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THE DEVELOPMENT OF MATHEMATICAL UNDERSTANDING AND ITS APPLICATION TO LIBYAN SECONDARY SCHOOL MATHEMATICS

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THE DEVELOPMENT OF MATHEMATICAL UNDERSTANDING AND ITS APPLICATION TO LIBYAN SECONDARY SCHOOL MATHEMATICS

by

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APPROVAL PAGE

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ABSTRACT

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It is unfortunate that many students in Libyan schools and elsewhere dislike mathematics and are even afraid of studying it. This is partially related to the fact that mathematics is presented in an extensive fashion in both scope and depth of content and does not provide opportunities for the student population to understand it.

The primary purpose of this study was to examine the problems of the method of presenting mathematics in Libyan secondary schools and to develop principles for understanding mathematics. There were four subordinate problems inherent in the main problems: 1) to study and analyze the characteristics of the new mathematics program in terms of content and method of presentation; 2) to study the problems of Libyan secondary school mathematics, specifically to determine weaknesses in the content and method of presentation; 3) to explore and develop criteria for understanding mathematics; and 4) to evaluate method of presenting mathematics in Libyan secondary schools in terms of the developed criteria.

Information for this study was gathered and presented by different methods. Chapter I provided a general background of the problem.

Chapter II included a general review of the international development of mathematics education. Chapter III presented

analysis and critique of the Libyan secondary school mathematics. Chapter IV attempted to develop criteria for understanding mathematics. Chapter V described the experimental procedures.

Six hypotheses were utilized for this study and were tested by analyzing response between form A and form B, using Binomial test of proportions procedures. The results indicated there were significant differences between the two presentations, and significantly more teachers preferred method of presentation B than method of presentation A.

Based on the analysis of the Libyan Mathematics program and the evaluation of method of presenting mathematics in the secondary school, this study reached the conclusion that the Libyan secondary school mathematics program needs to be modified to provide opportunities for secondary school students to understand mathematics and to be in accord with international school mathematics.

The writer made 17 recommendations to the Libyan mathematics education specialists to undertake a revision of the entire secondary school mathematics program in the country, using this study as a possible guide.

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CHAPTER I

BACKGROUND TO THE STUDY

The Problem

Most teachers become aware through their classroom experiences that there are important differences between the student who has learned by rote and the student who has learned with understanding or "meaningfully." There are differences in what these two kinds of students have learned, and there are consequent differences in what they can do with what they have learned. The belief that mathematics should be learned by understanding rather than by rote was one of the main motivations leading to the development and instruction of the new mathematics programs into primary and secondary schools.

The revision made in Libyan secondary school mathematics in 1969 required introduction of new topics to the mathematics program. In algebra, new topics such as sets, algebraic structure, group theory, field theory, ring theory, isomorphism, etc., were included. In geometry transformation, three dimensional geometry and vectors took a great part of the curriculum. Probability and statistics were introduced for the first time to grade 12.

However, the Libyan reformers presented the materials in an extensive fashion in both scope and depth of content and did not provide opportunities for the student population to understand them.

Methods of presenting mathematics in both textbook and classroom are still focusing on how to do mathematics, and the main job of the teacher is to tell the students statements and explain processes to memorize on the examination day.

Bruce Meserve said that "mathematics as a mass of memorized facts is deadly; mathematics is a study of systems of symbols capable of presenting important properties of the physical universe and can be a vibrant and living subject" (Meserve, 1958, p. 716).

Professor Braunfeld of the Comprehensive School Mathematics program added, "A student has simply been short-changed if after nine or twelve years' study of mathematics he leaves school with the notion that mathematics consists of a large collection of get-correct answer-to certain, usually contrived questions" (Braunfeld, n.d., p. 8). Too much emphasis on content has caused mathematics teachers to lose sight of the student.

In analyzing the method of presenting mathematics in Libyan mathematics textbooks, which reflects method of instruction, the investigator noticed that the method of presentation is of an expository nature with no attempt to include opportunity for any kind of participation from the student. Skills are given greater emphasis and no attention is given to the development of the concept. As an example a student easily can calculate the standard deviation but he is not able to apply it to a real situation or to explain what the answer means. The investigator noticed this fact when he made personal observations in the classroom.

It is unfortunate that many students in Libyan schools and elsewhere dislike mathematics and are even afraid of it. This is partially related to three important factors: 1) The curriculum contains a great quantity of material which has to be covered in a short time at the expense of quality; 2) Students are forced to learn a certain body of knowledge which school systems or the Ministry of Education feel they ought to know; 3) Because students are given limited chances to understand the materials, they may be able to produce answers to problems or questions by heart or by following rules, but they have little if any understanding of the principles.

Mathematics must be presented in a way to ensure understanding of the 'why' as well as the 'how' of mathematics. More emphasis must be placed on motivating students, generating positive attitudes, discovering, participating, and encouraging students to be creative.

Professor Holt said, "Pupils are bored because the things they are given and told to do in school are so trivial, so dull, and make such limited and narrow demands on the wide spectrum of their intelligence, capability, and most of the torrent of words that pours over them in school makes little or no sense" (Holt, 1964, pp. xiii - xiv).

Statement of the Problem

The purposes of this study are to examine the problems of the method of presenting mathematics in Libyan secondary school, to explore principles for understanding mathematics, and to develop criteria to evaluate the method of presenting mathematics in Libyan secondary school.

Subordinate Problems:

The subordinate problems inherent to the main problem are listed below:

1. To study and analyze the characteristics of the new mathematics program in terms of content and method of presentation.
2. To study the problems of Libyan secondary school mathematics, specifically to determine weaknesses in the method of presentation of mathematics in both textbook and classroom.
3. To explore and develop criteria for understanding mathematics.
4. To evaluate the method of presenting mathematics in Libyan secondary school in terms of the criteria identified in Chapter IV.

Definition of Terms

As used in this study, certain specific or technical terms are defined as follows:

Preparatory school: Preparatory school is a high school level consisting of seventh, eighth, and ninth grades.

Secondary school: Secondary school is a high school level consisting of tenth, eleventh, and twelfth grades.

Method of Presentation: By method of presentation is meant both method of presenting mathematics in textbooks and method of instruction.

Expository Teaching: The expository approach of teaching mathematics is based on exposing the students to certain phases of the subject matter.

All other specific or technical terms used in this study are defined as and where used.

Collection of Materials

The following primary and secondary source materials were used in this study.

The investigation of primary source material was accomplished as follows:

1. The investigator visited Libya many times to study and inspect the method of presentation of mathematics in secondary schools.
2. Personal observations in the classroom were made to get information about the method of teaching mathematics.
3. Personal interviews were conducted in Libya with mathematics teachers and administrators.
4. A questionnaire was made for Libyan secondary school mathematics teachers to get a general opinion about mathematics education in Libya.
5. An experimental study was conducted to evaluate the method of presentation of mathematics in Libyan secondary school mathematics program.

The secondary source materials were utilized as follows:

1. Investigation of many international school mathematics projects was undertaken.

2. Relevant books, journals and articles were used to provide a general background information to the study.
3. Theses, dissertations, and papers of material dealing with teaching mathematics and learning mathematics were examined through the Dissertation Abstracts and the Educational Resources Information Center (ERIC).

Method and Procedures

Data for each chapter have been obtained by a variety of methods each unique to the chapter.

In Chapter II, Review of the Related Literature, a survey of materials including books, periodicals, professional journals, newsletters, and unpublished papers was used to investigate the new mathematics curriculum.

The research for Chapter III was mainly in the Libyan secondary school mathematics program. This included a study of the traditional and the present mathematics programs. An analysis and critique of the present mathematics program were made by investigating the official mathematics textbooks and by a questionnaire distributed throughout the country to secondary school mathematics teachers.

Chapter IV attempts to develop principles for a more effective way to teach mathematics in secondary school. These principles are called criteria for understanding mathematics. The criteria are identified and developed from the information that was obtained from Chapters II, II, and IV.

In Chapter V experimental procedures were conducted to evaluate a selected topic from the Libyan secondary school program in terms of some of the criteria that developed in Chapter IV.

Chapter VI provides concluding recommendations based on materials presented in the preceding chapters.

Limitation

The findings and conclusions of this study were limited to the Libyan secondary school level, grades 9-12. However, this was a deliberate choice on the part of the investigator who wished to concentrate on the secondary-school level.

Design of the Study

This chapter describes the problem of this investigation, the statement of the problem and the subordinate problems are stated. Technical terms are defined. The sources of information are explained. An overview of the experimental method is outlined. The design of the problem is explained.

The second chapter is devoted to a review of the related literature which includes an introduction to the major causes of the revolution of the new mathematics that took place in the mid 1950's, the characteristics of the new mathematics, discussion of some mathematics projects in several countries, followed by a brief summary of the chapter.

An analysis and a critique of the method of presentation of mathematics in Libyan secondary school is presented in Chapter III.

In Chapter IV, principles for understanding mathematics are identified and discussed.

Chapter V includes the methodology of the experiment. It includes Form A which represents a translation of the topic -- sum of 1 + 2 + 3 + ... + n -- as it appeared in the Libyan book, and Form B which represents a method of presentation of the same topic that was developed by the investigator, and also includes the questions to be answered by the subject.

The final chapter contains the investigator's summary, conclusion, and recommendations.

CHAPTER II

REVIEW OF THE RELATED LITERATURE

Introduction

This review consists of five sections. It begins first with a discussion of the international development of mathematics education from 1900 to 1950. The second section focuses on the revolution of the new mathematics and its causes. The third section of the review provide analysis of some international mathematics programs. The fourth section presents the main characteristics of the new mathematics. Criticism of the new mathematics is the topic of concern for the fifth and last section of the review.

International Development of Mathematics Education

During the past eighty years mathematicians have struggled for educational reform. In that period one of the major objectives of teaching mathematics was to teach children some rules, formulas, and facts and make them able to memorize them.

In school mathematics of that period, arithmetic and learning mathematics meant spending hundreds of hours on drilling computational skills, memorizing what the teacher said in the classroom.

The summary of the historical development of mathematics education from 1900 to 1950 can be sketched as follows:

In 1908 the Committee of Fifteen on Geometry syllabus was organized to serve two purposes: 1) the tendency to present a long chain of theorems with few applications, and 2) the placing of an extensive number of exercises at the end of the book.

In 1911 the Report of the International Commission on the Teaching of Mathematics came as a result of the International Congress of Mathematicians meeting and recommending: 1) to survey the status of mathematics and the teaching of mathematics in various countries; 2) some geometric content was being omitted; 3) the sequence of topics in algebra was being rearranged; 4) utilitarian aims were becoming increasingly more important, and 5) the formal-discipline concept of education was being questioned (Osborne & Crosswhite, 1970, p. 183).

In 1916 the National Committee on Mathematics Requirements (NCMR) was organized to accomplish this purpose "the teaching of mathematics should be to develop those powers of understanding and analyzing relations of quantity and space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of individuals" (NCMR, 1923, p. 10).

In 1920 the National Council of Teachers of Mathematics was organized to promote better teaching, especially at the elementary and secondary school.

In 1923 the Report of the National Committee on Mathematics Requirements stressed five major areas of emphasis: 1) it defended the purpose of mathematics in secondary education; 2) the importance of transfer of learning; 3) the function concept was recognized as a unifying idea of a mathematics course; 4) the report stated content requirements for mathematics courses, and 5) model curricula from the United States and abroad.

In 1940 the Commission on the Secondary School Curriculum of the Progress of Education presented a report emphasizing the following areas: First, the teacher must guide the student in restricting the mathematical problem; second, the student must learn how to collect and analyze data; third, the student must learn approximation quantities along with the proper understanding of words such as precision, accuracy and rounding off; fourth, the student must learn and understand the concept of function to see how the concept has contributed to scientific field; fifth, understanding the nature of operations involved in arithmetic and algebra must be developed; sixth, the concept of proofs should be presented to students in the broadest sense, they must appreciate that some things are impossible to prove and that proofs of an inductive nature are valid; and seventh, students must know that symbols have a variety of meanings.

The second part of the Commission of Postwar Plans, published in 1945, covered grades 1 to 12 and the junior colleges listed thirty-four theses for the improvement of school mathematics. The following are some of the published theses: (1) The school should guarantee functional competence mathematics to all who can possibly achieve it. (2) More emphasis and much more careful attention must be given to the development of meanings. (3) It must be realized that readiness for learning arithmetical ideas and skills is primarily the product of relevant experience, not the effect of merely becoming older. (4) Drill (repetitive practice) must be administered much more wisely.

(5) Learning in arithmetic must be evaluated more comprehensively than is common practice. (6) The mathematics for grades 7 and 8 should be planned as a unified program and should be built around a few broad categories. (7) The mathematics program of grades 7 and 8 should be so organized as to enable the pupils to achieve mathematical maturity and power. (8) The sequential courses should be reserved for those pupils who, having the requisite ability, desire or need such work. (9) Teachers of the traditional sequential courses must emphasize functional competence in mathematics. (10) The main objective of the sequential courses should be to develop mathematical power. (11) The work of each year should be organized into a few large units built around key concepts and fundamental principles. (12) Simple and sensible applications to many fields must appear much more frequently in the sequential courses than they have in the past. (13) New and better courses should be provided in the high schools for a large fraction of the school's population whose mathematical needs are not well met in the traditional sequential courses. (14) The small high school can and should provide a better program in mathematics. (15) The junior college should offer at least one year of mathematics which is general in appeal, flexible in purpose, challenging in content, and functional in service. (16) The teacher of mathematics should have a wide background in the subjects he will be called upon to teach. (17) It is desirable that a mathematics teacher acquire a background of experience in practical fields where mathematics is used. (18) Provision should be made for the continuous education of teachers in service. (19) The resourceful teacher of mathematics should be given competent guidance in the production, selection, and use of slides and films (Bidwell & Glason, 1970, pp. 618-652).

After the Second World War more attention has been focused on renovating school mathematics; from this renovation the "new math" was born.

The Revolution of the New Mathematics

A real change in mathematics education has taken place in the mid 1950's, marked by the involvement of (UNESCO), the Organization for European Economic Cooperation (OEEC), and the Organization for Economic Cooperation and Development (OECD), with the participation of mathematicians, mathematics educators, curriculum specialists, psychologists, researchers, and classroom teachers. The major causes of this revolution as presented in the summary and conclusions of the Conference of the Organization for European Economic Cooperation were:

1. Mathematics is an "exploding" field of knowledge. More new mathematics has been developed during the past decade than in any period of its entire history. "The new development in graduate and research mathematics implies a necessary shift in emphasis for secondary school mathematics. New topics such as subtract algebra, vector spaces, theory of sets, etc. will enter the school program causing a changed point of view on what mathematics is today" (OEEC, 1961, p. 107).

2. As a result of creating new topics, new applications of mathematics suggest introducing "new problem material. Probability, statistical inference, finite mathematical structure, linear programming, numerical analysis--all indicate expansion in use applications of mathematics" (OEEC, 1961, p. 107).

3. Today, mathematics is not only applied to the sciences, but also to the humanities, e.g., psychology, social sciences, linguistics, and even to the arts. This wide-spread application of mathematics has partly been made feasible by the development of the electronic computer.

4. Expansion in the use of mathematics creates new standards of accuracy. "The development of a new standard of accuracy and clarity of statements and the emphasis on mathematical structures indicate a need for reconsideration of the concepts embodied in the classical treatment of mathematics" (OEEC, 1961, p. 107).

5. Traditionally, teaching methods in mathematics tended to generate dislike for mathematics, whereas the new methods will promote interest and competence by stressing the importance of understanding and individual achieving.

When these causes become known, international organizations such as UNESCO, the Organization for European Economic Cooperation (OEEC), the Organization for Economic Cooperation and Development (OECD) and other organizations such as the National Council of Teachers of Mathematics (NCTM), and the Commission on Mathematical Institute (CMI) organized seminars and conferences on the reform of mathematics education. The first active report was by the College Entrance Examination Board (CEEB) in 1959 and recommended that content in mathematics must be appropriate to the level of maturity of the student.

Students studying college preparatory mathematics should be taught in groups with similar interests and similar intellectual abilities.

CEEB believes that the content of mathematics must include applications of mathematics in social science, as well as applications of mathematics in industry. CEEB suggested the following content for Grades 7-12.

Arithmetic

Fundamental operations and numeration. Mastery of the four fundamental operations with whole numbers and fractions, written in decimal notation and in the common notation used for fractions; an understanding of the rationale of the computational processes; understanding of a place system of writing numbers, with use of binary notation (and perhaps other bases) to reinforce decimal notation; the meaning and use of an arithmetic mean. In addition, a knowledge of square root and the ability to find approximate values of square roots is desirable. (The process of division and averaging the divisor and quotient-Newton's method is suggested.)

Ratio. Understanding of ratio as used in comparing sizes of quantities of like kind, in proportions, and in making scale drawings. Per cent as an application of ratio; understanding of the language of per cent (rate), percentage, and base. In particular the ability to find any one of these three designated numbers, given the other two; ability to treat with confidence per cents less than 1 and greater than 100; applications of per cent to business practices, interest, discount, and budgets should be given moderate treatment.

Geometry

Measurement. The ability to operate with and transform the several systems of measure, including the metric system of length, area, volume, and weight; geometric measurements, including length of a line segment, perimeter of a polygon, and circumference of a circle, areas of regions enclosed by polygons and circles, surface areas of solids, volumes of solids, measure of angles (by degrees); the use of a ruler and protractor. The student should know the difference between the process of measuring and the measured quantity. Ability to apply measurement to practical situations. Use of measurement in drawing to scale and finding lengths indirectly.

Relationships among geometric elements. These include the concepts of parallel, perpendicular, intersecting, and oblique lines (in a plane and in a space); acute, right, obtuse, complementary, supplementary, and vertical angles; scalene, isosceles, and equilateral triangles; right triangles and the Pythagorean relations; sum of the interior angles of a triangle; the use of instruments in constructing figures; ideas of symmetry about a point and a line.

Algebra and Statistics

Graphs and formulas. Use of line segments and areas to represent numbers. Reading and construction of bar graphs, line graphs, pictograms, circle graphs, and continuous line graphs; meaning of a scale formulas for perimeters, areas, volumes, and per cents-introduced as generalizations as these concepts are studied. Use of symbols in formulas as placeholders for numerals arising in measurement; simple expressions and sentences involving "variables."

The second important movement of the revolution of the new mathematics was the International Seminar on Mathematics Education which was held at Budapest, Hungary, in September 1962 under the sponsorship of UNESCO, with participants from seventeen countries (Australia, Belgium, Canada, Denmark, U.S.A., Sweden, Switzerland, Czechoslovakia, France, Hungary, Italy, Japan, Netherlands, Poland, Romania, England, and U.S.S.R.). The discussion of this international meeting focused on three main topics (Fehr, 1966, p. 37).

The Materials of Instruction. The means of organizing modern instruction; number of hours of study devoted to mathematics: the conditions for working in the class; the subject matter of instruction; a common trunk; goals relative to basic concepts; research on materials for instruction; the need for experimentation; use of pilot experimental classes; resume of past experiments.

The Teaching of Mathematics. Steps leading to the learning of mathematics; conditions favorable to the learning of mathematics; motivation and learning; ways and means of learning; the development of mathematical thought-problems of conceptualization, the proof used in the instruction, and the role of problems; materials and games; the use and role of symbolism; cooperation among mathematicians, educators, and psychologists.

The Training and Retraining of Instructors. The mathematical preparation of teachers; the pedagogical preparation of teachers; the continued education of teachers; the shortage of instructional personnel in mathematics.

Recent International Development in School Mathematics

In this section, an attempt will be made to discuss and analyze some international mathematics programs. To make this presentation more convenient, these projects were analyzed and discussed in terms of goals and objectives, content, and method of presentation.

The University of Illinois Committee on School Mathematics (UICSM)

In December 1951, the Colleges of Education, Engineering, and Liberal Arts and Sciences established a committee (UICSM) to improve the high school mathematics. The mathematics project of the University of Illinois Committee on School Mathematics was directed by Max Beberman. The main objectives of the Illinois program were: (1) A consistent presentation of high school mathematics can be devised, (2) Students are interested in ideas, (3) Manipulative tasks should be used mainly to allow insight into basic concepts, (4) The language should be as unambiguous as possible, (5) The organization of materials should provide for student discovery of many generalizations, and (6) The student must understand his mathematics (Beberman, 1958, p. 4).

The program consisted of four courses to be used in grades 9-12. The content of the courses was as follows (Studies in Mathematics, 1960, 33-34). First Course: The first course included distinction between numbers and numerals, real numbers, principles of real numbers (associativity, commutativity and distributivity), inverse operation, relations of inequality, numerical variables ("pronumerals"), generalizations about real numbers, notation and some concepts of the algebra of sets, solution of equations, linear and quadratic, solution of "worded" problems, and ordered pairs of numbers.

Second Course: The second course consisted of sets and relations, linear and quadratic functions, systems of linear equations, measures of internals, arcs, angles, and plane regions, elementary properties of angles, polygons, and circles, and further study of manipulations of algebraic expressions.

Third Course: The third course included mathematical induction (generalizations, hereditary properties, recursive definitions, progressions, sigma-notation), exponents and logarithms (continuity and the limit concept, geometric progressions, the binomial series), complex numbers (field properties, systems of quadratic equations), and polynomial functions (the factor theorem, synthetic division, curve tracing).

Fourth Course: The fourth course consisted of circular functions (winding functions, periodicity, evenness and oddness, monotoneity, "analytical trigonometry" rather than "triangle solving," inverse circular functions), and deductive theories (abstraction of postulates from a model deduction of theorems from these postulates without reference to a model, reinterpretation of the theory to yield information about other models).

One of the fundamental concepts of this program was to help students to explore mathematics and to guide them to discover and generalize the mathematical idea. In Unit 6 the text strongly emphasized the presentation of a sequence of activities from which students may come to independently recognize the desired knowledge. In some units, UICSM placed great emphasis upon the development of concepts.

The development of understanding of concepts occupied more attention than the development of skills, The development of skills was not neglected, however, in Unit 7 and Unit 8 the pattern of presentation used in the development of an idea followed by a large number of exercises to provide theoretical mathematical problems.

Although UICSM tried to include applications over many practical and real life situations, routine problems appeared in many units as a major application. For example, in Unit 5, functions and relations were not presented in a real life situation, and in Unit 7 and Unit 8, mathematical induction and sequences did not use any kind of motivation or practical situations. It seems to be that UICSM did not consider real situations and practical problems as a major objective at this level.

In 1960 the material of the University of Illinois Committee on School Mathematics was used experimentally in 25 states by 200 teachers and 10,000 pupils (Beberman, 1962, p. 4). In 1962, the UICSM developed three new areas, a junior high school informal geometry course, a vector geometry course, and an arithmetic of fractions sequence for seventh and eighth grade low achievers.

<u>School Mathematics Study Group (SMSG)</u>

In 1958 the School Mathematics Study Group (SMSG) was organized by the National Council of Teachers of Mathematics and the National Science Foundation under the direction of E. G. Begle. The primary purpose of SMSG, was "to foster research and development in the teaching of school mathematics" (SMSG, 1963, p. 6). In connection with its general objective of improving mathematics teaching in schools, SMSG stated three specific objectives (1) to offer students not only the basic mathematic skills but also a deeper understanding of the basic concepts and structure of mathematics; (2) to attract and train more of those students who are capable of studying mathematics with profit; (3) to provide needed assistance for teachers who are preparing to teach mathematics (SMSG, 1963, p. 7). The chapter headings of the books on the high school program are as follows

Volume I: Volume I consists of the following eight chapters (SMSG, 1965, Part

Organizing Geometric Knowledge: Inductive and deductive reasoning, definitions and undefined terms, historical background, the Pythagorean theorem, the connection postulates, the linear measurement postulates, the separation postulates, the angle measurement postulate.

Concepts and Skills in Algebra: Introduction to postulates of real numbers, identity postulates and additive inverse, polynomials and factoring, division of polynomials, multiplicative inverse, rational expressions.

Formal Geometry: Perpendicularity, right angles, and congruence of angles, congruence, proof analysis and synthesis, perpendiculars: existence and uniqueness proof, an important inequality theorem, conditions which guarantee parallelism, more conditions which guarantee parellelism corresponding angles, the parallel postulate, triangles, quadrilaterals in plane, polygonal regions, areas of triangles, and quadrilaterals, the Pythagorean theorem.

Equations, Inequalities, and Radicals: Equality and equations, order and inequalities, radicals, solving quadratic equations, applications, suggested test items.

Circles and Spheres: Basic inequality theorems, basic definitions for circles and spheres, tangent lines, the fundamental theorem for circles, tangent planes, the fundamental theorem for spheres, arcs of circles, lengths of tangent and secant segments.

Complex Number System: Comments on the introduction to complex numbers, complex numbers, addition, multiplication and subtraction, standard form of complex numbers, division, quadratic equations, graphical representation absolute value, complex conjugate, polynomial equations.

Equations of the First and Second Degree in Two Variables: The straight line, the general linear equation $Ax + By + C = 0$, the parabola, the general definition of the conic, the circle and the ellipse, the hyperbola.

Systems of Equations: Systems of linear equations in two variables, systems of one linear and one quadratic equation, other systems, an equation of a plane in a three-dimensional coordinate system, the solution set of an equation in three variables, the graph of a first-degree. equation in three variables, algebraic representation of the line of intersection of two intersecting planes, the solution set of a system of three first-degree equations in three equations, miscellaneous exercises, equivalent systems of equations in three variables.

Volume II: Volume II includes the following twenty-eight chapters (SMSG, 1960, Parts 1 and 2).

Geometry: Lines and points, planes, intersections, intersections of lines and planes, betweenness and segments, separation, angles, locating positions and points, coordinates, coordinates in the plane, graphs in the plane.

Functions: Travel by car, falling objects, some examples of functions, ways of representing functions, some special functions, new kinds of functions.

Informal Algorithms and Flow Charts: Changing a flat tire, algorithms, flow charts, and computers, assignment and variables, input and output using as a counter, decisions and branching, flow charting the division algorithm.

Problem Formulation: Focusing on a problem, follow-through on a problem, mathematical modeling.

Number Theory: The division algorithm, divisibility, prime numbers, the number of primes, prime factorization, common divisors and common multiples, the whole numbers.

The Integers: The opposite function, the absolute value function, addition of integers, structure and addition, subtraction of integers, structure and multiplication, properties of multiplication and their use, introduction to equations.

The Rational Numbers: The non-negative numbers, multiplication and division, addition, the negative rational numbers, order and density, more on equations, another look at structure.

Congruence: Congruent segments and congruent angles, addition and subtraction properties of segments, addition and subtraction properties for angles, copying triangles, congruent triangles and correspondence, some applications of congruence, bisection of segments and angles, perpendicular lines, right angles, finding bisectors and perpendiculars, a shortest path problem.

Equations and Inequalities: Translation of English sentences into mathematical sentences, solutions of equations, equations of the form $ax + b = c$; equations of the form $ax + b = cx + d$, the use of graphs to solve equations, inequalities, solving inequalities.

Decimal Representation for Rational Numbers: Ratios and percents, decimal notation, decimal representation for rational numbers, rounding off and computing with decimals.

Probability: Uncertainty, fair and unfair games, finding probabilities, outcomes and events, counting outcomes; tree diagrams, estimating probabilities, probability of the event "A or B", probability of the event "A and B."

Measurement: Length, the British system of units, the Metric system, perimeter, angle measurement, circumference of a circle, approximation and precision.

Perpendiculars and Parallels (I): How parallelism and perpendicularity are related, rectangles, transversals and parallelism, parallelograms, general triangles, parallels and the circumference of the earth.

Similarity: Scale drawings, similar triangles, some applications of ratios, how a photo enlarger works, parallels and similarity, ratios, similar triangles and percent, similarities in right triangles, the Pythagorean theorem, trigonometric functions.

The Real Numbers: Square roots, a closer look at $\sqrt{2}$, calculating square roots, operations with radicals, real numbers and operations, the structure of the real number system.

Area, Volume, and Computation: Area of a rectangular region, areas of other polygons, area of a circle, volume and surface area, computation involving measurements, other measurements.

Perpendiculars and Parallels (II): Properties of triangles, circles and perpendiculars, parallels in space, perpendicular planes and lines, parallel planes.

Coordinate Geometry: Distance and midpoints, algebraic description of subsets of the line, distance between points in a plane, midpoint of a segment in a plane, separation of the plane by lines parallel to the axes, linear functions and subsets of the plane, graphs of certain non-linear functions, slope of a line, slope-intercept form: $y = mx + b$, parallel and perpendicular lines, coordinate proofs of geometric properties, three dimensional coordinate systems, algebraic description of subsets of space.

Problem Solving: Using drawings or diagrams, using mathematical symbols to represent verbal statements, organizing information in tables, using a wild guess, deductive reasoning, solving simpler problems, guessing from special cases.

Solution Sets of Mathematical Sentences: Solving simple equations, solving simple inequalities, simplifying expressions, solving fractional euqations, inequalities involving fractions, equations involving factors whose product is 0, factoring.

Rigid Motions and Vectors: Rigid motions in a plane, translations, reflections, rotations, composition of rigid motions, making rigid motions, compositions of translations, a little exploration, rigid motions using coordinates, vectors.

Computers and Programming: Some recent history and uses of the computer, organization of a digital computer, an assembly language, a procedural language.

Quadratic Functions: Functions of the type $x \to ax^2$, $a \neq 0$, translations of the parabola $x \to ax^2$, completing the square, solving quadratic equations, "Falling Body" functions, the use of quadratics in solving other equations.

Statistics: Cumulative and relative frequency, averaging, the mean, scatter, grouped data, using the mean and standard deviation.

Systems of Sentences in Two Variables: A decision problem, the mathematical model, some related problems, solution sets of systems of equations, parallel and coincident lines, solution by substitution, graphical solutions of systems of inequalities, applications.

Exponents and Logarithms: Integers as exponents, rational numbers as exponents, the function $\exp_2$, the function $\log_2$, logarithms to the base 10, computing with logarithms, the slide rule.

Logic: Simple and compound statements, symbolizing statements, truth value and truth tables, logical equivalence, rules of logical argument, proving a conditional statement, mathematical proofs, proof by contradiction, quantifiers and negation.

Probability and Statistics: Random sampling, distribution of sample means, confidence intervals, Bernoulli Trials, Pascal's Triangle, the binomial distribution, statistical hypotheses.

In analyzing the school mathematics study group project concludes that there is a balance between skills and concepts. The multiplication of polynomials, for example, is presented as an instance of the distributive property which develops understanding and eliminates the need for many special cases.

There are sufficient applications taken from physical situations as well as from real life such as family budget, commissions, and discount. In algebra, Volume I and Volume II, applications are used to introduce new ideas from which mathematical concepts are developed.

The method of presentation is the lecture-discussion approach, although in some sections of the materials, opportunity is provided to the learner to form his own generalizations by asking leading questions such as, "What do we find?", "What do we know?", "How could we write this?", but in general the text materials are not the self-teaching type.

Many topics of SMSG which are traditionally taught are treated in a different manner. Additional topics, such as mathematical system, statistical topics, elementary mathematics theory, and scientific notation are introduced.

Solid geometry is not presented by SMSG as a separate course; it is introduced early in tenth grade geometry to help develop the student's space perception. In some cases the formal proofs in solid geometry are integrated with plane geometry, and in other cases, they are presented in special chapters. Algebra and geometry are frequently integrated. While intuitive insight is encouraged, emphasis is placed on using precise vocabulary, and exact statements of definition. On the whole it is the opinion of the writer that structure of the materials and topics used in the project are appropriate and well placed.

The Madison Mathematics Project

The Madison Project was begun in 1961, under the direction of Dr. R. Davis. The project emphasized the creative informal exploration by the pupils, and rote drill is avoided entirely. The goals of the project were (1) to create experiences in mathematics which would be better than those children usually encounter, (2) to carry on this activity as much as is possible, and (3) to gain such understanding of curriculum and instruction as can be gleaned from this type of creative 'curriculum innovation activity' (Davis, 1967, p. 5).

The project stressed the discovery method as a method of presentation of mathematics in the classroom or in the textbook. There were many examples and illustrations to encourage the students to think out the solution for themselves. The project was designed to help the students and to guide them to explore mathematics and scientific questions, but never to lecture them or to tell them. The following five guidelines are suggested for the teachers to follow when they use the program.

Ask questions. In using the Student Discussion Guide, avoid exposition as much as possible. You should rarely, if ever, tell the students what to do or how to do it. Instead, ask questions.

The student learns by thinking through the question himself (the discovery method), or, in some cases, by imitating the behavior of the teacher or of other students.

Concentrate on concepts. Try to get the student to think about the basic concepts as early as possible. Avoid the use of extensive vocabulary since names and also calculations tend to obscure the concepts; it is the concepts that the student should think about. He can learn names and calculations later on, after he has thought about the concepts in a creative way for some time. (By then he is ready for names and calculations, and they will not hurt him.)

Conduct conversations, not lectures. In conducting classes, emphasize a maximum of student participation. If a student makes an error, it is better to wait until some other student notices the error and corrects it. Every effort should be made to get the students thinking and talking, not listening and accepting.

Provide success experiences. Nearly every student answer has some merit, if only as a courageous try. Attempt to respond to every student answer as a scientist might-it is an answer and it deserves respect. If it turns out to be right, it helps with the work. But even if it turns out to be wrong, it usually adds to our understanding and contributes to the task at hand.

Try to avoid moral judgments and words that suggest moral judgments. Saying "Yes, that works" (if it turns out to) is preferable to saying "Good" or "Right." If an answer turns out not to work, it is usually better to say something like this, "Well that doesn't work. Do you have any other suggestions?"

Engender the feeling that students and teacher are partners in an experience of intellectual discovery. Do not make them feel that you are standing over them waiting to pass judgment on them.

Practice the light touch. The Madison Project introduces mathematical topics years earlier than most curricula. This gives you the advantage of being under absolutely no pressure. It is not necessary to be tedious or comprehensive. You and the children may explore the fascinating concepts of mathematics as long as they continue to fascinate. When they threaten to become routine, turn to a new topic. After all, the real purpose is to get children to think, on their own, about the basic concepts of mathematics. Let them enjoy it. It is far better to end a lesson while the students are still begging for more.

The general thrust of the Madison Project was to expose students to have meaningful learning and deep understanding of mathematics as evidenced by this summary (Davis, 1964, p. 2).

> We want children to enjoy mathematics.
>
> We want children to have successful experiences with mathematics.
>
> We want children to approach mathematics problems creatively, and not to think in terms of following rote procedure.
>
> We want children to approach mathematics problems with determination, with persistence and with optimism.
>
> We want children to approach mathematics problems, even hard problems, with confidence.

We want children to have extensive <u>experience</u> with mathematical ideas and materials.

We want them to have <u>the familiarity</u> which comes from such experience.

We want to proceed carefully in <u>building readiness</u> for future mathematical experiences. In the past, traditional mathematics programs have expected this readiness to appear suddenly out of nowhere, and it has not tended to do so.

We want younger children who are <u>creative in</u> dealing with abstract ideas, who are <u>eager</u>, original, and very <u>honest</u> in their logic, to preserve this ability and enthusiasm. Under traditional curricula it has usually been lost, and the high-school student or college freshman no longer exhibits the clever resourcefulness of the fifth grader when dealing with abstract mathematical ideas.

The project consisted of five main parts to be used in grade 9-12, or even as early as 6-12 (Davis, 1967, pp. iii-v).

<u>Part One - Variables, Graphs, and Signed Numbers</u>

Variables, the Cartesian product of two sets, open sentences with more than one variables, signed numbers, postman stories, graphs with signed numbers, using names and variables in mathematics.

<u>Part Two - Logic</u>

Logic (by observing how people use words), logic (by making agreements), some complicated formulas in logic, logic (by thinking like a mathematician), inference schemes, the game of clues.

Part Three - Measurement Uncertainties

Measurement uncertainties, identities, making up some "big" identities by putting together "little" ones, shortening list axioms, and theorems), how we shall write derivations, subtraction and division?

Part Four - Function

Practice in making up your own derivations, extending system "Lattices" and exponents, guessing functions, guessing function (forms vs. numbers), where do functions come from, the notation f(x), some operations on equations, some operations on inequalities, "variables" vs. "constants," hints on how to solve problems, all the quadratic equations in the world, some history.

Part Five - Matrices

The idea of "mapping" or "correspondences," candy-store arithmetic, Rickey's special matrix, matrices (a new mathematical system), matrices and transformations, matrices and space capsules, simultaneous equations, new ways of writing old numbers, the hesitant search for new numbers, determinants, matrix inverse (a research problem).

Early in the project, the planners found that seventh graders were more engineering-oriented than science-oriented. The students prefered to manipulate physical objects rather than construct abstract models. Hence units involving statics mechanics and tests of breaking strength of samples of yarn were combined with abstract units about algebraically-constructed vectors and matrix algebra.

