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This thesis presents a class of narrow-stencil finite difference methods for approximating the viscosity solution of second-order linear elliptic Dirichlet boundary value problems. The methods are simple to motivate and implement. This thesis proves admissibility and stability results for the simple narrow-stencil finite difference methods as well as optimal convergence rates when the underlying solution to the partial differential equation (PDE) is sufficiently smooth. The results in this thesis extend the analytic techniques first developed by Feng and Lewis when approximating viscosity solutions of fully nonlinear elliptic PDEs using the Lax-Friedrich's-like method. Numerical tests are presented to gauge the performance of the methods and to validate the convergence results of the thesis.

NARROW-STENCIL FINITE DIFFERENCE METHODS FOR LINEAR SECOND
ORDER ELLIPTIC PROBLEMS OF NON-DIVERGENCE FORM

by

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CHAPTER 1: INTRODUCTION

In this thesis we develop and analyze various narrow-stencil finite difference (FD) methods for approximating the solution to the following linear elliptic second-order Dirichlet boundary value problem

$$\mathcal{L}[u] \equiv -A : D^2u + cu = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $\partial\Omega$ denotes its boundary, and $\Omega \cup \partial\Omega = \overline{\Omega}$ denotes the domain and boundary together. Notationally, D^2v denotes the Hessian matrix of the function v , $A \in [L^\infty(\overline{\Omega})]^{2 \times 2}$ is a symmetric negative-definite matrix, $c \in L^\infty(\overline{\Omega})$ is non-negative, and $g \in C^0(\partial\Omega)$. For transparency, we assume Ω is a rectangle, i.e., $\Omega = (a, b) \times (c, d)$.

This thesis proves admissibility and stability results for a class of simple narrow-stencil FD methods for approximating the solution u . Convergence rates will follow when u is sufficiently smooth. In the more general case, the approximations are guaranteed to converge locally uniformly to the viscosity solution u when (1.1) satisfies a comparison principle using the results in [6].

Since (1.1) is linear, more specialized methods have been developed and analyzed based on the properties of the coefficient matrix A . Problem (1.1) is a special case of the more generally fully nonlinear problem $F[u] = F(D^2u, \nabla u, u, x) = 0$. See [4] and [13] for recent survey papers that consider the state of the art for approximating viscosity solutions of fully nonlinear problems.

If A is differentiable, then (1.1) can be rewritten in divergence form since $\operatorname{div}(A\nabla u) = \operatorname{div}(A) \cdot \nabla u + A : D^2u$ giving $A : D^2u = \operatorname{div}(A\nabla u) + \mathbf{b} \cdot \nabla u$ for $\mathbf{b} = \operatorname{div}(A)$. As such, standard Galerkin-based approximation methods can be used. If A is continuous, then some finite element-based methods can be used such as the methods of Feng, Neilan, and Schnake [8]. An indirect vanishing moment approach paired with a conforming method can also be used based on the results of Feng, Lewis, and Schnake [7].

When A is not continuous, viscosity solution theory must be used unless A satisfies stronger assumptions such as a Cordés condition. When A does satisfy the Cordés condition, methods such as [19] can be used. Monotone methods are the standard approach for approximating viscosity solutions using the convergence theory of Barles and Souganidis [1]. There are many monotone schemes that follow this framework, such as semi-Lagrangian methods [3,5], meshless methods [10,15], two-scale methods [14], and those in [11,16,18,19]. Unfortunately, by [12], for a fixed grid it can be shown that there exist problems of the form (1.1) where the matrix A is chosen such that no monotone method exists. Furthermore, when monotone methods can be constructed, they often require the use of wide stencils.

In order to construct narrow-stencil schemes, we must abandon the monotone framework of Barles and Souganidis. Feng and Lewis recently proposed a Lax-Friedrich's-like narrow-stencil FD scheme that was proven to converge to the viscosity solution of fully nonlinear elliptic problems [6]. The narrow-stencil scheme, as shown in Figure 1.1b, is highly practical due to the fact that it is much easier to formulate and implement than wide-stencil schemes as illustrated in Figure 1.1a. A wide-stencil FD scheme must locally resolve radial directions using “nearest” neighbors to form three-point difference operators along all (relevant) radial directions. The 2D Lax-

Friedrich's-like narrow-stencil FD scheme simply incorporates the $2h$ nodes in the Cartesian directions to form a stabilization term that helps overcome the lack of monotonicity.

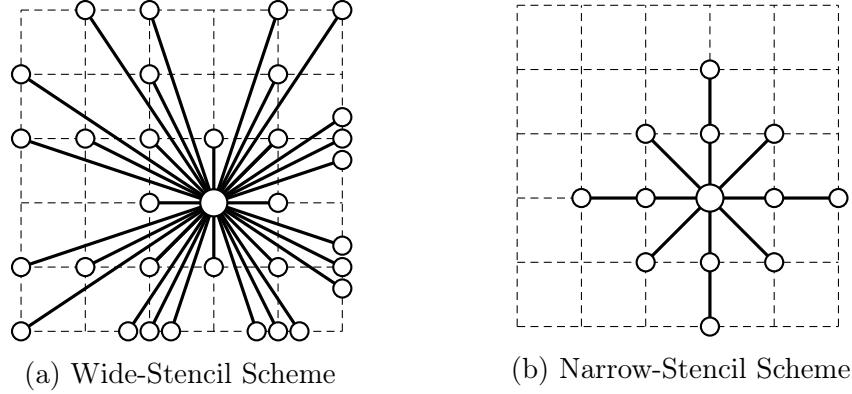


Figure 1.1. Comparison of local stencils for wide-stencil methods versus the Lax-Friedrich's-like method

CHAPTER 2: PRELIMINARIES

In this chapter, we introduce the notation that will be used, the assumptions used to define solutions to (1.1), and introduce the notation for defining meshes/grids of the domain $\bar{\Omega}$.

Standard function and space notation as in [2] and [9] will be adopted in this thesis. Let $\mathcal{S}^{2 \times 2} \subset \mathbb{R}^{2 \times 2}$ denote the set of symmetric real-valued matrices. For any bounded function $v \in B(\Omega)$, we define the upper and lower semicontinuous envelopes of v , respectively, by

$$v^*(\mathbf{x}) \equiv \limsup_{\mathbf{y} \rightarrow \mathbf{x}} v(\mathbf{y}), \quad v_*(\mathbf{x}) \equiv \liminf_{\mathbf{y} \rightarrow \mathbf{x}} v(\mathbf{y}).$$

2.1. Definitions

Definition 2.1. For matrices A and B , the FROBENIUS INNER PRODUCT, $A : B$, is defined as

$$A : B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}.$$

Definition 2.2. An $n \times n$ symmetric real matrix is said to be

- (i) POSITIVE-DEFINITE (SPD) if $\mathbf{x}^T A \mathbf{x} > 0 \forall \mathbf{x} \in \mathbb{R}^n$ for $\mathbf{x} \neq \vec{0}$.
- (ii) NON-NEGATIVE-DEFINITE if $\mathbf{x}^T A \mathbf{x} \geq 0 \forall \mathbf{x} \in \mathbb{R}^n$.

Definition 2.3. Assume $c \geq 0$. Then the PDE operator in (1.1) is UNIFORMLY ELLIPTIC if A is symmetric positive-definite, and it is DEGENERATE ELLIPTIC if A is symmetric non-negative-definite.

Definition 2.4. For a domain $\Omega \in \mathbb{R}^2$, we define the HESSIAN MATRIX OF u as

$$D^2u = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}.$$

2.2. Solution Concepts

There have been three main theories for the existence and uniqueness of solutions to (1.1): classical, strong, and viscosity solution theory. Due to the weaker assumptions of A , we cannot use strong or weak solution theory.

Classical solution theory assumes $D^2u(x)$ is defined for all $x \in \Omega$ so that (1.1a) is satisfied pointwise with $u = g$ on $\partial\Omega$. Strong and viscosity solution theory focuses on what it means to satisfy a second-order PDE when a classical solution u does not exist due to a lack of two classical derivatives. Strong solutions are solutions in $H^2(\Omega)$ that satisfy the PDE almost everywhere. All classical solutions are viscosity solutions, but the only way a viscosity solution would be considered a classical solution is if it is in C^2 . In general, viscosity solutions are defined in $C(\Omega)$ or $B(\Omega)$. Note that weak solution theory for defining solutions in $H^1(\Omega)$ does not apply when A is not differentiable because the PDE cannot be written in divergence form.

As long as A , c , f , and g are sufficiently smooth and $\partial\Omega$ is sufficiently nice, then classical solutions exist in $C^2(\Omega)$. However, when the coefficient matrix A is less smooth, A may only have viscosity solutions. For $A \in C(\Omega)$ and Ω sufficiently smooth, there exists a unique strong solution. However, since we only assume $A \in L^\infty(\Omega)$, we cannot ensure the existence of a strong solution. Thus, we focus on viscosity solution theory, where a viscosity solution is defined as follows.

Definition 2.5. A locally bounded function $u : \bar{\Omega} \rightarrow \mathbb{R}$ is called a

- (i) VISCOSITY SUBSOLUTION of (1.1) if $\forall \varphi \in C^2(\Omega)$, when $u_* - \varphi$ has a local

maximum at $x_0 \in \Omega$ with $\varphi(x_0) = u^*(x_0)$, there holds $\mathcal{L}[\varphi](x_0) \leq 0$.

- (ii) VISCOSITY SUPERSOLUTION of (1.1) if $\forall \varphi \in C^2(\Omega)$, when $u^* - \varphi$ has a local minimum at $x_0 \in \Omega$ with $\varphi(x_0) = u^*(x_0)$, there holds $\mathcal{L}[\varphi](x_0) \geq 0$.
- (iii) VISCOSITY SOLUTION of (1.1) if u is both a viscosity subsolution and a viscosity supersolution of (1.1).

To guarantee uniqueness of the viscosity solution to (1.1), we assume that the comparison principle holds. Note that the comparison principle is guaranteed to hold if $c \geq k_0 > 0$ for some $k_0 \in \mathbb{R}$. If $c = 0$, then there are examples where (1.1) does not have a unique solution. See [17] for more details.

Definition 2.6. Problem (1.1) is said to satisfy a COMPARISON PRINCIPLE if for any upper semicontinuous function u and lower semicontinuous function v on $\overline{\Omega}$ such that u is a viscosity subsolution and v is a viscosity supersolution of (1.1), then $u \leq v$ on $\overline{\Omega}$.

2.3. Defining a Grid/Mesh

Throughout this thesis, we are assuming Ω is a 2-rectangle meaning $\Omega = (a_x, b_x) \times (a_y, b_y)$. Only grids that are uniform in each (x, y) coordinate are considered. Define $\mathbf{h} = (h_x, h_y) \in \mathbb{R}^2$ such that

$$h_x = \frac{b_x - a_x}{N_x - 1}, \quad h_y = \frac{b_y - a_y}{N_y - 1}, \quad h = \max\{h_x, h_y\}$$

for integers $N_x \geq 2$ and $N_y \geq 2$, and let

$$N = N_x N_y, \quad \mathbb{N}_N = \{\alpha = (\alpha_x, \alpha_y) \mid 1 \leq \alpha_x \leq N_x, 1 \leq \alpha_y \leq N_y\}.$$

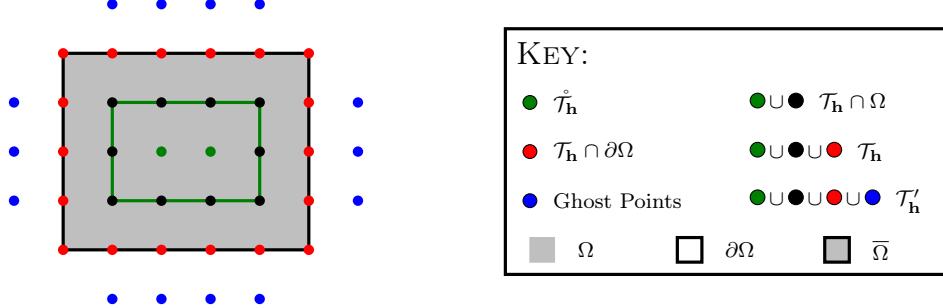


Figure 2.1. Illustration of the mesh/grid

Then, $|\mathbb{N}_N| = N$. We partition Ω into $(N_x - 1)(N_y - 1)$ sub-2-rectangles with grid points $\mathbf{x}_\alpha = (a_x + h_x(\alpha_x - 1), a_y + h_y(\alpha_y - 1))$ for each multi-index $\alpha \in \mathbb{N}_N$.

We call $\mathcal{T}_h = \{\mathbf{x}_\alpha\}_{\alpha \in \mathbb{N}_N}$ a mesh (set of nodes) for $\bar{\Omega}$, and we introduce an extended mesh \mathcal{T}'_h which extends \mathcal{T}_h by a collection of ghost grid points that are at most one layer exterior to $\bar{\Omega}$ in each coordinate direction. In particular, for $(x_0, y_0) \in \mathcal{T}_h \cap \Omega$, we choose grid points (x_1, y_0) or (x_0, y_1) adjacent to the boundary such that

$$(x_1, y_0) = (x_0 \pm 2h_x, y_0) \quad \text{or} \quad (x_0, y_1) = (x_0, y_0 \pm 2h_y).$$

We set $N'_x = N_x + 2$ and $N'_y = N_y + 2$, and then we define \mathbb{N}'_N by replacing N_x by N'_x and N_y by N'_y in the definition of \mathbb{N}_N while removing the extra multi-indices that would correspond to ghost grid points that are not in the set \mathcal{T}'_h to ensure $|\mathbb{N}'_N| = |\mathcal{T}'_h|$. Lastly, we define an interior mesh \mathcal{T}_h^* that removes a single boundary layer from \mathcal{T}_h .

Figure 2.1 is a visual representation of meshes and nodes with $N_x = 4$ and $N_y = 3$. The \bullet points are determined by an auxiliary boundary condition involving $\delta_{x_i, h_i}^2 U_\alpha$ for adjacent nodes $\mathbf{x}_\alpha \in \mathcal{T}_h \cap \partial\Omega$.

CHAPTER 3: DIFFERENCE OPERATORS

Now, we introduce several difference operators for approximating first and second-order partial derivatives. The multiple difference operators will be used to motivate and define our simple narrow-stencil finite difference methods.

3.1. First-Order Difference Operators

The following first-order difference operators are used to approximate first-order derivatives. They will also serve as building blocks for approximating second-order derivatives. We call (3.1a) the forward difference operator and (3.1b) the backward difference operator.

$$\delta_{x,h_x}^+ v(x, y) \equiv \frac{v(x + h_x, y) - v(x, y)}{h_x} \quad (3.1a)$$

$$\delta_{x,h_x}^- v(x, y) \equiv \frac{v(x, y) - v(x - h_x, y)}{h_x} \quad (3.1b)$$

$$\delta_{y,h_y}^+ v(x, y) \equiv \frac{v(x, y + h_y) - v(x, y)}{h_y} \quad (3.1c)$$

$$\delta_{y,h_y}^- v(x, y) \equiv \frac{v(x, y) - v(x, y - h_y)}{h_y} \quad (3.1d)$$

The central difference operator below is found by averaging the forward and backward operators.

$$\bar{\delta}_{x,h_x} v(x, y) \equiv \frac{1}{2} \delta_{x,h_x}^+ v(x, y) + \frac{1}{2} \delta_{x,h_x}^- v(x, y) = \frac{v(x + h_x, y) - v(x - h_x, y)}{2h_x} \quad (3.2a)$$

$$\bar{\delta}_{y,h_y} v(x, y) \equiv \frac{1}{2} \delta_{y,h_y}^+ v(x, y) + \frac{1}{2} \delta_{y,h_y}^- v(x, y) = \frac{v(x, y + h_y) - v(x, y - h_y)}{2h_y} \quad (3.2b)$$

Note that the central difference operators are second-order accurate while the forward and backward difference operators are only first-order accurate.

3.2. Second-Order Difference Operators

Using the first-order difference operators defined in the previous section, we can naturally build four difference operators for approximating second-order derivatives using composition. Thus, we can naturally define four discrete Hessian operators by

$$D_{\mathbf{h}}^{\mu\nu} = \begin{bmatrix} \delta_{x,h_x}^\nu \delta_{x,h_x}^\mu & \delta_{x,h_x}^\nu \delta_{y,h_y}^\mu \\ \delta_{y,h_y}^\nu \delta_{x,h_x}^\mu & \delta_{y,h_y}^\nu \delta_{y,h_y}^\mu \end{bmatrix} \quad \text{for } \mu, \nu \in \{+, -\}.$$

To build symmetric second-order accurate discrete Hessian operators, we introduce the following second-order difference operators. The various components of the discrete Hessian operators will be denoted using δ^2 . In particular, we use the following conventions:

$$\tilde{D}_{\mathbf{h}}^2 \equiv \frac{1}{2} (D_{\mathbf{h}}^{--} + D_{\mathbf{h}}^{++}) = \left[\tilde{\delta}_{x_i,x_j;h_i,h_j}^2 \right]_{i,j=1}^2 \quad (3.3a)$$

$$\hat{D}_{\mathbf{h}}^2 \equiv \frac{1}{2} (D_{\mathbf{h}}^{+-} + D_{\mathbf{h}}^{-+}) = \left[\hat{\delta}_{x_i,x_j;h_i,h_j}^2 \right]_{i,j=1}^2 \quad (3.3b)$$

$$\bar{D}_{\mathbf{h}}^2 \equiv \frac{1}{2} (\tilde{D}_{\mathbf{h}}^2 + \hat{D}_{\mathbf{h}}^2) = \frac{1}{4} (D_{\mathbf{h}}^{++} + D_{\mathbf{h}}^{+-} + D_{\mathbf{h}}^{-+} + D_{\mathbf{h}}^{--}) = \left[\bar{\delta}_{x_i,x_j;h_i,h_j}^2 \right]_{i,j=1}^2 \quad (3.3c)$$

$$\left[\overline{\overline{D}}_{\mathbf{h}}^2 \right]_{ij} \equiv \begin{cases} \left[\hat{D}_{\mathbf{h}}^2 \right]_{ij}, & i = j \\ \left[\bar{D}_{\mathbf{h}}^2 \right]_{ij}, & i \neq j \end{cases} = \begin{cases} \left[\tilde{\delta}_{x_i,x_j;h_i,h_j}^2 \right]_{i,j=1}^2, & i = j \\ \left[\bar{\delta}_{x_i,x_j;h_i,h_j}^2 \right]_{i,j=1}^2, & i \neq j \end{cases} \quad (3.3d)$$

