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MATHEMATICAL METAPHORS AND PHILOSOPHICAL STRUCTURES

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MATHEMATICAL METAPHORS AND PHILOSOPHICAL STRUCTURES

by

Dan Harvey Wishnietsky

A Dissertation Submitted to
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This dissertation has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Dissertation Adviser [signature]

Committee Members [signature]

[signature]

[signature]

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The purpose of this study was to examine relationships between mathematics and philosophy. The first part of the study examined the history and basic doctrines of idealism, realism, pragmatism, and existentialism. This was a basic overview which would familiarize the reader with the teachings of each philosophical system. Mathematical topics and structure were then used to model and evaluate each of the philosophies. By using mathematical metaphors to evaluate each philosophical structure, the reader could decide which beliefs would have worth to his or her life.

The second part of the study addressed the problem of choice. The belief that humans have few choices and that only one of those choices would bring success was evaluated using the binomial distribution to mathematically model the Greek dialectic. The belief that humans have an infinite number of choices was evaluated using Georg Cantor's mathematical argument that there are infinitely many decimal fractions on the finite line segment between zero and one.

The final section of the study illustrated how Kurt Godel, by mathematical investigation, discovered that no formal system can be both complete and consistent. By applying Godel's discovery, known as Godel's Theorem, to philosophy, religion, or any other school of thought, it was realized that no individual or system has complete truth.

Godel's work verified that every person was free to make their own decisions and determine what was best for their lives.

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Chapter I

Introduction

Purpose of Study

During my numerous years of schooling, one of my major objectives consisted of persuading the teacher that I knew the one correct answer. It did not take long to determine that high grades and honors were obtained by writing tests and papers which restated the same ideas taught by the instructors and the books. With each grade level, my thinking and creativity were increasingly replaced with the memorization and rewording of someone else's thoughts. The book and teacher were considered the infallible source of knowledge and this knowledge I assumed to be truth. Within the school, students were given a finite list of facts to memorize, there seemed to be an answer to every question, and the voice of the teacher was like the voice of God. School became life's basic training camp where my classmates and I were taught to blindly follow those in authority.

I learned to succeed in school and, by high school, my interests and grades enabled me to enter a mathematics and engineering track. These were considered exact sciences where finding the correct answer was always the goal and the answer was always in the book or the lecture. By listening to the teacher and reading the textbook, I was able to learn the correct answer and score well on exams. My high grades qualified me to enter a prestigious university where I continued listening to lectures, reading books, learning the correct answer, and scoring well on exams. My undergraduate classes became graduate classes, but the learning pattern did not change. Graduate classes in mathematics and engineering might have included more discussion than lower level classes, but the correct answers were still found in the readings and the instructor.

An unsettling change occurred in my life when I became a university mathematics instructor. I had always believed that teachers knew all the answers, and here was I, an instructor without complete knowledge. Adding to my dilemma was the observation that my students would not question me or the book, even when we were obviously wrong. In one instance, the answer given in the book for a statistical mean was not even within the range of the data. When I disclosed the book's mistake, the students were in shock. Here was their instructor, whom they believed infallible, telling them that the book, which they also believed infallible, was wrong. Instead of testing the conflicting claims, the students were filled with dread at the prospect of choice.

To try to find answers, I enrolled in graduate level philosophy, sociology, and education courses. The instructors of these courses taught me there was no one answer and the subject of importance was not the correct answer, but the correct question. Discussions filled class time, but it seemed after three hours of talking, nothing was accomplished. At least in mathematics class, my students solved a few problems in three hours. In philosophy, the students could not even agree if something was a problem. I soon became disillusioned with the speculative sciences for they did not solve my original dilemma. My students still believed the book and their instructor infallible, they were still unable to evaluate conflicting claims, and my knowledge was still incomplete.

The answer to my dilemma and the motivation for this work is a synthesis of the exact and the speculative sciences. My belief is that the speculative sciences, which strive for the correct question, and the exact sciences, which strive for the correct answer, do not oppose but complement each another. The objective of this work is to evaluate philosophy, a speculative science, with metaphors from mathematics, an exact science, and determine what knowledge or truth is found. By blending mathematics and philosophy, people can accept the concept of incomplete knowledge and also have a method for evaluating conflicts. The goal is to use mathematical metaphors as a tool for evaluating philosophical conflicts and as an aid in decision making.

Mathematics and Philosophy: An Interrelationship

When humans attempt to understand themselves and the universe, packets of knowledge or disciplines emerge. Two such disciplines are mathematics and philosophy. The traditional definitions of mathematics, "science of quantity" and "science of discrete and continuous magnitude" (Courant, 1941) imply that mathematics is an exact science which accurately measures the universe. Objects have number, form, arrangement, and other associated relations which can be rigorously defined using literal, numerical, and operational mathematical symbols. Mathematics seems to follow a defined course which leads to a specific destination. In contrast, philosophy is thought of as a speculative discipline. Instead of measuring the universe, philosophy explores the essence of individual life. It is still a science, for philosophers are engaged in the scientific activities of observation, identification, description, experimental investigation, and theoretical explanation. There is, however, no marked course. The science of philosophy can lead anywhere in the universe and anyone who desires to understand himself or herself, others, and the universe is a philosopher (Marti-Ibanez, 1964).

Mathematics and philosophy are not mutually exclusive disciplines. Both sciences use contemplation and speculation as an investigative method. Through contemplation and speculation, Einstein discovered his formula, $E=mc^2$, Leibnitz discovered calculus, and Poincare discovered proposed resolutions to logical paradoxes. Truths exist and are permanent, but like sand dollars hidden under the sand, they are unseen, waiting for a discoverer (Marti-Ibanez, 1964). This is the course mathematicians and philosophers share, the search for knowledge and truth. Having this goal, it is not surprising that many of the prominent people in philosophy are also renowned in mathematics. When Plato established his school in 387 B. C., it was called the school of mathematics and philosophy. From his school a mathematical model of knowledge developed which suggested that ethical truths can be deduced from self-evident axioms. Although Plato never employed deductive reasoning for specific ethical problems, his work guided many subsequent philosophers (Putnam, 1971). A primary example is Spinoza, who derived with formal precision the principles of ethics from metaphysical axioms (Ratner, 1954). The relationship between mathematics and philosophy is also suggested in Aristotle's work Topics. Here, Aristotle divded knowledge into the areas of the theoretical, the practical, and the productive. Considered theoretical were the disciplines of philosophy, mathematics, and physics, while ethics and politics were labelled as practical. The divisions soon blurred as Aristotle, in many of his works, interrelated philosophy, mathematics, physics, ethics, politics, and art (McKeon, 1941).

The interrelationship is enhanced even more by Descartes. Descartes believed that in the search for truth "the first precept was never to accept a thing as true until I knew it as such without a single doubt." In Meditations, Descartes outlined an analytical method of inquiry which was intended for use in scientific, philosophical, and all other rational disciplines. He believed in the unity of all philosophical and scientific knowledge. This is symbolized by his image of the Tree of Knowledge, where the roots are metaphysics, the trunk is physics, and the branches are the other sciences (Descartes, 1967). The image acknowledges the belief that all disciplines are interrelated and the way to understand one subject is to understand its relationship with other disciplines.

The Problem of Rationalism

The merging of the exact science of mathematics with the speculative science of philosophy caused a major problem. Instead of being conjectural when reflecting upon the different philosophies, many philosophers sought more positive aims (Robertson, 1957). In the eighteenth century, during the European enlightenment, many attempted to make reason the absolute ruler of human life. Part of this effort was the development of a theory of rationalism where the methods of mathematics were introduced into philosophy. The goal was to find the one superior philosophy which would provide its followers with abundant life. The influence of rationalism has remained, and today, it seems as if all schools of thought claim their way is superior and all others are ordinary. They have instituted a formal system which must be followed if one is to obtain success. People are protected from confusion and difficult choices, for only one way is presented and choice is abolished.

The problem created by rationalism is illustrated by George Berkeley's statement, "truth is the cry of all, but the game of the few" (Jessop, 1952). All people desire to know truth, but instead of testing the claims of people and systems, most individuals blindly believe what people say and the systems they represent. Rationalism, instead of promoting testing and evaluation by each individual, grants a few people the power to determine what is superior for all. When Aristotle wrote, "I have gained this by philosophy: that I do without being commanded what others do only from fear of the law" (McKeon, 1941), he presented the importance of all people having the spirit of philosophy and making their own choices. Aristotle could decide his own choices because he had studied claims, determined what he believed to be true, and knew why he believed it.

Many nineteenth century philosophers who refuted rationalism, such as Sir William Hamilton, William James, and Arthur Schopenhauer, spoke contemptuously of mathematics. Mathematics was blamed for rationalism because some philosophers who saw man as a machine tried to use mathematics to explain the machine (Robertson, 1957). This is like blaming mathematics for the atomic bomb. When Albert Einstein wrote $E=mc^2$, he illustrated in mathematical language a construction of the universe. It is not proper to blame mathematics for the nuclear arms race because some physicists used the knowledge contained in a mathematical formula to construct a nuclear bomb. The same formula used to create the atomic bomb was also used to discover cures for certain types of cancer. By the same logic, mathematics should not be blamed for rationalism. The same mathematics, which was thought to be the cause of rationalism, actually refuted rationalism in 1931 when Godels's Theorem proved all formal systems to be incomplete, inconsistent, or both (Hofstadter, 1979).

The criticism between mathematics and philosophy does not originate from mathematics or philosophy, but from ignorance. Between the spirit of mathematics and the spirit of philosophy there is no discord or strife. They are friendly rivals, perhaps even partners, in their pursuits and goals. William James realized this, for after attacking mathematics, he became aware of his ignorance, wrote of his errors, and confessed his mistakes (James, 1917). He understood that essential and significant relations do transpire between mathematics and philosophy. The works of Plato, Aristotle, Descartes, Spinoza, and others show how interrelated the two disciplines are when developing and refining ideas.

The Use of Mathematical Metaphors

By utilizing the language of both mathematics and philosophy, people can comprehend ideas, symbols, meanings, and relationships. If an idea is defined as something imagined or pictured in the mind, what is needed is a symbol of that idea so it can be communicated to others. This sentence is an example of how symbols are used to communicate ideas. The writer of this sentence discovered ideas and relationships which he wanted to communicate, so he translated the ideas into English and used the printed word to communicate his thoughts. In mathematics, symbols are also used to communicate ideas. An example is "8 + 5 = 13," a mathematical sentence which communicates familiar ideas and relationships learned in elementary school. When mathematical and philosophical symbols are used in isolation, the context of inquiry remains in their respective discipline. By using mathematical metaphors for philosophical structure, the contexts of mathematics and philosophy can be superimposed upon each other. The result is an interaction where the context of one discipline can be used to better understand the context of the other. The disciplines are no longer separate, but in interaction, allowing us to use what we know from one discipline to understand what we do not know in the other (Belth, 1977).

Mathematical metaphors are tools for evaluating and understanding many philosophical structures. Consider the previous symbols of "8", "5", "13", "+", and "=". An argument for idealism would be the mathematical metaphor that each of these symbols stand for ideas which were discovered by humans, but not created by them. The statement "8 + 5 = 13" is a statement expressing a relationship among ideas. The statement was created by humans; the relationship of ideas was discovered. The idealist says an idea in itself is an eternal thing and relationships among ideas are also eternal. If ideas are eternal, they are also unchangeable. Although language may speak of ideas changing, this is figurative speech that, if taken literally, will lead to scientific and philosophic disaster. An old idea may be replaced by a new and similar one, but the original idea is not transformed into the new one. The ideas and their relationships are increate and indestructible (Dampier, 1961).

To refute the idealist, a phenomenologist can use the mathematical metaphor that eight plus five equals fourteen, not thirteen. The addition, however, must be done in base nine instead of base ten. It is the individual who chooses the base for performing the addition which determines whether the answer is thirteen or fourteen. As the person's choice of mathematic's base determines the answer, the choice of people determines the ideas which are created, changed, or destroyed (Dampier, 1961). Phenomenologists believe that ideas are not fixed and eternal, but are developed and changed by the inner lives of each individual. Terms, such as temporal, mutable, capable of growth, decay, or destruction are words phenomenologists use to describe the characteristics of an idea (Kneller, 1984). The idealist can counter that nothing has changed. Thirteen, base ten, is the same idea as fourteen, base nine. Only the symbol has changed, not the idea. The argument could continue with the strength of mathematical metaphors helping to evaluate each point.

In addition to numerics, mathematical metaphors can be expressed in set theory notation. Consider the statement that P has the property q and whatever has the property q has the property q′, then P has the property q′. Statements such as these have long been the basis of logic theory; however, they can also apply to general statements rich in concrete applications. For example, if humans are by nature builders of social structure, and if all builders of social structure inherit the work of past generations and deliver it to future generations, then humans stand in relationship to both the dead and the unborn, uniting past, present, and future in one living, growing reality (Putnam, 1971). The example is of the same form as the set theory statement and is logically correct. What has not been shown true is the initial assumption that all humans are by nature builders of social structure. This remains a hypothesis and illustrates the limitation of mathematical metaphors, for by the use of mathematical metaphors, a person can only test and evaluate beliefs, not prove or disprove them.

When James, Hamilton, and Schopenhauer criticized rationalism, they had legitimate cause. The rationalists were saying that human behavior and philosophical truth could be explained only by reason and the qualitative element was not needed. The error of those criticizing rationalism was that they also criticized mathematics. Mathematicians do not seek to eliminate the intuitive and the qualitative. Humans do have many transcendental insights which can not be explained (Wilber, 1983). The aim of mathematical metaphors is to bring the strength of reason and logical rigor to the intuitive ideas of philosophy, not to prove which philosophy is correct, but to be a tool helping people evaluate which ideas are proper for their lives.

All humans have to deal with ideas on some level. Ideas are part of one's world or they are, in fact, the world. It is the world of ideas which forms the foundation of ethics, philosophy, mathematics, government, religion, education, and any other subject. Ideas are what give human beings a basis for theories and conduct of individual or community life. Every philosophy has humans in the world of ideas. The idealist believes ideas are apart from humanity needing to be discovered; the phenomenologist believes the ideas to be within humans. Choices differ, but it is our choices which make us who we are. Once the choice of theory is made, each of us is bound by the consequences of that theory. It is as if destiny has given a set of consequences, beyond our power to control, which we must follow, unless the choice of a new set of principles is made (Marti-Ibanez, 1964).

Because consequences follow choices, it is crucial that all people be able to recognize and evaluate their many choices. The power of mathematics in philosophy is seen when the consequences of certain choices are written using mathematical symbols. Often, it is easier to evaluate philosophical choices when they are written in mathematical symbols than in words. This does not enslave the intellect, but frees it, for intellectual freedom is the ability to think within the nature of ideas and in accordance with their relationships. The partnership between mathematics and philosophy can flourish because shared understanding promotes widening inquiry (Belth, 1977). When the context of philosophy is overlaid with the context of mathematics, new knowledge, perceptions, and expression become possible.

Choices, which are hidden when mathematics and philosophy are viewed separately, now come into view. The choices may be difficult or confusing, but an abundance of choice can help turn a person from error to truth (Belth, 1977).

Plato said, "the just retribution of him who errs is that he be set right" (Richards, 1966). People who have a genuine interest in both mathematics and philosophy have the benefit of studying subjects which correspond in outlook, temper, attainment, and limitation. This interrelationship will prevent both the philosopher and mathematician from error. Mathematics is characterized more by its method than by its subject matter, causing mathematical considerations to be accepted without enough thought or explanation. The nature of mathematics is quantitative and often the qualitative aspect of the subject is ignored. Mathematicians can adduce too lightly or too freely without considering the subject being studied. Philosophy will restrain an easy acceptance by forcing an explanation when pure mathematical thought requires none (Lodge, 1920). Mathematics can help the philosopher by discovering philosophical limitations. The language of philosophy can blind the philosopher to the limitations of an argument, especially in the areas of logic and reasoning. When the same argument is placed into mathematical terms, weaknesses can be discovered and errors prevented.

Before mathematics can symbolize philosophical thought, some knowledge of mathematics and philosophy must be present. The question can be asked about how much mathematical knowledge is needed for philosophy. First, it must be remembered that a philosopher is a human and the proper equipment for a philosopher includes as much mathematical training as is essential for all men and women. This does not make the question any easier to answer, for the amount of mathematics acquired during, for example, the first collegiate year is very meager compared to the existing body of knowledge. In respect to content, however, the information acquired in the freshman year is far more than Thales, Pythagoras, Plato, or Galileo had. The goal is not to find some magical minimum standard but to grasp the importance of continued learning. A person who understands only the concepts of arithmetic can only form metaphors based on those concepts. As one's knowledge increases to include the concepts of algebra, geometry, or calculus, one can form metaphors based on the newly learned concepts. These metaphors might be no better than the ones based on arithmetic, but they are now available for use.

Increased knowledge is beneficial, but of greater importance for understanding are open-mindedness, logical acumen, philosophical insight, and intellectual maturity (Sprinthall, 1977). Part of intellectual maturity, however, is the insight that important facts and principles from all the basic subjects are needed for intellectual growth. An educational problem of recent years is the lack of mathematical knowledge being imparted to students. Many secondary schools and colleges have reduced the mathematics requirements as to practically abolish the subject from the general education curricula. As society has become more industrial and technical, the most important facts and principles are commonly lost. People have become very specialized in their knowledge and have lost the benefits of a general education. General mathematical knowledge is one of the victims of this technocracy. As early as 1920, Sir Oliver Lodge noted:

> the mathematical ignorance of the average educated person has always been complete and shameless. One ought not, I suppose, to be too much astonished if in a vast, crude, formless, sprawling democracy like ours, a way to educational leadership is sometimes found by men whose innocence, not only of mathematics but of the other great subjects is complete and shameless (Lodge, 1920).

Aristotle placed the problem in its proper prospective when he wrote "educated men are as much superior to uneducated men as the living are to the dead" (McKeon, 1941). What has been lost, which is more important than the loss of facts, is the loss of the sense of relationship among subjects. When using mathematical metaphors for philosophical structure, mathematical and philosophical knowledge are not acquired in the usual sense. Scholars in both areas agree that there is only one way to become a mathematician or a philosopher, and that is through years of study. Using mathematical metaphors enables people to acquire insight into the essential nature of mathematics and philosophy as a distinctive type of thought, and also into the relationships between them. The great concepts and spiritual significance of both these subjects provide the understanding which can connect mathematics and philosophy with the other sciences, arts, and forms of human activity (Dampier, 1961).

Not only are mathematics and philosophy interrelated, but all the great subjects have aspects in common. When a general education is lost, so is the ability to synthesize (Dampier, 1961). A simple example is how mathematics is used in rendering clear the quantitative aspects of the world. When we describe, quantity is often part of the description. When a nation is called large, the question is how large. If an element is scarce, how scarce? Quantity can not be avoided even in the arts of poetry or music. Quantity and number are in the rhythms and octaves, for a subject without quantity is only half developed. The importance of the qualitative is equally great and a subject is complete only when the qualitative and quantitative are present. This conjuncture of the qualitative and the quantitative is what makes the bond between philosophy and mathematics so pleasant.

The Necessity of Evaluation

Individuals have in common instincts, powers, impulses, and traits which are shared with lower forms of life, but what makes humans a higher life form is the infinite variety of activities which are distinctively human. Through the history of human experience, the nature of our common humanity has been characterized by the mental capability for language, speech, and literature (Belth, 1977). Mathematical metephors are a language, just as the words of a philosophy book are a language. A valid mathematical metaphor forms when the same idea which is found in words and sentences is expressed in mathematical symbols. The idea, whether in words or mathematical symbols, can be communicated through speech and literature, allowing humanity, as a unit, the opportunity to understand and test the idea.

People, as they acquired understanding and knowledge, developed the areas of science, mathematics, philosophy, and religion. These disciplines embody the human search. Science holds a sense for the future, for prediction, and for natural law; mathematics provides the structure for logic and rigorous thinking; and philosophy holds a sense for wisdom, world harmony, and cosmic understanding. The religious faculty explores the mystery of divinity and, therefore, affects all areas of humanity (Dampier, 1961). These areas are part of what makes humans a higher class of beings than other animals. The activities are distinctively human and all humans, whatever their status, are as humans forced to participate. Each activity is interrelated, yet distinct, with a form uniquely its own. Within each activity and person is a desire for knowledge and truth. The person's philosophy is inconsequential, for the desire to know the truth is a calling that can overpower any philosophy, authority, or force.

