem, one must utilize their knowledge of content, or content knowledge. This question focuses on the mathematics content covered in the mathematics problem. Although answering this question does not require reviewing the student written work, it may be foundational to a strengths-based interpretation of the work. Knowing the math that is covered in the mathematics problem will help us to make sense of the student work, while establishing the level of understanding that is demonstrated by the student. In my study, responses to *What is the solution?* was categorized as *content knowledge*. Content knowledge was made obvious when the participants employed ideas, concepts and or procedures foundational to quadratic functions to reveal the solution or solutions to a problem, prior to interpreting the student work. If we do not know the solution to the given mathematics problem or know multiple approaches to reaching a solution for the problem, if applicable, it may impact our ability to discern the mathematical understandings present in the student written work. If this is the case, answering the question *What is the solution?* becomes an opportunity to expand our own MKT. We will expand our content knowledge by acquiring the understandings, concepts, ideas, and/or skills necessary to complete the mathematics problem.

What Mathematical Concepts Informed the Student Work and/or What Will the Student's Work Inform?

Continuing with a math focus, this question too requires content knowledge, but in a slightly different way. The knowledge required to answer *What mathematical concepts informed the student work and/or what will the student's work inform?* is not the same exact knowledge needed to complete the presented mathematics problem. Rather, it is knowledge of the mathematical concepts that will inform the content knowledge needed to complete the mathematics problem, and it is the knowledge of the mathematical concepts that will be informed as a result of completing the mathematics problem. For instance, knowing how to factor a quadratic expression will inform knowing how to algebraically determine the zeros of a quadratic function; knowing how to determine the zeros of a quadratic function will inform knowing how to find zeros of functions of degree three or higher. Essentially, answering *What mathematical concepts informed the student work and/or what will the student's work inform?* connects prior, current, and future mathematical ideas and concepts.

Answers to this question, categorized as *connections* in my study, was made evident when participants discussed the mathematical concepts that could inform a student's work and how a student's work could inform other mathematical concepts. Acknowledging the connections across the secondary mathematics curriculum and beyond can be an impactful demonstration of MKT in action. Connections inform instructional planning in ways that ensure precursory mathematical concepts are understood and current concepts are presented in a way that will foster and not hamper future learning.

What Mathematical Understandings or Ideas is the Student Demonstrating?

Specifically identifying the mathematical understandings and/or ideas embedded in the student written work, regardless of the accuracy of the work, is a hallmark of a strengths-based approach to engaging with student written work. This question focuses on the student. The work of extracting mathematical ideas, categorized as *interpretations* in my study, allows us to seek out the mathematical meaning or meanings that the student is communicating through writing. Interpretations were made evident when participants carefully studied the students written work and remarked "I think this student is saying…," "I think this student is doing…," or "I think this student understands…."

While interpretations are hypothetical in nature as they are dependent on our own understandings, interpretations demonstrate valuing of the student's work and sets the ground for the next step. The next step could be exploring how the student meaningfully arrived at their solution or determining the instructional move that will promote student learning. With either step, the act of interpreting is necessary to build upon the student's understandings and ideas.

How Could the Student Have Arrived Meaningfully at This Solution?

Still focusing on the student, the entity of *anticipations* in my study encompasses the expressed expectancy of, or student reasoning for specific methods, strategies, procedures, misconceptions, etc. While anticipations occurred at various stages of the engagement with student written work, answering *How could the student have arrived* *meaningfully at this solution?* brings deliberate focus to the actual work demonstrated by the student. Often comments such as "I have seen students…" or "students will…," communicated an expectancy of students and proceeded the in-depth examination of written work. These statements frequently provided insights into the participants' experiences through their numerous years of teaching. However, when directly focusing on the written work at hand, participants contemplated the reasoning of the student. This type of anticipation allowed participants to further focus on the work of the student and possibly gain greater insight into the student's solution by considering the student's thinking.

Anticipations not only allow us to prepare for instruction by considering the various questions, strategies or difficulties that students may encounter when engaging with a mathematics problem, but it can also provide a time of reflection after the problem has been completed by the student. Our consideration of the student's reasoning for his or her work can link the student's demonstration of knowledge to various influences, such as the instruction received, the resources used, or the experiences provided. Practicing the act of anticipating broadens our MKT as we consistently learn what students do and why they do it by engaging with their work in productive ways.

What Could I do?

This question may be the most common question that we, teachers, ask ourselves. The focus is now on us, the teachers. Occasionally, this may be the very first question that comes to mind when viewing student work, especially work that demonstrates partial or no understanding. However, a strengths-based approach encourages this question to be answered only after the mathematical meaning or understanding has been identified in the student written work. In this order, we can build upon the student's demonstrated understandings when answering *What could I do?* Study participants, after carefully reviewing and interpreting the student work, suggested what they would do next to increase student learning. These suggestions were categorized as *instructional moves*.

Developing the best possible instructional move in response to the analysis of a student's mathematical understanding is one of the many tasks of teaching (Ball & Bass, 2000). Consistently using student work to inform instructional moves by answering the question *What could I do?* takes a strengths-based approach to broadening our instructional possibilities, hence expanding MKT.

What Resources are Involved?

When interpreting student written work and discussing the next instructional moves, participants frequently identified instructional materials and resources, such as technology or curricula. These materials and resources, deemed *resources* in my study, were identified as resources that could aid in student understanding. Possessing knowledge of the various educational resources that can support learning of specific content is an important part of MKT. Engaging with student work in a strengths-based way can provide additional opportunities to identify resources that can specifically address the needs of the students, based on their written work.

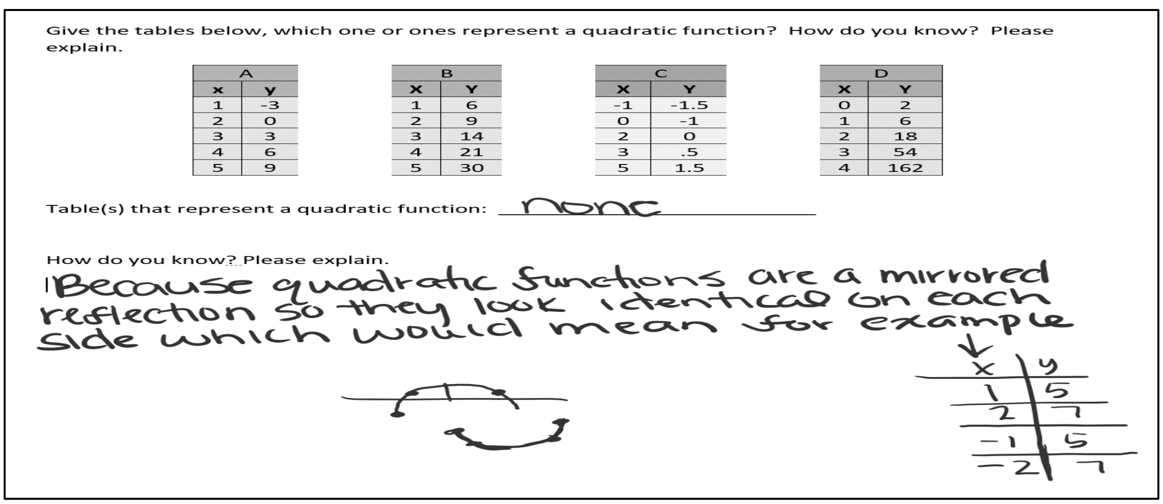
These six questions not only encourage a thoughtful and productive engagement with student written work but embody the work of teaching. The questions create a path to further student learning by building upon student knowledge while simultaneously providing opportunities to explore and/or expand our own MKT. To further explore the utility of the guiding questions, in the next section we will review participant response to Student Sample 2 (Figure 5.2).

Guiding Questions Answered

To take a strengths-based approach when viewing student written work, while possibly exploring and expanding MKT, we can begin to ask ourselves the very questions that my participants answered in their stream of consciousness when they engaged with the student written work. Samples of the participants' responses based on Student Sample 2 (Figure 5.2) follow.

Figure 5.2

Student Sample 2



Give the tables below, which one or ones represent a quadratic function? How do you know? Please explain.

A		B		C		D	
x	y	X	Y	X	Y	X	Y
1	-3	1	6	-1	-1.5	0	2
2	0	2	9	0	-1	1	6
3	3	3	14	2	0	2	18
4	6	4	21	3	.5	3	54
5	9	5	30	5	1.5	4	162

Table(s) that represent a quadratic function: none

How do you know? Please explain.

Because quadratic functions are a mirrored reflection so they look identical on each side which would mean for example

x	y
1	5
2	7
-1	5
-2	7

What is the Solution?

When study participants examined Student Sample 2, they identified several approaches to determining which table or tables represented a quadratic function.

Amongst the approaches were studying patterns of covariation, utilizing the second differences strategy, and graphing. Since participants knew that quadratic functions have a linear rate of change, all participants identified Table B as the only quadratic function represented in the tabular data. They also identified the function families that the non-quadratic tables belonged to. Table 5.2 summarizes the various knowledge of content that was demonstrated as participants established the solution to the problem.

Table 5.2

Content Knowledge Demonstrated

Content Knowledge	Cameron	Christian	Jamie	Jeremy	Kurin	Rena
Patterns of covariation	X	X	X	X	X	X
Linear rate of change	X	X	X	X	X	X
2nd difference strategy	X	X	X	X	X	X
Parabolic shape	X	X			X	
Symmetrical tabular data	X			X	X	

What Mathematical Concepts Informed the Student Work and/or What Will the Student's Work Inform?

According to Kurin,

> Having experiences with patterns is necessary - patterns is something students can easily see. If students know linear functions, they could recognize, a constant rate of change by looking at the differences between the y-values as x grows, for consecutive x-values, seeing the pattern, then looking at the difference in the rate of change of change, identifying the pattern.

Like most of the participants (four of the six), Kurin identified patterns in data and linear functions as concepts that would inform a student's ability to engage with the mathematics problem shown in Student Sample 2. In addition, three participants acknowledged that the ability to graph table data and recognize function types from the graph would inform student work.

Looking to future learning, all participants agreed that mathematical activities that would encourage the use of multiple function representations, like the math problem in Student Sample 2, would enlighten later mathematical concepts. Jamie elaborated,

> Using multiple representations, I think it definitely impacts them as they move through their mathematics career. When you think about calculus, we use multiple representations - these are different representations of the same thing. Moving from table data, to looking at the relationship between the second differences, or the third difference, or whatever, and then making that connection to derivatives, graphically, or algebraically… In calculus, using multiple representations is very impactful for students, provided they've experienced it. I think this all leads very naturally to Calculus. And if that is our intent, preparing students for other mathematics courses, just think about statistics and all those representations. Yes, multiple representations are extremely important.

What Mathematical Understandings or Ideas is the Student Demonstrating?

After examining the student's graph, along with the written table and verbal explanation, participants were able to identify possible mathematical understandings and ideas demonstrated by Student Sample 2. All six participants acknowledge that a mathematical understanding represented in the student's work is that of symmetry. According to Jeremy,

> The student is looking for the same on the left side of the y-axis and on the right side. They are looking for it to be symmetrical and none of the tables shows symmetry across the y-axis. They know the pattern of symmetry, the positive and the negative x-values will have the same y-value. So, that's what they are showing with (1, 5), (-1, 5), (2, 7), (-2, 7). This shows you the student's thinking and that's what we want to see.

While all participants were able to identify the mathematical understanding demonstrated in the Student Sample 2's work, the work was not what the participants had expected. Although the student did draw a graph, the participants predicted that students would identify the quadratic function by graphing the data in the tables. With the student's unexpected demonstration of knowledge, the participants were left to reason about the student's work, leading to the next question.

How Could the Student Have Arrived Meaningfully at This Solution?

Christian's rationale behind the work in Student Sample 2 is representative of the other five participants' reasoning. Christian stated,

> So, this student understands that quadratic functions have a line of symmetry. Since they did not see that symmetry in any of the tables, they are going to answer that there are no quadratic functions represented in the table data.

In addition, Jeremy speculated that the student's solution, while based on the absence of obvious symmetry, could be a result of the student's experiences. Jeremy elaborated,

> Maybe everything they have seen, ever done, or simply remember, when it comes to quadratic functions, looks the same on the left side of the y-axis as it does on the right side of the y-axis. It could be the only quadratic examples that they have ever been shown, you could see the symmetry. To them, to be quadratic, the table data must show the symmetry and none of these tables show that symmetry.

After gaining an understanding of the mathematics represented in the student's work and then contemplating how the student arrived at their solution, the focus of the questions being answered shifted to the teacher. Desiring to extend or enhance student learning, participants focused on their role as teacher by answering *What could I do?*

What Could I do?

Though all participants voiced instructional moves in their own way, all moves were built from the student's demonstrated understanding of symmetry. Jamie's response to Student Sample 2 exemplifies that of all participants:

> I would encourage the student to take the values that were given in the tables and sketch graphs, and then have them talk about what kinds of images they created. Since the student is looking for this pattern of mirrored images, or symmetry, I would have them, looked at the images and then think about how they could extend the table data. By extending the table data, and then the graphs, we could discuss and verify that the table data does or does not have this reflection that they are looking for. In fact, I might have them extend the table data before I ask them to look at a graph.

Rena, like the other participants, identified questions that she would also ask: "Do quadratic functions always have a vertical line of symmetry," and "Is it possible that there would be a line of symmetry not at the y-axis?" Since the student's work was embedded on the mathematical idea of symmetry, Rena wished to build upon that knowledge by possibly expanding the student's concept of symmetry in quadratics.