For notation brevity, we use the following convention:

$$\delta_{x,h_x}^2 \equiv \delta_{x_1,x_1;h_1,h_1}^2 = \delta_{x,x;h_x,h_x}^2 \quad (3.4a)$$

$$\tilde{\delta}_{x,h_x}^2 \equiv \tilde{\delta}_{x,x;h_x,h_x}^2, \quad \hat{\delta}_{x,h_x}^2 \equiv \hat{\delta}_{x,x;h_x,h_x}^2, \quad \bar{\delta}_{x,h_x}^2 \equiv \bar{\delta}_{x,x;h_x,h_x}^2 \quad (3.4b)$$

$$\delta_{y,h_y}^2 \equiv \delta_{x_2,x_2;h_2,h_2}^2 = \delta_{y,y;h_y,h_y}^2 \quad (3.4b)$$

$$\tilde{\delta}_{y,h_y}^2 \equiv \tilde{\delta}_{y,y;h_y,h_y}^2, \quad \hat{\delta}_{y,h_y}^2 \equiv \hat{\delta}_{y,y;h_y,h_y}^2, \quad \bar{\delta}_{y,h_y}^2 \equiv \bar{\delta}_{y,y;h_y,h_y}^2$$

$$\delta_{x,y;\mathbf{h}}^2 \equiv \delta_{x_1,x_2;h_1,h_2}^2 = \delta_{x_2,x_1;h_2,h_1}^2 = \delta_{x,y;h_x,h_y}^2 \quad (3.4c)$$

$$\tilde{\delta}_{x,y;\mathbf{h}}^2 \equiv \tilde{\delta}_{x,y;h_x,h_y}^2, \quad \widehat{\delta}_{x,y;\mathbf{h}}^2 \equiv \widehat{\delta}_{x,y;h_x,h_y}^2, \quad \bar{\delta}_{x,y;\mathbf{h}}^2 \equiv \bar{\delta}_{x,y;h_x,h_y}^2$$

A simple computation shows the operators correspond to

$$\begin{aligned} \tilde{\delta}_{x,y;\mathbf{h}}^2 v(x,y) &= \frac{1}{2h_x h_y} \left[v(x+h_x, y+h_y) - v(x+h_x, y) - v(x, y+h_y) + 2v(x, y) \right. \\ &\quad \left. + v(x-h_x, y-h_y) - v(x-h_x, y) - v(x, y-h_y) \right] \end{aligned} \quad (3.5a)$$

$$\begin{aligned} \widehat{\delta}_{x,y;\mathbf{h}}^2 v(x,y) &= \frac{1}{2h_x h_y} \left[-v(x+h_x, y-h_y) + v(x+h_x, y) + v(x, y-h_y) - 2v(x, y) \right. \\ &\quad \left. - v(x-h_x, y+h_y) + v(x-h_x, y) + v(x, y+h_y) \right] \end{aligned} \quad (3.5b)$$

$$\begin{aligned} \bar{\delta}_{x,y;\mathbf{h}}^2 v(x,y) &= \frac{1}{4h_x h_y} \left[v(x+h_x, y+h_y) - v(x+h_x, y-h_y) - v(x-h_x, y+h_y) \right. \\ &\quad \left. + v(x-h_x, y-h_y) \right] \end{aligned} \quad (3.5c)$$

for approximating mixed partial derivatives,

$$\tilde{\delta}_{x,h_x}^2 v(x,y) = \frac{v(x+2h_x, y) - 2v(x+h_x, y) + 2v(x, y) + v(x-2h_x, y) - 2v(x-h_x, y)}{2h_x^2} \quad (3.6a)$$

$$\widehat{\delta}_{x,h_x}^2 v(x,y) = \frac{v(x+h_x, y) - 2v(x, y) + v(x-h_x, y)}{h_x^2} \quad (3.6b)$$

$$\bar{\delta}_{x,h_x}^2 v(x,y) = \frac{v(x+2h_x, y) - 2v(x, y) + v(x-2h_x, y)}{4h_x^2} \quad (3.6c)$$

for approximating non-mixed partial derivatives with respect to x , and

$$\tilde{\delta}_{y,h_y}^2 v(x,y) = \frac{v(x, y+2h_y) - 2v(x, y+h_y) + 2v(x, y) + v(x, y-2h_y) - 2v(x, y-h_y)}{2h_y^2} \quad (3.7a)$$

$$\widehat{\delta}_{y,h_y}^2 v(x,y) = \frac{v(x, y+h_y) - 2v(x, y) + v(x, y-h_y)}{h_y^2} \quad (3.7b)$$

$$\bar{\delta}_{y,h_y}^2 v(x,y) = \frac{v(x, y+2h_y) - 2v(x, y) + v(x, y-2h_y)}{4h_y^2} \quad (3.7c)$$

for approximating non-mixed partial derivatives with respect to y .

Figure 3.1 is a visual representation of the local stencils for the various discrete Hessians. Note that the operators are all symmetric. Also note that ghost points may be needed when using $\tilde{\delta}_{xx}^2$ or $\bar{\delta}_{xx}^2$ since they involve nodes two steps away.

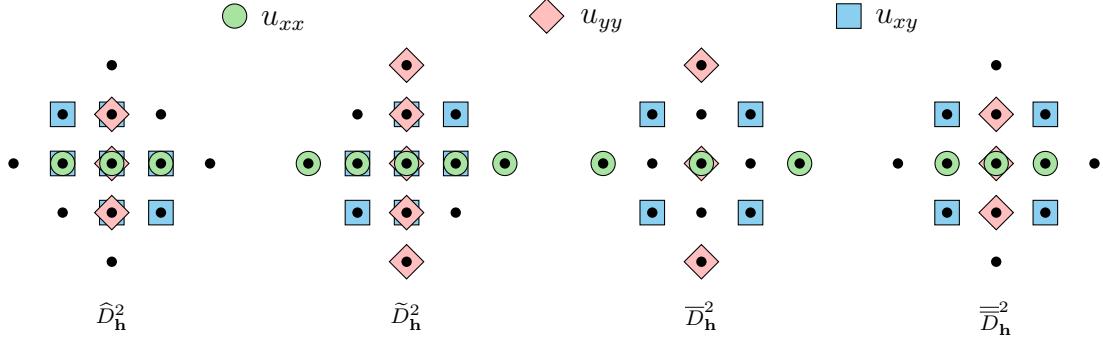


Figure 3.1. Illustration of the discrete difference operators

3.3. Verifying the Accuracy of the Discrete Hessians

In the following section, we verify the second-order accuracy of the discrete Hessian operators defined by (3.3a) and (3.3b). We first define the following expansions below for each type of shift used in the discrete operators. The expansion follows from Taylor's remainder theorem. Depending on the shift,

$$\xi_i \in (x - 2h_x, x) \text{ or } (x, x + 2h_x) \quad \text{and} \quad \eta_i \in (y - 2h_y, y) \text{ or } (y, y + 2h_y).$$

The remainder terms are all based on D^4u .

* CENTER POINT (NO SHIFT)

$$u(x, y) = u(x, y)$$

* HORIZONTAL SHIFTS

$$u(x + h_x, y) = u(x, y) + h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y)$$

$$u(x - h_x, y) = u(x, y) - h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y)$$

* TWO-STEP HORIZONTAL SHIFTS

$$u(x + 2h_x, y) = u(x, y) + 2h_x u_x(x, y) + 2h_x^2 u_{xx}(x, y) + \frac{4h_x^3}{3} u_{xxx}(x, y) + \frac{2h_x^4}{3} u_{xxxx}(\xi_3, y)$$

$$u(x - 2h_x, y) = u(x, y) - 2h_x u_x(x, y) + 2h_x^2 u_{xx}(x, y) - \frac{4h_x^3}{3} u_{xxx}(x, y) + \frac{2h_x^4}{3} u_{xxxx}(\xi_4, y)$$

* VERTICAL SHIFTS

$$u(x, y + h_y) = u(x, y) + h_y u_y(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_1)$$

$$u(x, y - h_y) = u(x, y) - h_y u_y(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_2)$$

* DIAGONAL SHIFTS

$$u(x + h_x, y - h_y) = u(x, y) + h_x u_x(x, y) - h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - h_x h_y u_{xy}(x, y) \\ + \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) - \frac{h_x^2 h_y}{2} u_{xxy}(x, y) + \frac{h_x h_y^2}{2} u_{xyy}(x, y) \\ - \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_5, y) - \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_6, \eta_3) \\ + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_7, \eta_4) - \frac{h_x h_y^3}{6} u_{xyyy}(\xi_8, \eta_5) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_6)$$

$$u(x - h_x, y + h_y) = u(x, y) - h_x u_x(x, y) + h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - h_x h_y u_{xy}(x, y) \\ + \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^2 h_y}{2} u_{xxy}(x, y) - \frac{h_x h_y^2}{2} u_{xyy}(x, y) \\ + \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_9, y) - \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{10}, \eta_7) \\ + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{11}, \eta_8) - \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{12}, \eta_9) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_{10})$$

$$u(x + h_x, y + h_y) = u(x, y) + h_x u_x(x, y) + h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) \\ + \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^2 h_y}{2} u_{xxy}(x, y) + \frac{h_x h_y^2}{2} u_{xyy}(x, y) \\ + \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_{13}, y) + \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{14}, \eta_{11}) \\ + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{15}, \eta_{12}) + \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{16}, \eta_{13}) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_{14})$$

$$u(x - h_x, y - h_y) = u(x, y) - h_x u_x(x, y) - h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) \\ + \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) - \frac{h_x^2 h_y}{2} u_{xxy}(x, y) - \frac{h_x h_y^2}{2} u_{xyy}(x, y) \\ - \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_{17}, y) + \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{18}, \eta_{15}) \\ + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{19}, \eta_{16}) + \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{20}, \eta_{17}) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_{18})$$

We now use the above relationships to verify the accuracy of the various discrete operators.

* EXPANSION 1: $\tilde{\delta}_{x,h_x}^2$

$$\begin{aligned}
& \tilde{\delta}_{x,h_x}^2 u(x, y) \\
&= \frac{u(x + 2h_x, y) - 2u(x + h_x, y) + 2u(x, y) + u(x - 2h_x, y) - 2u(x - h_x, y)}{2h_x^2} \\
&= \frac{1}{2h_x^2} \left[\left(u(x, y) + 2h_x u_x(x, y) + 2h_x^2 u_{xx}(x, y) + \frac{4h_x^3}{3} u_{xxx}(x, y) + \frac{2h_x^4}{3} u_{xxxx}(\xi_3, y) \right) \right. \\
&\quad - 2 \left(u(x, y) + h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) \right) \\
&\quad + 2 \left(u(x, y) \right) \\
&\quad + \left(u(x, y) - 2h_x u_x(x, y) + 2h_x^2 u_{xx}(x, y) - \frac{4h_x^3}{3} u_{xxx}(x, y) + \frac{2h_x^4}{3} u_{xxxx}(\xi_4, y) \right) \\
&\quad - 2 \left(u(x, y) - h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) \right) \left. \right] \\
&= \frac{1}{2h_x^2} \left[u(x, y) + 2h_x u_x(x, y) + 2h_x^2 u_{xx}(x, y) + \frac{4h_x^3}{3} u_{xxx}(x, y) + \frac{2h_x^4}{3} u_{xxxx}(\xi_3, y) \right. \\
&\quad - 2u(x, y) - 2h_x u_x(x, y) - h_x^2 u_{xx}(x, y) - \frac{h_x^3}{3} u_{xxx}(x, y) - \frac{h_x^4}{12} u_{xxxx}(\xi_1, y) \\
&\quad + 2u(x, y) + u(x, y) - 2h_x u_x(x, y) + 2h_x^2 u_{xx}(x, y) - \frac{4h_x^3}{3} u_{xxx}(x, y) \\
&\quad + \frac{2h_x^4}{3} u_{xxxx}(\xi_4, y) - 2u(x, y) + 2h_x u_x(x, y) - h_x^2 u_{xx}(x, y) + \frac{h_x^3}{3} u_{xxx}(x, y) \\
&\quad \left. - \frac{h_x^4}{12} u_{xxxx}(\xi_2, y) \right] \\
&= \frac{1}{2h_x^2} \left[2h_x^2 u_{xx}(x, y) - \frac{h_x^4}{12} u_{xxxx}(\xi_1, y) - \frac{h_x^4}{12} u_{xxxx}(\xi_2, y) + \frac{2h_x^4}{3} u_{xxxx}(\xi_3, y) \right. \\
&\quad \left. + \frac{2h_x^4}{3} u_{xxxx}(\xi_4, y) \right] \\
&= u_{xx}(x, y) - \frac{h_x^2}{24} u_{xxxx}(\xi_1, y) - \frac{h_x^2}{24} u_{xxxx}(\xi_2, y) + \frac{h_x^2}{3} u_{xxxx}(\xi_3, y) + \frac{h_x^2}{3} u_{xxxx}(\xi_4, y)
\end{aligned}$$

Thus $\tilde{\delta}_{x,h_x}^2 u$ is a second-order accurate approximation of u_{xx} .

* EXPANSION 2: $\widehat{\delta}_{x,h_x}^2$

$$\begin{aligned}
& \widehat{\delta}_{x,h_x}^2 u(x, y) \\
&= \frac{u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)}{h_x^2} \\
&= \frac{1}{h_x^2} \left[\left(u(x, y) + h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) \right) \right. \\
&\quad \left. - 2 \left(u(x, y) \right) \right. \\
&\quad \left. + \left(u(x, y) - h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) \right) \right] \\
&= \frac{1}{h_x^2} \left[u(x, y) + h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) - 2u(x, y) \right. \\
&\quad \left. + u(x, y) - h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) \right] \\
&= \frac{1}{h_x^2} \left[h_x^2 u_{xx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) \right] \\
&= u_{xx}(x, y) + \frac{h_x^2}{24} u_{xxxx}(\xi_1, y) + \frac{h_x^2}{24} u_{xxxx}(\xi_2, y)
\end{aligned}$$

Thus $\widehat{\delta}_{x,h_x}^2 u$ is a second-order accurate approximation of u_{xx} .

Similarly, $\widetilde{\delta}_{y,h_y}^2 u$ and $\widehat{\delta}_{y,h_y}^2 u$ are second-order accurate approximations of u_{yy} .

* EXPANSION 3: $\widetilde{\delta}_{x,y;\mathbf{h}}^2$

$$\begin{aligned}
& \widetilde{\delta}_{x,y;\mathbf{h}}^2 u(x, y) \\
&= \frac{1}{2h_x h_y} \left[u(x + h_x, y + h_y) - u(x + h_x, y) - u(x, y + h_y) + 2u(x) \right. \\
&\quad \left. + u(x - h_x, y - h_y) - u(x - h_x, y) - u(x, y - h_y) \right] \\
&= \frac{1}{2h_x h_y} \left[\left(u(x, y) + h_x u_x(x, y) + h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) \right. \right. \\
&\quad \left. + \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^2 h_y}{2} u_{xxy}(x, y) + \frac{h_x h_y^2}{2} u_{xyy}(x, y) \right. \\
&\quad \left. + \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_{13}, y) + \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{14}, \eta_{11}) \right. \\
&\quad \left. + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{15}, \eta_{12}) + \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{16}, \eta_{13}) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_{14}) \right) \\
&\quad - \left(u(x, y) + h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) \right)
\end{aligned}$$

$$\begin{aligned}
& - \left(u(x, y) + h_y u_y(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_1) \right) \\
& + 2 \left(u(x, y) \right) \\
& + \left(u(x, y) - h_x u_x(x, y) - h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) \right. \\
& \quad + \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) - \frac{h_x^2 h_y}{2} u_{xxy}(x, y) - \frac{h_x h_y^2}{2} u_{xyy}(x, y) \\
& \quad - \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_{17}, y) + \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{18}, \eta_{15}) \\
& \quad \left. + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{19}, \eta_{16}) + \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{20}, \eta_{17}) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_{18}) \right) \\
& - \left(u(x, y) - h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) \right) \\
& - \left(u(x, y) - h_y u_y(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_2) \right) \\
= & \frac{1}{2h_x h_y} \left[u(x, y) + h_x u_x(x, y) + h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) \right. \\
& \quad + \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^2 h_y}{2} u_{xxy}(x, y) + \frac{h_x h_y^2}{2} u_{xyy}(x, y) + \frac{h_y^2}{6} u_{yyy}(x, y) \\
& \quad + \frac{h_x^4}{24} u_{xxxx}(\xi_{13}, y) + \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{14}, \eta_{11}) + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{15}, \eta_{12}) \\
& \quad + \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{16}, \eta_{13}) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_{14}) - u(x, y) - h_x u_x(x, y) \\
& \quad - \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) - \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) - u(x, y) - h_y u_y(x, y) \\
& \quad - \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_y^3}{6} u_{yyy}(x, y) - \frac{h_y^4}{24} u_{yyyy}(x, \eta_1) + 2u(x, y) + u(x, y) \\
& \quad - h_x u_x(x, y) - h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) \\
& \quad - \frac{h_x^3}{6} u_{xxx}(x, y) - \frac{h_x^2 h_y}{2} u_{xxy}(x, y) - \frac{h_x h_y^2}{2} u_{xyy}(x, y) - \frac{h_y^3}{6} u_{yyy}(x, y) \\
& \quad + \frac{h_x^4}{24} u_{xxxx}(\xi_{17}, y) + \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{18}, \eta_{15}) + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{19}, \eta_{16}) \\
& \quad + \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{20}, \eta_{17}) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_{18}) - u(x, y) + h_x u_x(x, y) \\
& \quad - \frac{h_x^2}{2} u_{xx}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) - \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) - u(x, y) + h_y u_y(x, y) \\
& \quad \left. - \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_y^3}{6} u_{yyy}(x, y) - \frac{h_y^4}{24} u_{yyyy}(x, \eta_2) \right] \\
= & \frac{1}{2h_x h_y} \left[2h_x h_y u_{xy}(x, y) - \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) - \frac{h_y^4}{24} u_{yyyy}(x, \eta_1) - \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{h_y^4}{24}u_{yyyy}(x, \eta_2) + \frac{h_x^4}{24}u_{xxxx}(\xi_{13}, y) + \frac{h_x^3 h_y}{6}u_{xxxy}(\xi_{14}, \eta_{11}) \\
& + \frac{h_x^2 h_y^2}{4}u_{xxyy}(\xi_{15}, \eta_{12}) + \frac{h_x h_y^3}{6}u_{xyyy}(\xi_{16}, \eta_{13}) + \frac{h_y^4}{24}u_{yyyy}(x, \eta_{14}) \\
& + \frac{h_x^4}{24}u_{xxxx}(\xi_{17}, y) + \frac{h_x^3 h_y}{6}u_{xxxy}(\xi_{18}, \eta_{15}) + \frac{h_x^2 h_y^2}{4}u_{xxyy}(\xi_{19}, \eta_{16}) \\
& + \frac{h_x h_y^3}{6}u_{xyyy}(\xi_{20}, \eta_{17}) + \frac{h_y^4}{24}u_{yyyy}(x, \eta_{18}) \Big] \\
= & u_{xy}(x, y) - \frac{h_x^3}{48h_y}u_{xxxx}(\xi_1, y) - \frac{h_y^3}{48h_x}u_{yyyy}(x, \eta_1) - \frac{h_x^3}{48h_y}u_{xxxx}(\xi_2, y) \\
& - \frac{h_y^3}{48h_x}u_{yyyy}(x, \eta_2) + \frac{h_x^3}{48h_y}u_{xxxx}(\xi_{13}, y) + \frac{h_x^2}{12}u_{xxxy}(\xi_{14}, \eta_{11}) \\
& + \frac{h_x h_y}{8}u_{xxyy}(\xi_{15}, \eta_{12}) + \frac{h_y^2}{12}u_{xyyy}(\xi_{16}, \eta_{13}) + \frac{h_y^3}{48h_x}u_{yyyy}(x, \eta_{14}) \\
& + \frac{h_x^3}{48h_y}u_{xxxx}(\xi_{17}, y) + \frac{h_x^2}{12}u_{xxxy}(\xi_{18}, \eta_{15}) + \frac{h_x h_y}{8}u_{xxyy}(\xi_{19}, \eta_{16}) \\
& + \frac{h_y^2}{12}u_{xyyy}(\xi_{20}, \eta_{17}) + \frac{h_y^3}{48h_x}u_{yyyy}(x, \eta_{18})
\end{aligned}$$

Thus $\tilde{\delta}_{x,y;\mathbf{h}}^2 u$ is a second-order accurate approximation of u_{xy} .