Truth is not found by memorizing facts or rules written in some book or expounded by some person. The finding of truth requires one to search for it as a miner searches for gold (Marti-Ibanez, 1964). The human activity which embodies the search for truth is the process of thinking. Thinking is one of the great types of distinctively human acts, perhaps the most human, for it is this act which allows people to handle ideas and form concepts. By combining concepts, higher and more complex concepts are formed and relations among them can be discerned. Relationships are used to form judgments and soon various doctrines regarding life and the world emerge. Thinking is essential for understanding human life and what it entails, and since all men and women are citizens of the world of ideas, it is imperative that all people reason for themselves (Belth, 1977).

A process of thinking is the formation of metaphors. Metaphors are simply a transfer, the treating of one event as if it were another. The transfer makes the event more familiar, simpler, or available. The purpose of using mathematical metaphors for philosophical structure is to make the philosophy more familiar and easier to understand in order to form and test philosophical doctrines (Belth, 1977). The special type of thought mathematics brings to philosophy is rigor, or as mathematicians call it, logical rigor. The qualities present within are clarity, precision, and coherence. Mathematical metephors are demanding, calling for perfect clarity of expression, perfect precision of ideas, and perfect allegiance to the laws of thought. Most of what constitutes human thought, however, is not rigorous, but nebulous, vague, and indeterminate. Even mathematics cannot be handled with the rigorous demands of logic. The ideal of logical rigor in thinking remains important, not only in mathematical thought, but in all thinking, even where precision is the least attainable. Without rigor, an important standard for critical thinking, self-criticism is lost and when their is no self-criticism, any thought can be called truth (Weyl, 1949).

Plato clearly stated the significance of evaluating one's beliefs when he said, "the life which is unexamined is not worth living" (Richards, 1966). Most people can be persuaded by every new doctrine which is presented. They are not sure what they believe or, more importantly, the why of their belief. Since everyone lives by some philosophy, it is crucial that people have the tools to evaluate their philosophy and understand their beliefs. This enables individuals, as they expand their knowledge, to know when and when not to change. An application of philosophical understanding producing different actions occurs in education. Many teachers have never examined their own philosophy or their philosophy of teaching. Without a basic structure of thought from which to make decisions, they try every new educational strategy. These fads last about one school year or until the book publishers present another scheme. A teacher with an examined life and a knowledge of his or her philosophy will not bend with every new tactic, but will be able to evaluate each new situation and decide if change is appropriate (Sprinthall, 1977).

Mathematics is an important tool for evaluating one's own philosophy, for philosophers in every important era have portrayed a noble tradition of mathematical competence. By experiencing the relationship of mathematics and philosophy, these philosophers were able to bring insight upon the universal interests of the human spirit. Plato knew the mathematics of his time and expressed its spiritual significance. Aristotle followed with great contributions to both philosophy and mathematics. His works include the nature of mathematical definition, hypothesis, axiom, postulate, and logic. Descartes, called the father of modern philosophy with his method of radical doubt, was also the chief inventor of analytical geometry. Gottfried Leibniz, the co-founder of the most powerful instrument of thought yet devised by man, infinitesimal calculus, also developed modern symbolic knowledge and the dawning consciousness philosophy. Spinoza tried to clothe ethical theory, perhaps the highest of human interests, with the strength of mathematical rigor. These people, who were both mathematic and philosophic personalities, illustrate that anyone who endeavors to think both mathematically and philosophically is in illustrious company (Bell, 1937).

This work reveals how people can use mathematical thinking to acquire insight and wisdom not gained in any other way. Of greater importance, this work will stress why each individual must establish his or her own beliefs, not depending on the decisions of others. Plato said:

> Until philosophers are kings, or the kings and princes of this world have the spirit and power of philosophy, and political greatness and wisdom meet in one, and those commoner natures who pursue either to the exclusion of the other are compelled to stand aside, cities will never rest from their evils - no, nor the human race, as I believe - and then only will this our state have a possiblity of life and behold the light of day (Richards, 1966).

Plato lived in an elitist society and never believed that all people could or should be philosophers and kings. Times have changed since ancient Greece. In The Paideia Proposal, Mortimer Adler brought Plato's warning to our society when he wrote:

> Democracy has come into its own for the first time in this century. Not until this century have we undertaken to give twelve years of schooling to all our children. Not until this century have we conferred the high office of enfranchised citizenship on all our people, regardless of sex, race or ethnic origin (Adler, 1982).

In a democracy, all the people are involved in the political process and have the role of king. Plato's warning affects all citizens, for without the spirit and power of philosophy in each individual, western civilization will not ascend to its full potential.

Chapter II

Mathematical Metaphors and Idealism

Historical Perspective

Called the philosophy of Plato, Idealism stresses that the fundamental values in the world are mind and spirit. Reality is basically mental, through and as ideas, and abstractions are more fundamental to reality than what is experienced by the senses. The two basic forms of Idealism, metaphysical Idealism and epistemic Idealism, were asserted by Plato. The former teaches the ideality of reality and is illustrated by Plato's statement, "The absolute natures or kinds are known severally by the absolute idea of knowledge" (Jowett, 1937). The latter holds that in the knowledge process objects are conditioned by their mental perceptibility. Plato illustrated this belief with the following analogy:

> Let us now suppose that in the mind of each man there is an aviary of all sorts of birds - some flocking together apart from the rest, others in small groups, others solitary, flying anywhere and everywhere.... We may suppose that the birds are kinds of knowledge, and that when we were children, this receptacle was empty; whenever a man has gotten and detained in the enclosure a kind of knowledge, he may be said to have learned

> or discovered the thing which is the subject of the knowledge: and this is to know (Jowett, 1937).

The word "idealism" comes from the Greek word "idea" which means something seen or the look of something. Plato used the word in his philosophy to mean a universal, such as bigness, in contrast to a particular, such as something big. Plato also used "idea" to mean an ideal standard such as absolute beauty as opposed to individual comparisons of more or less beauty. This is illustrated in Philebus's discussion of beauty where beauty is described as an intrinsic property of objects and these objects, by their very nature, are always beautiful (Jowett, 1937). The objects, because of their intrinsic beauty, arouse within the beholder a pleasure which is unique. Two examples of intrinsic beauty are purity and symmetry. Purity guarantees the stability of beauty by eliminating dissimilar ingredients, for when an object is contaminated, so is its beauty. Symmetry gives beauty to an object by supplying it with form and structure. Plato's examples of things with intrinsic beauty included symmetrical objects produced with a carpenter's rule and square. Plato believed an Idea, or Form, when caught by the intellect, is not bound by time, but has always existed and always will exist. Thus, an Idea such as beauty is eternal, intrinsic, and more real than temporal objects.

The word "idea" has been defined in various ways. In medieval philosophy, ideas and forms were thought to have existed in the mind of God and created by Him. People received truth through the church as God gave direct revelation. By the early part of the seventeenth century, "idea" came to mean the thoughts within the minds of men. Divine revelation was often questioned and many philosophers encouraged people to think and decide for themselves. Rene Descartes wrote that "it was not enough to have a good mind. The main thing is to use it well" (Descartes, 1967). Descartes also used "idea" for the effects external objects acting on the sense organs had on the mind. He believed that external objects act like a stamp, pressing a shape or idea upon the soft material of the brain. This inspired John Locke's essay about human understanding (Locke, 1961) where he used "idea" to mean qualities conveyed into the mind by the senses which enable the mind to reflect about its own operation. Writing that "no man's knowledge here can go beyond his experience," Locke believed the mind could not go beyond those ideas which sense or reflection have offered. George Berkeley repeated Locke's view when he wrote that by our senses "we have the knowledge only of our sensations, ideas, or those things that are immediately perceived by sense" (Jessop, 1952).

It was not until the second year of the eighteenth century that "idealism" was used as a philosophical term. In a response to people like Epicurus and Thomas Hobbes, who believed that the soul is material, Gottfried Leibniz called philosophers like Plato and himself, who uphold an antimaterialistic view, idealists (Ortega y Gasset, 1971). The term became popular with philosophers who were critical of the antimaterialist metaphysic and soon idealism became synonymous for people whose thesis was that there was no such thing as material substance. Immaterialism became prominent in idealist theory because that was thought to be the most effective way of disproving materialism.

The main arguments against materialism are the metaphysical arguments of Leibniz and the epistemological arguments of Berkeley. Leibniz believed in an idealist system which had a series of realms of being, with God, the supreme uncreated spirit, in the highest realm. All members of the created realm were active and immaterial and the substances with self-consciousness were the creations made in God's image. Substances that were perceiving beings, whether or not conscious or self conscious, Leibniz called monads. Monads were identified with the metaphysical individuals or souls which were conceived as active, indivisible, and indestructible substances related in a system of pre-established harmony. Matter, as opposed to monads, could not be independently real because matter must be informed by the spiritual soul of monads (Ortega y Gasset, 1971). Georg Hegel echoed Leibniz's argument when he wrote that "the main principle of philosophy is the ideality of the finite and that every genuine philosophy is on that account idealism" (Friedrich, 1953). What is finite, such as matter, is not real but formed in the mind, and the true philosophy, idealism, recognizes this.

The best known epistemological argument for idealism was expounded by Berkeley. He said that what we immediately perceive are sensations or ideas and these ideas are objects of perception. What we call physical objects, such as dirt, wood, or desks, are actually orderly ideas and are mind dependent like the ideas which compose them. If sense experience is basic and reliable, then matter is rejected on the basis that the senses inform us of ideas but not of the material substances to which the ideas belong. To separate ideas from the notion of a material substance is, according to Berkeley, inconceivable. Berkeley continued by saying the idealist view is compatible with common sense, for common sense tells us that physical things are immediately perceived and have their perceived characteristics. The materialists, however, believe that what is immediately perceived are the ideas produced in the mind by physical objects and of these objects we can only have indirect knowledge. An indirectly perceived object does not have certain characteristics, such as color and hardness, which common sense says it has. This led Berkeley to conclude that material substances, even if they were conceivable, would be problematic existents. Their perception would defy common sense, thus a belief in materialism would cause skepticism about the existance of the familiar. In contrast, immaterialism, with its belief that physical things are ideas and immediately perceived, does not evoke skepticism (Jessop, 1952).

Basic Doctrines

Although there are several types of idealism which can be classified by culture and branch of philosophy, what distinguishes Idealism from other philosophies can be understood through its basic doctrines, questions, and arguments. First and foremost to idealism is the centrality of mind in knowledge and being. While other philosophies identify mind with matter and reduce the higher level of reality to the mathematical physics of atomic particles, Idealism defends the principle that matter can be explained by mind but mind can not be explained by matter. Arthur

Schopenhauer, in his work The World as Will and Idea, begins with the supposition that, "the world is my idea" (Schopenhauer, 1955). The work is a refutation of materialism where the inseparableness of subject and object is shown to be a primary fact of consciousness.

The prominence of mind expanded to the concept of the Absolute Mind or God, the perfect, uncreated spirit who has created everything else and is thus more fundamental than any of the matter he created. The idealist's concept of God is accurately portrayed by Pantheism, where nothing exists except God, so the material objects must be a part of God. What is true, according to Idealists, is the concrete universal or system created by the Absolute Mind (Boas, 1969). The purpose of the collective human spirit of intellectual inquiry is to discover the concrete systems of God's creation which are present in nature. The different systems, defined as the disciplines and sciences, were discovered over the long period of time called history and even before recorded history, it is well established that our human ancestors created languages, religions, and other institutions (Dampier, 1961).

While other philosophies focus on contemporary matters, Idealists seek the wide spread of epochs and eras, viewing the contemporary world in the aspect of eternity. Idealists claim their philosophy transcends time and cultural boundaries, for Idealisms have been discovered in all major cultures. These Ideals are the universal truths or concrete realities, such as mankind or literature. Because of limited knowledge, contradictions can develop as mankind searches for truth. Idealists overcome contradiction by discovering new knowledge about the overall coherent system of truth. New knowledge is synthesized with earlier discoveries, forming a higher degree of truth than that present in the earlier knowledge. The discovery and developing of these truths as an inherent part of the collective intelligence is the spiritual force Idealists call the spirit of philosophy (Boas, 1969).

An example of idealists eliminating contradiction is found in interpretion. This involves clearing the mind of prejudice, for it is the existance of prejudices which prevent people from understanding the ultimate clarities. Descartes believed that knowledge exists in the intuition as clear and distinct natures, and after prejudices are removed, one can see the world as it really is. The mind is pictured as a mirror which can only reflect what is there after it has been cleaned. Descartes believed that inquiry ends only in revelation and people must wipe their mirror clean in order to receive undistorted visions (Descartes, 1967).

Kant's method for eliminating contridiction differed from Descartes and is called the dialectic method of Idealism (Kant, 1949). Kant believed that the mind approaches the world with its own concepts and presuppositions, and instead of reflecting the world, the mind tries to understand and interpret it. Contradiction is overcome by penetrating into the overall coherent system of truth and discovering new truth. New truth is integrated with earlier discoveries, leading to a synthetic judgment without the contradiction. The key point of Kant, however, is the same as Descartes. There is an ultimate truth beyond the common sense and the ordinary sense experience and this truth must be discovered. Truth involves the existance of some ultimate spiritual reality for without an ultimate spiritual reality, it is impossible to eliminate contradiction. This makes it essential for seekers of knowledge to understand that there is ultimate truth and that mind is central in knowledge and being.

Idealists believe mathematics to be a well ordered part of ultimate truth discovered by the collective human spirit of intellectual inquiry (Descartes, 1967). Basic to mathematics is the idea of number. When a person sees the word "eight" or the number "8", a mental image of one's idea of eight is formed. The image might be eight dollars, a number line with "8" situated midway between "7" and "9", or this many dots "::::". Numbers can be made in various ways and there is no one definition. What becomes important are the relationships between numbers and the internal structure of the number system. People might conceive "four" and "two" in a multitude of ways, but the addition of "four" and "two" must equal "six" according to the internal structure of the number system. Symbolically the relationship can be written "four plus two equals six", "4 + 2 = 6" or ":: + : = :::", but in each case the symbols which represent the same numbers when added together equal the same answer.

Many agree with the Idealist's belief that the number system was not human developed, but discovered, and is part of a universal order. This was illustrated when, in 1977, the United States launched Voyager 1 and 2, probes designed to provide information about the earth to beings outside our solar system. On its information plate are the symbols ".", "..", and "...". The symbols represent counting the first three positive integers and are intended to demonstrate human intelligence. If the structure of the number system was designed by humans instead of a discovered universal, no alien intelligence would recognize the symbolism. Placing the dots on the probe implies that the United States believes the number system to be part of the Idealist's universal order (Morrison, 1979).

The qualities of intrinsic beauty found in mathematics inspire Idealists to believe mathematics is part of the universal order. Many mathematical operations display the form and symmetry of the number system. When the distance from zero to six is measured, the distance is two times the distance measured from zero to three. This is because six is two times three. Symmetry is also seen when the distance from zero to a positive number is measured. If the same distance is measured on the opposite side of zero, the same number is encountered, except the sign is negative. The perfect symmetry of the number line allows the mathematical operations of addition, subtraction, multiplication, and division to be illustrated using distances from zero. The number line can be extended into more than one dimension and still keep its symmetry. When a line perpendicular to the original number line is constructed through the "zero" of the number line, a two dimensional space is formed. The "zero" is now on the horizontal and vertical number line. If the same scale of the horizontal line is used for the vertical line, the symmetry becomes two dimensional. By using a similar construction, the space can be extended into three dimensions. Most mathematics and geometry books do not advance beyond three dimensional space, but mentally, a symmetrical space of more than three dimensions can be conceived.

Mathematical Metaphors

The language of Idealism is found within the structure of the number system. In the number system it is not possible to obtain the square root of a negative number. The square root of positive four is positive two or negative two because when positive or negative two is multiplied by itself, the answer is four. No number, when multiplied by itself will render a negative number, for a positive number multiplied by a positive number will be positive and a negative number multiplied by a negative number will also be positive. The mind, however, can conceive of taking the square root of a negative number. This idea is symbolized by a class of numbers called imaginary numbers. A new, undefined, symbol "i" was developed such that when "i" is multiplied by itself, the product is -1 (Shenk, 1977). The square root of negative numbers can now be symbolized by numbers with "i" as part of its form. The square root of -1 is "i" because "i" times "i" equals -1. The square root of -49 is 7i because 7i times 7i equals -49. The numbers are imaginary but their symbols are as valid as the symbols for the system of real numbers.

Euclidean geometry is another mathematical system which uses the language of Idealism for it consists of a set of objects with assumed relations and properties. The set of objects is called space, a mathematical idealization or abstraction of the three dimensional world. Objects in space are called points and are idealizations of positions in space. Two other idealizations in geometry are lines and planes. A line consists of an infinite number of points, straight and extending infinitely far in both directions, and a plane models a flat surface of infinite extent in all directions. Points, lines, and planes only exist as mental concepts and certain relations about space, points, lines, and planes are accepted in geometry without proof. These are referred to as postulates and are the pure a priori intuitions of space. Once the postulates are accepted and organized, provable statements about space, called theorems, can be developed (DeLacy, 1963).

It is the transcendental idealism of Immanuel Kant which is modeled by geometry. Kant believed that knowledge of the world can not be gained by using rational thought or sense experience alone (Kant, 1949). Unless perceptions were organized into pure a priori intuitions of space and time, knowledge of the objective world would be impossible. The perceptions of Kant pattern the idealizations and postulates of geometry. Within the a priori framework, it is possible, by using mental perception, to refer to things in causal relation with one another. Without the a priori intuitions and categories of understanding, the senses could give no knowledge of the world. As theorems can not exist without a priori postulates, knowledge of the world can not exist without pure a priori insight.

An idealism of the undefined mathematical concept, division by zero, is the basis for infinitesimal calculus, believed by many to be the most powerful instrument of thought devised by man. Students learn early in arithmetic that expressions such as 7/0 are undefined because the fraction line means division and division by zero is impossible. Not being able to divide by zero does makes intuitive sense because it is impossible to divide a certain number of objects, such as seven, into zero parts. The concept expands into algebra when unknowns are introduced. The expression 7/x is valid only if "x" is not equal to zero. In the expression $(X^2-9)/(X-3)$, "X" can not equal three since that would make the denominator equal zero.

The idealism of division by zero is formed by thinking of the denominator becoming close to zero without actually equaling zero. In the previous example, $(X^2-9)/(X-3)$, even though "X" can not equal three, the fraction does approach a number as the value of "X" becomes closer and closer to three. One way to test the idea is to replace "X" with numbers which approach three and evaluate the fraction. The following table shows the results:

Value of X	Value of fraction
3.1	6.1
3.01	6.01
3.00	undefined
2.99	5.99
2.9	5.9

The table shows that even though the fraction is undefined at three, the value of the fraction seems to approach six as the value of "X" approaches three. This can be verified by elementary algebra. The numerator factors into $(X+3)(X-3)$ making the fraction $(X+3)(X-3)/(X-3)$. The $(X-3)$ in the numerator cancels the $(X-3)$ in the denominator and the fraction reduces to $(X+3)$. The number "three" can now be sustituted for "X" giving the answer "six".

The concept of evaluating an expression as the unknown approaches a number without actually becoming that number is called limit theory and is the basis for calculus (Shenk, 1977). This branch of mathematics enables the calculation of variations. Before limit theory and calculus, only constant values could be calculated. An example is that the velocity of an object could be determined by the formula "velocity = distance traveled / time elapsed" only if the velocity remained constant. Calculus eliminated the restriction of constant velocity by being able to calculate the velocity of an object even if the velocity is changing. By using limit theory, the denominator of the fraction, time elapsed, can approach zero and the velocity calculated. Even when the velocity varies over the elapsed time, calculus is able to calculate the velocity at each specific instant. Similar calculations can be made for the slope of curves at a particular point, areas which need to be maximized or minimized, and in other situations when quantities vary.

Zeno's paradox of the arrow is an example of limit theory providing understanding (Chappell, 1962). Zeno wrote that the arrow occupies a given position, being at a place just equal to its own dimensions. The arrow can not move in the place in which it is not, but neither can it move in the place in which it is for this is a place equal to itself. Everything is always at rest when it is at the place equal to itself, and since the flying arrow is always at the place in which it is, it is always at rest. The paradox is that a flying arrow is not at rest, but Zeno's logic says it is. The error of the reasoning is in the concept of time. Zeno's logic is valid only for a time instant of zero duration. A flying arrow, when photographed with a fast film and shutter speed, appears at rest, for the camera has placed the arrow in a time span of zero. If the time duration is greater than zero, the arrow is in motion and has a velocity. The velocity of a flying arrow at a time span of zero can not be calculated using the formula "velocity = distance traveled/ time elapsed" because when the time elapsed equals zero, the fraction is not defined. Limit theory can obtain the instantaneous velocity by calculating the value the formula approaches when the time elapsed approaches zero.