In 1967 Cambridge conference report stated that, as a result of this experience, the need for better and more prepared written material was evident.

Evaluation of Madison Project is limited and not found in the literature.

Cambridge Conference on School Mathematics (CCSM)

During the summer of 1963 a group of twenty-five mathematicians and scientists met in Cambridge, Massachusetts to review school mathematics and establish goals for mathematics education. The results of these meetings were published in a report known as the Cambridge Report. The report included three parts: the first part consisted of chapters dealing with goals and techniques. The second part outlines two different proposed curriculum for grades 7-12. The third part of the report consisted of some of the working papers. The goals of this project as reported in the Educational Service Incorporated (ESI) were (1) Acquisition of skills by means of integrating drill into problems that lead to new concepts; (2) Parallel development of geometry and arithmetic-algebra; (3) Familiarizing the student with mathematics by means of a spiral curriculum; (4) Building self-confidence. It was stated that "even modestly endowed students can recreate a large part of mathematics if they can remember a few basic ideas." Students must be convinced that they can rely on their own analytical thinking; (5) Precise language, notation and symbolism as necessary for communication with precision; (6) Balance between pure and applied mathematics; and (7) Understanding both the power and the limitations of mathematics (ESI, 1963, p. 2).

CCSM believes that a student who has worked through the full thirteen years of mathematics in grades K-12 should have a level of training comparable to three years of top level college training today, that is, we shall expect him to have the equivalent of two years of calculus, and one semester each of modern algebra and probability. CCSM suggested that the method of presentation of mathematics should be intuitive as much as possible. The emphasis was on developing creative and independent thinking in the individual. It suggested the mathematical idea represented in the following form, "Here is a situation - think about it - what can you say?" The topical outline of the first proposal for grades 7-12 was as follows (Aichele & Reye, 1977, pp. 54-57).

Algebra: Review of properties of numbers, ring of polynomials over a field, polynomial functions, rational forms and functions, quadratic equations, iterative procedures, difference polynomials, Euclidean algorithm, diophantine equations, modular arithmetic, complex numbers as residue classes of polynomials mod $x^2 + 1$, derivative of a polynomial.

Probability: Review of earlier experience with probability, basic definitions in probability theory for finite sample spaces, sampling from a finite population, unordered sampling, ordered sampling without and with replacement, conditional probability, independence, random variables and their distributions, expectation and variance, Chebychev's inequality, joint distribution of random variables and independent variables, Poisson distribution, tatistical estimation and hypothesis testing.

Geometry: Intuitive and synthetic geometry to the Pythagorean theorem, Cartesian plane and space, lines, planes, circles, and spheres, motions in Euclidean space, groups of motions, matrices and linear transforming, vectors, linear independence, rotations in the plane and in space, complex numbers and rotations in the plane, trigonometry, vector space of n dimensions, conics and quadrics, projective geometry, transformation laws, tensors.

Geometry: Geometry of complex numbers, linear fractional transformations, mappings by elementary functions, stereographic projections, neighborhoods, continuous functions, fundamental theorem of algebra, winding number, location of roots.

Linear Algebra: Simultaneous linear equations, linear mappings, matrices, subspaces and factor spaces, equivalences of matrices, change of bases, and matrices of a transformation, triangular form of matrices, invariant subspaces, diagonal form of symmetric matrices and quadratic forms, determinants, Cayley-Hamilton theorem, inner products and orthogonal transformations.

Analysis: Real numbers, sequences and series, probability for countable sample spaces, limits of functions, continuous functions, derivatives, mean value theorem, antiderivatives, simple differential equations, exponential and logarithmic functions, trigonometric functions, linear differential equations with constant coefficients, differential geometry of curves, definite and indefinite integrals, areas, Taylor series, indeterminate forms, probability for continuous distributions, calculus for functions of several variables.

The topical outline of the second proposal for grades 7-12 was as follows (Aichele & Reye, 1977, pp. 56-57).

Algebra and Geometry: Review of properties of numbers, logic of open statements and quantifiers, linear equations and inequalities, systems of n linear equations in m variables, flow charts, logic of formal proofs discussed, exiomatic development of Euclidean geometry of two and three dimensions, analytical geometry, lines, circles, parabolas, quadratic equations, functions - composite, inverse; functional equations, polynomial functions, geometry of circles and spheres, trigonometric functions, vectors in two and three dimensions, omplex numbers, possible introduction to logarithms.

Probability: Binomial theorem, combinatorial problems, review of earlier experience with probability, basic definitions in probability theory for finite sample spaces, sampling from a finite population, unordered sampling, ordered sampling with and without replacement, conditional probability, independence, random variables and their distributions, expectation and variance, Chebychev's inequality, joint distribution of random variables and independent variables, Poisson distribution, statistical estimation and hypothesis testing.

Calculus: Limits of functions and continuity (lightly), derivative, slope of tengent line, velocity, derivatives of polynomials, sines and cosines, sums and products, applications, curve tracing, maxima and minima, rate problems, Newton's method for finding roots of polynomials, antiderivatives, definite integral and area, the Mean Value theorem, fundamental theorem of calculus, applications.

Algebra and Geometry: Volumes of figures (prisms, pyramids, cylinders, cones, spheres), linear equations and planes, rigid motions of space, linear and affine transformations, matrices, determinants, solutions of linear systems, quadratic forms, diagonalization, conics, and quadrics, numerical methods.

Analysis, Probability, and Algebra: Infinite sequences and series of real and complex numbers, absolute and unconditional convergence, power series, probability for countable sample spaces, linear algebra, subspaces, bases, dimension, coordinates, linear transformations and matrices, systems of equations, determinants, quadratic forms, diagonalization.

Analysis: Limits of functions, continuity, rules for differentiation, mean value theorem and its consequences, definite integral, its existence for continuous functions, logarithmic and exponential functions, trigonometric functions, hyperbolic functions, applications, techniques of integration, Taylor series, indeterminate forms, interpolation, difference methods, differential equations, probability for continuous distributions, differential geometry of curves in space, multidimensional differential and integral calculus, Boundary value problems, Fourier series, integral equations, Green's functions, variational and international methods.

The algebra in the first proposal began with a review of the real numbers and proceeded with the study of the ring of polynomials over a field and then algebra, and complex numbers. The corresponding courses in the second proposal emphasized polynomial functions instead of the ring approach, and complex numbers were introduced as ordered pairs of real numbers instead of as residue classes in the ring of polynomial mod $x^2 + 1$.

Both geometry proposals used the synthetic approach up to the Pythagorean theorem, and similarity. Transformations, Euclidean space, and matrices were presented also.

The second proposal introduced calculus in the ninth grade in the nature of heuristic approaches. Both proposals included a complete and extensive study of calculus in grades 11 and 12. Probability was also presented in both proposals by intuitive methods and with the use of calculus.

Although it was not the primary purpose of the conference to engage in a project, it was later considered important to develop some teaching materials for classrooms.

Secondary School Mathematics Curriculum Improvement Study (SSMCIS) - Unified Mathematics Program

The Secondary School Mathematics Curriculum Improvement Study (SSMCIS) was begun in 1966 under the direction of Fehr.

SSMCIS opposed the classical viewpoint that mathematics consists of four branches - arithmetic, algebra, geometry, and analysis, each considered a closed and separate field of study. SSMCIS believes that

> Mathematics as a branch of knowledge, no longer holds to this classical point of view. From the developments in understanding numbers, organizing algebraic systems, and

> creating new geometric spaces, along with the emergence of set theory, formalism, and the concept of structure, a new contemporary viewpoint of mathematics has come into being. As early as the 1930's mathematicians recognized that certain fundamental concepts underpinned all the branches of mathematics and that structural concepts gave possibilities for organizing all mathematics into a unified body of knowledge. As a consequence contemporary mathematics uses the basic notion of sets, relations, and mappings; the algebraic structures of group, ring, field, and vector space; and topological structures. This structuring permits greater understanding and efficiency in learning and uncovers concepts and theories previously hidden by the traditional separation (Fehr, 1974, p. 29).

A group of mathematicians and mathematics educators from Europe and the United States was convened in 1966 to state and formulate goals, scope, and sequence of the program for students in grades 7-12. At the base of the program were the ideas of sets, relations, mappings, and operations. On these fundamental concepts were built the structures group, ring, field, and vector space. The goals of the program were as follows:

1. The mathematics we teach to our students today should be relevant to their needs in the society of tomorrow in which they live. Thus the mathematics we teach should reflect the manner in which the subject is conceived for future use. To this end we must first of all be concerned with the development of the intellect - the ability to do cognitive thinking.

2. The mathematics we teach and the way we teach it should develop the human mind in its capacity to understand and interpret numerical, spatial, and logical situations occurring in the physical universe and life within it, and to approach problems with a scientific, questioning, and analytic attitude.

3. All our students must come to know mathematics as it is conceived today, what material it deals with, what types of thinking (not only axiomatic) it uses, what it accomplishes, and how it is invading almost all other domains of human activity.

4. Our instruction must also have an "informational and skill" dimension, inasmuch as it is charged with transmitting from one generation to the next that inherited knowledge, and the skill to use it that will be considered basic in the years ahead. This information should be acquired during the process of developing mathematical thinking. This target of instruction permits us to drop a great deal of the traditional content no longer considered useful and to choose more general and more unifying concepts as a foundation for instruction.

5. It is the usefulness of our subject that has maintained it as a principal discipline of educational endeavor. Our instruction serves to develop the capacity of the human mind for the observation, selection, generalization, abstraction, and construction of models for use in solving problems in the other disciplines. Unless the study of mathematics can operate to clarify and to solve human problems, it has indeed only narrow value.

The program consists of five courses and five booklets and teachers' commentaries which present suggestions for teaching, estimated times for the study of each topic, solutions to all exercises, and suggested tests for each chapter. The outline of the program is (Fehr, 1974, p. 31).

Course 1: Finite number systems, sets and operations, mathematical mappings, integers and addition, probability and statistics, multiplication of integers, lattice points in a plane, sets and relations, transformations of the plane, segments, angles, isometries, elementary number theory, the rational number, some applications of the rational numbers, algorithms and their graphs.

Course 2: Mathematical language and proof, groups, an introduction to axiomatic affine geometry, fields, the real number system, coordinate geometry, real functions, descriptive statistics, transformations of the plane, isometries, length, area, and volume, appendix A mass points.

Course 3: Introduction to matrices, linear equations and matrices, algebra of matrices, graphs and functions, combinatorics, probability, polynomials and rational functions, circular functions I, informal space geometry.

Course 4: Programming in BASIC, quadratic equations and complex numbers, circular functions II, probability: conditional-probability and random variables, algebra of vectors, linear programming, sequences and series, exponential and logarithmic functions, vector spaces and subspaces.

Course 5: Introduction to continuity, more about continuity, limits, linear approximations and derivatives, properties of derivatives, further study of the derivative, linear mappings and linear programming, probability: expectation and Markov chains, integration.

Course 6: Infinity, conics, circular functions analytic properties, exponential and logarithmic functions-analytic properties, integration techniques and applications, probability: infinite outcome.

Booklets:

A. Introduction to Statistical Inference

B. Determinants, Matrices and Eigenvalues

C. Algebraic Structures, Extensions and Homomorphisms

D. An Introduction to Differential Equations

E. Geometry Mappings and Transformations

The developers of SSMCIS tried to design a program to provide more applications and less abstraction which could be used by the majority of high school students.

It can be noticed that the goals of the project were the development of mathematical thinking and skills, and the development of the capacity to apply mathematics in other situations.

The topics were presented by a spiral approach with emphasis on the unifying concepts of sets, relations, functions, and structure.

The Unified Science and Mathematics for Elementary Schools (USMES)

The Unified Science and Mathematics for Elementary Schools (USMES) was initiated in 1970 and organized by the staff of the Educational Development Center in Newton, Massachusetts. The main objective of USMES is to present mathematics in both real and practical situations to children. The problem is real in that it applies to some aspects of school or community life. It is practical because children's work represents useful achievement in that it may lead to some improvement in the situation being investigated. This expectation of useful accomplishment gives the children a commitment to finding a solution to the challenge. The project consisted of nine main resources for grades one through eight (USMES, 1973).

The USMES Guide: This book describes the USMES project, real problem solving, classroom strategies, the design lab, the units, and the support materials as well as ways that USMES helps students learn basic skills.

Teacher Resource Books (one per unit): Each of these guides to using USMES units describes a broad problem, explains how students might narrow that problem to meet their particular needs, recommends classroom strategies, and presents logs from teachers whose classes have worked on the unit.

Design Laboratory Manual: This guide helps teachers and administrators set up, run, and use a design lab --a place with tools and materials where students can build things they need for their work on USMES units. A design lab may be a corner of a classroom, a portable cart, or a separate room.

Background Papers: These papers, correlated with the "How To" Series, provide teachers with information and hints that do not appear in the student materials.

Curriculum Correlation Guide: By correlating the 26 USMES units with other curriculum materials, this book helps teachers integrate USMES with other school activities and lessons.

Primary "How To" Series: This series helps children learn skills like designing an opinion survey and choosing the appropriate measuring tool.

Its cartoon-style format helps younger children and those with reading difficulties acquire the skills and knowledge they need to do things like redesign their classroom, find the best buy in potato chips, or run a school store.

Intermediate "How To" Series: This magazine-style series covers in more detail essentially the same materials as the cartoon-style series with a few booklets on additional skills. This series gives students a chance to read something they have a need to read.

Design Lab "How To" Series: These illustrated cards help children learn how to use tools safely and effectively. They will be available in primary and intermediate versions.

Teacher Resource Books:

Advertising, Bicycle Transportation, Classroom Design, Classroom Management, Consumer Research, Describing People, Designing for Human Proportions, Design Lab Design, Eating in School, Getting There, Growing Plants, Manufacturing, Mass Communications, Nature Trails, Orientation, Pedestrian Crossings, Play Area Design and Use, Protecting Property, School Rules, School Supplies, School Zoo, Soft Drink Design, Traffic Flow, Using Free Time, Ways to Learn/Teach, Weather Predictions.

The USMES classroom method of teaching is based on the participation of the children, and the teacher is expected to be a guide but not to direct, to ask stimulating questions but not to supply the answers. The teacher presents the children with a challenge or several challenges to which children may arise from discussing issues in a broader context. In USMES program the children work in groups, each group may work on a different aspect of the problem depending on interest and ability.

There are seven stated teacher responsibilities (USMES, 1973, pp. 1-8). Teachers must first introduce the challenge in a meaningful way which allows the children to relate to it and open up general avenues of approach. They must be coordinators, collaborators, and assistants in the projects. They must help the children get involved in the challenge and provide opportunity for them to work on it two to three times a week. The teachers must make materials and tools available. They must let the children make their own mistakes and find their own way and be ready to point out sources of help for specific information. The teachers need to provide frequent opportunities for group reports and student exchange of ideas in class discussions, letting the children improve the ideas of the class. Finally they should ask appropriate questions to stimulate student's thinking and to increase the depth of investigation and analysis of data.

Comprehensive School Mathematics Project (CSMP)

The Comprehensive School Mathematics Project was based on the recommendations of the Cambridge Report in 1963 and was initiated in 1973 under the direction of Burt Kaufman. The main purpose of the experimental project was to develop an individualized mathematics program to fit the needs and abilities of each student. The project consists of two programs, one for the secondary school and the other for the elementary school.

CSMP believes that it is not important that the calculations be carried out according to prescribed standard algorithms; what is important in basic mathematics concepts and which is often forgotten are the following points.

> a) The ability to know which calculations are appropriate to the given problem or situation. Clearly, the ability to calculate is almost totally worthless to those who cannot tell to which situations the calculations apply.
> b) The ability to make reasonable estimates and appropriate approximations. In many, if not most cases in everyday life, it is absurd to make "precise" calculations on the basis of imprecise data. The real trick is to teach people to make quick, simple calculations that will give an answer good enough for the case at hand.
> c) Some elementary knowledge of probabilistic and statistical statements. Much of the mathematics of everyday life and common experience has to do with probabilistic and statistical inferences. Something about these ideas should be in the intellectual equipment of everyone (Braunfeld, 1973, p. 4).

CSMP pointed out six principles for learning mathematics. These principles are based on what mathematics is about, knowing the kinds of problems mathematicians tackle, the kind of formulations of a problem that make it likely to be tractable, the kind of methods that mathematics uses to solve problems, the kinds of standards by which correctness in mathematical argumentation is judged. The principles of CSMP are:

<u>Mathematical proof.</u> It is essential to understand what does and does not constitute a sound mathematical argument, and to have seen some sound arguments in a reasonable variety of mathematical areas - geometry, algebra, finite mathematics, probability, number theory, etc. It is essential to understand that all mathematical propositions are conditional -

that they assert merely that _if_ such and so _then_ such and such follows. It is also important to appreciate mathematics as a unified, deductive system and to understand that a relatively small number of basic axioms suffice to serve as a foundation of very broad and elaborate disciplines.

The precision of mathematics discourse. The languages of mathematics are much more precise and explicit than that of ordinary English. It is important that the student have experience and some facility with the use of these very precise languages, and that he have some appreciation of the benefits derived in mathematics from the very precise formulations of problems and results that mathematics demands.

Nontrivial problems. The mathematically literate person will have had occasion to see the precise language of mathematics and the rigorous standards of mathematical argumentation used to solve a number of nontrivial problems. It is not really important what specific problems are chosen for this purpose. What is important is simply to see in a variety of contexts that mathematics "pays off," i.e. that mathematics yields results that are not immediately obvious. In order to gain a deeper appreciation of the power of mathematics, it is useful if the student has been given opportunities to try his own hand at the solution of nontrivial perhaps even difficult problems. It is not essential that he succeed in his efforts to solve such problems, much less that he become "good" at solving difficult problems. It is only important that he get a _feeling_ for how to tackle problems and what mathematical inquiry is like.

Mathematical structures. Mathematics is deeply concerned with discovering likenesses between things that appear at first blush very different.

It does this by abstracting from the difference. Perhaps the most classical and well-known example is Descartes' discovery that every geometrical problem can be formulated algebraically and conversely. The search for similarities, the effort to classify and sort the objects of mathematical study are perhaps most prominent in modern algebra. Students should have an opportunity to become acquainted with some of the more basic structures of modern algebra, e.g., groups, rings, vector spaces, and fields. They should develop some appreciation of the power and elegance derived from a study of structure.

Computers and algorithms. The importance of computers in our society need hardly be discussed in detail here. Students should have an appreciation for the kinds of things computers can and cannot do. There must be no mystery whatever, in principle, about what the computer does and how it does it. For this reason, the idea of algorithmic procedures should be firmly in hand.

Applications. Mathematics has had an immeasurable impact on almost all the sciences: astronomy, physics, chemistry, biology, psychology. It is impossible to have more than a few examples of such application in the school curriculum. Nevertheless, it is vital that the student have seen at least a few nontrivial, nonmechanical applications of mathematics. Some knowledge of mathematical model building is a prerequisite of mathematical literacy. Again it is probably not important that the student has seen this or that specific application. What is important is simply that he has seen them in a variety of different fields.

CSMP believes that the content must include a variety of interesting mathematics, and the student should have been exposed to and worked through these different interesting principles. The following topics and areas will be included in the CSMP program.

<u>The complete ordered field of real numbers.</u> Students should have a good understanding of the algebra of this field and some acquaintance with its elementary topological properties. They should know as much as possible about the elementary functions defined on the reals, specifically the polynomial functions, the circular functions and the logarithmic and exponential functions. Ideally, they should have an understanding of limits, the derivative, and the Riemann integral, their properties, and some of the elementary uses to which these ideas may be applied.

<u>Geometry.</u> Some knowledge of elementary geometry, preferably including a transformation or linear algebra point of view, is surely essential for mathematical literacy.

<u>Probability and Statistics.</u> Probability theory is intrinsically a rich and elegant discipline within mathematics. It can therefore serve well as one of the examples to show what mathematics <u>is</u> (discussed above). Moreover, its applications, both elementary and advanced, are so enormous and varied that it appears clear that a knowledge of some parts of this subject is essential to mathematical literacy.

<u>Number Theory.</u> The elements of number theory are easily accessible and number theory can thus be used to illustrate very early some of the notions of proof, precision of language and problem solving techniques discussed above. Moreover, number theory has played a large and important role in the history of mathematics, giving rise to many more

advanced theories in algebra and analysis. Being a discipline that has found relatively few applications, it is an example of the kinds of problems that arise in "pure" mathematics. CSMP is now in the process of developing material for use by all students. There was a specific summer-school work shop for teachers permitted to teach this project.

Boston University Mathematics Project (BUMP)

The Boston University Mathematics Project was created in 1975, under the direction of U. H. Schaim. Some of the materials incorporated in this project were developed with support of National Science Foundation.

The purpose of the Boston University Mathematics Project is to serve as a vehicle for the development of relevant computational skills, mathematics reasoning, and geometric perceptions in three dimensions. The application of mathematics was considered to be an important factor in the selection of the materials. The method of presentation of mathematics encouraged individuals as well as groups, and strong emphasis was placed on student activities. The content of BUMP consisted of ten units for grades 7 and 8. Each unit included five types of sections: 1) activities by the whole class, small groups or individuals; 2) short reading sections to be assigned and discussed, or to be read in class; 3) questions to be worked out at home or in class; 4) sections intended to help students with weaker backgrounds, and 5) sections provid extra challenge and pleasure for the strongly motivated student.

The following are the unit outlines for the Boston University Mathematics Project (BUMP, 1975, pp. 12-16).

<u>The Cube</u>: Visualizing the cube, patterns for cubes, unfolding the cube, cross sections of the cube.

<u>Volume</u>: Length and area, units of volumes, volumes and surface areas, volumes of right prisms and right cylinders, volumes of the irregular solids, a useful property of liquids, liters, and cubic meters.

<u>Powers of Ten</u>: Large numbers, visualizing large numbers, approximating products of whole numbers, exponential notations, significant digits and standard notations, multiplying and dividing numbers in exponential notation, significant digits in products and in quotients.

<u>Signed Numbers</u>: Numbers less than zero, signed numbers and vectors, adding signed numbers, total and average numbers, multiplying and dividing signed numbers, subtracting signed numbers, change and percent change.

<u>Planet Earth</u>: Continents and oceans, reporting sums of measured quantities, ratios, area and population maps, negative exponents, orders of magnitude, the maning of "per," population densities.

<u>Variables and Functions</u>: What is the rule?, variables, ways to write functions, finding values of functions, circles and cylinders, permutations.

<u>Indirect Measurements</u>: Combining measurements and calculation, indirect measurements of height, your personal "Range Finder," other three-step calculations.

<u>Sampling</u>: Coded messages, frequency of letters, percent, finding a percent of a whole, adding samples, decoding using frequency of common letters, using percent to find the whole, using samples to count whales.

The Nuffield Mathematics Project

The Nuffield Project was begun in 1964 in England and supported by the Nuffield Foundation. The project consists of a series of booklets which present new ways for mathematics teachers to teach mathematics. The project is based on the work of Piaget, emphasizing learning through discovery rather than memorization. Learning should take place inside the classroom as well as outside the classroom. However, the manual, I Do I Understand clearly states:

> The mere provision of materials is not quite sufficient. The situation must be carefully structured by the teacher if the children are to make real discoveries.
>
> Whenever new materials are introduced there seem to be three separate states through which children must pass. At first the child needs a period of free experimentation with the material...The second stage involves the introduction of the necessary vocabulary...The third stage sees the emergence of a problem--probably some question that has arisen during the discussion (Nuffield, 1967, pp. 16-17).

There are many materials that are used in the classroom. "Topic Books" are available to be used by the students as a source of information and facts. These books do not present topics by topic and page by page as in a regular textbook, but they rather serve as reference materials. Steps to construct materials, tools, and measuring devices to be used in classroom or in the school yard. Another important element of the project is the assignment card, which is designed to question the students about the work they do. The assignment card is intended to suggest direction for investigation and take the form of an open question, "What do you notice?" or "Can you see a pattern?" Finally, the children put down all the information related to their work and experience, including drills or algorithms and give to the teacher for evaluation.

Characteristics of the New Mathematics Curriculum

In examining the goals, the content, and the presentation of mathematics of the different programs that are discussed in the previous section of this chapter, the writer noticed the following characteristics.

A. Content

One of the major characteristics of today's programs is a tendency to push down to the high school level topics that were studied on the university level. Topics such as set theory, modern algebra, non-Euclidean geometry, theorem of probability, topology and symbolic logic occupy a prominent place in today's high school mathematics.

Some reformers of mathematics believe that set theory is the core of all mathematics and that it should be taught to elementary school children. Professor McShane of the University of Virginia said:

> The idea of a set is ancient and obvious...There is certainly nothing unusual in forming the mental concept that results from thinking of several things simultaneously, as a coherent whole. As far back as recorded language goes, it has contained collective nouns, such as a "swarm" of bees and a "flock" of sheep. And a set (or class or collection--we use them as synonyms) is just this mental concept formed by thinking of several things as forming a coherent single. Perhaps it is because the idea of set seemed so simple and obvious that it did not emerge as a clear-cut part of mathematics until less than a century ago. But by now the habit of thinking in terms of sets has become routine for most mathematicians, and has also become part of the everyday thinking of many other scientists too (McShane, 1957, p. 36).

On the elementary level, set theory is introduced to treat addition, subtraction, multiplication and division of numbers as well as fractional numbers. On the high school level, set concept is used very widely to clarify the subjects of relations, mapping, one-to-one correspondence, function, group, field, ring, isomorphism. Figure 1 illustrates applications of the concepts of sets.

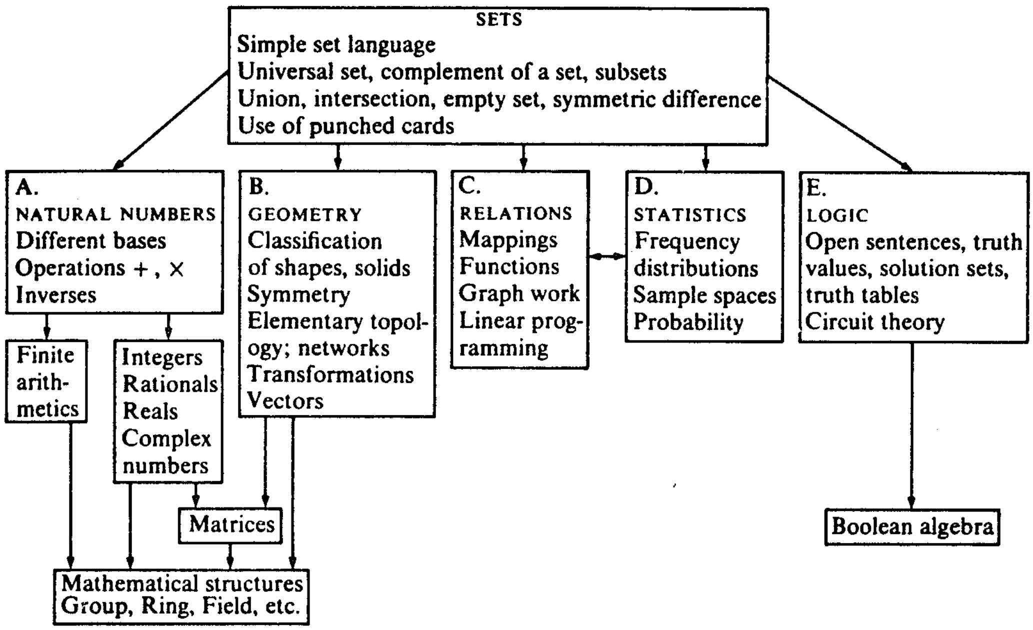


Figure 1: Applications of the Concept of Sets

Algebra for the new high school mathematics deals with the familiar operations of addition and multiplication, but combines these dealings with the observation that addition and mulitplication apply not just to numbers but to other objects as well. These other subjects may be permutations, sets, groups, rings, fields, isomorphism...etc.

Geometry of the new mathematics curricula emphasized axiomatic methods, transformation. Reformers believe that geometry is the best means to expose students to axiomization and deductive reasoning.

The concept of "vector" and "vector space" is suggested and used in many high school programs. The purpose of including the "vector" concept in geometry is that "vector algebra, a vector analysis not only had important applications to physics and engineering but became a powerful method for the study of geometry" (Prenowitz, 1977, p. 100).

Application of probability and statistics is used widely in the new curriculum. applications of statistics inside and outside school life, industry, and other cases is emphasized.

B. Learning Mathematics

By 1950 there was a recognized need for understanding "learning mathematics." The main concern focused on what learning theorists and research can tell about how students learn mathematics. Three important theories of learning have been modified to fit mathematical learning situations.

The cognitive development theory (Piaget). This theory is based on the belief that the child's mental activity is organized into structures. Mental acts are related to each other and grouped together in clusters called "schemas" or patterns of behavior. According to this theory, mental growth is a social process where the child does not interact with his physical environment as an isolated individual. He interacts with it as a part of a social group. As the child progresses from infancy to maturity, his ways of acting and thinking are changed. Piaget finds that the child passes through four different stages of mental growth which he calls the sensory-motor stage, the pre-operational stage, the stage of concrete operation, and the formal operation stage.

The sensory-motor stage, which extends through the first eighteen months of the life of an infant, mainly states that the child's activity is accompanied by mental activity based on the mental representation of objectives and mental anticipation of activities. He is able to think of an object that he does not see. As an example, if we show a nine-month old infant a toy, he will reach out and pick it up. If you drop a cloth over the toy before it has been picked up, the infant will remove the cloth and pick up the toy.

The second stage, the pre-operational stage, ranges from the age of eighteen months to about six or seven years. It marks the beginning of language, using symbols, trial and error, thinking, and procedures.

The concrete stage may start at about the age of six or seven. During this period, the child separates the concept of mass from the concept of length. He knows that the mass of an object remains unchanged when its form is changed.

The formal operational stage, or the hypothetical-deductive operational level, is the last stage. This stage usually does not occur until eleven to twelve years of age. The child, in this period, becomes capable of reasoning with ideas rather than objects in the physical world as a basis for his thinking.

The analytical learning theory (Gagne). Gagne believes that teaching should begin with a task analysis of the instructional objective. The teachers always ask the question, "What is it you want the learner to know?" After identification of the capability, the teacher should ask, "What would you need to know in order to do that?" (Gagné, 1977, p. 185). Let us say that a student could not complete the task unless he could first perform prerequisite task a and task b, but in order to perform task a, a student must be able to perform task c and d. (See Figure 2.)

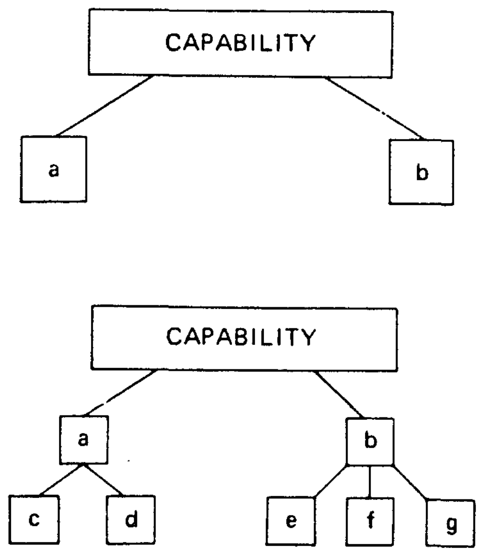


Figure 2: Identification of the Capabilities

Gagné identified four stages of learning: prerequisites of learning, conditions of learning, conditions for retention, and learning style.

Prerequisites of Learning. The teacher determines the prerequisites of learning as he establishes the sequence of performances a student must follow so that learning can take place. Gagne says that a learning event takes place when the stimulus situation affects the learner in such a way that his performance changes from a time before being in that situation to a time after being in it (Gagné, 1970, p. 5).

The first stage of this theory emphasized that in order for a student to attain a certain performance objective, he must know the prerequisites to that certain performance. For example, the mathematics teacher must ask himself questions such as these: What does a student have to know before he can find the roots of a quadratic equation? What does the student have to know before he develops proof of the Pythagorean Theorem? What does the student have to know before he can solve a problem dealing with the volume of a rectangular prism?

Conditions of Learning. Gagné suggested that there are eight types of learning. These eight types may be used to analyze the psychological nature of curriculum. They are (1) signal learning, (2) stimulus-response, (3) motor learning, (4) verbal association, (5) discrimination learning, (6) concept learning, (7) principle learning, and (8) problem solving (Gagné, 1970, p. 8).

1. The signal of learning is similar to the classical conditioning experiments of the Russian scientist, Ivan Pavlov.

2. The stimulus-response learning came from experiments conducted by Edward L. Thorndike, who concluded that learning consists of connections between stimuli and responses and that repetition is essential to learning (Thorndike, 1898, p. 3). In teaching mathematics, if the teacher teaches the child to learn to say the symbol for example "seveness," the teacher should reward the child whenever he/she says "seven." The reward may be a verbal comment such as "Good," "Excellent," or a smile or hug, etc.

3. The motor chaining learning is described by Gagné as putting together a set of several stimulus-responses in an appropriate order such as writing numerals in correct forms or drawing of geometric forms.

4. The verbal association learning deals with the verbal level where meaning is not necessarily a part of the verbal association.

5. The discrimination learning deals with the selection of an appropriate stimulus from a set of stimuli which includes asking the child to point out various categories of objects such as round things, red things, wooden things, and so on in a room or a play yard. Then ask the children to pick out specific objects such as a particular book, picture, or toy. In teaching mathematics, the student must learn to discriminate between symbols which may appear similar (+,x), the Greek letter π which is approximately equal to 3.14 when it is used in calculating areas of geometric figures and $\pi = 180°$ when it is used in trigonometry.

6. Concept learning deals with understanding of an idea or an abstraction. Gagné claims that if a child is asked to identify several cut-out shapes of squares, triangles, and circles then the child can choose all the shapes of triangles, circles, and squares only if he knows what a triangle is.

7. Principle learning involves studying of two or more concepts together to form a new concept. In principle learning students are expected to learn various rules and generalizations to be able to learn a new concept.

8. Problem-solving learning. Gagné believes that in problem-solving learning students use rules, facts and generalizations which have been learned previously and apply them in new situations.

Conditions for Retention. Gagné claims that in order for the learning to be of value to students, they must retain what they learn.

He described four factors for retention: (1) Motivation, (2) Concrete, (3) Verbal chains, (4) Amount of practice during initial learning.

Learning Style. Gagné says that the teacher must have guidelines for his teaching method. When the learner is confronted with a task to be performed, he is unlikely to be able to perform it unless those enabling tasks or parts of the final task have been accomplished. Gagné believes that, first of all, we must analyze the task into all necessary subordinate tasks. Second, we order these analyses into a validated sequence. Thirdly, we start to ask the question, "What does the learner know?" starting from the final level.

Gagne gives a useful instructional sequence to be used in rule learning (Gagné, 1970, p. 203).

Step 1: Inform the learner about the form of the performance to be expected when learning is completed.

Step 2: Question the learner in a way that requires the reinstatement (recall) of the previously learned concepts that make up the rule.

Step 3: Use verbal statements (cues) that will lead the learner to put the rule together, as a chain of concepts in the proper order.

Step 4: By means of a question, ask the learner to "demonstrate" one or more concrete instances of the rule.

Step 5: (optional, but useful for later instruction) By a suitable question, require the learner to make a verbal statement of the rule.

Bruner on the learning of mathematics. Bruner believes that a student will learn something when he needs it. That is as if you would teach someone how to swim by throwing him into deep water. Bruner says that one starts with the complex and plans to learn the simple components in the context of working with the complex.

For Gagne, the learning begins from the bottom up. One starts with the simplest capability and moves to the complex one. For Bruner, the same diagram may be appropriate, but the direction of the learning process will be an opposite one. Bruner believes that learning starts with the complex capability and then the learner will learn the simple capability. (See Figure 3.)

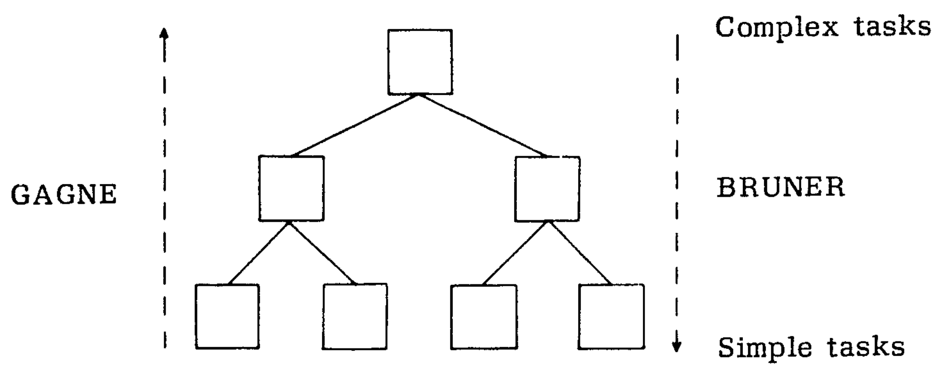


Figure 3: Analyzing a Task into Subordinate Tasks

Bruner has used mathematics in his development of theorems for instruction and has provided four theorems on the learning of mathematics: the construction theorem, the notation theorem, the theorem of construction and variation, and the theorem of connectivity (Bruner & Kenney, 1963, pp. 63-64).