* EXPANSION 4: $\widehat{\delta}_{x,y;\mathbf{h}}^2$

$$\begin{aligned}
& \widehat{\delta}_{x,y;\mathbf{h}}^2 u(x, y) \\
= & \frac{1}{2h_x h_y} \left[-u(x + h_x, y - h_y) + u(x + h_x, y) + u(x, y - h_y) - 2u(x, y) \right. \\
& \quad \left. - u(x - h_x, y + h_y) + u(x - h_x, y) + u(x, y + h_y) \right] \\
= & \frac{1}{2h_x h_y} \left[- \left(u(x, y) + h_x u_x(x, y) - h_y u_y(x, y) + \frac{h_x^2}{2}u_{xx}(x, y) - h_x h_y u_{xy}(x, y) \right. \right. \\
& \quad \left. + \frac{h_y^2}{2}u_{yy}(x, y) + \frac{h_x^3}{6}u_{xxx}(x, y) - \frac{h_x^2 h_y}{2}u_{xxy}(x, y) + \frac{h_x h_y^2}{2}u_{xyy}(x, y) \right. \\
& \quad \left. - \frac{h_y^3}{6}u_{yyy}(x, y) + \frac{h_x^4}{24}u_{xxxx}(\xi_5, y) - \frac{h_x^3 h_y}{6}u_{xxxy}(\xi_6, \eta_3) \right. \\
& \quad \left. + \frac{h_x^2 h_y^2}{4}u_{xxyy}(\xi_7, \eta_4) - \frac{h_x h_y^3}{6}u_{xyyy}(\xi_8, \eta_5) + \frac{h_y^4}{24}u_{yyyy}(x, \eta_6) \right) \\
& \quad + \left(u(x, y) + h_x u_x(x, y) + \frac{h_x^2}{2}u_{xx}(x, y) + \frac{h_x^3}{6}u_{xxx}(x, y) + \frac{h_x^4}{24}u_{xxxx}(\xi_1, y) \right) \\
& \quad + \left(u(x, y) - h_y u_y(x, y) + \frac{h_y^2}{2}u_{yy}(x, y) - \frac{h_y^3}{6}u_{yyy}(x, y) + \frac{h_y^4}{24}u_{yyyy}(x, \eta_2) \right)
\end{aligned}$$

$$\begin{aligned}
& -2 \left(u(x, y) \right) \\
& - \left(u(x, y) - h_x u_x(x, y) + h_y u_y(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - h_x h_y u_{xy}(x, y) \right. \\
& \quad + \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^2 h_y}{2} u_{xxy}(x, y) - \frac{h_x h_y^2}{2} u_{xyy}(x, y) \\
& \quad + \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_9, y) - \frac{h_x^3 h_y}{6} u_{xxx}(x, y) \\
& \quad \left. + \frac{h_x^2 h_y^2}{4} u_{xyy}(x, y) - \frac{h_x h_y^3}{6} u_{yyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, y) \right) \\
& + \left(u(x, y) - h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) \right) \\
& + \left. \left(u(x, y) + h_y u_y(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, y) \right) \right] \\
= & \frac{1}{2h_x h_y} \left[-u(x, y) - h_x u_x(x, y) + h_y u_y(x, y) - \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) \right. \\
& \quad - \frac{h_y^2}{2} u_{yy}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) + \frac{h_x^2 h_y}{2} u_{xxy}(x, y) - \frac{h_x h_y^2}{2} u_{xyy}(x, y) \\
& \quad + \frac{h_y^3}{6} u_{yyy}(x, y) - \frac{h_x^4}{24} u_{xxxx}(\xi_5, y) + \frac{h_x^3 h_y}{6} u_{xxx}(x, y) - \frac{h_x^2 h_y^2}{4} u_{xyy}(x, y) \\
& \quad + \frac{h_x h_y^3}{6} u_{yyy}(x, y) - \frac{h_y^4}{24} u_{yyyy}(x, y) + u(x, y) + h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) \\
& \quad + \frac{h_y^3}{6} u_{xxx}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) + u(x, y) - h_y u_y(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) \\
& \quad - \frac{h_y^3}{6} u_{yyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, y) - 2u(x, y) - u(x, y) + h_x u_x(x, y) \\
& \quad - h_y u_y(x, y) - \frac{h_x^2}{2} u_{xx}(x, y) + h_x h_y u_{xy}(x, y) - \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_x^3}{6} u_{xxx}(x, y) \\
& \quad - \frac{h_x^2 h_y}{2} u_{xxy}(x, y) + \frac{h_x h_y^2}{2} u_{xyy}(x, y) - \frac{h_y^3}{6} u_{yyy}(x, y) - \frac{h_x^4}{24} u_{xxxx}(\xi_9, y) \\
& \quad + \frac{h_x^3 h_y}{6} u_{xxx}(x, y) - \frac{h_x^2 h_y^2}{4} u_{xyy}(x, y) + \frac{h_x h_y^3}{6} u_{yyy}(x, y) \\
& \quad - \frac{h_y^4}{24} u_{yyyy}(x, y) + u(x, y) - h_x u_x(x, y) + \frac{h_x^2}{2} u_{xx}(x, y) - \frac{h_x^3}{6} u_{xxx}(x, y) \\
& \quad + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) + u(x, y) + h_y u_y(x, y) + \frac{h_y^2}{2} u_{yy}(x, y) + \frac{h_y^3}{6} u_{yyy}(x, y) \\
& \quad \left. + \frac{h_y^4}{24} u_{yyyy}(x, y) \right] \\
= & \frac{1}{2h_x h_y} \left[2h_x h_y u_{xy}(x, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_1, y) + \frac{h_y^4}{24} u_{yyyy}(x, y) + \frac{h_y^4}{24} u_{yyyy}(x, y) \right. \\
& \quad \left. + \frac{h_x^4}{24} u_{xxxx}(\xi_2, y) + \frac{h_x^4}{24} u_{xxxx}(\xi_5, y) - \frac{h_x^3 h_y}{6} u_{xxx}(x, y) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_7, \eta_4) - \frac{h_x h_y^3}{6} u_{xyyy}(\xi_8, \eta_5) + \frac{h_y^4}{24} u_{yyyy}(x, \eta_6) \\
& - \frac{h_x^4}{24} u_{xxxx}(\xi_9, y) + \frac{h_x^3 h_y}{6} u_{xxxy}(\xi_{10}, \eta_7) - \frac{h_x^2 h_y^2}{4} u_{xxyy}(\xi_{11}, \eta_8) \\
& + \frac{h_x h_y^3}{6} u_{xyyy}(\xi_{12}, \eta_9) - \frac{h_y^4}{24} u_{yyyy}(x, \eta_{10}) \Big] \\
= & u_{xy}(x, y) + \frac{h_x^3}{48h_y} u_{xxxx}(\xi_1, y) + \frac{h_y^3}{48h_x} u_{yyyy}(x, \eta_2) + \frac{h_y^3}{48h_x} u_{yyyy}(x, \eta_1) \\
& + \frac{h_x^3}{48h_y} u_{xxxx}(\xi_2, y) + \frac{h_x^3}{48h_y} u_{xxxx}(\xi_5, y) - \frac{h_x^2}{12} u_{xxxy}(\xi_6, \eta_3) \\
& + \frac{h_x h_y}{8} u_{xxyy}(\xi_7, \eta_4) - \frac{h_y^2}{12} u_{xyyy}(\xi_8, \eta_5) + \frac{h_y^3}{48h_x} u_{yyyy}(x, \eta_6) \\
& - \frac{h_x^3}{48h_y} u_{xxxx}(\xi_9, y) + \frac{h_x^2}{12} u_{xxxy}(\xi_{10}, \eta_7) - \frac{h_x h_y}{8} u_{xxyy}(\xi_{11}, \eta_8) \\
& + \frac{h_y^2}{12} u_{xyyy}(\xi_{12}, \eta_9) - \frac{h_y^3}{48h_x} u_{yyyy}(x, \eta_{10})
\end{aligned}$$

Thus $\widehat{\delta}_{x,y;\mathbf{h}}^2 u$ is a second-order accurate approximation of u_{xy} .

Combining the above results, we can see that the discrete Hessians $\widetilde{D}_{\mathbf{h}}^2$, $\widehat{D}_{\mathbf{h}}^2$, $\overline{D}_{\mathbf{h}}^2$, $\overline{\overline{D}}_{\mathbf{h}}^2$ are all second-order accurate since they are all defined in terms of $\widetilde{\delta}_{xx}^2$, $\widehat{\delta}_{xx}^2$, $\widetilde{\delta}_{yy}^2$, $\widehat{\delta}_{yy}^2$, $\widetilde{\delta}_{xy}^2$, $\widehat{\delta}_{xy}^2$.

CHAPTER 4: NUMERICAL METHODS

This chapter formulates a class of narrow-stencil methods as well as a more general class of methods inspired by the generalized monotone methods of Feng and Lewis in [6]. In total, we are going to consider five different methods for approximating solutions to (1.1).

4.1. A Particular Class of Narrow-Stencil Methods

The first class of methods corresponds to simply replacing the Hessian with a discrete Hessian in (1.1). In particular, we seek a grid function $U_\alpha : \mathbb{N}'_N \rightarrow \mathbb{R}$ such that

$$\mathcal{L}_\mathbf{h} U_\alpha \equiv -A(\mathbf{x}_\alpha) : D_\mathbf{h}^2 U_\alpha + c(\mathbf{x}_\alpha) U_\alpha = f(\mathbf{x}_\alpha) \quad \text{for } \mathbf{x}_\alpha \in \mathcal{T}_\mathbf{h} \cap \Omega, \quad (4.1a)$$

$$U_\alpha = g(\mathbf{x}_\alpha) \quad \text{for } \mathbf{x}_\alpha \in \mathcal{T}_\mathbf{h} \cap \partial\Omega, \quad (4.1b)$$

$$\delta_{x_i, h_i}^2 U_\alpha = 0 \quad \text{for } \mathbf{x}_\alpha \in \mathcal{S}_{h_i} \subset \mathcal{T}_\mathbf{h} \cap \partial\Omega, \quad (4.1c)$$

where $D_\mathbf{h}^2 \in \{\widehat{D}_\mathbf{h}^2, \overline{D}_\mathbf{h}^2, \overline{\overline{D}}_\mathbf{h}^2\}$ and the set \mathcal{S}_{h_i} is defined by

$$\mathcal{S}_{h_i} \equiv \{\mathbf{x}_\alpha \in \mathcal{T}_\mathbf{h} \cap \partial\Omega \mid \mathbf{x}_\alpha + h_i \mathbf{e}_i \in \mathcal{T}_\mathbf{h} \cap \Omega \text{ or } \mathbf{x}_\alpha - h_i \mathbf{e}_i \in \mathcal{T}_\mathbf{h} \cap \Omega\} \quad (4.2)$$

for all $i \in \{1, 2\}$. The auxiliary boundary condition (4.1c) is used to define ghost values when choosing $D_\mathbf{h}^2 = \overline{D}_\mathbf{h}^2$. We can also use (5.1b), discussed in the next chapter, to attempt to partially eliminate the boundary layer error associated with (4.1c) whenever $u_{x_i x_i} \neq 0$ along $\partial\Omega$.

- * METHOD 1: Choose $D_{\mathbf{h}}^2 = \widehat{D}_{\mathbf{h}}^2$, defined by (3.5b).
- * METHOD 2: Choose $D_{\mathbf{h}}^2 = \overline{D}_{\mathbf{h}}^2$, defined by (3.5c).
- * METHOD 3: Choose $D_{\mathbf{h}}^2 = \overline{\overline{D}}_{\mathbf{h}}^2$ defined by (3.3d), which is built using the operator $\widehat{\delta}_{x_i, h_i}^2$ on the diagonal and $\overline{\delta}_{x, y; \mathbf{h}}^2$ otherwise.

4.2. A General Class of Narrow-Stencil Methods

For more generality, we will also consider schemes with $\mathcal{L}_{\mathbf{h}}$ defined by

$$\mathcal{L}_{\mathbf{h}} U_{\alpha} \equiv -A(\mathbf{x}_{\alpha}) : D_{\mathbf{h}}^2 U_{\alpha} + c(\mathbf{x}_{\alpha}) U_{\alpha} + M(\mathbf{x}_{\alpha}) : (\widetilde{D}_{\mathbf{h}}^2 - \widehat{D}_{\mathbf{h}}^2) U_{\alpha}, \quad (4.3)$$

where $M : \Omega \rightarrow \mathbb{R}^{2 \times 2}$ is a symmetric matrix with non-negative components. The term $M(\mathbf{x}_{\alpha}) : (\widetilde{D}_{\mathbf{h}}^2 - \widehat{D}_{\mathbf{h}}^2) U_{\alpha}$ is called a numerical moment, and it acts as a stabilization term. We apply this to our last two methods.

- * METHOD 4: Choose $D_{\mathbf{h}}^2 \equiv \check{D}_{\mathbf{h}}^2$, for $\check{D}_{\mathbf{h}}^2$ defined by

$$[\check{D}_{\mathbf{h}}^2]_{ij}(\mathbf{x}_{\alpha}) \equiv \begin{cases} [\widetilde{D}_{\mathbf{h}}^2]_{ij}(\mathbf{x}_{\alpha}), & \text{if } a_{ij} > 0, \\ [\widehat{D}_{\mathbf{h}}^2]_{ij}(\mathbf{x}_{\alpha}), & \text{if } a_{ij} \leq 0 \end{cases} \quad \forall \mathbf{x}_{\alpha} \in \mathcal{T}_{\mathbf{h}} \cap \Omega.$$

We refer to this method as Godunov-like. Note that this method also corresponds to choosing $D_{\mathbf{h}}^2 = \overline{D}_{\mathbf{h}}^2$ and $M(\mathbf{x}_{\alpha}) = \max |A_{ij}(\mathbf{x}_{\alpha})|$ in (4.3). This method naturally fits into the generalized monotone framework of Feng and Lewis in [6].

- * METHOD 5: Choose $D_{\mathbf{h}}^2 = \overline{D}_{\mathbf{h}}^2$ and the numerical moment with

$$M = \gamma \mathbf{1}_{2 \times 2}, \quad \text{where} \quad \gamma \geq \frac{1}{2} \sup_{(x, y) \in \Omega} \|A_{ij}(x, y)\|_{\max}.$$

This method corresponds to the Lax-Friedrich's-like method of Feng and Lewis in [6].

CHAPTER 5: NUMERICAL TESTS

We seek to test the accuracy of the various proposed methods. We will use two auxiliary boundary conditions in our numerical tests to see how much of an effect the potential boundary layer error has on the interior when using \bar{D}_h^2 and defining the required ghost values. To this end, we assume one of the two following boundary conditions for all $\mathbf{x}_\alpha \in \mathcal{S}_{h_i}$,

$$\delta_{x_i, h_i}^2 U_\alpha = 0, \quad \text{or} \quad (5.1a)$$

$$\delta_{x_i, h_i}^2 U_\alpha = \delta_{x_i, h_i}^2 U_{\alpha'}, \quad (5.1b)$$

where $\mathbf{x}_{\alpha'}$ is the immediate neighbor in the x_i direction with $\mathbf{x}_{\alpha'} \in \mathcal{T}_h \cap \Omega$. We will refer to (5.1a) as BC 1, and (5.1b) as BC 2. Since we cannot set $\delta_{x_i, h_i}^2 U_\alpha = u_{x_i, x_i}(\mathbf{x}_\alpha)$, the boundary conditions can create a boundary layer error that propagates into the interior. We will measure errors on $\mathcal{T}_h \cap \Omega$, as well as the restricted mesh \mathcal{T}_h° to see if the increased error mostly restricts itself towards the boundary. As shown in Section 8.5, BC 2 yields smaller truncation errors than BC 1. Note that the \hat{D}_h^2 and \bar{D}_h^2 operators on their own do not require any ghost points; thus, an auxiliary boundary condition is not required.

We consider the norm $\|\cdot\|_2$ defined by

$$\|V\|_2 \equiv \left(h_x h_y \right)^{\frac{1}{2}} \|V\|_{\ell^2(\mathcal{T}_h \cap \Omega)} \quad (5.2)$$

for any grid function V . We also consider the error when removing nodes adjacent to

the boundary by measuring

$$\|V\|_{\dot{2}} \equiv \left(h_x h_y \right)^{\frac{1}{2}} \|V\|_{\ell^2(\mathcal{T}_h \cap \Omega)} . \quad (5.3)$$

All tests will correspond to $\Omega = (-1, 1) \times (-1, 1)$ and $c = 0$. We choose various solutions and various coefficient matrices A . We form a non-singular linear system $MU = F$ that is solved using MATLAB's backslash command. The matrix M is formed using sparse storage. We measure the rate of convergence in $\|\cdot\|_2$ by increasing $N_x = N_y$. We also consider the truncation errors for the various difference operators. We mention the five different exact solutions we will consider in Section 5.1 and the various matrices A in Sections 5.2 – 5.5. Test results can be found in Chapter 8. Overall, we observe optimal or near optimal performance when A is SPD.

5.1. Solutions

In this section we mention the five different solutions we will consider in the numerical tests.

- * SOLUTION 1: $u(x, y) = \sin\left(\frac{\pi}{2}(x + y)^2\right)$

Using this solution, the methods based on \widehat{D}_h^2 are of (or very close to) order 2, while methods based on \overline{D}_h^2 have reduced order because of the boundary issues. BC 2 for methods based on \overline{D}_h^2 gives us a rate that is much closer to being second-order. Truncation error results for this solution can be found in Section 8.5.

- * SOLUTION 2: $u(x, y) = x^2 + 3xy + \frac{1}{2}y^2 + 3$

Since u is quadratic, we see that all methods based on \widehat{D}_h^2 are exact, whereas the methods based on \overline{D}_h^2 only achieve exact results when using BC 2. Truncation error results for this solution can be found in Section 8.5.

* SOLUTION 3: $u(x, y) = e^{xy+2y}$

This solution gives the same general results as found for Solution 1. Truncation error results for this solution can be found in Section 8.5.

* SOLUTION 4: $u(x, y) = \frac{x^3}{18} \left(3 \log(x^2) - 11 \right) + \left(y - \frac{1}{2} \right)^{\frac{8}{3}} \sqrt{\left| x + \frac{1}{5} \right|^5}$

This solution has less regularity due to the third derivative blow-up, and thus does not achieve second-order convergence. Truncation error results for this solution can be found in Section 8.5.

* SOLUTION 5: $u(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$

The solution is not C^2 and must be understood in the viscosity sense. The solution is only used for one particular test problem. As expected, the convergence rates are sub-optimal. See Section 5.5 for more information.

5.2. Test 1: Degenerate Matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

A is chosen so that it is symmetric non-negative-definite. Further, plugging A into (1.1), this test corresponds to solving

$$-u_{xx} - 2u_{xy} - u_{yy} = f.$$

We consider using solutions 1–4 as our choices for the exact solution u . The results for this test can be found in Section 8.1. Note that the analysis in Chapter 6 does not fully apply since A is not strictly positive-definite. However, the problem is elliptic since it corresponds to solving $-u_{\eta\eta} = f$ for η in the direction $x + y$.