Mathematics is still being discovered as truth is sought in every intuitive thought. Descartes analyzed the procedure used in part two of his <u>Discource on Method</u> (Descartes, 1967). Descartes discovered that mathematics began with simple and clear ideas that the mind could understand and know with absolute certainty. Knowledge then advanced, one step at a time, toward more refined truth, making sure each step of the argument could not be disputed. Descartes believed that the mind understood initial truth through intuition and all subsequent truth through deduction. Intuition, for Descartes, was a divine vision of such clarity that the receiver had no doubt of its truth. Deduction consisted of clear and certain conclusions which proceeded from what was obvious and simple to what was complex and remote (Engel, 1981). Descartes believed his procedure was valid for discovering knowledge in any discipline and described it as follows:

> The first was never to accept anything for true which I did not clearly know to be such...to comprise nothing more in my judgment than was presented to my mind so clearly and distinctly as to exclude all ground of doubt. The second, to divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution. The third, to conduct my thoughts in such order that by commencing with objects the simplest and easiest to know, I might ascend by little and little, and, as it were, step by step, to the knowledge of the more complex... And the last, in every case to make enumerations so complete, and reviews so general, that I might be assured that nothing was omitted (Descartes, 1967).

Chapter III

Mathematical Metaphors and Realism

Historical Perspective

The transition from Idealism to Realism began with Aristotle, Plato's most famous student. Believing that "education is the best provision for old age" (Randall, 1960), Aristotle studied the Platonic doctrine of ideas for almost twenty years and then ammended it with the doctrine of forms. According to Plato, ideas are a timeless essence, independent of the physical world in which they take place, and physical objects are only the imperfect manifestations of these ideas. Aristotle opposed this doctrine, believing that every object in the sense world consisted of the interconnected concepts of matter and form. The form of an object consisted of the succession of its material embodiments which gave it intelligible structure (McKeon, 1941). Using mathematical terminology, Aristotle is saying the object is the sum of its parts and the sum of its parts has the potential for being the object. Matter is the material embodiments of an object which always have the potential for being formed into the object. This potential Aristotle called the purpose of nature:

> If purpose, then, is inherent in art, so is it in Nature also. The best illustration is the case of a man being his own physician, for Nature is like that - agent and patient at once (McKeon, 1941).

Aristotle believed that the material embodiments of an object exist independently of the mind and can be made the focus of scientific study. Since an object's form does not depend on Mind, it is possible to obtain faithful and direct knowledge of the real world. Aristotle's descriptions of ways to obtain direct knowledge evolved into the scientific method of concept formation, experiments, observations, and validation of the hypothesis (Randall, 1960). It is within the scientific method that the transition from idealism to realism is most apparent. The first step in the scientific method, concept formation, is someone stating what he or she believes to be truth. It is a purely mental process which, according to realists, must be tested by observation. Experiments are designed to test the concept, and the results of the experiments are observed. If the observations confirm the hypothesis, the assumption is accepted. Concepts which are not confirmed by observation are rejected. The mind forms the concepts, but in realism, the senses control what is truth.

This transition can be illustrated mathematically by using the concept of vectors. A vector is defined as a quantity completely specified by a magnitude and a direction (Shenk, 1977). When a person says one town is fifty miles northeast of another town, the direction and the distance can be symbolized as a vector. The magnitude or length of the vector represents the distance between the two towns, fifty miles, and vector's orientation on a coordinate plane signifies its direction, northeast. This is an example of a displacement vector. Other vectors which are often used are position vectors, force vectors, and velocity vectors. Position vectors give the position of an object relative to some origin. On a two dimensional coordinate plane, the origin is symbolized (0,0), and the position of any object can be written in the form (x,y) where x is the distance from the Y-axis and y is the distance from the X-axis. Force vectors symbolize the force applied to an object and the direction from which the force is applied. Velocity vectors give an objects speed and direction of motion.

Mathematicians developed an arithmetic, algebra, and calculus of vectors that, presumably, would model the effect of force and velocity on objects in space. The concepts of vector analysis were tested by experiment and observation. The concepts were validated and current calculus textbooks present the vector analysis which models the observed behavior of objects. The concepts were first an ideal and then verified by realism. There is also vector analysis which involves four or more dimensional space. This analysis is still only an ideal since a space of more than three dimensions has not been observed.

Like the definitions of most words, the meaning of "realism" evolved over the centuries. In medieval thought realism was the doctrine that universals have a real, objective existance. Modern philosophy uses the term for the point of view that material objects exist independently of mental process (Butler, 1968). Realists reject the claim that material objects do not exist independently of the mind and have strived to show that knowledge of physical objects is obtained directly or through sensation. Realism thus rejects the idealist view of material objects or external realities existing only within the mind. G. E. Moore, in his paper, "Refutation of Idealism," (Moore, 1922) rejected the view that things which are unperceived cannot exist. He said that the idealist who agrees with the thesis, to be is to be perceived, has not differentiated between the act and the object in sensation. The sensation of heat is not the same as the hot object. Heat from a sun lamp gives the same sensation as heat from an oven. The act is the same, but the objects are different. When the object is separated from the awareness of it, there is no reason to deny the existence of an unperceived object.

Idealism was the prominent Western philosophy at the end of the nineteenth century, but the twentieth century heralded an upsurge of realism in the United States and Britain. Today, few English speaking philosophers espouse idealism as the current thought climate honors common sense and science (Butler, 1968). To many, realism seems so obvious that idealism as a philosophy does not seem plausible. What is often forgotten is that each generation possesses its own idioms, issues, and logical presuppositions. The current thought climate has been so ingrained into people that a different thought structure is almost considered heresy. Not only realists, but almost everyone else agrees that material objects are independent of one's perception.

Basic Doctrines

Among realists, accounts of perception vary and cause serious divisions. A major division of realism is direct realism (Snow, 1978), the view that perception is a direct confrontation with an external object. The simplest form of direct realism is naive realism, often referred to as the innocent prejudice of the simple person which has to be overcome if progress is to be made. Naive realism believes all the qualities felt by the senses are correct and these qualities are the intrinsic properties of material objects. By sight people observe various colored, shaped expanses that are thought to be the surfaces of material objects. Sounds that are heard are believed to emanate from such objects and the sense of touch gives the knowledge of the object's smoothness and hardness. The claim of direct realism can be shown as false by comparing two observations. When person "A" sees a table from above, the table is round. Person "B" observes the table from a distance and sees an ellipse. The shape of the table is an intrinsic property; therefore, the table cannot be both round and elliptical. Other examples include the color-blind person who sees a black shape instead of a red book and the drunk who sees the snake-like shapes that are not real.

New realism and the selective theory tries to eliminate the contradictions of naive realism caused by conflicting data (Snow, 1978). New realists believe all the appearances of an object are its intrinsic properties and are directly comprehended by the person. A table which looks round to one person and elliptical to another person is both round and elliptical. A mountain which looks green when near and blue at a distance is both green and blue. These are not private observations, for they can be photographed and observed by others. Objects have many sets of properties and it is the function of a person's senses to reveal one property from each set of properties.

It is not a contradiction to say the mountain is both blue and green when near it is green and at a distance it is blue. A problem occurs when there is a conflict in the sensory data, for if people were always aware of the actual characteristics of an object, there would be no talk of errors or misconceptions. Objects would also have to be very complex if they comprised all the qualities which correspond to human observations. Another problem concerns the strange qualities objects possess when the observer has taken drugs. It is still not clear why the nervous system responds to or selects one of the many characteristics an object can possess when certain drugs are in a person's system. This is particularly true when the different appearances are the results of differences in the participant and not in the pattern of light waves (Hart, 1983).

Another area of direct realism which tries to reconcile the objection of objects having contradictory qualities is perspective realism. This theory stresses that shapes, colors, and other qualities are not intrinsic but relative qualities (Snow, 1978). The table is round when viewed from this position and elliptical from another position. The mountain is green in one type of light and blue in another type of light. Since shape and color are not intrinsic, but relative properties, no contradiction occurs. Sensible qualities become contingent on the perceiver's point of view. The perspective may be temporal, spatial, or illuminative, with each viewpoint perceiving the object in a different way. The intrinsic properties of the object do not change, for the object still has an intrinsic shape, color, and other qualities at its own location. Physical objects simply appear different from different positions.

The theory still has the weakness of not being able to separate the perceived from the intrinsic. To solve the problem, the sense-datum theory of direct realism assumes that if an object is seen directly, it is seen as it really is (Snow, 1978). When a round table is seen as an ellipse, it is not seen directly. What is seen is the eliptical datum belonging to it. Where the perspective realist treats all perceptions of an object as equally valid, the sense-datum realist says it is reasonable to treat some appearances as more valid. The more valid appearances are the ones which perceive the object as it actually is. Finding the valid appearances is aided by the fact that objects do seem to have real measured shapes and volumes not relative to a viewpoint. The distance from the center of the table top to the outside edge can be measured. If the distance is always the same, it would be more valid to accept the observation which sees the table as round. If the distance from the center to the edge varied, the eliptical observation would be accepted as correct.

Many realists, because of their study of causal and psychological processes in perception, reject direct realism. They believe it is important to distinguish between external public objects and the brain activity produced by the action of the objects on the sense organs (Hart, 1983). This general view originated with the representative realism of Descartes and Locke, and it is still maintained in principle by many scientists. What is called seeing a table is actually light rays reflected from the table striking the eye. This causes chemical changes in the retina, sending impulses along the optic nerve. The brain then interprets the signals and perceives the shape, color, and other visual properties of the table. The other senses can be given a similar account. Perceiving has become the direct awareness of sensa and perceiving external objects is redefined as perceiving the sensa caused by the objects. Because of the part played by both the object and the sensa, representative realism is part of dualist realism and not direct realism (Snow, 1978).

The difference between the idealist's and realist's concept of mental perception is that the realist believes it is illogical to infer that nothing exists outside of mind. Because one cannot discover X does not mean X does not exist or that it is unreasonable to believe X exists. The object "X" could range from subatomic particles to life on other planets. The idealist's problem of no existance apart from the mind escalates when "X" is another person. The difficulty is the implication that self is the only thing which can be known or verified and that self is the only reality. A person can never know anything which is not part of his or her private experience. This, however, denies the ordinary belief that people are aware of other people and external public objects (Butler, 1968). Ludwig Wittgenstein has argued the realist position from a linguistic perspective (Bartley, 1973). If people had only private experiences, it would be impossible to speak about them. Language implies rules which can be communicated and then checked with respect to public objects. Communication reveals that people view different objects differently and a degree of distortion is introduced by a person's mind when trying to perceive external public objects. No one can know the total truth about external objects. One can, however, try to discover the degree of distortion and eliminate it by comparing results obtained by different methods of knowing.

Mathematical Metaphors

For the realist, mathematics is a logical, symbolic discipline created by humans for communicating knowledge obtained directly or through sensation. Questions like "how many?" and "how long?" needed quantitative answers and the number system was developed to provide the information. When an observer sees a quantity of objects, the number system allow the communication of the exact number of objects present. The numerical operations are also based on sense data. Groups of objects are placed with other groups of objects and the total amount of objects present is obtained by adding the number of objects in one group with the objects in the other group. Subtraction is obtained by removing objects from a group and counting the number of objects remaining. If groups of the same size are observed, the number of objects can be obtained by multiplying the number of objects in one of the groups by the total number of groups. Division is obtained by separating one group into several equal parts. The number of objects in each part is calculated by dividing the number of objects in the original group by the number of equal parts. All mathematical operations, according to realists, are not mental abstactions, but based on actual sense data caused by external objects (Butler, 1968).

The realist's view is pictured in the science of measurement. In measurement, numerical value is ascribed to an object based on the number of times some given quantity is contained in the object. If a table top is rectangular, an observer can communicate the length and width of the table by using a standard measure of distance. The measure can be in inches, feet, meters, or any other distance known by the observer and the audience. If three feet can be contained in the length with no distance left over, the length of the table is three feet. Any distance above a whole foot unit can be expressed in a fraction of a foot or in inches. The width of the table can be measured in the same way. A new measurement, area, can be formed by multiplying the length of a rectangle by its width. A square unit of area measurement has now been defined and can be used to descibe the area of all shapes. If an object is three dimensional, measurements to determine volume can be made. Many tools have been developed to measure distance and many formulas have been developed to calculate area, volume, and the length of unknown sides. Verifying the calculations is based on direct observation, not mental abstractions. When using the same tools and measuring the same object, all people with competent measuring skills should calculate the same measurements. When contradictions occur, measurements and calculations can be repeated in order to determine which ones are accurate.

Quantification has been accomplished for most sensory data. Sound can be measured in decibels, light in lumens, and touch by a hardness scale. Realists maintain that as humans developed mathematics to communicate the intrinsic qualities of objects, these qualities had to be quantified for mathematics to be used. The most useful of the scales used for quantification is the ratio scale. Ratio scales have a true zero, and as a result, the scale values are multipliable quantities (Kidder, 1981). Once data is quantified, mathematical operations can be performed and measurements compared. Even the so called imaginary numbers communicate intrinsic qualities. In an electrical circuit, the positive imaginary numbers measure inductance and the negative imaginary numbers measure resistance (Shenk, 1977).

Limit theory, which gives a value for the undefined operation of division by zero, developed because observable data conflicted with mental conclusions. An example is Zeno's paradox of the arrow, where the conclusion reached is that the arrow flying through the air is at rest. Visual observation contradicts Zeno's conclusion and limit theory mathematically explains his error in logic. As cited in the chapter on Idealism, Zeno's logic is true only when the time elapsed equals zero. Since "time elapsed" is the denominator of the fraction in the velocity formula, "time elapsed" can not equal zero, so limit theory is used to calculate the velocity the arrow approaches as the denominator approaches zero (Shenk, 1977). The velocity, at "time elapsed" equals zero, is called the instantaneous velocity and was introduced because observation indicated that a flying arrow must have a velocity greater than zero, even when elapsed time is zero.

The mathematics of limit theory, calculus, is also based on observations. Few values are constant over time and a mathematics was needed to measure the instantaneous value of observed change. Calculus is able to relate rates of change, calculate length of curves, areas of closed planes, volumes of solids, variations of pressure, work, density, weight, and other areas where change occurs. These calculations can only be estimated without limit theory and calculus. It was inevitable to the realist a mathematics had to be developed which would give an exact value for any spacial or temporal change. The estimated value illustrates indirect observation of perspective realism and the value from calculus is the direct observation which shows the object as it actually is (Shenk, 1977).

That objects have real shapes, areas, and volumes, not relative to individual viewpoints, is what gives mathematics its universal acceptance. A person's theory will be rejected if it does not concur with general observation and follow accepted mathematical structure. Calculus is accepted because its results agree with observation. So do the results of arithmetic, algebra, and geometry. The development of new mathematics is not an abstract mental exercise, according to realists, but a way to better understand public objects. Changes which occurred in geometry during the nineteenth century illustrate mathematical development. Euclidean geometry is the model most people use to visualize the physical universe. It is taught in most high schools and comes from a text written by Euclid about 300 B.C. Non-Euclidean geometries arose out of a deeper understanding of parallelism. Where Euclidean geometry states that parallel lines are always the same distance apart, in the nineteenth century, alternative geometries were proposed in which space is hyperbolic and the distance between two parallel lines can increase or decrease. The observations that inspired these alternative geometries are the basis of Albert Einstein's special theory of relativity and are needed to study the shape of the universe (Foster, 1981).

Mathematics does have a mental component, but the purpose is to better understand the material universe. When a mathematician uses matrix theory to calculate velocity in four dimensional space, it is a theoretical value. This presents no problem for the realist. Just because four dimensional space has not been discovered does not mean it does not exist. If and when the fourth dimension is observed, mathematical calculations can be compared with observation and any refinements needed to make the answers agree with observation can be made. Discovering truth in mathematics or any discipline is not based on some mental ideal, but on observation. If a mathematics instructor believes students will learn more from computer assisted instruction than from a traditional classroom situation, the belief is not accepted as fact. Educators will test the hypothesis and observe how well students learn the subject using the computer. The experiment will be repeated in different situations, and if observation confirms the hypothesis, it will be accepted as truth until contradictory observations are found. The goal of the realist is to discover new truth through observation, for the senses, not revelation, verify knowledge (Butler, 1968).

Chapter IV

Mathematical Metaphors and Pragmatism

Historical Perspective

The philosophical movement of the eighteenth century known as the Enlightenment brought a new direction to realism. Before this time most realists tried to discover by observation the truth of God's creation. The publication of Origin of the Species and the spread of evolutionary thought caused some philosophers to question previously accepted doctrines about God. The idea was entertained that truth was not only observable, but changeable. The early formation of this concept was the pragmatic realism of Charles Peirce (Butler, 1968). Influenced by Kant's work, Critique of Pure Reason, Peirce believed that the growth of psychological and biological knowledge would influence how all knowledge was declared valid. Thinking was seen as but one step in the production of habit and action, and by using biological and psychological knowledge, metaphysical obscurities could be readily understood. Peirce asserted that:

> In order to ascertain the meaning of an intellectual conception, one should consider what practical consequences might conceivably result by necessity from the truth of that conception (Peirce, 1940).

For Peirce, the truth of a proposition lay in its logical or physical consequences and if consequences change, so does truth.

Peirce's pragmatic realism developed into pragmatism within the "Metaphysical Club" of Cambridge, Massachusetts. The club, founded in the 1870s by Charles Peirce, William James, and others, is a rare example of a philosophy club actually producing something philosophical (Butler, 1968). Pragmatic thought was not the creation of one mind but an evolved philosophical movement which rejected the traditional academic philosophy of the late nineteenth century and sought to establish new positive aims. Because pragmatism was the product of several people, pragmatists often had different interpretations of what is meant by pragmatism. Charles Peirce is given credit for first developing pragmatism in the 1870s, but years later Peirce asked William James, "Who originated the term 'pragmatism'?, Where did it first appear in print?, What do you understand by it?" James gave Peirce full credit for inventing the term "pragmatism", but the two men often gave very different accounts of the pragmatic philosophy (James, 1917). The rift became so great that Peirce renamed his philosophy "pragmaticism". For Peirce, pragmatism was a technique for the successful communication of intellectual problems while

James applied pragmatism to issues of moral value and religious belief. James and Peirce actually developed different approaches to different philosophical problems and it was James's respect for Peirce which led him to call his philosophy "pragmatism" and cite Peirce as the developer (Butler, 1968).

The problem of defining a philosophical doctrine for pragmatism occurs because its associated ideas and attitudes developed over a period of time by several different people. Under the influences of Peirce, James, and John Dewey, pragmatism experienced reformulations and directional shifts. In 1908 Arthur Lovejoy distinguished thirteen possible forms of pragmatism. This was only the tip of the iceberg, for F.C.S. Schiller, in a humorous vein, said there were as many forms of pragmatism as there were pragmatists (Moore, 1922). The people who supported pragmatism also found many philosophers from the past were pragmatists. Suddenly, Socrates, Protagoras, Aristotle, Francis Bacon, Spinoza, Locke, Berkeley, Hume, Kant, and Mill all were called pragmatists. To avoid meaningless debate over definition, pragmatism is said to be a theory of meaning developed by Charles Peirce in the 1870s, revieved and reformulated in 1898 by William James as a theory of truth, and further developed in the twentieth century by John Dewey and F.C.S. Schiller.

The study of the phenomenology of human thought and the use of language inspired Charles Peirce to formulate his view of pragmatism (Moore, 1922). Peirce believed the way to investigate claims, assertions, beliefs, and ideas was through the understanding of signs. A sign was anything which stands for something else and permits communication. Peirce's desire was to develop a general theory of signs which would classify and analyze the types of signs and sign relationships which make communication possible. Signs presuppose a society with minds in communication with other minds, and for Peirce, signs were how the mind of one person communicated with the mind of another. For understanding to occur, signs must be socially standardized by a community into a system of communication. Peirce's pragmatism is a procedure of successful communication, based on linguistic and conceptual clarity, which can be used when people have intellectual problems. The emphasis is on method and Peirce often remarked that pragmatism is not a philosophy or a theory of truth, but a technique for solving philosophical or scientific problems. Signs, such as ideas, concepts, and language, must have a clear, precise meaning. If a meaning is not clear, pragmatism has a method for bringing distinctness. Unclear meanings are simply replaced with clearer ones by employing a condition with an unclear sign.