The constructive theorem states that learning involves movement and manipulation of concrete objects by the learner. Bruner believes that presenting an idea in a concrete fashion helps the student not only understand the idea, but makes him provide his own example of it. Bruner suggested that the teacher should encourage students to construct rules rather than to "give" them. If a concept such as "eightness" is to be learned, the child would get eight objects, enumerate them, rearrange them into various sub-groups, and match them in one-to-one relationships with sets also having eight objects.

The notation theory states that mathematics can be learned by two levels: picture level (iconic) or symbolic level. According to Bruner, the iconic picture level is associated with images in the mind's eye or actual pictures. The symbolic level deals with such symbols for ideas as words or mathematical symbols including numbers.

In the picture level, we present the mathematical ideas as pictures, for example, a set of three chairs on the blackboard. The symbolic level is more abstract than the picture level.

The contrast and variation theorem. Bruner claims that the process of moving from concrete or special representation to more abstract or general representation involves two operations, contrast and variation. A mathematical idea can be learned by contrast. For example, an odd number is contrasted to an even number or a square is contrasted to a rectangle, or a negative number is contrasted to a positive number. The learning of a generalization of a mathematical concept requires the move from the concrete representation to the abstract ideas.

The theorem of connectivity states simply that no concept or operation in a formal system is entirely disconnected from other concepts and operations within the system (Bruner & Kenney, 1963, pp. 63-64). In other words, learning mathematics can be accomplished in the form of spiral curriculum, which means ideas are reported within increased depth as the child develops intellectually and gains a greater background in the subject. The theorem of connectivity has been exemplified by the development of SMSG, The Illinois Project, and The University of Maryland Project.

In 1960, Bruner presented a paper before the National Council of Teachers of Mathematics in Salt Lake City in which he gave four important aspects to his theories which are related to the teaching and learning of mathematics. The four aspects are discovery, intuition, readiness, and structure.

Discovery teaching presents mathematics in a way that makes sense to the learner. It is an instructional process in which the learner is placed in a situation where he is free to explore and manipulate materials, to investigate, and conclude. The discovery method will be discussed in detail in the next section.

Intuition implies the act of grasping the meaning or significance of a problem without explicit reliance on the analytic apparatus of one's craft. If the intuitive mode yields hypotheses quickly, that produces interesting combinations of ideas before their worth is known

(Bruner, 1977, p. 174). The following are two examples showing how intuition can be accomplished.

Example 1: One of the problems in teaching mathematics is making the definition $x^\circ = 1$, when $x \neq 0$ meaningful to the mathematics student who faces it for the first time.

$x^n = x^{n+\circ}$ by the Additive Property of Zero

$x^n = x^n \cdot x^\circ$ by $x^{n+m} = x^n \cdot x^m$

Therefore, for x^n to equal $x^n \cdot x^\circ$, x° must be one because of the Multiplicative Property of one. Hence, we defined x° to be 1 (Curtiss, 1967, p. 91).

Example 2: By writing several columns of numerals similar to the following points out what develops as you proceed down the column to the zero component:

.	.
.	.
.	.
$2^5 = 32$	$3^5 = 243$
$2^4 = 16$	$3^4 = 81$
$2^3 = 8$	$3^3 = 27$
$2^2 = 4$	$3^2 = 9$
$2^1 = 2$	$3^1 = 3$
$2^0 = \boxed{}$	$3^0 = \boxed{}$

It can be noticed that as the exponents decrease, the corresponding products for the 2's are divided by 2. Since $2^1 = 2$, this implies 2^0 is 1.

According to Bruner, readiness is a function not so much of maturation, which is not to say that maturation is not important, but rather of our intentions and our skill at translation of ideas into the language and concepts of the age we are teaching. Bruner used Piaget's principle of conversation to support his idea by giving this example: "A six-year old child will often doubt that there is the same amount of fluid in a tall, thin glass jar as there was in a flat, wide one even though he has seen the fluid poured from the latter into the former" (Bruner, 1977, p. 179).

Bruner believes that a student who learns the structure of a discipline is able to retain what he has learned to create interest in the subject for himself. "Teaching specific topics on skill without making their context in the broad fundamental structure of a field of knowledge is uneconomical in several deep senses" (Bruner, 1960, p. 30). Bruner explains the importance of teaching structure by saying "knowledge has been acquired without sufficient structure to tie it together. Knowledge likely to be forgotten is an unconnected set of facts which has a pitiably short half-life in memory. Organizing facts in terms of principles and ideas from which they may be inferred is the only known way of reducing the quick rate of loss of human memory" (Bruner, 1960, p. 31).

C. Method of Teaching

Many new methods of teaching were created to meet the goals and objectives of the new programs. The developing of the new methods tried to replace the old method of teaching where the teacher lectures and the student memorizes and repeats by different new methods such as discovery method, laboratory method, and individual method. The new methods considered the student is the main part of the teaching process, and provide opportunities for students to be active and to participate through doing and using. As a result, there was a great change of focus from the traditional teacher-dominated ones to pupil-centered instruction. However, expository method is still used in many programs because "curricula are heavily crowded and students must pass official examinations which determine their getting jobs or entering the universities" (UNESCO, 1972, p. 100). The following are some of the new methods.

Discovery method. This is considered one of the most important methods associated with the new school mathematics. Discovery is particularly appropriate to the new mathematics because of the great emphasis on structure and logic in the new programs. Professor Polya said that the best way to learn anything is to discover it by yourself.

Discovery approach is opposite of the method in which the teacher tells students how to proceed in a certain situation or gives them a rule to follow. Discovery usually comes from an inductive approach while a rule-given procedure results from a deductive approach. Students may use objective materials or visual aids to organize a pattern. After using supplementary aids, students may discover the pattern. The opposite of this inductive approach is to tell the students the rule or procedure to learn and apply to examples.

Biggs describes five different types of discovery learning (Biggs, 1971, pp. 278-285):

1. Impromptu learning results when a situation stimulates the pupil to find an answer. The teacher neither initiates the problem nor aids in the solution.

2. Free exploratory discovery is different from impromptu learning because the teacher initiates the discovery by providing the kinds of materials that are effective in creating a situation for discovery.

3. Guided discovery also originates with the teacher who supplies the materials needed or asks questions that direct the learning.

4. Directed discovery is more structured or controlled than guided discovery, because the materials and questions are planned in advance by the teacher and are usually in the form of worksheets.

5. Programmed learning leads the pupil step by step to the conclusion of an activity. The work is highly structured and offers little opportunity for free-ranging discovery.

The laboratory method. This method consists of the performance of the work in the mathematical classroom (laboratory) which should be equipped with games to be played, tools and equipment for experimentation, blocks for building structures, patterns, concepts, boxes and barrels, shapes, scales and balances, rulers, measuring devices, etc. The laboratory method emphasizes the abilities to:

1. The ability to relate mathematical symbols and vocabulary to models.

2. The ability to manipulate mathematical symbols (the student should be able to work with symbols that represent ideas).

3. The ability to perceive the structure of mathematics.

4. The ability to think creatively about mathematics.

The individual method. This method aims to teach students individually, so that there may be individual progress according to individual strength. The basic components of individualized instruction are a set of objectives to be achieved, diagnostic pretests for each student for each unit of study, a prescription to guide students through the objectives to be attained, posttest and an opportunity to restudy to mastery.

D. Deduction

Some reformers felt that if mathematics were presented in a deductive reasoning form, students perceiving the structure of mathematical ideas would become more effective learners and users of the subject. Another purpose for using the deductive approach in the new mathematics is that most of the students seek from mathematics a collection of well established concepts and methods that can be applied to problems outside of mathematics.

Because of these two reasons, new programs emphasized the study of mathematical structure and the role of axioms and definitions. This has been accompanied by more use of logic and more emphasis on the functions of mathematics and the interrelations between many branches of mathematics.

Most of the programs that have been discussed tried to present the structure of mathematics by selecting some undefined terms called "Primitive terms." These terms supply basic words for communication. Thus, in geometry the word "Point" is used as an undefined term. Next, certain basic assumptions are called axioms are stated. These axioms are statements that provide relationships between the basic elements of the structure. Finally, certain theorems are stated and proved by a sequence of statements, each of these statements is either a definition, an axiom, or a previously proved theorem.

E. Symbolism and Terminology

Because of the introduction of the new abstract topics to school mathematics, more similarities and terminology have to be studied. One of the prominent manifestations of abstraction in mathematics is the symbolism used to convey ideas. Thus closed sentence, binary operation, closure, uniqueness of inverse, associativity, computativity, distributivity, empty set, line segments basis, group, field, ring, vector, vector space and several hundred other terminology are defined formally and included in the new mathematics curriculum. In general, the language of the new mathematics was extremely precise and clear to avoid confusion and to encourage insight. Table 1 shows the difference between

Table (1)

The Difference between Old and New Mathematical Language

Old Mathematics	New Mathematics	
AB	$\overline{AB}$	A segment
	AB	Measure of a segment
	$\overleftrightarrow{AB}$	A line
	$\overrightarrow{AB}$	A ray
<ABC	<ABC	An Angle
	M°<ABC	Measure of an angle in degrees
	M_r<ABC	Measure of an angle in radians

Criticism of the New Mathematics

One of the leading critics of the "new" mathematics is Professor Morris Kline of New York University. Professor Kline believes that students must do the building of the mathematical idea to understand what it means. As he pointed out, "Mathematics must be developed not deductively, but constructively. We must build up the concepts, techniques, and theorems from the simplest cases to the slightly more involved, and then to the still more involved" (Kline, 1966, p. 323).

The second major criticism of the new mathematics is its abstract presentation. Professor Kline believes that mathematics must be, as far as possible, presented in concrete situations, to provide enough experience to the student to participate, to be active, to gain understanding, and to develop and make his own definition of that particular topic.

The third criticism of the new mathematics is that mathematics is presented rigorously. On April 11, 1958, Professor Kline addressed the thirty-sixth annual meeting of the National Council of Teachers of Mathematics at Cleveland, Ohio. He said that "the presentation of mathematics in rigorous form is ill-advised on other counts. Mathematics must be understood intuitively in physical or geometrical terms... when this is achieved, it is proper to formulate the concepts and reasoning in as rigorous a form as young people can take" (Kline, 1966, p. 324). Another point came up as a rejection of rigorous presentation of mathematics "the capacity to appreciate rigor must be developed. The capacity to appreciate rigor is a function of the age of the student and not of the age of mathematics" (Kline, 1966, p. 324).

The wide use of symbolism and terminology is objected to by some mathematics users who believe that school mathematics must have as few symbols as possible, use common words, and keep the new terminology to a minimum.

Kline pointed out four principles which he felt provide more understanding of mathematics (Kline, 1958, pp. 425-427).

1. Mathematics teachers must try to arouse interest in the subject. If this is agreed upon, then we should select material which will serve the purpose of arousing interest.

2. Mathematics teachers must supply motivation and purpose to the mathematics they teach. This means that they must motivate each topic with a genuine problem and show that the mathematics does something to solve that problem.

3. Mathematics is primarily a series of great intuitions. The way to make the meaning of an idea clear is to present it in the intuitive setting that led to its creation or in terms of some simple modern equivalent. Physical or geometrical illustrations or interpretations will often supply this meaning.

4. Mathematics in every age has been part of the broad cultural movement of the age; we must then relate the mathematics to history, science, philosophy, social science, art, music, literature, logic, as well as to any other development which the topic at hand permits.

Summary

From the discussion of the various international conferences, and from the implementation of the mathematics projects discussed in this chapter, the investigator noticed that the new mathematics program have many common characteristics. However, it should be pointed out that not all programs give the same emphasis to each of these characteristics. The following is a summary of these qualities.

1. Mathematics curriculum should be designed for what the students are interested in.

2. Lecture method of teaching, rote learning, and memorization are the poorest of all kinds of learning.

3. Introduction of new topics and removal of other topics taught to be less significant.

4. Placement of topics using the spiral approach.

5. Presenting topics as ideas and concepts to unify and interrelate the different branches of mathematics.

6. Introduction of new methods of teaching where the student is the center of the teaching process.

7. Emphasis on discovery in teaching mathematics.

8. The role of teachers is to direct the students to the right sources and provide guidance.

9. Emphasis on teaching structure of mathematics.

10. Emphasis on precise language, symbolism, and terminology.

11. Emphasis on practical problems, and real life situations in the applications of mathematics.

12. Emphasis on instructional materials, modeling, concrete materials, films, slides, etc.

CHAPTER III

ANALYSIS OF THE LIBYAN SECONDARY SCHOOL MATHEMATICS PROGRAM

Introduction

It was not until the latter part of 1968 that real progress began to take place in the development of mathematics in Libyan schools. In 1969, Libya was the first Arabic country to introduce "the new mathematics" in all its secondary level schools (senior high school) under direction from UNESCO. In 1972, a committee consisting of mathematics professors from Fatah University in Tripoli and mathematics supervisors from the division of mathematics supervision in the Ministry of Education organized to revise the mathematics content in the preparatory level (junior high school).

Special summer training programs were held at the School of Education, Fatah University, for all preparatory and secondary school teachers. The main purpose of the program was to teach the teachers the new mathematics topics that would be included in the new content. The mathematics program included algebra topics such as set theory, system of real numbers, system of rational numbers, linear inequalities, linear programming, group theory, field theory, ring theory, and isomorphism. Geometry involved the study of vector geometry, geometric transformation, geometry in three dimensions, solid analytical geometry. Other topics such as logic and probability were included in the program, too.

In order to present a clear picture of the mathematics content, the outline of the two programs will be presented.

Mathematics Program of the First Period (1951-1968)

In 1951, Libya applied the Egyptian mathematics program with slight changes to fit the nature of Libya's educational system. The syllabus did not contain new topics and the method of teaching was largely a matter of showing students how to do algebra, for instance, and how to use it to solve problems, rather than understanding the underlying concepts and operations.

The program consisted of mathematics covering grades 10-12. Grade 10 (first year) included algebra and geometry. Grade 11 (second year) included algebra, geometry, mechanics, and trigonometry. Grade 12 (third year) included algebra, geometry, mechanics, and calculus. The following are the chapter headings of this program (Ministry of Education, 1959).

First Year (Grade 10)

Algebra (2 periods per week)

Chapter 1: Factorization, Greatest Common Factor, Least Common Factor

Chapter 2: Fractions, Addition of Fractions, Subtraction of Fractions, Multiplication of Fractions, Division of Fractions, Change of Sign

Chapter 3: Solution of Second Degree Equations that Contains One Unknown

Chapter 4: Equations of Two Unknowns of Second Degree

Chapter 5: Graphs, Plotting a Point, Drawing a Straight Line, Drawing of the Line, Solution of Equations of One Unknown Graphically

Geometry (2 periods per week)

Chapter 1: Pythagorean Theorem, The Converse of the Pythagorean Theorem

Chapter 2: Definition of Perpendicular Lines, Right Angles, Proofs of the Following Theorems:

(1) If two straight lines intersect, the vertical angles are equal.

(2) If two straight lines in the same plane are perpendicular to the same straight line, they are parallel.

(3) If two lincs are parallel to the same line, they are parallel to each other.

Chapter 3: Similarity and Figures, Ratio and Proportion, Similar Triangles, Applications of Similarity

Second Year (Grade 11)

Algebra (2 periods per week)

Chapter 1: Exponents, Zero Exponent, Negative-Integer Exponents, Rational Exponents, Radical Exponents

Chapter 2: Surd, Historical Background, Definition of a Surd, Multiplication of Surds, Some Operations on Surds

Chapter 3: Logarithms, Meaning of Logarithms, Basic Properties of Logarithms, Computing with Logarithms, Exponential and Logarithms Equations

Chapter 4: General Applications

Co-ordinate Geometry (2 periods per week)

Chapter 1: Distance Between Two Points, Division of Lines from Outside and from Inside

Chapter 2: Slope of Straight Lines, Slope of Straight Line Passed Through Two Points

Chapter 3: System of Linear Equations

Chapter 4: General Applications

Mechanics (2 periods per week)

Chapter 1: Scaler Quantities, Vector Quantities, Displacement, Results of Two Placement, General Applications

Chapter 2: Uniform Motion, Curves of Distance and Time, Curves of of Velocity and Time, General Applications

Chapter 3: Motion with Uniform Acceleration in Straight Line, General Applications

Chapter 4: Motion Under Gravity of Earth, With Applications

Trigonometry (1 period per week)

Chapter 1: Angles, The Sine and the Cosine of Angles, Trigonometric Ratios from 0° to 360°

Third Year (Grade 12)

Algebra (2 periods per week)

Chapter 1: Arithmetic Progression (A.P.), First Term of A.P. , General Term of A.P. , Common Difference of A.P.

Chapter 2: Geometrical Progression G.P. , General Term of Common Ratio of (G.P.), Mean, Geometrical Mean, Arithmetic Mean, Applications

Chapter 3: Annuity, Deferred Annuity, Amount of an Accumulated Annuity

Chapter 4: Permutation, Combination, General Applications

Chapter 5: Binomial Theorem

Chapter 6: Complex Quantities

Calculus (1 period per week)

Chapter 1: Limits, Constants, Variables, Functions, Limit Theorems

Chapter 2: Differentiation, Functions and Their Graphs

Chapter 3: Geometrical Applications

Chapter 4: Maximum and Minimum of Function

Chapter 5: Integration

Co-ordinate Geometry (1 period per week)

Chapter 1: Basic Definitions, Equation of Circle, General Form of Equation of Circle

Chapter 2: Tangent Line, Contact of Two Circles, Intersection of Two Circles

Mechanics (2 periods per week)

Chapter 1: Mass, Weight, Gravity, Units of Mass, First Law of Newton, System of Bodies, Second and Third Law of Newton, General Applications

Chapter 2: Work, Power, Energy, Applications

Chapter 3: Impulses

The Present Mathematics Program

The present mathematics program started in 1969 when the Ministry of Education called upon university mathematicians to organize the mathematics curriculum. The content of mathematics included new topics such as sets, algebraic structure, group theory, ring theory, field theory, and isomorphism. In geometry, topics such as transformation, matrix transformation, determinants, and solid analytical geometry were included. In statistics, the content included probability, regression of correlation and the different methods to calculate central tendency and dispersion. The following is the outline of the content as presented in the official textbooks of the Ministry of Education.

First Year. This year, called the first year of "Thanaweya," the program included algebra, geometry, and trigonometry.

Algebra:

Algebra was taught twice per week. It includes the following topics:

Sets. What a set is, finite sets, writing sets, sets of numbers, subsets, empty sets, equal sets, equivalent sets, complement, universal sets, intersection of sets, union of sets, improper subsets, proper subsets, Venn diagrams, cardinal numbers, commutative property. associative property, distributive property, zero property, unity.

Real Numbers. Set of counting numbers, set of real numbers, system of rational numbers, properties of rational numbers, set of irrational numbers, properties of irrational numbers, geometrical presentation for the square roots, addition and subtraction of roots, multiplication of roots, equations containing roots.

Rational Phrases. Addition of rational phrases, subtraction of rational phrases with different denominator, complex fractions, function of second degree, concept of function, value of function, degree of function, function of second degree and its equation, the function ax^2, the function $ax^2 + c$, the function $ax^2 + bx + c$, maximum and minimum values, addition and subtraction of roots, exponents and logarithms, law of exponents, $x^m \cdot x^n = x^{m+n}$, $x^m \div x^n = x^{m-n}$, $(x^m)^n = x^{mn}$, meaning of zero exponent x°, meaning of exponent $x^{a/b}$, meaning of exponent x^{-n}, theory of logarithms, system of logarithms, computation by means of logarithms, linear inequalities, axioms of inequalities,

if $a > b$, then $a + c > b + c$

if $a > b$, then $a - c > b - c$

if $c > o$ and $a > b$, then $ac > bc$

if $a > b$ and $c < o$, then $ac < bc$.

It also includes the study of

- one linear inequality in one variable
- one linear inequality in two variables
- system of linear inequalities containing one variable
- system of linear inequalities containing two variables
- inequalities of second degree

Linear programming, meaning of linear programming, general applications of linear programming (Bialah & Humdi, 1975).

Geometry:

Geometry is taught twice per week during the whole year. It includes vectors, displacement, concept of displacement graph of displacement, the result of two displacements, basic operations on vectors, physical applications on vectors, applications of vectors in the field of geometry, ordered pairs, magnitude of a vector, arithmetical operations on vectors as ordered pairs, basic concepts in the analytical geometry, linear equations and the straight line, directed distance, distance between two points, division of straight segment, linear equation, slope of line, linear equation of the form $\{(x,y) : y = mx + c,\}$ equation of the line when the slope and a point are known, parallel lines, perpendicular lines. The transformation chapter includes symmetry, reflection, pictures, transformation, and translation (Bialah, et al., 1974).

Trigonometry:

There is just one chapter for trigonometry. The chapter deals with special angles, area of a parallelogram, definition of Sin, Cos, Tan, Cot, Sec, Csc, graphs of $r = \frac{1}{60}(\theta)$, $r = (1 + \cos\theta)$, $r = 5 \cos 2\theta$, radian measure, proof of Law of Sines $\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$, proof of Law of Cosines $a^2 = b^2 + c^2 - 2bc \text{ Cos } A$, and proofs of some equalities (Bialah, et al., 1974).

Second Year. The mathematics program of the second year of "Thanaweya," which is equivalent to Grade 11 of senior high school, consists of pure mathematics and applied mathematics. The pure mathematics include algebra, geometry, calculus, and trigonometry.

Algebra:

Algebra is taught two periods per week and deals with the algebraic system of structure, number systems, the counting number system, the whole number system, the integer system, rational number system, the real number system, complex number system, modular arithmetic, algebraic structures, binary operation, unary operation, commutative properties, associative properties, distributive properties, closure, neutral element, inverse element, group theory, ring theory, integral domain, field theory, Abelian group, isomorphism, matrices, addition of matrices, subtraction of matrices, scaler multiplication, zero matrix, 2 X 2 matrices, inverse of matrices, commutative properties, associative property, and simultaneous equations. The chapter on series and sequences includes finite series, geometrical series, geometric progression, arithmetic progression, finite sequence, infinite sequence, limit of a sequence, sums of infinite sequences, convergent and divergent sequences. Mathematical induction deals with proof of some statement using mathematical induction method. The chapter of permutation and combination includes the meaning of permutation, meaning of combination, the binomial theorem, and proof of the binomial theorem (Ab -Yousef & Humdi, 1976).

Geometry:

Geometry is taught two periods per week. It consists of four chapters. Chapter I includes open sentences, intersection of sets graphically. The second chapter consists of angle between two lines, area of a triangle using determinants, the general case of area of a triangle. The third chapter presents circles, equation of circles with (0,0) center, equation of circles with (K,H) centers, parametric equation of circle, equation of tangent to circle, and exercises.

Chapter 4 includes the loci, equation of a point moving in a plane, and geometry in three dimensions (Ben-Hamed Lath & Humdi, 1976).

Trigonometry:

Trigonometry is taught one period per week and includes the following topics.

Proof of $\text{Sin } (A + B) = \text{Sin } A \text{ Cos } B + \text{Cos } A \text{ Sin } B$

$$\text{Cos } (A + B) = \text{Cos } A \text{ Cos } B - \text{Sin } A \text{ Sin } B$$

$$\text{Tan } (A + B) = \frac{\text{Sin } (A + B)}{\text{Cos } (A + B)} = \frac{\text{Tan } A + \text{Tan } B}{1 - \text{Tan } A \text{ Tan } B}$$

$$\text{Cos } (A - B) = \text{Cos } A \text{ Cos } B + \text{Sin } A \text{ Sin } B$$

$$\text{Sin } (A - B) = \text{Sin } A \text{ Cos } B - \text{Cos } A \text{ Sin } B$$

$$\text{Tan } (A - B) = \frac{\text{Tan } A - \text{Tan } B}{1 + \text{Tan } A \text{ Tan } B}$$

$$\text{Sin } C + \text{Sin } D = 2 \text{ Sin } \frac{c + d}{2} \text{ Cos } \frac{c - d}{2}$$

$$\text{Sin } C - \text{Sin } D = 2 \text{ Cos } \frac{c + d}{2} \text{ Sin } \frac{c - d}{2}$$

The second chapter includes proof of some trigonometric statements, trigonometric equations, relationships between the sides of a triangle when its sides are known, two sides and an angle known, two angles and one side known, and practical applications (Ben-Hamed, et al., 1976).

Calculus:

Calculus is taught twice per week. The topics of this subject are: binary operations and functions, variables, constants, one-to-one correspondence, the real variables, intervals, infinity, binary relations, definition of binary operations, domain and range, inverse relations functions, presentation of a function, $f = \{(x,y) : y = f(x)\}$, constant functions, independent and dependent variables, odd functions, even functions, the continuous functions, mappings (mapping, image, preimage), the meaning of onto, surjection mappings, bijection mappings. The chapter of limits deals with meaning of a limit, theories of limits and thus proofs.

$$\lim_{x \to a} (cx + k) = ca + k \qquad \forall\ c, k \in R$$

$$\lim_{x \to a} f_1(x) \pm f_2(x) = \lim_{x \to a} f_1(x) \pm \lim_{x \to a} f_2(x)$$

$$\lim_{x \to a} f_1(x) \div f_2(x) = \lim_{x \to a} f_1(x) \div \lim_{x \to a} f_2(x) \text{ if } \lim_{x \to a} f_2(x) \neq 0$$

The chapter on rate of change and the first derivative includes the rate of change, first derivative, increasing and decreasing function sign of the first derivative, and the theorems of differentiation:

if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$ for any rational number n.

if $y = k$, then $\frac{dy}{dx} = o$ where k is constant.

Let y be a differential function and c be a constant; thus the function is differentiable and

if $g = c.f$ then $\frac{dg}{dx} = c\,\frac{df}{dx}$.

If f and g are differentiable on an open interval (a,b), then

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}.$$

Product rule states that if the function f and g are differentiable on an interval I, then the product function $f \cdot g$ is differentiable on I. That is, if $y = f . g$, then $\frac{d}{dx}(f \cdot g) = m\,\frac{df}{dx} + n\,\frac{dg}{dx}$.

The chapter on integration includes the definite integrals, integration as sum,

definition (1) $\int_{a}^{b} f(x)\,dx = \lim_{x \to \infty} \sum_{r=1}^{n} f(x_r) \cdot x_r$ (Riemann Sums)

definition (2) $\int_{a}^{a} f(x)\,dx = 0$

definition (3) $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$.

geometrical meaning of the definite integral, the indefinite integral, and theorems of integration:

if n is any rational number, n = -1, then

$$\int x^n \, d\,x = \frac{1}{n+1} \quad x^{n+1} + C \qquad n \neq -1$$

$$\int K \; f(x)\,dx = K \int f(x)\,dx$$

$$\int (f_1(x) + f_2(x))\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx.$$

Fundamental Theorem of Calculus (without proof), geometrical application, plane areas, general exercises (Hamza & Ab-Yousef, 1976).

Applied Mathematics (Statics)

This branch of mathematics was taught twice a week. The content of mechanics includes in its first chapter, vectors, vectors and scalers, geometric presentation of vectors, equality of two vectors, kinds of vectors, addition of vectors, subtraction of vectors, null vector or zero vector, properties of vectors, commutative properties, associative roperties, cartesian presentation of vectors, polar presentation of vectors, matrices presentation of vectors, the relationships between the cartizian presentation and the polar presentation, multiplication of vector by scaler quantities, unit vector, position vectors, relative position, and resolution of vectors.

In the second chapter, there is presentation of force, length, mass, time, dimensions of units, reaction, statics, particle of two forces, analytical method to get the resultant of two forces, resultant of more than two forces.

Chapter three includes statics of a particle and system of particles, equilibrium, conditions of equilibrium, and some experiments.

Chapter four deals with moments and couples, line of action of a force, definition of moments, units of moments, some theorems, determination of line of action of a resultant of many forces, parallel forces, like forces, unlike forces, definition of couples, properties of couples, equivalent couples, equilibrium couples, experiments, and simple couple.

Chapter five includes equilibrium of rigid bodies, equilibrium of a rigid particle under action of two forces, experiment, friction, force of friction, coefficient of friction, angle of friction, cone of friction, experiment, mass center dealing with definition of mass center, some experiments and general applications (Khashaba, Lath, & Humdi, 1976).

<u>Third Year</u>. (Grade 12) This year is the last year of the "Thanaweya" level. The program of this year includes two branches of mathematics, pure mathematics and applied mathematics. The pure mathematics consists of algebra, geometry, trigonometry, calculus, and statistics. The applied mathematics (Dynamics) deals with two branches, Kinematics and Kinetics.

<u>Algebra</u>:

Algebra is taught twice per week. The algebra content includes definition of determinants, some properties of determinants, the minor and co-factor of determinants, kinds of determinants, symmetric determinants, skew symmetric determinants, multiplication of determinants, solving first degree equations using determinants, solving equations of two and three variables using determinants.

The chapter on matrices includes writing a matrix, dimension of a matrix, singular matrix, non-singular matrix, subtraction of matrices, equal matrices, multiplication of matrices, the reciprocal matrix, square matrix, the transpose of a matrix, the transpose of a product, symmetric and skew-symmetric matrices, the single column matrix.

Chapter three includes complex number systems, definitions, field of complex numbers, set of real numbers in a subset of the set of complex numbers, isomorphism, geometrical presentation of the complex numbers, proof of $(\cos e + i \sin e)^n = \cos n e + i \sin n e$.

The last two chapters deal with theories of rings and fields, integral domains, logic, propositions, compound propositions, connectives, logical statements, truth tables, conditional propositions, tautology, quantifiers, and algebra of propositions (Hadad, Chawdy, & Humdi, 1976).

Geometry:

Geometry includes geometric transformations, point transformation, translation, rotation, isometrics. The second chapter includes matrices transformation and rotation with an angle.

The third chapter presents conic sections, the parabola, the ellipse, the hyperbola.

Chapter four includes geometry in three dimensions, regular solids, prisms, parallelopipeds, cubes, pyramids, regular pyramids, right circular cylinders, right circular cones, and spheres.

The last chapter deals with solid analytical geometry, meaning of one dimension, two dimensions, three dimensions, distance between two points in space, distance between two points using the vector method, Cartesian product in three dimensions, law of division of a distance between two points (using vectors), equation of a plane passed through three points, conditions of intersection of three planes (Rajab & Humdi, 1976).

Calculus:

Calculus is taught twice per week. It includes types of functions, explicit functions, inverse functions, function of a function, and parametric functions.

The second chapter includes first derivative for transcendental functions, second derivative of functions, function of functions, (composite functions), implicit differentiation, differentiation of inverse functions.

Chapter three includes transcendental functions, first derivative of trigonometric functions, exponential functions, and logarithmic functions.

Chapter four includes applications on differentiation, applications using chain rules, velocity and acceleration, maximum and minimum functions increasing and decreasing functions, graphs, concavity, and inflection points.

The last chapter includes integration and its methods, review of methods of integration, integration of trigonometric functions, integration of exponential functions, integration by parts, applications on using integrations, volumes of uniform solids, and volumes of solids of revolution (Rajab, Hamza, & Hadad, 1976).

Statistics:

Statistics is taught one period per week. It includes data, presentation of data, description of data, tables, graphs, organizing data, bar graphs, histograms, central tendency, the mean, the mode, the median, quartiles, uses of the various means, dispersion, range, quartile deviation, mean deviation, standard deviation, skewness, regression and correlation, demographic statistics, census, bio statistics, probability, basic concepts of probability, sample space, events, finite equi-probable sample, spaces conditional probability, and independence (Dash & Humdi, 1976).

The Applied Mathematics (Dynamics):

Applied mathematics is taught twice per week for the whole year. The first chapter includes velocity and acceleration, movement of particles, translation, average velocity, instantaneous velocity, speed, relative velocity, average acceleration, instantaneous acceleration, applications, projectiles, time of flight.

The second chapter deals with circular motion, angular velocity, instantaneous center.

Chapter three includes simple harmonic motion, periodic time. In Chapter four, the student studies Newton's Laws, absolute units, gravitational units.

Chapter five includes three topics: work, power, and energy. The work is defined as $W = \vec{F} \cdot \vec{D}$, where F is force and D is distance. Power is defined as force times velocity. The energy part emphasized study of two forms of energy, potential energy and kinetic energy, the principle of conservation of energy and constrained motion.

The last chapter deals with impulse and impact, conservation of momentum and direct impact (Khashaba, Lath, & Humdi, 1976.)

Analysis of the Program

The Questionnaire.

In order to understand the problems of the mathematics program and to make the analysis more objective, it was necessary to obtain the opinions of mathematics teachers in Libyan secondary schools.

The questionnaire contains 19 questions concerning different aspects of the program, such as participation of teachers and students in the creating of the curriculum, quantity of material, method of teaching, applications, instructional materials, and method of teaching. In order to give the mathematics teachers the full opportunity to answer the questions, it was necessary to write the questionnaire in the native language, Arabic (a copy of the questionnaire is reported in Appendix A).

The Questions

1. Do you think that mathematics teachers should participate in setting the mathematics curriculum?

2. Do you think the opinions of the students should be considered in the development of the school of mathematics?

3. Do you think the Applied Mathematics (Statics and Dynamics) should be omitted from the secondary school mathematics?

4. Do you prefer to teach some topics that are not mentioned in the official textbooks?

5. Do you prefer to teach mathematics as one unified subject each related to the other, or as separate subjects, such as algebra, geometry, or calculus?

6. Do you think that the time devoted to the mathematics program of the first year is sufficient?

7. Do you think that the time devoted to the mathematics program of the second year is sufficient?

8. Do you think that the time devoted to the mathematics program of the third year is sufficient?

9. Does the method of teaching that you use now provide opportunity for the students to participate in the classroom?

10. Does the method of teaching mathematics that you use now help the students to develop mathematical concepts?

11. Does the method of teaching mathematics that you use now provide opportunity for the students to discover mathematical ideas?

12. Do you think that the program should include non-routine problems?

13. Do you think that the program should include real-life prolems?

14. Do you think that the program includes instructional materials and visual aids?

15. Do you think that the textbooks should be accompanied by teacher's commentaries which contain review items, test questions, and elaborations on certain difficult topics?

16. Do you think that the placement of topics in the program is appropriate?

17. Do you think that the programs include devices to motivate the students?

18. Do you think the program provides opportunity for students to learn mathematical generalizations?

19. Do you think the program provides opportunity for students to construct mathematical proofs?

Results of the Questionnaire

The number of secondary school teachers in Libya is about 217. The number of responses was 128 from secondary school teachers. The responses included representation from different regions of the country. For the purpose of this study, analysis of the questionnaire item by item will be helpful.

1. Out of 128, 128 (100%) agreed that a composite of secondary school teachers should participate in setting the mathematics curriculum.

2. Out of 128, 71 (55%) of the respondents agreed that the opinion of the students should be considered in the development of the school mathematics.

3. Out of 128, 100 (78%) agreed that the subject of Mechanics (Applied Mathematics) should be omitted from the secondary school program.

4. Out of 128, 70 (55%) of the respondents prefer to teach some topics that are not mentioned in the official textbooks.

5. Out of 128, 84 (66%) of the respondents prefer to teach mathematics as one subject whereas 44 out of 128 (34%) of the respondents prefer to teach mathematics as separate subjects. Most of these two groups emphasized the unity of some topics from algebra and calculus.

6. Out of 128, 84 (66%) of the respondents agreed that the time devoted to the first year mathematics program is sufficient.

7. Out of 128, 48 (37%) of the respondents agreed that the time devoted to the second year mathematics program is sufficient.

8. Out of 128, 19 (15%) of the respondents agreed that the time devoted to the third year mathematics program is sufficient.

9. Out of 128, 80 (62%) of the respondents agreed that the method of teaching that they use provides opportunity to the students to participate in the classroom.

10. Out of 128, 58 (45%) of the respondents agreed that the method of teaching they use helps the students to develop mathematical concepts.

11. Out of 128, 59 (46%) of the respondents agreed that the method of teaching that they use provides opportunity for the students to discover mathematical ideas.

12. Out of 128, 95 (74%) of the respondents agreed that the present program should include some non-routine problems.

13. Out of 128, 89 (69%) of the respondents agreed that the program should include real-life problems.

14. Out of 128, 92 (42%) of the respondents agreed that the program should include instructional material or visual aids.

15. Out of 128, 108 (84%) of the respondents agreed that the textbooks should be accompanied by teacher's commentaries.

16. Out of 128, 93 (73%) of the respondents agreed that the placement of topics in the program is appropriate.

17. Out of 128, 101 (79%) of the respondents indicated that the mathematics program does not include devices for motivating the students.