5.3. Test 2: Discontinuous Matrix

$$A(x, y) = \begin{bmatrix} 2 + |\sin(6\pi x)| & -|xy| - |x+y| - \sqrt{|x-y|} \\ -|xy| - |x+y| - \sqrt{|x-y|} & 4 + 4(e^x + e^y + e^{-x} + e^{-y}) - \text{sign}(\cos(6\pi(x+y))) \end{bmatrix}$$

Each component of this matrix is non-differentiable, and there are several points where the A_{22} component is discontinuous. Because this matrix varies so much, when using method 5 we set $\gamma = 18$. The results for this test using solutions 1-4 can be found in Section 8.2.

5.4. Test 3: Wide-Stencil Problem

We choose a matrix $A = Q\Lambda Q^T$ that does not align with any of our test grids.

Define Λ by

$$\Lambda(x, y) = \begin{bmatrix} 2 - \sin(e^x) \cos(e^{-y}) & 0 \\ 0 & 2 - \text{sign}(\cos(6\pi x) \sin(6\pi y)) \end{bmatrix},$$

where $\text{sign}(\cos(6\pi x) \sin(6\pi y)) \in \{-1, 0, 1\}$.

Based on our finest mesh, with $N_x = N_y = 140$, we choose $h_x = h_y = \frac{2}{141}$.

Define

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{h_y}{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} h_x \\ 5h_y \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 10h_x \\ h_y \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} \frac{h_x}{2} \\ 2 \end{bmatrix},$$

and let $\mathbf{q}_1^{(i)} = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|_2}$ for $i = 1, 2, 3, 4$. Note that \mathbf{v}_1 and \mathbf{v}_4 do not align with grid points while \mathbf{v}_2 requires nodes five layers away and \mathbf{v}_3 requires nodes ten layers away.

Choose $\mathbf{q}_2^{(i)}$ to be unit length and orthogonal to \mathbf{v}_i , and define the orthogonal matrices

$Q_i = [\mathbf{q}_1^{(i)}, \mathbf{q}_2^{(i)}]$ for $i = 1, 2, 3, 4$. Finally, we define

$$A(x, y) = \begin{cases} Q_1 \Lambda(x, y) Q_1^T, & \text{for } x \geq 0, y \geq 0, \\ Q_2 \Lambda(x, y) Q_2^T, & \text{for } x < 0, y \geq 0, \\ Q_3 \Lambda(x, y) Q_3^T, & \text{for } x < 0, y < 0, \\ Q_4 \Lambda(x, y) Q_4^T, & \text{otherwise.} \end{cases}$$

By the choice of Q_1 or Q_4 no monotone method exists for this problem. The results for this test can be found in Section 8.3 where we observe optimal performance for our narrow-stencil methods.

5.5. Test 4: Low Regularity Solution

$$A(x, y) = \frac{16}{9} \begin{bmatrix} x^{2/3} & -x^{1/3}y^{1/3} \\ -x^{1/3}y^{1/3} & y^{2/3} \end{bmatrix}$$

and $f(x)$ and Dirichlet boundary data are chosen such that the solution is given by solution 5, i.e. $u(x, y) = x^{4/3} - y^{4/3}$. Even values for $N_x = N_y$ are used in order to avoid $x = 0, y = 0$ as grid components so that A is not the zero matrix. The problem is degenerate elliptic, and the exact solution is not in $C^2(\Omega)$. Note that the auxiliary boundary condition has little to no effect on the performance. The results for this test can be found in Section 8.4.

CHAPTER 6: ANALYSIS OF THE SCHEMES

6.1. Admissibility

The goal of this chapter is to show that the proposed narrow-stencil schemes (4.1) and (4.3) have a unique uniformly bounded solution in a weighted ℓ^2 -norm. The idea for proving the well-posedness and stability of the methods is to equivalently reformulate the proposed schemes as a fixed point problem and to prove the mapping is contractive in the ℓ^2 -norm. To this end, let $S(\mathcal{T}'_{\mathbf{h}})$ denote the space of all grid functions on $\mathcal{T}'_{\mathbf{h}}$, and introduce the mapping $\mathcal{M}_\rho : S(\mathcal{T}'_{\mathbf{h}}) \rightarrow S(\mathcal{T}'_{\mathbf{h}})$ defined by

$$\widehat{U} \equiv \mathcal{M}_\rho U, \quad (6.1)$$

where the grid function $\widehat{U} \in S(\mathcal{T}'_{\mathbf{h}})$ is defined by

$$\widehat{U}_\alpha = U_\alpha - \rho [\mathcal{L}_{\mathbf{h}} U_\alpha - f(\mathbf{x}_\alpha)], \quad \text{if } \mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \Omega, \quad (6.2a)$$

$$\widehat{U}_\alpha = g(\mathbf{x}_\alpha), \quad \text{if } \mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \partial\Omega, \quad (6.2b)$$

$$\delta_{x_i, h_i}^2 \widehat{U}_\alpha = 0 \quad \text{if } \mathbf{x}_\alpha \in \mathcal{S}_{h_i} \subset \mathcal{T}_{\mathbf{h}} \cap \partial\Omega \quad (6.2c)$$

for $\rho > 0$ an undetermined constant. The fixed-point technique is used to account for the lack of symmetry caused by the coefficient matrix A when it is not constant-valued. The technique will also yield an immediate stability result. For transparency, we will only consider the auxiliary boundary condition (5.1a). The result in this chapter can be extended to the alternate auxiliary boundary condition (5.1b).

6.1.1. Admissibility for Constant-Valued Coefficient Matrices

Assume A in (1.1) is constant-valued over Ω . We show that the scheme is equivalent to solving a linear system $MU = F$ where the matrix M is symmetric positive-definite. Notationally, we define the partial ordering for symmetric matrices by $M_1 \geq M_2$ if $M_1 - M_2$ is non-negative-definite for any symmetric matrices M_1, M_2 .

Let $\lambda_0 > 0$ denote the smallest eigenvalue of A . Define $A_0 \equiv A - \lambda_0 I$. Then, A_0 is symmetric non-negative-definite. Thus, there exists an eigenvalue decomposition $A_0 = Q\Lambda Q^T = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T$, where $\lambda_k \geq 0$ for $k \in \{1, 2\}$ and $Q = [\mathbf{q}_1, \mathbf{q}_2]$ for $\{\mathbf{q}_1, \mathbf{q}_2\}$ an orthonormal basis for \mathbb{R}^2 . Define $\mathbf{q}_k = \sum_{i=1}^2 b_i^{(k)} \mathbf{e}_i$, and observe that

$$\mathbf{q}_k \mathbf{q}_k^T = \sum_{i=1}^2 \left(b_i^{(k)} \right)^2 \mathbf{e}_i \mathbf{e}_i^T + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 b_i^{(k)} b_j^{(k)} \mathbf{e}_i \mathbf{e}_j^T.$$

Thus,

$$\begin{aligned} A_0 &= \sum_{k=1}^2 \lambda_k \left[\sum_{i=1}^2 \left(b_i^{(k)} \right)^2 \mathbf{e}_i \mathbf{e}_i^T + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 b_i^{(k)} b_j^{(k)} \mathbf{e}_i \mathbf{e}_j^T \right] \\ &= \sum_{i=1}^2 \left[\sum_{k=1}^2 \lambda_k \left(b_i^{(k)} \right)^2 \right] \mathbf{e}_i \mathbf{e}_i^T + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 \left[\sum_{k=1}^2 \lambda_k b_i^{(k)} b_j^{(k)} \right] \mathbf{e}_i \mathbf{e}_j^T, \end{aligned}$$

and it follows that

$$[A_0]_{ii} = a_{ii} - \lambda_0 = \sum_{k=1}^2 \lambda_k \left(b_i^{(k)} \right)^2, \quad [A_0]_{ij} = a_{ij} = \sum_{k=1}^2 \lambda_k b_i^{(k)} b_j^{(k)}$$

for all $i, j \in \{1, 2\}$ with $j \neq i$.

Let $N_0 = |\mathcal{T}_h \cap \Omega|$, $D_i \in \mathbb{R}^{N_0 \times N_0}$ denote the matrix representation of $\bar{\delta}_{x_i, h_i}$, $D_i^\pm \in \mathbb{R}^{N_0 \times N_0}$ denote the matrix representation of δ_{x_i, h_i}^\pm , $L_1 \in \mathbb{R}^{N_0 \times N_0}$ denote the matrix representation of $-\sum_{i=1}^2 \delta_{x_i, h_i}^2$, $L_2 \in \mathbb{R}^{N_0 \times N_0}$ denote the matrix representation of $-\sum_{i=1}^2 \delta_{x_i, 2h_i}^2$, and $C \in \mathbb{R}^{N_0 \times N_0}$ denote the matrix representation of $c(\mathbf{x}_\alpha) \geq 0$.

Then $(D_i^\pm)^T = -D_i^\mp$, $(D_i)^T = -D_i$, and C is a diagonal matrix with non-negative entries. Furthermore, there exists diagonal matrices $B_i^{(1)}, B_i^{(2)} \in \mathbb{R}^{N_0 \times N_0}$ for $i \in \{1, 2\}$ with non-negative components such that

$$\begin{aligned} L_1 &= -\sum_{i=1}^2 \left[\frac{1}{2} (D_i^+ D_i^- + D_i^- D_i^+) - B_i^{(1)} \right] \\ &\geq -\sum_{i=1}^2 \frac{1}{2} (D_i^+ D_i^- + D_i^- D_i^+) = \sum_{i=1}^2 \frac{1}{2} \left((D_i^-)^T D_i^- + (D_i^+)^T D_i^+ \right) \end{aligned}$$

and

$$L_2 = -\sum_{i=1}^2 (D_i D_i - B_i^{(2)}) \geq -\sum_{i=1}^2 D_i D_i = \sum_{i=1}^2 (D_i)^T D_i.$$

The positive components of $B_i^{(1)}$ and $B_i^{(2)}$ correspond to nodes \mathbf{x}_α near the boundary. Such corrections are needed in the matrix form to account for the values of $\delta_{x_i, h_i}^\pm U_{\alpha'}$ when $\mathbf{x}_{\alpha'} \in \mathcal{T}_h \cap \partial\Omega$. For example, suppose $\mathbf{x}_\alpha - h_i \mathbf{e}_i \in \partial\Omega$. Then

$$\begin{aligned} \delta_{x_i, h_i}^+ \delta_{x_i, h_i}^- U_\alpha &= \frac{1}{h_i} \delta_{x_i, h_i}^+ (U_\alpha - U_{\alpha-\mathbf{e}_i}) = \frac{1}{h_i} \delta_{x_i, h_i}^+ U_\alpha - \frac{1}{h_i} \delta_{x_i, h_i}^+ U_{\alpha-\mathbf{e}_i} \\ &= \frac{1}{h_i} \delta_{x_i, h_i}^+ U_\alpha - \frac{1}{h_i^2} U_\alpha + \frac{1}{h_i^2} u(\mathbf{x}_\alpha - h_i \mathbf{e}_i). \end{aligned}$$

However, when computing using the matrix representation, D_i^- treats the boundary value $U_{\alpha-\mathbf{e}_i}$ as a known value in its representation and removes it when calculating $D_i^- U$. Consequently, D_i^+ does not act on the boundary node leading to a smaller coefficient for the adjacent interior node involved in the calculation of $\delta_{x_i, h_i}^+ U_{\alpha-\mathbf{e}_i}$. Thus, L_1 would contain the contribution $\frac{1}{h_i^2}$ to the coefficient for U_α , while $-D_i^+ D_i^-$ would not. The representation for L_2 has a similar issue when accounting for the boundary values defined by (4.1b) and ghost values defined by (4.1c). Indeed, (4.1c)

ensures that the ghost value $U_{\alpha \pm 2\mathbf{e}_i}$ satisfies

$$\frac{1}{2h_i^2}U_{\alpha \pm 2\mathbf{e}_i} = \frac{1}{h_i^2}u(\mathbf{x}_{\alpha \pm \mathbf{e}_i}) - \frac{1}{2h_i^2}U_\alpha$$

for $\mathbf{x}_{\alpha \pm \mathbf{e}_i} \in \mathcal{T}_h \cap \partial\Omega$, where $\frac{1}{2h_i^2}U_{\alpha \pm 2\mathbf{e}_i}$ is directly involved in the computation of $\tilde{\delta}_{x_i, h_i}^2 U_\alpha$.

Lastly, note that correction terms are not needed when considering the relationship of $\delta_{x_i, h_i}^\pm \delta_{x_j, h_j}^\mp$ to $D_i^\pm D_j^\mp$ for $i \neq j$ since the computation of $\delta_{x_i, h_i}^\pm U_{\alpha'}$ would only include boundary nodes whenever $\mathbf{x}_\alpha \pm h_j \mathbf{e}_j \in \mathcal{T}_h \cap \partial\Omega$.

Suppose D_h^2 is chosen to be \bar{D}_h^2 in (4.1). Observe that

$$\begin{aligned} -A_0 : \bar{D}_h^2 &= -\sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} \bar{\delta}_{x_i, h_i} \bar{\delta}_{x_j, h_j} \\ &= -\sum_{i=1}^2 (a_{ii} - \lambda_0) \bar{\delta}_{x_i, h_i} \bar{\delta}_{x_i, h_i} - \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 a_{ij} \bar{\delta}_{x_i, h_i} \bar{\delta}_{x_j, h_j} \\ &= -\sum_{i=1}^2 \sum_{k=1}^2 \lambda_k \left(b_i^{(k)} \right)^2 \bar{\delta}_{x_i, h_i} \bar{\delta}_{x_i, h_i} - \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 \left[\sum_{k=1}^2 \lambda_k b_i^{(k)} b_j^{(k)} \right] \bar{\delta}_{x_i, h_i} \bar{\delta}_{x_j, h_j} \\ &= -\sum_{k=1}^2 \lambda_k \left[\sum_{i=1}^2 \left(b_i^{(k)} \right)^2 \bar{\delta}_{x_i, h_i} \bar{\delta}_{x_i, h_i} + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 b_i^{(k)} b_j^{(k)} \bar{\delta}_{x_i, h_i} \bar{\delta}_{x_j, h_j} \right] \\ &= -\sum_{k=1}^2 \lambda_k \left(\left[\sum_{i=1}^2 b_i^{(k)} \bar{\delta}_{x_i, h_i} \right] \left[\sum_{i=1}^2 b_i^{(k)} \bar{\delta}_{x_i, h_i} \right] \right), \end{aligned}$$

and it follows that

$$\begin{aligned} 0_{N_0 \times N_0} &\leq \sum_{k=1}^2 \lambda_k \left(\left[\sum_{i=1}^2 b_i^{(k)} D_i \right]^T \left[\sum_{i=1}^2 b_i^{(k)} D_i \right] \right) \\ &= \sum_{i=1}^2 \sum_{k=1}^2 \lambda_k \left(b_i^{(k)} \right)^2 D_i^T D_i + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 \left[\sum_{k=1}^2 \lambda_k b_i^{(k)} b_j^{(k)} \right] D_i^T D_j \\ &= \sum_{i=1}^2 (a_{ii} - \lambda_0) D_i^T D_i + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 a_{ij} D_i^T D_j = \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} D_i^T D_j. \end{aligned}$$

Using the fact that $D_i^T = -D_i$, there holds

$$\begin{aligned}
M &\equiv - \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} D_i D_j + \sum_{i=1}^2 a_{ii} B_i^{(2)} + C \\
&= -\lambda_0 \sum_{i=1}^2 \left(D_i^2 - B_i^{(2)} \right) - \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} D_i D_j + \sum_{i=1}^2 (a_{ii} - \lambda_0) B_i^{(2)} + C \\
&= \lambda_0 L_2 + \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} D_i^T D_j + \sum_{i=1}^2 (a_{ii} - \lambda_0) B_i^{(2)} + C \\
&\geq \lambda_0 L_2 > 0_{N_0 \times N_0}.
\end{aligned} \tag{6.3}$$

Similarly, for $D_{\mathbf{h}}^2 = \widehat{D}_{\mathbf{h}}^2$ in (4.1), there holds

$$\begin{aligned}
-A_0 : \widehat{D}_{\mathbf{h}}^2 &= -\frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} \delta_{x_i, h_i}^+ \delta_{x_j, h_j}^- - \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} \delta_{x_i, h_i}^- \delta_{x_j, h_j}^+ \\
&= -\frac{1}{2} \sum_{k=1}^2 \lambda_k \left(\left[\sum_{i=1}^2 b_i^{(k)} \delta_{x_i, h_i}^- \right] \left[\sum_{i=1}^2 b_i^{(k)} \delta_{x_i, h_i}^+ \right] \right) \\
&\quad - \frac{1}{2} \sum_{k=1}^2 \lambda_k \left(\left[\sum_{i=1}^2 b_i^{(k)} \delta_{x_i, h_i}^+ \right] \left[\sum_{i=1}^2 b_i^{(k)} \delta_{x_i, h_i}^- \right] \right).
\end{aligned}$$

Then

$$\begin{aligned}
0_{N_0 \times N_0} &\leq \sum_{k=1}^2 \lambda_k \left(\left[\sum_{i=1}^2 b_i^{(k)} (D_i^\pm) \right]^T \left[\sum_{i=1}^2 b_i^{(k)} D_i^\pm \right] \right) \\
&= \sum_{i=1}^2 \sum_{k=1}^2 \lambda_k \left(b_i^{(k)} \right)^2 (D_i^\pm)^T D_i^\pm + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 \left[\sum_{k=1}^2 \lambda_k b_i^{(k)} b_j^{(k)} \right] (D_i^\pm)^T D_j^\pm \\
&= \sum_{i=1}^2 (a_{ii} - \lambda_0) (D_i^\pm)^T D_i^\pm + \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 a_{ij} (D_i^\pm)^T D_j^\pm = \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} (D_i^\pm)^T D_j^\pm,
\end{aligned}$$

and it follows that

$$M \equiv - \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} \frac{1}{2} \left(D_i^- D_j^+ + D_i^+ D_j^- \right) + \sum_{i=1}^2 a_{ii} B_i^{(1)} + C \tag{6.4}$$

$$\begin{aligned}
&= -\lambda_0 \sum_{i=1}^2 \left[\frac{1}{2} (D_i^- D_i^+ + D_i^+ D_i^-) - B_i^{(1)} \right] + \sum_{i=1}^2 (a_{ii} - \lambda_0) B_i^{(1)} \\
&\quad - \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} (D_i^- D_j^+ + D_i^+ D_j^-) + C \\
&= \lambda_0 L_1 + \sum_{i=1}^2 (a_{ii} - \lambda_0) B_i^{(1)} \\
&\quad + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} \left[(D_i^+)^T D_j^+ + (D_i^-)^T D_j^- \right] + C \\
&\geq \lambda_0 L_1 > 0_{N_0 \times N_0}
\end{aligned}$$

using the fact that $(D_i^\pm)^T = -D_i^\mp$.