An example of an unclear sign is the phrase, "the man is tall". A man who is considered tall in Japan might be considered short by professional basketball players. A condition which would clarify the sign is that "if the height of the man is measured, he would be more than two yardsticks tall." The unclear sign "tall" is replaced and clarified pragmatically with a conditional statement in which a definite operation will produce a definite result. The operation is measuring the man's height. If the result of the measurement shows the man is taller than two yardsticks, he is described as tall. If the height is less than two yardsticks, the sign "tall" does not characterize the man. Even though Peirce's definition of "sign" encompassed all types of thought, his pragmatic method only applied when ascertaining the meaning of difficult words and abstact or intellectual concepts. These were areas where no consensus of thought is found, and in order to ease communication, pragmatism suggested that the words are not precisely defined or were being used in different ways. No real problem was solved, but by carefully defining words, pragmatism showed the problem never existed.

Peirce was given credit for inventing pragmatism, but it was the leadership of William James which expanded the pragmatic philosophy. Peirce and James, through their friendship, exerted much intellectual influence upon each other, but their versions of pragmatism were very different. James sought meaning, not in Peirce's schema of general concepts and formulas of action, but in experienced facts and plans of action. He believed that:

> We are spinning our own fates, good or evil, and never to be undone. Every smallest stroke of virtue or of vice leaves its never so little scar...Nothing we ever do is, in the strict scientific literalness, wiped out (James, 1917).

James's pragmatism emphasizes moral interests and moral values (James, 1917). The testing ground for intellectual efforts was the immediate, the concrete, and the practical. He believed that philosophy should discover what definite difference a certain idea, thought, or experience would make in the life of an individual at a definite moment. By examining "the definite differences at the definite moments" (James, 1917), it became possible to evaluate its meaning and truth. Meaning and truth were included in James's more fundamental category of value. When an experience was useful, workable, and has practical consequences, then for James, it had value. Thoughts of greater value enabled a person to move from one part of an experience to another more confidently, more satisfactorily, and with less labor than thoughts of lesser value.

A concern of James was whether people's lives were enriched by their beliefs and concepts. Beliefs which had value provided clues for action, formed by immediate experience and practical consequences. The correct behavior became the key to a higher level of life experience, for James believed most people were living very restricted lives:

> Most people live, whether physically, intellectually, or morally, in a very restricted circle of potential being. They make use of a very small portion of their possible consciousness, and of their soul's resources in general, much like a man who, out of his whole bodily organism, should get into a habit of using and moving only his little finger. Great emergencies and crisis show us how much greater our vital resources are than we had supposed (James, 1917).

The enrichment of life also provided justification for the moral and religious belief in James's pragmatism. His view was not that of a theologian, but of a psychologist or moralist. For James, when a person had a belief which answered or satisfied a need, the benefit supplied to the person by the belief justified the belief (James, 1917). An example is the many beliefs about life after death. If a person has one of these beliefs which causes him or her to live with less fear, the belief is justified. James said his justification procedure is valid only when the belief of the individual at any given time is part of the person's psychological and physiological behavior, the evidence for or against the belief is equal, and the belief makes a positive impact on the person's behavior. A positive impact on one person's life does not cause the belief to be universally true. A belief which has positive effects on one person in one situation can be detrimental to another person in the same or a different situation. Truth can also change over time for the same person as the person and situations change. The positive influence of a belief is not constant and the truth of a belief can only be tested for the present time and situation by observing the person's behavior in the current setting.

That truth changes because of circumstances led to the concept that if people could alter events, they could change truth. The fast increasing technology of the early nineteenth century, Darwin's theory of evolution, and the doctrine of the inevitability of progress confirmed the belief that humans had the ability to modify the future. The first person to develop these concepts into a philosophy was Auguste Comte (Comte, 1971). His desire was to reform society with a new positivist philosophy. In his writings, Comte demonstrated the need for social planning and preached that moral transformation must precede any desired social improvement. Comte also established a theory of social causation where individual acts and motives were determined by institutional settings. By manipulating the settings, any desired behavior could be produced and the ideal positivist society established. The positivist concept found many adherents in England and men such as John Mill and Herbert Spencer developed the doctrine of utilitarianism (Butler, 1968). This was an ethic philosophy which considered actions to be morally right if they were useful or promoted happiness. The goal of all public action was to be the greatest happiness of the greatest number. This would make progress inevitable and morality would be linked to a system that could transform society.

The best known of the utilitarian philosophies was elaborated by Karl Marx (Caute, 1967). Marx believed the task of philosophers was not to understand the world but to change it.

> Nothing can have value without being an object of utility. If it be useless, the labor contained in it is useless, cannot be reckoned as labor, and cannot therefore create value (Caute, 1967).

His book, Das Kapital, was a guide to social action that would effect a class war. England's industrialism brought about a distict class difference between the owners of industrial plants and the workers. Believing that capitalism could never adequately provide for the workers, Marx advocated the overthrow of capitalism in favor of state ownership of capital and land. The envisioned communism would change human nature itself. There would be no scarcity, frictions and war would cease, society would be planned so all could benefit, and people would be free to establish their own destiny. People would contribute according to their abilities and receive according to their needs. Eventually the state would die and a noble, free, and classless society would remain. Marx's utopia has yet to develop, but the effect of his works control the lives of millions of people.

Pragmatic thought in the United States did not have perfection as its final goal. Having been influenced by Darwinism, John Dewey developed a philosophy which emphasized that the process of perfecting, maturing, and refining was the aim of living (Dewey, 1939). Although he preferred to call his philosophy experimentalism or instumentalism, Dewey is considered a leading exponent of pragmatism. He was a disciple of James who believed that the most important part of a claim was its active, dynamic function. When a claim is acted upon, it leads in a true or false direction where truth is defined as collections of events which receive comfirmation in their consequences. Simply stated, a hypothesis which works is true. By finding what works, Dewey hoped to discover the knowledge and means of making all that is excellent, especially the making of goods, secure in experienced existance.

Dewey believed experimentation could determine the truth of every proposition (Dewey, 1939). Ideas, thoughts, and hypothesis were subjected to tests, the results of the tests were compared with previous knowledge, and the hypothesis was classified as true or false. As experimentation increased, the knowledge gained brought people a better approximation of reality. Many of the tests were scientific predictions in the form of an if-then proposition. The "if" section was the operations which were performed and the "then" section was the phenomena which should have been observed after the "if" operations were executed. If the consequences of the forcast occurred, the if-then statement was true. The hypothetical character of if-then statements illustrated that the results were not final or complete, but intermediate and instrumental. Absolute truth was never found in one experiment, but endless enquiry would approximate the ideal limit of reality.

It was in education that Dewey had his greatest influence (Winn, 1959). In the latter part of the nineteenth century, most psychologists believed children were passive creatures which had to be forced to learn. Using the slogan "learn by doing", Dewey addressed how children were not passive, but active, curious, and exploring people who instinctively learn. In The Child and the Curriculum, Dewey wrote:

> We believe in the mind as a growing affair, and hence as essentially changing, presenting distinctive phases of capacity and interest at different periods (Dewey, 1939).

Dewey often criticized the education that dominated the American schools of his time for its rigid and formal approach to learning. For Dewey, education was to be a reconstruction of experience where immature experience developed into seasoned skills and habits of intelligence. The school had to provide the proper environment which encouraged the habits and dispositions which constituted intelligence. Believing that the school was the most important medium for strengthening and developing a genuine democratic community, Dewey's schools were to be a miniature society which could bring social reform. The controlled social environment of the school made it possible to encourage the development of creative individuals who could work to eliminate existing evils and produce good. A free and humane school would provide an atmosphere which allowed all students to participate in systematic and open inquiry and help develop a society with greater harmony and aesthetic quality.

Mathematical Metaphors

Mathematics and the number system provide an almost perfect illustration of Peirce's pragmatism. Peirce was concerned with the use of language and signs. The number system achelves Peirce's dream of being a general theory of signs which classifies and analyzes types of numbers and number relationships. The number system is a language and each number is a sign for a thought or a concept. The operations of mathematics reveal sign relationships which connect one sign to another and make communication possible. Within mathematics, the number system is called a well ordered integral domain. This means that numbers and number relations follow certain rules of logical rigor. An example is that the number system has one and only one multiplicaticative identity. The only number which a person can multiply a number "A" by and obtain an answer of "A" is one. This is illustrated mathematically by the algebraic equation "A X 1 = A" or "1 X A = A", since multiplication is commutative. By following the rules of a well-ordered integral domain, it is possible to prove there is no other identity for the operation of multiplication. Suppose there is another multiplicative identity called "I", then "A X I = A" and "I X A = A". Since both "A X 1" and "A X I" equal "A", they must equal each other and "A X 1 = A X I". When this equation is solved for "I", the identity "I" equals one and the proof is complete. There can be no other identity except "one".

Because mathematical symbols and operations are universally accepted, mathematicians are able to satisfy Peirce's goal of linguistic and conceptual clarity in order to solve intellectual problems. Clarity, however, was not always a feature of mathematics. In the sixteenth century, when the concept of an unknown in an equation was first utilized, the equation $x^3 + 6x = 20$ was written "Cubus p 6 rebus aequalis 20" (Shenk). The notation improved to 1C + 6N aequalis 20 by the late sixteenth century and single letter unknowns and positive integer powers appeared in the seventeeth century. The current notation was popularized by Descarte in 1637 in the appendix of a tratise called "La Geometrie" (Shenk, 1977). The evolving of algebraic notation illustrates the importance Peirce placed on replacing unclear signs with clear ones. After new signs are developed, a way must be established for standardizing the signs within the community in order to facilitate communication. Descartes' article fulfilled the purpose in 1637. Today, standardization and communication are achieved through articles in mathematical journals.

Mathematics, like Peirce's pragmatism, is not a solution or an answer to any problem, but a technique to finding solutions of a philosophical or scientific nature. In the example where unclear concepts are replaced with clearer ones, mathematics goes beyond Peirce's explanation. Peirce, when he wanted to clarify a sign, would provide a conditional statement of a given situation which would produce a definite result. "The man is tall if he is more than two yardsticks high" illustrates a conditional statement. Since the number system is continuous, tall does not have to be based on one condition. People can be measured and the height, in inches, recorded. With the quantified data, people can be listed according to height or placed in many categories. This provides more meaning and empirical significance to language than Peirce's conditional statements.

William James's pragmatism emphasizes the importance of immediate experience, practical consequences, and clues to action (James, 1917). This is symbolized in mathematics by algorithms and proofs. An algorithm is a method or process of calculation, according to a set of fixed rules, which yields the solution of a problem or some class of problems. If a person wants to solve a linear equation, such as 3(X + 7) = 2X - 4, there is a set of rules that must be followed. First the grouping symbols on the left side of the equation are removed by multiplying "X + 7" by three. The equation then becomes 3X + 21 = 2X - 4. To solve linear equations, the terms with the unknown must be on one side of the equation and the terms without the unknown must be on the other side. To accomplish this "2X" and "21" can be subtracted from both sides of the equation showing that X = -25. The same result can be accomplished by subtracting "3X" and adding "4" to both sides of the equation. The result is 25 = -X. Solving for "X" by multipling both sides by negative one attains -25 = X, the same answer as before. The algorithm is the immediate experience in mathematics. Methods are taught in class in order to give students clues of action for solving problems. Most equations can be solved in more than one way, but the algorithm shows the practical consequences of each step. For solving linear equations, the practical consequence is that if a person uses the same operation on both sides of an equation, the new equation has the same answer as the original. If a different operation is performed on each side, the answer is lost.

Proofs illustrate the practical consequences of James and the precise logical theory of concepts presented by Dewey. A proof is a method of validating a proposition by using specific rules, assumptions, axioms, and sequentially derived conclusions (Shenk, 1977). In plane geometry certain assumptions are made about points, lines, and planes. From those assumptions, specific axioms can be proved, and these axioms can be used to form and prove other axioms. An example would be the assumption that a triangle can have at most one angle equal to or more than ninety degrees. The proof would consist of the fact that a triangle has three angles and the sum of these angles equals one-hundred eighty degrees. If one of the angles equals ninety or more degrees, ninety degrees or less is left for the other two angles. If the second angle equals ninety degrees, there would be no measure left for the third angle and a triangle must have three angles. This proves, in a non-rigorous manner, that a triangle can only have one angle of ninety degrees or more.

In addition to practical consequences, Dewey was concerned about the experimental determination of future consequences. Mathematics accomplishes this in probability theory. Mathematical probability theory is concerned with the determination of the likelihood of any event when there is insufficient data to determine with certainty its occurrence or failure. The three major interpretations of probability are classical, frequency, and subjective. Classical probability is used when the set of events can be counted without doing an experiment. Probabilities of a coin toss can be calculated using the classical interpretation. If one coin is tossed, there are two possible outcomes, heads or tails. To calculate the probability of a head occurring a fraction is made. The denominator is the total number of possible outcomes and the numerator is the number of outcomes which satisfy the probability requirement. The probability of a head is 1/2 since there are two possible outcomes and only one of the outcomes satisfies the condition of one head. When the coin is tossed two times, the probability of both tosses being a head can be calculated using the classical method. There are four possible outcomes, head-head, head-tail, tail-head, and tail-tail, so four becomes the denominator. Only one outcome satisfies the condition of two heads; therefore, the probability of two heads is 1/4 or 0.25.

The frequency interpretation is used when the experiment is actually performed. Suppose two coins were flipped together one hundred times and the desired information was the probability of obtaining two heads each flip. The number of times the experiment is performed, in this case 100, becomes the denominator. The numerator is the number of times two heads occur. Notice that the numerator can not be known until the experiment is completed. If two heads occurred 27 times the frequency probability would be 27/100 or 0.27. As the frequency of an experiment increases, the frequency probability should approach the classical probability (Langley, 1971).

Subjective probability is used when trying to predict some future event (Langley, 1971). A politician might use subjective probability to predict how people might react to a political vote. Classical interpretation can not be used here since there are no a priori facts such as the possible outcomes when flipping coins. The politician can use the frequency interpretation of probability by polling the voters, but even that is unreliable because views change over time. The politician makes a subjective decision based on what he or she knows about the voters attitudes, feelings, and beliefs. The actual voter response, however, will not be known until the decision is made. Even though the subjective probability can not be known with absolute certainty, it would fit Dewey's instrumentalism because inquiry was initiated in conditions of doubt and produced a judgment based on logic and reason.

Logic and reason is the center of a mathematical metaphor for the positivist's utopian society. In the utopia there would be no shortages and everyone would have what they needed. The study of mathematics revealed that mathematics needed a complete codification of the universally accepted modes of human reasoning as they applied to mathematics. Two mathematicians, Bertrand Russell and Alfred Whitehead, claimed to have accomplished what would be a mathematical utopia. The said that their work, Principia Mathematica, would derive all mathematics from logic and without contradiction.

> Mathematics takes us still further from what is human, into the region of absolute necessity, to which not only the actual world, but every possible world must conform (Russell, 1964).

All utopias are questioned and this one was no different. The German mathematician, Dave Hilbert, asked the world community of mathematicians to demonstrate rigorously that the methods described by Russell and Whitehead contained without contradiction all of mathematics. Instead of proving utopia, a mathematician by the name of Kurt Godel proved utopia to be an illusion (Hofstadter, 1979).

Chapter V

Mathematical Metaphors and Existentialism

The Individual

The philosophies previously discussed place major significance on society and the external environment. It is Existentialism which emphasizes solitary existence and the importance of the individual.

> Man can will nothing unless he has first understood that he must count on no one but himself; that he is alone, abandoned on earth in the midst of his infinite responsibilities, without help, with no other aim than the one he sets himself, with no other destiny than the one he forges for himself on this earth (Sartre, 1943).

An example of how the individual is lost in most societies and philosophies is illustrated by statistics, the mathematics of the collection, organization, and interpretation of numerical data. The process begins with each individual providing a data point. The data point can be any numerical value such as a test score or shoe size. Instead of reporting each individual's value, statistics reports scores which depicts the data of all the individuals. The most common statistics are mean and standard deviation. If the data points are test scores, the statistic reported is the mean or average test score for the group. Also reported is the standard deviation, which tells how widely spread are the data points. If the test scores are distributed in a bell shaped curve, sixty-eight percent of the scores are within one standard deviation of the mean, ninety-five percent of the scores are within two standard deviations of the mean, and almost one-hundred percent of the scores are within three standard deviations of the mean. The group is well described by these statistics, but any one individual score has lost most of its importance (Langley, 1971).

The Existentialist would ask the statistician what would be the effect on each individual when decisions are made based on group averages. In an educational setting, the content of a curriculum might be chosen based on the average score of a group of students. Even if a person believed the chosen curriculum was the best choice for the individuals scoring within one standard deviation of the mean, there are still thirty-two percent of the students who are being ignored. There is also the question of making choices based only on quantified external data. Existential philosophy does not separate the internal and external world. When all phenomena are examined psychologically, it has its existance in the states of the mind. The worth of knowledge is not determined by an external observation, but on the biological value of the data contained in one's consciousness (Kaufmann, 1975). The existentialist believes in the importance of all the students who took the test and would desire the best curriculum for each.

Historical Perspective

Although its proponents claim Pascal, St. Augustine, and Socrates were existentialists, the philosophy was formulated in the nineteenth century by Soren Aabye Kierkegaard in order to relate and defend his concept of true Christianity (Kierkegaard, 1971). After Kierkegaard, many philosophers with various beliefs claimed the existentialist label. While there is no meaningful structure which will define or encompass existentialism, the important themes associated with existentialism are recurrent. Existentialism focuses on the uniqueness and isolation of the individual in an indifferent or hostile universe, the questions of human existance, freedom of choice, and responsibility for the consequences of action (Kaufmann, 1975).

The key to existentialism centers in the epitaph Kierkegaard chose for himself, "that individual." The individual as primary contrasts with other philosophies which emphasize the concept of philosophical system or the concept of society (Kaufmann, 1975). A philosophical system, for Kierkegaard, was a conceptual structure which would try to understand individual existence as a part of the whole universe. He believed that "all essential knowledge relates to existence, or only such knowledge as had an essential relationship to existence is essential knowledge" (Kaufmann, 1975). By exhibiting logically necessary connections between every individual and the universe, reasons for every person's existence is provided.

The existentialist contrasts the concept of the individual from the concept of a philosophical system and people in society living stereotype roles. The mass of people understand themselves in terms of their views or beliefs, not as individuals. In both the philosophical system and society, the individual is secondary to the embodied concept. In contrast, existentialism believes that what exists is primary and concepts are deficient attempts to understand individual existence. Concepts must fail to provide adequate answers for individual existence because "man is not the sum of what he has but the totality of what he does not yet have, of what he might have" (Contat, 1974). The individual will always evade complete conceptualization.

For the existentialist, a conceptual system consists of a complete set of truths derived by deduction from an axiomatic starting point. Kiergegaard believed no concept can fully picture existence because existence is not a property of an object. What entails existance, such as action and choice, can be understood only if viewed as an agent and not as a spectator. A person can only understand his or her own existence and no one elses, for there is no order in the social universe, and any established connection between objects can rupture at any time. For this reason, philosophical system building must be eliminated if existence is to be understood. No individual has a rational scheme for understanding and mastering the universe and reason only leads to generalizations which will eventially fail. This is illustrated in the writings of Dostoyevsky, who is often called the forerunner of existentialism, because he stressed the unpredictablity of the universe and examined how individuals act when faced with choice (Kaufmann, 1975).

The existentialist claim that the individual can not be understood within a rational system is not as radical as it first appears. Existentialism is not committed to irrationalism but to the limitations of reason. Some existential philosophers even argue for the limits of reason on rational grounds and they usually explain that rationalism is valid in the natural sciences and mathematics. The German existentialist Karl Jaspers even accepted positivism as a valid version of the sciences, in error only when it tries to explain the activity of reasoning. Jasper's existentialism did not discredit reasoning, but demanded that reason be understood in a less restrictive way (Jaspers, 1971).