18. Out of 128, 90 (70%) of the respondents agreed that the mathematics program does not provide opportunity for students to learn mathematics generalization.

19. Out of 128, 70 (55%) of the respondents agreed that the mathematics program does not provide opportunities for the students to construct a mathematical proof.

Critical Analysis of the Program

A. Organization that Determines the Curriculum

A committee was appointed by the Mathematics Supervision Department in the Ministry of Education to organize the high school mathematics curriculum. The committee consisted of university mathematics professors and mathematics education supervisors, but seldom were mathematics teachers included. The writer feels that in order for Libyan mathematics teachers to be able to cope with difficult situations, and to be sensitive to the needs of the students, they must participate in this committee since they are the ones who are practicing the teaching and carrying out the recommendations of this committee. As evidenced by 128 out of 128 (100%) of the responses, it was agreed that the teachers should participate in the setting of the curriculum and 55% agreed that the opinion of the students should be considered (Questions 1 and 2).

B. Quantity of Material

The mathematics program of Libyan secondary schools contains a great quantity of material which is written in detail. The so-called Mechanics consists of topics which are often included in the physics course or are unnecessary and irrelevant to the needs of the students as evidenced by 78% of the respondents agreeing that the subject of Mechanis should be omitted from the secondary school program (Question 3).

Because teachers are not given sufficient time to cover all the material, students are trained to get the information passively from the teacher and repeat what they have heard with little chance for understanding or independent thinking. About two months before the end of the school year, students are asked to come in the afternoon for extra mathematics lessons because they have not covered the whole book which has to be covered before the examination. Too much emphasis on content has made mathematics teachers neglect students and their participation in the classroom. This generates negative attitudes toward mathematics creating a dislike for mathematics and making students even afraid of it.

From the questionnaire (Questions 6, 7, 8), we see that 84 out of 128 (66%) agreed that the time devoted to the first year program is sufficient; 34% agreed that the time devoted to the second year is sufficient, and only 15% agreed that the time devoted to the third year is sufficient.

The philosophy that forces students to learn a certain body of knowledge which school systems such as the Ministry of Education feel they ought to know is an invalid philosophy because the goals of the program must not concentrate only on preparation for more advanced mathematics courses. It must concentrate on understanding those courses as well.

C. Placement of Topics

The spiral approach to the placement of topics is a major characteristic of the Libyan secondary program.

Topics are presented at different levels with differing degrees of difficulty. For example, regarding the concept of the algebraic structure in Algebra I (Grade 10), the chapter starts with the set of counting numbers {1, 2, 3, . . . }, the set of whole numbers being defined as the set containing zero and the counting numbers. The rational numbers are defined as the set of all numbers that can be be expressed as the quotient of two integers. In the second algebra book, Grade 11 (Ab-Yousef, et al., 1976, pp. 27-31), definitions and examples of the following properties are presented:

Binary operation - A binary operation is a rule which assigns to each ordered pairs of elements from a set S a uniquely defined element of a set T.

Closure - An operation is said to be closed if the element assigned to the ordered subset of S is also an element of the set S.

Commutativity - An operation * is commutative with respect to a set of elements S if $a * b = b * a \quad \forall\, a, b \in S$.

Associativity - An operation * is associative with respect to a set of elements S if $(a * b) * c = a * (b * c)$.

Distributivity - An operation * is said to be distributive over an operation o with reference to a set S if $a, b, c \in S$

$a * (b \circ c) = (a * b) \circ (a * c)$ left distributive

$(b \circ c) * a = (b * a) \circ (c * a)$ right distributive

Identity element - The element N is said to be the identity element of N if $e * a = a \quad \forall\, a \in N$.

Inverse element - the element a^{-1} (read an inverse) of a if $a * a^{-1} = e$.

In the third algebra book Grade 12 (Haddad et al., 1976, pp. 167-177), students apply the properties that are studied in Grade 11 to more advanced algebraic structure such as group theory, field theory, ring theory, isomorphism, and integral domains.

In general, the material seems to be placed in logical sequence compared to other mathematics programs. About 73% of the respondents agreed that the placement of topics in the program is appropriate (Question 16).

D. Applications

The program put too much emphasis on routine problems and ignored completely the non-routine problems. In the end of every topic, a list of direct problems has been listed as "Exercises." Most of the problems do not provide opportunity for the applications of mathematics to problems arising from real life situations. There are two types of problems included in the programs: non-verbal problems and verbal problems. Most of the problems listed as exercises are non-verbal problems as in the following examples:

> Find the equation of the line that passed through the points (2,3), (5,6). Geometry, Grade 12 (Ben-Hamed et al., 1976, p. 70).
>
> Use matrices to solve:
>
> $2X + 3Y - Z = 5$
>
> $3X - Y + Z = 4$
>
> $5X + 2Y - Z = 3$, Algebra Grade 12 (Haddad et al., 1976, p. 120).

The second type of problems included in the program are verbal problems as in the following example:

> A man (A) walked from the point w with displacement (4,3) every day for 4 days). On the fifth day, he reached to the point K with displacement (1,-6). A man (B) walked from the point W with displacement (2,6), then walked with displacement (3,-5) until he reached the point T. Find the displacement TK and the distance TK. Geometry, Grade 10 (Bialah et al., 1975, p. 71).

The solution is demonstrated in the book as follows:

$$\overline{WT} = 4(4,3) + (1,-6)$$

$$= (16,12) + (1,-6) = (17,6)$$

$$\overline{WK} = (2,6) + (3,-5) = (5,1)$$

$$\overline{TK} = \overline{WK} - \overline{WT}$$

$$= (5,1) - (17,6) = (-12,-5)$$

$$\text{The distance TK} = \sqrt{(-12)^2 + (-5)^2}$$

$$= \sqrt{144 + 25} = \sqrt{169} = 13$$

The writer, with the support of the teachers, believes very strongly that the program must include problems arising from real life situations. There were 95 out of the 128 responses (74%) who agreed that the present program should include non-routine problems (Question 12), and 89 out of 128 (69%) agreed that the program should include real life problems (Question 13).

E. Skills vs. Concepts.

The program puts too much attention on rules and definitions. In algebra, the program focuses on operations and procedures with expressions and solution of equations. For example, in the Algebra Book II (Ab-Yousef et al., 1976, p. 187), the authors presented the topic of Binomial Theorem in a way that emphasizes memorization of facts, neglecting the developing of the idea. The following is the presentation of Binomial Theorem in the official textbook:

> The Binomial Theorem, discovered by Newton, is to be a very important event in the history of mathematical science. We will study a special case of this theorem.

If n is a positive integer, then

$$(x + y)^n = x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 \; . \; . \; . + y^n$$
$$= \sum_{r=0}^{n} {}^nC_r y^r \, x^{n-r} \qquad . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

Proof: we may examine the validity of this theorem by using the mathematical induction method.

Let n = 1: $\text{R.H.S.} = (x + y)^1 = x + y$

$\text{L.H.S.} = x^1 + {}^1C_1y \, x^{1-1} = x^1 + y.x^0 = x + y$

Therefore the theorem is valid for n = 1.

Suppose the theorem is valid when n = k

i.e., $(x + y)^k = x^k + {}^kC_1x^{k-1}y + {}^kC_2x^{k-2}y^2 + {}^kC_3x^{k-3}y^3$

$$+ \; . \; . \; . \; {}^kC_rx^{k-r}y^r + \; . \; . \; . + y^k \text{ ------------ } (1)$$

Now, multiplying both sides by (x + y),

$$(x + y)^{k+1} = (x + y) \;\; x^k + {}^kC_1x^{k-1}y + {}^kC_2x^{k-2}y^2 + \; . \; . \; .$$
$${}^kC_rx^{k-r}y^r + \; . \; . \; . + y^k$$

By grouping the factors of x^{k+1}, $y\,x^k$, $y^2\,x^{k-2}$, $y^3\,x^{k-2}$ . . .
we find that,

$$(x + y)^{k+1} = x^{k+1} + ({}^kC_1 + 1)\; y\,x^k + ({}^kC_2 + {}^kC_1)\; y^2\,x^{k-1}$$
$$+ \; . \; . \; . + ({}^kC_r + {}^kC_{r-1})\; y^r\,x^{k+1-r} + \; . \; . \; . + y^{k+1}$$
$$= x^{k+1} + {}^{k+1}C_1\; y\,x^k + {}^{k+1}C_2\; y^2\,x^{k-1} + \; . \; . \; .$$
$$+ {}^{k+1}C_ry^rx^{k+1-r} + \; . \; . \; . + y^{k+1}$$

and by using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$,

we see that the expansion of $_k\;(x + y)^{k+1}$ has the same form as the expansion of $(x + y)^k$.

So, if the theory is valid when n=k, then it is valid when n = k + 1. But it is valid when n=1, therefore it is valid when n=2, n=3, n=4, and it is valid when n is any positive integer.

In the opinion of the investigator, this method of presenting the topic of Binomial Theorem is skill oriented. It emphasizes memorization of the expansion of $(x + y)^n$, void of true understanding. As evidenced, 45% of the respondents believed that the program does not help students to develop mathematical concepts (Question 10).

The Binomial Theorem can be best introduced by using inductive methods and Passcal's triangle to develop the expansion of $(x + y)^n$, demonstrate specific cases such as $(x + y)^o$, $(x + y)^1$, $(x + y)^2$, etc., which will lead to the generalization of $(x + y)^n$.

F. Method of Teaching

The method of teaching is a lecture method with no attempt to include any form of discovery or participation from the students. This can be seen from the analysis of presentation of the binomial theorem above and from the analysis of the evaluated topic presented in Chapter V. Although 62% of the respondents believe that teachers provide opportunity to the students to participate in the classroom (Question 9), 46% of the respondents agreed that the method of teaching they use now provides opportunity to discover mathematical ideas (Question 11).

G. Language

Vocabulary and symbols which are used throughout the whole program are formal but not precise. As an example, the first two Arabic letters of the word sum (مجـ) is used to mean sum of 1 + 2 + 3 + ... +n. (Σ) which means S also used to represent the sum of 1 + 2 + 3 + ...+n. However, two different notations for the same concept may be confusing to students upon first exposure to them.

An example of formality may be considered the definition of the cartisian product as seen in Algebra Book I, p. 39.

$$x \times y = \{(x,y) : x \varepsilon X, y \varepsilon Y\}$$

In addition to the formal language, emphasis is placed on the symbolism. In algebra for example, the following symbols are in the program. ($a \varepsilon B$) reads "a is a member of a set B," ($\emptyset$) represents the empty set, (U) means union, ($\cap$) means Intersection.

H. Instructional Material

Visual aids and concrete situations in teaching mathematics are considered to be a very important factor in any mathematics program because they add the variety, the depth, and the breadth which make the learning process pleasant and meaningful.

Solid geometry of the third year, Grade 12 (Rajab, & Humdi, 1976, pp. 149-170), involves the study of five solids.

Prisms: definition of prism, total area of prism, lateral area of prism, volume of prism.

Pyramids: definition of pyramid, regular pyramid, types of pyramids (triangular pyramid, square pyramid, hexagonal pyramid, octogonal pyramid), volume of pyramids.

Cylinders: definition of a cylinder, right cylinder, an oblique cylinder, the volume of a cylinder.

Cones: definition of cone, types of cones (right cone, oblique cone), lateral area of cone, total area of cone, volume of cone.

Sphere: definition of sphere, volume of a sphere, area of circular regions, surface area of a sphere.

Although the chapter of solid geometry covers much mathematics and consists of many examples and exercises, it does not provide opportunity for the student to see the solids or encourage them to make some. From the questionnaire, we can see that 88% of the respondents believed that the present mathematics program does not include instructional material to help the students to visualize the mathematical idea.

The writer suggests very strongly that the program must provide models, or the teacher encourage students to make models from cardboard as shown in Appendix B.

Summary

The Libyan Secondary School Mathematics Program was analyzed in terms of participation of teachers in creating the program, quality of material, placement of topics, applications, skills vs. concepts, method of teaching, and instructional material.

The method of presentation in both classrooms and textbooks that are used by the majority of the mathematics teachers consisted of traditional lecture-type approaches.

CHAPTER IV

CRITERIA FOR UNDERSTANDING MATHEMATICS

Introduction

The purpose of this chapter is to identify and develop criteria for understanding mathematics. The criteria are developed and presented under headings that are closely related to general ideas or categories. Each category is divided into subcategories which are considered part of its general idea of that category. These headings are: computational skills, discovery, comprehension, analysis, applications, instructional materials, and motivations.

Each criterion has an introductory statement which either clarifies what the writer means by the headings or indicates the significance of the ideas encompassed by it.

Sense of Understanding

As with many concepts that are basic or primitive to a discipline, "understanding" is very difficult to define. The reason for that difficulty is that understanding can never be seen directly, but it can only be determined by the behavior of the students.

The term "understanding" is a hypothetical term inferred from observations of behavior which take forms of performances, abilities, and behavior.

John Holt, in his book How Children Fail, said, "I feel I understand something if and when I can do some, at least, of the following:

(1) state it in my own words; (2) give examples of it; (3) recognize it in various guises and circumstances; (4) see connections between it and other facts or ideas; (5) make use of it in various ways; (6) foresee some of its consequences; (7) state its opposite or converse" (Holt, 1976, p. 104).

Understanding a mathematical idea occurs when deep insight and meaning are given to that idea. These insights and meanings require practice, manipulation of concrete representation materials and constructive activities resulting in intellectual or material products.

Understanding requires participation and communication between teachers and students to help students discover facts, rules, patterns, and make generalizations. It requires applications of mathematics in our real world, making relationships between elements of mathematical problems, and analyzing data from algebraic form to verbal form and vice versa.

The 45th yearbook of the National Society for the Study of Education (NSSE) presents the following characteristics of understanding:

1. A pupil understands when he is able to act, feel, or think intelligently with respect to a situation.

2. The completeness of understanding to be sought varies from situation to situation and varies in any learning situation with a number of factors.

3. The pupil must develop worthwhile understanding of the world in which we live as well as of symbols associated with this world.

4. Understanding should be verbalized, but verbalizations may be relatively devoid of meanings.

5. Understandings develop as the pupil engages in a variety of experiences rather than through doing the same thing over and over again.

6. Successful understanding comes in large part as a result of the method employed by the teacher (NSSE, 1946, pp. 28-37).

Developing Criteria for Understanding Mathematics

Understanding mathematics, which is part of learning mathematics, is a behavioral outcome usually determined by means of statements called "objectives." These objectives show the kind of performance that will be accepted as evidence that the learner has achieved the objectives.

Three sources have been used in this study for developing the criteria for understanding mathematics. The first source has been an analysis of objectives and methods of presenting mathematics in both the classroom and textbook of mathematical programs as presented in Chapter II. The second source has been the analysis and study of the problems of the Libyan mathematical program as presented in Chapter III. The third source has been a survey of objectives and goals of some organizational framework that has contributed to mathematical education. The following is some of that organizational framework.

1. One of the most useful classifications of objectives in mathematics achievement is the Taxonomy of Educational Objectives, Bloom's Taxonomy, Handbook 1 (Bloom, 1965). This handbook presented eight important areas of achievement in mathematics.

Knowledge: Knowledge of specific facts, knowledge of terminology.

Means of dealing with specifics: Knowledge of conventions, knowledge of trends and sequences, knowledge of classes and categories, knowledge of criteria, knowledge of methodology.

Knowledge of universals and abstractions: Knowledge of principles and generalizations, knowledge of theories and structure.

Comprehension: Transition, interpretation, extrapolation.

Applications: Problems in a real world.

Analysis: Analysis of elements, analysis of relationships, analysis of organizational principles.

Synthesis: Production of a unique communication, production of a plan or proposed set of operations, derivation of a set of abstract relations.

Evaluation: Judgement in terms of internal evidence, judgment in terms of external criteria.

2. When the experimental curriculum materials of the SMSG, IUSCM, UUMAP, and GCM mathematics projects were completed, the National Longitudinal Study of Mathematical Abilities began a series of achievement tests. The following criteria illustrate levels of cognitive behavior in mathematics.

Knowing: Knowing terminology.

Translating: Changing from one language to another, expressing ideas in verbal, symbolic, or geometric form codifying patterns.

Manipulating: Carrying out algorithms using techniques.

Choosing: Making comparisons, selecting appropriate facts and techniques, guessing, estimating, changing one's approach, selecting new symbolism.

Analyzing: Analyzing data, finding differences, recognizing relevant and irrelevant information, seeing patterns, isomorphisms, and symmetrics, analyzing proofs, recognizing need for additional information, recognizing need for proof or counter examples.

Synthesizing: Specializing and generalizing, conjecturing, formulating problems, constructing a proof or a problem.

3. The broad goals model of teaching mathematics is considered an important source for the criteria. This model consists of four general goals, utilitarian goals, social goals, cultural goals, and personal goals. Each category of the four broad goals previously stated includes general statements which are described in more specific statements of objectives. The particular model of instructional objectives described here is developed by P. G. Scope, University of Cambridge. The model includes four main categories (Scope, 1973, pp. 11-20).

Utilitarian Goals

(a) Mathematics for everyday life: Number and number operations, measurement and approximation, basic geometry, graphs and relations.

(b) Tool for other subjects: Basic algebra, calculus, trigonometry, statistics, vectors, matrices, etc.

(c) Foundation for further study in mathematics.

Social Goals

(a) Methods of investigations: Discovery method, inductive method, deductive method, individual method, laboratory method.

(b) Work with others: Organization, care of equipment, community rights, social motivations.

Cultural Goals

(a) Historical developments: Original thought processes, developments based on these examinations of structure.

(b) Mathematics as a language: Shapes, sizes, and changes in the power of symbolism, mathematical models.

(c) Mathematics and logic: To think logically, to analyze a complex situation, recognize the logical relations among interdependent factors.

(d) Aesthetic appreciation: Beauty in geometrical forms found in nature and art, appreciation of the power of mathematics and its contribution to the development of civilization and, in particular, science.

Personal Goals

(a) Character building through: Active involvement, personal successes, work with others.

(b) Opportunities for stimulating curiosity, self-expression, self-criticism.

4. The National Council of Teachers of Mathematics (NCTM) presented recommendations for school mathematics for the 1980's. The following are some of the recommendations (NCTM, 1970, p. 7).

Mathematics programs should give students experience in the application of mathematics, in selecting and matching strategies to the situation at hand. Students must learn to:

- formulate key questions
- analyze and conceptualize problems
- define the problem and the goal
- discover patterns and similarities
- seek out appropriate data
- experiment
- transfer skills and strategies to new situations
- draw on background knowledge to apply mathematics.

Students should be encouraged to question, experiment, estimate, explore, and suggest explanations. Problem solving, which is essentially a creativity, cannot be built exclusively on routines, recipes, and formulas.

The identification of basic skills in mathematics is a dynamic process and should be made in order to reflect new and changing needs.

There should be increased emphasis on the following activities:

- locating and processing quantitative information
- collecting data
- organizing and presenting data

- interpreting data
- drawing inferences and predicting from data
- estimating measure
- measuring using appropriate tools
- mentally estimating results of calculations
- calculating with numbers rounded to one or two digits
- using imagery, maps, sketches, and diagrams as aids to visualizing and conceptualizing a problem
- using concrete representations and puzzles that aid in improving the perception of spatial relationships.

The Criteria

In this section, criteria for understanding mathematics are presented as general headings, each heading consisting of specific statements that are illustrated with examples.

I. Computational Skills

Computational skills are considered to be a very important part of learning and understanding in mathematics because they provide sources for insight into the structure of our mathematical system and they promote productive thinking in problem solving. In 1959, a study made by the College Entrance Examination Board indicated that skills are surely needed but they must be based on understanding and not merely on rote memorization. Once meaning has been achieved, then drill should be provided to establish skills.

Another study made in 1961 by Organization for European Economic Co-operation stated that children should know how to do simple and rapid mental computations; they must be accustomed to finding very quickly the order of magnitude for a sum or a product.

The 1978 Yearbook of the National Council on Teachers of Mathematics (NCTM) presented four important purposes of teaching computational skills: (1) it facilitates the teaching of many topics, (2) it helps children to understand both the meaning and the significance of the mathematics they are learning, (3) it facilitates the exploration of topics and the recognition of generalization, and (4) it has social utility (NCTM, 1978, pp. 1-12).

There are three subcategories of the computational skills level:

A. Knowledge of specific facts.

B. Knowledge of terminology.

C. Ability to carry out an algorithm.

Students in the secondary school have been exposed to many facts through the high school such as: 4 is an even number; π is irrational; the measure of a right angle is 90°.

A. <u>Knowledge of Specific Facts</u>

This subcriterion refers to mathematical knowledge such as formulas and relationships. Secondary school students are expected to recall or recognize material that they learned before. The following examples illustrate this subcriterion.

<u>Example 1</u>: Name the following properties:

a. $a + b = b + a$ d. $a \cdot 1 = a$

b. $a + 0 = a$ e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

c. $a + (b + c) = (a + b) + c$ f. $a \cdot b = b \cdot a$

<u>Example 2</u>: An axiom is:

a. A proof c. A proposition to be proved

b. An undefined term d. An assumption

<u>Example 3</u>: The formula for the area of a circular region with a radius r is:

a. $2\pi r$ b. 2π c. πr^2 d. $2\pi^2 r$

B. <u>Knowledge of Terminology or Symbolism</u>

This can be rephrased as familiarity with the language of mathematics. In other words, the shorthand used by mathematicians to express a certain mathematical idea, mathematical relationship, or mathematical operations. The following are some examples.

<u>Example 1</u>: Which of the following is a <u>prime number</u>?

a. 6 b. 39 c. 11 d. 51

<u>Example 2</u>: The <u>additive inverse</u> of 7 is:

a. $\frac{1}{7}$ b. 7 c. -7 d. $\frac{-1}{7}$

<u>Example 3</u>: The absolute value of a number N is written as:

a. $-N$ b. $|N|$ c. $\vec{N}$ d. $\sqrt{N}$

<u>Example 4</u>: 5! means:

a. $5 + 5 + 5 + 5 + 5$ c. $5 \times 4 \times 3 \times 2 \times 1$

b. $5 \times 5 \times 5 \times 5 \times 5$ d. $\sqrt[5]{5}$

C. Ability to Carry out Algorithms

An algorithm is a sequence of steps toward the solution of a particular type of problem. These steps are always precise and well defined. The students can easily memorize or recall algorithms once they understand the process and can perform each individual step. The following example shows how students perform routine manipulations with understanding.

Example 1: A company manufactures three models of T.V.'s, A, B, and C. Each model needs certain numbers of tubes and speakers as follows:

	Model A	Model B	Model C
Number of tubes	13	18	20
Number of speakers	2	3	4

Suppose orders were received in January and February as follows:

	January	February
Model A	12	6
Model B	24	12
Model C	12	9

To determine the number of tubes and speakers required to manufacture these orders in each month, the following procedures must be done.

The number of tubes required in January

$$13(12) + 18(24) + 20(12) = 828$$

The number of speakers in January

$$2(12) + 3(24) + 4(12) = 144$$

The number of tubes and speakers for February are, respectively

$$13(6) + 18(12) + 20(9) = 474$$

$$2(6) + 3(12) + 4(9) = 84$$

The four sums can be arranged as follows:

	January	February
Number of tubes	828	474
Number of speakers	144	84

The equation can be presented in equation form

$$\begin{bmatrix} 13 & 18 & 20 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 12 & 6 \\ 24 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 828 & 474 \\ 144 & 84 \end{bmatrix}$$

After students understand the meaning behind the steps, the multiplication of two matrices, a general algorithm can be presented by:

The product of two matrices A and B is as follows:

$$A \cdot B = \begin{bmatrix} \text{1st row of A x 1st column of B} & \text{1st row of A x 2nd column of B} \\ \text{2nd row of A x 1st column of B} & \text{2nd row of A x 2nd column of B} \end{bmatrix}$$

In teaching computational skills, teachers should encourage students to:

- memorize material they reasonably understand
- memorize basic facts soon after they develop an understanding of symbolism and terminology by (a) recognize embodiment of the fact, (b) understand the concept of the fact, (c) use the fact in simple exercise.

Practice is a major part of teaching computational skills, which can be effective by using the following:

- participation of students in drill with intent to memorize
- during drill session teachers should emphasize remembering - not lengthy explanations
- making practice time short, and have assignment almost every day
- give instruction to students, show them how to practice
- present practice in a variety of activities such as games, contests, puzzles.

II. <u>Discovery</u>

Because free discovery learning is difficult, if not impossible to achieve, guided discovery has emerged as a valuable element of understanding mathematics. In using guided discovery the students are encouraged to think for themselves and to draw conclusions from situations which may be contrived by the teacher. "The strategy of guided discovery encourages students to think on their own, to learn on their own, and to become independent of the teachers" (Krulik & Weise, 1975, p. 138). The role of the teacher in this technique as a guide helps the students to make conjectures on what appears to be true and then to explain these conjectures to other students. An example of guided discovery techniques will be given in Chapter V - Method of Presentation B.

There are three important subcriteria of discovery.

A. Guided discovery.

B. Participation in the classroom.

C. Exploration of the materials.

III. Comprehension

Many scholars agree that comprehension level behavior is more complex than computational behavior because it involves knowledge of some abstractions, structure, knowledge of generalizations, and decision making. There are six subcategories of this level: (A) Knowledge of concepts, (B) Knowledge of rules and generalizations, (C) Knowledge of mathematical structure, (D) Ability to translate problems or sentences into algebraic or graphic forms, and conversely the ability to interpret algebraic or graphic representation, (E) Ability to follow a line of reasoning, (F) Ability to recognize a pattern.

A. Knowledge of Concepts

Van Engen, in the Twenty-first Yearbook of the National Council of Teachers of Mathematics, The Learning of Mathematics: Theory and Practice, pointed out that, "The confusion has not been materially reduced; one cannot always tell what a person means by a concept" (Conny, Davis, & Henderson, 1974, p. 80). Bloom added that "A concept is made up of a set of attributes, each of these attributes are facts. Therefore, one may look upon a concept as a set of related specific facts" (Bloom, 1971, p. 669). The following example illustrates the idea of a concept.

Example 1: The concept of isomorphic groups is best introduced by counter examples. The listing below is arranged in such a way that m and 2^m are on two different rows.

m	0	1	-1	2	-2	3	-3	4	-4	5	-5
2^m	1	2	.5	4	0.25	8	.175	16	.0625	32	0.03125

Now, consider the circled values. Add the elements in the upper row and multiply the elements in the lower row.

$3 + (-4) = -1$ $\qquad\qquad$ $(8) \times (.0625) = .5$

The answer (-1 and .5) lie in the same column. This is the characteristic of isomorphic groups.

Definition of the Concept of Isomorphic Group: Two groups (S,*) and (T,#) are isomorphic if and only if there is a one-to-one correspondence between the elements of S and T so that

if a in S corresponds to x in T

and b in S corresponds to y in T

then a * b corresponds to x # y.

The general form can be proved as follows:

given $m \to 2^m$ for any integer m,

$$n \to 2^n$$

and certainly

$$m + n \to 2^{m+n}$$

but we know

$$2^{m+n} = 2^m \cdot 2^n$$

so

$$m + n \to 2^m \cdot 2^n.$$

From the previous example, it can be seen that the concept of isomorphic group is a combination of two other concepts [G (m,+) and G $(2^m$,x)]. Each group satisfies specific facts called closure, associative, identity element, and inverse.

It can be seen how these two groups are related to each other by showing

$$3 \in m \rightarrow 8 \in 2^m$$

$$-4 \in m \rightarrow .025 \in 2^m$$

then

$$3+(-4) \rightarrow (8)(.0625)$$

$$-1 \rightarrow .5$$ which is true.

Teaching concepts is an intellectual process that involves reading, listening, writing, thinking, abstracting, and generalizing. In order for teachers to present concepts effectively, they must help and encourage students to meet these activities.

- Teacher should provide <u>background</u> for students about the concept. This background might include information such as definition that they studied before and related to the concept, or stating sufficient conditions.
- Student should be encouraged to see how combinations of concepts may form a new concept.
- Teacher should provide appropriate concrete material to make learning concepts more understandable.

B. <u>Knowledge of Rules and Generalization</u>

This item involves knowledge of some facts about the mathematical idea and connecting these facts to draw a general conclusion on generalization. In the following example, students try to predict the number of subsets that could be made from any given set. Students were asked to complete Table 2 and make generalizations that tell how the number of elements in a set is related to the number of subsets. (See Tables 2 & 3).

Table 2

Number of Subsets Made from a Given Set

Set	Number of Elements	Subsets	Number of Subsets
{ }	0	{ }	1
{*}	1	{ },{*}	2
{*,0}	2	{ },{*},{0},{*,0}	4
{*,0,#}	3	{ },{*},{0},{#},{*,0} {*,#},{#,0},{*,0,#}	8
⋮	⋮	⋮	⋮

Table 3

Generalization of Numbers of Subsets

Number of Elements	Number of Subsets	Generalization Applied
0	1	2^0
1	2	2^1
2	4	2^2
3	8	2^3
4	16	2^4
5	32	2^5
.	.	.
.	.	.
.	.	.
10	?	2^{10}

In teaching rules and generalization, students should be encouraged to master the following points:

- Before students start work, they must have sufficient data such as: what is a subset, know how to find the number of subsets from a given set.
- Teachers should list data in a certain way to help students see the pattern. For example,

number of elements	2	3	4
number of subsets	4	$\boxed{}$	16

- Students should be asked to guess the number of subsets, such as what is the number of subsets of a 3 element set.
- Students need to verify their answers, i.e., if a student guesses the number of subsets of a 3 element set is 9 (according to the pattern he has). Teachers should ask student to check his answer with Table 3
- Teachers should encourage students to give another response by representing the number of subsets as follows

number of elements	2	3	4
number of subsets	2^2	$\boxed{}$	2^4

C. Knowledge of Mathematical Structure

Structure of mathematics plays an important role in most of the new mathematics programs. Johnson and Rising pointed out three reasons for the importance of structure. When one learns the structure of mathematics, he should have an easier time learning new topics or even new branches of mathematics. When certain basic properties (e.g., when the distributive property) are known, they can be used to explain a variety of new situations. The principle $a(b + c) = ab + ac$ can furnish a rationale for factoring problems, $3x + 6y = 3(x + 2y)$ or $3 \times 21 = 3\ (20 + 1) = 60 + 3 = 63$. When one understands the structure of a subject, he should improve his retention of ideas (Johnson, 1967, pp. 68-69). For example, the identity element for multiplication $a \cdot 1 = 1 \cdot a = a$ can provide one with a basis for reconstructing many individual algorithms such as

$$\frac{a}{b} = \frac{?}{bc}$$

$$\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{c}{c} = \frac{ac}{bc}.$$

The following example shows the relationship between sets of numbers and their relation to the structure of the algebraic system.

Natural Numbers	$N = \{ 1,2,3,4, \ldots \}$
Whole Numbers	$W = \{ 0,1,2,3,4, \ldots \}$
Integers	$I = \{ \ldots, -3,-2,-1,0,1,2,3, \ldots \}$
Rational Numbers	$Q = \{ x:x = \frac{a}{b} \quad a \in I,\ b \in N\}$
Irrational Numbers	$IR = \{ x:x \neq \frac{a}{b} \quad a \in I,\ b \in N\}$
or	IR = all numbers which may be written as infinite non-periodic decimal expressions.
Real Numbers	$R = Q \cup IR$

The Venn diagram in Figure 4 shows how these sets are related.

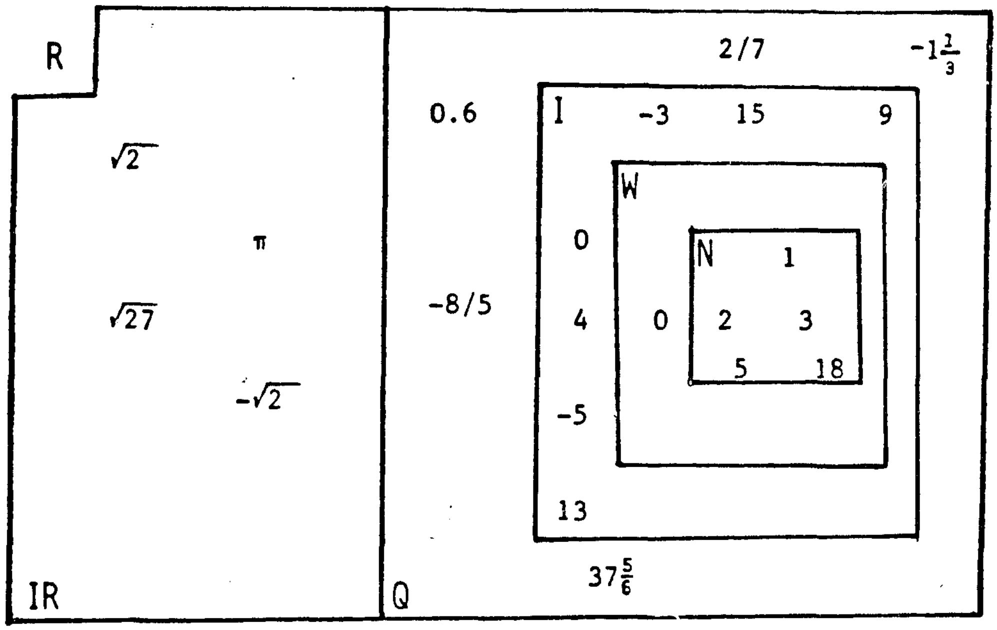


Figure 4: Relationships between sets of numbers

The theories of groups, fields, and rings refer to abstract mathematical systems which consist of sets of elements and one or two binary operations, usually called addition and multiplication, satisfying the properties of closure, identity, inverses, commutativity, associativity and distributivity. Table 4 shows the structure of these three theories with respect to different operations.

Table 4

Structure of Algebraic System

Property	Operation	Group	Ring	Field
Closure	+	G_1	R	F
Closure	X	G_2	R	F
Associative	+	G_1	R	F
Associative	X	G_2	R	F
Identity Element	+	G_1	R	F
Identity Element	X	G_2		F
Inverse	+	G_1	R	F
Inverse	X	G_2		F
Commutative	+		R	F (Except skew fields)
Commutative	X			
Distributive	X over +		R	F

G_1 is a group with respect to addition
G_2 is a group with respect to multiplication

In teaching theories of algebraic structure, teachers should use concrete situations by first presenting concrete examples, then the abstract method can be introduced by stating the definitions and properties on the abstract level. Using the concrete approach requires students to participate and solicit them in arriving at conclusions. The following is an example of teaching the group theory using concrete situations. The teacher asks some students to stand in front of the class and explain to them the military commands of left face, right face, about face, and stand steady (attention; don't move). The rest of the class will be given a workout sheet consisting of the following table.

*	LF	RF	AF	SS
LF	AF	SS	RF	LF
RF	SS	AF	LF	RF
AF	RF	LF	SS	AF
SS	LF	RF	AF	SS

The students will be asked to fill out the table as the teacher gives the military commands to the standing students. After the table is completed, students will be asked to check the properties

- closure is obvious
- LF * AF = AF * LF thus it is commutative
- (LF * RF) * AF = LF * (RF * AF)

  SS * AF = LF * LF

  AF = AF

  hence it is associative.
- SS never changes one of the other commands, thus SS is the identity element. SS * $\square = \square$
- the inverse of LF is RF and AF is AF and SS is SS., $LF^{-1} = RF$; $RF^{-1} = LF$, $AF^{-1} = AF$ and $SS^{-1} = SS$

  $LF * LF^{-1} = LF * RF = SS$

  thus inverse property held.

D. <u>The Ability to Translate Problems into Algebraic or Graphic Forms and Conversely</u>

Many teachers have said the most difficult part of teaching algebra is the word problems, because it requires special ability in translating word problems, because it requires special ability in translating word problems to mathematical problems. In solving word problems, students need to do three things: (1) translate the problem from language form to mathematical form, (2) store the mathematical form, (3) interpret the solution and state the conclusion in words.

Example 1: The sum of two numbers is 10. Find these numbers if their product is to be as large as possible.

Solution: Let x be one number, then 10 - x is another number. The product of the two numbers is x(10-x). This product is negative when $x < 0$ or $10 < x$ and positive when $0 < x < 10$.

x	9	8	7	6	5	4	3	2	1
x(10-x)	9	16	21	24	25	24	21	16	9

From the table, the maximum is reached when x = 5. This can be seen better by drawing the graph of y = x (10-x) (Figure 5).