Lastly, we consider the choice $D_h^2 = \overline{\overline{D}}_h^2$. By [6], there holds

$$-\frac{1}{2} (D_i^- D_i^+ + D_i^+ D_i^-) > -\frac{1}{2} (D_i^+ D_i^+ + D_i^- D_i^-).$$

Thus,

$$-\left(D_i^- D_i^+ + D_i^+ D_i^- \right) > -\frac{1}{2} (D_i^+ D_i^+ + D_i^- D_i^- + D_i^+ D_i^- + D_i^- D_i^+) = -2D_i^2,$$

and it follows that $-\frac{1}{2} (D_i^- D_i^+ + D_i^+ D_i^-) > -D_i^2$. Combining this with (6.3), we have

$$\begin{aligned}
M &\equiv - \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} \left[\overline{\overline{D}}_h^2 \right]_{ij} + C \tag{6.5} \\
&= \lambda_0 L_1 - \frac{1}{2} \sum_{i=1}^2 [A_0]_{ii} (D_i^- D_i^+ + D_i^+ D_i^- - 2B_i^{(1)}) - \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 [A_0]_{ij} D_i D_j + C \\
&\geq \lambda_0 L_1 + \sum_{i=1}^2 (a_{ii} - \lambda_0) B_i^{(1)} - \sum_{i=1}^2 [A_0]_{ii} D_i^2 - \sum_{i=1}^2 \sum_{\substack{j=1 \\ j \neq i}}^2 [A_0]_{ij} D_i D_j + C \\
&= \lambda_0 L_1 + \sum_{i=1}^2 (a_{ii} - \lambda_0) B_i^{(1)} + \sum_{i=1}^2 \sum_{j=1}^2 [A_0]_{ij} (D_i)^T D_j + C
\end{aligned}$$

$$\geq \lambda_0 L_1 > 0_{N_0 \times N_0}.$$

Therefore, (4.1) has a unique solution whenever A is constant-valued, and the matrix representation $MU = F$ yields a symmetric positive-definite matrix M .

Remark. The operators $\widehat{D}_{\mathbf{h}}^2$ and $\overline{D}_{\mathbf{h}}^2$ naturally appear when decomposing $A : D_{\mathbf{h}}^2$ in terms of difference operators only defined along Cartesian directions to show

$$\sum_{i=1}^2 \sum_{j=1}^2 A_{ij} D_i^T D_j \geq 0_{N_0 \times N_0}$$

and

$$\frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 A_{ij} ((D_i^+)^T D_j^+ + (D_i^-)^T D_j^-) \geq 0_{N_0 \times N_0}$$

for a constant matrix A that is symmetric non-negative-definite. In contrast, a wide-stencil method tries to directly approximate the central difference operators along the various \mathbf{q}_k directions.

6.1.2. Proving \mathcal{M}_ρ is a Contraction

To show that the mapping \mathcal{M}_ρ has a unique fixed point in $S(\mathcal{T}'_{\mathbf{h}})$, we first establish a lemma that specifies conditions under which \mathcal{M}_ρ is a contraction in ℓ^2 . The result will utilize the following result found in [6]:

Lemma 6.1. *Let $B, F \in \mathbb{R}^{N \times N}$ such that B is symmetric non-negative-definite and F is symmetric positive-definite. Define $R \in \mathbb{R}^{N \times N}$ such that R is upper triangular and $F = R^T R$. Then*

$$\|\sigma I - FB\|_2 \leq \sigma$$

for all positive constants σ such that $\sigma I > RBR^T$.

Lemma 6.2. *Assume $D_h^2 = \widehat{D}_{\mathbf{h}}^2$ in $\mathcal{L}_{\mathbf{h}}$ defined by (4.3). Suppose the operator $A \in [L^\infty(\overline{\Omega})]^{2 \times 2}$ in (1.1) is uniformly elliptic with $0 < \lambda \vec{\xi} \cdot \vec{\xi} \leq \vec{\xi} \cdot A(\mathbf{x}) \vec{\xi} \leq \Lambda \vec{\xi} \cdot \vec{\xi}$ for all*

$\vec{\xi} \neq \vec{0}$ and $\mathbf{x} \in \Omega$ and $c \geq k_0 \geq 0$ is uniformly bounded. Choose $U, V \in S(\mathcal{T}'_{\mathbf{h}})$, and let $\widehat{U} = \mathcal{M}_\rho U$ and $\widehat{V} = \mathcal{M}_\rho V$ for \mathcal{M}_ρ defined by (6.1) and (6.2). Then there holds

$$\|\widehat{U} - \widehat{V}\|_{\ell^2(\mathcal{T}_{\mathbf{h}})} \leq \left(1 - \rho \frac{\lambda\kappa}{4} - \rho \frac{k_0}{4}\right) \|U - V\|_{\ell^2(\mathcal{T}_{\mathbf{h}})}$$

for all $\rho > 0$ sufficiently small, where $\kappa > 0$ denotes the minimal eigenvalue of the matrix representation of $-\sum_{i=1}^2 \delta_{x_i, h_i}^2$ and $1 - \rho \frac{\lambda\kappa}{4} - \rho \frac{k_0}{4} > 0$.

Proof. Let $W \equiv V - U$ and $\widehat{W} \equiv \widehat{V} - \widehat{U}$. Then, by (6.2b), we can assume $\widehat{W}_\alpha = 0$ for all $\mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \partial\Omega$. By the linearity of $\mathcal{L}_{\mathbf{h}}$ and (6.2a), there holds

$$\begin{aligned} \widehat{W}_\alpha &= W_\alpha - \rho \mathcal{L}_{\mathbf{h}} W_\alpha && (6.6) \\ &= W_\alpha - \rho \left[-A(\mathbf{x}_\alpha) : \widehat{D}_{\mathbf{h}}^2 W_\alpha + c(\mathbf{x}_\alpha) W_\alpha + M(\mathbf{x}_\alpha) : (\widetilde{D}_{\mathbf{h}}^2 - \widehat{D}_{\mathbf{h}}^2) W_\alpha \right] \\ &= W_\alpha - \rho \left[-\left(A(\mathbf{x}_\alpha) - \frac{\lambda}{2} I_{2 \times 2}\right) : \widehat{D}_{\mathbf{h}}^2 W_\alpha - \frac{\lambda}{2} I_{2 \times 2} : \widehat{D}_{\mathbf{h}}^2 W_\alpha + c(\mathbf{x}_\alpha) W_\alpha \right. \\ &\quad \left. + M(\mathbf{x}_\alpha) : (\widetilde{D}_{\mathbf{h}}^2 - \widehat{D}_{\mathbf{h}}^2) W_\alpha \right] \\ &= (1 - \rho c(\mathbf{x}_\alpha)) W_\alpha + \rho \frac{\lambda}{2} \sum_{i=1}^2 \delta_{x_i, h_i}^2 W_\alpha \\ &\quad - \rho \left[-\left(A(\mathbf{x}_\alpha) - \frac{\lambda}{2} I_{2 \times 2}\right) : \widehat{D}_{\mathbf{h}}^2 W_\alpha + M(\mathbf{x}_\alpha) : (\widetilde{D}_{\mathbf{h}}^2 - \widehat{D}_{\mathbf{h}}^2) W_\alpha \right] \end{aligned}$$

for all $\mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \Omega$.

Let $N_0 = |\mathcal{T}_{\mathbf{h}} \cap \Omega|$, $L_1 \in \mathbb{R}^{N_0 \times N_0}$ denote the matrix representation of $-\sum_{i=1}^2 \delta_{x_i, h_i}^2$, and $C \in \mathbb{R}^{N_0 \times N_0}$ denote the matrix representation of $c(\mathbf{x}_\alpha) \geq 0$. Then $L_1 \geq \kappa I$ and $C \geq k_0 I$.

We next utilize the frozen coefficient technique. Define the matrices $A_\alpha, M_\alpha \in \mathbb{R}^{N_0 \times N_0}$ as the matrix representations of $-(A(\mathbf{x}_\alpha) - \frac{\lambda}{2} I_{2 \times 2}) : \widehat{D}_{\mathbf{h}}^2$ and $M(\mathbf{x}_\alpha) : (\widetilde{D}_{\mathbf{h}}^2 - \widehat{D}_{\mathbf{h}}^2)$, respectively. Then A_α and M_α are symmetric non-negative-definite, where the properties of A_α follow from Section 6.1.1 and the properties of M_α follow from [6] which

shows that each component of $(\tilde{D}_{\mathbf{h}}^2 - \hat{D}_{\mathbf{h}}^2)$ corresponds to a symmetric positive-definite operator as well as the assumption that the coefficient matrix $M(\mathbf{x}_\alpha)$ has all non-negative components.

Define $E_k \in \mathbb{R}^{N_0 \times N_0}$ by $[E_k]_{ij} = 1$ only if $i = j = k$ and 0 otherwise. Notationally, we let $\alpha(k)$ be the multi-index corresponding to the single-index k . Then, (6.6) can be written in matrix form by

$$\widehat{\mathbf{W}} = (I - \rho C)\mathbf{W} - \rho \frac{\lambda}{2} L_1 \mathbf{W} - \rho \sum_{k=1}^{N_0} E_k (A_{\alpha(k)} + M_{\alpha(k)}) \mathbf{W} = G\mathbf{W}$$

for the iteration matrix $G \in \mathbb{R}^{N_0 \times N_0}$ defined by

$$G \equiv (I - \rho C) - \rho \frac{\lambda}{2} L_1 - \rho \sum_{k=1}^{N_0} E_k (A_{\alpha(k)} + M_{\alpha(k)}).$$

Choose $\epsilon > 0$. Let $N' = 2 + 2N_0$. Observe that $E_k + \epsilon I$ is symmetric positive-definite for all k . Choose K such that $0_{N_0 \times N_0} \leq \sum_{k=1}^{N_0} (A_{\alpha(k)} + M_{\alpha(k)}) \leq KI$ using the fact that $A_{\alpha(k)}$ and $M_{\alpha(k)}$ are symmetric non-negative-definite for all indices k . Then, by Lemma 6.1, there holds

$$\begin{aligned} \|G\|_2 &\leq \frac{1}{2} + \left\| \frac{1}{2N'} I - \rho C \right\|_2 + \left\| \frac{1}{2N'} I - \rho \frac{\lambda}{2} L_1 \right\|_2 \\ &\quad + \sum_{k=1}^{N_0} \left\| \frac{1}{2N'} I - \rho(E_k + \epsilon I) A_{\alpha(k)} \right\|_2 + \sum_{k=1}^{N_0} \left\| \frac{1}{2N'} I - \rho(E_k + \epsilon I) M_{\alpha(k)} \right\|_2 \\ &\quad + \rho \epsilon \sum_{k=1}^{N_0} \|A_{\alpha(k)} + M_{\alpha(k)}\|_2 \\ &\leq \frac{1}{2} + \frac{1}{2N'} - \rho k_0 + \frac{1}{2N'} - \rho \frac{\lambda}{2} \kappa + 2 \sum_{k=1}^{N_0} \frac{1}{2N'} + \rho \epsilon K \\ &= \frac{1}{2} + \frac{2 + 2N_0}{2N'} - \rho k_0 - \rho \frac{\lambda}{2} \kappa + \rho \epsilon K \\ &< \frac{1}{2} + \frac{1}{2} - \rho \left(k_0 + \frac{\lambda \kappa}{2} - \frac{k_0 + \lambda \kappa}{4} \right) < 1 - \rho \frac{3}{4} k_0 - \rho \frac{1}{4} \lambda \kappa \end{aligned}$$

for all $\rho > 0$ sufficiently small and $\epsilon < \frac{k_0 + \lambda\kappa}{4K}$ chosen independently of ρ . The bound

$$\|\widehat{W}\|_{\ell^2(\mathcal{T}_h \cap \Omega)} \leq \left(1 - \rho \frac{\lambda\kappa}{4} - \rho \frac{k_0}{4}\right) \|W\|_{\ell^2(\mathcal{T}_h \cap \Omega)}$$

follows since $\|\widehat{\mathbf{W}}\|_2 \leq \|G\|_2 \|\mathbf{W}\|_2$, and the bound over \mathcal{T}_h follows since $W_\alpha = 0$ over $\mathcal{T}_h \cap \partial\Omega$. The proof is complete. \square

Remark. The proof can trivially be extended to the choices $D_h^2 = \overline{D}_h^2$ and $D_h^2 = \overline{\overline{D}}_h^2$. When choosing $D_h^2 = \overline{D}_h^2$, the value for κ is determined by the minimal eigenvalue of the matrix representation of $-\sum_{i=1}^2 \delta_{x_i, 2h_i}^2$.

6.2. Stability

6.2.1. Existence, Uniqueness, and Stability

We now use the result in Section 6.1.2 to show that the simple finite difference scheme has a unique solution that is uniformly bounded in the weighted ℓ^2 -norm. As an immediate corollary to Lemma 6.2, we have the following well-posedness result by the Contractive Mapping Theorem.

Theorem 6.3. *Suppose the operator $A \in [L^\infty(\bar{\Omega})]^{2 \times 2}$ in (1.1) is uniformly elliptic with $0 < \lambda \vec{\xi} \cdot \vec{\xi} \leq -\vec{\xi} \cdot A(\mathbf{x}) \vec{\xi} \leq \Lambda \vec{\xi} \cdot \vec{\xi}$ for all $\vec{\xi} \neq \vec{0}$ and $\mathbf{x} \in \Omega$ and $c \geq k_0 \geq 0$ is uniformly bounded. The scheme (4.1) with \mathcal{L}_h defined by either (4.1a) or (4.3) for approximating problem (1.1) has a unique solution.*

Remark. We emphasize that the scheme has a unique solution whenever $k_0 > 0$ or $\lambda > 0$.

Using the analogous proof to Theorem 4.3 in [6], we have the following stability result that also follows from Lemma 6.2.

Theorem 6.4. Suppose the operator $A \in [L^\infty(\bar{\Omega})]^{2 \times 2}$ in (1.1) is uniformly elliptic with $0 < \lambda \vec{\xi} \cdot \vec{\xi} \leq -\vec{\xi} \cdot A(\mathbf{x}) \vec{\xi} \leq \Lambda \vec{\xi} \cdot \vec{\xi}$ for all $\vec{\xi} \neq \vec{0}$ and $\mathbf{x} \in \Omega$, and suppose that $c \geq k_0 \geq 0$ is uniformly bounded. If $g = 0$, then the solution U to the scheme (4.1) with \mathcal{L}_h defined by either (4.1a) or (4.3) for approximating problem (1.1) satisfies the uniform bound

$$\|U\|_{\ell^2(\mathcal{T}_h \cap \Omega)} \leq \frac{4d}{\lambda\kappa + k_0} \|f(\cdot)\|_{\ell^2(\mathcal{T}_h \cap \Omega)}.$$

If f is uniformly bounded over Ω , then there holds

$$\begin{aligned} (h_x h_y)^{\frac{1}{2}} \|U\|_{\ell^2(\mathcal{T}_h \cap \Omega)} &\leq \frac{4d}{\lambda\kappa + k_0} (h_x h_y)^{\frac{1}{2}} \|f(\cdot)\|_{\ell^2(\mathcal{T}_h \cap \Omega)} \\ &\leq \frac{4d|\Omega|^2}{\lambda\kappa + k_0} \sup_{\mathbf{x} \in \Omega} |f(\mathbf{x})|. \end{aligned}$$

Remark. The weighting $(h_x h_y)^{\frac{1}{2}}$ is consistent with using the L^2 -norm for f in the limit as $h \rightarrow 0$.

We now extend Theorem 6.4 to account for $g \neq 0$.

Theorem 6.5. Suppose the operator $A \in [L^\infty(\bar{\Omega})]^{2 \times 2}$ in (1.1) is uniformly elliptic with $0 < \lambda \vec{\xi} \cdot \vec{\xi} \leq -\vec{\xi} \cdot A(\mathbf{x}) \vec{\xi} \leq \Lambda \vec{\xi} \cdot \vec{\xi}$ for all $\vec{\xi} \neq \vec{0}$ and $\mathbf{x} \in \Omega$, and suppose that $c \geq k_0 \geq 0$ is uniformly bounded. Then the solution U to the scheme (4.1) with \mathcal{L}_h defined by either (4.1a) or (4.3) for approximating problem (1.1) satisfies the uniform bound

$$(h_x h_y)^{\frac{1}{2}} \|U\|_{\ell^2(\mathcal{T}_h \cap \Omega)} \leq C,$$

where C is a positive \mathbf{h} -independent constant which depends on Ω , the lower (proper) ellipticity constants λ and k_0 , $\|f(\cdot)\|_{C^0(\bar{\Omega})}$, and $\|g\|_{C^0(\partial\Omega)}$.

Proof. Define the function $v \in C^0(\bar{\Omega}) \cap H^2(\Omega)$ to be the solution to

$$-\Delta v = 0 \quad \text{in } \Omega, \tag{6.7a}$$

$$v = g \quad \text{on } \partial\Omega, \quad (6.7b)$$

and define $V : \mathcal{T}'_{\mathbf{h}} \rightarrow \mathbb{R}$ by $V_\alpha = v(\mathbf{x}_\alpha)$ for all $\mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \overline{\Omega}$ and introduce ghost values so that the auxiliary boundary condition holds for V . Then, there holds

$$\mathcal{L}_{\mathbf{h}}(U_\alpha - V_\alpha) = f(\mathbf{x}_\alpha) - \mathcal{L}_{\mathbf{h}}(V_\alpha) \equiv \tilde{f}(\mathbf{x}_\alpha)$$

for all $\mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \Omega$ with $U_\alpha - V_\alpha = 0$ for all $\mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \partial\Omega$. Thus, by Theorem 6.5, there holds

$$(h_x h_y)^{\frac{1}{2}} \|U - V\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)} \leq \frac{4d}{\lambda\kappa + k_0} (h_x h_y)^{\frac{1}{2}} \|\tilde{f}(\cdot)\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)},$$

and it follows that

$$\begin{aligned} (h_x h_y)^{\frac{1}{2}} \|U\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)} &\leq (h_x h_y)^{\frac{1}{2}} \|V\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)} \\ &\quad + \frac{4d}{\lambda\kappa + k_0} (h_x h_y)^{\frac{1}{2}} \|f(\cdot)\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)} \\ &\quad + \frac{4d}{\lambda\kappa + k_0} (h_x h_y)^{\frac{1}{2}} \|\mathcal{L}_{\mathbf{h}} V\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)} \\ &\leq C\|v\|_{L^2(\Omega)} + C \sum_{i=1}^2 \sum_{j=1}^2 \|a_{ij} v_{x_i x_j}\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega)} \end{aligned}$$

for some constant $C > 0$ independent of \mathbf{h} . The result follows by the assumptions for f , the boundedness of A , and the regularity of v . The proof is complete. \square

6.2.2. ℓ^∞ -Stability

Using the techniques in [6], it can be shown that $\delta_{x_i, h_i}^2 U_\alpha$ and $\delta_{x_i, 2h_i}^2 U_\alpha$ are ℓ^2 -stable for all $i \in \{1, 2\}$ for U the solution to (4.1). Consequently, by the discrete Sobolev embedding result in Theorem 5.4 of [6], the approximation U is ℓ^∞ -stable.

6.3. Convergence

We first apply the results in [6] to guarantee convergence to the underlying viscosity solution. We then apply the stability result in Section 6.2.1 to derive optimal rates of convergence when the solution to (1.1) is sufficiently smooth.

Let $u_{\mathbf{h}}$ denote the piecewise constant extension for U the solution to (4.1). Then, the proof technique in [6] can be adapted to yield the following convergence result.

Theorem 6.6. *Suppose the operator $A \in [L^\infty(\bar{\Omega})]^{2 \times 2}$ in (1.1) is uniformly elliptic with $0 < \lambda \vec{\xi} \cdot \vec{\xi} \leq -\vec{\xi} \cdot A(\mathbf{x}) \vec{\xi} \leq \Lambda \vec{\xi} \cdot \vec{\xi}$ for all $\vec{\xi} \neq \vec{0}$ and $\mathbf{x} \in \Omega$, and suppose that $c \geq k_0 \geq 0$ is uniformly bounded. Also suppose that g is continuous on $\partial\Omega$ and f is bounded over Ω . If problem (1.1) satisfies the comparison principle of Definition 2.6, then $u_{\mathbf{h}}$ converges to u locally uniformly as $\mathbf{h} \rightarrow \mathbf{0}^+$, where u is the unique continuous viscosity solution of (1.1).*

Suppose $u \in C^4(\bar{\Omega})$ is the unique solution to (1.1). Then, we can prove optimal rates of convergence for the approximation (4.1).