Basic Doctrines

Existentialist do strive for knowledge. In their attempt to discover what are emotions, beliefs, and acts of will, many existentialist philosophers use a conceptual method derived from the phenomenologists Franz Brentano and Edmund Husserl (Kaufmann, 1975). Brentano isolated the individual in order to describe accurately the central features of believing, feeling. and acting and Husserl placed awareness of oneself as a primary role of consciousness. Their scheme said there is always an object for emotions, beliefs, and acts. The belief is belief that, an emotion such as anger is anger about, and an act is an act toward. The object of belief or emotion is not in the external world, for a person's belief might be false or the anger might be about something that never happened. The object is internal to the person's belief or emotion. The language of phenomenology calls the object of emotion or belief the intentional object, but the emphasis remains on the concept of "that individual."

Jean-Paul Sartre illustrated intentionality as the difference between one's knowledge of self and one's knowledge of others (Contat, 1974). Others are viewed not as they are, but as intentional objects of an individual's perceptions, beliefs, and emotions. Not supprisingly, Sartre wrote that "hell is - other people." The paradox here is that an individual views the self as a person and others as objects. People are never objects to themselves and they refuse to be objects to others. If others regard an individual as an object, the individual says their view is wrong. Existentialists do not say that because beliefs have intentional objects, the beliefs are false or a person is committed to viewing other people as things. There is always an additional premise to the existentialist's claim that making others objects is to view them as other than what they are. By removing the additional premise a person can view the other as a person. This was illustrated by Martin Buber when he wrote about the I-It and the I-Thou relationship (Buber, 1970). The I-It is a person looking at another person as an object while the I-Thou relationship is person to person. Person to person relationships occur when individuals confront each another with their whole being and with no ulterior motive.

The main thesis of existentialism is that the central truth of human nature is the possibility of choice (Kaufmann, 1975). People do not have fixed natures that limit or determine choice, but the choices which people make is what brings their nature into being. This means, for the existentialist, that existence precedes essence, choice is everywhere, and all actions imply choice. Even when a person does not choose explicitly, which is true in most cases, the action implies an implicit choice. For Kierkegaard, a person's action shows a choice between three coherent lifestyles, the aesthetic in which pleasure is pursued, the ethical in which principles are treated as binding, and the religious in which God is obeyed. Kierkegaard felt that among these three a choice must be made. Sartre provided a fourth alternative by saying no choice is a choice. Even when a person does not choose, the person has chosen not to choose.

Present choices are governed by previous choices. Many actions appear to be governed by criteria, but these criteria are chosen. When a person chooses to get married, work for a certain company, or attend school, the criteria forming by one's lifestyle is the result of choosing that lifestyle. For the existentialist, there is no rational reason for such choices. Not only is there no rational reason for choice, there is also no causal explanation for actions. If human actions could be causally explained, determinism would be true because causality excludes the possibility of humans being responsible and free. It is the fact of freedom that Sartre believes brings people to despair. There has always been a fear of the dark, the nothing to confront. People do not want to make choices in an unmade future. They want someone or something else to make their decisions.

Because the existentialist believes in the sovereignty of individual choice in each situation, the other person can not be addressed in the same manner as in other philosophies. All people must make their own choices based on their own experiences. Argument is powerless unless the other chooses to agree with the speaker's premise. In their effort to eliminate self-assertion, many existentialist writers argue with the reader or frame their arguments in a hypothetical way. Kierkegaard often wrote, "If you choose this starting point, then that logically follows..." He also wrote under different names so the reader would be confronted with many points of view instead of a single argued case (Kaufmann, 1975). By presenting many choices, Kierkegaard hoped his readers would think for themselves and make their own choices.

Mathematical Metaphors

Mathematics is not a subject which is known for its freedom of choice. Most people believe each step of a mathematical algorithm must be explicitly followed and that their are certain mathematical conventions which must be obeyed. Most mathematical texts have the same algorithms and the same conventions because they follow the pragmatic philosophy of solving a problem satisfactorily in the fewest possible steps (Shenk, 1977). The logical rigor of mathematics does not have to include the pragmatic constraints, for it is possible to solve problems without following the usually stated method, but by choosing whatever steps one desires within certain criteria. An example is solving the algebraic equation "3X+7=X+11". In most cases, the problem would be solved by first subtracting "X" from both sides of the equation giving the equivalent equation "2X+7=11". The next step would be to subtract "7" from each side yielding "2X=4". The problem is completed by dividing each side by "2" producing the answer "X=2". The problem has been solved in three steps, but freedom of choice is even present in the three step process. The problem can also be solved in three steps by first subtracting "7" from both sides, then subtracting "X", and finally dividing by "2". There are two other ways to solve this equation in three steps, but most mathematicians would not recommend the process because it involves the use of negative numbers.

There is an infinite number of ways to solve the equation "3X+7=X+11," but the solution will take more than three steps. One solution taking more than three steps involves adding "7X" to both sides of the equation. This gives "10X+7=8X+11". Next subtract "11" from both sides to obtain "10X-4=8X". By subtracting "10X" from both sides, the equation becomes "-4=-2X". The solution is obtained by dividing both sides by "-2", giving the answer, "2=X". The algorithm now has four steps, but the answer remains the same. It is possible to solve the equation using four steps or five hundred steps if certain criteria are followed. Criteria are always present, as there is no absolute freedom in mathematics or in existentialism. When solving linear equations with one unknown, the limitation is the criteria that the same mathematical operation must affect both sides of the equation and division by zero is not allowed. Within this criteria, there is freedom and choice.

That existentialists are not committed to irrationalism metaphors the criteria which are necessary in mathematics. When solving equations, if different operations were performed on each side of an equation, the equation would become inconsistent. A simple example is the equation "7=7". If "3" was added to the left side and subtracted from the right side, the equation would become "10=4", a incorrect mathematically statement. Inconsistency also occurs when there is division by zero. The following "proof" that "2=1" illustrates the problem.

$x=y$	Given
$x^2=xy$	Multiply both sides by "x"
$x^2-y^2=xy-y^2$	Subtract y^2 from both sides
$(x+y)(x-y)=y(x-y)$	Factor both sides
$(x+y)(x-y)/(x-y)=y(x-y)/(x-y)$	Divide both sides by (x-y)
$x+y=y$	Quotient
$y+y=y$	Substitute x for y since x=y
$2y=y$	Combine like terms
$2y/y=y/y$	Divide both sides by y
$2=1$	Quotient

The proof followed all the rules of algebra, but "2" and "1" are not equal. The error is in the step where both sides are divided by (x-y). Since "x" and "y" are equal, "x-y" must equal zero and division by zero is not allowed within the algebraic system. When Karl Jaspers said that existentialist must not reject reason but understand reason in new and less restrictive ways, he was providing a lesson for the discipline of mathematics. Mathematics must have logical rigor and constraints, but the logic of mathematics, as the reason of existentialism, must be understood in new and less restrictive ways.

Using the standard mathematical conventions is not a requirement of the discipline. When a number line is written, the positive numbers are placed to the right of zero and the negative numbers to its left. An acceptable mathematical structure is obtained if a person decides to reverse standard convention and place the negative numbers to the right of zero. The choice of how to label a number line can be extended to an x-y axis. On a Cartesian coordinate system, the positive values of "x" are to the right of the y-axis and the positive values of "y" are above the x-axis. This system could be modified by reversing the positive and negative numbers on either axis. The convention of having each axis intersect at a ninety degree angle in not required. Where the axes intersect become the origin and as long as the angle between them is less than one-hundred eighty degrees and more than zero degrees, the coordinate system is mathematically valid. It should be noted that the mathematics is more elementary when the axes do intersect at ninety degrees (Shenk, 1977).

Kierkegaard's definition of philosophical systems, people in the mass, and the individual are also pictured by mathematics. For Kierkegaard, the philosophical system was an attempt to understand existence within a framework which would logically connect every part of the universe (Kierkegaard, 1971). In mathematics, there are systems in which logical connections are made. The system can be a number line, where the logical connections between numbers are the operations of addition, subtraction, multiplication, and division, or the more complicated systems of two or three dimensional space. The operations connecting the systems can range from arithmetic to geometry to calculus. Three dimensional space is an often used system because the world is believed to be three dimensional. Some examples of using three dimensional space as a system involve the calculation of work, velocity, acceleration, density, and weight. Mathematical systems are not limited to three dimensions or less. There are many mathematical models of what would occur in a space of more than three dimensions.

Kierkegaard's people in the mass, who live out stereotyped roles are also seen in mathematics classes. Most students can only solve problems using previously presented conventional systems. When a problem is presented, these people follow a step by step memorized solution process. The answer is obtained, but not understood. If a new concept or problem is presented, the majority of students can not understand the mathematics. They listen to some teacher or read some book which will tell them how to solve the problem. If the teacher or book is incorrect, most students are unconcerned. What is important is that an answer is obtained and life is not disturbed. For students, the undisturbed life is receiving a high grade in the course. The problem occurs when someone says those in authority, such as a teacher or a book is wrong. Since the student's lives are secondary to the mathematical authority as Kierkegaard's mass is secondary to the system, most students refuse to accept that the authority is mistaken.

There are some individuals who refuse to be secondary. In mathematics, these are the people who strive to understand and place themselves above the concepts. When they are not satisfied with the stated concepts, they develop new ones. People such as Galileo, Newton, and Copernicus questioned those in authority and developed new concepts. Often individuals such as these pay dearly for their rebellion. The Existentialist believes being an individual merits the cost, for the individual must never be subordinate to concepts. Mathematics can honor this belief when individuals claim their preeminence and strive for knowledge and understanding.

Chapter VI

The Dialectic and the Binomial Distribution

Historical Perspective

The term "dialectic" is found in the writings of many philosophers. Although the word originated from a Greek expression portraying the art of conversation, the dialectic has many philosophical definitions and a universal meaning would probably be meaningless. It is the Greek example of the dialectic which finds its metephor in the binomial distribution. The sense of the Greek dialectic was to refute the hypothesis of another by showing the unacceptable conclusions of that hypothesis (Randall, 1960). A classical example is the fifth century B.C. paradox of Zeno of Elea. Here Achilles is going to race a tortoise. The tortoise, however, is given a one-hundred yard head start. If Achilles can run ten times as fast as the tortoise, in the time it takes Achilles to run the one-hundred yards to the tortoise's starting position, the tortoise has run ten yards and is still in the lead. When Achilles runs that ten yards, the tortoise runs one yard and remains in the lead. Zeno's paradox says that Achilles will never pass the tortoise because in the time it takes Achilles to run to the tortoise's original position, the tortoise has moved to a new position. The tortoise will always remain in the lead, even if it is only by an infinitesimal amount (Salmon, 1970). Aristotle probably had in mind this paradox when he stated explicitly the Greek dialectic and created the science of formal logic. It is not an acceptable consequence that Achilles never overtakes the tortoise; therefore, any hypothesis that leads to this conclusion must be accepted as false. Stated logically if "p" implies "q" and "q" is false, then "p" is false (Randall, 1960).

The dialectic was a source of controversy to the early Greek philosophers. Using the dialectic to defeat opponents through indirect logical arguments was used by Zeno, Aristotle, and Plato for serious philosophical purpose. In the hands of the Sophists, however, the dialectic became a way for winning a dispute. The Sophist Protagoras claimed he could make an inferior argument appear to be the better. If this were the aim, dialectic is more rhetoric than philosophy. Plato called this aim a degenerate form of dialectic and named it "eristic" after the Greek word meaning strife. Plato refuted the deliberate use of invalid argument in his dialogue Euthydemus (Jowett, 1937). Aristotle also answered the Sophists in his book Sophistical Refutations (McKeon, 1941). Aristotle believed the dialectic was a positive activity and he clearly separated eristic from dialectic. Where the purpose of the eristic was the winning of the argument, the purpose of the dialectic was the search for truth.

Plato used Socrates as a person who stands in contrast to the Sophists. An irony is that Socrates, in his search for truth, also enjoyed winning an argument. This is called the "elenchus" and it is a major part of the Socratic dialectic. Socrates' "elenchus" is actually a synthesis of dialectic and eristic. This synthesis might have developed from a lost work of Protagoras, which some people believe begins with the claim that "there are two sides to every question." If the book continued by considering the truth of statements and counterstatements, then Protagoras should be given credit for the Hegelian dialectic and not eristic (Boas, 1969).

For Socrates the dialectic was a prolonged examination where the opponent's original thesis was refuted by drawing, through a series of questions and answers, a consequence that is unacceptable. The procedure is logically valid since it corresponds to the logical law that if "p" implies "q", and "q" is false, then "p" is false. The philosophical method of repeated questioning to obtain truths remains popular and is called the Socratic method. The search for truth did not end with a particular case. Socrates led his opponents to a generalization by getting them to accept a set of propositions about a certain instance as a universal truth. Aristotle credited Socrates with two innovations regarding the dialectic, logical argument and universal definition (Randall, 1960).

The Mathematical Metaphor

Testing a hypothesis using a binomial distribution is a mathematical parallel to the Greek dialectic. First a hypothesis is presented. It is called the null hypothesis because the hypothesis is implied to be no different from the truth. An example is when a manufacturer claims that ninety percent or more of the bolts which he sells meets a certain stress test. The claim is assumed true until evidence is obtained to discredit the hypothesis. This is the similar to the innocent till proven guilty assumption that is made in a courtroom. If the evidence shows the null hypothesis false, then what is called the alternative hypothesis is accepted. Continuing with the example, the alternative hypothesis is that less than ninety percent of the bolts will meet the stress test. Notice how the null hypothesis and the alternative hypothesis are the two opposite statements from the lost work of Protagoras. There are only two alternatives available in a binomial experiment and one has to be accepted as true. Either ninety percent or more of the bolts pass the stress test or less than ninety percent pass (Langley, 1971).

The question then becomes, how does one decide which hypothesis to accept as true? One solution would be to give every bolt a stress test. A bolt which passes the stress test is a success and a bolt which does not pass is a failure. If there are ninety percent or more successes, the manufacturer's claim would be validated. This would give the answer, but the cost and time of such a test would prohibit its use. Another possibility is to take a random sample of bolts, give them a stress test, and determine what percentage of the random sample passed. This would give evidence, but not the complete evidence which was found in the first solution. Ninety percent can not be the magic number for choosing which hypothesis is correct because random samples have random errors. It is possible that if all the bolts were tested, ninety percent or more of the bolts would pass, but in the random sample the passing rate would be only eighty-five percent. This is like tossing a fair coin ten times. A person would expect to obtain five heads and five tails. It is possible, however, to toss a fair coin ten times and record ten heads. The event is highly unlikely, and anyone would question the fairness of the coin, but the occurrence is possible. In fact, probability theory says a fair coin tossed ten times will record ten heads approximately one time in a thousand (Bradley, 1976).

In the example of the bolts, suppose a random sample of one-thousand bolts is obtained. If the manufacturer's claim is true, one would expect at least nine-hundred bolts to pass the test. Would the manufacture's claim be rejected if only 899 passed the test? Suppose only 880 passed the test or 850 passed? Where will the boundary be placed so if less than that number of bolts from the random sample failed to pass the test, the manufacture's claim will be rejected and the alternative hypothesis accepted? This fits perfectly with the logical law of "p" implies "q". "P" is the manufacturer's claim and "q" is the results of testing the random sample. If the results of the random sample is contrary to the manufacture's claim, then "p" must be false.

Logic says there are two possibilities for "p", either "p" is true or it is false. From the results of "q", a statement will be made regarding the truth of "p". This gives the following four possible outcomes: (1) "P" is actually true and from the results of "q" the correct decision is made that "p" is true, (2) "P" is actually true but from the results of "q" an erroneous decision is made that "p" is false, (3) "P" is actually false and from the results of "q" it is correctly decided that "p" is false, and (4) "P" is false but is thought to be true because of the results of "q". Binomial probabability says the truth of "p" can not be known positively, but the truth does exist. This is like the Platonic notion of ultimate truth that needs to be ascertained. Since the truth of "p" can not be known with one-hundred percent certainty, a decision needs to be made about what percent of the time we are willing to be wrong. Since we are assuming that "p" is true until proven false, the choice of error is how often are we willing to say "p" is false when actually it is true. This percentage will determine the boundary for "q". If the result of "q" is on one side of the boundary, "p" will be assumed true. If the result is on the boundary's opposite side, "p" will be declared false.

Returning to the example of the bolts problem, we will state that if "p" is true, we want "q" to declare "p" to be true ninety-five percent of the time. Using statistical data, it is found that the decision boundary is between 884 and 885. This means that if 885 or more bolts from the random sample of 1000 bolts pass the stress test, we will accept the manufacturer's claim as truth. If less than 885 pass, the claim is rejected. The ninety-five percent determines the boundary because probability theory says that if the manufacturer is telling the truth, ninety-five times out of one-hundred a random sample of 1000 bolts will have 885 or more bolts pass. We will, however, be wrong five percent of the time if "p" is true. Suppose instead of being wrong five percent of the time if "p" is true, we are only willing to be wrong one percent of the time. The statistical data now puts the decision boundary between 877 and 878. Fewer bolts are required to pass the test because we have increased the amount of evidence needed to say the manufacturer is wrong.

By placing the error percentage at the control of the statistician, binomial probability can, as the Sophist Protagoras claimed, make an inferior claim appear to be better. Suppose two groups desire to test the bolts strength. One group, a consumer affairs group, is willing to be in error ten percent of the time and the other group, an industry lobbyist group, is willing to be in error only one percent of the time. If 878 or more bolts out of the one-thousand pass the stress test, the lobbyist accepts the claim. For the consumer group over 887 bolts must pass the test before the claim is accepted. There is no problem if the sample has more than 887 bolts or less than 878 bolts pass the test. If more than 887 bolts pass, both groups will accept the manufacturer's claim and both groups reject the claim if less than 878 bolts pass. The problem occurs when the number of bolts passing is between 878 and 887. In this cass, the lobbyists accept the claim and the consumer group rejects it. Plato well named this form of the dialectic "eristic" or strife. Anyone who reads the Congressional Record understands the strife which occurs as business and consumer affair groups reach different conclusions, using the same data, just by choosing different error factors. Within the study of statistics, one's choice of error is considered no more ethical than anothers if the error factor is clearly stated. When the facts are known, the readers can then judge for themselves the validity of the conclusions (Langley, 1971).

Problems occur when people manipulate the data to make sure a certain conclusion is obtained. This often occurs on television advertisements when the announcer says that in a recent survey three out of four dentists recommended brand Z of toothpaste. What the announcer fails to mention is that it was not until the tenth group of four dentists that three out of four dentists recommended the brand which was being advertised. Of the forty dentists from the ten groups of four, perhaps only ten thought brand Z worthy of recommendation. The fact that one group of four gave three positive responses was all the advertiser needed. No lie was said, but the wrong conclusion was implied. Most people listening would commit an error of generalization and assume that three out of four or seventy-five percent of all dentists recommended brand Z.

It has already been noted in the bolt example that it is impossible to test all the bolts, so a sample of 1000 bolts was chosen and tested. The results of the sample were then generalized to the whole population of bolts. With the dentists, the manufacturers of brand Z want people to generalize their sample of four dentists to all dentists. Clearly, a person must be cautious when deciding which population generalizes from the sample. It would be foolish to make decisions regarding the economic status of Blacks in the United States by only sampling Blacks who live in Beverly Hills. The characteristics of the sample must be the same as the characteristics of the population if the generalizations are to be valid. When Socrates led his students to a generalization, he had them accept a set of propositions about a specific case. These axioms had to be true for both the particular case and the generalization. The same has to be true in binomial probability. The set of propositions which we attribute to the sample must be true of the population for the generalization to be valid.

Another concept of binomial probability parallels the changeability of truth stressed by pragmatic philosophy (Butler, 1968). While the traditionalism of Plato believes in an absolute truth which needs to be discovered, pragmatists believe truth changes with time. What was true yesterday might not be true today, and it is the responsibility of each new generation to discover and interpret their own truth. When a hypothesis is accepted or rejected using binomial probability, the decision is believed correct at that moment. The truth may be different the next month, day, or hour. This is best observed when popularity polls are taken during an election campaign. Suppose on Monday, a sample of eligible voters was asked if they prefer candidate A or candidate B. The victor of the poll can not be sure of winning on election day because at that time the voters might choose differently. Even our bolt example shows the changability of truth. On a certain day 1000 bolts were tested and enough of them passed to validate the manufacturer's claim. The next day an inferior steel was used, and if 1000 bolts were tested from that batch, a different conclusion would be reached. The manufacturer could also manipulate the situation to create a truth. If it is known when the bolts are going to be tested, the manufacturer can use a higher grade of steel during that time period and after the sample is tested return to the lower grade of steel.