What Resources are Involved?

In discussing their teaching moves, participants frequently discussed resources that could aid in instruction. For example, Jamie identified one resource as a graphing calculator. Accordingly, Jamie suggested, "they could put the data in the calculator and see the plot on their calculator. Using a calculator might be a little more efficient than having them plot the data by hand." Similarly, all participants identified some type of graphing utility, whether a specific type of calculator or computer software, that would be a useful resource in instruction. In addition, Jeremy, acknowledging the necessity to use different representations (e.g., tables, graphs, etc.), suggested curricular materials that encourage the use of multiple representations as a viable resource. He stated "when I teach using XXX materials, I always get to show multiple representations. Using problems from the XXX curricula with the multiple representations, would help here."

Discussion

As seen from the participants' responses, the guiding questions set a path for a strengths-based approach to engaging with student work. The questions require us to focus on and build from the mathematical ideas and understandings demonstrated by the student. By doing this, we can determine actions and/or interactions that will advance student understanding by responding directly to the thoughts of the student. Through a strengths-based approach, we can increase student learning.

As students are always learning, so are we, the teachers. We can learn from the work of students. Student written work, often considered "performances of understanding" (McDonald, 2002, p. 121), is an ideal tool for exploring and expanding one's ability to comprehend the thinking of students, thus providing a way to enhancing our own MKT. In my study, participants displayed evidence of their MKT as they engaged with student work in meaningful and useful ways. Learning from the participants, I propose that using the six questions as a guide to studying student written work will support a strengths-based approach while also providing opportunities to explore and expand one's own MKT:

1. What is the solution?
2. What mathematical concepts informed the student work and/or what will the student's work inform?
3. What mathematical understandings or ideas is the student demonstrating?
4. How could the student have arrived meaningfully at this solution?
5. What could I do?
6. What resources are involved?

Given the simplistic nature of the guiding questions, the use of the questions is applicable at various levels of teacher enhancement or professional development activities. To start, individual teachers can use the question as they review and learn from their own student work. Teachers can privately identify and address their areas of need, whether it be learning different methods to arrive at the same solution or identifying resources that could enhance student learning. Considering how much we can learn from others, the guiding questions will also be ideal for use in professional learning communities (PLCs) and larger professional development activities. Through collaboration with fellow educators, groups of teachers can use the questions to increase their strengths-based approach to student work while expanding their own MKT by learning from others.

CHAPTER VI

RESEARCH COMMENTARY: MOVING SECONDARY MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT) FORWARD

Abstract: Lack of agreement on the structure of the knowledge needed to teach secondary mathematics has impeded the development of a unified conception of secondary MKT (mathematical knowledge for teaching). This lack of agreement is evident in the variations of study approaches and frameworks produced at the secondary level. However, if we wish to ensure that all students have access to teachers who possess the knowledge needed to prepare them for careers and post-secondary education, a unified conception of secondary MKT is necessary. Therefore, in this commentary, I highlight secondary MKT advancements and barriers, with the purpose of challenging the mathematics education community to work toward a unified conception of secondary MKT.

Mathematical Knowledge for Teaching

Given the intense research interest in teacher knowledge, more specifically, mathematical knowledge for teaching (commonly referenced as MKT) over the past decade, the field has yet to reach a consensus on what MKT is for secondary mathematics teachers. Research has shown that teachers must know the mathematics that they plan to teach, but the other components of teacher knowledge, the knowledge that supports teaching and student learning, is still not well understood (Hill et al., 2007). Despite

Shulman's introduction of pedagogical content knowledge and the wide range of studies that followed, the mathematics education community has endeavored. For years, scholars have worked to determine the exact content and structure of the knowledge that is needed to effectively teach mathematics. While progress has been made, most notably at the elementary level, work at the secondary level is still evolving. For instance, at the elementary level, MKT is defined well enough for researchers to develop and employ multiple forms of the Learning Mathematics for Teaching (LMT) assessment in numerous program evaluations and studies of relationships and effects (Hoover et al., 2016). Whereas, the lack of agreement on the structure of secondary MKT has led to diverse study approaches and numerous frameworks, which some deem "distressing" (Hill et al., 2007, p. 131).

However, through the possibly distressing variations at the secondary level, there are some things about MKT that transcend grade level. First, this knowledge is unique. As stated by the Conference Board of the Mathematical Sciences, "the mathematical knowledge needed for teaching differs from that of other professions" (2012, p. xii). Further, MKT is necessary. "All mathematics teachers rely on mathematical knowledge for teaching" (National Research Council, 2010, pp. 114–115). Finally, MKT is not inherent. "All teachers need continuing opportunities to deepen and strengthen their mathematical knowledge for teaching" (Conference Board of the Mathematical Sciences, 2012, p. 68). Given that MKT is unique, necessary, and not inherent, it is vital that we continue to work towards a well-defined conception of secondary MKT to ensure that all students have access to teachers whom possess the knowledge needed to prepare them for careers and post-secondary education.

To make progress in developing a well-defined conception of secondary MKT it is important that we begin by understanding what work has already been accomplished. With this commentary, I challenge the field to work toward a more unified conception of secondary MKT. To do so, I survey the literature, highlighting the progress that has been made and the barriers that are preventing advancement. Finally, I present ideas that can help reevaluate the differences in the literature and move the mathematics community to a more unified conception of secondary MKT.

From Teacher Knowledge to Mathematical Knowledge for Teaching

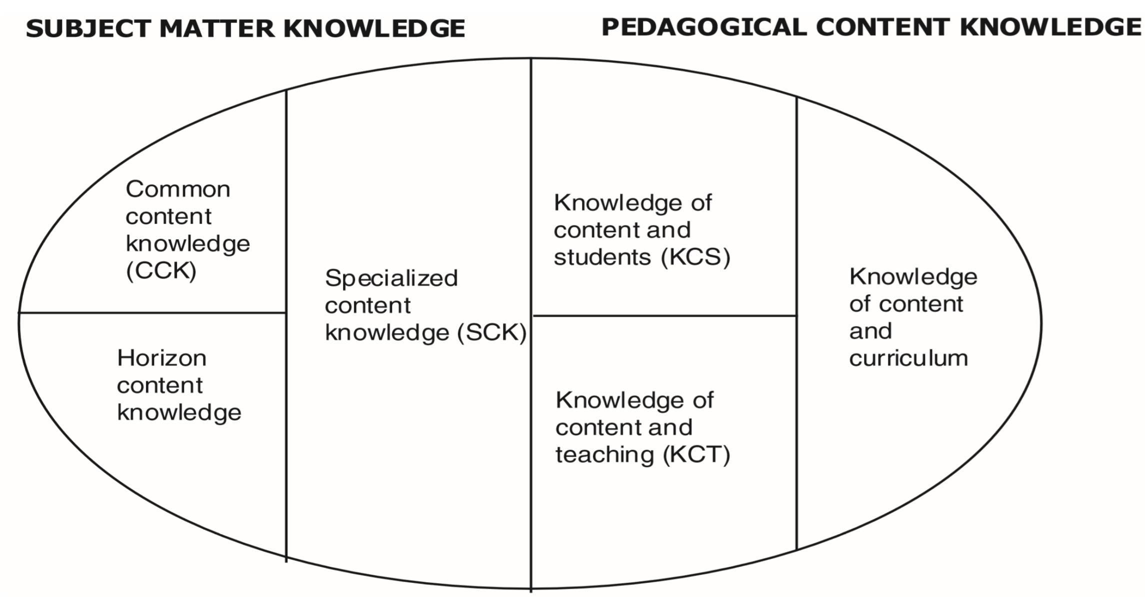
Most agree that teachers must possess sound knowledge of the subject, or in other words, “teachers must know the subject they teach” (Ball et al., 2008, p. 404). However, an early study of mathematics teachers’ knowledge conducted by Edward Begle (1972) concluded that advanced subject matter knowledge does not guarantee student achievement. Investigating teacher content knowledge with a meta-analysis of empirical literature, he found that the relationship between teachers’ content knowledge and student outcomes was not significant (Begle, 1979). Almost 20 years after Begle’s study, a study of teachers’ characteristics and student scores on the National Assessment of Educational Progress (NAEP) supported Begle’s earlier findings (Monk, 1994). In examining the NAEP, Monk found that student learning during the 10^{th} and 11^{th} grade year of high school was modestly impacted by the number of undergraduate courses taken by the teacher, but the impact leveled off after five undergraduate courses. While subject matter knowledge is the foundational knowledge needed for teaching, the efforts of Begle and Monk have empirically shown that the knowledge needed to teach mathematics goes beyond an in-depth understanding of content. Their studies have identified a clear need to expand the conception of the knowledge needed for teaching.

After Beagle but before Monk, Lee Shulman (1986) was addressing the need to expand the conception of teacher knowledge through his introduction of pedagogical content knowledge. Pedagogical content knowledge, described by Shulman as a type of professional knowledge unique to teaching, pushes content and pedagogical understanding to a level that explicitly addresses the disciplinary demands of teaching a specific subject. By identifying new views of teacher knowledge, this notion of pedagogical content knowledge influenced researchers to conceptualize the knowledge that teachers possess as a knowledge that extends well beyond content understanding as Begle's earlier work had suggested. Pedagogical content knowledge allows teachers to successfully connect new concepts to students' current understandings, plan for potential obstacles, and engage students in ways that will create paths for continued mathematical learning (Wilson et al., 2014).

Reconfiguring and expanding upon the earlier pedagogical content knowledge work of Shulman, Ball and colleagues (2008), through the work of the Learning Mathematics for Teaching (LMT) research group, developed a framework for conceptualizing the knowledge needed specifically for teaching elementary mathematics. The LMT research group took a practice-based approach, examining the work of teaching, to gain an understanding of the mathematical knowledge used by elementary teachers. This effort produced the widely referenced Mathematical Knowledge for Teaching Framework (see Figure 6.1). This practice-based framework details ways of knowing mathematics useful for teaching. Within this framework, the knowledge needed to carry out the work of teaching mathematics is categorized into two broad domains, subject matter knowledge (SMK) and pedagogical content knowledge (PCK).

Figure 6.1

Mathematical Knowledge for Teaching Framework (Adapted from Ball et al., 2008)



The domain of subject matter knowledge (SMK) includes common content knowledge, specialized content knowledge, and horizon content knowledge. Common content knowledge (CCK), is the knowledge and skills of mathematics that can be used in settings outside of teaching (Ball et al., 2008). For example, having the skill or knowing the algorithm to multiply two-digit numbers is content knowledge that is not limited to instruction. Knowledge that allows the teacher to help students connect the multiplication algorithm to other concepts in mathematics, for example place value, division, or exponentiation, is specialized content knowledge (SCK). SCK is considered knowledge and skill that is unique to teaching (Ball et al., 2008). Horizon content knowledge (HCK) is knowledge of how mathematical concepts span the mathematical careers of students. For example, knowing that the algorithm for multiplication will later inform binomial multiplication is an aspect of HCK.

The domain of PCK is composed of knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. Knowledge of content and students (KCS) is a combination of teachers knowing their students and knowing about mathematics (Ball et al., 2008). For example, knowing that a common misconception among students is to compute $12^2 = 12 \times 2 = 24$ instead of $12^2 = 12 \times 12 = 144$ is an aspect KCS. Knowing which strategies to use in instruction to prevent common student misconceptions or knowing how to design instruction to facilitate deep understanding of mathematical concepts that can inform later learning, requires that teachers possess knowledge of the content and knowledge of teaching. This is considered knowledge of content and teaching (KCT) in the Mathematical Knowledge for Teaching Framework. Lastly, in the pedagogical domain, knowledge of content and curriculum (KCC), is the knowledge of the programs and instructional materials available to teach certain topics and the value or effectiveness of the programs and materials (Ball et al., 2008).

The Mathematical Knowledge for Teaching Framework, commonly used to measure the knowledge of practicing teachers (Blömeke & Delaney, 2014) and design for teacher learning, has been cited more than 6000 times, with a large portion of this scholarship focused on elementary mathematics teaching. The ability to categorize and describe the knowledge needed to teach mathematics has enabled researchers to link elementary teachers' mathematical knowledge to student achievement (Hill et al., 2005) and to the quality of instruction (Hill, Ball, et al., 2008). Although grounded in the work of elementary math teachers, the utilization of the Mathematical Knowledge for Teaching Framework to gain a more in-depth understanding of teacher knowledge has not gone unnoticed by secondary mathematics scholars.

Secondary Use of Mathematical Knowledge for Teaching Framework

While the generalization of the Mathematical Knowledge for Teaching Framework to other grade levels has never been a recommendation of the LMT research group, the framework has been used in various ways at the secondary level. Some researchers have used the Mathematical Knowledge for Teaching Framework exactly as defined by Ball and colleagues (e.g., Campbell & Lee, 2017; Khakasa & Berger, 2016), while others have expanded upon the framework to specifically address secondary content (Hatisaru & Erbas, 2017; Steele et al., 2013; Taşdan & Koyunkaya, 2017). A review of this literature reveals three primary foci of secondary MKT research. Aspects of this area of research can be described as evaluative, accessing teachers' mathematical knowledge (Khakasa & Berger, 2016; Steele et al., 2013; Taşdan & Koyunkaya, 2017), developmental, primarily focused on designing measures of secondary teacher knowledge (Herbst & Kosko, 2014; Howell et al., 2016; Steele, 2013), and investigative, examining links between teacher knowledge and practice (Campbell & Lee, 2017; Hatisaru & Erbas, 2017; Steele & Rogers, 2012).