Theorem 6.7. *Suppose the operator $A \in [L^\infty(\bar{\Omega})]^{2 \times 2}$ in (1.1) is uniformly elliptic with $0 < \lambda \vec{\xi} \cdot \vec{\xi} \leq -\vec{\xi} \cdot A(\mathbf{x}) \vec{\xi} \leq \Lambda \vec{\xi} \cdot \vec{\xi}$ for all $\vec{\xi} \neq \vec{0}$ and $\mathbf{x} \in \Omega$, and suppose that $c \geq k_0 \geq 0$ is uniformly bounded. If problem (1.1) has a unique solution $u \in C^4(\bar{\Omega})$ and $\mathcal{T}_{\mathbf{h}}$ is quasiuniform, then*

$$(h_x h_y)^{\frac{1}{2}} \|e_{\mathbf{h}}\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)} \leq Ch^2$$

for $e_{\mathbf{h}}(\mathbf{x}_\alpha) \equiv U_\alpha - u(\mathbf{x}_\alpha)$ and C independent of \mathbf{h} .

Proof. We have the error $e_{\mathbf{h}}$ satisfies

$$\begin{aligned}
\mathcal{L}_{\mathbf{h}} e_{\mathbf{h}}(\mathbf{x}_\alpha) &= f(\mathbf{x}_\alpha) - \mathcal{L}_{\mathbf{h}} u(\mathbf{x}_\alpha) \\
&= f(\mathbf{x}_\alpha) - \mathcal{L} u(\mathbf{x}_\alpha) + \mathcal{L} u(\mathbf{x}_\alpha) - \mathcal{L}_{\mathbf{h}} u(\mathbf{x}_\alpha) \\
&= f(\mathbf{x}_\alpha) - f(\mathbf{x}_\alpha) + \mathcal{L} u(\mathbf{x}_\alpha) - \mathcal{L}_{\mathbf{h}} u(\mathbf{x}_\alpha) \\
&= \mathcal{L} u(\mathbf{x}_\alpha) - \mathcal{L}_{\mathbf{h}} u(\mathbf{x}_\alpha).
\end{aligned}$$

Furthermore, $e_{\mathbf{h}}(\mathbf{x}_\alpha) = 0$ for all $\mathbf{x}_\alpha \in \mathcal{T}_{\mathbf{h}} \cap \partial\Omega$. Thus, by Theorem 6.4, there holds

$$\left(h_x h_y \right)^{\frac{1}{2}} \|e_{\mathbf{h}}\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)} \leq \frac{4d}{\lambda\kappa + k_0} \left(h_x h_y \right)^{\frac{1}{2}} \|\mathcal{L} u - \mathcal{L}_{\mathbf{h}} u\|_{\ell^2(\mathcal{T}_{\mathbf{h}} \cap \Omega)}.$$

The result follows since all of the second-order difference operators in Section 3 have second-order local truncation errors for $u \in C^4(\bar{\Omega})$ and quasiuniform meshes. The proof is complete. \square

Remark. Note that the approximation U defined by (4.1) with $D_{\mathbf{h}}^2 = \bar{D}_{\mathbf{h}}^2$ may have a boundary layer error due to the auxiliary boundary condition that creates an $\mathcal{O}(1)$ error in $\mathcal{L} u - \mathcal{L}_{\mathbf{h}} u$ for the nodes adjacent to the boundary. The error could be reduced using interpolation to specify a more accurate auxiliary boundary condition. In Chapter 8 we observe optimal rates of convergence for tests using $\bar{D}_{\mathbf{h}}^2$ and the simple auxiliary boundary condition (4.1c).

CHAPTER 7: CONCLUSION

In this thesis we presented a narrow-stencil finite difference method for approximating the viscosity solution of second-order linear elliptic Dirichlet boundary problems. We considered the Lax-Friedrich's-like method of Feng and Lewis as well as several other simple narrow-stencil methods. We have proven admissibility and stability results for simple narrow-stencil finite difference methods for approximating the solution. Numerical tests were presented to gauge the performance of the methods and to validate the convergence results of the thesis. Going further, we would like to extend results to fully nonlinear problems, study degenerate elliptic problems more systematically, and extend the results to the more general PDE $A : D^2u + \mathbf{b} \cdot \nabla u + cu = f$ that includes first-order terms.

CHAPTER 8: NUMERICAL TEST RESULTS

In this chapter, we record results for the tests mentioned in Chapter 5. In particular, we consider the four different tests based on Sections 5.2 – 5.5, for the five solutions introduced in Section 5.1 and the five methods introduced in Sections 4.1 and 4.2. Each table will be labeled as TEST A.B, where A refers to the test number, and B refers to solution number. We record the errors measured in $\|\cdot\|$ defined by (5.2) and $\|\cdot\|_2$ defined by (5.3), as well as rates. We record truncation errors for the various solutions in Section 8.5.

8.1. Test 1 Results

$$\text{SOLUTION 1: } u(x, y) = \sin\left(\frac{\pi}{2}(x + y)^2\right)$$

METHOD 1

TEST 1.1		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.60e-02	—	5.39e-01	—	4.81e-01	—	5.39e-01	—
20	9.52e-02	2.39e-02	1.79	2.26e-01	1.35	2.05e-01	1.32	2.26e-01	1.35
40	4.88e-02	6.63e-03	1.91	9.06e-02	1.37	8.39e-02	1.34	9.06e-02	1.37
60	3.28e-02	3.02e-03	1.98	5.21e-02	1.39	4.94e-02	1.33	5.21e-02	1.39
80	2.47e-02	1.72e-03	1.99	3.51e-02	1.39	3.38e-02	1.34	3.51e-02	1.39
100	1.98e-02	1.11e-03	1.99	2.58e-02	1.40	2.50e-02	1.36	2.58e-02	1.40
120	1.65e-02	7.74e-04	1.99	1.99e-02	1.42	1.94e-02	1.39	1.99e-02	1.42
140	1.42e-02	5.71e-04	1.99	1.60e-02	1.45	1.56e-02	1.42	1.60e-02	1.45

METHOD 2 USING BOUNDARY CONDITION 1

TEST 1.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.35e-01	—	4.42e+00	—	4.25e+00	—	3.20e+00	—
20	9.52e-02	6.34e-02	2.58	1.21e+00	2.01	1.04e+00	2.17	8.35e-01	2.08
40	4.88e-02	1.57e-02	2.09	4.60e-01	1.44	4.16e-01	1.37	4.12e-01	1.06
60	3.28e-02	7.03e-03	2.02	2.69e-01	1.35	2.51e-01	1.27	2.55e-01	1.21
80	2.47e-02	3.98e-03	2.01	1.90e-01	1.22	1.82e-01	1.14	1.84e-01	1.14
100	1.98e-02	2.55e-03	2.01	1.50e-01	1.08	1.45e-01	1.02	1.47e-01	1.03
120	1.65e-02	1.78e-03	2.00	1.26e-01	0.96	1.23e-01	0.91	1.24e-01	0.93
140	1.42e-02	1.31e-03	2.00	1.11e-01	0.86	1.09e-01	0.82	1.09e-01	0.84

METHOD 2 USING BOUNDARY CONDITION 2

TEST 1.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.87e-01	—	4.62e+00	—	4.54e+00	—	4.62e+00	—
20	9.52e-02	1.10e-01	1.95	1.81e+00	1.45	1.80e+00	1.43	1.81e+00	1.45
40	4.88e-02	2.81e-02	2.04	1.03e+00	0.84	1.03e+00	0.84	1.03e+00	0.84
60	3.28e-02	1.26e-02	2.02	6.32e-01	1.23	6.31e-01	1.22	6.32e-01	1.23
80	2.47e-02	7.13e-03	2.01	4.31e-01	1.35	4.31e-01	1.35	4.31e-01	1.35
100	1.98e-02	4.58e-03	2.01	3.16e-01	1.40	3.16e-01	1.40	3.16e-01	1.40
120	1.65e-02	3.19e-03	2.00	2.44e-01	1.43	2.44e-01	1.43	2.44e-01	1.43
140	1.42e-02	2.35e-03	2.00	1.96e-01	1.45	1.96e-01	1.44	1.96e-01	1.45

METHOD 3

TEST 1.1		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.13e-01	—	2.31e+00	—	2.25e+00	—	2.31e+00	—
20	9.52e-02	6.27e-02	1.89	9.05e-01	1.45	8.72e-01	1.47	9.05e-01	1.45
40	4.88e-02	1.68e-02	1.97	3.27e-01	1.52	3.16e-01	1.52	3.27e-01	1.52
60	3.28e-02	7.60e-03	1.99	1.79e-01	1.51	1.75e-01	1.49	1.79e-01	1.51
80	2.47e-02	4.32e-03	1.99	1.17e-01	1.51	1.15e-01	1.49	1.17e-01	1.51
100	1.98e-02	2.79e-03	1.99	8.36e-02	1.52	8.24e-02	1.50	8.36e-02	1.52
120	1.65e-02	1.95e-03	1.99	6.34e-02	1.53	6.26e-02	1.52	6.34e-02	1.53
140	1.42e-02	1.44e-03	1.99	5.01e-02	1.55	4.95e-02	1.54	5.01e-02	1.55

METHOD 4

TEST 1.1		$\ U - u\ _2$		$\ \delta_{x,h_x}^2 U - u_{xx}\ _2$		$\ \delta_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \delta_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.60e-02	—	5.39e-01	—	4.81e-01	—	5.39e-01	—
20	9.52e-02	2.39e-02	1.79	2.26e-01	1.35	2.05e-01	1.32	2.26e-01	1.35
40	4.88e-02	6.63e-03	1.91	9.06e-02	1.37	8.39e-02	1.34	9.06e-02	1.37
60	3.28e-02	3.02e-03	1.98	5.21e-02	1.39	4.94e-02	1.33	5.21e-02	1.39
80	2.47e-02	1.72e-03	1.99	3.51e-02	1.39	3.38e-02	1.34	3.51e-02	1.39
100	1.98e-02	1.11e-03	1.99	2.58e-02	1.40	2.50e-02	1.36	2.58e-02	1.40
120	1.65e-02	7.74e-04	1.99	1.99e-02	1.42	1.94e-02	1.39	1.99e-02	1.42
140	1.42e-02	5.71e-04	1.99	1.60e-02	1.45	1.56e-02	1.42	1.60e-02	1.45

METHOD 5 USING BOUNDARY CONDITION 1

TEST 1.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.38e-01	—	6.01e+00	—	3.65e+00	—	2.92e+00	—
20	9.52e-02	8.72e-02	1.56	4.51e+00	0.44	1.94e+00	0.98	1.89e+00	0.67
40	4.88e-02	2.38e-02	1.94	2.95e+00	0.64	8.63e-01	1.21	1.23e+00	0.64
60	3.28e-02	1.11e-02	1.91	2.29e+00	0.63	5.01e-01	1.37	8.96e-01	0.80
80	2.47e-02	6.49e-03	1.89	1.94e+00	0.58	3.30e-01	1.48	7.18e-01	0.78
100	1.98e-02	4.26e-03	1.91	1.72e+00	0.55	2.35e-01	1.54	6.14e-01	0.71
120	1.65e-02	3.00e-03	1.94	1.56e+00	0.53	1.77e-01	1.56	5.46e-01	0.65
140	1.42e-02	2.22e-03	1.96	1.44e+00	0.52	1.39e-01	1.57	4.97e-01	0.61

METHOD 5 USING BOUNDARY CONDITION 2

TEST 1.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.13e-01	—	4.75e+00	—	3.62e+00	—	4.75e+00	—
20	9.52e-02	9.73e-02	1.21	2.55e+00	0.96	1.90e+00	1.00	2.55e+00	0.96
40	4.88e-02	3.90e-02	1.37	1.04e+00	1.34	7.46e-01	1.39	1.04e+00	1.34
60	3.28e-02	2.07e-02	1.60	5.72e-01	1.51	3.97e-01	1.59	5.72e-01	1.51
80	2.47e-02	1.26e-02	1.75	3.71e-01	1.53	2.48e-01	1.65	3.71e-01	1.53
100	1.98e-02	8.39e-03	1.84	2.66e-01	1.50	1.72e-01	1.65	2.66e-01	1.50
120	1.65e-02	5.96e-03	1.89	2.04e-01	1.47	1.28e-01	1.63	2.04e-01	1.47
140	1.42e-02	4.44e-03	1.93	1.64e-01	1.45	1.00e-01	1.60	1.64e-01	1.45

$$\text{SOLUTION 2: } u(x, y) = x^2 + 3xy + \frac{1}{2}y^2 + 3$$

METHOD 1

TEST 1.2		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	9.01e-15	—	8.89e-14	—	4.26e-14	—	9.10e-14	—
20	9.52e-02	4.29e-14	-2.41	4.81e-13	-2.61	2.24e-13	-2.57	4.80e-13	-2.57
40	4.88e-02	4.20e-13	-3.41	3.13e-12	-2.80	2.22e-12	-3.43	3.17e-12	-2.82
60	3.28e-02	2.69e-13	1.12	3.85e-12	-0.53	2.20e-12	0.03	3.73e-12	-0.41
80	2.47e-02	1.96e-13	1.11	6.64e-12	-1.92	4.01e-12	-2.12	6.93e-12	-2.18
100	1.98e-02	5.59e-13	-4.74	1.32e-11	-3.10	9.08e-12	-3.70	1.29e-11	-2.82
120	1.65e-02	2.67e-13	4.09	1.52e-11	-0.78	1.01e-11	-0.57	1.78e-11	-1.77
140	1.42e-02	7.54e-13	-6.79	2.28e-11	-2.66	1.56e-11	-2.87	2.48e-11	-2.18

METHOD 2 USING BOUNDARY CONDITION 1

TEST 1.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.12e-02	—	1.53e-01	—	1.50e-01	—	1.33e-01	—
20	9.52e-02	8.80e-03	1.96	1.15e-01	0.44	1.14e-01	0.42	1.08e-01	0.31
40	4.88e-02	2.34e-03	1.98	8.42e-02	0.47	8.39e-02	0.46	8.19e-02	0.42
60	3.28e-02	1.06e-03	1.99	6.95e-02	0.48	6.94e-02	0.48	6.83e-02	0.46
80	2.47e-02	6.05e-04	1.99	6.06e-02	0.49	6.05e-02	0.48	5.98e-02	0.47
100	1.98e-02	3.90e-04	1.99	5.44e-02	0.49	5.43e-02	0.49	5.38e-02	0.48
120	1.65e-02	2.72e-04	1.99	4.97e-02	0.49	4.97e-02	0.49	4.93e-02	0.48
140	1.42e-02	2.00e-04	2.00	4.61e-02	0.49	4.61e-02	0.49	4.58e-02	0.48

METHOD 2 USING BOUNDARY CONDITION 2

TEST 1.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.98e-15	—	1.26e-13	—	1.18e-13	—	1.24e-13	—
20	9.52e-02	2.33e-14	-2.39	2.45e-12	-4.59	2.43e-12	-4.68	2.43e-12	-4.61
40	4.88e-02	1.85e-13	-3.09	7.60e-11	-5.14	7.61e-11	-5.15	7.62e-11	-5.15
60	3.28e-02	1.47e-12	-5.22	1.36e-09	-7.26	1.36e-09	-7.26	1.36e-09	-7.26
80	2.47e-02	1.27e-12	0.53	2.06e-09	-1.46	2.06e-09	-1.46	2.06e-09	-1.46
100	1.98e-02	2.58e-12	-3.23	6.56e-09	-5.25	6.56e-09	-5.25	6.57e-09	-5.25
120	1.65e-02	9.44e-12	-7.17	3.45e-08	-9.18	3.45e-08	-9.18	3.45e-08	-9.18
140	1.42e-02	7.20e-12	1.77	3.57e-08	-0.23	3.57e-08	-0.23	3.57e-08	-0.23

METHOD 3

TEST 1.2		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.22e-15	—	8.93e-14	—	3.02e-14	—	8.72e-14	—
20	9.52e-02	8.62e-15	-1.52	4.29e-13	-2.43	2.31e-13	-3.15	4.34e-13	-2.48
40	4.88e-02	1.16e-13	-3.89	2.36e-12	-2.55	1.19e-12	-2.45	2.35e-12	-2.53
60	3.28e-02	1.40e-13	-0.46	3.68e-12	-1.12	2.09e-12	-1.41	3.71e-12	-1.14
80	2.47e-02	2.10e-13	-1.43	6.37e-12	-1.93	3.46e-12	-1.78	6.07e-12	-1.74
100	1.98e-02	4.18e-14	7.32	1.02e-11	-2.15	5.65e-12	-2.23	1.01e-11	-2.31
120	1.65e-02	8.46e-13	-16.64	1.65e-11	-2.64	1.20e-11	-4.15	1.70e-11	-2.89
140	1.42e-02	1.17e-12	-2.11	2.41e-11	-2.49	1.58e-11	-1.82	2.22e-11	-1.74

METHOD 4

TEST 1.2		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	9.01e-15	—	8.89e-14	—	4.26e-14	—	9.10e-14	—
20	9.52e-02	4.29e-14	-2.41	4.81e-13	-2.61	2.24e-13	-2.57	4.80e-13	-2.57
40	4.88e-02	4.20e-13	-3.41	3.13e-12	-2.80	2.22e-12	-3.43	3.17e-12	-2.82
60	3.28e-02	2.69e-13	1.12	3.85e-12	-0.53	2.20e-12	0.03	3.73e-12	-0.41
80	2.47e-02	1.96e-13	1.11	6.64e-12	-1.92	4.01e-12	-2.12	6.93e-12	-2.18
100	1.98e-02	5.59e-13	-4.74	1.32e-11	-3.10	9.08e-12	-3.70	1.29e-11	-2.82
120	1.65e-02	2.67e-13	4.09	1.52e-11	-0.78	1.01e-11	-0.57	1.78e-11	-1.77
140	1.42e-02	7.54e-13	-6.79	2.28e-11	-2.66	1.56e-11	-2.87	2.48e-11	-2.18

METHOD 5 USING BOUNDARY CONDITION 1

TEST 1.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	5.68e-02	—	7.04e-01	—	9.54e-02	—	1.91e-01	—
20	9.52e-02	1.82e-02	1.76	5.45e-01	0.40	5.11e-02	0.97	1.67e-01	0.21
40	4.88e-02	5.10e-03	1.90	4.03e-01	0.45	2.66e-02	0.97	1.29e-01	0.38
60	3.28e-02	2.35e-03	1.94	3.34e-01	0.47	1.80e-02	0.98	1.09e-01	0.44
80	2.47e-02	1.35e-03	1.96	2.92e-01	0.48	1.36e-02	0.99	9.54e-02	0.46
100	1.98e-02	8.74e-04	1.97	2.62e-01	0.49	1.09e-02	0.99	8.61e-02	0.47
120	1.65e-02	6.12e-04	1.98	2.40e-01	0.49	9.15e-03	0.99	7.90e-02	0.47
140	1.42e-02	4.52e-04	1.98	2.23e-01	0.49	7.86e-03	0.99	7.34e-02	0.48