The Problem of Choice

Philosophical decisions are not as simple as deciding if a bolt will pass a stress test. An example of a more complex situation is a teacher deciding which model of control will be used in a classroom (Sprinthall, 1977). The teacher desires an atmosphere which promotes learning. One method of control is the obedience by control method where all transgressions are confronted. Another method is the permissive model. In this classroom, the teacher is indifferent to student misbehavior and does not seem to mind when students talk, leave their seats, or are not prepared. A statistical experiment can be designed in which two classes are used to test which method is preferable. In one class, the model used is obedience by control and the other class used the permissive model. At the beginning of the school year, each student is given a standardized test to measure their knowledge. The same test is given each student at the end of the school year and the scores recorded. The difference in scores can be defined as the amount of learning. Using statistical techniques similar to those used with binomial probabilities, the model which better promotes learning can be discovered by comparing the scores from each class.

Several problems must be addressed. A decision has to be made about how one measures learning. It is simple to determine if a bolt can pass a stress test by placing the bolt under the desired amount of stress and observing whether the bolt deforms. Measuring learning is more complex and there is no agreement of method among educators. Standardized tests have been accused of being biased, not reliable, and not valid as a measure of learning. Even if a perfect measure of learning could be designed, other variables besides the method of discipline affect the learning process. The statistician would have to control variables such as textbooks, teacher personality, and classroom environment. Every facet of the two classrooms, except how students are disciplined, would have to be identical for the experiment to be valid. Also to be addressed is the question of making a decision based solely on quantified data. There are many ways, other than experimental research studies, to evaluate. Some non-mathematical methods of evaluating programs are professional judgment, decision-oriented studies, policy studies, and connoisseur based studies. The complexity of most problems requires that several methods of evaluation be utilized before a conclusion is reached (Scriven, 1980).

The complexity of philosophical choices caused the Idealist Georg Hegel to modify the Greek dialectic. Instead of believing one hypothesis was unacceptable and the other was true, Hegel believed truth existed on both sides of every question (Hegel, 1975). For Hegel, the most universal of all relations was that of contrast. The truth which was on one side of a question or thesis would always lead to its opposite or antithesis. Since nothing was eternally changeless, the thesis and antithesis interacted and formed a more complex whole or synthesis. Every contradiction was actually a relationship. In education, the synthesis of obedience by control and teacher permissiveness would be the discipline techniques of teacher effectiveness training developed by Thomas Gordon (Sprinthall, 1977). The change from thesis and antithesis to synthesis becomes the primary relationship of life. The synthesis becomes the new thesis and the cycle is repeated as every condition becomes a necessary stage in the evolution of thought.

People often forget how many choices they have. They enjoy being told that there are only two ways, with one way being superior to the other. The world becomes tidy and simple where confusion and difficult choices are eliminated. Because of its simplicity, the binomial distibution is often used to decide which choice is correct. There are only two choices and the choices can examined using quantitative measures. After using college level mathematics to compare the effects of each choice, the calculations will reveal the single choice which will provide success. Those following the selected choice will be prosperous while those on the other path will find failure.

The simplicity of the binomial distribution is also its defect. The world is not inscribed with only two choices, but with diversity and variety. Life is not neat and simple, but filled with confusion and difficult choices. Because individuals wish to avoid life's difficulties, those who speak of many possibilities are often rejected. George Moore's statement, "The difficulty in life is the choice" (Moore, 1922) states the philosophical problem. People believe that by rejecting complexity they insulate themselves from truth. Philosophers wish to eliminate this false sense of safety by revealing that life is insecure and no level of national properity or personal security can eliminate its perils (Marti-Ibanez, 1964). Anyone at anytime can lose health, peace, freedom, wealth, and love. The only real security life offers is the dynamic security from within. The security derived when a person has infinite flexibility of mind and an infinite valued orientation. A person then becomes a lover of wisdom, not as one who already knows, but as one who wants to know.

The complexity of most decisions suggests the need for sound evalutions and decisions. Binomial probability is an important decision making tool; however, the user must understand its limitations. No decision can be made with absolute certainty as there is always a posibility that the wrong choice was made. It is possible to bias the decision by manipulating the error factor, choosing a sample that is different from the population, or by temporarily making a change during the time of testing. Even if a correct hypothesis is chosen, there is still a danger when generalizing from the particular case to the universal. The qualities of the sample must be the same as the whole population for the generalization to be valid. The limitations of binomial probability do not invalidate its method of decision making any more than the limitations of logic invalidate the laws of Aristotle. Mathematics is just one tool of evaluation. Other tools can and must be used. With the information provided by each method of evaluation, people can discern for themselves which decision should be made (Scriven, 1980).

Chapter VII

Infinity

Historical Perspective

The concept of infinity was found early in the annals of Western thought when questions involving whether the world, time, or anything could be infinite in extent or infinitely divisible were discussed by early philosophers. Basic questions concerning infinity effected the question of whether the idea of something being infinite was internally coherent and consistent (Snow, 1978). The basic problem was the lack of understanding about infinity. People questioned whether things were really infinite, or was the human conception of infinity formed when something increased indefinitely in some aspect while the thing itself remained finite? The first major work to discuss questions about infinity was Aristotle's Physics (McKeon, 1941). Other discussions regarding infinity are found in the writings of Descartes, Spinoza, Hobbes, Locke, Hume, Kant, and Hegel.

The first Western philosopher who speculated about infinity was Anaximander (Brumbaugh, 1964). His infinity was the limitless substance which formed the limited things of the world. The substance was limitless, or infinite, because it was eternal, not having a beginning or end; it was inexhaustable, having a never ending supply; and it lacked internal boundaries and distinctions, having the ability to be everywhere. Infinity, however, was not spatially unlimited or qualitatively indeterminate. Anaximander believed space to be a sphere filled with nature's basic elements in a fused state. Air was considered by Anaximander to be the basic constituent of the universe and a primary example of limitless substance.

The Pythagoreans adopted Anaximander's concept of infinity, but their main contribution was to postulate the existence of the limit as a concept giving structure to the limitless (Brumbaugh, 1964). This limit had a geometric interpretation with the limitless once limited giving a point, twice limited giving a line, thrice limited giving a plane, and four times limited giving a solid. Each limit represents a point in space. Two points in space determine a line; three points, not collinear, determine a plane; and four points, not coplanar, determine a solid. The line, plane, and solid can be thought of as infinite in extent, meaning only the point is limited.

Plato's thoughts about infinity are contained in his work <u>Philebus</u> (Jowett, 1937). Infinity was part of a fourfold classification Plato gave to all things which now exist in the universe. The things which make the world can be viewed as unlimited, limit, mixture, and the cause of the mixture. The basis of Plato's theory is that the nature and the good of anything must consist of intelligible order. The universe structures the world by mixing the limit with the unlimited. The unlimited stands for each aspect of the universe, consisting as a collection of conflicting opposites, such as hot-cold or dry-moist. Limit consists of whatever ends the conflict between the unlimited. For Plato, the introduction of number can end the conflict by stating how hot, how cold, how dry, or how moist.

The moral aspect of humanity also used Plato's limit. Human pleasures tend to unlimited or infinite excess and must be controlled by the limit of law and order (Richards, 1966). Limit produces order and order is good, for without limit and order, the world would be a formless, unintelligible chaos. This logic prevented Plato from describing God or the divine as infinite. If God is perfect, the principle of limit must be present. It was God's task to take pre-existent matter and place upon it intelligible form, thereby making an ordered whole. Without divine limits the world would be formless, void, and evil. By saying matter has to have limit to be good, Plato believed it would be contradictory to say God is good and unlimited.

The gap between Plato and current Christian theologians is filled by the writings of Plotinus. Plotinus said God could be infinite if the concept of infinity, or unbounded, is applied to two categories of existence. First, infinity is applied to matter. Here, infinity is evil because matter tends to formlessness. This is stated by the physical law that entropy is increasing, where entropy is the measure of the randomness, disorder, or chaos of a system. Plotinus also applied infinity to the divine. The divine mind is infinite because of its endless power, complete unity, and self-sufficiency. The divine mind, unlike matter, does not tend to chaos; therefore, infinity when applied to God is not evil (Plotinus, 1977). A current application of Plotinus is the Biblical concept that all things are held together by God. The world was formless and void until God created the earth. After the creation, all created matter tended to disorder and entropy increased. God, being infinite mind, was able to limit entropy and maintain creation in an ordered state. For maintaining matter in an ordered state, the divine Mind has to be described as the good.

The concept of the divine Mind is consistent with idealism, the view that mental and spiritual values are fundamental in the world. The material world is believed to be an appearance of God since nothing exists except God and his attributes. Truth exists only within the the divine Mind and it is the goal of individuals to understand the mind of God. The Greek concept of infinity illustrates the impossibility of complete understanding (Brumbaugh, 1964). God's mind is thought to be infinite in extent. If knowledge is acquired through mental process, God still has more knowledge that must be understood. The individual can continue the quest for truth, but there is always more truth to be gained. It is as if there is a law of eternal progression. A person can progress in obtaining knowledge, but the quest is never complete. The divine Mind always has more to give.

A question which can be asked concerns God's knowledge. Is God's mind in a state of eternal progression causing the amount of truth to continually expand? If Truth is static, then it would not be infinite in extent and it might be possible for a person to understand all truth. If God's knowledge is expanding, is He really God by the traditional definition? What has developed can be called a divine paradox. If God is God, then He knows all truth and the amount of truth is not infinite. If truth is infinite, God must be learning more and causing truth to continually expand. One answer to the paradox might be that an infinite mind can hold infinite knowledge. If both God and truth are infinite, God's mind can understand the infinite amount of truth. People, because they are not infinite, can not perceive the total infinity of truth or totally understand it. Another answer is that truth is finite, but God is infinite. God, being infinite, can fit all truth into His Mind. Humans, because of limited mental capacity, can not and their understanding of truth remains limited.

The realist view of infinity contrasts with the idealist view. The idealist is trying to understand a mind which is infinite in extent; the realist is trying to divide space and matter into infinitely many parts. By studying each of the individual parts, the realist believes truth can be found and as each part is divided into smaller parts, more truth can be known (Snow, 1978). If it were possible to divide matter into infinitely many parts, complete truth could be discovered. This concept was used by seventeenth century mathematicians to develop infinitesimals, quantities which are supposed to be infinitely small and yet not zero (Shenk, 1977). The use of infinitesimals brought mathematics philosophical questions concerning the notion of infinitely large and infinitely small. Many questioned the idea of infinitely small, nonzero numbers, but the concept was accepted because of its effectiveness as a mathematical tool. The use of infinitesimals by the German philosopher and mathematician Gottfried Leibniz is the basis for modern calculus theory (Shenk, 1977). Leibniz also related infinitesimals to idealism with the theory that the world consists of infinitesimal, indivisible, and indestructible spiritual atoms called "Monads". Realists would agree with Leibniz except they would say the monads are material and not spiritual.

An example of the realist's view of infinity is seen when body parts are replaced with prosthetic devices. Suppose technology is able to develop an exact duplicate of the hand. The human hand could then be replaced with the prosthesis and the person would not notice any difference. Knowledge continues to increase and artificial arms, lungs, blood vessels, and brain regions are transplanted. The question is, when does the person cease to be a person and become an android? Perfect replication was made by dividing the human body into infinitesimal parts. What can be replaced by the artificial and what of the human must remain for the person to remain a person? People of different philosophies have different answers. The realist who believes that conscicusness is physically based would say people are already like androids and if perfect replication were possible, everything could be replaced. People who believe in a soul would probably say everthing could be replaced except the part of the body which houses the soul. Others believe a person is not the same if just one part is replaced because the human body is able to change and a prosthesis is a static device (Wilber, 1983).

The pragmatist, instead of dividing matter into infinitely many parts, is trying to make time infinitely divisible. Each segment of time is spent learning ideas, beliefs, and concepts which have value. Pragmatism tries to enrich daily life and raise the level of life experience by studying ideas, beliefs, and concepts which take people from one experience to another satisfactorily, securely, simply, and with less labor (James, 1917). A way to measure how well an idea accomplishes its goal is to time the task. A person can sew a dress with a needle and thread in ten hours. With a sewing machine, the same dress can be made in three hours. The idea of a sewing machine has value because it accomplished the task with less labor, simply, and satisfactorily. Another example is a person attending a university. The person with a college degree can obtain a better paying job than the person without the degree and feels more satisfied and secure. By dividing the time periods into more segments, the pragmatist can better prepare for a better life in the future.

The pragmatist's paradox parallels Zeno's paradox of Achilles racing the turtle. When Achilles runs to the turtle's beginning position, the turtle is still in the lead. In the time period it takes Achilles to move from the turtle's beginning position to the turtle's second position, the turtle has again moved a distance ahead. The time it takes for Achilles to reach the turtle's new position decreases with each run, but according to Zeno, the turtle can never be passed. Pragmatists have the same paradox. Each new concept can decrease the amount of time needed for a task, but the goal of less labor is never complete. A new concept is sought which will continue to decrease the time. As Achilles can never finish the race with the turtle because the turtle is in the lead, even if it is by an infinitesimal amount, pragmatists can not enjoy the present because they continue to work for a future time which never arrives. The future time always seems closer, but it is always in the future, even if only by an infinitesimal amount.

An existentialist's concept of infinity can be thought of as time and distance which is infinite but bounded. An example is an eliptical race track. The runners can theoretically continue around the track for an infinite length of time and run an infinite distance, but they are always bound to the course. Existentialists believe that individuals are bound to themselves. They can venture in many directions, but they can not break the bonds of self. The road of the existentialist can be pictured as the mathemmatical symbol for infinity, a horizontal figure eight. The center where the lines cross represents what Kierkegaard calls "that individual." The path of existence can lead away from the center point, but the path always returns. The distance traveled may be infinite, but the distance from the center point always returns to zero.

Mathematical Definitions

The concept of infinity offered by philosophy has produced some mathematically false notions of infinity. The question which has not been answered concerns what it means to say something is infinite. Philosophy has provided intuitive explanations which can be found in unabridged dictionaries. A typical entry states that something is infinite if it has no limit, and is boundless, unlimited endless, or immeasurably great in extent or duration. The dictionary definitions use infinity to describe God, space, and time. The mathematical definition of infinity, found in dictionaries, says that a quantity is infinite if it has no limit or is greater than any assigned quantity. These definitions tend to be unclear and in a logical sense wrong. There are many things called finite or infinite which do not have in any ordinary sense a limit or end. Common examples which illustrate vagueness of definition are phrases as "unlimited credit" or "unlimited sunshine." The vagueness and uncertainty extended into the sciences, for at one time space was believed to be infinite. The argument said if space was finite it would have spatial boundaries, but then there would be space on each side of the boundary. Modern physics solved the dilemma by stating that space is finite, but unbounded. The dilemma may be solved logically, but the uncertainty is still present as a recently published astronomy book contains seven modern cosmologies of space (Hartmann, 1985).

It was the middle of the nineteenth century before mathematicians endeavored to explain infinity. Many theories have been presented and today there is still no consensus of opinion. Two theories which demonstrate mathematical explanations of infinity were presented by Bernard Bolzano and Georg Cantor. Bolzano used the concept of classes and numbers to define infinite in extent (Bolzano, 1972). Two classes are said to be equivalent when the members of one class can be paired with those in the other so that each member of each class is paired with one and only one of the other. This is formally called a one-to-one mapping. Intuitively, two equivalent classes must have the same number of members. Cardinal numbers, then, are determined by families of classes with the property that any two classes in the same family are equivalent. If a person asks "How many?", the answer must be the same for all classes in the same group. The definition of finite or infinite begins with a nonempty class "A". Let A(1), A(2), ... be a sequence of classes determined as follows: A(1) contains some random member of "A", and each succeeding class contains everything in its predecessor plus something new chosen from "A". The sequence may terminate because some class A(k) has contains all the members of "A", so its successor can not be constructed. The class "A" is then finite. If, however, every class in the sequence has a successor, "A" is infinite. This is very similar to the idealist concept of infinite truth. The is a class of truth "T". The sequence begins with initial knowledge T(1). Each succeeding class has the knowledge of the previous class plus a new truth from "T". Currently, humanity is proceeding along the sequence. What is not known is whether there will be a class T(k) which contains all the members of "T" or if "T" is infinite.

Cantor presented an argument for being infinitely divisible by demonstating that the finite interval from zero to one contains infinitely many real numbers (Cantor, 1952). The essence of the argument is that if a list of the real numbers between zero and one were made, it can be shown that another number, not on the list, could be added. This is true even if the list were of infinite size. Suppose, for the discussion, that an infinite list could be constructed in which each positive integer "N" is matched with a real number r(N) between zero and one, and each real number between zero and one occurs somewhere on the list. Since real numbers are infinite decimals, the beginning of the list might be as follows:

```
r(1): .3  5  8  0  4  6  3  3  . . . . . . . .
r(2): .6  7  8  9  0  2  5  6  . . . . . . . .
r(3): .7  0  0  0  0  0  0  0  . . . . . . . .
r(4): .6  6  6  6  6  6  6  6  . . . . . . . .
```

No matter how long the list, Cantor developed a method of constructing a new number, r(k), which is not on the list. The construction consists of changing the digits which are "N" places after the decimal point. In r(1), the "3" would be changed because it is the first digit after the decimal point. The "7" in r(2) is changed because it is the second digit after the decimal and so on. The numbers which are to be changed form a diagonal and Cantor called this the diagonal argument. After the numbers are changed, they are prefixed by a decimal point and r(k) is constructed. The proof that r(k) is not on the list is shown by r(k)'s first digit is not the same as the first digit of r(1), r(k)'s second digit is not the same as the second digit of r(2) and so on. Hence r(k) is different from r(1), r(2), and so on meaning r(k) was not on the original list. Applying this argument to the realist trying to construct a perfect prosthesis, the task would be impossible because there is always a point not found and duplicated. If knowledge is gained by dividing matter into infinitely many parts, perfect knowledge is impossible by Cantor's argument because there are always parts which are not known.

Cantor's argument can be used by philosophers to refute the concept of a simplistic world as both mathematics and philosophy require infinite flexibility of mind to understand the infinite number of choices. Alfred Whitehead stated that:

> Our minds are finite, and yet even in these circumstances of finitude we are surrounded by possibilities that are infinite, and the purpose of human life is to grasp as much as we can out of that infinitude (Whitehead, 1977).

If each number is labeled as an idea, by constructing an infinite amount of numbers within a finite line segment, Cantor illustrated an infinte number of ideas. When the non-mathematical thoughts are added to the numerical numbers, the infinity is indeed large. With the wealth of facts and knowledge which are in the world, the Greek meaning of the word "philosopher" becomes appropriate. The Greek implies the person is a lover of wisdom, not one who already knows, but one who wants to know. The goal of knowing all truth is not what interests the philosopher, but the road to it. The poet, Christopher Marlowe, echoed the philosopher's goal when he wrote:

> Nature that framed us of four elements,
> Warring within our breasts for regiment,
> Doth teach us all to have aspiring minds:
> Our souls, whose faculties can comprehend
> The wondrous Architecture of the world:
> And measure every wandering planet's course,
> Still climbing after knowledge infinite,
> And always moving as the restless Spheres,
> Will us to wear ourselves and never rest,
> Until we reach the ripest fruit of all,
> That perfect bliss and soul felicity,
> The sweet fruition of an earthly crown.
> (Marlowe, 1967)

If people will walk the road of knowledge, they might discover that the quest for truth is not as complex or as frightening as they believe. They might even acquire the internal security that can not be removed.

Chapter VIII

Godel's Theorem

Axiomatization

Mathematics was a discipline for many centuries before mathematicians reflected on its nature, methods, and results. From ancient Greece through the first half of the nineteeth century, most mathematicians, believing that Euclid's geometry and Aristotle's syllogisms modeled the real world, used many of the fundamental concepts of mathematics in a naive manner. This changed after 1851 when non-Euclidean geometries were discovered (Hofstadter, 1979). Both Mathematicians and philosophers began to question whether even the basic theories of mathematics, such as the study of whole numbers, had a solid foundation. The study of mathematics, known as metamathematics, undertook the task of determining the true nature of mathematical reasoning so mathematicians could distinguish correct from incorrect procedure. Part of the problem was language. Mathematical reasoning had always engaged the language of normal communication causing words to have different meanings to different people. It became imperative to establish a single uniform notation which would allow mathematicians to resolve disputes over the validity of proofs. This required the establishment of a universal code of the accepted modes of human reasoning.