Evaluative Developmental Secondary Studies

Seeking to appraise the level of teacher knowledge, Khakasa and Berger (2016) used all six subdomains in the Mathematical Knowledge for Teaching Framework to assess and categorize secondary teachers' proficiency across a breadth of mathematics content. Through a task and opinion questionnaire, interviews and lesson observations, the proficiency levels of 117 teachers were evaluated. Practicing teachers were deemed fluent, partially fluent, or inadequate based on their demonstrated level of synergy across all six Mathematical Knowledge for Teaching subdomains within five mathematical content areas. A fluent teacher understood the content (CCK), its development and application (SCK and HCK) and could provide appropriate instruction for successful student learning (KCT and KCS) through the effective interpretation of curriculum and resources (KCC). Results of the study indicated that overall participants were partially fluent. However, researchers noted that higher marks of fluency were achieved with content that was articulated in the participants' curriculum.

In Taşdan and Koyunkaya's (2017) evaluation of teacher knowledge, four of the six Mathematical Knowledge for Teaching Framework's subdomains were examined. Focusing solely on the MKT for functions, Taşdan and Koyunkaya elaborated the CCK, SCK, KCS, and KCT subdomains to reflect focusing on a specific secondary content area. HCK and KCC were omitted from the study because the researchers deemed the subdomains connected to real classroom experiences, which the preservice teachers lacked. By analyzing the preservice teachers' design and implementation of a function-based lesson plan amongst their methods course peers, Taşdan and Koyunkaya concluded that these preservice teachers possessed limited knowledge regarding teaching functions and needed more experience to expand all of the Mathematical Knowledge for Teaching subdomains.

Similarly, Steele and colleagues (2013) focused on functions, but only included two subdomains in their study, CCK and SCK. To determine if a graduate level course could provide opportunities to develop MKT for functions, Steele and colleagues examined the growth of preservice and practicing teachers. Through written assessments, interview data, and discourse analysis, the researchers were able to track the changes in MKT of all 21 participants. Though the researchers did not specifically categorize the knowledge level of the participants, they concluded that the content-focused methods course provided the experiences needed to support the development of MKT.

Although these three evaluative studies vary in what subdomains of MKT were examined and how the subdomains were defined (broadly or content specific), there is an idea that permeates all three studies. The studies elude to the role that experience may have in the level of MKT. Whether the experience was familiarity with curriculum (Khakasa & Berger, 2016), authentic classroom teaching (Taşdan & Koyunkaya, 2017), or engagement in a graduate level course (Steele et al., 2013), the scholars' measures of MKT associated experiences with increased MKT. The role of experiences in developing MKT should be considered as efforts are put forth to develop a representative conception of secondary MKT. In addition, the measures that are used in such work could provide useful information. Since measures were not the primary outcomes of the evaluative studies, I now turn to secondary developmental studies to gain insight into studies focused on designing measures of secondary MKT.

Developmental Studies

A second category of studies focused on the development of MKT measures. Across these studies, researchers demonstrated the utility of items designed to measure aspects of MKT. Working to develop secondary measures of MKT, Herbst and Kosko (2014) considered four subdomains from the Mathematical Knowledge for Teaching Framework, CCK, SCK, KCS, and KCT. Through piloting cycles, Herbst and Kosko constructed the MKT-G instrument (G for geometry), a tool comprised of 34 tasks aimed at measuring secondary geometry teachers' MKT. The multiple choice and multiple response items were developed based on Ball and colleagues' conception of MKT, but with a focus on high school geometry. For example, an item developed to access KCS might explore a teacher's awareness of students' misconception of angle bisectors. According to Herbst and Kosko, "the instrument worked relatively well and pilot data show correlations between scores in each of the [sub]domains CCK, SCK, KCS, and KCT and the number of years of experience teaching geometry" (2014, p. 42). The researchers concluded that the results of their work support using the Mathematical Knowledge for Teaching Framework as a means of conceptualizing secondary teacher knowledge. It allowed for the design of items that focused on the nature of the expertise that is needed to teach geometry.

Similarly, Steele (2013) developed and piloted items to measure the MKT for geometry. However, Steele only focused on two subdomains, CCK and SCK in his open-response item development. The subdomains of CCK and SCK were elaborated specifically to focus on the knowledge needed to teach length, perimeter, and area. For example, an aspect of SCK was defined as knowing the "affordances and constraints of different formulas related to length, perimeter, and area" (Steele, 2013, p. 251). Steele's development and piloting of tasks that were grounded in the practice of teaching, designed to illustrate the relationship between CCK and SCK, and capture knowledge beyond correct and incorrect answers, further revealed that items could be designed to explore and distinguish important aspects of teacher knowledge.

Further in the arena of item development, Howell et al. (2016) designed assessment items aimed at obtaining evidence of MKT at the secondary level. With the intent of determining if the Mathematical Knowledge for Teaching Framework would extend to the secondary level, think-aloud cognitive interviews with 23 prospective and practicing teachers revealed that the items measured aspects of MKT as designed. While this research could also be considered investigative in nature as it explores the extendibility of the Mathematical Knowledge for Teaching Framework, it is the development and validation of the measures that informed the scholars' results. For the assessment items, Howell and colleagues broadly defined MKT, without giving specifics for the six subdomains. MKT was considered the knowledge that is not acquired through conventional means, such as through undergraduate and/or graduate mathematics courses. The researchers concluded that their validation of the MKT assessment items demonstrated the ability to measure secondary teacher knowledge that extends beyond conventional mathematics knowledge. In addition, their efforts demonstrate that the design principles used in Ball and colleagues' exploration of elementary MKT can be applied at the secondary level.

In these developmental studies, the work of Herbst & Kosko (2014) and Steele (2013) provide insight into the utility of the Mathematical Knowledge for Teaching Framework at the secondary level. Building from select subdomains of the Mathematical Knowledge for Teaching Framework, these scholars were able to design topic-specific items to measure teacher knowledge. Though Howell and colleagues' validation of measures differed from these developmental efforts by broadly approaching both the construct of MKT and secondary mathematics content, their work further supports the usefulness of the Mathematical Knowledge for Teaching Framework at the secondary level.

With the capability of designing measures and assessing levels of secondary MKT, a few scholars have sought to identify links between MKT and practice. To gain insight into these efforts, I now review secondary studies that I deem investigative in nature.

Investigative Secondary Studies

The final category of MKT secondary studies focuses on investigating the relationships between MKT and teacher practice and other variables of interest. For example, in Campbell and Lee's (2017) study to identify the impact of professional activities on teacher knowledge, the researchers cited Ball and colleagues when discussing MKT but did not provide additional details on their conceptualization of the construct. Presumably, the scholars are bridging the six MKT subdomains to present a broad perspective of teacher knowledge. The scholars analyzed the interactions of practicing teachers engaged in professional learning communities (PLCs) at two high schools. Conversations during the PLCs were coded as no mathematics, mathematics reference, or mathematics discussion. Those mathematics discussions that focused on the work of teaching were regarded as potential opportunities for developing teachers' mathematical knowledge. Since conversations amongst PLC members rarely focused on mathematics content, Campbell and Lee found that interactions "hindered the potential development of MKT" (2017, p. 124).

Taking a slightly different investigative approach, Hatisaru and Erbas (2017) examined two practicing teachers and focused on the MKT for the function concept. In their analysis of teacher knowledge, they conceptualized teacher knowledge based on only three subdomains of the Mathematical Knowledge for Teaching Framework: SCK, KCS, and KCT. However, the scholars elaborated the subdomains to relate them specifically to the function concept. For example, KCT was defined to be "adapting different definitions, explanations, representations, and examples in teaching; communicating with definition(s) of the function; awareness of the limitation and strengths of different representations; knowing students' existing conceptions and using them to make adjustments" (2017, p. 707). Observations, interviews, and written assessments revealed that a complex relationship exists between the teacher's MKT and their students' learning outcomes. The teacher's MKT influences instruction, which in turn influences student learning. Although not straightforward, the learning outcomes were greater for the students of the teacher with the higher level of MKT.

Finally, Steele and Rogers (2012) studied the relationship between teacher knowledge and the practice of teaching proof in high school. Their MKT-P framework for teaching proof, broadly focusing on subdomains of CCK and SCK, was built from literature on student learning and teachers' knowledge of proof. Steele and Rogers acknowledge that the MKT-P framework had limitations, but stated that the framework "provides a fruitful starting point grounded in previous research for investigating teachers' mathematical knowledge" (2012, p. 161). Using the MKT-P framework to identify evidence of teacher knowledge, the scholars analyzed interviews, written assessments, and teaching observations of two contrasting cases, a novice teacher, and an expert teacher. Results indicated that MKT evident in clinical settings (i.e., written assessment and interviews) played out differently in the classroom. While both the novice and expert demonstrated a wealth of knowledge in clinical settings, the expert's classroom observation revealed greater utilization of this knowledge in practice.

As seen through these investigative studies, efforts have been put forth at the secondary level to link MKT to aspects of practice. Though variations are still evident in investigative studies, what stands out to me in this work is how more detailed elaborations of MKT, like that of Hatisaru and Erbas (2017), were more productive in revealing links. This too could be informative as we find ways to transgress towards a unified conception of MKT.

Though providing insight into secondary MKT efforts, the differences among the evaluative, developmental, and investigative studies (e.g., MKT broadly defined or content specific; MKT as one domain or MKT as a combination of subdomains), are noteworthy. As seen from the review of the secondary MKT studies, the use of Ball and colleagues' Mathematical Knowledge for Teaching Framework has allowed progress at the secondary level. Though progress is being made, the existing studies do not easily lead to or promote a consolidated understanding of secondary MKT. Quite possibly, the concerns regarding the use of the Mathematical Knowledge for Teaching Framework at the secondary level are reflected in these secondary MKT research efforts.

Concerns at the Secondary Level

While the evaluative, developmental, and investigative work of these scholars has contributed to our understanding of secondary MKT, efforts using Ball and colleagues' Mathematical Knowledge for Teaching Framework does not go without question. In reviewing the literature, I have also noted the critiques and concerns regarding the use of Ball and colleagues' framework at the secondary level. Two of those concerns, CCK for secondary mathematics teachers and the challenge of discerning between the subdomains, I will discuss here.

Secondary CCK

CCK is considered by some as the "problematic part of the MKT framework for secondary school mathematics teachers" (Keskin et al., 2018, p. 336). Although Ball and colleagues say by using *common* in CCK, "we do not mean to suggest that everyone has this knowledge" (Ball et al., 2008, p. 399), this subdomain frequently brings to question to whom the knowledge is common for and what mathematical concepts should be considered common. While *adding*, an elementary mathematics concept, to determine the score of a basketball game after a made free-throw may be common to most adults, is *determining the parabolic flight path* of the basketball into the hoop (a secondary mathematical concept) common also? The subdomain CCK, possibly due to the layman's connotation of *common*, is simply challenging to operationalize at the secondary level.

While some secondary researchers have tried to bring more clarity to the CCK subdomain by specifying for whom the knowledge is common, for example "well-educated" adults (Steele & Rogers, 2012) or "professionals in other mathematically intensive fields" (Khakasa & Berger, 2016, p. 423), some do not address the subdomain at all. Further complicating the construct, Speer, King, and Howell (2015) question if what is defined as SCK for elementary teachers should actually be considered CCK for secondary mathematics teachers. This brings to question what exactly should be considered common knowledge for secondary teachers. For example, Speer and colleagues posit "recognizing the mathematical accuracy of a definition, considered part of SCK for elementary teachers, is CCK for those with more mathematics education" (2015, p. 114).

The lack of clarity and agreement surrounding CCK is evident in the work of secondary scholars. Scholars have either not explicitly examined the CCK subdomain (Campbell & Lee, 2017; Hatisaru & Erbas, 2017; Howell et al., 2016; Steele & Rogers, 2012) or generated content-specific definition of CCK for their studies (Herbst & Kosko, 2014; Steele, 2013; Steele et al., 2013; Taşdan & Koyunkaya, 2017). Explicitly defining

CCK for a study is an assumed necessity as one strives to develop detailed measures or evaluate levels of teacher knowledge. While the scholars pursued various agendas and content areas (see Table 6.1), some similarities span their CCK definitions. For example, knowing definitions is an aspect of CCK according to three of the four scholars. Additionally, from the CCK definitions, it may be inferred that the ability to complete a secondary mathematics problem is also an aspect of CCK.

Interestingly, one group of secondary scholars that did not explicitly examine CCK, did discuss content knowledge, but refrained from deeming the subdomain as *common*. In their efforts to develop measures to assess the extendibility of elementary teacher knowledge models to the secondary level, Howell and colleagues (2016) considered knowledge in two categories: MKT and conventional mathematical knowledge. Here, MKT is the knowledge that is required by the "mathematical work of teaching" (Ball & Bass, 2002, p. 9) and it is not acquired through conventional means. Conversely, conventional mathematics knowledge is "mathematics that is likely to be taught and learned in undergraduate institutions" (Howell et al., 2016, p. 18). Additionally, conventional knowledge is necessary but not sufficient for mathematics teaching. Considering that foundational concepts, such as definitions and procedures, are taught in undergraduate math courses, the idea of conventional knowledge could be like what other secondary scholars describe as common. However, Howell and colleagues set a standard of whom should possess this knowledge; those that have been taught and learned the concepts in undergraduate institutions.