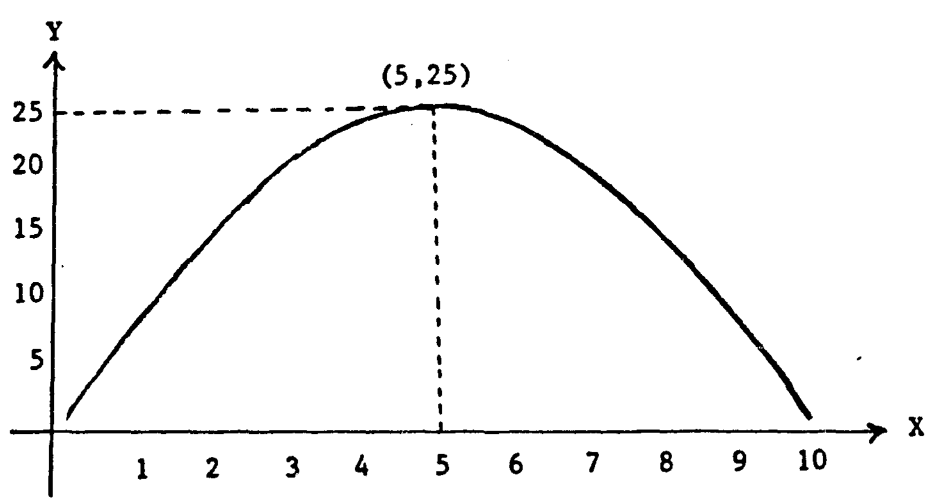


Figure 5: A graph of x(10-x)

The following points are considered important steps in solving word problems:

- the teacher might read the problem first, then ask students to read it part by part with interpreting each part to mathematics
- after translating the whole problem from word problem to mathematical terms, the teacher should encourage students to make the calculations
- students should use diagrams or tables to organize the information of the problem
- teachers should help students to draw a conclusion and state it in their words, such as "the two numbers that their sum is 10 and their product is as large as possible are 5 and 5."

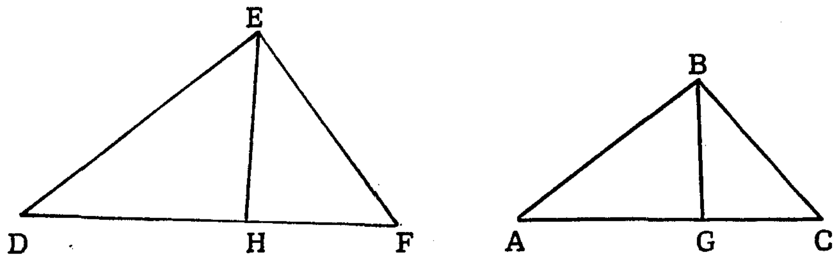
E. Ability to Follow a Line of Reasoning

One of the goals of studying mathematics in general and geometry and logic in particular is to develop the ability to think critically and follow a line of reasoning. The main purposes of reasoning are to ensure understanding the "why" behind every step in the proof and to encourage students to create their own thinking. The following are some examples:

Example 1: Corresponding altitudes of similar triangles have the same ratio as the corresponding sides.

Hypothesis: $\overline{BG}$ and $\overline{EH}$ are altitudes of ΔABC and ΔDEF, respectively: $\Delta ABC \sim \Delta DEF$.

Conclusion: $\frac{BG}{EH} = \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.



Solution: Proof.

Statements	Reasons
1. $\overline{BG}$ and $\overline{EH}$ are altitudes of $\triangle ABC$ and $\triangle DEF$	Hypothesis
2. $\overline{BG} \perp \overline{AC}$; $\overline{EH} \perp \overline{DF}$.	An altitude of a triangle is perpendicular to the line of the opposite side.
3. $\angle AGB$ and $\angle DHE$ are right angles.	Perpendicular lines form right angles.
4. $\angle AGB = \angle DHE$.	All right angles are equal.
5. $\triangle ABC \sim \triangle DEF$.	Hypothesis.
6. $\angle A = \angle D$.	Corresponding angles of similar triangles are equal.
7. $\triangle BGA \sim \triangle EHD$.	A.A. Similarity Theorem.
8. $\frac{BG}{EH} = \frac{AB}{DE}$	Corresponding sides of similar triangles are proportional.
9. Also, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.	Same reason as step 8 (from step 5).
10. $\frac{BG}{EH} = \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.	Transitive

Example 2: Prove if a whole number is divisible by 4, then it is an even number.

Statements	Reasons
1. A whole number is divisible by 4.	1. Hypothesis.
2. If a whole number is divisible by 4, then it is divisible by 2.	2. Hypothesis.
3. $\therefore$ it is divisible by 2.	3. 1,2 M.P.
4. If a number is divisible by 2, then it is an even number.	4. Hypothesis.
5. $\therefore$ it is an even number.	5. 3,4 M.P.

In teaching mathematics in general and geometry and logic in particular, reasoning is considered to be a very important factor. The following points help teachers as well as students in teaching or learning reasoning

- teachers should help students to develop analytic proofs as well as synthetic proofs in their reasoning
- teachers should provide the necessary background, well organized to help students search for theorems or postulates which might be helpful in a given situation
- students should be encouraged to start their line of reasoning by information obtained from the problem such as "hypothesis" or "given"
- teachers should guide students to avoid falling into circular reasoning
- teachers should never require students to memorize the reasoning, but they should be encouraged to follow the logical chain of the reasoning
- students should use drawings and sketches of figures if necessary

F. Ability to Recognize or Discover a Pattern

Developing number awareness is considered to be a very important aspect of teaching mathematics. Most of the time, students need to see a chain of steps consisting of countable examples or results to help them to recognize or discover a pattern.

Example 1: With 4 cups on the bottom
10 cups in all (Figure 6A)

With 5 cups on the bottom;
14 cups in all (Figure 6B)

With 6 cups on the bottom;
18 cups needed in all (Figure 6C)

How many cups are there when there are 12 cups on the bottom?

Can you find out how many cups there are when n cups are on the bottom? (Justine, 1980).

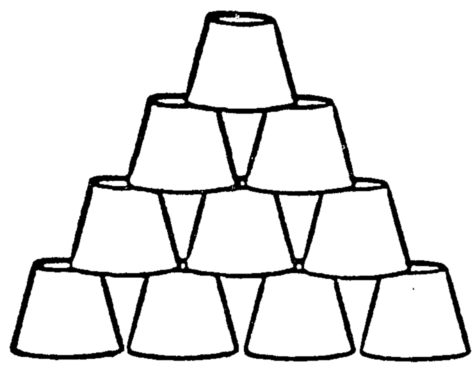


Figure 6A

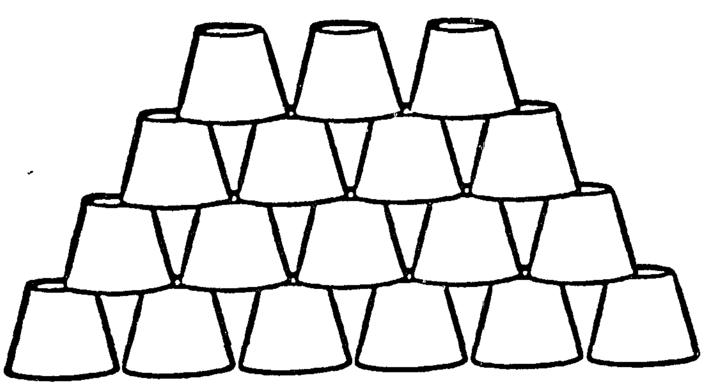


Figure 6C

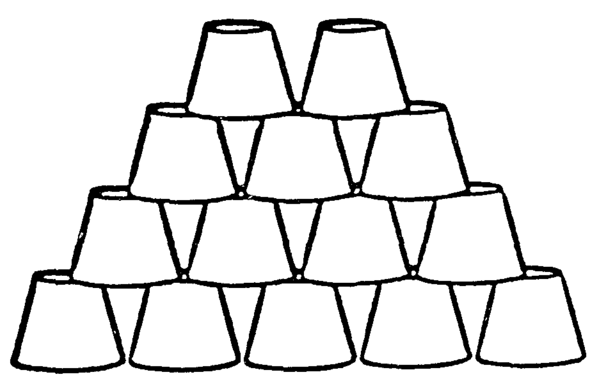


Figure 6B

The generalization can be seen from the following steps:

<u>Cups in Bottom</u>	<u>Total Cups</u>
4	4 + 3 + 2 + 1 = 10
5	5 + 4 + 3 + 2 = 14
6	6 + 5 + 4 + 3 = 18
7	7 + 6 + 5 + 4 = 22
8	8 + 7 + 6 + 5 = 26
.	.
.	.
.	.
n	$n+(n-1)+(n-2)+(n-3) = \boxed{}$

In case there are n cups on the bottom, the total cups will be:

$$n + (n-1) + (n-2) + (n-3) = 4n-6$$

<u>Example 2</u>: Building squares around square: (1) draw a square; draw a second square around the first square and count the additional number of squares built around the first square; (3) repeat the same procedures until you have built five or six squares as shown in figure 7.

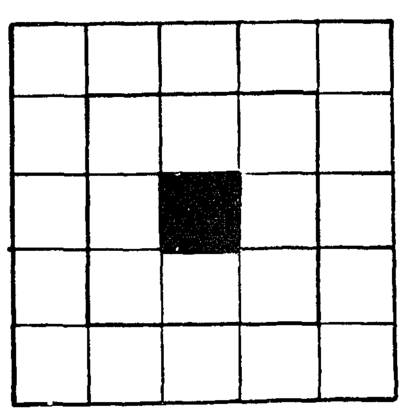


Figure 7: Building squares around square

Squares	Number of Additional Squares
2nd	$8 \longrightarrow 1 \times 8$
3rd	$16 \longrightarrow 2 \times 8$
4th	$24 \longrightarrow 3 \times 8$
5th	$32 \longrightarrow 4 \times 8$
6th	$40 \longrightarrow 5 \times 8$
.	.
.	.
.	.
nth	$\longrightarrow (n-1) \times 8 \longrightarrow 8n-8$

It can be seen that when n squares are built around the square, the total number of squares can be determined by $8n-8$.

The previous two examples can be used to help students to recognize or discover a pattern. The following are some suggestions that might be used in teaching this ability.

- the students should be encouraged to construct enough individual examples to observe the pattern
- the students should organize the information or the observation in tables such as:

n		Total
4	cups on the bottom	$4 + 3 + 2 + 1 = 10$
5	cups on the bottom	$5 + 4 + 3 + 2 = 14$
6	cups on the bottom	$6 + 5 + 4 + 3 = 18$

- the students should be asked to write the total of cups when n is reasonably large (this suggestion will encourage the students to look back to the pattern)

n	Total
12	12 + 11 + 10 + 9 = 44
99	99 + 98 + 97 + 96 = 390

- students should examine the specific cases to speculate on some general formula. These general formula trials should be checked against the unknown data - until finally a reliable formula is created.

IV. Analysis

The analysis level behaviors consist of the following abilities: (A) Ability to solve non-routine problems (problem solving), (B) Ability to construct a proof, (C) Ability to criticize a proof, and (D) Ability to formulate and validate generalization.

A. Ability to solve non-routine problems (problem solving)

In the introduction to 1980 NCTM yearbook Problem Solving in School Mathematics, Polya said, "solving a problem is finding the unknown means to a distinctly conceived end, solving problems is the specific achievement of intelligence, and intelligence is the specific gift of man" (NCTM, 1980, p. 1).

Polya gives a definite technique for helping learn how to solve a problem. This technique is divided into four aspects.

1. Understanding the problem. This aspect requires students to read and think in order to know the unknown of that problem. It requires the student to pick out the data and organize it. It requires him to find out what conditions the data must satisfy.

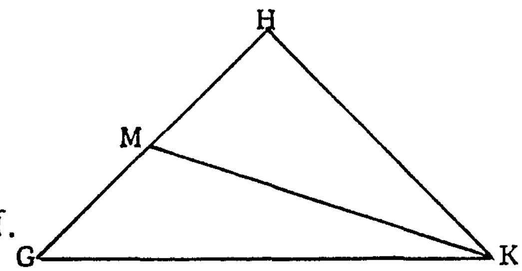
2. Thinking of a plan. When students are facing difficult problems, they ask for help, which translates as a request to be given a plan. Polya suggests that a plan can be introduced by asking students if they have solved a similar or simpler problem.

3. Carrying out the plan. This aspect requires students to identify the steps of the plan, to check each step of the plan, and to prove that each step is correct.

4. Looking back. After students carry out the plan or solve the problem, they need to check the solution to make sure the used method is correct.

B. Ability to Construct a Proof

According to Kolb and Bassler, one important step in developing the ability to construct a proof is for the student to concentrate on figuring out an argument before he writes down any part of a formal proof (Otto & Kolb, 1971, pp. 359-378). He should write the statements and reasons of a formal proof only after he has an idea about the way the proof might proceed. One way to get the student to develop this skill is to have him practice. This practice can be done by providing information such as the following:



Given: ΔGHK, $\overline{GK} = \overline{HK}$

M between G and H,
$\angle GKM = \angle HKM$

Conclusion: M is the midpoint of $\overline{GH}$.

Statements	Reasons
1. GK = HK	(1) ____________
2. $\angle GKM \cong \angle HKM$	(2) Given
3. ____________	(3) Identity
4. $\Delta GKM \cong$ ________	(4) ____________
5. ____________	(5) ____________
6. ____________	(6) Definition of midpoint

Teachers should encourage students to use the following points when they teach proofs.

- Make a list of guesses and conjectures, both correct and incorrect.
- Draw figures and try examples to test their guesses.
- Select the order in which the proofs of the conjectures will be attempted.
- Devise an informal argument to justify the conjectures orally.
- Write a formal proof with reasons.
- Determine if the proof holds for a specific case or special figures.
- Determine if the proof holds for a general case.

C. Ability to Criticize a Proof

The following example might encourage students to learn not to accept any proof, unless its steps are correct and well stated.

<u>Example</u>: Let us try to prove 1 = 2.

Let $x = y$

$$x^2 = xy = y^2$$

$$x^2 - y^2 = x^2 - xy$$

$$(x-y)(x+y) = x(x-y)$$

$$x + y = x$$

$$2x = x$$

$$2 = 1 \qquad \text{since } x = y$$

In this kind of proof, the students should be encouraged not to accept any step in the proof until they make sure it is mathematically correct.

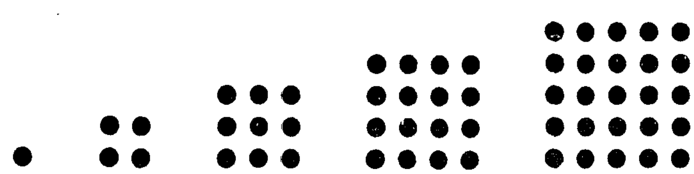
In this example, they should understand that if $x = y$ then $x - y = 0$, and then $x\,(x-y)$ divided by $(x-y)$ does not equal x.

i.e., $\frac{x(x-y)}{(x-y)} \neq x$ when $(x-y) = 0$

D. <u>Ability to Formulate and Validate Generalizations</u>

This behavior level requires students to formulate and generalize the idea, then validate the relationship by discovering and proving the mathematical statement. As an example:

<u>Example 1</u>: Square Numbers



- Copy the figures shown, then sketch the next one.
- Count the number of dots in each square number and fill the following table.

Square	Number of dots	Sum
1	1	1
2	1 + 3	4
3	1 + 3 + 5	9
4	1 + 3 + 5 + 7	16
.	.	.
.	.	.
.	.	.
n	1 + 3 + 5 + 7 + ... +2n-1	n^2

- Write the number of dots of the fourth, fifth, sixth, and seventh square.
- Find the sums of the dots of each square.
- What is the nth square number?
- Generalize for the nth square number.
- Prove your generalization by mathematical induction.

$$1 + 3 + 5 + 7 \ldots +(2n-1) = \sum_{k=1}^{n} (2k-1) = n^2$$

V. Applications

Students studying mathematics need to see how content and concepts make sense. This need can be met by showing some applications of mathematics. Applications usually serve three relatively distinct ends. They may illumine situations in everyday life; they may help in the development of some other discipline, and they may be of value in some other branch of mathematics. There are three subcategories of this behavior level: (A) Ability to solve routine problems, (B) Ability to make comparisons, (C) Ability to analyze data.

A. Ability to Solve Routine Problems

Thomas Butts, in the 1980 Yearbook of the National Council of Teachers of Mathematics, gives five kinds of problems. The following is a brief summary of each kind (NCTM, 1980, pp. 23-24).

Recognition exercises: These type of exercises require recall of definitions, basic facts, and theorems. These exercises usually are in the form of fill-in-the blank, multiple choice, true-false type problems.

Algorithmic exercises: These type of problems require students to follow a certain type of steps to solve the problem.

Example 1: Solve for x and y (elimination by addition or subtraction) the equations.

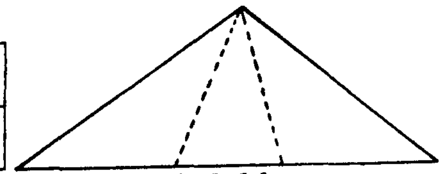
$$x + 3y = -3$$
$$56x - 7y = 19$$

Application problems: The source book on applications of MAA and NCTM gives three important criteria for application problems. They are: (1) data should be realistic, (2) the unknown in the problem should be reasonably expected to be unknown in reality, (3) the answer to the problem should be a quantity someone might plausibly have a reason to need.

Open-search problem: These types of problems encourage guessing which is considered to be a very important factor in understanding mathematics.

Example 2: Repeatedly fold a triangle through one of its vertices.

Number of folds	0	1	2	3	4	5	7
Number of triangles	1	3	6	10	15	21	?



Problem situation: This approach adopted by the USMES project, presents a problem in a situation and tells students, "Here is a situation, think about it."

Example 3: Design a parking lot. Possible problems to consider could include the following: How large should each space be? At what angle should each space be placed? How much should be charged per car per hour?

Since the teaching techniques of solving routine problems are similar to the techniques suggested in Criterion I "Computational Skills," it will not be necessary to repeat them here. Those suggestions will be applicable on this subcriterion.

B. Ability to Make Comparison

This item requires that students determine a relationship between two sets of information, make some calculations, recall some facts, and formulate a decision. The following two examples illustrate this ablity.

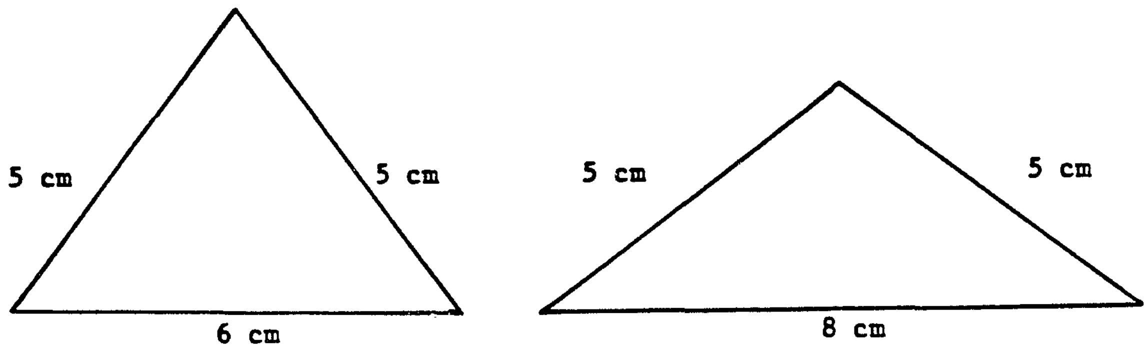
Example 1: The following figures represent the maximum speeds during a race.

Car 1: 180 km/hr　　Car 3: 175 km/hr

Car 2: 160 km/hr　　Car 4: 178 km/hr

How many speeds are greater than the mean?

Example 2: Compare the areas of the two triangles.

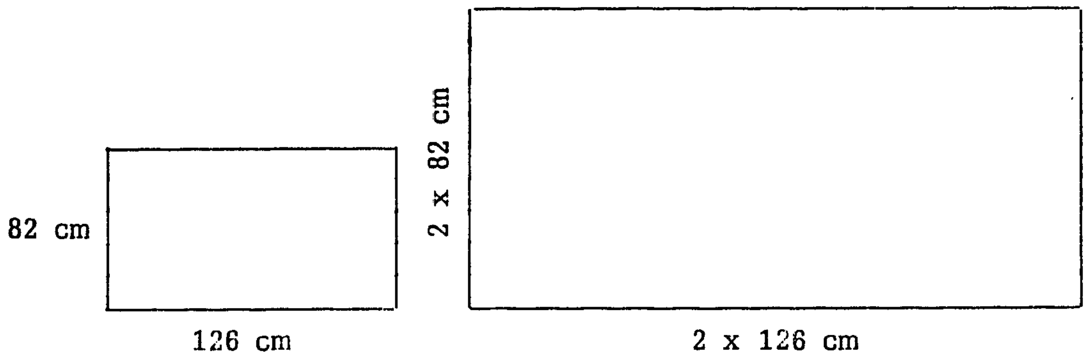


Teaching this item required teachers to encourage students to recall some definitions or rules that are necessary for the solution of the problem such as mean, area of triangle, ...etc. The students should be guided to organize and to tabulate the information to help them to make the comparison. The teachers might use the suggestions that are presented in teaching subcriterion E, criterion III (Ability to Follow a Line of Reasoning).

C. Ability to Analyze Data

This item deals with reading and interpreting information, manipulating that information. "The student may be required to distinguish relevant from irrelevant information, to assess what information is required, to determine what related problems may be examined" (Bloom, 1971, p. 678).

Example 1: A rectangle has a length of 126 cm and a width of 28 cm. Its area is 126 x 82 cm^2. (Do not do the multiplication). If a new rectangle is twice as long and twice as wide, its area is (2 x 126) x (2 x 82) cm^2, what does this say about the area of the two rectangles?



In teaching students to analyze data, they should be encouraged to read the problem, to translate it from words to mathematics, to make calculations, and to draw a conclusion. Teachers might encourage students to guess the answer. In the previous example, the students should be encouraged to think or guess the answer before they start the calculations. The students may think that the area of the large rectangle is twice the area of the small rectangle, but they will be surprised when they find out from the calculations that it is not true.

VI. Instructional Material

Learning mathematics is based not only on seeing and hearing, but also on the sense of touch and tactile movement. Visualizing a mathematical idea helps students to bridge the gap between the concreteness to the abstract. The following are some of the goals of using instructional material that was presented by the National Council of Teachers of Mathematics (NCTM) in its 34th yearbook (NCTM, 1973, p. 3).

1. Motivating the student to begin learning.

2. Presenting a sequence of stimuli that is structured to keep the student engaged in learning.

3. Giving the student directions to do something, look at something, or start some activity.

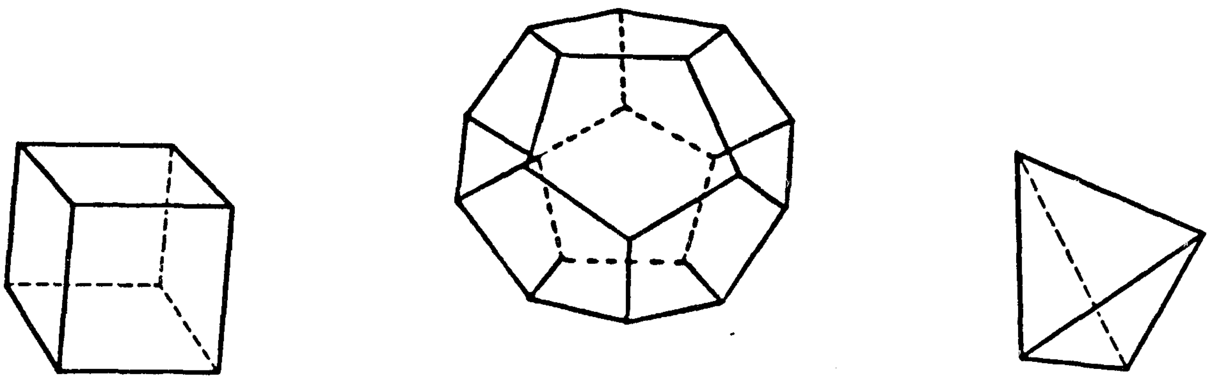
4. Communicating the structure of the subject, i.e., relating concept, definition, assumption, and theorem.

5. Providing opportunities for practice.

According to Heddens, there are four situations in teaching mathematics: (a) concrete situation, (b) semi-concrete situation, (c) semi-abstract situation, (d) abstract situation.

A. Concrete Situation

Concrete situations involve presenting a mathematical idea in a real form, in other words, using real or tangible objects. Models, for example, are considered to be very important factors in teaching geometry and elementary topology. "Using models helps to enlarge the totality of sensations and to improve the quality of sensations received by the learner. These two phenomena contribute to improvement in learning" (NCTM, 1973, p. 235). A successful lesson of models may be started by having each student show a variety of polygons and then construct them (Figure 8). After making sure that every student knows the meaning of vertices (V), edges (E), and faces (F), the teacher asks each student to count and record the number of vertices, edges, and faces. As the results are tabulated, after a short period of exploration, a relationship is discovered and expressed by the formula $V + F = E + 2$ (see Table 5).



Cube | Dodecahedron | Pyramid

Figure 8: Mathematical Solids

Table 5

The Relationships Between V, F, and E

	V	F	E	V + F = E + 2
Cube	8	6	12	8 + 6 = 12 + 2
Dodecahedron	20	12	30	20 + 12 = 30 + 2
Pyramid	5	5	8	5 + 5 = 8 + 2

A second example of using models is giving geometrical interpretations to algebraic operations. Asking students to give geometrical interpretations to algebraic expressions can lead to challenging projects. The students who produced the cubical block in Figure 9 used it to give a geometrical interpretation to the proof of $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

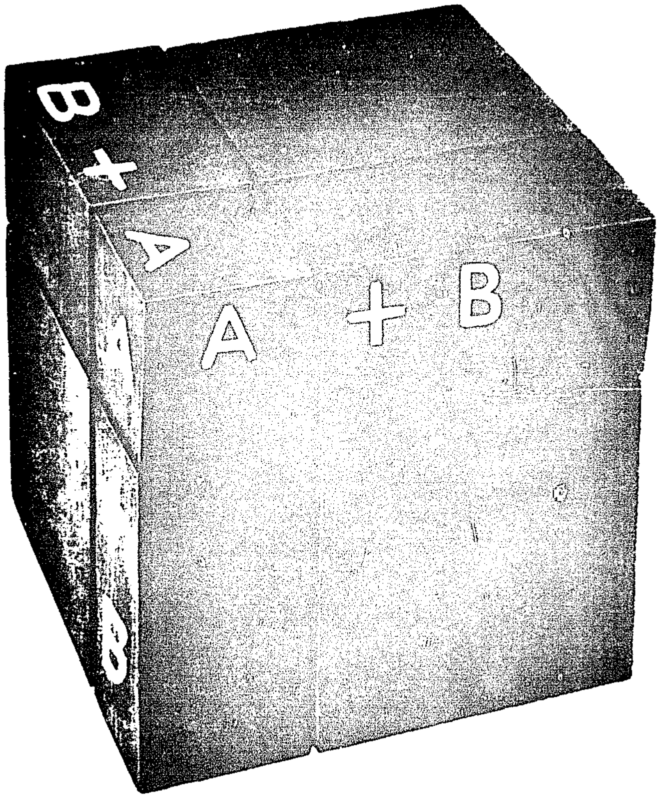


Figure 9. Cubical Block

B. Semi-Concrete Situation

A semi-concrete situation is expressed by drawing or showing pictures of the real objects which may be accomplished by using:

Video tapes: Using television and video tapes helps students to see and hear the lesson more than one time in order to catch points missed during the first presentation.

Overhead projectors: Overhead projectors are very useful in teaching mathematics, especially in complicated drawings which cannot be made quickly in the classroom and when one wants to superimpose one drawing on another.

<u>Films</u>: Films provide students with a convenient means of mastering some concepts. The 16mm and 8mm films are widely used now in schools.

C. <u>Semi-Abstract Situation</u>

A semi-abstract situation involves use of <u>tallies</u> to represent objects for mathematical ideas. A common example of such representation is 卌 ||. The following is an example of this representation.

<u>Example</u>: Table 5 shows frequency distributions of 25 scores having the same range of 50-75. Every tally represents a score of a student.

Table 6.

Frequency Distribution

Score	Test A	Test B	Test C	Test D
75	卌 卌	|	卌 卌 卌	|
70	|||	|	|||	卌
65	卌	|	|	卌 卌 |
60	卌 |	卌		卌
55	|	||||		||
50		卌 卌 卌	|	|

D. <u>Abstract Situation</u>

In this situation, a mathematical idea can be described by using abstract symbols such as $\int$, $\Longrightarrow$, $\wedge$, $\vee$, $\neg$, $\cup$, $\cap$, $\sqrt{}$ Σ, ...etc.

<u>Example 1</u>:

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$$

Example 2: If $A \subset B$ and $B \subset C$, then $A \subset C$. $\forall A, B, C \in N$

Proof: If $A \subset B$, then there is some set of natural numbers X such that $A \cup X = B$.

If $B \subset C$, then there is some set of natural numbers Y such that $B \cup Y = C$

Hence, $B \cup Y = (A \cup X) \cup Y$

$B \cup Y = A \cup (X \cup Y)$

$B \cup Y = C$.

Using instructional material involves two kinds of instruction. Small group instruction and large group instruction.

Large group instruction can be employed in abstract situations or semi-abstract situations where the teacher lectures, writes on the blackboard. The student's role in large group instruction is usually restricted to listening and talking, observing, and demonstration.

Small group instruction can be used in concrete situations, where the classroom is divided into small groups (about 5 in each group). The small group instruction encourages students to become actively involved in the learning process through interaction with each other and with the teacher.

VII. Motivation

Motivation is not only one of the central problems of psychology but one of education as well. Among the kinds of questions most frequently asked by teachers are these:

How can I get my student to pay attention?

How can I get them to do their assignments?

How can I get them to behave themselves?

How can I get them to learn?

How can I get them to at least try?

In each of these questions, by the word "get" I presume that they mean motivate (Kolesnik, 1978, pp. 1-2). The majority of the students have the ability to learn mathematics, but they do not always have the desire, the inclination, or the will to use their abilities in the way that their teachers would like them to (Kolesnik, 1978, p. 2). The following are some devices which might be used to motivate students in learning mathematics.

A. Introducing the Topic

One of the most common problems in teaching mathematics is that the students study mathematics without knowing the reason or reasons for studying it. Many teachers try to satisfy students by saying "You are studying this topic because you need it later." The following are some ways a mathematics teacher can initiate a lesson. (1) Giving reasons for studying a mathematical topic by stating the importance (applications) of the topic on our lives. (2) Stating the importance of the topic in other fields of science. (3) Stating the goals the lesson will provide in some direction for study. (4) Using historical materials to show the students the development of the topics historically (this item will be discussed in detail in subcriterion C). (5) Presenting a problematical situation. This gains the students' attention or arouses their interest in what is to come (Runion, 1972).

B. Mathematics Recreations

Mathematics recreations play a very important role in understanding mathematics. The Thirty-third Yearbook of The National Council of Teachers of Mathematics, The Teaching of Secondary School Mathematics, pointed out "that interest is easily aroused by mathematical recreation: games, jokes, puzzles, tricks, paradoxes, fallacies, and illusions. Such readily available enrichment materials give students practice in maintaining skills and concepts, and they often provide an opportunity for student discovery of the mathematical explanation of some mystery" (NCTM, 1970, p. 150). Interest can be stimulated by posing questions, presenting fresh ideas or arousing curiosity. Students frequently think of mathematics as dull, boring, impractical, and difficult. The teacher can do much to affect these feelings by creating interesting ideas such as drawing and displaying optical illusions on the bulletin board or presenting games and puzzles or showing students how mathematics developed historically, etc. These are some items of mathematics recreations.

Optical Illusion

The following are some examples of optical illusions which can be exhibited for fun or they can be used to convince students to rely on reasons in their proofs rather than relying on figures.

Example 1: Figure 10 shows a remarkable illusion by artist M. Escher. The picture shows a stream of water which perpetually runs downhill.

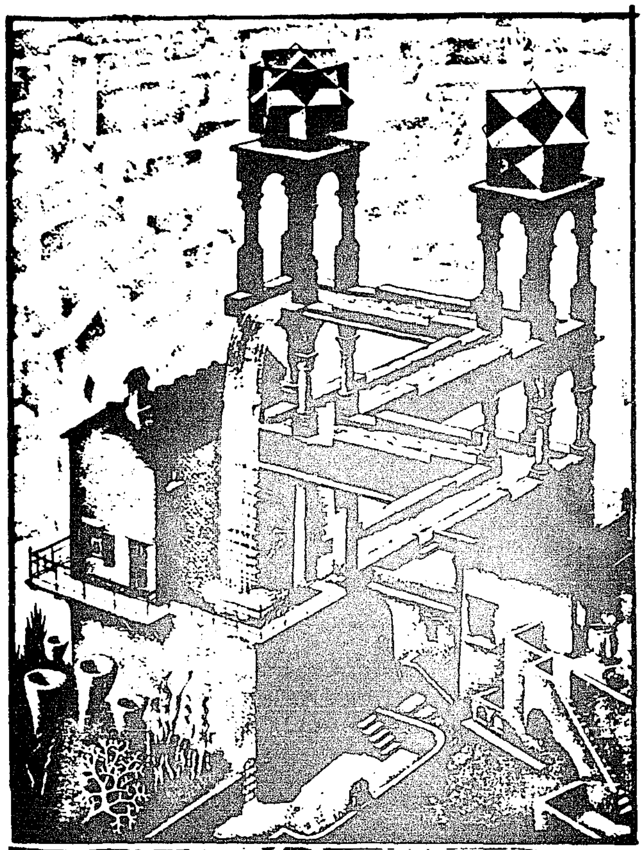


Figure 10: A stream of water runs downhill

Example 2: Are the long lines parallel in Figure 11?

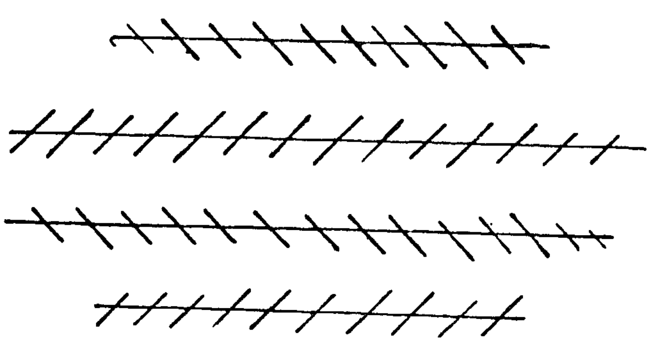


Figure 11: Parallel lines

<u>Example 3</u>: Do you believe that Figure 12 is a photograph of a wooden crate used for shipping?

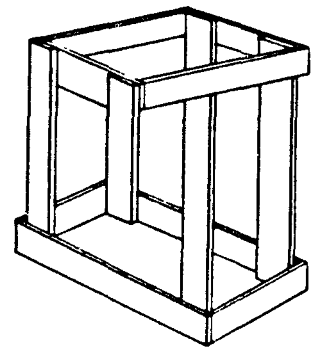


Figure 12: A wooden crate

<u>Example 4</u>: Are the sides of the triangle in Figure 13 bent or straight? Are the two heavy lines in Figure 14 parallel?

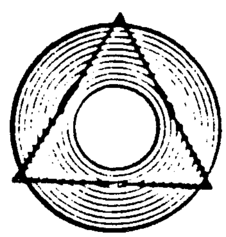


Figure 13: Bent or straight

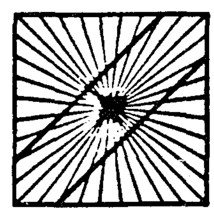


Figure 14: Bent or straight

<u>Example 5</u>: In Figure 15, which segment is longer, $\overline{AB}$ or $\overline{CD}$?