METHOD 5 USING BOUNDARY CONDITION 2

TEST 1.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	5.79e-15	—	4.91e-14	—	3.04e-14	—	5.13e-14	—
20	9.52e-02	1.88e-14	-1.82	2.09e-13	-2.24	1.52e-13	-2.49	2.19e-13	-2.24
40	4.88e-02	4.49e-14	-1.30	1.43e-12	-2.87	8.28e-13	-2.54	1.15e-12	-2.48
60	3.28e-02	5.42e-14	-0.48	2.73e-12	-1.63	1.71e-12	-1.82	2.28e-12	-1.73
80	2.47e-02	5.90e-14	-0.30	5.44e-12	-2.43	4.01e-12	-3.01	5.01e-12	-2.77
100	1.98e-02	2.73e-13	-6.94	1.05e-11	-2.99	9.28e-12	-3.80	1.11e-11	-3.62
120	1.65e-02	1.97e-13	1.80	1.34e-11	-1.33	1.18e-11	-1.31	1.49e-11	-1.62
140	1.42e-02	2.62e-13	-1.86	1.92e-11	-2.38	1.54e-11	-1.75	1.87e-11	-1.50

$$\text{SOLUTION 3: } u(x, y) = e^{xy+2y}$$

METHOD 1

TEST 1.3		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.22e-01	—	7.77e-01	—	5.36e-01	—	4.63e-01	—
20	9.52e-02	3.96e-02	1.74	3.34e-01	1.31	2.61e-01	1.11	2.43e-01	1.00
40	4.88e-02	1.14e-02	1.86	1.24e-01	1.48	1.06e-01	1.35	1.02e-01	1.30
60	3.28e-02	5.32e-03	1.92	6.66e-02	1.56	5.89e-02	1.47	5.74e-02	1.45
80	2.47e-02	3.07e-03	1.94	4.23e-02	1.60	3.82e-02	1.53	3.74e-02	1.51
100	1.98e-02	1.99e-03	1.96	2.96e-02	1.62	2.70e-02	1.56	2.66e-02	1.55
120	1.65e-02	1.40e-03	1.97	2.20e-02	1.64	2.03e-02	1.58	2.00e-02	1.57
140	1.42e-02	1.03e-03	1.97	1.71e-02	1.65	1.59e-02	1.60	1.57e-02	1.59

METHOD 2 USING BOUNDARY CONDITION 1

TEST 1.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _{\hat{2}}$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.63e-01	—	4.88e+00	—	4.20e+00	—	3.60e+00	—
20	9.52e-02	1.16e-01	1.76	3.95e+00	0.33	3.54e+00	0.27	3.26e+00	0.15
40	4.88e-02	3.25e-02	1.90	2.87e+00	0.48	2.68e+00	0.41	2.57e+00	0.35
60	3.28e-02	1.50e-02	1.95	2.34e+00	0.51	2.23e+00	0.46	2.17e+00	0.43
80	2.47e-02	8.58e-03	1.97	2.02e+00	0.51	1.95e+00	0.48	1.91e+00	0.45
100	1.98e-02	5.55e-03	1.98	1.81e+00	0.51	1.75e+00	0.48	1.72e+00	0.46
120	1.65e-02	3.88e-03	1.98	1.65e+00	0.51	1.60e+00	0.49	1.58e+00	0.47
140	1.42e-02	2.86e-03	1.98	1.52e+00	0.51	1.49e+00	0.49	1.47e+00	0.48

METHOD 2 USING BOUNDARY CONDITION 2

TEST 1.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.42e-01	—	2.72e+00	—	2.50e+00	—	2.63e+00	—
20	9.52e-02	1.12e-01	1.73	1.48e+00	0.94	1.42e+00	0.88	1.45e+00	0.91
40	4.88e-02	3.19e-02	1.88	6.65e-01	1.20	6.50e-01	1.16	6.59e-01	1.18
60	3.28e-02	1.48e-02	1.93	3.93e-01	1.32	3.87e-01	1.30	3.91e-01	1.31
80	2.47e-02	8.51e-03	1.95	2.66e-01	1.37	2.63e-01	1.36	2.65e-01	1.37
100	1.98e-02	5.51e-03	1.97	1.96e-01	1.40	1.94e-01	1.39	1.95e-01	1.40
120	1.65e-02	3.86e-03	1.97	1.51e-01	1.42	1.50e-01	1.41	1.51e-01	1.42
140	1.42e-02	2.85e-03	1.98	1.22e-01	1.43	1.21e-01	1.42	1.21e-01	1.43

METHOD 3

TEST 1.3		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.16e-01	—	1.42e+00	—	1.32e+00	—	1.40e+00	—
20	9.52e-02	6.66e-02	1.82	5.94e-01	1.35	5.67e-01	1.31	5.98e-01	1.32
40	4.88e-02	1.86e-02	1.91	2.19e-01	1.49	2.12e-01	1.47	2.21e-01	1.49
60	3.28e-02	8.58e-03	1.95	1.18e-01	1.56	1.15e-01	1.55	1.19e-01	1.57
80	2.47e-02	4.92e-03	1.96	7.48e-02	1.60	7.33e-02	1.58	7.54e-02	1.60
100	1.98e-02	3.18e-03	1.97	5.23e-02	1.62	5.14e-02	1.61	5.27e-02	1.62
120	1.65e-02	2.22e-03	1.98	3.90e-02	1.63	3.84e-02	1.62	3.93e-02	1.63
140	1.42e-02	1.64e-03	1.98	3.03e-02	1.64	2.99e-02	1.63	3.05e-02	1.64

METHOD 4

TEST 1.3		$\ U - u\ _2$		$\ \delta_{x,h_x}^2 U - u_{xx}\ _2$		$\ \delta_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \delta_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.22e-01	—	7.77e-01	—	5.36e-01	—	4.63e-01	—
20	9.52e-02	3.96e-02	1.74	3.34e-01	1.31	2.61e-01	1.11	2.43e-01	1.00
40	4.88e-02	1.14e-02	1.86	1.24e-01	1.48	1.06e-01	1.35	1.02e-01	1.30
60	3.28e-02	5.32e-03	1.92	6.66e-02	1.56	5.89e-02	1.47	5.74e-02	1.45
80	2.47e-02	3.07e-03	1.94	4.23e-02	1.60	3.82e-02	1.53	3.74e-02	1.51
100	1.98e-02	1.99e-03	1.96	2.96e-02	1.62	2.70e-02	1.56	2.66e-02	1.55
120	1.65e-02	1.40e-03	1.97	2.20e-02	1.64	2.03e-02	1.58	2.00e-02	1.57
140	1.42e-02	1.03e-03	1.97	1.71e-02	1.65	1.59e-02	1.60	1.57e-02	1.59

METHOD 5 USING BOUNDARY CONDITION 1

TEST 1.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.47e-01	—	1.81e+00	—	1.49e+00	—	7.07e-01	—
20	9.52e-02	1.27e-01	1.04	1.63e+00	0.17	1.25e+00	0.27	8.13e-01	-0.22
40	4.88e-02	5.01e-02	1.39	1.16e+00	0.51	9.33e-01	0.44	6.49e-01	0.34
60	3.28e-02	2.67e-02	1.59	8.90e-01	0.66	7.26e-01	0.63	5.10e-01	0.61
80	2.47e-02	1.66e-02	1.68	7.28e-01	0.71	5.91e-01	0.73	4.16e-01	0.72
100	1.98e-02	1.13e-02	1.74	6.20e-01	0.72	4.97e-01	0.79	3.51e-01	0.77
120	1.65e-02	8.20e-03	1.77	5.44e-01	0.72	4.28e-01	0.82	3.04e-01	0.80
140	1.42e-02	6.23e-03	1.80	4.88e-01	0.72	3.76e-01	0.85	2.68e-01	0.82

METHOD 5 USING BOUNDARY CONDITION 2

TEST 1.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.05e-01	—	9.13e-01	—	8.54e-01	—	3.33e+00	—
20	9.52e-02	9.16e-02	1.24	5.82e-01	0.70	4.46e-01	1.00	2.34e+00	0.54
40	4.88e-02	3.38e-02	1.49	2.87e-01	1.06	2.34e-01	0.96	1.27e+00	0.92
60	3.28e-02	1.76e-02	1.65	1.74e-01	1.26	1.49e-01	1.13	8.11e-01	1.13
80	2.47e-02	1.07e-02	1.73	1.19e-01	1.36	1.05e-01	1.25	5.73e-01	1.22
100	1.98e-02	7.24e-03	1.78	8.68e-02	1.41	7.82e-02	1.32	4.32e-01	1.28
120	1.65e-02	5.22e-03	1.82	6.68e-02	1.45	6.10e-02	1.38	3.40e-01	1.32
140	1.42e-02	3.93e-03	1.84	5.33e-02	1.48	4.91e-02	1.41	2.77e-01	1.35

$$\text{SOLUTION 4: } u(x, y) = \frac{x^3}{18} \left(3 \log(x^2) - 11 \right) + \left(y - \frac{1}{2} \right)^{\frac{8}{3}} \sqrt[5]{\left| x + \frac{1}{5} \right|^5}$$

METHOD 1

TEST 1.4		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.60e-02	—	2.37e-01	—	1.94e-01	—	2.20e-01	—
20	9.52e-02	1.36e-02	1.88	8.10e-02	1.66	6.53e-02	1.68	7.73e-02	1.62
40	4.88e-02	3.86e-03	1.88	2.52e-02	1.75	2.04e-02	1.74	2.49e-02	1.69
60	3.28e-02	1.83e-03	1.87	1.26e-02	1.74	1.03e-02	1.71	1.28e-02	1.68
80	2.47e-02	1.08e-03	1.86	7.78e-03	1.70	6.44e-03	1.66	8.00e-03	1.65
100	1.98e-02	7.19e-04	1.85	5.40e-03	1.66	4.52e-03	1.61	5.60e-03	1.62
120	1.65e-02	5.16e-04	1.84	4.03e-03	1.62	3.41e-03	1.56	4.21e-03	1.58
140	1.42e-02	3.90e-04	1.83	3.17e-03	1.58	2.70e-03	1.52	3.32e-03	1.55

METHOD 2 USING BOUNDARY CONDITION 1

TEST 1.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	6.59e-02	—	9.76e-01	—	9.31e-01	—	8.80e-01	—
20	9.52e-02	1.88e-02	1.94	6.42e-01	0.65	6.34e-01	0.59	6.06e-01	0.58
40	4.88e-02	4.99e-03	1.99	4.43e-01	0.56	4.41e-01	0.55	4.30e-01	0.51
60	3.28e-02	2.31e-03	1.94	4.01e-01	0.25	4.00e-01	0.24	3.94e-01	0.22
80	2.47e-02	1.31e-03	2.00	3.43e-01	0.55	3.43e-01	0.54	3.39e-01	0.53
100	1.98e-02	8.61e-04	1.91	2.97e-01	0.65	2.97e-01	0.65	2.94e-01	0.65
120	1.65e-02	5.98e-04	2.02	2.61e-01	0.72	2.60e-01	0.72	2.58e-01	0.72
140	1.42e-02	4.51e-04	1.85	2.57e-01	0.09	2.57e-01	0.08	2.56e-01	0.07

METHOD 2 USING BOUNDARY CONDITION 2

TEST 1.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.89e-02	—	3.85e-01	—	3.37e-01	—	3.21e-01	—
20	9.52e-02	7.37e-03	2.11	2.44e-01	0.70	2.36e-01	0.55	2.35e-01	0.48
40	4.88e-02	1.80e-03	2.10	1.40e-01	0.83	1.39e-01	0.80	1.39e-01	0.79
60	3.28e-02	8.15e-04	2.00	1.33e-01	0.12	1.33e-01	0.10	1.33e-01	0.10
80	2.47e-02	4.67e-04	1.96	1.17e-01	0.46	1.17e-01	0.46	1.17e-01	0.46
100	1.98e-02	3.08e-04	1.89	9.55e-02	0.92	9.54e-02	0.92	9.54e-02	0.92
120	1.65e-02	2.20e-04	1.85	7.43e-02	1.39	7.42e-02	1.39	7.42e-02	1.39
140	1.42e-02	1.68e-04	1.76	8.07e-02	-0.55	8.07e-02	-0.55	8.07e-02	-0.55

METHOD 3

TEST 1.4		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.48e-02	—	1.75e-01	—	1.39e-01	—	1.44e-01	—
20	9.52e-02	4.92e-03	1.70	6.83e-02	1.45	5.58e-02	1.41	5.93e-02	1.37
40	4.88e-02	1.52e-03	1.76	2.36e-02	1.59	1.97e-02	1.55	2.14e-02	1.52
60	3.28e-02	7.58e-04	1.75	1.24e-02	1.62	1.05e-02	1.59	1.15e-02	1.56
80	2.47e-02	4.65e-04	1.72	7.83e-03	1.62	6.67e-03	1.59	7.39e-03	1.57
100	1.98e-02	3.19e-04	1.70	5.49e-03	1.61	4.71e-03	1.58	5.24e-03	1.56
120	1.65e-02	2.36e-04	1.68	4.13e-03	1.59	3.56e-03	1.55	3.97e-03	1.54
140	1.42e-02	1.83e-04	1.67	3.25e-03	1.56	2.82e-03	1.52	3.15e-03	1.51

METHOD 4

TEST 1.4		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.60e-02	—	2.37e-01	—	1.94e-01	—	2.20e-01	—
20	9.52e-02	1.36e-02	1.88	8.10e-02	1.66	6.53e-02	1.68	7.73e-02	1.62
40	4.88e-02	3.86e-03	1.88	2.52e-02	1.75	2.04e-02	1.74	2.49e-02	1.69
60	3.28e-02	1.83e-03	1.87	1.26e-02	1.74	1.03e-02	1.71	1.28e-02	1.68
80	2.47e-02	1.08e-03	1.86	7.78e-03	1.70	6.44e-03	1.66	8.00e-03	1.65
100	1.98e-02	7.19e-04	1.85	5.40e-03	1.66	4.52e-03	1.61	5.60e-03	1.62
120	1.65e-02	5.16e-04	1.84	4.03e-03	1.62	3.41e-03	1.56	4.21e-03	1.58
140	1.42e-02	3.90e-04	1.83	3.17e-03	1.58	2.70e-03	1.52	3.32e-03	1.55

METHOD 5 USING BOUNDARY CONDITION 1

TEST 1.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.26e-01	—	1.34e+00	—	3.75e-01	—	6.31e-01	—
20	9.52e-02	4.55e-02	1.57	1.06e+00	0.37	2.27e-01	0.78	4.01e-01	0.70
40	4.88e-02	1.42e-02	1.74	8.07e-01	0.40	1.20e-01	0.95	2.79e-01	0.54
60	3.28e-02	6.83e-03	1.84	6.78e-01	0.44	8.20e-02	0.96	2.30e-01	0.48
80	2.47e-02	4.02e-03	1.87	5.96e-01	0.45	6.23e-02	0.97	2.01e-01	0.48
100	1.98e-02	2.65e-03	1.89	5.38e-01	0.46	5.03e-02	0.97	1.81e-01	0.48
120	1.65e-02	1.88e-03	1.90	4.94e-01	0.47	4.22e-02	0.98	1.66e-01	0.48
140	1.42e-02	1.40e-03	1.90	4.60e-01	0.48	3.63e-02	0.98	1.54e-01	0.48

METHOD 5 USING BOUNDARY CONDITION 2

TEST 1.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.65e-01	—	6.91e-01	—	3.58e-01	—	6.34e-01	—
20	9.52e-02	5.24e-02	1.77	2.92e-01	1.33	1.57e-01	1.28	2.55e-01	1.41
40	4.88e-02	1.50e-02	1.87	1.31e-01	1.19	5.98e-02	1.44	9.50e-02	1.47
60	3.28e-02	7.03e-03	1.91	8.50e-02	1.10	3.33e-02	1.47	5.42e-02	1.41
80	2.47e-02	4.08e-03	1.92	6.28e-02	1.07	2.20e-02	1.47	3.69e-02	1.36
100	1.98e-02	2.66e-03	1.93	4.98e-02	1.05	1.59e-02	1.46	2.76e-02	1.32
120	1.65e-02	1.88e-03	1.93	4.13e-02	1.04	1.22e-02	1.45	2.18e-02	1.29
140	1.42e-02	1.40e-03	1.93	3.53e-02	1.03	9.83e-03	1.44	1.80e-02	1.27

8.2. Test 2 Results

$$\text{SOLUTION 1: } u(x, y) = \sin\left(\frac{\pi}{2}(x + y)^2\right)$$

METHOD 1

TEST 2.1		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.72e-02	—	8.33e-01	—	3.18e-01	—	8.91e-02	—
20	9.52e-02	1.24e-02	2.06	3.44e-01	1.37	1.30e-01	1.39	4.02e-02	1.23
40	4.88e-02	3.24e-03	2.01	1.45e-01	1.29	5.32e-02	1.33	1.60e-02	1.38
60	3.28e-02	1.47e-03	2.00	8.15e-02	1.45	2.85e-02	1.57	8.48e-03	1.60
80	2.47e-02	8.31e-04	2.00	5.23e-02	1.57	1.77e-02	1.67	5.23e-03	1.70
100	1.98e-02	5.35e-04	2.00	3.62e-02	1.67	1.21e-02	1.74	3.55e-03	1.75
120	1.65e-02	3.73e-04	2.00	2.66e-02	1.71	8.76e-03	1.78	2.57e-03	1.80
140	1.42e-02	2.74e-04	2.00	2.03e-02	1.75	6.64e-03	1.81	1.94e-03	1.83

METHOD 2 USING BOUNDARY CONDITION 1

TEST 2.1		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \overline{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \overline{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.14e-01	—	2.94e+00	—	1.54e+00	—	1.22e+00	—
20	9.52e-02	5.43e-02	2.12	1.17e+00	1.43	6.23e-01	1.40	7.64e-01	0.73
40	4.88e-02	1.40e-02	2.02	4.52e-01	1.42	2.08e-01	1.64	3.43e-01	1.20
60	3.28e-02	6.34e-03	2.00	2.56e-01	1.43	1.04e-01	1.75	2.00e-01	1.36
80	2.47e-02	3.60e-03	2.00	1.68e-01	1.48	6.21e-02	1.80	1.34e-01	1.41
100	1.98e-02	2.31e-03	2.00	1.20e-01	1.53	4.14e-02	1.84	9.70e-02	1.46
120	1.65e-02	1.61e-03	2.00	9.05e-02	1.56	2.97e-02	1.85	7.40e-02	1.50
140	1.42e-02	1.19e-03	2.00	7.10e-02	1.59	2.23e-02	1.87	5.85e-02	1.54

METHOD 2 USING BOUNDARY CONDITION 2

TEST 2.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.73e-01	—	3.62e+00	—	1.50e+00	—	3.73e-01	—
20	9.52e-02	4.71e-02	2.02	1.15e+00	1.78	6.08e-01	1.39	1.38e-01	1.54
40	4.88e-02	1.25e-02	1.98	5.43e-01	1.12	2.22e-01	1.51	6.06e-02	1.23
60	3.28e-02	5.73e-03	1.97	3.21e-01	1.32	1.16e-01	1.63	3.36e-02	1.49
80	2.47e-02	3.26e-03	1.98	2.09e-01	1.51	7.20e-02	1.69	2.10e-02	1.65
100	1.98e-02	2.11e-03	1.98	1.46e-01	1.63	4.90e-02	1.75	1.43e-02	1.73
120	1.65e-02	1.47e-03	1.99	1.08e-01	1.70	3.55e-02	1.78	1.04e-02	1.79
140	1.42e-02	1.09e-03	1.99	8.23e-02	1.75	2.69e-02	1.81	7.84e-03	1.83