Table 6.1

CCK Definitions in Content Specific Studies

Scholars	**Content Area**	**Common Content Knowledge (CCK)**
Herbst & Kosko (2014)	Geometry	• definitions, properties, and constructions of plane figures, including triangles, quadrilaterals, and circles; parallelism and perpendicularity; transformations; area and perimeter; three-dimensional figures; surface area and volume; and coordinate geometry.
Steele (2013)	Geometry	• state a definition of function • create and classify examples and non-examples of functions
Steele, Hillen, & Smith (2013)	Functions	• calculate the perimeter and area of shapes given length measurements • demonstrate a conceptual understanding of the relationships between lengths, perimeter, and area, including: the non-constant relationship between perimeter and area the impact of changes to one-dimensional attributes on perimeter and area
Taşdan & Koyunkaya (2017)	Functions	• know central definitions and properties of functions • know connections between the concept and other mathematical concepts • know relevant applications of functions in and outside mathematical contexts • can successfully complete secondary school students' problems involving the concept and identify incorrect answers or inaccurate definitions of the concept of a function • use terms and notation correctly • know the material they teach

Discerning Subdomains

The work of Howell et al. (2016) also brings attention to another concern regarding the use of the Mathematical Knowledge for Teaching Framework, the challenge of discerning subdomains. This may be why Howell and colleagues, like other secondary scholars (Campbell & Lee, 2017; Steele & Rogers, 2012) did not explicitly parse the Mathematical Knowledge for Teaching Framework as subdomains in their study. This could also explain why some scholars only studied certain subdomains (see Figure 6.2).

Along with others (Hill, Ball, et al., 2008; Howell, 2012), the ability to distinguish between subdomains was also a concern of Ball and colleagues. According to these scholars, “it is not always easy to discern where one of our categories divides from the next” (Ball et al., 2008, p. 403). Seemingly, some secondary scholars have avoided this dilemma by looking at MKT as one entity. For example, Steele and Rogers (2012) used their MKT-P framework to investigate how a teacher’s MKT for proof is evident in classroom or clinical settings. While the MKT-P framework outlined components of proof knowledge, such as defining proof, identifying proofs and non-proofs, creating proofs, and understanding the roles of proof in mathematics, the components were not clearly identified as facets or attributes of specific MKT subdomains. However, the scholars did note that the components were more closely aligned to Ball and colleagues’ CCK and SCK subdomains, but they did not further discuss which components were CCK or which components were SCK.

Another possible outcome of not being able to discern subdomains is the study of only select subdomains. Of the secondary studies discussed here, most of the scholars did not examine all six subdomains. Even in their examination of certain subdomains, the scholars provided clear components of the subdomains, essentially enabling demarcation of the subdomains. For example, Hatisaru and Erbas (2017) only examined subdomains SCK, KCS, and KCT. However, in their study, they provided detailed components of each subdomain. For instance, knowing multiple representations of functions was SCK, awareness of student difficulties with verbal representations was KCS, and adapting representations in teaching was KCT. While each of these knowledge aspects relate to representations of functions, the scholars have drawn lines between the subdomains by explicitly defining the components of each.

Figure 6.2

Secondary Studies Using the Mathematical Knowledge for Teaching Framework

Scholars	Nature of Study	MKT studied as subdomains	Subdomains Studied (☑ *provided study specific subdomain definition*)						Teacher Participants	
			CCK	SCK	HCK	KCS	KCT	KCC	Prospective	Practicing
Campbell & Lee, 2017	Investigative	No								9
Hatisaru & Erbas, 2017	Investigative	Yes		☑		☑	☑			2
Herbst & Kosko, 2014	Developmental	Yes	☑	☑		☑	☑			
Howell et al., 2016	Developmental	No							3	20
Khakasa & Berger, 2016	Evaluative	Yes	✓	✓	✓	✓	✓	✓	117	
Steele, 2013	Developmental	Yes	☑	☑					12	13
Steele & Rogers, 2012	Investigative	No								2
Steele et al., 2013	Evaluative	Yes	☑	☑					15	6
Taşdan & Koyunkaya, 2017	Evaluative	Yes	☑	☑		☑	☑		3	

The issues that are created by the concerns that surround the subdomains of the Mathematical Knowledge for Teaching Framework, whether it be in how it is titled (i.e., common) or how it is defined (i.e., what is considered SCK versus KCS) will continue to be challenging in our work if we do not move towards a unified vision of secondary MKT. While this may be acceptable for some, the issue inherently lies at the preparation and continued development of our secondary mathematics teachers. Returning to the quote, “All teachers need continuing opportunities to deepen and strengthen their mathematical knowledge for teaching” (Conference Board of the Mathematical Sciences, 2012, p. 68), how can we genuinely provide these opportunities to all teachers if we as a mathematics education community do not share an agreed upon conception of what secondary MKT is? It is time that we take steps to move forward.

Moving Forward

While the loosely defined terminology (Howell, 2012) has prompted questions regarding CCK and essentially made consistent discernment of the subdomains difficult, the Mathematical Knowledge for Teaching Framework is still one of the most widely used tools for exploring teacher knowledge. Evident from the work of secondary scholars, the framework has been instrumental in contributing to the understanding of teacher knowledge and can serve as a unifying tool. However, as a field, secondary mathematics education scholars still lack a unified vision of teacher knowledge. Without a unified vision, it is challenging to inform teacher preparation programs, licensure/certification procedures, or professional development agendas. I believe that continuing to build and learn from the work of others, including that of Ball and colleagues, is key to moving toward to a more unified vision and requires a different approach. A more exploratory approach will allow the Mathematical Knowledge for Teaching Framework to be used as a guide; a guide that requires one to fill in the details of secondary teacher knowledge.

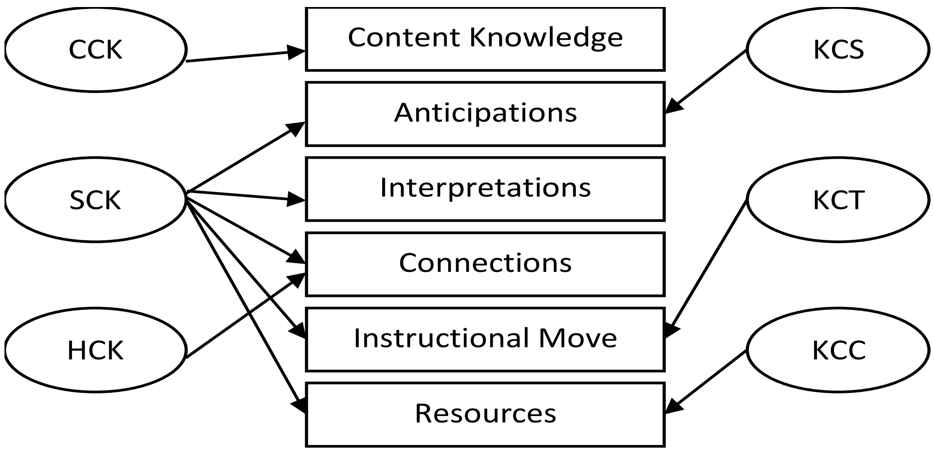
In my exploration of the knowledge needed for teaching quadratic functions (Zimmerman, 2020), I veered from the evaluative, developmental, or investigative approaches evident in several secondary MKT studies. Within these types of studies, scholars have frequently predefined the knowledge of select subdomains based on their expert opinion and/or literature on student learning. These predefined components are then used to investigate or evaluate participants in various stages of their mathematics education career although it has been noted that experience impacts MKT (Herbst & Kosko, 2014; Leinhardt & Smith, 1985; Steele & Rogers, 2012; Taşdan & Koyunkaya, 2017). Therefore, in my exploratory approach I engaged with highly experienced mathematics educators to gain an in-depth view of the components of MKT evident in their work. It was through the thematical categorization of data that I was able to identify connections to all six of Ball and colleagues' Mathematical Knowledge for Teaching Framework subdomains, consider the issues concerning the subdomains, and then produce a vision of secondary MKT that was representative of my data.

With no predefined components, my exploratory approach precipitated the categorizations of the data by recurrent themes. This played a key role in filling in the details of the participants' demonstrated knowledge. For example, participants in my study frequently discussed how they thought students would engage with math problems. If predetermined codes based on Ball and colleagues' Mathematical Knowledge for Teaching Framework had been used, these instances could have been deemed KCS. However, before being linked to the Mathematical Knowledge for Teaching Framework, the descriptive term that evolved for this representation of teacher knowledge was *anticipations*. Notably, other scholars have included anticipating students' thinking and actions as a part of the KCS subdomain (Hatisaru & Erbas, 2017; Herbst & Kosko, 2014; Taşdan & Koyunkaya, 2017). However, considering the way that SCK is defined in Ball and colleagues' Mathematical Knowledge for Teaching Framework, "mathematical knowledge not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400), anticipations could also be an aspect of SCK. What other professionals would need to anticipate how students will complete a math problem?

Like anticipations, other entities of teacher knowledge identified in my study (connections, interpretations, instructional moves, and resources) could be considered both an aspect of SCK and an aspect of another subdomain in Ball and colleagues' Mathematical Knowledge for Teaching Framework (see Figure 6.3). This highlights how the subdomains can easily overlap and why providing descriptive names to entities of teacher knowledge is an important first step in developing a unified vision of secondary teacher knowledge. Providing descriptive names may help us decide how to better discern the subdomains or prompt a conversation considering the necessity of subdomains at the secondary level.

Figure 6.3

Secondary MKT Entities

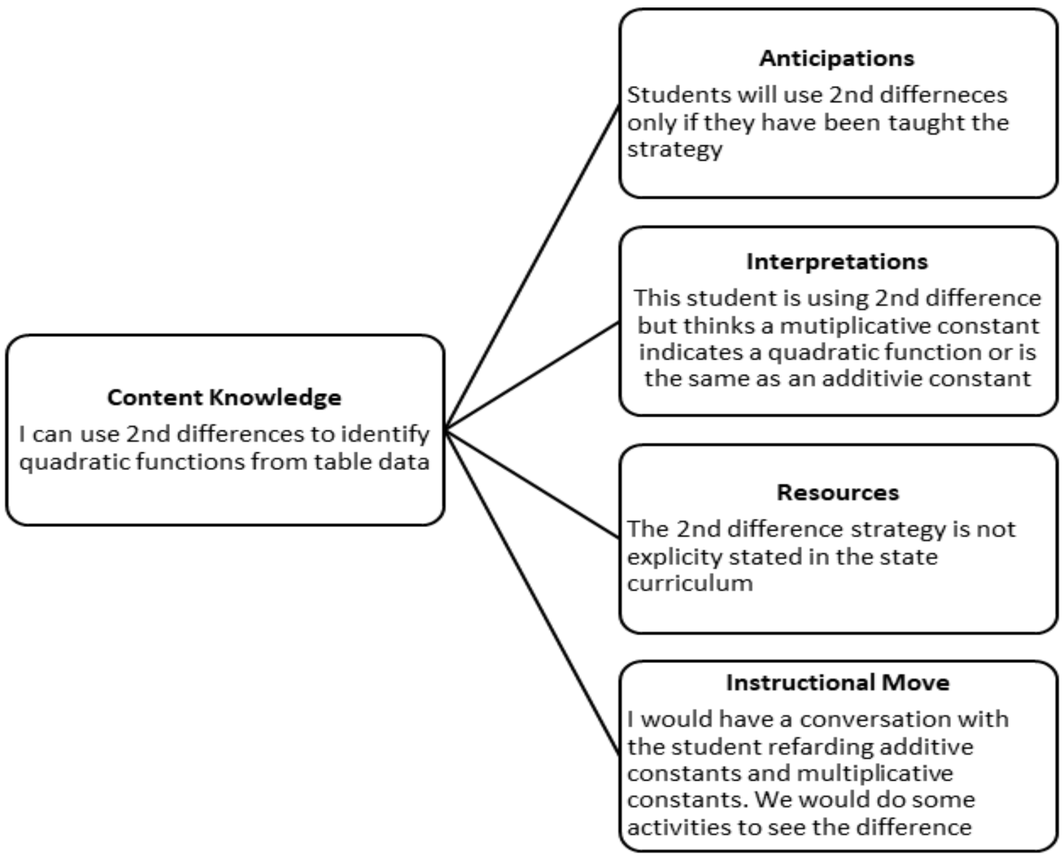


Notably, one entity of teacher knowledge revealed in my exploratory examination, content knowledge, I did not consider an aspect of SCK, as it was not *only* used for teaching. Content knowledge, specifically the knowledge of quadratic functions and equations, was not categorized as SCK but could possibly be considered by some as CCK. Although content knowledge was common across my study participants, reflecting on the earlier discussed concerns regarding the use of the term "common," I do not consider "common" a useful descriptor. Likewise, using the description of "conventional," as used by Howell et al. (2016) to categorize knowledge obtained from undergraduate college courses, is also not appropriate. Participants in my study revealed that the acquisition of content knowledge was not limited to undergraduate courses. Content knowledge was gained by participants from various experiences, including but not limited to, graduate courses, professional development activities, and the work of teaching itself.