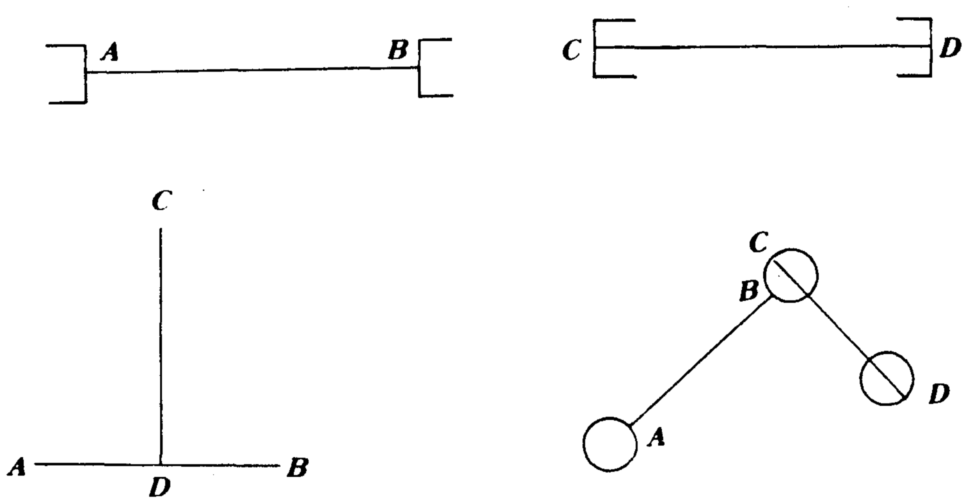
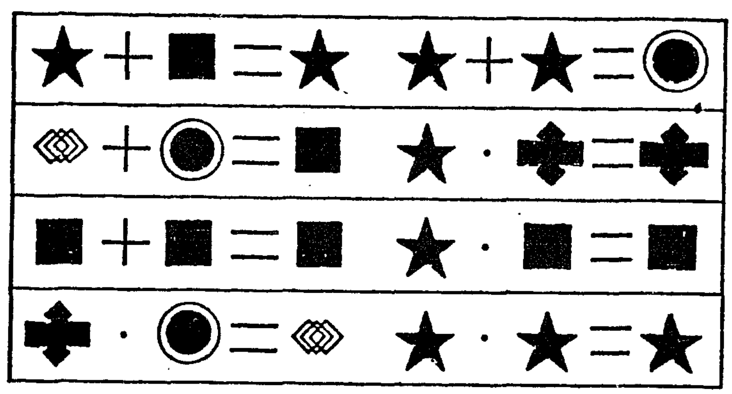


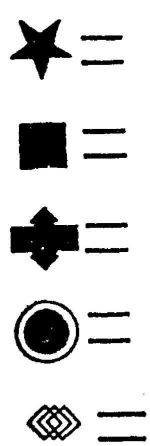
Fig. 15: Which segment is longer

Games and Puzzles

The following collection is a sample of suitable games and puzzles for the classroom.

Example 1: Each symbol represents one of these numbers: 0, 1, 2, -1, -2. Use the properties of integers to determine what number each symbol represents (Justine, 1980).





Example 2: TA - Yen Chinese

1. Think of a number between 1 and 30.
2. Divide the number by 2 and tell me the remainder.
3. Divide the number by 3 and tell me the remainder.
4. Divide the number by 5 and tell me the remainder.
5. Your number is ________.

How the trick works:

1. Multiply the remainder from 2 by 15.
2. Multiply the remainder from 3 by 10.
3. Multiply the remainder from 4 by 6.
4. Add the products to discover the number.
5. If the number of products is greater than 30, subtract 30 to find the number.

The theory behind the "trick" is modular arithmetic.

Example 3: Loosely tie together the hands of two people as shown in Figure 16 with one strip looped around the other. Now get them apart without untying or cutting the string. It can be done.

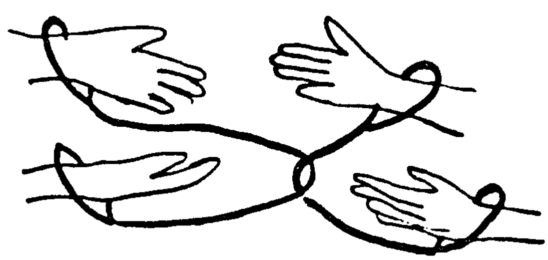


Figure 16: The tied hands

C. Historical Topics

The history of science, of which the history of mathematics is a branch, is still often considered a collection of results of scientific inquiry through the ages. History of mathematics is considered a very important subject in school mathematics because it helps students gain interest and generates positive attitudes especially when solutions of the problems and geometrical demonstrations are interspersed with historical remarks. Teaching history of mathematics not only reminds us of what we have, but may also teach us how to increase our information and provide us with greater understanding of the way in which thought builds through the ages. The following examples illustrate the Arab achievements in mathematical science.

Example 1: It is stated in many records that the Arabs designed the numeral system based on the number of angles in every numeral as shown in Figure 17.

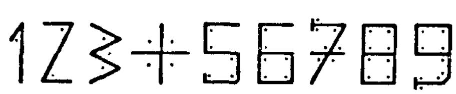
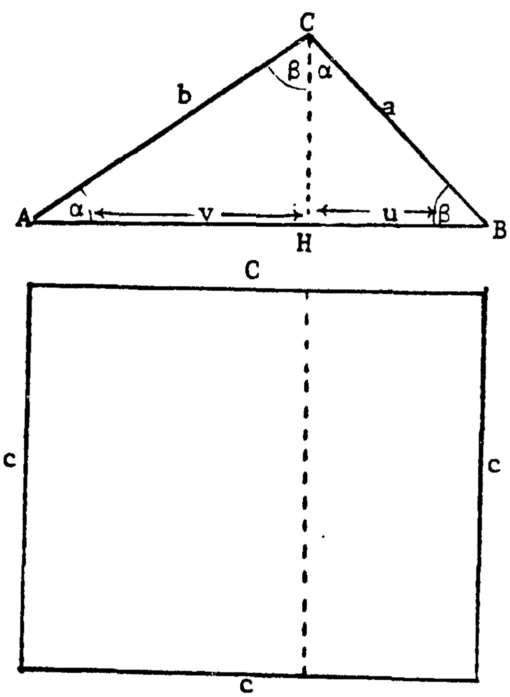


Figure 17: Arabic Numerals

Example 2: Al-Khowarizmi was considered to be the greatest mathematician at the court of the Al-Momun. He wrote On Arithmetic which was translated into Latin by Robert Chester under the title of Algoitmide Numero Indonum (Nikula, 1970, pp. 3-6). He also wrote a very good book on integration and comparison kept at the Bodleian Library in Oxford. He wrote on proportion and astronomy under the title Ibn-Al-Muthanna's Commentary on the Astronomical Tables of Al-Khwarizmi. He used the following argument in his translation of the Pythagorean Theorem (NCTM, 1980).



$\because \ \Delta HBC \cong \Delta CBA$

$\therefore \ \frac{HB}{CB} = \frac{BC}{BA} \implies \frac{u}{a} = \frac{a}{c}$

$\therefore \ a^2 = uc \longrightarrow 1$

$\because \ \Delta AHC \cong \Delta ACB$

$\therefore \ \frac{AH}{AC} = \frac{AC}{AB} \implies \frac{v}{b} = \frac{b}{c}$

$\therefore \ b^2 = vc \longrightarrow 2$

from 1 & 2

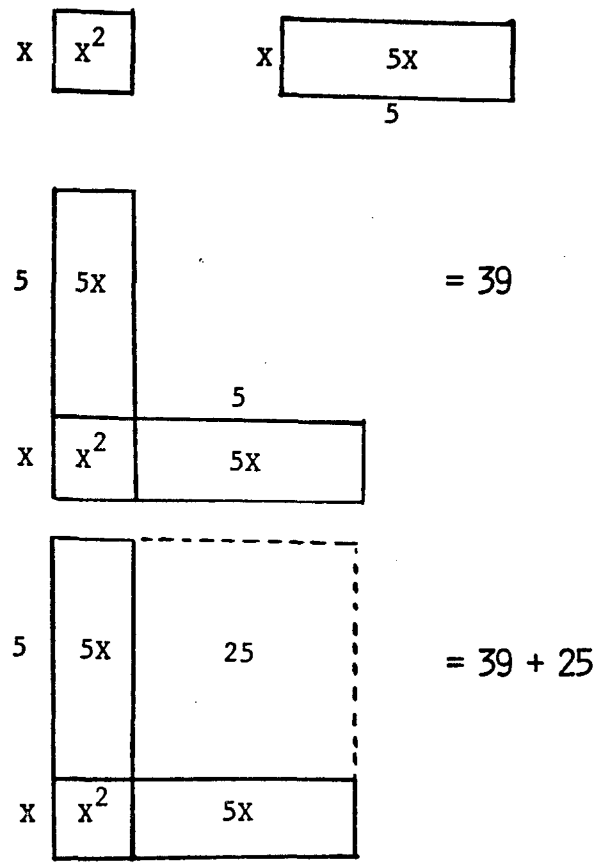
$$a^2 + b^2 = uc + vc$$

$$= c(u+v)$$

$$= c \cdot c \quad \text{since } (u + v = c)$$

$\therefore \ a^2 + b^2 = c^2$

Example 3: A meaningful method to solve equations of the second degree using completion of the square is given by Al-Khowarizmi. To solve $x^2 + 10x = 39$, Al-Khowarizmi drew a small square to represent x^2, and two rectangles to represent 10x (NCTM, 1980).



$$(x + 5)^2 = 64$$

$$x + 5 = 8$$

$$x = 3$$

D. Mathematics in Nature

The topic presented here is the development of Fibonacci sequence. Because of the space limit, a brief introduction to the development of the sequence is presented. Properties and applications of the sequence in nature are found in Appendix D.

The development of Fibonacci sequence: Someone placed a pair of rabbits in a certain place enclosed on all sides by a wall to find out how many pairs of rabbits will be born in one year, it being assumed every month a pair of rabbits produces another pair and that rabbits begin to bear two months after they are born. Fibonacci listed the total pairs of rabbits at the end of each month and created the following sequence of numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233. Upon realizing the sequence, he realized that each number was the sum of the two preceding numbers. In other words, 1 + 1 = 2, 1 + 2 = 3, 2 + 3 = 5, 3 + 5 = 8, ... , 89 + 144 = 233. Figure 18 illustrates the breeding of rabbits.

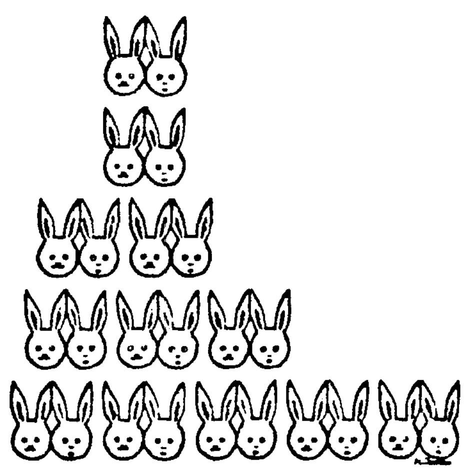


Figure 18: The breeding of rabbits

Summary

The criteria for understanding mathematics which have been developed and discussed in this chapter are summarized below. As the next step in this chapter are summarized below, the writer will evaluate a selected topic from the Libyan secondary mathematics program in terms of some of the developed criteria. The criteria are:

I. Computational Skills

- A. Knowledge of Specific Facts
- B. Knowledge of Terminology and Symbolism
- C. Ability to Carry out Algorithms

II. Discovery

- A. Guided Discovery
- B. Participation in the Classroom
- C. Explorating the Materials

III. Comprehension

- A. Knowledge of Concepts
- B. Knowledge of Rules and Generalization
- C. Knowledge of Mathematical Structure
- D. The Ability to Translate Problems into Algebraic or Graphic Forms and Conversely, the Ability to Interpret Algebraic or Graphic Representation
- E. Ability to Follow a Line of Reasoning
- F. Ability to Recognize or Discover a Pattern

IV. Analysis

- A. Ability to Solve Non-routine Problems
- B. Ability to Construct a Proof

C. Ability to Criticize a Proof

D. Ability to Formulate and Validate Generalization

V. Applications

A. Ability to Solve Routine Problems

B. Ability to Make Comparisons

C. Ability to Analyze Data

VI. Instructional Material

A. Concrete Situation

B. Semi-Concrete Situation

C. Semi-Abstract Situation

D. Abstract Situation

VII. Motivations

A. Introducing the topics

B. Mathematical Recreations

C. Historical Topics

D. Mathematics in Nature

CHAPTER V

METHODOLOGY OF THE EXPERIMENT

Introduction

The purpose of this chapter is to examine the method of mathematics presentation in Libyan secondary school mathematics. In particular, the investigator will evaluate the topic, "Finding a formula for 1 + 2 + 2 + ... + n," that is presented in the Libyan secondary school program, Book 2, (Ab-Yousef, & Humdi, 1975, p. 142.)

Subjects

Secondary school mathematics teachers participated in the experiment. Seventy-five teachers were randomly selected from different secondary schools throughout the country. These teachers taught this mathematics program and this topic in their secondary schools.

Instrumentation

Two forms were prepared to be used in this experiment. Form A represents the method of presentation of the topic, Finding the sum of 1 + 2 + 3 + ... + n" as it appears in the Libyan textbook, Algebra II (Ab-Yousef, Humdi, 1976, p. 192). Form B represents the method of presentation of the same topic that was developed by the investigator to meet some of the criteria identified in Chapter IV. A list of questions contains 17 items to be answered by the subjects. Questions 1 through 11 are used to test Six Criteria Motivations, Discovery, Comprehension, Analysis, Instructional material, and Applications. Questions 12 through 15 are not related directly to the criteria, but they are used to see which presentation is more preferable. Questions 16 and 17 are used to obtain the personal reaction for both presentation A and presentation B.

Form A - Method of Presentation A

Introduction

There are two logical methods used to find results or new facts. These methods are:

a) Deductive Method.

b) Inductive Method.

In the deductive method we start with a general case such as a proposition or a theorem, to reach a special case. In the inductive method, we start with a special case to get to a general case, as in the following example.

Find a formula for the following terms:

$1 + 2 + 3 + 4 + \ldots + n.$

When $n = 1$	1	$= 1 = \frac{1}{2} \cdot 1 \cdot 2$
$n = 2$	$1 + 2$	$= 3 = \frac{1}{2} \cdot 2 \cdot 3$
$n = 3$	$1 + 2 + 3$	$= 6 = \frac{1}{2} \cdot 3 \cdot 4$
$n = 4$	$1 + 2 + 3 + 4$	$= 10 = \frac{1}{2} \cdot 4 \cdot 5$
$n = 5$	$1 + 2 + 3 + 4 + 5$	$= 15 = \frac{1}{2} \cdot 5 \cdot 6$

From the previous numerical facts, we can suggest a general formula for the sum of $1 + 2 + 3 + 4 + 5 + \ldots + n$.

Thus $1 + 2 + 3 + 4 + 5 + 6 + \ldots + n = \frac{1}{2}(n)(n + 1)$

Problem 1:

Find a law for the following terms:

$1 + 3 + 5 + 7 + \ldots + (2n - 1)$

Let S_1 be the sum of the first term

S_2 be the sum of the first two terms

S_3 be the sum of the first three terms, and so on.

$S_1 = 1 \qquad = 1 = 1^2$

$S_2 = 1 + 3 \qquad = 4 = 2^2$

$S_3 = 1 + 3 + 5 \qquad = 9 = 3^2$

$S_4 = 1 + 3 + 5 + 7 \qquad = 16 = 4^2$

$S_5 = 1 + 3 + 5 + 7 + 9 \qquad = 25 = 5^2$

From the previous facts, we can predict a law for n terms, which is n^2.

Thus $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$

Problem 2:

Find a formula for the following terms:

$2 + 4 + 6 + 8 + 10 + \ldots + 2n$

$S_1 = 2 \qquad = 2 = 1^2 + 1$

$S_2 = 2 + 4 \qquad = 6 = 2^2 + 2$

$S_3 = 2 + 4 + 6 \qquad = 12 = 3^2 + 3$

$S_4 = 2 + 4 + 6 + 8 \qquad = 20 = 4^2 + 4$

$S_5 = 2 + 4 + 6 + 8 + 10 \qquad = 30 = 5^2 + 5$

From the previous facts, we can say that $S_n = n^2 + n$

Thus, $2 + 4 + 6 + 8 + 10 + \ldots 2n = n(n + 1)$

Critical Analysis of Presentation A

In teaching Presentation A, the teacher started the lesson by telling the students "we want to find a formula for 1 + 2 + 3 + ... n." Then the teacher writes the statement $1 + 2 + 3 + \ldots + n = ?$ on the blackboard and explains the lesson as follows:

$$\begin{array}{lll} \text{When } n = 1 & 1 & = 1 = \tfrac{1}{2} \cdot 1 \cdot 2 \\ n = 2 & 1 + 2 & = 3 = \tfrac{1}{2} \cdot 2 \cdot 3 \\ n = 3 & 1 + 2 + 3 & = 6 = \tfrac{1}{2} \cdot 3 \cdot 4 \\ n = 4 & 1 + 2 + 3 + 4 & = 10 = \tfrac{1}{2} \cdot 4 \cdot 5 \end{array}$$

Then the teacher turns to the students and says, "from the previous facts we can see very easily that the formula for $1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 2)$."*

This kind of presentation encourages students to memorize the formula without understanding the meanings behind the steps of the proof, the students will know that $1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1)$ because the teacher told them, but they do not know why the formula works or how the teacher got it.

Having no personal part in developing the proof of the mathematical statement is painful, generates negative attitudes and makes students afraid of mathematics.

Lack of understanding this proof kills curiosity and appreciation no matter what content is being taught. Herbert Spencer pointed

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*This analysis is based on personal observations made by the observer in many mathematics classes in Libyan Secondary Schools.

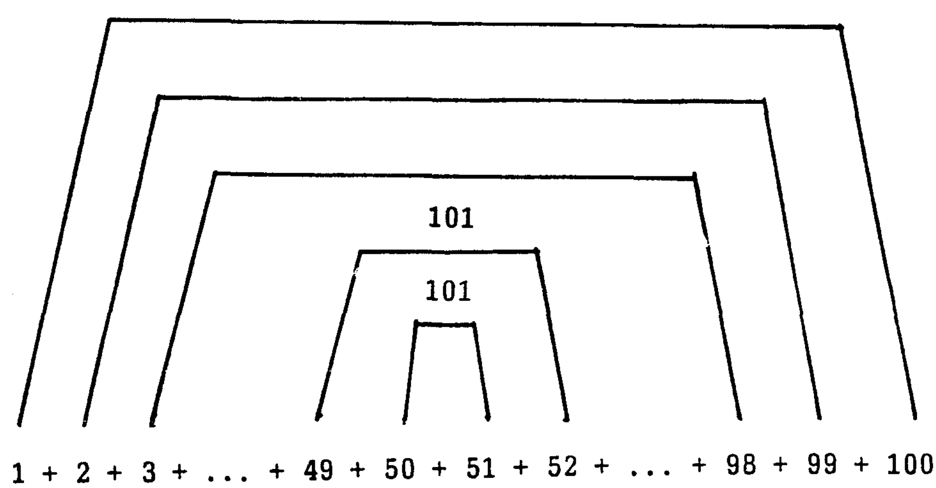
out that "...to give the net product of inquiry without the inquiry that leads to it is found to be enervating and inefficient" (Jones, 1970, pp. 501-503).

Presenting mathematics cannot be done by "telling." The teacher should provide a learning situation that will provoke discussion, participation, discovery, and motivations. The teacher should encourage the students to be active in the teaching-learning process by encouraging them to formulate a pattern and draw conclusions by themselves. As much as they possibly can, the teachers should use diagrams, illustrations, and concrete materials to provide a sequence of stimuli to keep students engaged in learning.

Form B - Method of Presentation B

Introduction

When the great mathematician Gauss was in the fifth grade, his teacher asked him to add 1 + 2 + 3 + 4 + ... + 100. To keep him suitably occupied for some time. Gauss gave the answer immediately. He found the sum of the first 100 counting numbers as included in this array.



Gauss said, I will have 50 pairs of 101, thus the sum is 50 times 101 which is equal to 5050.

In this lesson, we will learn a general formula for 1 + 2 + 3 + 4 + ... + n.

If we take a congruent copy of the original figure (Figure 19) and place it on top and to the right of the given figure see Figure 20.

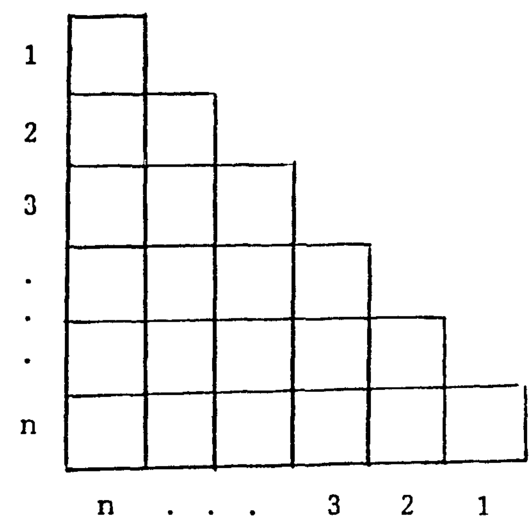


Figure 19

T*. What does the entire figure represent?

S*. A rectangle.

T. Good, what is the area of this rectangle?

S. n (n + 1)

T. Good, why?

S. Because it is a rectangle with n and (n + 1) sides.

T. Very good, what do we do to get the area of the figure that represents 1 + 2 + ... + n?

S. Take off the added figure.

T. O.k. Is there another answer? I mean without taking off the added figure.

---

*T represents Teacher

*S represents Student

S. Divide n (n + 1) by 2, will give you the area of the figure that represents 1 + 2 + 3 + ... + n.

T. Excellent; therefore, the sum of

$$1 + 2 + 3 + 4 + \ . \ . \ . + n = \frac{n\ (\ n + 1\ )}{2}$$

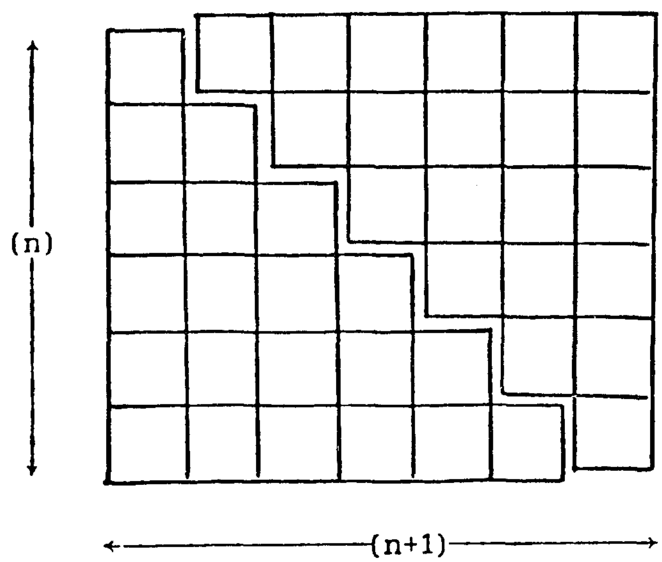


Figure 20

Application 1: Squares from squares

Cut some 1 cm x 1 cm squares and fill out Table 7, by constructing squares from the squares you cut.

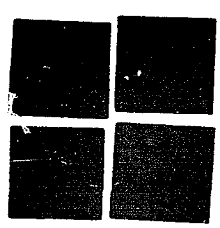
Table 7

Squares from Squares

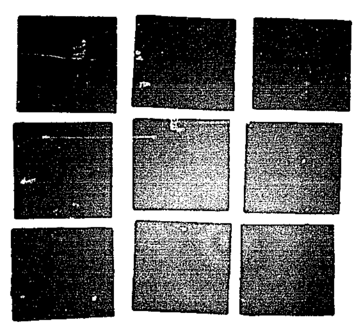
Trial	Tiles added	Area added	Total area recorded as the sum of areas	Total area	Dimension
1	1	1	1	1	1 X 1
2	3	3	1 + 3	4	2 X 2
3	5	5	1 + 3 + 5	9	3 X 3
4	7	7	1 + 3 + 5 + 7	16	4 X 4
5	9	9	1 + 3 + 5 + 7 + 9	25	5 X 5
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
n	2n-1	2n-1	1 + 3 + 5 + 7 +...+(2n-1)	n^2	n X n

As students start to construct the squares and fill out the table, the teacher walks around and supervises the students' work. After about five or six trials, the teacher might check if the students can recognize the pattern, and then guide them to discover the formula.

1



1 + 3



1 + 3 + 5

Thus, $1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2$

Application 2 Rectangles from rectangles

Cut rectangles 1 cm x 2 cm and fill out the following table by constructing new rectangles from the ones you cut out.

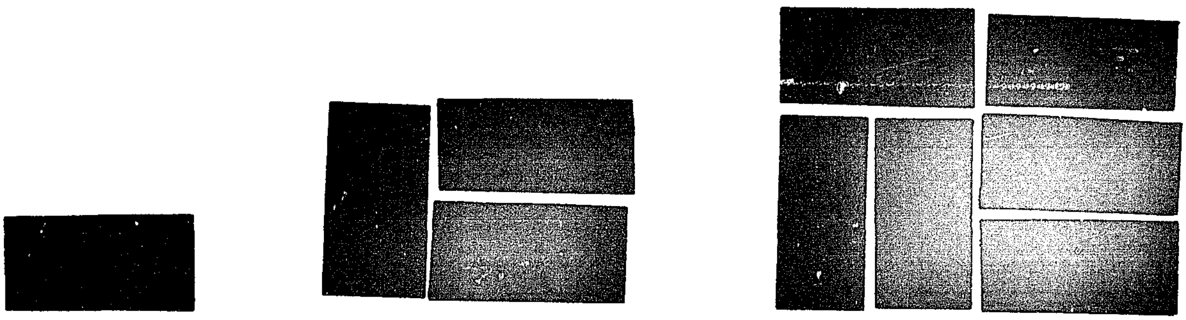


Table 8

Rectangles from Rectangles

Trial	Tiles added	Area added	Total areas recorded as the sum of areas	Total areas	Dimensions
1	1	2	2	2	1 X 2
2	2	4	2 + 4	6	2 X 3
3	3	6	2 + 4 + 6	12	3 X 4
4	4	8	2 + 4 + 6 + 8	20	4 X 5
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
n	n	2n	2 + 4 + 6 + ... + 2n	n(n+1)	n (n+1)

Thus, $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

Criteria Used in Presentation B

In developing method of presentation B, the following criteria are used.

Motivation: The presentation starts with historical introduction to show the students how Gauss discovered the formula of "1 + 2 + 3 + ... + n" when he was in the fifth grade. Diagrams and concrete materials are also used to make the lesson more interesting. These criteria are measured by questions 1 and 11.

Discovery: The major teaching technique used in presentation B is guided discovery which emphasizes 1) participation, 2) discussion, and 3) exploration. These criteria are measured by questions 2, 3, and 4.

Comprehension: Two subcriteria of comprehension are used in presentation B: 1) helping the students to recognize or discover a pattern, and 2) helping the students to derive the formula and generalize it. These criteria are measured by questions 8 and 10.

Analysis: In applying these criteria two subcriteria are used: 1) helping the student to formulate and validate generalization, and 2) helping the students to construct a mathematical proof. These criteria are measured by questions 6 and 7.

Instructional materials and applications: A semi-concrete situation is used in presentation B. This includes diagrams, squares and rectangles made from cardboard.

Two applications are included in presentation B: 1) to help the students use the instructional material, and 2) to recognize a pattern and 2) to recognize a pattern and to discover the formula by themselves. These criteria are measured by questions 5 and 9.

Preference the presentation: Questions 12, 13, 14, and 15 measured which presentation teachers might prefer in terms of student likes, meaningful learning, teachers' choice, and students retention of learning.

Testing the Criteria

The Questions

Answer the following questions by circling one answer only:

1) Which presentation has the more meaningful introduction?
2) Which presentation provides clearer steps to the students to discover the formula?
3) Which presentation generates more student participation?
4) Which presentation provides more opportunity for the student to explore the material?
5) Which presentation uses more diagrams and concrete materials?
6) Which presentation helps the student more to formulate and validate generalization?
7) Which presentation helps the student more to construct a mathematical proof?
8) Which presentation helps the student more to recognize or discover a pattern?
9) Which presentation includes more applications to help the student use instructional materials, organize a pattern, and make generalization?

10) Which presentation helps the student more to derive the formula and generalize it?

11) Which presentation motivates the student more?

12) Which presentation do you think students might prefer?

13) Which presentation will result in better learning?

14) I will choose presentation A, presentation B, to teach.

15) Students will retain learning longer with presentation A presentation B.

16) What is your personal reaction to presentation A?

17) What is your personal reaction to presentation B?

Hypotheses

In this experimental study, five elements (criteria) of "understanding" mathematics are applied and tested. The following hypothesis will measure each element (criteria) independently.

Research Hypothesis I: Significantly more teachers will prefer method of presentation B than will prefer method of presentation A for motivating the students (as measured by QM_1 and QM_{11}).

Research Hypothesis II: Significantly more teachers will prefer method of presentation B than will prefer method of presentation A for discovering the formula (as measured by QD_2, QD_3, QD_4).

Research Hypothesis III: Significantly more teachers will prefer method of presentation B than will prefer method of presentation A for comprehension (as measured by QC_8 and QC_{10}).

Research Hypothesis IV: Significantly more teachers will prefer method of presentation B than will prefer method of presentation A for analysis (as measured by QA_6 and QA_7).

Research Hypothesis V: Significantly more teachers will prefer method of presentation B than will prefer method of presentation A for the instructional materials and applications (as measured by QIA_5 and QIA_9).

Research Hypothesis VI: Significantly more teachers will prefer method of presentation B than will prefer method of presentation A with respect to students likes, meaningful learning, teacher's choice, and student retention of learning (as measured by QP_{12}, QP_{13}, QP_{14}, QP_{15}).

Procedures

Subjects were divided (randomly) into two groups; group I was asked to read form A, while group II was asked to read form B. Then group I was given form B to read, and group II was given form A to read. After both groups had finished reading both forms, they were asked to respond to the list of 17 questions.

Statistical Design

Binomial test of proportions was used to test each item individually. The level of significance, α is chosen at .05, which represents the type I error or probability of accepting the research hypothesis when Z is larger than $Z_{1-\alpha}$. The following terms and formulas are used:

N_A = Number selecting presentation A

N_B = Number selecting presentation B

P_1 = Proportion selecting presentation B

P_2 = Proportion selecting presentation A

$$P_1 = \frac{N_B}{N} \quad \text{and} \quad Z = \frac{P_1 - 1/2}{\sqrt{\tfrac{1}{2}\,(1-\tfrac{1}{2})/N}}$$

Analysis of Data and Statistical Results

Table 9 shows the analysis of data in terms of the number of teachers who selected method of Presentation A (N_A), the number of teachers who selected method of Presentation B (N_B), the proportion who selected method of Presentation B (P_1). The total number of responses was 75 for all questions.

Table 9

Item Analysis of Two Methods of
Mathematical Presentations

Question	N_A Presentation A	N_B Presentation B	P_1 Proportion of Presentation B
1	29	46	.61
2	25	50	.67
3	17	58	.77
4	27	48	.64
5	11	64	.85
6	27	48	.64
7	27	48	.64
8	24	51	.68
9	5	70	.93
10	16	59	.79
11	9	66	.88
12	26	49	.65
13	17	58	.77
14	12	63	.84
15	10	65	.87

The Sol-20 Microcomputer was used to analyze the data. Table 10 shows the statistical results of the Binomial test of proportions.

Table 10

Analysis of Binomial Test of Proportions

Question	Criteria	Question Identity	Z
1	Motivation	QM_1	1.96299
2	Discovery	QD_2	2.88675
3	Discovery	QD_3	4.73427
4	Discovery	QD_4	2.42487
5	Instructional Materials & App.	QIA_5	6.11991
6	Analysis	QA_6	2.42487
7	Analysis	QA_7	2.42487
8	Comprehension	QC_8	3.11769
9	Instructional Materials & App.	QIA_9	7.50555
10	Comprehension	QC_{10}	4.96521
11	Motivation	QM_{11}	6.58179
12	Preference	QP_{12}	2.65581
13	Preference	QP_{13}	4.73427
14	Preference	QP_{14}	5.88897
15	Preference	QP_{15}	6.35085

Analysis of Data and Statistical Results for Each Criteria

The analysis of data and statistical results of the Binomial test of proportions for each criteria are shown in the following tables.

Table 11

Analysis of Binomial Test of Proportions for Motivation

	N_A	N_B	N	P_1	Z*
QM_1	29	46	75	.61	1.96
QM_{11}	9	66	75	.88	6.58

*Significant at .05 level.

Table 12

Analysis of Binomial Test of Proportions for Discovery

	N_A	N_B	N	P_1	Z*
QD_2	25	50	75	.67	2.89
QD_3	17	58	75	.77	4.73
QD_4	27	48	75	.64	2.42

*Significant at .05 level.

Table 13

Analysis of Binomial Test of
Proportions for Comprehension

	N_A	N_B	N	P_1	Z*
QC_8	24	51	75	.68	3.12
QC_{10}	16	59	75	.79	4.96

*Significant at .05 level.

Table 14

Analysis of Binomial Test of
Proportions for Analysis

	N_A	N_B	N	P_1	Z*
QA_6	27	48	75	.64	2.42
QA_7	27	48	75	.64	2.42

*Significant at .05 level.

Table 15

Analysis of Binomial Test of Proportions
for Instructional Materials & Applications

	N_A	N_B	N	P_1	Z*
QIA_5	11	64	75	.85	6.12
QIA_9	5	70	75	.93	7.50

*Significant at .05 level.

Table 16

Analysis of Binomial Test of Proportions for Preference

	N_A	N_B	N	P_1	Z*
QP_{12}	26	49	75	.65	2.65
QP_{13}	17	58	75	.77	4.73
QP_{14}	12	63	75	.84	5.89
QP_{15}	10	65	75	.87	6.35

*Significant at .05 level.

The number of times each person preferred method of presentation B (out of 15 questions) was computed. For the 75 subjects, the mean of the times that method of presentation B was preferred is 11.19.

Interpretation of Data

Hypothesis I predicted that significantly more teachers will prefer method of presentation B than will prefer method of presentation A for motivating the students as measured by item 1 and item 11. The results of the Binomial test of proportions shown in Table 11 indicated that the value of $Z = 1.96$ for QM_1 and the value of $Z = 6.58$ for QM_{11} (The Motivation Criteria) are greater than $Z_{.05} = 1.64$. Therefore, Hypothesis I was accepted.

From this test it can be concluded that mathematics teachers prefer method of presentation B developed by this investigator for motivating the students to learn mathematics.

Hypothesis II predicted that significantly more teachers prefer method of presentation B than will prefer method of presentation A for discovery as measured by item 2, item 3, and item 4. The results of the Binomial Test of Proportions shown in Table 12 indicated that the value of $Z = 2.89$ for QD_2, the value of $Z = 4.73$ for QD_3, and the value of $Z = 2.42$ for QD_4 (The Discovery Criteria) are greater than $Z_{.05} = 1.64$. Therefore, Hypothesis II was accepted.

From this test it can be concluded that mathematics teachers prefer method of Presentation B developed by this investigator as a method of improving students' discovery of mathematics ideas.

Hypothesis III predicted that significantly more teachers will prefer method of Presentation B than will prefer method of Presentation A for comprehension as measured by item 8 and item 10.

The results of Binomial Test of Proportions shown in Table 13 indicated that the value of $Z = 3.12$ for QC_8 and the value of $Z = 4.96$ for QC_{10} (The Comprehension Criteria) are greater than $Z_{.05} = 1.64$. Therefore, Hypothesis III was accepted. From this test, it can be concluded that mathematics teachers prefer method of presentation B developed by this investigator as a method of improving students' comprehension of mathematics.

Hypothesis IV predicted that significantly more teachers will prefer method of presentation B than will prefer method of presentation

A for Analysis as measured by item 6 and item 7. The results of Binomial Test of Proportions shown in Table 14 indicated that the value of $Z = 2.42$ for QA_6 and the value of $Z = 2.42$ for QA_7 (The Analysis Criteria) are greater than $Z_{.05} = 1.64$. Therefore, Hypothesis IV was accepted.

From this test it can be concluded that mathematics teachers prefer method of presentation B developed by the investigator as a method of improving students' analysis of mathematics.

Hypothesis V predicted that more teachers will prefer method of presentation B than will prefer method of presentation A for the instructional materials and applications as measured by item 5 and item 9. The results of Binomial Test of Proportions shown in Table 15 indicated that the value of $Z = 6.12$ for QIA_5 and the value of $Z = 7.50$ for QIA_9 (The Instructional Materials and Applications Criteria) are greater than the value of $Z_{.05} = 1.64$. Therefore, Hypothesis V was accepted.

From this test it can be concluded that mathematics teachers prefer method of presentation B developed by this investigator as a method of including applications and using instructional materials.

Hypothesis VI predicted that significantly more teachers will prefer method of presentation B than will prefer method of presentation A with respect to students' likes, meaningful learning, teachers' choice and students' retention of learning as measured by item 12, item 13, item 14, and item 15. The results of the Binomial Test of Proportions shown in Table 16 indicated that the value of $Z = 2.65$

for QP_{12}, the value of $Z = 4.73$ for QP_{13}, the value of $Z = 5.89$ for QP_{14}, and the value of $Z = 6.35$ for QP_{15} as measured by QP_{12}, QP_{13}, QP_{14}, and QP_{15}. Therefore, Hypothesis VI was accepted.

From this test it can be concluded that mathematics teachers would prefer method of presentation B developed by this investigator as being more responsive to students' like, being more meaningful learning, being a more likely teacher's choice, and having a higher likelihood of student retention of mathematical knowledge.

Summary of Reactions to Questions 16 and 17

Out of 75 respondents, 18 teachers gave a personal reaction to method of presentation A, and 21 teachers gave a personal reaction to method of presentation B. Summarizations of the reactions listed for method of presentation A follow:

- Method of presentation A does not provide opportunity for student to participate in the classroom.
- Method of presentation A emphasized lecture approach of teaching mathematics.
- Method of presentation A does not encourage students to discover the general formula.
- Method of presentation A is straight forward, and lacking in motivation.
- Method of presentation A is easier to teach.