In 1879, a critical evaluation of mathematics was undertaken by Gottlob Frege and mathematicians began the process of axiomatization (Bell, 1937). Axiomatization or the axiomatic method is the process of constructing a deductive system in which all statements except a specified few are logically derived by specified rules. The specified few which are not deduced are called axioms or postulates. Axiom, derived from the Greek word meaning fitting or worthy, is often thought of as a self-evident truth. The mathematician uses the axioms to derive provable theorems using the language of mathematics. The procedure for advancing theorems from axioms is ordinary logic, which permits any believable argument. Many of the arguments, such as mathematical induction, are esoteric to the discipline. The theorems, when placed together, form a formalized system or theory. The formalized theory introduces signs for propositions, relations, logical connections, and individuals. Statements can be transformed into formulas and one set of formulas can infer other formulas according to certain specific rules. The application of the rules do not require meaning, only the physical recognition of the sign's shape (Hofstadter, 1979).

A simple formal system could be the HT-system. The HT-system consists of three distinct symbols, "H", "T", and "-". The system begins with the definition that aH-Ta- is an axiom whenever "a" is composed only of hyphens. If a=--, then --H-T--- is an axiom. A rule for producing theorems could be that if "a", "b", and "c" are strings of hyphens and aHbTc is a theorem, then aHb-Tc- is a theorem. If a=-, b=--, and c=---, then if -H--T--- is a theorem, -H---T---- is a theorem. The HT-system can become meaningful if "H" is defined as addition, "T" is defined as equal, and the number of hyphens represent the corresponding integer. The axiom then becomes the equation 2+1=3. The theorems can be tested using the rules of addition and the system reveals a reality known to second graders.

Formal Systems

The notion of formal system was widely accepted in the 1920s due to the work of A. N. Whitehead and Bertrand Russell. Their work, Principia Mathematica, contained a system where signs were manipulated according to rules and the meanings of the signs were ignored (Russell, 1964). The signs were simple marks written one after another forming formulas satisfying certain conditions based on their shapes and occurances. The axioms, theories, and proofs were the well-formed formulas which satisfied certain perceptual conditions. The German mathematician and metamathematician, David Hilbert, believed that all mathematics reduced to a formalized theory of well-formed formulas. Whitehead and Russell claimed their work was that theory which derived all of mathematics from logic and without contradiction. Russell believed that:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. (Russell, 1964)

Although their work was widely accepted, Whitehead and Russell's claim was questioned. It was not clear that all mathematics was contained in Principia Mathematica or if the methods used were contradiction free. To test their claim, Hilbert challenged mathematicians to demonstrate rigorously that the methods of Whitehead and Russell were without contradiction and that every true statement of number theory could be derived. Many mathematicians during the first thirty years of the twentieth century accepted Hilbert's challenge. Some tried to prove Principia Mathematica to be consistent and complete by using the methods outline in the book. This type of proof was criticized because the methods used in the proof are the same ones to be proved. To eliminate circular reasoning, Hilbert suggested that the proof be based only on "finistic" modes of reasoning, the small set of reasoning methods accepted by most mathematicians. The search for proof ended in 1931 when Kurt Godel published his paper, "On Formally Undecidable Propositions in Principia Mathematica and Related Systems I." In this paper, Godel proved that Whitehead and Russell's axiomatic system was inconsistent, and more generally, that all axiomatic systems were either incomplete, inconsistent, or both (Hofstadter, 1979).

Incompleteness and Inconsistency

Godel's discovery, known as Godel's Theorem, is based on the philosophical paradox stated by Epimenides, a Cretan who made the celebrated statement: "All Cretans are liars." The paradox, called the liar's paradox, violates the usual practice of defining statements as true or false. If the statement is believed true, then Epimenides, being a Cretan, could not tell the truth. Epimenides, however, is a Cretan who is saying a true statement which shows the statement is false. If the statement is believed false, then the Cretan

Epimenides must be telling the truth. If he is telling the truth, the statement cannot be false. Another example of the liar's paradox occurs when a person says, "I am lying." The paradox occurs when people refer to themselves or try to be introspective (Hofstadter, 1979). Godel connected the paradox to mathematics by using mathematical reasoning to explore mathematical reasoning. Godel showed how self-referential mathematical statements produce the same paradoxes found in the self-referential statements of language. Godel not only discredited the work of Whitehead and Russell, but showed that any axiomatic system is incomplete and that all truth is not provable.

When Godel's paper was published in 1931, the notion of formal system was the accepted standard of precision in mathematical foundations. Although Russell and Whitehead was questioned, many thought the Aristolelian ideal of perfect deduction from first principles had been attained (Robertson, 1957). Aristotle and other ancient Greeks believed reasoning was a patterned process governed by certain laws. In an attempt to structure the thought process, Aristotle codified syllogisms, a form of deductive reasoning containing a major premise, a minor premise, and a conclusion. The liar's paradox restated as a syllogism would be "All Cretans are liars; Epimenides is a Cretan;

therefore, Epimenedes is a liar." In the nineteenth century, logicians again tried to codify deductive reasoning patterns. People such as George Boole, Augustus De Morgan, Gottlob Frege, and David Hilbert went considerably further than Aristotle and developed the discipline of formal logic. Boole, in his work, The Laws of Thought, investigated the structure of propositions and deductive reasoning using a method which abstracts from the content of propositions and deals only with their logical form (Boole, 1953). The propositions were written in a mathematical language, which allowed the logician to test the validity of any inference regardless of subject content. Many of the social sciences, such as sociology and philosophy, accepted logical reasoning and tried to develop formal systems for their discipline that would be complete and consistent (Kidder, 1981).

Aristotle's ideal was shattered when in 1931 when Godel discovered that mathematics can not be bound by a formal system. The failure of perfect deduction within mathematics also affected other disciplines. Mathematics is regarded as the standard of rational knowledge for all the sciences. By proving the deductive system of mathematics inadequate, Godel illustrated that deductive systems in all areas were deficient. The argument is presented that since all the sciences, except for mathematics, are so remote from complete formalization, Godel's work should have little influence outside mathematics (Rosser, 1937). It is for precisely this reason that Godel's work must be remembered in all the sciences. Many ill-structured sciences, philosophies, and religions try to convince people they have found the one, true structure with all the correct answers. Answers are found in any book store and sell for $15.95. These books are best sellers because people believe there is a simple formula which will solve their problems. When Godel proved the formalization of mathematics to be incomplete and inconsistent, he also revealed that disciplines with less formalization will experience the same deficiency.

The essence of Godel's logically rigorous proof can be comprehended by studying the design of the "perfect" speaker. The designer wants the speaker to reproduce any and all sounds perfectly. It will be complete, able to reproduce all sounds, and consistent, able to reproduce the sounds without error. The speaker produces sound by converting electrical impulses from a radio, television, or phonograph, into vibratory disturbances in the air. The vibrations hit the eardrum causing it to vibrate. These vibrations are then converted by the ear and brain into sound. What is often forgotten is that the vibrations which hit the eardrum also hit the speaker which produced them. The speaker has a resonance frequency, the frequency at which it naturally vibrates. The effect of an object being struck by its resonance frequency is seen when two identical tuning forks are in the same locality. If one is vibrating, the air disturbances will soon cause the other to vibrate. If the speaker is struck with its resonance frequency, the speaker, like the tuning fork, will begin to vibrate at that frequency. The sound, when allowed to continue, will cause the speaker to vibrate with greater intensity until the speaker is destroyed. When the speaker produces its own resonance frequency, there is inconsistency, for the speaker destroys itself. To keep this from happening, the designer can make it impossible for the speaker to reproduce its resonance frequency. There is now incompleteness because there is some sound the speaker cannot produce. Godel proved mathematically what the designer discovered acoustically; no formal system can be both complete and consistent.

A verification of Godel's Theorem comes from an unusual source, the scriptures of Judaism and Christianity (Nave, 1921). Although many ministers and evangelists claim their denomination or religion has all the answers, the Bible they claim to believe says their knowledge is incomplete. The Hebrew Scriptures quote God as saying His thoughts are higher than the thoughts of the people. That

God has thoughts that humans do not have reveals the incompleteness of any spiritual person who claims to have all the answers. This was echoed by the Apostle Paul when he wrote to the Corinthians that his knowledge was only partial and will only be complete when he is with Christ. Jesus also showed incompleteness when He was on earth. When asked when He would return to establish the kingdom of God on earth, Jesus said no one knows except the Father. The admission of incompleteness by Jesus, Paul, or any person must not be confused with inconsistency. Even though Jesus and Paul exhibited incompleteness, all they did say could be true and consistent. What Godel demonstrated and the Bible echoes is that no one on earth has complete truth.

The failure of formalization created a deficiency in mathematical structure. The school of mathematical intuitionism tries to fill the deficiency by demonstrating how mathematical concepts and inferences occur regularly in ordinary thinking (Weyl, 1949). The existence of mathematical objects, which are "grasped" by mind, are independent of experience and provide mathematics a structure beyond formalization. To these mathematical objects, existence can not be independent of thought. An example is the natural numbers. No formal system can uniquely determine the natural numbers, but in the human mind they are stable, unambiguous objects. They are obtained by beginning at zero and repeatedly taking the following integer: 0,1,2,3,... This is obvious to most people and few are in doubt about what is and what is not a natural number. No one confuses a natural number with other mathematical objects, such as a radical, or with a nonmathematical object. Where formalization has failed, the human mind succeeds.

The flaw of intuitionism is in its lack of definition and its failure in areas where formalization succeeds. The intuitive notion of natural number does not define the general notion of number. If a person counted a finite number of integers, there is still the question of the integers beyond that point. The mind can conceive of a finite amount of integers and operations, but it is formalization which can prove for all cases. This occurs in Cantor's concept of an infinite number of points between zero and one. Intuitively, one would assume that a finite line segment would have a finite number of points. Cantor developed the previously shown method of finding a new number between zero and one, no matter how many numbers have already been found. The solution to the dilemma between formalization and intuitionism is to limit the notion of a formal system. Godel never said all axioms within a formal system are wrong. He only proved the formal system to be incomplete, inconsistent, or both.

Everyone lives by some system, for without structure, life would be chaos. The opposite of no structure is a complete formal system, also known as jail. Godel freed people from the jail of formal systems with his mathematical proof, for if no formal system has all the answers, there must be a place for choice and intuition. Since it is impossible to have a system which is both complete and consistent, the best humans can develop is a system which is consistent, but known to be incomplete. The designer of the previously mentioned speaker does not want the speaker to destroy itself, so the speaker was designed with the inability to reproduce its resonance frequency. The system is now consistent, but incomplete. The designer must also use intuition when designing the speaker. If the resonance frequency is middle C, the speaker has a severe limitation. The designer should design the speaker so the resonance frequency is a seldom used or inconsequential note. The system is still incomplete, but the limitation is less severe.

When people develop a system of living, they must understand its incompleteness. The people and philosophical systems which claim to have all the answers only place people in bondage. Plato's statement of a life unexamined is not worth living and Godel's theorem are intertwined. A system of living needs to be examined for inconsistencies and the inconsistencies removed, leaving an incomplete, but consistent system. A person is always examining philosophies and axioms. The ones which are consistent with a person's philosophy may be added to the system, always remembering that the system is incomplete. Intuition is involved because there are many consistent, but incomplete systems. Each individual must choose which beliefs to add or eliminate in order to maintain consistency. The philsophers' hope is that the choices will provide each person with a life worth living. As Ben Johnson wrote,

> True happiness
> Consists not in a multitude of friends,
> But in the worth and choice (Johnson, 1981).

Chapter IX

Meaning

Meaning, Communication, and Inquiry

The formal systems of mathematics consist of signs which are manipulated according to rules which ignore the interpretations of the signs and simply deal with them as marks written one after another (Rosser, 1937). These signs are grouped into formulas, strings of marks which satisfy certain conditions relating to their shape and occurence. Proofs consist of well-formed formulas which also satisfy certain perceptual conditions. When analyzing philosophy, meaning and interpretation can not be ignored. Philosophical communication does not consist of marks written one after another, but it involves the meaning of words or some other meaningful element of language. These words must have clarity of meaning if philosophers are to communicate their ideas, for a truth which can only be understood by the one stating it is only a half truth (Marti-Ibanez, 1964).

Even mathematicians need explicit definition when a formal system is based upon a portion of reality. The mathematical expression "3 + 6" would be meaningless if some people took the sign "+" to mean subtraction, and others interpreted the sign as division or multiplication. In mathematics, "+" is explicitly defined as the operation of addition, but this does not eliminate all problems and further definition is often necessary. Even when the operations of multiplication and subtraction are explicitly defined, some people could evaluate the "3 X 4 - 3" by first multiplying three times four and then subtracting three, giving an answer of nine. Others might subtract three from four before multiplying and obtain an answer of three. To eliminate the ambiguity, the order of operations is explicitly defined and multiplication is performed before subtraction.

Without meaning there can be no inquiry. A basic task for philosophy is to make explicit the meaning of words and provide a conceptual foundation for philosophical exploration (Marti-Ibanez, 1964). The task in philosophy is much more complex than in mathematics. When using a language of words instead of signs and shapes, ambiguity of meaning often occurs. The meaning of many words can be unclear, being influenced by intention, purpose, designation, reference, definition, translation, causal antecedents, or consequences. Some reasons for uncertainty of definition are a constrast in standpoints between the speaker and the interpreter, the difference in meaning between a specific utterance and a word's general use, and giving attention to the expressive instead of the referential use of language. Often the ambiguities can be eliminated by giving attention to the context in which the word occurs. This involves determining what a word linguistically means, what the speaker means by the word, what the word means to the interpreter, and what the word means in the original language of the speaker.

The goal of definition and meaning is to provide the speaker and the listener with the same mental picture (Russell, 1980). In mathematics, when people read "3 X 4 - 3", the goal is to picture the answer as nine. When a person says the word "man", a specific image is in the mind of the speaker. Many diverse mental pictures can exist within the listener. Some people will picture the human race and others will picture an individual person. With certain adjectives, such as "a tall Indian man", the mental pictures begin to coincide. There is still uncertainty and further definition is necessary. Indian might mean from India or American Indian, while tall can imply different heights to different people. By defining and redefining, mental pictures can coincide making meaningful analysis possible.

Inquiry usually reveals a need for the meaning of words to be more explicitly defined (Russell, 1980). Often a philosopher will encounter competing claims which can not be resolved using the accepted meaning of a word. Present are considerations which direct the inquiry in two or more directions. To resolve the conflict, the philosopher must be more explicit about meaning. Because of the vocabulary used by philosophers, the explicit meaning of certain words is not obvious. Words like "dog" and "walk" correspond to a thing or an action which is easily observable and in most cases easily defined. Providing explicit definition for words like "exist" and "belief" have challenged many philosophers and an explicit definition may not exist. The meaning of such words is not an observable feature like length, color, or other physical characteristics, for no one can see or sense something's existence or belief. When philosophers do try to make more explicit definitions, they seldom agree, as evidenced when people even within the same school of philosophy disagree about the meaning of various terms.

Theories of Meaning

The attempts to overcome the difficulty of determining meaning can be classified into three major theories, referential, ideational, and stimulus-response. It should not be surprising that the three theories of meaning parallel three of the major philosophies; idealism, realism, and pragmatism. Referential theory symbolizes realism and describes how most people think about meaning (Russell, 1980). A word refers to something or someone and every meaningful expression has a referent. If there is the word "Bill", there is also the person so named. The concept can be generalized to say that for any word to have meaning, it has to name, designate, or refer to something other than itself. As in realism, a physical something must exist. One of the problems with referential theory occurs when two expressions have the same referent but different meaning. The classic example involves the expressions "the morning star" and "the evening star." Both these expressions refer to the same entity, the planet Venus, but they have different meaning. It is not possible to know that the evening star and the morning star refer to the same object just by understanding the meaning of the phrases. It was an astronomical discovery which showed the morning and the evening star are the same.

The theory of meaning which parallels idealism is the ideational theory. In ideational theory, language is the instrument for communicating thought (Russell, 1980). Thought is a mental process which consists of a sequence of ideas in a person's consciousness, the ideas being directly accessible only to the individual. When trying to communicate ideas, people will use publicly observable sounds and marks to represent their thoughts. Successful communication occurs when one person's utterances arouse in another the idea trying to be communicated. Every word is associated with an idea, and since the words of philosophy often do not have a referent and are unseen and unsensed, ideation is often used by philosophers when they try to communicate. The lack of consensus among philosophers reveals the failure of this process. When a word does not have a referent, people seem unable to connect the appropriate idea with the linguistic expression. People have a vague sense of the word's meaning but there in no one-to-one correspondance between the word, its associated image, and meaning. Many words of different meaning can be associated with the same image and one word can evoke many different images.

Although the exact meaning of many words is uncertain, there is a public consensus about the general meaning of most words. Agreement about general definition suggested to many philosophers that meaning involved publicly observable actions of language. This belief was reinforced when psychologists began to explain certain aspects of behavior in terms of stimulus-response connections (Russell, 1980).

Connecting the meaning of words and sentences with the publicly observable features of a communication situation formed the foundation of the stimulus-response theory of meaning. Several forms of this theory evolved. The simplest stated that the meaning of a linguistic form is the situation in which the speaker utters it and the response it calls forth from the hearer. If this were true, all words would have a multitude of meanings, for the same word, when uttered in many different situations, evokes various responses. The situations in which the word is spoken have nothing in common which would give the word a distinctive meaning.

More sophisticated forms of the stimulus-response theory were developed by psychologists as Charles Osgood and behaviorally oriented philosophers as Charles Morris (Morris, 1955). They focused on how people responded to utterances and seemed to ignore the environment in which the utterances were made. Language was treated the same as natural signs which are not intentially produced. When a car makes an unusual noise, the noise is a natural sign which can be interpreted. To the trained mechanic, the noise has an explicit meaning. In the same manner, Osgood and Morris believed the explicit meaning of words were determined by how people interpreted them and responded to their utterance. Two problems which still remained were that on different occasions the same utterance used in the same sense produced very different responses and sometimes there was no response at all.

An analysis of meaning which aims to avoid the deficiencies of referential theory, ideation theory, and stimulus response theory was developed by Ludwig Wittgenstein and his followers at Cambridge University (Wittgenstein, 1980). The theory is based on a pragmatic view of the nature of language and is encapsulated in the slogan, "Do not look for the meaning, look for the use." Wittgenstein believed that words, phrases, and sentences were abstract entities consisting of time ordered sound types to which particular soundings may more or less approximate. Meaning is not attached to any particular sound or word but to the action the sound or word elicits in the speaker or listener. The sentence is usually thought of as the smallest linguistic unit which can evoke action. Some of the many types of action that people can perform when speaking sentences are informing, persuading, and frightening. Saying the sentence becomes the locutionary act and the produced effect is the perlocutionary act. The same sentence, said in different ways, can evoke different responses. A cook saying the food is on the table might be informing the audience that it is time to eat and the people will sit at the table. A parent shouting to a child that the food is on the table might be trying to persuade the child to come home and eat. The sentence can frighten if it is spoken harshly to the child who refuses to return home and eat. The uses of language are many and the meaning of any word or sentence can not be found in any one realm of being. It is precisely because of the many uses of language that the meaning of a words, especially words esoteric to philosophy, can not be explicitly defined by Wittgenstein's theory. Words like "exist" and "belief" are not designed to elicit responses, but to provide understanding. It could be argued that the actions of a person's life could be used to define "exist" and "belief", but then each person would have a different definition since no two lives are exactly alike.

The individual life provides the basis for meaning in existentialism. A general meaning might exist within society, but only the individual can give explicit meaning to words and symbols. Explicit meaning depends on the point of view of the hearer, the hearer's general understanding of the specific utterance, and the disposition of the hearer when the language was received (Russell, 1980). That there are as many explicit definitions as people and that these definitions change over time does not bother the existentialist. The general meaning of words provides what is necessary for communication but only the individual can supply inner understanding. The person ceases to be only an observer, peering unobtrusively at the world, trying to find the one right meaning. He or she manipulates and participates in what is described and in doing so brings about changes of meaning. It is the interaction between the consciousness of the hearer and physical reality which dominates the existentialist theory of meaning. As Anais Nin wrote in her diary:

> There is not one big cosmic meaning for all, there is only the meaning we each give to our life...To seek a total unity is wrong. To give as much meaning to one's life as possible is right to me (Nin, 1966).