Fully delineating who possesses content knowledge (i.e., common or not) or the extensiveness to how it was obtained (i.e., conventional) was beyond the scope of my study. However, my exploration did start to address another question regarding MKT; what is the relationship or relationships between the subdomains, or in my case, knowledge entities? Namely, I deemed content knowledge as the enabler of the other entities of teacher knowledge. Based on evidence, these entities, which were considered aspects SCK and at least one other MKT subdomain, were all enabled by content knowledge. Essentially, content knowledge facilitated participants anticipating student moves, interpreting student work, making instructional moves, and/or identifying resources. For example (see Figure 6.4), a study participant identified ways that she could identify quadratic functions from table data, including using the 2nd differences strategy (*content knowledge*). (In the 2nd differences strategy, the rate of change of the rate of change is calculated from table data.) The participant suspected that students would not use the 2nd differences strategy unless they had been explicitly taught the approach (*anticipation*s), acknowledging that the 2nd differences strategy was not a part of her state's curriculum (*resources*). However, when reviewing student work, the participant noticed a student's attempt at using the strategy that revealed a misunderstanding regarding constant change (*interpretations*). The participant said that she would have a conversation and design activities clarifying additive and multiplicative constants that would aid in the student's enhanced understanding (*instructional move*). Here, the participant's content knowledge enabled her anticipation of students' strategies, acknowledgement of a resource, interpretation of student work, and identification of an instructional move.

Figure 6.4

Content Knowledge as the Enabler



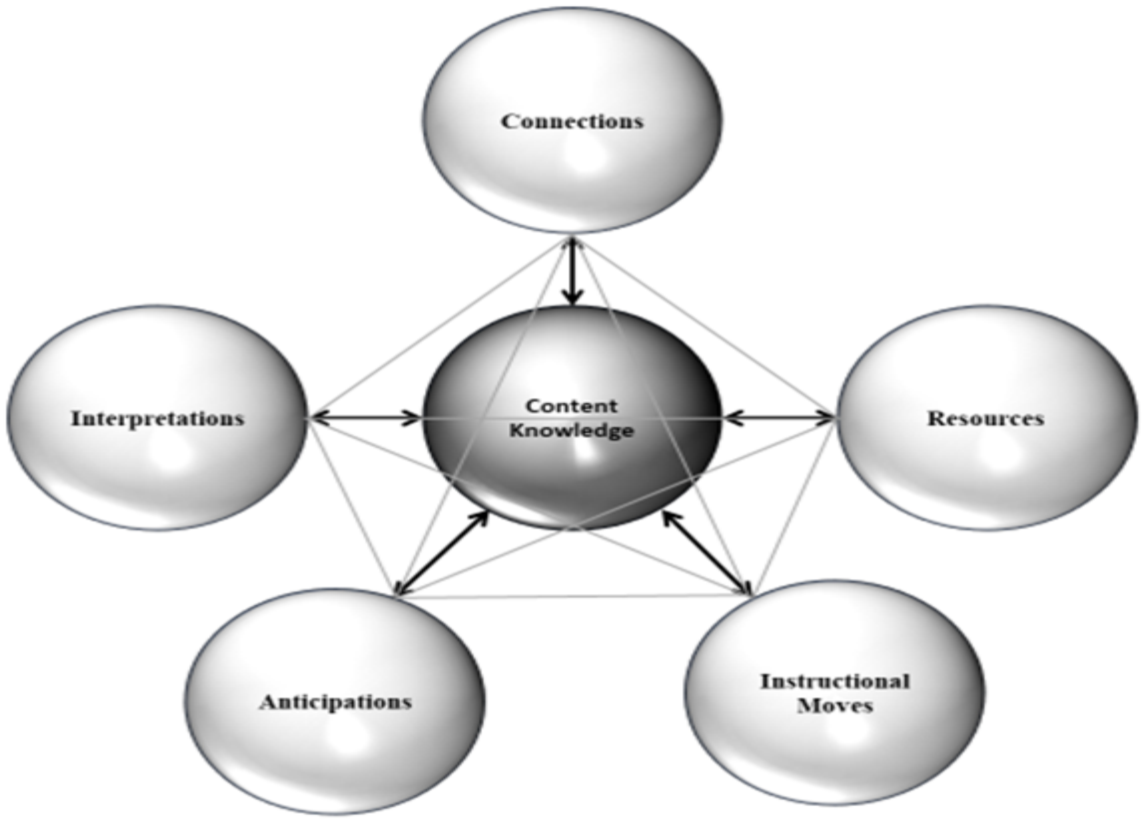
Hence, possessing content knowledge, what I described as the foundational knowledge of quadratics, enabled the other entities to exist. Further, I also noted that entities of teacher knowledge were both complementary and informative to other entities. For example, the ability to interpret the mathematics in a student's work, aided in one's ability to make an instructional move, or continually interpreting the mathematics in student work, leads to an expanded range of anticipations. Additionally, participants revealed that the work of teaching also resulted in the expansion of content knowledge. For example, when discussing a formula that can be used to determine the vertex of a parabola, a participant responded, “I learned that from teaching it. I did not learn it in high school, I can guarantee that. While I may have been exposed to it in high school, it stuck when I was teaching.”

The dynamic relationships found to exist between the six entities of teacher knowledge revealed in my study are illustrated in Figure 6.5. This visual presentation illustrates the five synergistic entities that are enabled by the entity of content knowledge. This new conception of secondary MKT provides an innovative way to think about secondary mathematics teacher knowledge.

While this conception of MKT is quite different from that of Ball and colleagues, it was informed by their work and the work of others. Evident by my study, building upon the work of others and learning from those that successfully do the work of teaching can expand our understanding of secondary MKT. For this reason, it is my hope that secondary scholars will continue exploratory efforts, possibly utilizing my proposed conception of secondary mathematics teacher knowledge, to help move the mathematics education community to a unified conception

Figure 6.5

MKT Quadratic Functions



CHAPTER VII

CONCLUSION

In this dissertation, I set out to explore the knowledge needed to teach quadratic functions with the purpose of contributing to a sparsely researched area, secondary MKT. As evident through my introduction, some progress has been made in studying secondary MKT, but the diversity and variety in study approaches have not yielded a unified conception of MKT at the secondary level. Motivated by the work of MKT scholars and the need to move towards a unified conception of secondary MKT, I utilized an exploratory multi-case design to investigate secondary teacher knowledge by investigating the question: *What is the nature of the knowledge expert mathematics educators use when engaging with student written work on quadratic function tasks?*

Efforts to answer this question resulted in three manuscripts. Collectively the three manuscripts provide insights into the potential impact of studying secondary MKT. First, by exploring, we gain empirical evidence of secondary MKT. This is evident in manuscript 1. Second, we see how empirical efforts can result in practical applications (manuscript 2). Finally, we are informed of the ways to keep moving secondary MKT forward (manuscript 3).

After a brief review of the three manuscripts, I will discuss the implications for researchers, mathematics teacher educators working with prospective and practicing teachers, teachers, and policy makers.

Discussion of the Three Manuscripts

The first manuscript specifically addressed the research question by exploring the knowledge demonstrated by the expert mathematics educators while engaging with student written quadratic function work. Here, we learn what MKT is for quadratic functions. Guided by the Mathematical Knowledge for Teaching Framework, results from the analysis of the experts' semi-structured task-based interviews reveal that the nature of mathematical knowledge for teaching quadratic functions can be characterized as six entities: content knowledge, connections, interpretations, anticipations, instructional moves, and resources. These entities, classified as math, student, or teacher focused, are linked. Namely, the five synergistic entities –connections, anticipations, interpretations, instructional moves, and resources – are enabled by the entity of content knowledge.

As a result of pursuing an empirical endeavor to learn about MKT directly from the experts, the knowledge we gain informs the researcher and the practitioner communities. The second manuscript details a set of six questions that emerged from my analysis. These questions guide the intentional focus on the mathematical understandings represented in written student work, encouraging a strengths-based approach to engagement with student work. In addition, the questions allow exploration of one's own MKT. These questions can provide meaningful learning experiences for individuals, professional learning communities, and large group professional development activities.

Finally, approaches taken in this empirical endeavor differ from the approaches of existing MKT studies. This approach has resulted in an innovative way to conceive secondary MKT that could establish a path for moving towards a unified vision of secondary MKT. Aiming to be a source of motivation, along with the work of others, the third manuscript addresses the advances that have been made in understanding secondary MKT and the barriers that could be hindering progress. I present ideas that help to reevaluate differences in the literature and encourage the mathematics education community to continue efforts to gain a greater understanding of secondary MKT.

Overall, this study provides additional evidence that secondary MKT exists, is observable, and can be understood using the Mathematical Knowledge for Teaching Framework. I was able to identify, document, and connect aspects of this knowledge. My work most closely aligns with the work of Ball and colleagues (Ball et al., 2004, 2008; Ball & Bass, 2002) as they worked to establish the Mathematical Knowledge for Teaching Framework. It was from the practice of teachers that their conception for elementary MKT evolved. My conception of secondary MKT for quadratics evolved from the work of my experts.

Though the Mathematical Knowledge for Teaching Framework was useful, I concur with the scholars that have noted its challenges (Hill et al., 2008; Howell, 2012; Howell et al., 2016) . Being used purely as an analytical tool at the secondary level will be problematic. Similar to Howell (2016), I found creating clear delineations between the six subdomains in the framework was impossible as a result of the loosely defined subdomains. For instance, I found all my entities related to SCK and at least one other subdomain. However, I do note that Ball and colleagues acknowledged the overlap of the domains - "it is not always easy to discern where one of our categories divides from the next" (Ball et al., 2008, p. 403). For this reason, it is quite possible that we should not be looking for separation between subdomains, but possibly the relationships among the subdomains.

Two relationships where identified among my entities: content knowledge as the enabler and synergy among the other five entities - connections, interpretations, anticipations, instructional moves, and resources. Quite notable, is content knowledge as the enabler. Through secondary teacher preparation programs and licensure processes, there has always been a high priority placed on content knowledge. Now, given the findings from my study, it becomes more apparent how content knowledge enables the overall work of teaching. While at the base level, content knowledge allows one to distinguish correct responses from incorrect responses, it is the enabling power of this knowledge that is vital to teaching. Future endeavors should investigate how content knowledge takes on enabling power.

Applying an appreciative lens to my exploration of secondary MKT allowed me to understand the knowledge that my experts possessed, not what they lacked. This is different from the secondary studies that have studied MKT with a predefined notation of MKT (e.g. Hatisaru & Erbas, 2017; Khakasa & Berger, 2016; Steele et al., 2013). By first valuing what teachers know, we can establish a pathway to developing a conception of secondary MKT that is truly representative of teacher knowledge and widely agreed upon. A shared conception of secondary MKT will guide our measures and establish consistency in the field, resulting in a solid knowledge base of secondary MKT.

Implications

Above, I discussed the direct impact of studying secondary MKT that was evident in the three manuscripts. Now, I go a step further, addressing the implications of the findings for researchers, mathematics teacher educators that are working with prospective and practicing teachers, teachers, and policy makers.

Researchers

The conception presented in Figure 1.3 provides a new way of viewing secondary MKT that may be useful for future research. First and foremost, findings from my study indicate that secondary teacher knowledge can be described in detail. The six entities, which provide descriptive names of the experts' demonstrated knowledge, are evident of how movement away from vague domains to more explicit, meaningful categorizations is possible. This is a necessary step in moving closer to a well-defined conception of secondary MKT.

In addition, findings detail the classes and relationships among the entities. Understanding the focus of the entity, such as math, student, or teacher, and understanding the connectedness between the entities, e.g. content knowledge as the enabler, provides yet another lens for researchers to explore secondary MKT. This conception of secondary MKT can be a useful tool for researchers as they continue to find ways to investigate, assess, and improve secondary MKT.

Mathematics Teacher Educators

Findings from my study can be informative for those that design preservice teacher education courses or programs and professional development activities. As mathematics teacher educators work to ensure the adequate preparation of aspiring secondary mathematics teachers and the ongoing professional development of practicing secondary mathematics teachers, gaining greater insight into MKT is needed. At the minimum, mathematics teacher educators can use the conception to explore and discuss the various entities that combine to represent MKT. Moving from there, the conception of MKT can serve as a road map for desired knowledge outcomes.

At a content specific level, we now know more about teaching quadratic functions. These findings can be instrumental in designing activities, whether for preservice teachers or practicing teachers, that specifically target expanding one's knowledge for teaching quadratic functions.

Teachers

At the heart of understanding MKT is the teacher. While the teacher is often the one being studied or evaluated by others, findings from my study, particularly the six questions detailed in the second manuscript, can provide the teacher with a means of self-assessment. Using the questions to engage with student written work in a strengths-based way allows one to gauge individually and privately the limits or extent of their own MKT. This work aligns with the appreciative inquiry approach that was utilized in the design of the study as the questions can be a tool to "lift up strengths at all levels" (Godwin, 2016, p. 27). The six questions, used as a tool, uplift the mathematical ideas of students while empowering teachers as they are in control of their individual MKT evaluation and plan for self-improvement.

Policy Makers

The last implication is for those in charge of creating and carrying policies. As evident from the data, many of the knowledge entities come through the work of teaching. While teacher preparation programs establish a base level of content knowledge, policy should consider and reflect the fact that teachers need time and space to develop entities of MKT. Such policies would provide teachers with adequate time early in their careers to develop robust MKT before being held accountable for having that knowledge. While my study did not address whether MKT can be developed by experience alone, other studies have shown that instructional practices do not naturally improve by experience (V. R. Jacobs et al., 2010). Therefore, policy makers should ensure teachers have spaces for professional learning that assist teachers in learning from students and practice to develop and refine their MKT.