Summarizations of the reactions to method of presentation B are as follows:

- Method of Presentation B has more students' participation and teachers can stimulate students' thinking step by step.
- Method of Presentation B encourages students to organize into a pattern and derive the formula.
- Method of Presentation B may require more time than method of Presentation A.
- Method of Presentation B includes drawings and tables which encourage students to understand the lesson more.

Many respondents chose method B because it was responsive to the qualities examined by the preceding questions.

CHAPTER VI

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

The purpose of this study was to develop criteria for understanding mathematics and to evaluate the method of presenting mathematics in Libyan secondary schools in terms of some of the developed criteria. Specifically the purposes were these:

1. To study and analyze the characteristics of the new mathematics program.
2. To study and analyze the Libyan secondary school mathematics program, especially to determine the weaknesses of the program.
3. To develop criteria for understand'ng mathematics in the secondary level.
4. To evaluate a selected topic from the Libyan secondary school mathematics program in terms of some of the developed criteria.

Data for the study were gathered by different methods. Chapter II (Review of the Related Literature) and Chapter IV (Criteria for Understanding Mathematics) involved a search of professional literature, books, journals and dissertation abstracts.

Data for Chapter III (Analysis of the Libyan Secondary Mathematics Program) were obtained from Libyan mathematics textbooks used in the program, by a questionnaire answered by Libyan teachers, by direct observation of the Libyan mathematics classes, by interviews with Libyan educators, and by reviewing literature published with the Libyan Ministry of Education.

Data for Chapter V (Methodology of the Experiment) were gathered by using experimental procedures to evaluate methods of presenting a mathematical topic in the Libyan secondary school mathematics program.

The Sol-20 microcomputer was used to compute the Binomial test of proportions to determine if significant differences existed between the method of presentation A (presentation of the topic as it appears in the Libyan Mathematics Program), and Presentation B (presentation of the same topic that was developed by the investigator).

Conclusions

Based on the analysis of the Libyan secondary school mathematics program LSSMP and the findings of the evaluation of the method of presenting the topic "Finding a formula for 1 + 2 + 3 + ... + n." This study reached the conclusion that the method of presenting mathematics in LSSMP does not provide opportunities for secondary school students to understand mathematics. The findings specifically indicated that:

1. Method of presenting mathematics in Libyan secondary school mathematics program does not include devices for motivating the students.

2. Method of presenting mathematics in the Libyan secondary school mathematics program does not help students to discover mathematical ideas, to participate in the classroom, and to explore the learned material.
3. On the comprehension level of behavior, the method of presenting mathematics in the Libyan secondary school mathematics program does not provide an opportunity for the students to recognize or discover a pattern and to make mathematical generalizations.
4. On the analysis level of behavior, the method of presenting mathematics in the Libyan secondary school mathematics program does not encourage the students to formulate and validate generalizations, and does not provide an opportunity for the students to construct a mathematical proof.
5. The method of presenting mathematics in the Libyan secondary school mathematics program does not include applications to encourage students to use concrete situations to discover a pattern, and to formulate generalizations.

Recommendations

In order for Libyan students to learn mathematics meaningfully and to develop understanding of mathematics in the secondary school level, methods of presenting mathematics in the Libyan secondary school mathematics program must be modified to include the following qualities:

1. Methods of presenting mathematics must be shifted from the teacher as the center of the curriculum to the students. This will enable Libyan students to learn more effectively and gain understanding. The role of Libyan teachers will be guided to aid the students in arriving at meaning and understanding. Three important factors are suggested to achieve these qualities: a) using guided discovery approach of teaching mathematics, b) encourage students to participate in the classroom, and c) encourage students to explore the material.
2. Although emphasis is placed on computational skills in the present Libyan secondary school mathematics program, the skills require rote memorization rather than understanding. It is this writer's opinion that computational skills that consist of a) knowledge of specific facts, b) knowledge of terminology, and c) ability to carry out algorithms must be learned meaningfully. This can be accomplished by a) using the history of mathematics to show the development or the creation of the terminology or the symbols, b) using concrete situations to demonstrate certain procedures, and c) emphasize the "why" as well as the "how" in carrying out the algorithms.
3. In order for Libyan students to achieve the comprehension level of behavior, the method of presenting mathematics in Libyan secondary school mathematics program should

encourage students a) to learn the meaning of the mathematical concepts, b) to learn how to derive formulas and make generalizations, c) to learn the mathematical structure, d) to be able to translate problems from one mode to another (for example, to translate verbal problems into algebraic or graphic forms and conversely); e) to learn how to follow a line of reasoning.

4. In order for the Libyan students to develop independent and original thinking, they must be able to think by themselves. This can be accomplished by including the following subcategories of the analysis level of behavior: a) including indirect problems (non-routine problems) to help the students learn the technique for problem solving; b) including and presenting topics to encourage students to learn how to construct proofs, c) including some fallacies or wrong proofs to make students aware that a proof is not correct unless the steps are correct and well stated, and d) including problems to help the students make generalizations and validate them.

5. Attention must be given to applications of mathematics to include not only a large number of direct problems (routine problems) but also to include a) different types of routine applications, such as recognition exercise, algorithmic exercises, application problems, open-search problems, problem situations; b) problems to encourage

students to learn how to compare data and determine the relationship between two sets of information, c) problems that deal with reading and interpreting information, and manipulating that information.

6. It is recommended that the program must include instructional material to help the students to visualize a mathematical idea and to bridge the gap between the concrete and the abstract. It should be pointed out that it is not possible to use concrete situations for every mathematical concept in the program. However, different situations can be used to accomplish this quality: a) concrete situations, b) semi-concrete situations, c) semi-abstract situations, and d) abstract situations.

7. It is recommended that the program must include motivation devices to arouse interest. This can be achieved by a) giving meaningful introduction to the topic showing the importance or the application of the topic in our life, or using the history of mathematics to show how the topic is created, b) using mathematical recreations such as puzzles, games, optical illusions, mathematics in nature.

Further Recommendations

There are several other items which the investigator believes will contribute to understanding mathematics.

1. The present program uses one textbook for each grade. It is recommended that the Ministry of Education

provide textbooks of different mathematics programs that are used in other Arab countries or those programs that are used internationally or encourage the Libyan mathematics education specialists and mathematics teachers to write textual materials. An alternative choice of textbooks will help teachers as well as students achieve different ways of presenting mathematics and then they will get more insight into the subject matter.

2. It is recommended that textbooks be accompanied by teachers' editions and commentaries. These teachers' editions might include different ways to solve problems, different approaches or strategies for teaching certain topics, more background on difficult concepts, sample test, etc.

3. It is strongly recommended that freedom be given in teachers to teach mathematical topics that are not presented in the official textbook or to encourage students to use other mathematics books, since these will provide opportunities for the students to become familiar with new methods of presenting the idea and different ways to solve the problems.

4. It is further recommended that each school be provided with school libraries, equipped with books, reference books, and professional journals in each subject. In mathematics the library might be equipped with different

international high school mathematics programs, as well as books and journals. These sources will not only help teachers to have more than one source for teaching the subject, but also will help them to find out the advantages and disadvantages of the methods of prsenting mathematics in the current Libyan program.

5. It is recommended that an Association for Libyan Mathematics Teachers be organized. The purposes of this association would be: a) to search for methods of presenting mathematics which focus on the students rather than the teacher, b) to discuss the problems of teaching mathematics in the country, c) to plan for meetings and conferences, and d) to produce a journal or journals in the field of mathematics education.

6. History of Arabic and Islamic mathematics should be included in the Libyan secondary school mathematics program. Topics of this type will not only motivate students to learn mathematics and generate positive attitudes but will help them understand the cultural significance of mathematics to the society.

7. In order for Libyan mathematics teachers to be aware of the development of mathematical science and to have a continuously strong foundation in mathematics education it is recommended that either professional development not stop upon their completion of the university, but it

continue during the in-service years. To achieve this recommendation, a program should be created to take the form of course study, seminars, research work, and workshops.

8. The success of the mathematics program and the improvement of instruction can be achieved by the cooperative effort of the supervisors and teachers. It is recommended that the mathematics supervisors not only aim to evaluate teachers in writing annual reports for their promotion, but also to serve as a source of information, leadership, and expert knowledge in the teaching of mathematics.

9. It is recommended that a conference or a meeting be held in the country annually to discuss the types and means of creating a new program or revising the present program. Invited to this conference should be a) secondary school mathematics supervisors from the department of supervision - Ministry of Education (b) mathematics education and curriculum specialists from the Schools of Education of the Libyan universities, c) mathematicians from the mathematics departments in the Libyan universities, and d) teachers of the discipline, both at the level of instruction and at the teachers' training institutions. At the end of the conference, a committee consisting of mathematicians, educators, experienced teachers, psychologists and supervisors of mathematics in secondary schools should be

selected to: a) follow up recommendations of the conference, b) modify or revise the program, and c) consider the opinions of all mathematics teachers and secondary school students regarding the present mathematics program.

10. Two important areas in which Libyan mathematics program fails are those of writing textual materials and textbook qualities. The investigator suggested the following qualities that were adopted from the 1965 National Council of Teachers of Mathematics (NCTM, 1965) to aid Libyan writers and mathematicians in writing textbooks. The qualities which were presented in the form of questions related to three different areas: presentation and content, the physical characteristics of a textbook, and the services provided by the publisher (Kostaki, 1978, pp. 50-52).

Qualities related to presentation and content included the following items.

Structure - Does each new topic presented fit into the whole structure of mathematics? How does it relate to it? What is its scope and depth?

Qualities related to physical characteristics of a book and publishers' services focused on:

General format - Is the layout and construction both attractive and functional in regard to book size, quality of paper and binding, type of print? Is color being used?

Index and references - Is the table of contents detailed? Is a glossary of symbols and definitions provided? Does this facilitate referral to ideas?

Usability - Is the book free of typographical errors? Is the purchase of additional materials necessary?

Publisher's services - Are the sales and promotional materials informative? Does the publisher include supplementary materials such as tapes, etc.?

Teacher's manuals - Do they clarify the text and are they of help to the teacher? Do they include diagnostic and follow-up tests, etc.?

Rigor - What is the nature of the development of the arguments? What kinds of justifications are used in proof? Is the level of rigor appropriate for the maturity of the students?

Vocabulary - Are the vocabulary and reading level appropriate for the students? Are ideas restated in different ways rather than by exact repetition?

Defined and undefined terms - Are there distinctions made between defined and undefined terms? Is the terminology used appropriate for students at this stage?

Correctness - Is the text free of statements most mathematicians would regard as errors?

Theorems and proofs - How are major generalizations used? By what arguments are the formal conclusions established? Is frequent occurrence of a generalization regarded as a substitute for proof?

Generalizations - Are opportunities to generalize provided?

Ordering - Is the sequence between and within topics arranged in a spiral form?

Tests, exercises and reviews - Are they adequate and self-evaluating? Are problems stated clearly? Are there any challenging exercises with the levels of difficulty being identified? Do review exercises point to further instructional needs?

Illustrative examples - Do they clarify and reinforce the presentation of concepts?

Teachability - Can the text be used by students on their own? Is the book's approach compatible with that of the teacher?

Optional topics - Can they be deleted with no loss of continuity?

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APPENDICES

APPENDIX A

Analysis of the Libyan Secondary School

Mathematics Program

The Questionnaire

Results of the Questionnaire

The Questionnaire

Dear Secondary School Mathematics Teacher:

The modern mathematics program has already been applied to the three years of the secondary level, but this program will not succeed unless we try, together, to study it, to analyze it, and to find out its deficiencies. Since the teacher's opinion is quite important for this study and this analysis, please respond to all the questions of this questionnaire with accuracy and precision.

Thank you.

The Questions

Write a check mark (√) beside the chosen answer.

1. Do you think that mathematics teachers should participate in setting of the mathematics curriculum?

   Yes ______ No ______

2. Do you think that the opinions of the students should be considered in the development of the mathematics curriculum?

   Yes ______ No ______

3. Do you think that the applied mathematics (statics and dynamics) should be omitted from the mathematics curriculum?

   Yes ______ No ______

4. Do you prefer to teach some topics that are not mentioned in the official textbooks?

   Yes ______ No ______

5. Do you prefer to teach mathematics as one unified subject related to each other or as separate subjects, algebra, geometry, or calculus?

   Yes ______ No ______

6. Do you think the time devoted to the mathematics curriculum of the first year is sufficient?

   Yes ______ No ______

7. Do you think that the time devoted to the mathematics curriculum of the second year is sufficient?

   Yes ______ No ______

8. Do you think that the time devoted to the mathematics curriculum of the third year is sufficient?

   Yes ______ No ______

9. Does the method of teaching that you now use provide opportunity for the students to participate in the classroom?

   Yes ______ No ______

10. Does the method of teaching that you use now help the students to develop mathematical concepts?

Yes ______ No ______

11. Does the method of teaching that you use now provide opportunities for the students to discover mathematical ideas?

Yes ______ No ______

12. Do you think that the program should include non-routine problems?

Yes ______ No ______

13. Do you think that the program should include real life problems?

Yes ______ No ______

14. Do you think that the program includes instructional materials or visual aids?

Yes ______ No ______

15. Do you think that the textbooks should be accompanied by teacher's commentaries which contained review items, test questions, and elaborations on certain difficult topics?

Yes ______ No ______

16. Do you think that the placement of the topics in the program is appropriate?

Yes ______ No ______

17. Do you think that the program includes devices to motivate the students?

Yes ______ No ______

18. Do you think that the program provides opportunities for the students to learn mathematical generalizations?

Yes ______ No ______

19. Do you think the program provides opportunities for the students to learn to construct a mathematical proof?

Yes ______ No ______

Table 17

Item Analysis of the Results of the Questionnaire

	Questions	Responses	(Percent)
1.	Do you think that mathematics teachers should participate in the setting of the mathematics curriculum?	Yes	100%
		No	——
2.	Do you think that the opinions of the students should be considered in the development of the mathematics curriculum?	Yes	55%
		No	45%
3.	Do you think that the applied mathematics (statics and dynamics) should be omitted from the mathematics curriculum?	Yes	78%
		No	22%
4.	Do you prefer to teach some topics that are not mentioned in the official textbook?	Yes	55%
		No	45%
5.	Do you prefer to teach mathematics as one unified subject related to each other or as separate subjects, algebra, geometry, or calculus?	Yes (One subject)	66%
		No (Separate subject)	34%
6.	Do you think the time devoted to mathematics curriculum of the first year is sufficient?	Yes	84%
		No	16%
7.	Do you think the time devoted to mathematics curriculum of the second year is sufficient?	Yes	37%
		No	63%
8.	Do you think the time devoted to mathematics curriculum of the third year is sufficient?	Yes	15%
		No	85%
9.	Does the method of teaching that you use now provide opportunity for the students to participate in the classroom?	Yes	62%
		No	38%
10.	Does the method of teaching that you use now help the students to develop mathematical concepts?	Yes	45%
		No	55%

Questions	Responses	(Percent)
11. Does the method of teaching that you use now provide opportunities for the student to discover mathematical ideas?	Yes	46%
	No	54%
12. Do you think that the program should include non-routine problems?	Yes	74%
	No	26%
13. Do you think the program should include real life problems?	Yes	69%
	No	31%
14. Do you think that the program includes instructional materials or visual aids?	Yes	42%
	No	58%
15. Do you think that the textbooks should be accompanied by teacher's commentaries which contain review items, test questions, and elaborations on certain difficult topics?	Yes	84%
	No	16%
16. Do you think that the placement of the topics in the program is appropriate?	Yes	73%
	No	27%
17. Do you think that the program includes devices to motivate the students?	Yes	21%
	No	79%
18. Do you think that the program provides opportunities for the student to learn mathematical generalizations?	Yes	30%
	No	70%
19. Do you think the program provides opportunities for the students to learn to construct a mathematical proof?	Yes	45%
	No	55%

APPENDIX B

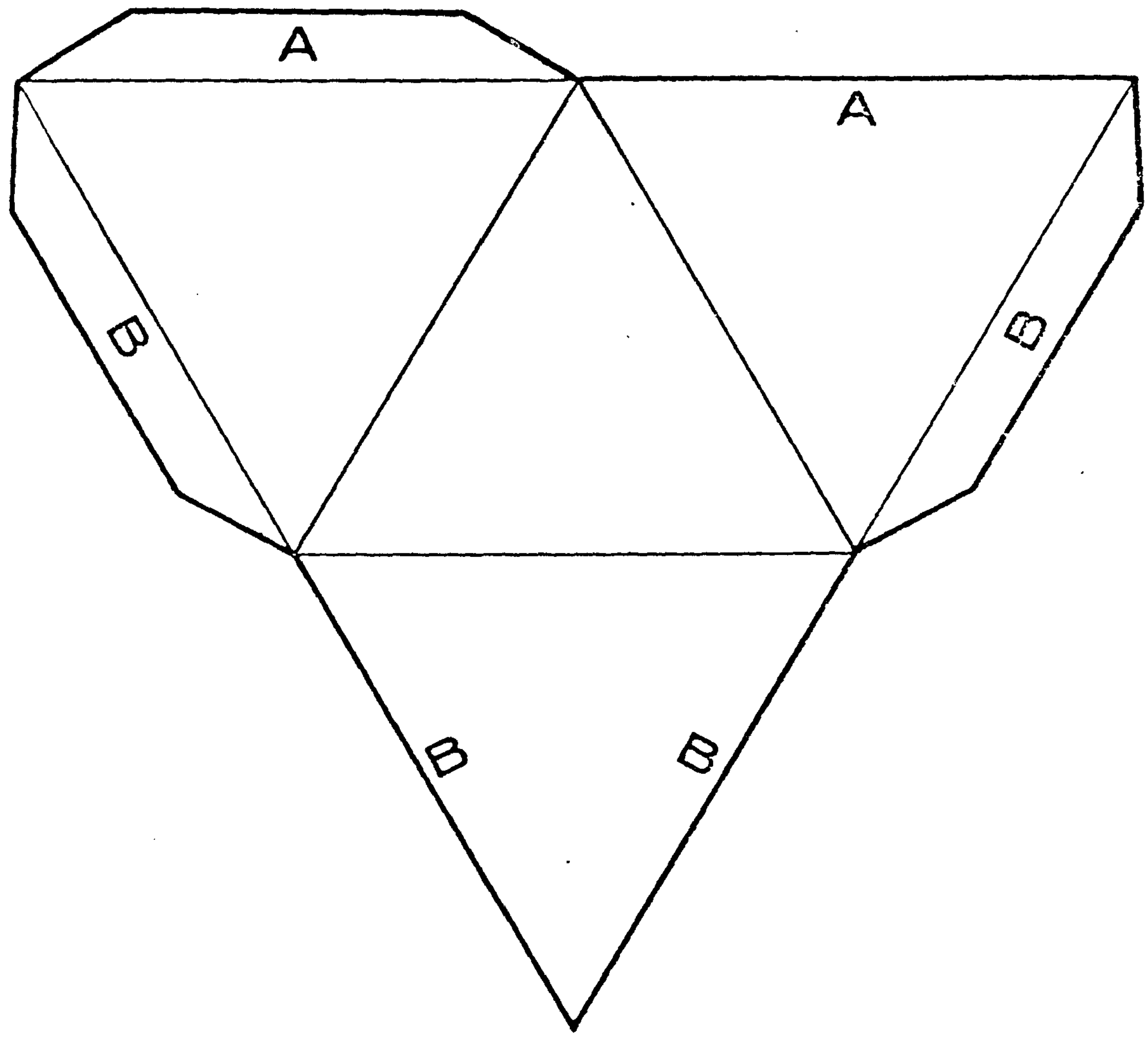
Instructional Materials

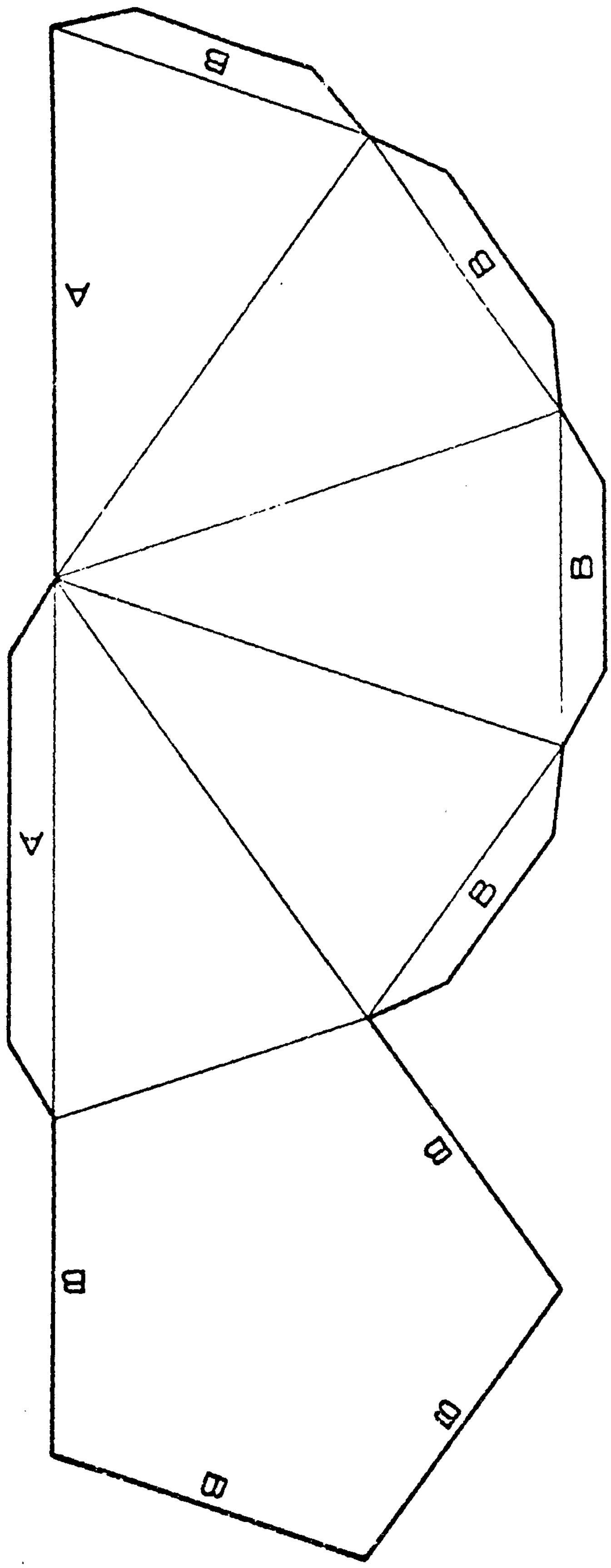
Making Mathematical Models

How to Make Good Models

1. Copy the model of your choice.
2. Cut out the model, keeping well away from the drawing.
3. Trim carefully around the edges.
4. Fold along each crease to get a clean edge. Do not start to glue until all the creases are done.
5. Glue. The glue always goes on the back (non-printed side) of each flap.

TETRAMID





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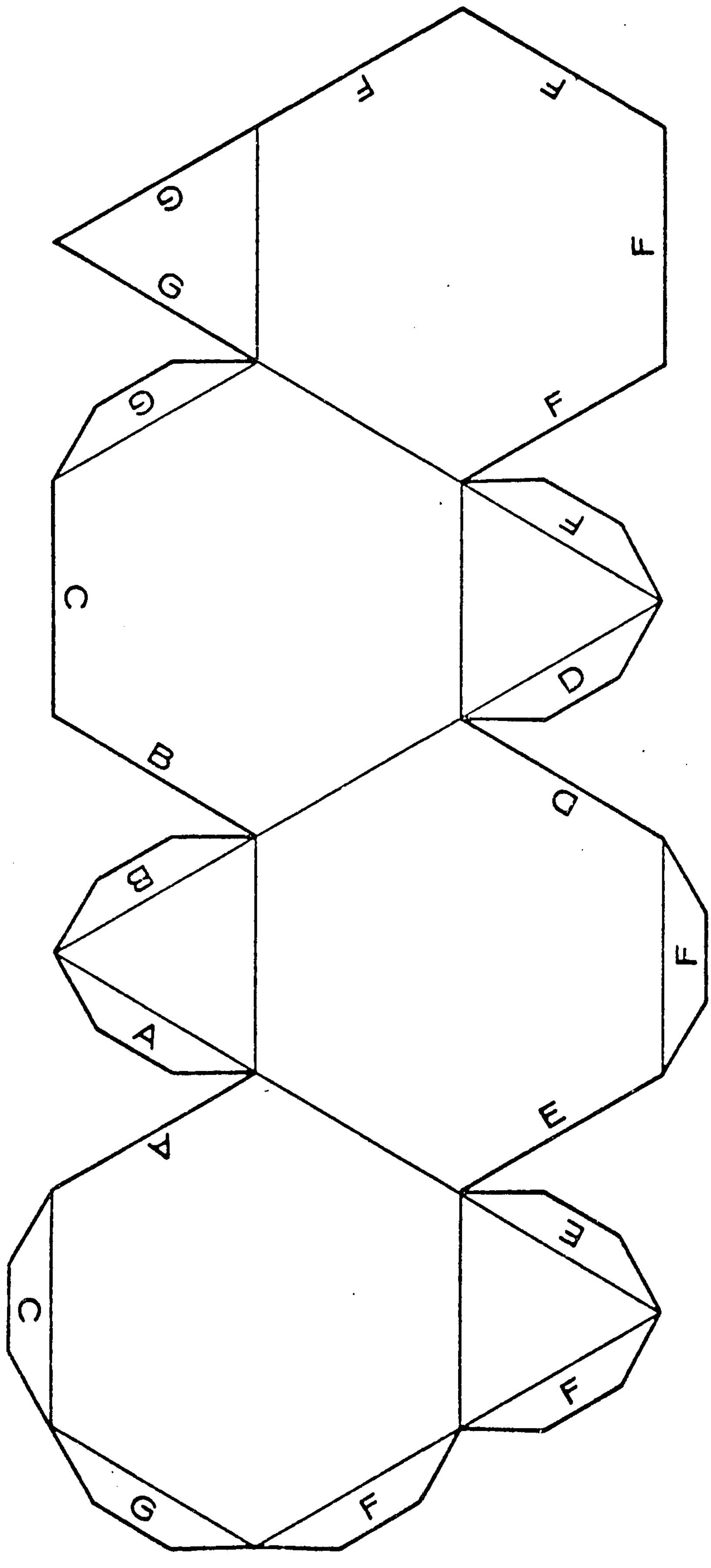
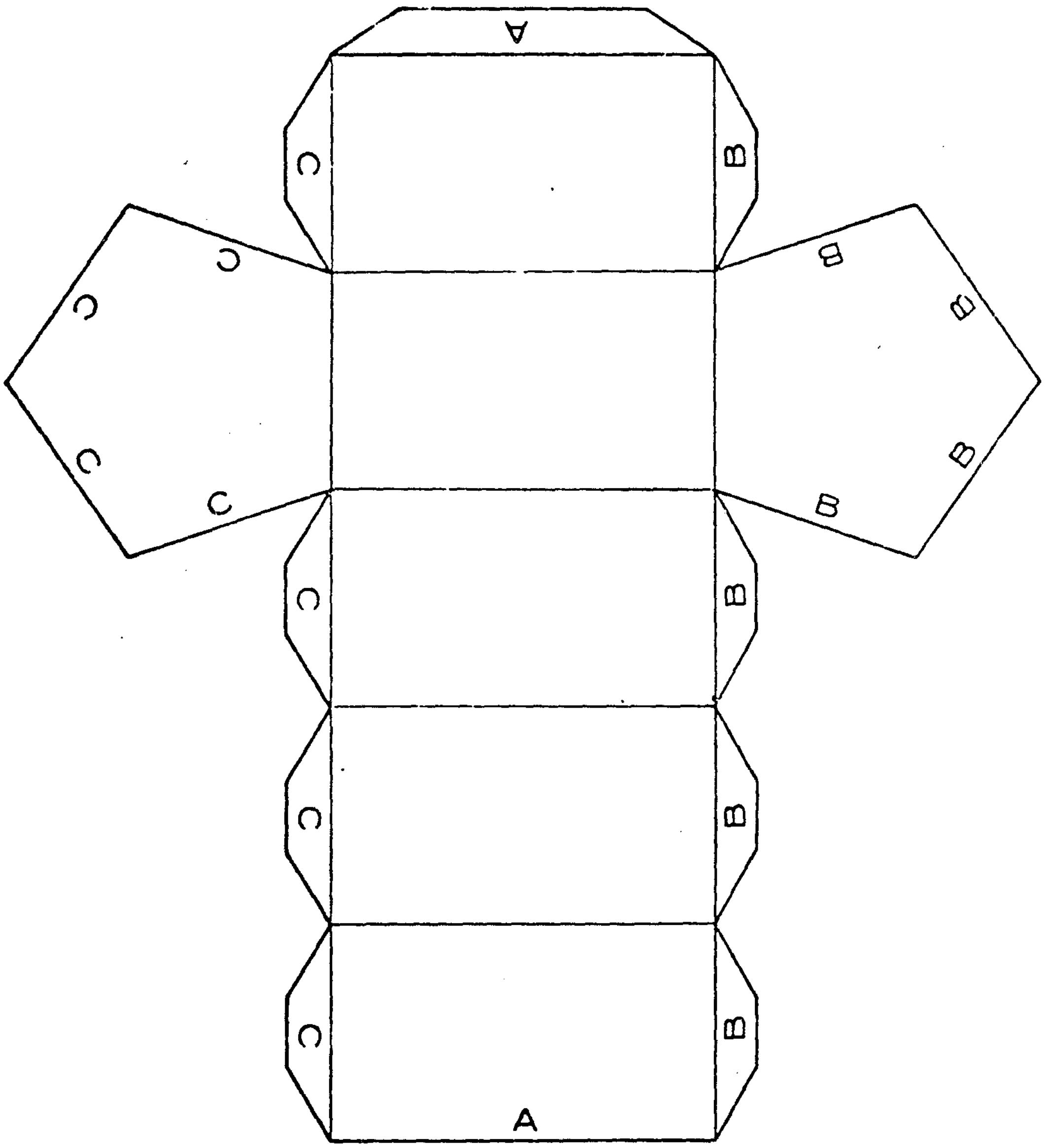
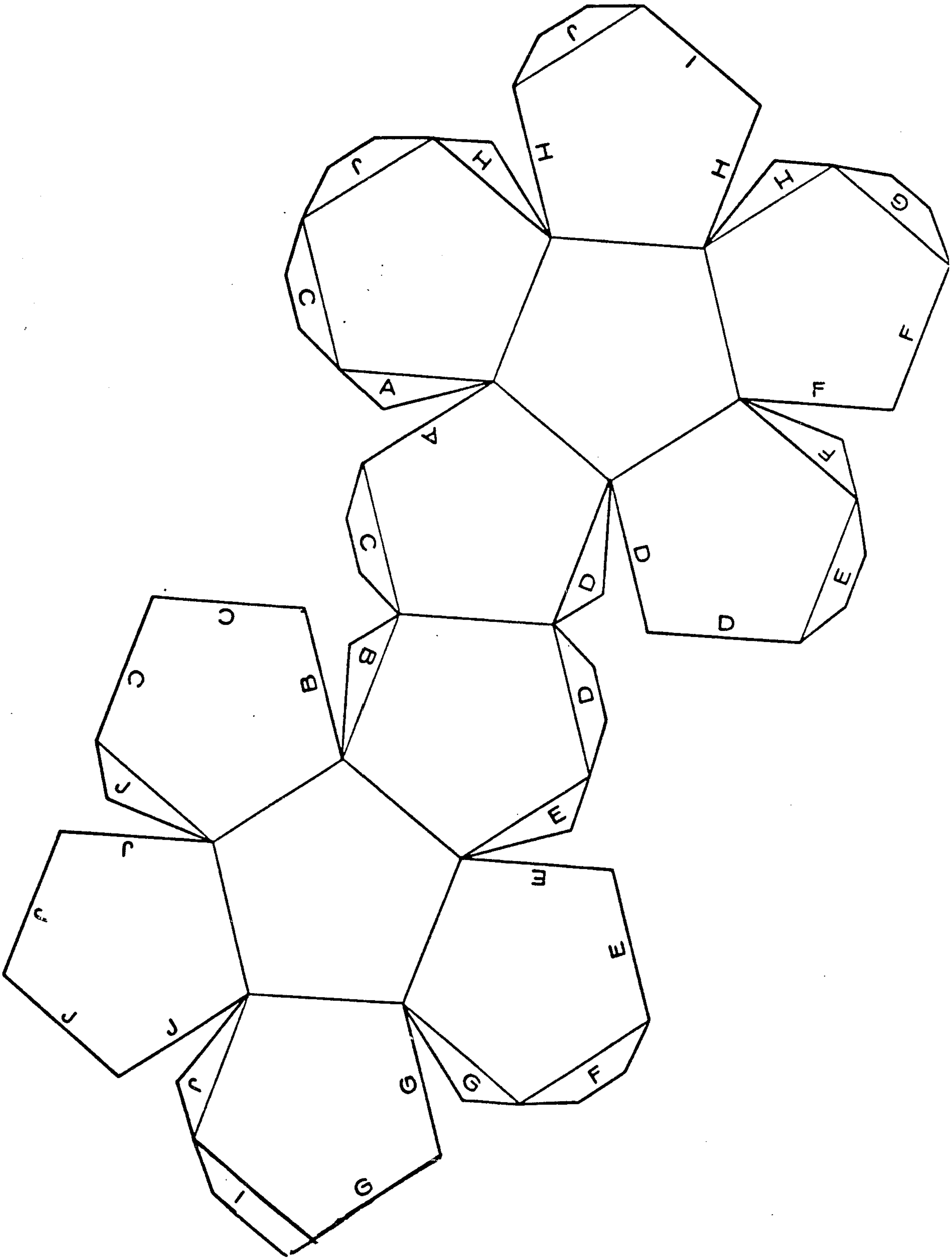