Many modern theories have descriptions individuals can understand only within themselves. Astronomers have a cosmological theory which states that the universe is finite but unbounded. The general meaning of finite and unbounded clash in this theory. When something is finite it has bounds and what is unbounded is infinite. The individual is given the responsibility of providing meaning. A metaphor might be an expanding balloon. The balloon is a finite space, but it is conceivable that it could continuously expand. Another example occurs in quantum mechanics (Harth, 1982). This theory states that electrons can not be localized in space and time with arbitrary precision. It seems that electrons can be in two places at once moving with an inherent fuzziness. This has come to be known as Heisenberg's uncertainty principle which allows one to only speculate about the location of an electron, never knowing where it "really" is. In both these examples, an explicit definition can not be provided by any theory because of inadequate knowledge and the reality of nature itself. The problem of finding meaning has become intrinsic to reality.

Inadequate theories of meaning, combined with Godel's Theorem, reemphasize the incompleteness and probable inconsistency of any philosophy. Although all philosophies agree that human inquiry and communication depend on shared meaning, there is debate about how this takes place. Each of the four extant major philosophies has a theory of meaning based on the respective philosophy. Trying to define words describing a philosophy with a meaning theory based on that philosophy is somewhat circular. It is like people who try to pull themselves up by their own bootstraps. Eventially the bootstraps break. What does seem consistent in each of the discussed philosophies is the existentialist's concept of each individual providing explicit meaning to words. This is demonstrated when philosophers identify themselves with the same philosophical school, use the same terminology, and believe very differently.

The lack of consistency in meaning does not invalidate the philosophy nor diminish the understanding it gives certain individuals. There is a freedom which would not be present if the meaning of words were exact. The inexact meanings can be compared to the inherent fuzziness of particles in quantum mechanics. People can speculate about the probable location of the electron, but they do not know where the electron "really" is. The theory says the electron has a certain percent chance of being in one place and a certain percent chance of being in another. Using the conventional notion, the electron is really at one place or the other. The person just does not know which. With words, people can speculate about their probable meaning without knowing what the words actually mean. This is illustrated by each person reading this book. The reader forms mental images prompted by the words on the page, but these images might or might not be the ones the author desired. There is no opportunity for comparison or redefinition because the author and the reader are not in active communication. Each reader interprets the words as he or she sees fit.

Redefinition does occur when people communicate with one another and discuss different topics. As individuals attempt to relate their specific interpretations of any subject, each person's explicit meaning is lost in the process. The explicit meanings within each of the individual are different, but they can not be communicated. People use the general meaning of words when they communicate, causing each person's interpretation to diffuse. It is diffusion of meaning which gives the appearance of a consensus within a group. What has actually occurred is a redefinition from explicit meaning to general meaning. There only appears to be consensus, as imperfect definitions give the appearance of agreement. Each person still has his or her unique interpretation which can not be communicated.

Quantum mechanics suggests that the fuzziness of word meaning might be due to the nature of reality and not inadequate communication. It has been noted that electrons move in an inherent fuzziness (Harth, 1982). The conventional notion was used to say that the electron is in one place or another. The exact location is just not known. This notion was rejected by Niels Bohr, who said that uncertainties are not merely inadequate knowledge or understanding, but concern nature itself. Reality becomes dependent on a person's knowledge, for when a theory says an electron has a seventy percent chance of being is one position and a thirty percent chance of being in another position, the electron is actually seventy percent in one location and thirty percent in the other. Only when the person actually looks is the electron forced into one position. The observer has not just recorded reality, he or she has changed it. Meanings operate in the same inherent fuzziness, for there is no actual meaning until a person makes an observation and declares one. The word enters the mind and the person makes the observation. A manipulation and a participation occurs as the person touches the world and causes unavoidable and unpredictable changes. By defining the word, the observer has caused the mind and the physical world to interact and the interaction has changed reality.

Chapter X

Freedom and Choice

Free-will or Determinism

If any one issue has dominated philosophical thought, it is the question about whether or not individuals have freedom and choice. Free-will or self-determination allows people to make decisions which are independent of external constraints and in accordance with their inner motives and ideals. The reality of freedom and the autonomy of rational beings was the belief of Immanuel Kant. His transcendental idealism had humans free from antecedent conditions, for regardless of a person's character, motives, or circumstances, there was always a freedom of alternative choices (Kant, 1949). Opposite of the doctrine of free-will is the doctrine of determinism. Often attributed to Thomas Hobbes, determinism is the doctrine that every fact in the universe is entirely guided by law; thus, the facts of human history are completely dependent upon and conditioned by their causes (Hobbes, 1974).

Both free-will and determinism are found in Plato's political idealism. In the Republic, Plato distinguished three classes of people, the philosopher-kings, the soldiers, and the workers (Richards, 1966). The philosopher-kings were destined to be the rulers because by nature and training they were best for the job. Because of their virtue, philosophical wisdom, and rational ability, the philosopher-kings were free and the only people qualified to make choices. The soldiers were trained to be the guardians of the state who ensured that the decisions of the rulers were instituted. Plato's largest class of people, the workers, obeyed the rulers, complied with their policies, and lived lives which were completely determined by the decisions of others.

Two modern examples of Plato's political idealism are college campuses and church hierachies. At the university, the philosopher-king is the professor. Because of training and wisdom, the professor has the right to teach and make decisions. The brighter students are the soldiers. They become teaching assistants or tutors who ensure the professor's instructions are followed. The workers are the majority of the students. They obey the rules and comply with the professor's policies. Within the church, the philosopher-king is the minister and other ecclesiastical leaders. Their virtue and wisdom allows them to speak for God, telling others how God wants them to live. The soldiers are the devout followers. They become elders and deacons, ensuring the decisions of the leaders are followed. Most of the church members are workers, obeying the rulers and complying with church policy.

The concepts of freedom and choice are also found in realism (Snow, 1978). Realism grants people the freedom to speculate and investigate. Data are obtained by examination, using instruments and the senses. The data become the objective realities which realists can manipulate with their hands and analyze with their minds. Freedom and choice end when the correct answer is observed with the senses. For the realist, truth must be observable. An example concerns questions about consciousness. The realist has the freedom to investigate consciousness on a physical basis. In most cases, it is the structure of the brain which is explored. The possibility of a person having a spirit is rejected because it is not matter and can not be observed. Here freedom ends, for anything which transcends matter is rejected.

Pragmatism has a narrow view of freedom and choice. Since a choice must be useful, workable, and practical, most pragmatists believe only one choice is correct, the one which is the most useful, the most practical, and the most workable. All other choices are inferior (Butler, 1968). An example is the educational system in the United States. Most people believe the correct choice for youth is to finish high school and attend college. When decisions are made concerning the curriculum; however, school administrators usually design a program which is useful, practical, and workable for the school, not the student. The emphasis is often on how much time, effort, and money will be saved or how many students will the program attract. Seldom is the effect on any one individual student discussed (Parelius, 1978). When pragmatists research the curriculum, they are searching for a more useful, workable, or practical method to replace the current method. The quantitative aspect of the program becomes primary and the qualitative aspect is often lost.

That choice is central to human nature is the thesis of existentialism (Kaufmann, 1975). Determinism is emphatically rejected by existentialists. They do not believe that people have fixed natures which determine their choices, but it is their choices which determine their nature. There is choice in every action, forcing every person to makes choices, and any action can be used as an example because every action is accompanied by choice. When the choice is made, the responsibility for the choice rests with the individual. No one can or has the right to make a choice for another. All individuals must decide for themselves.

A poignant contribution to the question of human freedom and choice is found in "The Grand Inquisitor" by Fyodor Dostoevsky (Dostoevsky, 1981). The story is centered in Seville, Spain, during the time of the Spanish Inquisition. The Grand Inquisitor has just burned almost one hundred heretics for God's glory when Jesus appears in the town. The town's people are drawn to Him and His infinite compassion. He loves them, heals their sick, and even raises a child from the dead. The Grand Inquisitor, seeing this, sends guards to arrest Jesus. The people tremble into obedience, open a path for the guards, and then bow down before the inquisitor. Without understanding why, the crowd has chosen to obey a man instead of the God they claim to worship.

The masses were not the only ones drawn to Jesus. The Grand Inquisitor was also drawn. Within him was a desire to make Jesus understand why the same people who claim to love Him would burn Him as a heretic. The answer is found by contrasting the desires of people with the desires of God. Jesus's desire was to make people free. When Jesus was hungry, Satan tempted Him by saying, "If you are God, turn the stones into bread." Jesus, knowing the worthlessness of freedom which could be bought with bread, rejected Satan's offer. When Satan tried to make Jesus prove He was the Son of God by jumping from the temple, Jesus again refused. Although Psalm 91 prophesied that the Messiah would not strike His foot against a stone, Jesus would not tempt God. He knew tempting God would eliminate the freedom of faith and enslave humanity to miracles. Satan then tempted Jesus with all the kingdoms of the world and their splendor. Jesus was promised all He saw if He would worship Satan. This Jesus also rejected. No amount of material possessions can compensate for forced worship and the loss of freedom.

The inquisitor praised Jesus for His answers, but also reminded Him that He is not man, but God. People can never be permanently free because the very things Jesus rejected are the very demands people make on life. According to the inquisitor, people demand from life bread, a sense of the miraculous , and someone to worship. These three demands are the weaknesses of humanity which prevent people from being free. If people are offered bread when hungry, they will gladly trade their freedom for bread. Since most people can not feed themselves, they look for someone who can. Gladly they trade their freedom for food, depending on the inquisitor or people like him for their survival. The need for the miraculous is also provided by the inquisitor. When people are told about truths which they can not understand, they are thankful to the inquisitor for being the mediator between God and the masses. Even when the miracle can be understood, people ignore the truth. When the inquisitor gives bread from God, it is actually the crowd's bread made from their own hands. This does not concern the masses. They are thankful for someone to handle their affairs. The crowd is convinced that they are weak, worthless, rebellious, and that they can never be free. The inquisitor fulfills the third need by being the object of worship. Complete submission is found in worship and the masses know the value of total submission. Freedom scatters in unknown paths and brings unhappiness, but in obedience there is comfort and joy.

For Dostoevsky, Christianity is an impossible ideal. Its demands are greater than the nature of man. Even the Grand Inquisitor admits to Jesus that he is following Satan by providing for the people what Jesus rejected. The inquisitor believes that his love for the masses is greater than Jesus's love. Jesus loved humanity for what it could be, free and limitless. The Grand Inquisitor loved people for what they are, with their limitations. The inquisitor is like Plato's elect, having the thinking mind which is able to reason and rule. Inquisitor means questioner and it is his ability to question which elevates the inquisitor to his position of authority. If the crowd could question, the

Grand Inquisitor would be overthrown. Humanity is the workers, needing to be told what to do and how to live. Without being told what to do, the masses could not survive. The bread which humanity produces would become stale if it was not given to the inquisitor for redistribution to the people. The crowd is thankful to the inquisitor for allowing them to survive and the inquisitor enjoys his position of authority. The inquisitor, like Plato's elect, also has his soldiers to protect the faithful from the heretics who think contrary to the rulers. When another questioner, like Jesus appears, those in authority send the soldiers to silence the questioning voice. Plato's theory about what is best for humankind is complete. The rulers, soldiers, and workers are fulfilled in their positions and those that are not pleased with their position are removed from society.

The counter-argument is presented by Jesus, not in words, for the inquisitor did all the speaking, but in actions. It was the inquisitor himself who presented what Jesus was trying to teach. The inquisitor understood the price the masses pay for losing their freedom. He and Jesus did not argue about the truth of Jesus's teaching. Jesus and the inquisitor both agreed that a person who exchanges freedom for bread, miracles, or someone to worship has paid too high of price. That is why the rulers remain free and only the soldiers and workers are slaves. The argument centered about the nature of the masses. Jesus believed humanity has the ability to be free. The Inquisitor believed human nature forced people to reject freedom with only a few elect destined to rule and be free. The question which must be posed is that if only the rulers can think and the crowd needs a leader to worship, why are soldiers necessary? The purpose of the soldiers in "The Grand Inquisitor" was to arrest a questioner. Soldiers in Plato's political ideal were to keep the masses in control. People who do not think can be controlled by the ruler. Soldiers are not needed. That soldiers are necessary to support the existentialist belief that people are thinking and making choices all the time. There is a part of each individual which is free and this freedom can be expanded. It is apparent that leaders fear the expansion of freedom when they use soldiers to hinder dissidents. The inquisitor does not love the people as he says. He is fearful of losing his power and position so he eliminates anyone deemed to be a threat.

When D. H. Lawrence first read "The Grand Inquisitor," he thought it was a worthless piece of cynical, satanical prose (Lawrence, 1955). He rejected the grand inquisitor's argument of people being weak, slavish, and self-deceptive, who gladly yield immortality, true freedom, and salvation.

An older Lawrence had a different view. His change in belief caused Lawrence to write, "My heart sinks right through my shoes. I hear the final unanswerable criticism of Christ... bourne out by long experience of humanity." With great reluctance, Lawrence concludes that the grand inquisitor is correct when he says Christ's demands are beyond human strength. The many centuries have shown that few people are strong enough to endure the sufferings of a free faith. Christ's desire for humanity is but an illusion for most people. It is the inquisitor's argument which has become reality.

Jacob Bronowski presented a more optomistic view of humanity. His book, The Ascent of Man, echoed Lawrence's belief that people must be free to make their own decisions (Bronowski, 1974). Bronowski believed that people can not maintain their integrity if they let others run the world for them. Where the two men differed is in their assessment of human destiny. Lawrence viewed human history and has most of humanity destined for slavery and self-deception. Bronowski viewed human history and believeed that we are entering an era where knowledge and integrity are crucial. He believed that if western civilization does not allow people to determine their own destiny, it will cease to exist in its present form. The ascent from slavery toward freedom will occur elsewhere.

> Man is a singular creature. He has a set of gifts which make him unique among the animals: so that, unlike them, he is not a figure in the landscape - he is a shaper of the landscape. In body and mind he is the explorer of nature, the ubiquitous animal, who did not find but has made his home on every continent...His imagination, his reason, his emotional subtlety and toughness, make it possible for him not to accept the environment but to change it (Bronowski, 1974).

It is the nature of human imagination which will allow people to transcend their fears and to have confidence in their future.

The Search for Freedom

If human nature does not condemn the masses to slavery and if the existentialist thesis that existence precedes essence is true, why are so few people free? Plato provided one answer when he related the importance of training. For Plato, birth determined who were destined to lead and who were destined to follow. Those born elite had to be trained for their role as philosophers and kings. They were born, not with knowledge, but with the capacity to acquire knowledge (Richards, 1966). The existentialist would argue that Plato's elite were not elite by nature, but that their training gave them the nature of the philosopher and king. The others were trained to be soldiers and slaves and it was this training which brought about their nature. Most educational systems train the masses to be followers. The book is always correct and the instructor is the source of all knowledge. If a book has an error, most people refuse to believe the book is mistaken. When an authority figure makes a command, few question the consequences. The few who are trained to think and question are mostly wealthy upper class students in private schools. What determines who leads and who follows has not changed. The elite are still determined by birth.

Another reason for a lack of freedom is what Kierkegaard calls a generalized dread (Kaufmann, 1975). The dread is not about anything specific, but of the unknown. Sartre believed that dread is due to the fact of freedom. Freedom means the future is unmade and an unknown future is frightening. Most people dread moving, changing jobs, and anything else which has unknown results. When stress tests are published, most anxiety is produced by unexpected events. The events can be positive or negative. A loved one's death, a divorce, and winning a sweepstakes all increase anxiety. It is when a person's life has convention, complacency, and conformity, that the dread of the unknown dissipates. The Grand Inquisitor gave the masses more than bread, mystery, and someone to worship. He gave them structure and eliminated the unknown future. It was for that reason the crowd was thankful. Today's leaders use the same technique. The populace is promised a future in which they know what to expect. They exchange their freedom because it is less fearful allowing a leader to make decisions than to choose for oneself.

There is an inconsistency in the arguments of the inquisitor and Plato. The inquisitor admits to Jesus that he is following Satan by allowing the crowd to surrender their freedom. Satan's temptations did not cause Jesus to lose His freedom, but the people will gladly lose their freedom to the same enticement. By tempting the masses with the Satan's promises, the inquisitor gains control of the crowd. The inquisitor says he does this because he loves the masses with a love greater Jesus's love. The inconsistency of the inquisitor argument is that he admits to following Satan, but he also equates Satan with destruction. The inquisitor understands that the price the crowd pays by becoming slaves is total invalidation. They cease to be individuals and become nonessential parts of the inquisitor's following. The final inconsistency of the inquisitor occurs when he releases Jesus instead of burning Him as a heretic. Others might believe that the inquisitor's criticism of Jesus is valid, but deep within himself the inquisitor knows Jesus is correct. It is possible for people do be free.

Plato presents an argument for freedom in the allegory of the cave (Richards, 1966). The cave is occupied by people who are bound in such a way that they can only see one wall of the cave. Behind them is a light source which causes shadows to form on the wall. On the roof of the cave is a hole through which light can be seen. One person is able to escape the bonds, leave the cave through the hole, and experience light. When the person returns and tells about the reality outside the cave, the others refuse to believe. The people who are bound think the shadows are real and there is no other reality. The allegory illustrates Plato's belief that only the elite can understand reality and the majority of people remain bound, fit only to be workers and slaves. Even when they are taught the truth, Plato believed the masses would not understand truth. Teaching them to be free would be a waste of time, for it is their nature to be followers, needing to be told how to act.

When Plato said that only an examined life is worth living, he was saying that only the life of the elite has worth. The elite understand reality, make the decisions, and have most of the freedom. The crowd never examines, but always follows. It becomes their task to add value to the life of the elite. The value of most individuals is declared worthless and all but a few are expendable. Existentialism has challenged elitism by saying all of humanity has worth for each individual is a logically necessary connection in the conceptual scheme of the universe. Even though different people have different talents, each person has the ability to examine his or her life and make choices. According to existentialism, none of the people in the cave are bound. All are free to advance toward the light, but most choose not to do so. The spirit of philosophy compels philosophers to compel the masses to search for truth. Instead of teaching only the elite, Plato could have taught all, giving each person the chance to receive as much light as he or she could understand.

Summary and Conclusion

Sartre said, "Man is not the sum of what he has but the totality of what he does not yet have, of what he might have" (Kaufmann, 1975). Without testing claims of truth, people can not attain what they do not have. They are trapped in Plato's cave. Using mathematical metaphors for philosophical structure has accomplished two goals. The first was to illustrate how mathematical metaphors can be used to test and evaluate systems of philosophy. A metaphor is literally a transfer, in which one object or idea is denoted by another when an analogy exists between them. Symbolizing philsophical structure with mathematical symbols is an example of a metaphor. In this type of thinking, one context of knowledge is placed over another. New knowledge, new perceptions, and new expression then become possible. When a system of philosophy claims to possess truth, the claim can be tested using the knowledge revealed through metaphorical thinking. Metaphors can be used at all levels of mastery, for mathematical structure from arithmetic to calculus has been used as metaphors for philosophy. People can use the metaphors they understand to test claims of truth and formulate their beliefs.

The second and more important perception was discovered when Godel's theorem was used to evaluate the formal systems of philosophy. If philosophers claim their system has all truth, Godel's Theorem refutes them by showing that no formal system is complete and consistent. This discredits all philosophers, kings, and grand inquisitors who claim to know all the answers. Mathematics, held responsible by Hamilton, James, and Schopenhauer for rationalism, proved how speculative all the sciences are. Even the so called exact sciences are not all knowing and contradiction free.

The mathematician admits uncertainty in probability theory. The physicist admits uncertainty in quantum mechanics. All of humankind is free to think because no individual, system, or philosophy has all knowledge and truth. People are free to speculate, examine, and decide what is true.

The freedom of decision making requires continuous speculation and examination. People must discover as many choices as possible and evaluate these choices. Different people will decide on different choices, but each person must remember that no choice is complete and some are not consistent. After the choice is made, each person must continue his or her search for new choices, evaluating the newly discovered choices, and then make any appropriate change. Anais Nin understood the need for continuous evaluation when she wrote:

> There are very few human beings who receive the truth, complete and staggering, by instant illumination. Most of them acquire it fragment by fragment, on a small scale, by successive developments, cellularly, like a laborious mosaic (Nin, 1966).

Nin believed that Plato was correct when he said a life unexamined is not worth living. He was wrong believing only the elite can do the examining. Each person can choose freedom, living what he or she believes to be wisdom from the many voices which are heard.

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