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APPENDIX A

INTERVIEW PROTOCOLS

Interview 1: Background
**Informed Consent forms will be distributed and collected from the participant prior to the start of the interviews.*

This study aims to gain a better understanding of the knowledge domains used by experts, like you, when engaging with student work. While the exploration of knowledge domains will occur over a series of interviews, the primary focus of this first interview is learning about your education and professional experiences in the mathematics field. The interview takes about one hour. If it is ok with you, I will be taking notes and audio-taping our conversation today. Are you ready to begin?

1. Please introduce yourself to me again.
2. Tell me about your education.
   Probes:
   a) degrees held
   b) field of study
3. Please describe your professional work experiences in the mathematics field. We can start with your current position.
   Probes:
   a. Current position:
   b. Responsibilities:
   c. Time in current position:
   d. Other professional experiences (repeating a- c)
4. Have you received any rewards and/or recognition for you work in the mathematics field? Please elaborate.
5. (If currently teaching) What level (grade or college year) students do you currently teach? What topics do you teach?
6. If required to do so, how would you categorize yourself – as a mathematician, a mathematics teacher educator, or a mathematics teacher? Or a combination of any of the above? Please explain
7. What does the phrase "mathematical knowledge for teaching", MKT, mean to you?
8. Is there anything else that you think I should know about you background or thoughts regarding MKT?

Interview 2: Exploring MKT Domains: Think A-louds

This study aims to gain a better understanding of the knowledge domains used by experts when engaging with student work. This think-aloud interview will allow me to gain a greater understanding into the knowledge domains used by you to complete the tasks and review student work. As we progress, I may interrupt, asking clarifying questions if that is ok. The interview takes about one hour. If it is ok with you, I will be taking notes and audio-taping our conversation today. I am very interested in learning from you. Are you ready to begin?

1. Present participant with task. Ask participant to think aloud.
   Probes
   a. You said …. what do you mean by that?
   b. You mention ... why is that important to you?
   c. What are you thinking?

   Note: this cycle of activity/questioning may be repeated for each task)

Interview 3: Exploring MKT Domains: Continued

Several days ago, you were provided a summary from Interview #2.

1. Based on the notes from our task-think-aloud session, is there anything that you would like to modify, add, remove? Please explain.

Your participation in the think-aloud last interview enabled me to gain insight into the knowledge domains used when engaging with tasks and student work relevant to quadratic functions. The purpose of my study is to better understand the knowledge domains used by experts, such as you, when engaging with a secondary mathematics topic, quadratic functions. Research at the secondary level, regarding mathematical knowledge for teaching is sparse. Since more fruitful work has been done at the elementary level, I am using a framework from the elementary level to help guide my work.

Deborah Ball (2008) and colleagues developed the mathematical knowledge for teaching (MKT) framework (present MKT framework). Within this framework, the knowledge needed to carry out the work of teaching mathematics is categorized into two broad domains, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). These domains are further divided into subdomains: SMK into common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK); PCK into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC).

Based on this framework, I have categorized a few of your statements from the previous interview, paying particular attention to the concept of quadratic functions (present categorization). What do you think of the categorizations?

Probes:

a. What should be moved?
b. Added?
c. Deleted?

2. What comes to mind when you look at the ways in which "knowledge" has been categorized?

Interview 4: Expert Focus Group

Welcome everyone. At our last interview, we discussed the summary I composed for you individually. Based on your input from our last meeting, I have updated the summary. Please take a few minutes to review.

1. As you read through the report, what is the first thing that came to mind?
2. Is there anything that needs to be added, revised, or deleted from the report?
3. Would you like to share any reflections on your individual report? Is there anything else?

Here is composite report which summarizes my findings across all experts.

1. What is your first impression?
2. Is there anything that stands out to you? Please explain?
3. What modifications or revisions need to be made?
4. With the recommended changes, do you feel that this composite report is representative of the knowledge used when engaging with student work around quadratics? Why or why not?
5. How could you see this information being used?
6. Is there anything else?

APPENDIX B

STUDENT WRITTEN WORK SAMPLES

Table Task

Given the tables below, which one or ones represent a quadratic function. How do you know? Please explain.

A	
x	y
1	-3
2	0
3	3
4	6
5	9

B	
X	Y
1	6
2	9
3	14
4	21
5	30

C	
X	Y
-1	-1.5
0	-1
2	0
3	.5
5	1.5

D	
X	Y
0	2
1	6
2	18
3	54
4	162

Table(s) that represent a quadratic function: ______________________________

How do you know? Please explain.

Table Task – Student A

Given the tables below, which one or ones represent a quadratic function. How do you know? Please explain.

A	
x	y
1	-3
2	0
3	3
4	6
5	9

B	
X	Y
1	6
2	9
3	14
4	21
5	30

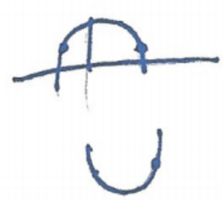
C	
X	Y
-1	-1.5
0	-1
2	0
3	.5
5	1.5

D	
X	Y
0	2
1	6
2	18
3	54
4	162

Table(s) that represent a quadratic function: none

How do you know? Please explain.

Because quadratic functions are a mirrored reflection so they look identical on each side which would mean for example



x	y
1	5
2	7
-1	5
-2	7

Table Task - Student B

Given the tables below, which one or ones represent a quadratic function. How do you know? Please explain.

A	
x	y
1	-3
2	0
3	3
4	6
5	9

+1 < ; >+3 ; >3

$m = \frac{3}{1}$

B	
X	Y
1	6
2	9
3	14
4	21
5	30

>3 ; >5 +2 ; >7 +2 ; >9 +2

C	
X	Y
-1	-1.5
0	-1
2	0
3	.5
5	1.5

+.5 ; -1 ; +.5 ; +1

D	
X	Y
0	2
1	6
2	18
3	54
4	162

>4 X3 ; >12 X3 ; >36 X3 ; >108

Table(s) that represent a quadratic function: Table B and D

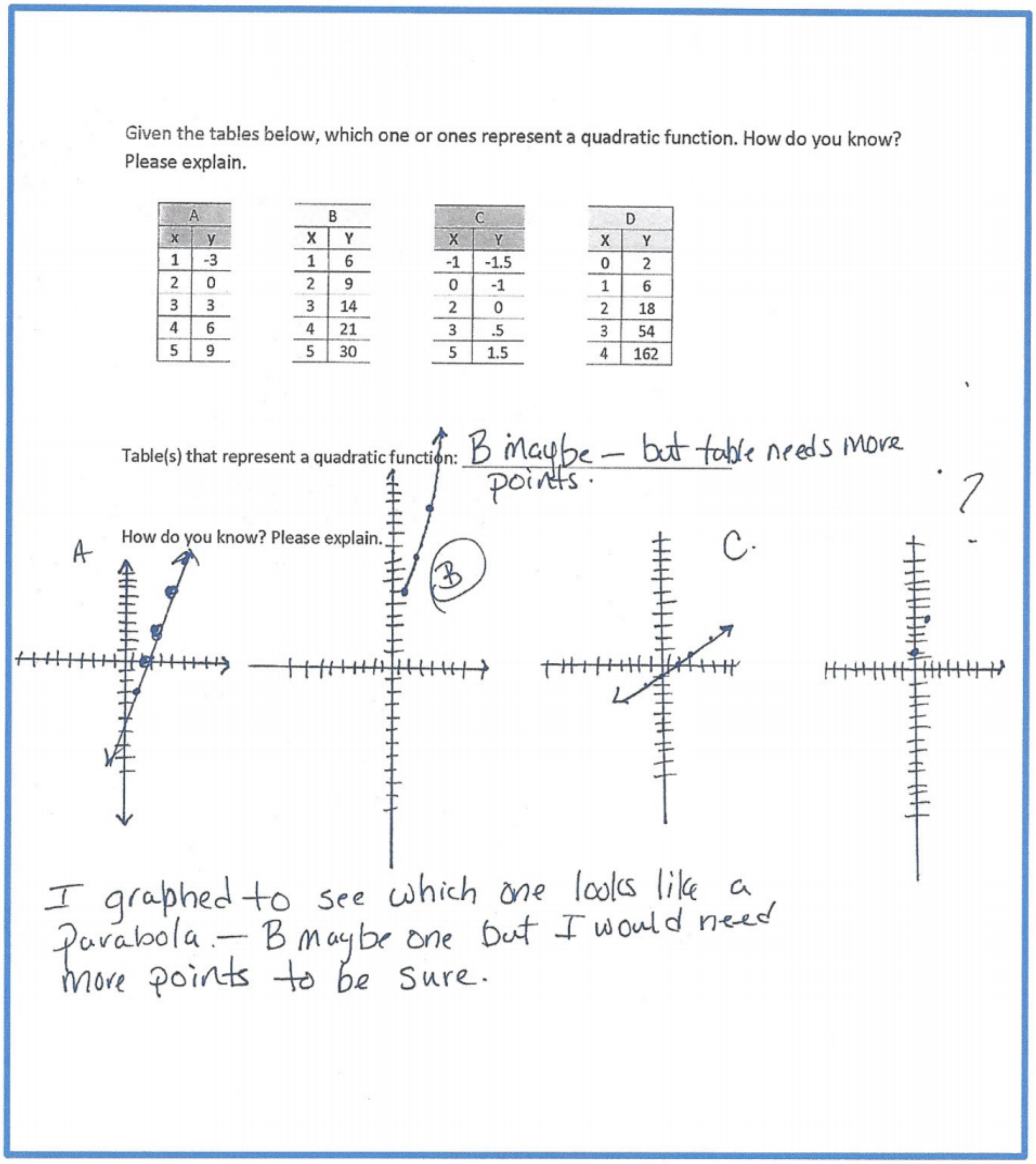
How do you know? Please explain.

The rate of Change from number to number is a Constant number

6 >3
9 ← +2
14 >5 ← +2
21 >7 ← +2
30 >9

2 >4 x3
6 >12 x3
18 >36 x3
54 >108
162

Table Task - Student C



Given the tables below, which one or ones represent a quadratic function. How do you know? Please explain.

A	
x	y
1	-3
2	0
3	3
4	6
5	9

B	
X	Y
1	6
2	9
3	14
4	21
5	30

C	
X	Y
-1	-1.5
0	-1
2	0
3	.5
5	1.5

D	
X	Y
0	2
1	6
2	18
3	54
4	162

Table(s) that represent a quadratic function: B maybe – but table needs more points.

How do you know? Please explain.

I graphed to see which one looks like a parabola. – B maybe one but I would need more points to be sure.

Review Task

1. What is an "axis of symmetry"?

2. How would you find the axis of symmetry for $f(x) = -4x^2 - 24x - 27$? Please explain (written description), then show your work.

3. Is there more than one way to find the axis of symmetry? Please explain.

4. How do you know if a function has a minimum or maximum?

5. Does $f(x) = -4x^2 - 24x - 27$ have

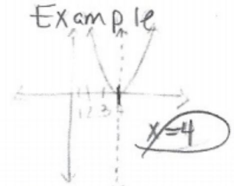
   a. a minimum or maximum?

   b. Why?

   c. What is it?

Review Task – Student D

Example



1. What is an "axis of symmetry"?

An axis of symmetry is the middle point of a graph on the x-axis

2. How would you find the axis of symmetry for $f(x) = -4x^2 - 24x - 27$? Please explain (written description), then show your work.

First I would find the vertex, using $\frac{-b}{2a}$

$\frac{24}{2(-4)} = \frac{24}{-8} = -3$

Then plug that into the x of the equation

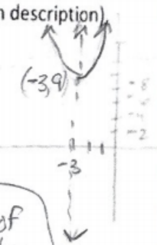
$f(-3) = -4(-3)^2 - 24(-3) - 27$

$f(-3) = -36 + 72 - 27$

$f(-3) = 9$

my x would be my axis of symmetry

axis of symmetry $x = -3$



3. Is there more than one way to find the axis of symmetry? Please explain.

Just by ~~finding~~ looking at the function and finding the center/midpoint

4. How do you know if a function has a minimum or maximum?

- First you have to plug in points to graph the function
- Then look to see if the graph has low or high points → the lowest point would be the minimum and the highest would be the maximum

5. Does $f(x) = -4x^2 - 24x - 27$ have

a. a minimum or maximum?

a minimum

b. Why?

Because (-3,9) is the lowest point on the graph

c. What is it?

$y = 9$

Review Task – Student E

1. What is an "axis of symmetry"?

a line of Symmetry for a graph. Where the sides divided can mirror each other. Normally indicated by red line.

2. How would you find the axis of symmetry for $f(x) = -4x^2 - 24x - 27$? Please explain (written description), then show your work.

h is opposite of sign

① isolate #
② half of square B
③ Add what you got to constant #
④ factor
⑤ the # in equation is axis of symmetry

$= -4x(x + 6x + 9\ ) - 27 + 36$

-36

$= -4x(x+3) + 9$

a.o.s $= x = -3$

3. Is there more than one way to find the axis of symmetry? Please explain.

yes you can find axis of symmetry by looking at the graph & it's the point that divides graph in 2 equal halves. The equation $x = -b/2a$ will help also.

4. How do you know if a function has a minimum or maximum?

If has a max the begining # will be negative. If has a min the begining # of function will be positive.

5. Does $f(x) = -4x^2 - 24x - 27$ have

a. a minimum or maximum?

maximum

b. Why?