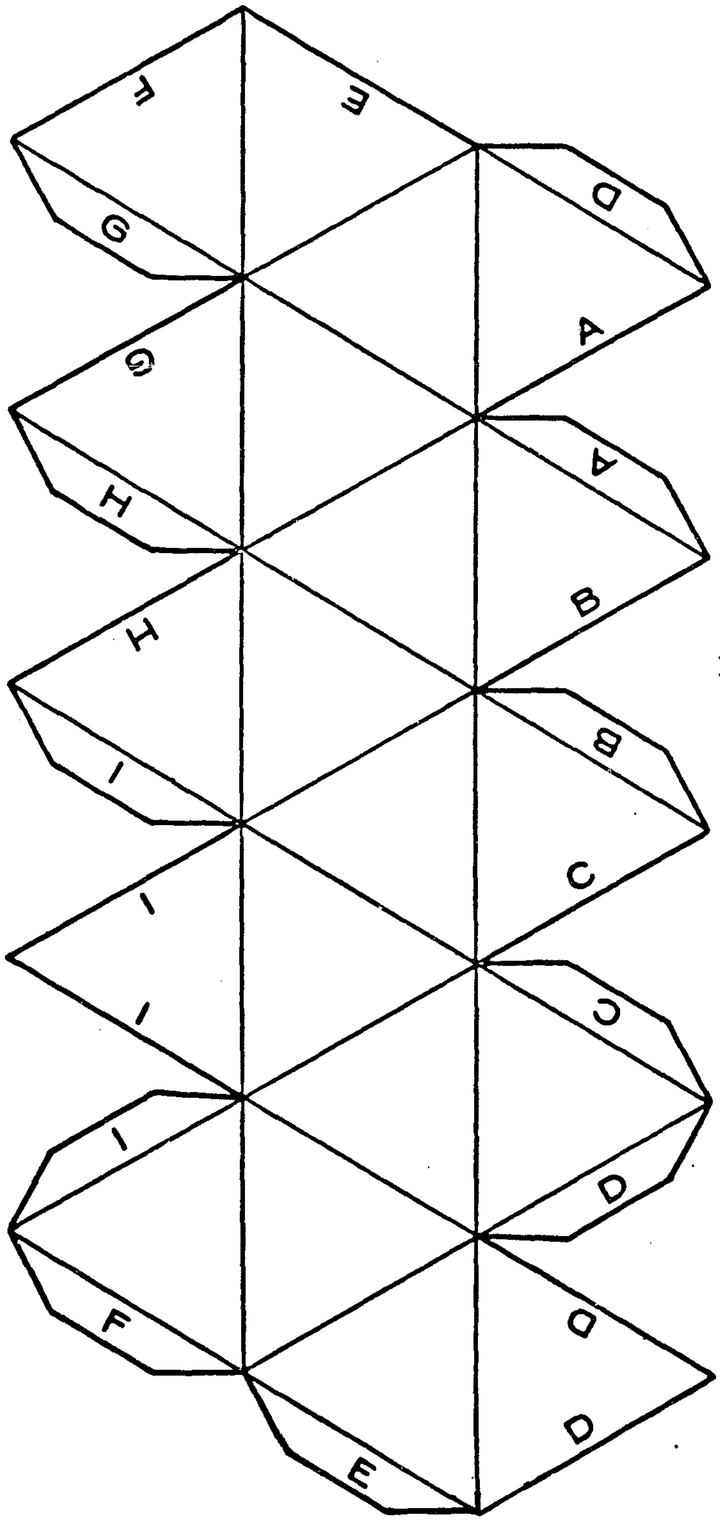
TRUNCACUBE

PENTACUBE



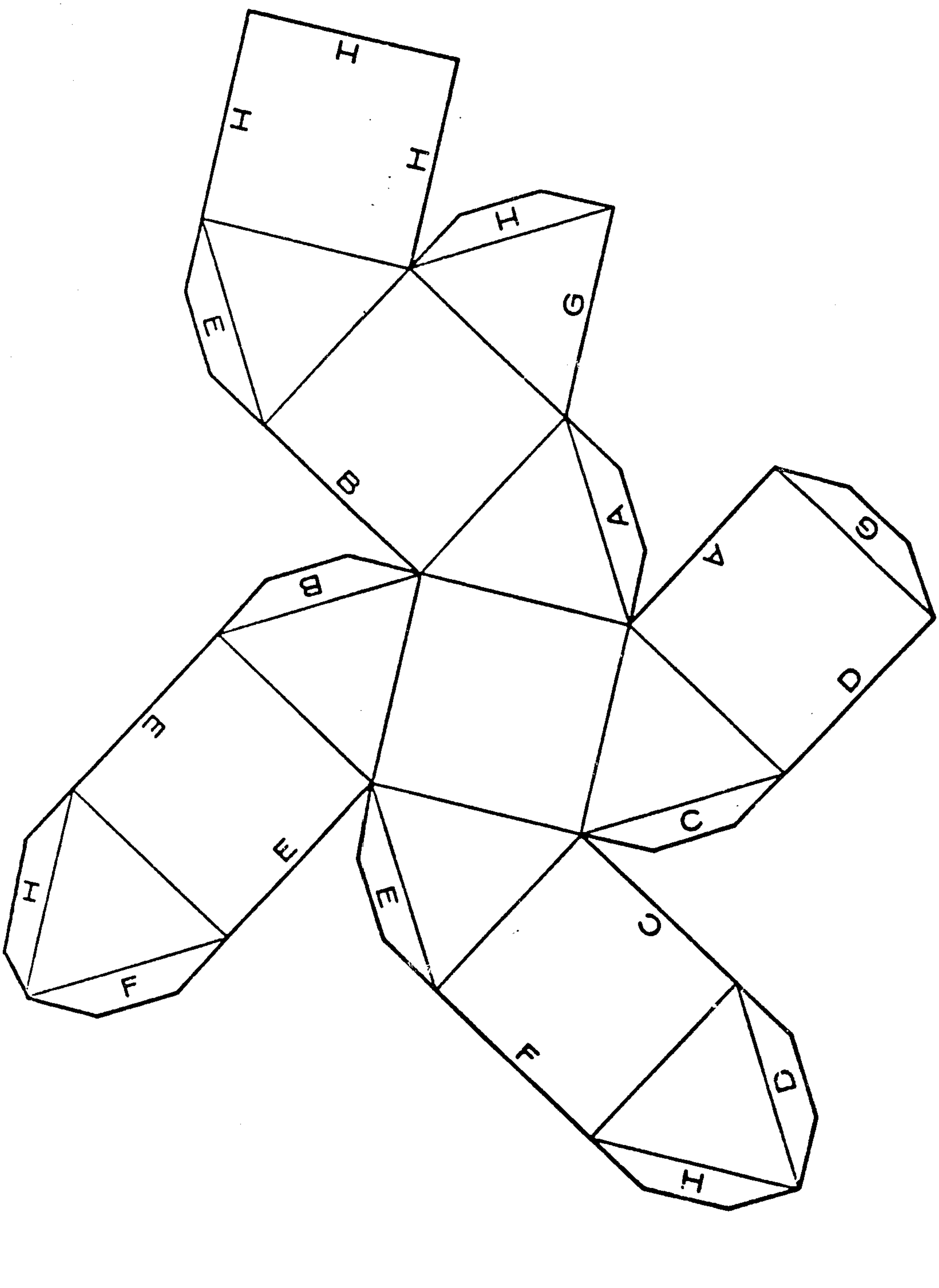
DODECUBE



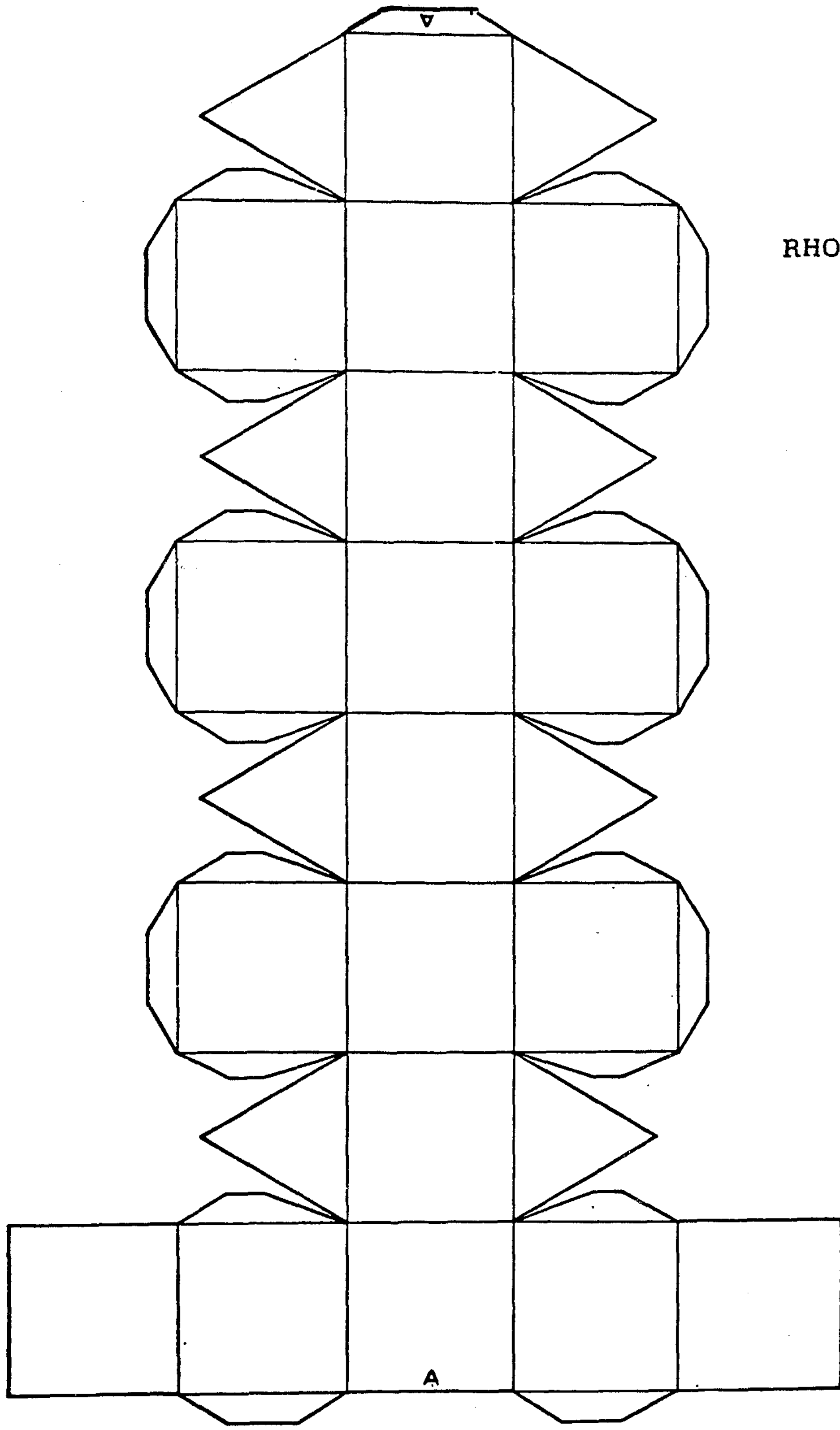


ICOSACUBE

CUBOCUBE



RHOMBICUBE



APPENDIX C

The Evaluation Material

Form - A (Method of Presentation A)

Form - B (Method of Presentation B)

The Questions

Dear Mathematics Teachers:

I have enclosed two forms representing two different methods of presenting a formula for "1 + 2 + 3 + ... + n."

Will you please read both forms and then answer the questions on the questionnaire?

Thank you.

Form A - Method of Presentation A

Form A - Method of Presentation A

Introduction

There are two logical methods used to find results or new facts. These methods are:

a) Deductive Method.

b) Inductive Method.

In the deductive method we start with a general case such as a proposition or a theorem, to reach a special case. In the inductive method, we start with a special case to get to a general case, as in the following example.

Find a formula for the following terms:

$1 + 2 + 3 + 4 + \ldots + n.$

When $n = 1$	1	$= 1 = \frac{1}{2} \cdot 1 \cdot 2$
$n = 2$	$1 + 2$	$= 3 = \frac{1}{2} \cdot 2 \cdot 3$
$n = 3$	$1 + 2 + 3$	$= 6 = \frac{1}{2} \cdot 3 \cdot 4$
$n = 4$	$1 + 2 + 3 + 4$	$= 10 = \frac{1}{2} \cdot 4 \cdot 5$
$n = 5$	$1 + 2 + 3 + 4 + 5$	$= 15 = \frac{1}{2} \cdot 5 \cdot 6$

From the previous numerical facts, we can suggest a general formula for the sum of $1 + 2 + 3 + 4 + 5 + \ldots + n$.

Thus $1 + 2 + 3 + 4 + 5 + 6 + \ldots + n = \frac{1}{2}(n)(n + 1)$

Problem 1:

Find a law for the following terms:

$1 + 3 + 5 + 7 + \ldots + (2n - 1)$

Let S_1 be the sum of the first term

S_2 be the sum of the first two terms

S_3 be the sum of the first three terms, and so on.

$S_1 = 1 \qquad = 1 = 1^2$

$S_2 = 1 + 3 \qquad = 4 = 2^2$

$S_3 = 1 + 3 + 5 \qquad = 9 = 3^2$

$S_4 = 1 + 3 + 5 + 7 \qquad = 16 = 4^2$

$S_5 = 1 + 3 + 5 + 7 + 9 \qquad = 25 = 5^2$

From the previous facts, we can predict a law for n terms, which is n^2.

Thus $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$

Problem 2:

Find a formula for the following terms:

$2 + 4 + 6 + 8 + 10 + \ldots + 2n$

$S_1 = 2 \qquad = 2 = 1^2 + 1$

$S_2 = 2 + 4 \qquad = 6 = 2^2 + 2$

$S_3 = 2 + 4 + 6 \qquad = 12 = 3^2 + 3$

$S_4 = 2 + 4 + 6 + 8 \qquad = 20 = 4^2 + 4$

$S_5 = 2 + 4 + 6 + 8 + 10 \qquad = 30 = 5^2 + 5$

From the previous facts, we can say that $S_n = n^2 + n$

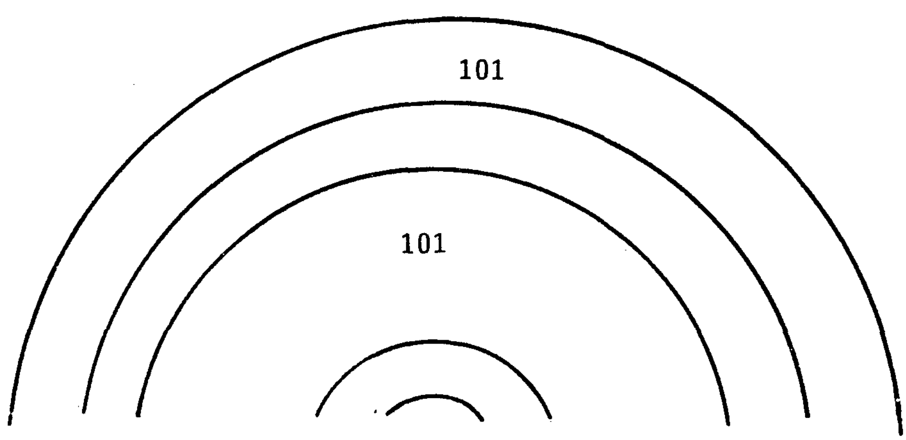
Thus, $2 + 4 + 6 + 8 + 10 + \ldots 2n = n(n + 1)$

Form B - Method of Presentation B

Form B - Method of Presentation B

Introduction

When the great mathematician Gauss was in the fifth grade, his teacher asked him to add 1 + 2 + 3 + 4 + ... + 100. To keep him suitably occupied for some time. Gauss gave the answer immediately. He found the sum of the first 100 counting numbers as included in this array.



1 + 2 + 3 + ... + 49 + 50 + 51 + 52 + ... + 98 + 99 + 100

Gauss said, I will have 50 pairs of 101, thus the sum is 50 times 101 which is equal to 5050.

In this lesson, we will learn a general formula for 1 + 2 + 3 + 4 + ... + n.

If we take a congruent copy of the original figure (Figure 18) and place it on top and to the right of the given figure see Figure 19.

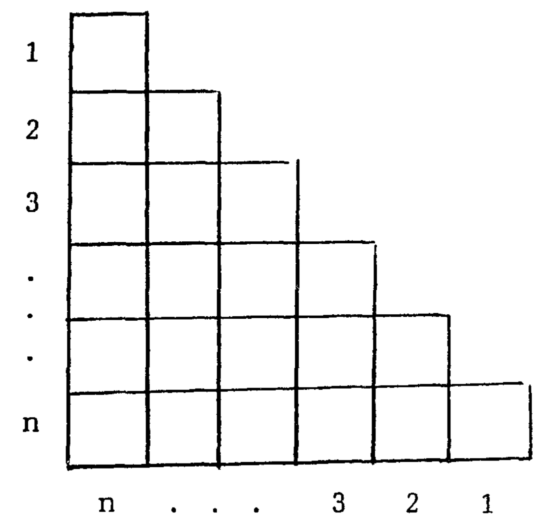


Figure 18

T*. What does the entire figure represent?

S*. A rectangle.

T. Good, what is the area of this rectangle?

S. n (n + 1)

T. Good, why?

S. Because it is a rectangle with n and (n + 1) sides.

T. Very good, what do we do to get the area of the figure that represents 1 + 2 + ... + n?

S. Take off the added figure.

T. O.k. Is there another answer? I mean without taking off the added figure.

---

*T represents Teacher

*S represents Student

S. Divide n (n + 1) by 2, will give you the area of the figure that represents 1 + 2 + 3 + ... + n.

T. Excellent; therefore, the sum of

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

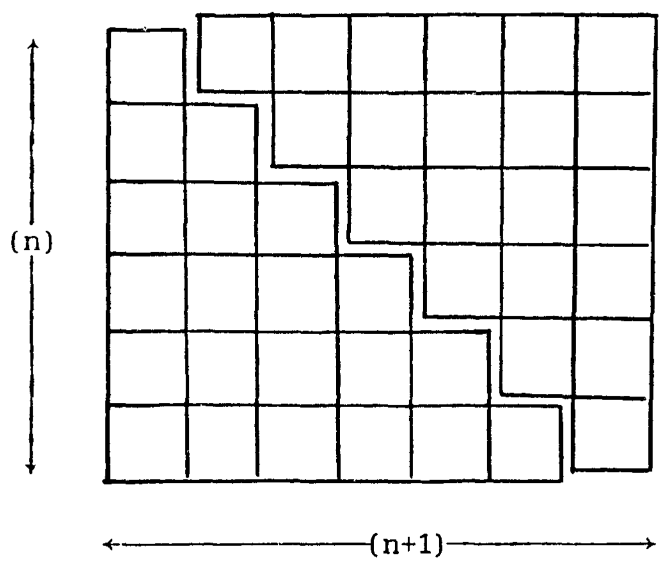


Figure 19

<u>Application 1</u>: Squares from squares

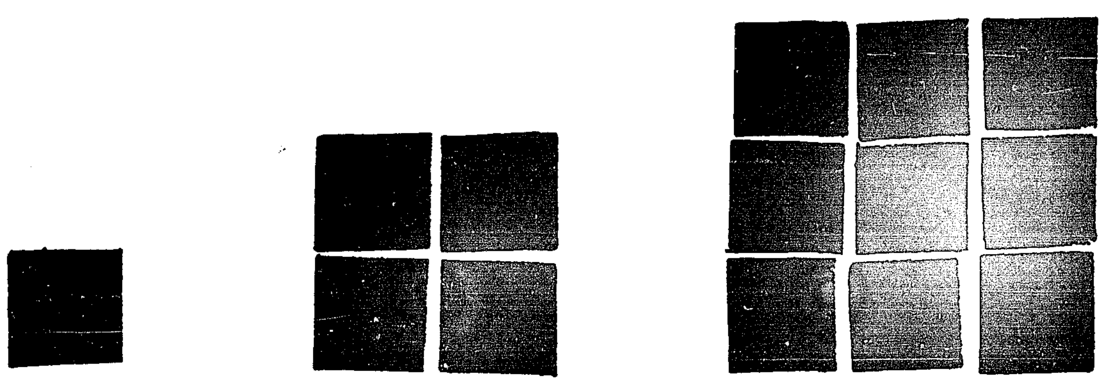
Cut some 1 cm x 1 cm squares and fill out Table 7, by constructing squares from the squares you cut.

Table 7

Squares from Squares

Trial	Tiles added	Area added	Total area recorded as the sum of areas	Total area	Dimension
1	1	1	1	1	1 X 1
2	3	3	1 + 3	4	2 X 2
3	5	5	1 + 3 + 5	9	3 X 3
4	7	7	1 + 3 + 5 + 7	16	4 X 4
5	9	9	1 + 3 + 5 + 7 + 9	25	5 X 5
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
n	2n-1	2n-1	1 + 3 + 5 + 7 +...+(2n-1)	n^2	n X n

As students start to construct the squares and fill out the table, the teacher walks around and supervises the students' work. After about five or six trials, the teacher might check if the students can recognize the pattern, and then he guided them to discover the formula.



1   1 + 3   1 + 3 + 5

Thus, $1 + 3 + 5 + 7 + . . . + (2n-1) = n^2$

Application 2 Rectangles from rectangles

Cut rectangles 1 cm x 2 cm and fill out the following table by constructing new rectangles from the ones you cut out.

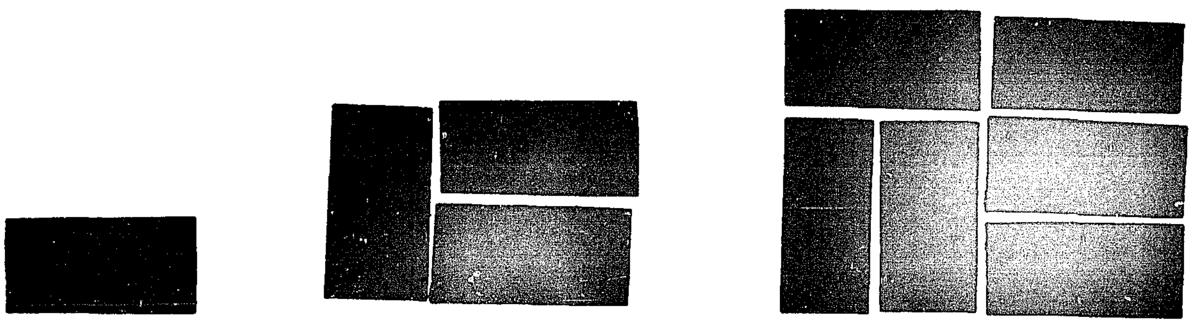


Table 8

Rectangles from Rectangles

Trial	Tiles added	Area added	Total areas recorded as the sum of areas	Total areas	Dimensions
1	1	2	2	2	1 X 2
2	2	4	2 + 4	6	2 X 3
3	3	6	2 + 4 + 6	12	3 X 4
4	4	8	2 + 4 + 6 + 8	20	4 X 5
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
n	n	2n	2 + 4 + 6 + ... + 2n	n(n+1)	n (n+1)

Thus, $2 + 4 + 6 + ... + 2n = n (n + 1)$

The Questions

Answer the following questions by circling one answer <u>only</u>:

1) Which presentation has the more meaningful introduction?

   Presentation A — Presentation B

2) Which presentation provides the clearer steps to the student to discover the formula?

   Presentation A — Presentation B

3) Which presentation generates more student participation?

   Presentation A — Presentation B

4) Which presentation provides more opportunity for the student to explore the material?

   Presentation A — Presentation B

5) Which presentation uses more diagrams and concrete materials?

   Presentation A — Presentation B

6) Which presentation helps the student more to formulate and validate generalization?

   Presentation A — Presentation B

7) Which presentation helps the student more to construct a mathematical proof?

   Presentation A — Presentation B

8) Which presentation helps the student more to recognize or discover a pattern?

   Presentation A — Presentation B

9) Which presentation includes more applications that help the student to use instructional materials, organize materials, organize a pattern, and make generalization?

   Presentation A — Presentation B

10) Which presentation helps the student more to derive the formula and generalize it?

Presentation A Presentation B

11) Which presentation motivates the student more?

Presentation A Presentation B

12) Which presentation do you think students might prefer?

Presentation A Presentation B

13) Which presentation will result in better learning?

Presentation A Presentation B

14) I will choose Presentation A, Presentation B, to teach.

15) Students will retain learning longer with:

Presentation A Presentation B

16) What is your personal reaction to Presentation A?

17) What is your personal reaction to Presentation B?

APPENDIX D

Mathematics in Nature

Fibonacci Numbers

Mathematics Characteristics of Fibonacci Number

The following sequence is called The Fibonacci Sequence (FS). It is named after the Italian mathematician, Leonardo Fibonacci, in 1202.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Each term of this sequence is called a Fibonacci number (FN). The following are some interesting characteristics of this sequence.

Exploration 1:

Label the Fibonacci numbers as follows:

$$\begin{array}{cccccccccccccc} 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & 89, & 144, & \cdots & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} & \cdots & F_n \; \cdots \end{array}$$

Notice that there is a pattern beginning with the third term.

$2 = 1 + 1$	$F_3 = F_1 + F_2$
$3 = 1 + 2$	$F_4 = F_2 + F_3$
$5 = 2 + 3$	$F_5 = F_3 + F_4$
In general	$F_n = F_{n-2} + F_{n-1}$

Exploration 2:

Square a term of (FS) and compare the result with product of (FN) on each side of it.

$2^2 - 1 \cdot 3 = 1$	$(F_3)^2 - F_2 \cdot F_4 = (-1)^2$
$3^3 - 2 \cdot 5 = -1$	$(F_4)^2 - F_3 \cdot F_5 = (-1)^3$
$5^2 - 3 \cdot 8 = 1$	$(F_5)^2 - F_4 \cdot F_6 = (-1)^4$
In general	$(F_n)^2 - F_{(n-1)} \cdot F_{(n+1)} = (-1)^{n-1} \quad n \geq 2$

Exploration 3:

If Pascal's triangle is written as shown below and diagonals are added, what numbers should go in the blanks?

	1	2	3	5	8	13 . . .
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Exploration 4:

If any F_n is divided by the next highest F_{n+1} $(n \geq 2)$, what would you discover?

$\frac{2}{3} =$ _____, $\frac{3}{5} =$ _____, $\frac{21}{34} =$ _____ $\frac{55}{89} =$ _____

In general $\frac{F_n}{F_{n+1}}$ = constant $n \geq 2$

This constant is approximately equal to .16 and called the golden ratio Ø.

Determination of the Golden Ratio

To obtain the golden ratio, consider a segment AB where c divides $\overline{AB}$, so that

$$\frac{AB}{AC} = \frac{AC}{AB - AC}$$

Since we are interested in the ratio AB/AC, let AB be any positive real number, i.e., AB = 1 and AC = x

$$\therefore \quad \frac{1}{x} = \frac{x}{1 - x}$$

$$1 - x = x^2$$

$$x^2 + x - 1 = 0$$

$$\therefore \quad x = \frac{-1 + \sqrt{5}}{2} \approx .618$$

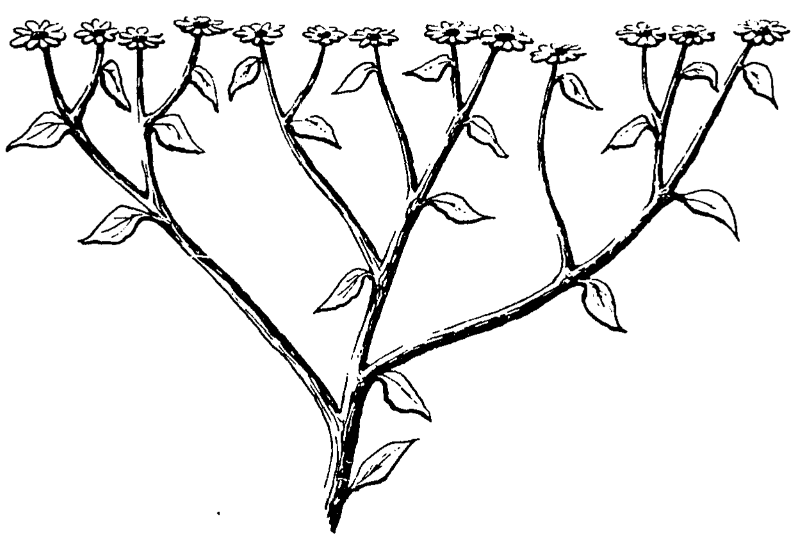
Fibonacci Number in Nature

Exploration 5:

If you measure the dimensions of many common rectangles, such as playing cards, windows, writing pads, and some book covers, you will find that on the average, their proportions were close to the golden ratio.

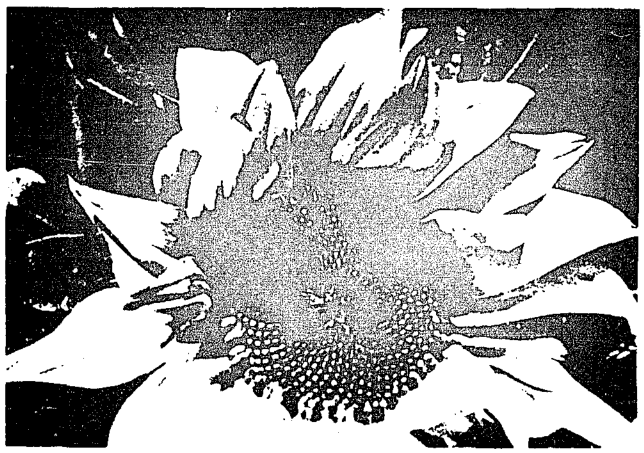
Exploration 6:

Fibonnacci numbers can be found in the arrangement of leaves on the stems of some plants. Can you find the Fibonacci numbers in this drawing of a plant?



Exploration 7:

In some sunflower heads, the number of spirals of seeds, to the left and to the right are formed Fibonacci numbers. Count the spirals (left and right). Do you get Fibonacci numbers? See the drawing below.



APPENDIX E

Results of the Experiment

Run Program ƵDAT1

Listing of Run Program ZDAT 1
Cases 1-75

```
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BBBBB/BBBBB/BBBBB////////

ID # ØØ2
ABBBB/BABBB/BABBB////////

ID # ØØ3
ABBAB/AABBB/BABBA////////

ID # ØØ4
ABBAB/BAABB/BBABB////////

ID # ØØ5
BBBBB/BBBBB/BBBBB////////

ID # ØØ6
AAAAA/AAAAA/AAAAA////////

ID # ØØ7
ABBAB/BABBB/BBBBB////////

ID # ØØ8
AABAA/BAABB/BAAAB////////

ID # ØØ9
AAAAB/BABBB/BBABB////////

ID # Ø1Ø
AAAAB/AABBB/ABABB////////

ID # Ø11
AABAB/AAABB/BBABB////////

ID # Ø12
AABAA/BBABB/BAAAB////////

ID # Ø13
AAAAB/AAABB/BAABB////////

ID # Ø14
AABAB/BAAAB/BAABA////////

ID # Ø15
AABBB/AABBB/BBBBB////////

ID # Ø16
BBABB/AAABB/BBBBB////////

ID # Ø17
BBBBB/BBBBB/BBBBB////////
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ID # Ø18
BBBBB/BAABB/BAABA////////

ID # Ø19
AAAAB/AAABA/AABAB////////

ID # Ø2Ø
ABBBB/BBBBB/BBBBB////////

ID # Ø21
BBABB/AAABB/BAABB////////

ID # Ø22
AAABB/BBBBB/BAABA////////

ID # Ø22
AAABB/BBBBB/BAABA////////

ID # Ø23
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ID # Ø24
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ID # Ø25
AABAB/AAABB/BBBAB////////

ID # Ø26
BBBBB/BABBB/BBBBB////////

ID # Ø27
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ID # Ø29
AAAAA/AAAAA/AAAAA////////

ID # Ø3Ø
ABBAB/ABBBA/BABBB////////

ID # Ø31
ABBBB/ABBBA/BBBBB////////

ID # Ø32
BBBBB/BBBBB/BBBBB////////

ID # Ø33
BABAB/ABABA/BABBB////////

ID # Ø34
ABABB/ABBBB/BABBB////////

ID # Ø35
AAAAA/AAABA/AAAAA////////

ID # Ø18
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ID # Ø19
AAAAB/AAABA/AABAB////////

ID # Ø2Ø
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ID # Ø22
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ID # Ø29
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ID # Ø33
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ID # Ø72
BBABA/BABBB/BBBBB////////

ID # Ø73
BBABB/ABBBB/BBBBB////////

ID # Ø74
ABBBA/BBBBB/BBBBB////////

ID # Ø75
BABAB/ABABA/BABBB////////

Listing of Numbers of Selecting Method
of Presentation B (Program ZDAT1)

```
QUESTION #  1   NUMBER OF B'S:  46
QUESTION #  2   NUMBER OF B'S:  50
QUESTION #  3   NUMBER OF B'S:  58
QUESTION #  4   NUMBER OF B'S:  48
QUESTION #  5   NUMBER OF B'S:  64
QUESTION #  6   NUMBER OF B'S:  48
QUESTION #  7   NUMBER OF B'S:  48
QUESTION #  8   NUMBER OF B'S:  51
QUESTION #  9   NUMBER OF B'S:  70
QUESTION #  10   NUMBER OF B'S:  59
QUESTION #  11   NUMBER OF B'S:  66
QUESTION #  12   NUMBER OF B'S:  49
QUESTION #  13   NUMBER OF B'S:  58
QUESTION #  14   NUMBER OF B'S:  63
QUESTION #  15   NUMBER OF B'S:  65
```

Listing of Proportions of Selecting Method of Presentation B and Computation of Z (Program ZDAT1)

QUESTION # 1	PROPORTION OF B'S: 0.613333	Z =	1.96299
QUESTION # 2	PROPORTION OF B'S: 0.666667	Z =	2.88675
QUESTION # 3	PROPORTION OF B'S: 0.773333	Z =	4.73427
QUESTION # 4	PROPORTION OF B'S: 0.640000	Z =	2.42487
QUESTION # 5	PROPORTION OF B'S: 0.853333	Z =	6.11991
QUESTION # 6	PROPORTION OF B'S: 0.640000	Z =	2.42487
QUESTION # 7	PROPORTION OF B'S: 0.640000	Z =	2.42487
QUESTION # 8	PROPORTION OF B'S: 0.680000	Z =	3.11769
QUESTION # 9	PROPORTION OF B'S: 0.933333	Z =	7.50555
QUESTION # 10	PROPORTION OF B'S: 0.786667	Z =	4.96521
QUESTION # 11	PROPORTION OF B'S: 0.880000	Z =	6.58179
QUESTION # 12	PROPORTION OF B'S: 0.653333	Z =	2.65581
QUESTION # 13	PROPORTION OF B'S: 0.773333	Z =	4.73427
QUESTION # 14	PROPORTION OF B'S: 0.840000	Z =	5.88897
QUESTION # 15	PROPORTION OF B'S: 0.866667	Z =	6.35085

Ø	4	1	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	4	2	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	4	3	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	4	4	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	4	5	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	4	6	B	B	B	B	A	A	B	B	B	B	A	B	B	B	B
4	3	7	B	A	B	A	A	B	B	B	B	B	B	B	B	B	B
Ø	4	8	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	4	9	B	A	A	B	B	A	B	A	A	B	B	A	B	B	B
Ø	5	Ø	A	A	A	A	B	B	B	B	B	A	A	A	A	A	A
Ø	5	1	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	5	2	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	5	3	B	A	B	B	B	B	A	B	B	A	A	B	B	B	B
Ø	5	4	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	5	5	B	B	B	B	B	A	B	B	B	B	B	A	B	A	B
Ø	5	6	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	5	7	B	B	B	B	B	B	B	B	B	B	B	A	B	B	B
Ø	5	8	A	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	5	9	B	A	B	A	B	A	B	A	B	A	B	A	B	B	B
Ø	6	Ø	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
Ø	6	1	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	6	2	B	A	B	A	A	A	A	A	B	A	B	B	B	A	B
Ø	6	3	A	B	B	A	B	B	B	A	B	B	B	B	A	B	B
Ø	6	4	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	6	5	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	6	6	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	6	7	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	6	8	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	6	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
Ø	7	Ø	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	7	1	B	A	B	A	B	A	B	A	B	A	B	A	B	B	B
Ø	7	2	B	B	A	B	A	B	A	B	B	B	B	B	B	B	B
Ø	7	3	B	B	A	B	B	A	B	B	B	B	B	B	B	B	B
Ø	7	4	A	B	B	B	A	B	B	B	B	B	B	B	B	B	B
Ø	7	5	B	A	B	A	B	A	B	A	B	A	B	A	B	B	B

Ø	Ø	1	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	Ø	2	A	B	B	B	B	B	A	B	B	B	B	A	B	B	B
Ø	Ø	3	A	B	B	A	B	A	A	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	B	A
Ø	Ø	4	A	B	B	A	B	B	A	A	B	B	B	B	A	B	B
Ø	Ø	5	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	Ø	6	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
Ø	Ø	7	A	B	B	A	B	B	A	B	B	B	B	B	B	B	B
Ø	Ø	8	A	A	B	A	A	B	A	A	B	B	B	A	A	A	B
Ø	Ø	9	A	A	A	A	B	B	A	B	B	B	B	B	A	B	B
Ø	1	Ø	A	A	A	A	B	A	A	B	B	B	A	B	A	B	B
Ø	1	1	A	A	B	A	B	A	A	A	B	B	B	B	A	B	B
Ø	1	2	A	A	B	A	A	B	B	A	B	B	B	A	A	A	B
Ø	1	3	A	A	A	A	B	A	A	A	B	B	B	A	A	B	B
Ø	1	4	A	A	B	A	B	B	A	A	A	B	B	A	A	B	A
Ø	1	5	A	A	B	B	B	A	A	B	B	B	B	B	B	B	B
Ø	1	6	B	B	A	B	B	A	A	A	B	B	B	B	B	B	B
Ø	1	7	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	1	8	B	B	B	B	B	B	A	A	B	B	B	A	A	B	A
Ø	1	9	A	A	A	A	B	A	A	A	B	A	A	A	B	A	B
Ø	2	Ø	A	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	2	1	B	B	A	B	B	A	A	A	B	B	B	A	A	B	B
Ø	2	2	A	A	A	B	B	B	B	B	B	B	B	A	A	B	A
Ø	2	3	B	B	A	A	B	A	A	A	B	A	B	B	B	B	A
Ø	2	4	A	B	B	B	B	B	A	B	B	B	B	B	B	B	B
Ø	2	5	A	A	B	A	B	A	A	A	B	B	B	B	B	A	B
Ø	2	6	B	B	B	B	B	B	A	B	B	B	B	B	B	B	B
Ø	2	7	B	B	B	B	B	B	A	B	B	B	B	B	B	B	B
Ø	2	8	A	B	B	B	B	B	B	B	B	B	B	A	B	B	B
Ø	2	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
Ø	3	Ø	A	B	B	A	B	A	B	B	B	A	B	A	B	B	B
Ø	3	1	A	B	B	B	B	A	B	B	B	A	B	B	B	B	B
Ø	3	2	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	3	3	B	A	B	A	B	A	B	A	B	A	B	A	B	B	B
Ø	3	4	A	B	A	B	B	A	B	B	B	B	B	A	B	B	B
Ø	3	5	A	A	A	A	A	A	A	A	B	A	A	A	A	A	A
Ø	3	6	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	3	7	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	3	8	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	3	9	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
Ø	4	Ø	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B