The begining of function starts w/a negative -4 so I can tell will have a maximum value.

c. What is it?

maximum value is $(-3, 9)$

Stretching Task

9. Circle the function that would produce the widest parabola.

$f(x) = 2x^2 + 6x - 3$ $\qquad$ $f(x) = -\frac{1}{7}(x+1)^2$ $\qquad$ $f(x) = -5(x-1)^2 + 7$

Stretching Task – Student F

Circle the function that would produce the widest parabola.

a. $f(x) = 2x^2 + 6x\text{-}3$ $\qquad$ **b.** $f(x) = -\frac{1}{7}(x+1)^2$ $\qquad$ **c.** $f(x) = -5(x\text{-}1)^2 + 7$

$[-\frac{15}{2}, \infty)$ $\qquad$ $(-\infty, 0]$ $\qquad$ $(-\infty, 7]$

Stretching Task – Student H

9. Circle the function that would produce the widest parabola.

$f(x) = 2x^2 + 6x - 3$ $\qquad$ $f(x) = -\frac{1}{7}(x+1)^2$ $\qquad$ $f(x) = -5(x-1)^2 + 7$

The graph is being expanded becouse of the $-\frac{1}{7}$, it expands it by 7 times.

Solutions Task

3. a. Use the graph to solve the equation $x^2 - 2x - 8 = 0$. Mark the solutions on the graph.

x = __________ or x = ____________

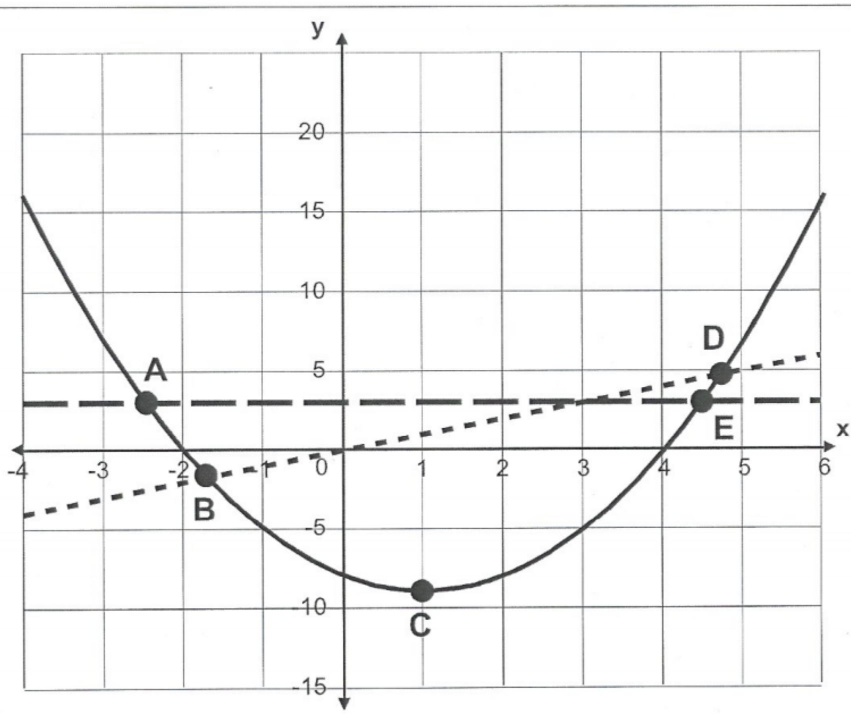
b. Use algebra to solve the same equation.

____

____

____

____



Solutions Task – Student F

3. a. Use the graph to solve the equation $x^2 - 2x - 8 = 0$. Mark the solutions on the graph.

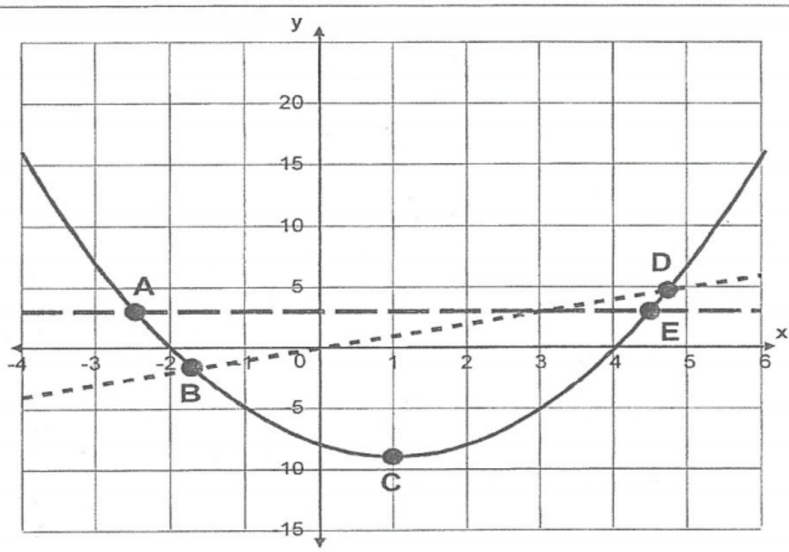
x = 4 or x = -2

b. Use algebra to solve the same equation.

$4^2 - 2(4) - 8 = 0$ $\quad$ $2^2 - 2(2) - 8 = 0$

$16 - 8 - 8 = 0$ $\quad$ $4 - 4 - 8 = 0$

$0 = 0$ $\quad$ $-8 = 0$



Solutions Task – Student L

3. a. Use the graph to solve the equation $x^2 - 2x - 8 = 0$. Mark the solutions on the graph.

x = -2.5 or x = 4.5

b. Use algebra to solve the same equation.

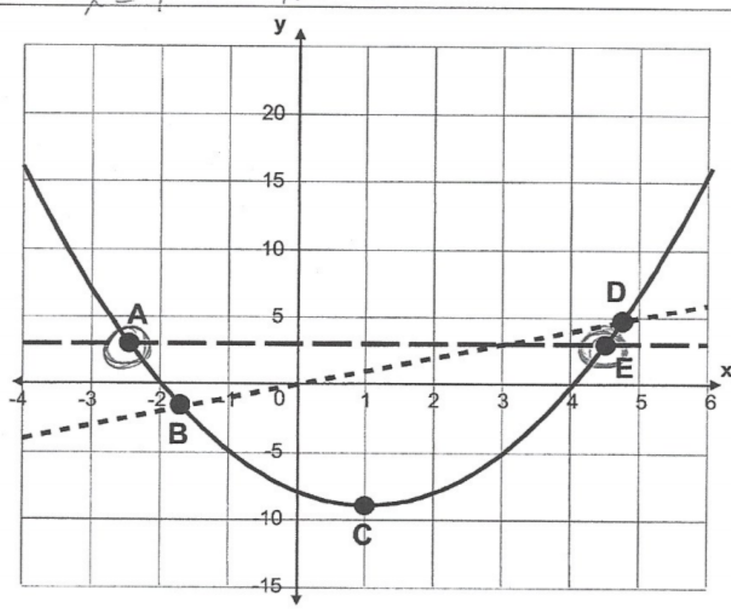
$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

check:
$x^2 + \underbrace{2x - 4x}_{-2x} - 8 = 0$ ✓

$x - 4 = 0$ $\quad$ $x + 2 = 0$

$x = 4$ $\quad$ $x = -2$



Lawnmower Task

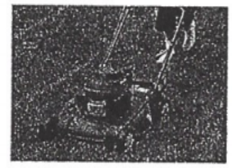
Mowing Lawns

Ali needs to save $700 by the end of the summer for senior year expenses, so she takes a job with a local lawn service. The lawn service has had trouble with teenagers quitting the job after a few days, so they devised an interesting way of paying them for summer work. First, the service only allows teenage employees to mow two lawns per day. On the first day of employment, they pay $2 per yard. On the second day, they will pay $3 per yard. On the third day, they will pay $4 per yard, and so on. Ali takes the job but she is wondering how long it will take her to earn $700. How much money has Ali made after n days of employment with the lawn service? How many days must she work to make $700?

Lawnmower Task – Student M

Mowing Lawns

Ali needs to save $700 by the end of the summer for senior year expenses, so she takes a job with a local lawn service. The lawn service has had trouble with teenagers quitting the job after a few days, so they devised an interesting way of paying them for summer work. First, the service only allows teenage employees to mow two lawns per day. On the first day of employment, they pay $2 per yard. On the second day, they will pay $3 per yard. On the third day, they will pay $4 per yard, and so on. Ali takes the job but she is wondering how long it will take her to earn $700. How much money has Ali made after *n* days of employment with the lawn service? How many days must she work to make $700?



Day	lawns	$ per lawn	$ per day	total $
1	2	2	4	4
2	2	3	6	10
3	2	4	8	18
4	2	5	10	28
5	2	6	12	40
6	2	7	14	54
7	2	8	16	70
8	2	9	18	88
9	2	10	20	108
10	2	11	22	130
11	2	12	24	154
12	2	13	26	180
13	2	14	28	208
14	2	15	30	238
15	2	16	32	270
16	2	17	34	304
17	2	18	36	340

day	1	2	3	4	5	6
$ (total)	4	10	18	28	40	54

Day	lawns	$ per lawn	$ per day	total
18	2	19	38	378
19	2	20	40	418
20	2	21	42	460
21	2	22	44	504
22	2	23	46	550
23	2	24	48	598
24	2	25	50	648
25	2	26	52	700

$\frac{700}{d} =$

Lawnmower Task – Student N

Mowing Lawns

Ali needs to save \$700 by the end of the summer for senior year expenses, so she takes a job with a local lawn service. The lawn service has had trouble with teenagers quitting the job after a few days, so they devised an interesting way of paying them for summer work. First, the service only allows teenage employees to mow two lawns per day. On the first day of employment, they pay \$2 per yard. On the second day, they will pay \$3 per yard. On the third day, they will pay \$4 per yard, and so on. Ali takes the job but she is wondering how long it will take her to earn \$700. How much money has Ali made after *n* days of employment with the lawn service? How many days must she work to make \$700?

day 1 4
day 2 6
3 8
4 10
5 12
6 14
7 16

20 + 20 + 20 = 70

4
6
8
10
12
14
16
18
20
22
24
26

30 30 30 30

6×30=180

1 4
2 6
3 8
4 10
5 12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
46
48
50
25 52

56

54

11×50=550

12×54=648

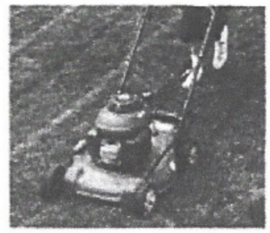
54
48
600
648

648+52=700

25 days

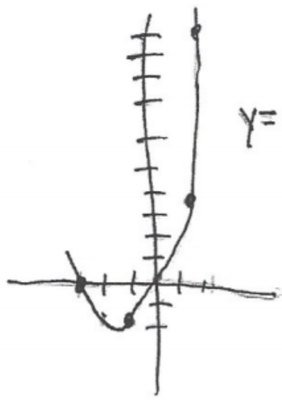
Lawnmower Task – Student O

Mowing Lawns



Ali needs to save $700 by the end of the summer for senior year expenses, so she takes a job with a local lawn service. The lawn service has had trouble with teenagers quitting the job after a few days, so they devised an interesting way of paying them for summer work. First, the service only allows teenage employees to mow two lawns per day. On the first day of employment, they pay $2 per yard. On the second day, they will pay $3 per yard. On the third day, they will pay $4 per yard, and so on. Ali takes the job but she is wondering how long it will take her to earn $700. How much money has Ali made after *n* days of employment with the lawn service? How many days must she work to make $700?

x	y	
1	4	4
2	6	10
3	8	18
4	10	28
5	12	40



$y = (x+0)(x+3)$

$x(x+3) = 700$

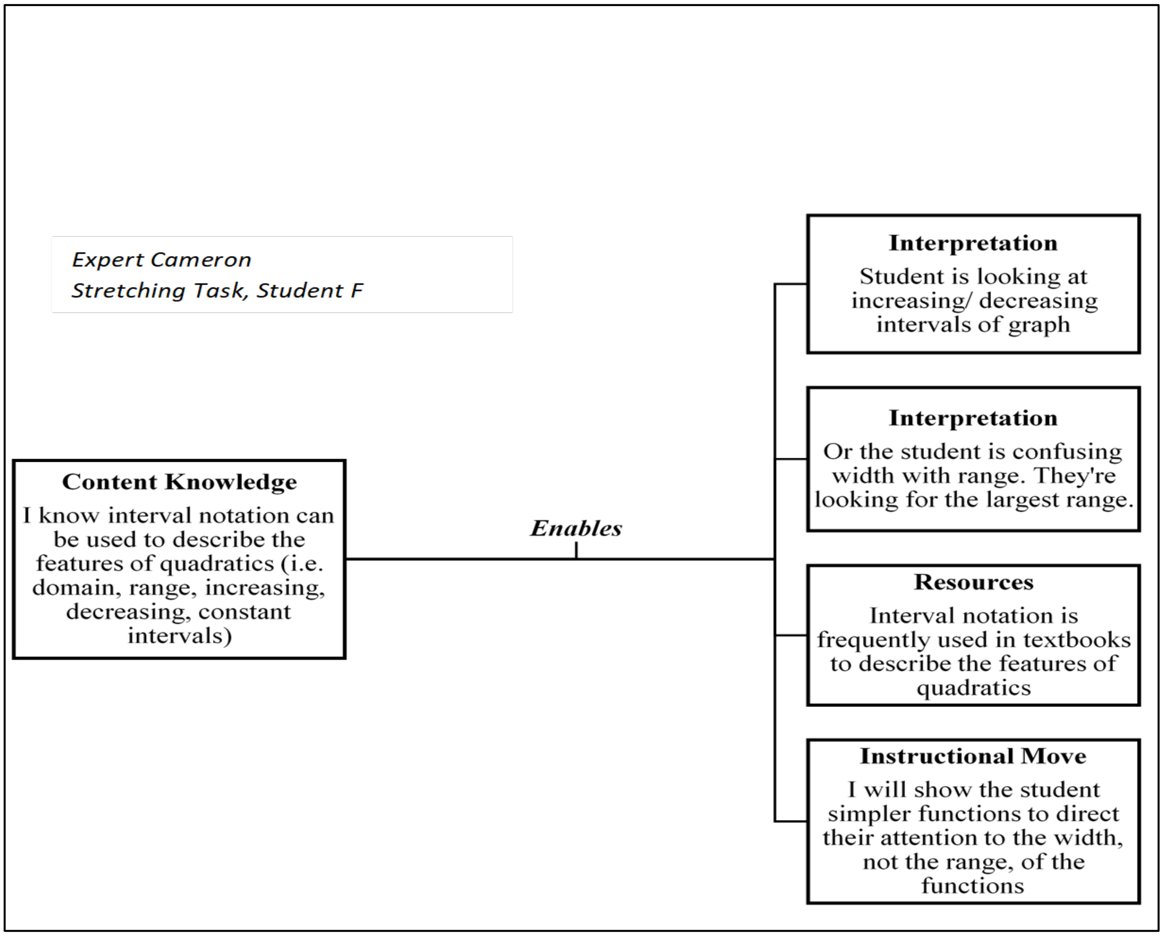
$x = 700 \qquad x+3 = 700$
$\quad -3$
$x = 697$

0	0	0
-1	-2	-2
-2	-1	-2
-3	0	0

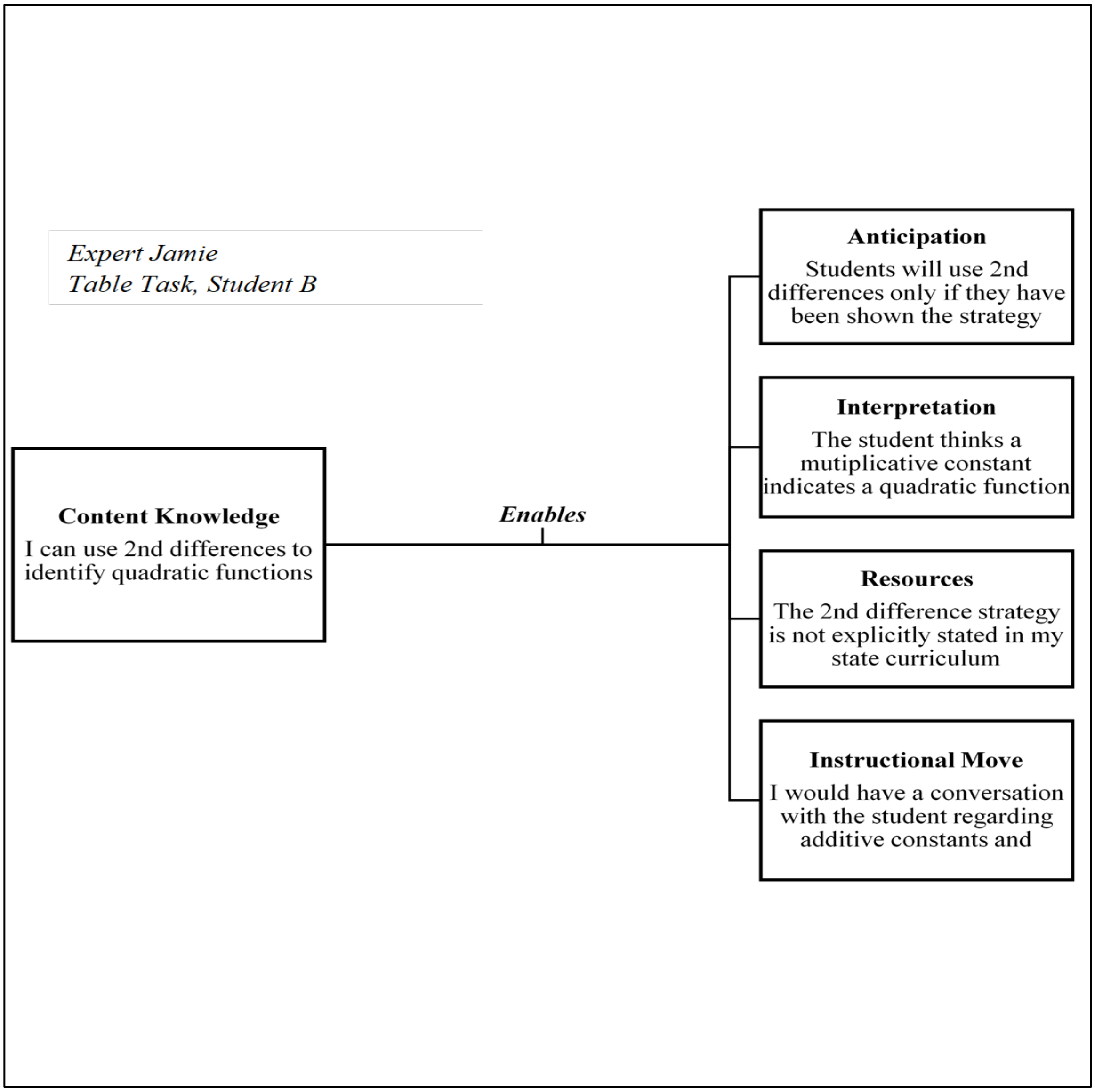
APPENDIX C

EXPERT CONTENT KNOWLEDGE ENABLER TRACES

Cameron

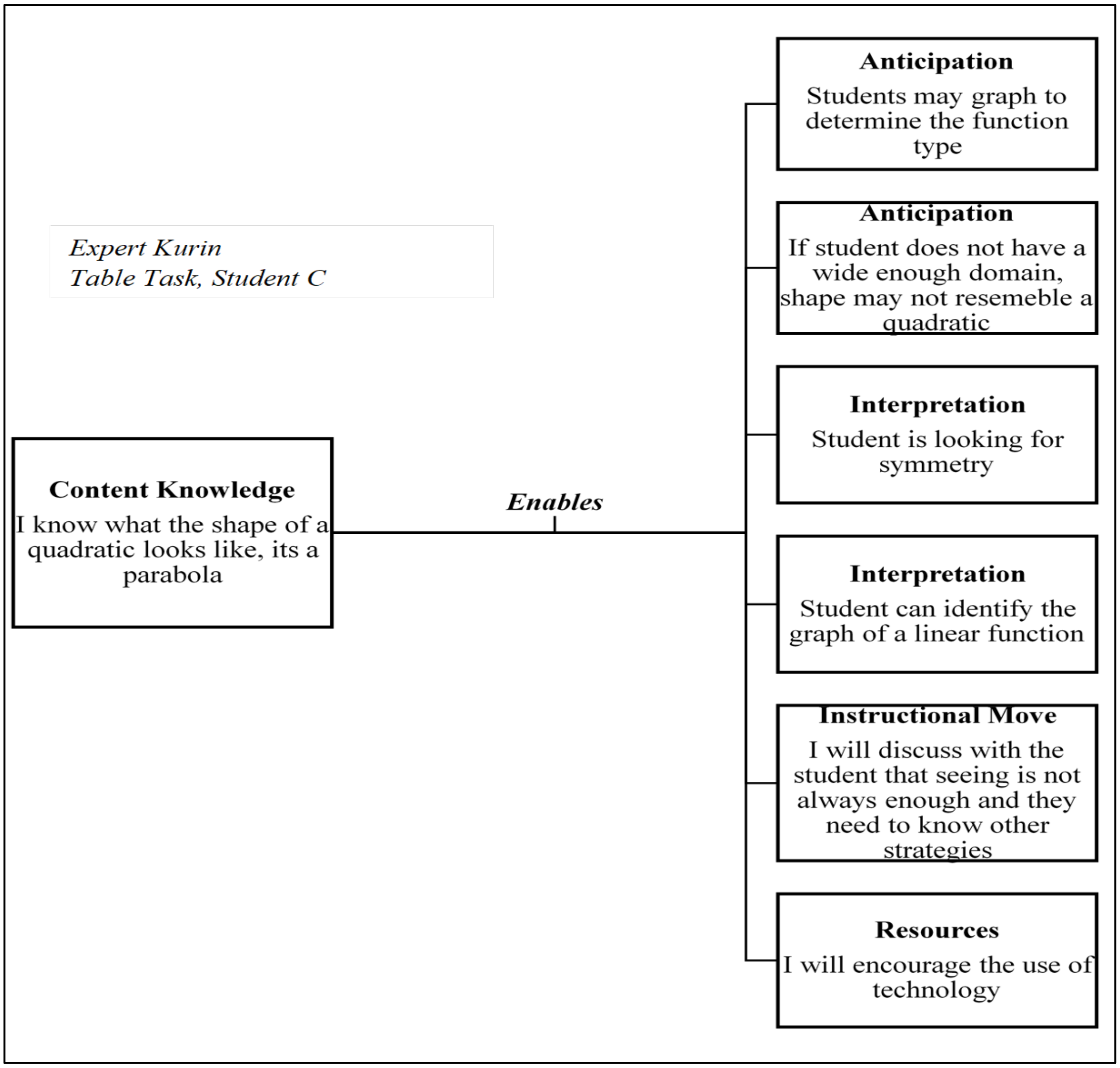


Jamie



Expert Jamie
Table Task, Student B
Content Knowledge
I can use 2nd differences to identify quadratic functions
Enables
Anticipation
Students will use 2nd differences only if they have been shown the strategy
Interpretation
The student thinks a mutiplicative constant indicates a quadratic function
Resources
The 2nd difference strategy is not explicitly stated in my state curriculum
Instructional Move
I would have a conversation with the student regarding additive constants and

Kurin


Anticipation
Students may graph to determine the function type
Expert Kurin
Table Task, Student C
Anticipation
If student does not have a wide enough domain, shape may not resemeble a quadratic
Interpretation
Student is looking for symmetry
Content Knowledge
I know what the shape of a quadratic looks like, its a parabola
Enables
Interpretation
Student can identify the graph of a linear function
Instructional Move
I will discuss with the student that seeing is not always enough and they need to know other strategies
Resources
I will encourage the use of technology

APPENDIX D

CODEBOOK FOR MKT FOR QUADRATIC ENTITIES

MATH	**Content Knowledge**	**Definition**	Knowledge of quadratics (functions and equations); the mathematical ideas, concepts, definitions, and procedures foundational to quadratics
		Examples	— *My definition of a quadratic is $f(x) = ax^2+bx+c$, where a is not equal to zero.* — *The parabolic shape has symmetry.* — *To recognize quadratics from table data, I would look for a constant second difference.* — *Based on the coefficient of x^2, whether it is positive or negative would indicate if the function is opening up or down.* — *I could use the quadratic formula, complete the square, or factoring to determine the zeros of the quadratic function.* — *Different kinds of shifts are based the coefficients*
	Connections	**Definition**	Knowledge of mathematical concepts that inform quadratic learning and understandings; knowledge of mathematical concepts informed by quadratics
		Examples	— *If students know linear functions, they could recognize a constant rate of change and determine which of the functions are not quadratic.* — *It is going to be important for a student to be able to work all three of these [quadratic] representations- the table, the graph, and the quadratic equation, because we are going to do that in Calculus.* — *Being able to identify patterns in data will help with identifying quadratic function table data.* — *Knowing the rate of change of a quadratic function is a linear function, a cubic function is quadratic function, quartic is a cubic, and so on, sets the needed foundation for understanding derivatives*

STUDENT	**Interpretations**	**Definition**	Knowledge of mathematical ideas/concepts represented in student work; interpreting/unpacking/explaining the mathematical ideas present in student work.
		Examples	— *This student computed the 1st and 2nd differences.* — *The student understands what a line of symmetry is from how they describe it.* — *This student used the graph to find the solution but evaluated at 2 instead of -2* — *The student is finding the sums and then doubling the sum.* — *This student is talking about the point, the vertex, as opposed to the maximum value of the function.*
	Anticipations	**Definition**	Knowledge of anticipated student methods, strategies, procedures, misconceptions, etc.; Knowledge of student justifications for methods, strategies, procedures, etc.
		Examples	— *Since they did not see that symmetry in any of the tables, they are going to answer that there are no quadratic functions represented in the table data.* — *A student might use the vertex formula to find the axis of symmetry.* — *They graphed this way probably because that is the way that they've seen parabolas most often graphed, opening up.* — *I would expect students to make a table first.* — *I don't think they would go to second differences unless that is something that was explicitly taught to them*
TEACHER	**Instructional Moves**	**Definition**	Knowledge of educative reasoning for next teaching move/learning technique to promote learning; reasoning and/or purposes regarding the given tasks, suggested next tasks, and/or next suggested instructional move

		Examples	— *I am going to help with the language around their solution by asking "what do you mean by middle point?"* — *I would ask them to graph and then confirm their solution.* — *I want the student to walk me through their backing up part.* — *I will start with a simpler function to determine the shifts and then move to more complex functions.* — *I am giving the student problems that connect solutions across multiple representations.*
	Resources	**Definition**	Knowledge of instructional materials and resources, such as technology and curricula, that aid in student understanding; suggested materials and resources to assist in student learning
		Examples	— *Using a calculator to graph would be more efficient.* — *Using XXX software is a great way to visually show shifts in the graphs.* — *The math task in YYY curriculum encourages the use of multiple quadratic function representations*