METHOD 3

TEST 2.1		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.51e-02	—	7.87e-01	—	2.22e+00	—	9.90e-02	—
20	9.52e-02	1.19e-02	2.07	3.22e-01	1.38	8.64e-01	1.46	4.47e-02	1.23
40	4.88e-02	3.09e-03	2.01	1.35e-01	1.29	2.67e-01	1.75	1.72e-02	1.43
60	3.28e-02	1.40e-03	2.00	7.59e-02	1.45	1.27e-01	1.87	8.99e-03	1.63
80	2.47e-02	7.93e-04	2.00	4.87e-02	1.57	7.41e-02	1.91	5.51e-03	1.73
100	1.98e-02	5.10e-04	2.00	3.37e-02	1.67	4.84e-02	1.93	3.72e-03	1.77
120	1.65e-02	3.55e-04	2.00	2.48e-02	1.71	3.41e-02	1.94	2.68e-03	1.82
140	1.42e-02	2.62e-04	2.00	1.90e-02	1.75	2.53e-02	1.95	2.02e-03	1.84

METHOD 4

TEST 2.1		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \delta_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.31e-02	—	7.43e-01	—	4.23e+00	—	1.17e-01	—
20	9.52e-02	1.13e-02	2.07	3.00e-01	1.40	1.63e+00	1.48	5.20e-02	1.26
40	4.88e-02	2.94e-03	2.01	1.26e-01	1.30	4.95e-01	1.78	1.91e-02	1.50
60	3.28e-02	1.33e-03	2.00	7.05e-02	1.45	2.34e-01	1.88	9.81e-03	1.67
80	2.47e-02	7.54e-04	2.00	4.52e-02	1.57	1.36e-01	1.92	5.96e-03	1.76
100	1.98e-02	4.85e-04	2.00	3.13e-02	1.67	8.86e-02	1.94	4.01e-03	1.80
120	1.65e-02	3.38e-04	2.00	2.30e-02	1.70	6.23e-02	1.95	2.87e-03	1.84
140	1.42e-02	2.49e-04	2.00	1.76e-02	1.75	4.62e-02	1.96	2.16e-03	1.86

METHOD 5 USING BOUNDARY CONDITION 1

TEST 2.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.38e-01	—	8.29e+00	—	4.99e+00	—	5.11e+00	—
20	9.52e-02	2.09e-01	1.15	6.79e+00	0.31	3.26e+00	0.66	4.30e+00	0.27
40	4.88e-02	7.74e-02	1.48	4.87e+00	0.49	1.71e+00	0.97	3.05e+00	0.51
60	3.28e-02	3.91e-02	1.72	4.00e+00	0.50	1.07e+00	1.17	2.49e+00	0.51
80	2.47e-02	2.35e-02	1.79	3.51e+00	0.46	7.53e-01	1.24	2.19e+00	0.45
100	1.98e-02	1.57e-02	1.84	3.19e+00	0.43	5.72e-01	1.25	2.01e+00	0.39
120	1.65e-02	1.12e-02	1.84	2.95e+00	0.42	4.57e-01	1.25	1.88e+00	0.37
140	1.42e-02	8.44e-03	1.87	2.77e+00	0.42	3.78e-01	1.23	1.78e+00	0.35

METHOD 5 USING BOUNDARY CONDITION 2

TEST 2.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.60e-01	—	8.45e+00	—	5.02e+00	—	6.80e+00	—
20	9.52e-02	2.17e-01	1.16	5.65e+00	0.62	3.10e+00	0.74	3.98e+00	0.83
40	4.88e-02	7.78e-02	1.53	3.01e+00	0.94	1.41e+00	1.18	1.68e+00	1.29
60	3.28e-02	3.84e-02	1.78	1.91e+00	1.15	7.83e-01	1.48	8.92e-01	1.59
80	2.47e-02	2.27e-02	1.85	1.33e+00	1.27	4.94e-01	1.63	5.50e-01	1.70
100	1.98e-02	1.49e-02	1.90	9.93e-01	1.33	3.39e-01	1.71	3.75e-01	1.74
120	1.65e-02	1.06e-02	1.91	7.74e-01	1.38	2.46e-01	1.76	2.73e-01	1.76
140	1.42e-02	7.87e-03	1.94	6.25e-01	1.40	1.87e-01	1.80	2.08e-01	1.77

$$\text{SOLUTION 2: } u(x, y) = x^2 + 3xy + \frac{1}{2}y^2 + 3$$

METHOD 1

TEST 2.2		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	8.09e-15	—	1.08e-13	—	3.91e-14	—	5.98e-14	—
20	9.52e-02	3.73e-15	1.20	4.30e-13	-2.14	1.29e-13	-1.84	1.91e-13	-1.79
40	4.88e-02	1.74e-14	-2.30	1.96e-12	-2.27	4.86e-13	-1.99	7.04e-13	-1.95
60	3.28e-02	2.44e-14	-0.85	4.01e-12	-1.80	1.15e-12	-2.17	1.62e-12	-2.10
80	2.47e-02	2.52e-14	-0.12	7.71e-12	-2.30	2.13e-12	-2.16	2.98e-12	-2.15
100	1.98e-02	2.09e-14	0.86	1.14e-11	-1.79	3.21e-12	-1.87	4.44e-12	-1.81
120	1.65e-02	1.88e-14	0.58	1.72e-11	-2.27	4.56e-12	-1.95	6.36e-12	-1.99
140	1.42e-02	3.02e-14	-3.11	2.27e-11	-1.81	6.30e-12	-2.10	8.66e-12	-2.02

METHOD 2 USING BOUNDARY CONDITION 1

TEST 2.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.12e-02	—	1.86e-01	—	2.59e-02	—	3.88e-02	—
20	9.52e-02	6.34e-03	1.87	8.77e-02	1.16	1.48e-02	0.87	2.92e-02	0.44
40	4.88e-02	1.70e-03	1.97	4.61e-02	0.96	7.21e-03	1.07	1.57e-02	0.93
60	3.28e-02	7.73e-04	1.98	3.19e-02	0.93	4.66e-03	1.10	1.01e-02	1.11
80	2.47e-02	4.40e-04	1.99	2.44e-02	0.95	3.44e-03	1.06	7.52e-03	1.04
100	1.98e-02	2.84e-04	1.99	1.98e-02	0.94	2.73e-03	1.06	6.07e-03	0.97
120	1.65e-02	1.98e-04	1.99	1.66e-02	0.96	2.27e-03	1.03	5.12e-03	0.94
140	1.42e-02	1.46e-04	2.00	1.44e-02	0.95	1.93e-03	1.03	4.45e-03	0.92

METHOD 2 USING BOUNDARY CONDITION 2

TEST 2.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.78e-15	—	3.55e-14	—	1.13e-14	—	1.93e-14	—
20	9.52e-02	3.57e-15	-1.08	1.37e-13	-2.08	4.30e-14	-2.07	5.77e-14	-1.69
40	4.88e-02	8.31e-15	-1.26	6.02e-13	-2.21	1.59e-13	-1.95	2.14e-13	-1.96
60	3.28e-02	1.16e-14	-0.83	1.26e-12	-1.87	3.87e-13	-2.24	4.57e-13	-1.91
80	2.47e-02	2.29e-14	-2.42	2.52e-12	-2.43	6.55e-13	-1.86	7.39e-13	-1.69
100	1.98e-02	1.22e-14	2.85	3.22e-12	-1.12	1.06e-12	-2.18	1.21e-12	-2.22
120	1.65e-02	1.75e-14	-2.00	6.07e-12	-3.50	1.46e-12	-1.75	1.61e-12	-1.60
140	1.42e-02	3.25e-14	-4.04	6.84e-12	-0.78	2.10e-12	-2.40	2.25e-12	-2.19

METHOD 3

TEST 2.2		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.85e-15	—	1.20e-13	—	2.24e-14	—	4.04e-14	—
20	9.52e-02	6.20e-15	-0.74	7.07e-13	-2.75	8.89e-14	-2.13	1.78e-13	-2.29
40	4.88e-02	2.65e-14	-2.17	2.51e-12	-1.89	2.49e-13	-1.54	6.46e-13	-1.93
60	3.28e-02	3.88e-14	-0.96	5.34e-12	-1.90	6.64e-13	-2.47	1.46e-12	-2.05
80	2.47e-02	6.12e-14	-1.61	1.07e-11	-2.46	1.08e-12	-1.71	2.54e-12	-1.96
100	1.98e-02	3.33e-14	2.76	1.35e-11	-1.03	1.86e-12	-2.48	4.15e-12	-2.22
120	1.65e-02	7.08e-14	-4.18	2.40e-11	-3.19	2.48e-12	-1.59	5.71e-12	-1.77
140	1.42e-02	8.95e-14	-1.54	2.75e-11	-0.90	3.56e-12	-2.35	8.06e-12	-2.25

METHOD 4

TEST 2.2		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	5.86e-15	—	1.61e-13	—	3.92e-14	—	5.35e-14	—
20	9.52e-02	5.43e-15	0.12	4.53e-13	-1.60	1.33e-13	-1.89	1.75e-13	-1.83
40	4.88e-02	8.58e-15	-0.68	2.04e-12	-2.25	5.13e-13	-2.02	7.15e-13	-2.11
60	3.28e-02	2.83e-14	-3.00	4.35e-12	-1.90	1.14e-12	-2.01	1.55e-12	-1.95
80	2.47e-02	2.97e-14	-0.17	7.37e-12	-1.86	2.02e-12	-2.01	2.86e-12	-2.16
100	1.98e-02	1.94e-14	1.92	1.17e-11	-2.08	3.13e-12	-2.00	4.39e-12	-1.94
120	1.65e-02	2.84e-14	-2.10	1.72e-11	-2.15	4.49e-12	-1.99	6.26e-12	-1.97
140	1.42e-02	2.22e-14	1.61	2.20e-11	-1.62	5.96e-12	-1.85	8.39e-12	-1.91

METHOD 5 USING BOUNDARY CONDITION 1

TEST 2.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	8.98e-02	—	9.57e-01	—	1.67e-01	—	3.86e-01	—
20	9.52e-02	3.23e-02	1.58	8.53e-01	0.18	1.21e-01	0.50	4.51e-01	-0.24
40	4.88e-02	1.12e-02	1.59	7.01e-01	0.29	7.56e-02	0.71	4.17e-01	0.11
60	3.28e-02	5.75e-03	1.67	6.07e-01	0.36	5.36e-02	0.86	3.76e-01	0.26
80	2.47e-02	3.50e-03	1.75	5.44e-01	0.39	4.13e-02	0.92	3.43e-01	0.32
100	1.98e-02	2.37e-03	1.77	4.98e-01	0.39	3.36e-02	0.93	3.19e-01	0.33
120	1.65e-02	1.72e-03	1.79	4.64e-01	0.40	2.85e-02	0.93	3.00e-01	0.34
140	1.42e-02	1.31e-03	1.79	4.36e-01	0.41	2.47e-02	0.93	2.85e-01	0.34

METHOD 5 USING BOUNDARY CONDITION 2

TEST 2.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.98e-15	—	1.43e-13	—	4.79e-14	—	1.03e-13	—
20	9.52e-02	2.30e-13	-5.20	7.82e-13	-2.63	5.27e-13	-3.71	6.24e-13	-2.79
40	4.88e-02	6.17e-14	1.97	2.04e-12	-1.43	7.22e-13	-0.47	8.60e-13	-0.48
60	3.28e-02	1.55e-13	-2.32	1.22e-11	-4.50	2.17e-12	-2.77	2.08e-12	-2.22
80	2.47e-02	6.75e-13	-5.18	2.42e-11	-2.41	4.08e-12	-2.23	4.26e-12	-2.53
100	1.98e-02	1.47e-12	-3.51	1.92e-11	1.05	6.17e-12	-1.88	6.65e-12	-2.02
120	1.65e-02	5.64e-13	5.29	4.93e-11	-5.23	9.80e-12	-2.56	8.75e-12	-1.52
140	1.42e-02	2.89e-13	4.37	3.90e-11	1.53	9.19e-12	0.42	1.03e-11	-1.06

$$\text{SOLUTION 3: } u(x, y) = e^{xy+2y}$$

METHOD 1

TEST 2.3		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.83e-02	—	1.21e+00	—	3.93e-01	—	1.21e-01	—
20	9.52e-02	2.15e-02	2.00	4.42e-01	1.56	1.64e-01	1.36	5.30e-02	1.28
40	4.88e-02	5.70e-03	1.98	1.52e-01	1.60	5.53e-02	1.62	1.74e-02	1.67
60	3.28e-02	2.59e-03	1.99	7.80e-02	1.68	2.76e-02	1.75	8.52e-03	1.79
80	2.47e-02	1.47e-03	2.00	4.75e-02	1.75	1.65e-02	1.81	5.06e-03	1.84
100	1.98e-02	9.45e-04	2.00	3.19e-02	1.80	1.10e-02	1.85	3.35e-03	1.86
120	1.65e-02	6.59e-04	2.00	2.30e-02	1.81	7.85e-03	1.86	2.38e-03	1.89
140	1.42e-02	4.85e-04	2.00	1.73e-02	1.84	5.88e-03	1.88	1.78e-03	1.90

METHOD 2 USING BOUNDARY CONDITION 1

TEST 2.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _{\hat{2}}$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.71e-01	—	5.32e+00	—	1.59e+00	—	1.43e+00	—
20	9.52e-02	1.41e-01	1.86	3.43e+00	0.68	8.47e-01	0.97	1.15e+00	0.33
40	4.88e-02	3.85e-02	1.94	2.01e+00	0.80	4.14e-01	1.07	7.20e-01	0.70
60	3.28e-02	1.76e-02	1.98	1.43e+00	0.86	2.68e-01	1.10	5.29e-01	0.78
80	2.47e-02	9.98e-03	1.99	1.12e+00	0.87	1.96e-01	1.10	4.24e-01	0.78
100	1.98e-02	6.43e-03	1.99	9.13e-01	0.91	1.54e-01	1.08	3.58e-01	0.77
120	1.65e-02	4.49e-03	1.99	7.74e-01	0.91	1.27e-01	1.07	3.08e-01	0.83
140	1.42e-02	3.30e-03	2.00	6.73e-01	0.92	1.08e-01	1.07	2.72e-01	0.81

METHOD 2 USING BOUNDARY CONDITION 2

TEST 2.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.40e-01	—	3.46e+00	—	1.21e+00	—	3.43e-01	—
20	9.52e-02	7.46e-02	1.81	1.47e+00	1.32	5.17e-01	1.32	1.73e-01	1.06
40	4.88e-02	2.12e-02	1.88	5.51e-01	1.47	1.77e-01	1.60	6.21e-02	1.53
60	3.28e-02	9.88e-03	1.92	2.91e-01	1.60	8.94e-02	1.72	3.15e-02	1.71
80	2.47e-02	5.68e-03	1.95	1.81e-01	1.69	5.39e-02	1.78	1.90e-02	1.78
100	1.98e-02	3.68e-03	1.96	1.23e-01	1.75	3.61e-02	1.82	1.27e-02	1.82
120	1.65e-02	2.58e-03	1.97	8.92e-02	1.77	2.59e-02	1.84	9.12e-03	1.85
140	1.42e-02	1.91e-03	1.98	6.77e-02	1.80	1.94e-02	1.87	6.85e-03	1.87

METHOD 3

TEST 2.3		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.73e-02	—	1.20e+00	—	8.88e-01	—	1.25e-01	—
20	9.52e-02	2.12e-02	2.00	4.37e-01	1.56	3.19e-01	1.58	5.41e-02	1.29
40	4.88e-02	5.64e-03	1.98	1.50e-01	1.60	9.83e-02	1.76	1.76e-02	1.68
60	3.28e-02	2.56e-03	1.99	7.70e-02	1.68	4.73e-02	1.84	8.64e-03	1.79
80	2.47e-02	1.45e-03	2.00	4.69e-02	1.75	2.77e-02	1.88	5.13e-03	1.84
100	1.98e-02	9.34e-04	2.00	3.15e-02	1.80	1.82e-02	1.90	3.40e-03	1.87
120	1.65e-02	6.51e-04	2.00	2.27e-02	1.81	1.29e-02	1.92	2.41e-03	1.89
140	1.42e-02	4.80e-04	2.00	1.71e-02	1.84	9.59e-03	1.93	1.80e-03	1.91

METHOD 4

TEST 2.3		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.64e-02	—	1.18e+00	—	1.41e+00	—	1.29e-01	—
20	9.52e-02	2.10e-02	2.00	4.31e-01	1.56	4.88e-01	1.64	5.53e-02	1.31
40	4.88e-02	5.57e-03	1.98	1.48e-01	1.60	1.46e-01	1.80	1.79e-02	1.68
60	3.28e-02	2.53e-03	1.99	7.60e-02	1.68	6.94e-02	1.88	8.77e-03	1.80
80	2.47e-02	1.44e-03	2.00	4.63e-02	1.75	4.04e-02	1.91	5.20e-03	1.84
100	1.98e-02	9.24e-04	2.00	3.11e-02	1.80	2.64e-02	1.92	3.44e-03	1.87
120	1.65e-02	6.44e-04	2.00	2.24e-02	1.82	1.86e-02	1.94	2.45e-03	1.90
140	1.42e-02	4.74e-04	2.00	1.69e-02	1.84	1.38e-02	1.94	1.83e-03	1.91

METHOD 5 USING BOUNDARY CONDITION 1

TEST 2.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.57e-01	—	1.51e+00	—	1.10e+00	—	4.81e-01	—
20	9.52e-02	7.15e-02	1.22	1.47e+00	0.04	7.95e-01	0.51	6.43e-01	-0.45
40	4.88e-02	2.49e-02	1.58	1.14e+00	0.38	5.24e-01	0.62	5.85e-01	0.14
60	3.28e-02	1.24e-02	1.75	9.86e-01	0.37	3.90e-01	0.74	5.31e-01	0.24
80	2.47e-02	7.39e-03	1.83	8.90e-01	0.36	3.15e-01	0.76	5.02e-01	0.19
100	1.98e-02	4.94e-03	1.83	8.15e-01	0.40	2.67e-01	0.74	4.78e-01	0.22
120	1.65e-02	3.51e-03	1.88	7.55e-01	0.42	2.33e-01	0.75	4.56e-01	0.26
140	1.42e-02	2.63e-03	1.90	7.03e-01	0.47	2.04e-01	0.87	4.31e-01	0.38

METHOD 5 USING BOUNDARY CONDITION 2

TEST 2.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.39e-01	—	2.11e+00	—	1.02e+00	—	3.21e+00	—
20	9.52e-02	1.63e-01	1.54	1.33e+00	0.72	5.12e-01	1.07	2.03e+00	0.71
40	4.88e-02	4.99e-02	1.77	6.32e-01	1.11	2.17e-01	1.28	1.02e+00	1.03
60	3.28e-02	2.35e-02	1.90	3.61e-01	1.41	1.18e-01	1.53	6.32e-01	1.20
80	2.47e-02	1.35e-02	1.94	2.33e-01	1.55	7.44e-02	1.64	4.39e-01	1.29
100	1.98e-02	8.83e-03	1.94	1.63e-01	1.60	5.15e-02	1.67	3.28e-01	1.32
120	1.65e-02	6.21e-03	1.95	1.20e-01	1.68	3.76e-02	1.74	2.57e-01	1.36
140	1.42e-02	4.60e-03	1.97	9.29e-02	1.70	2.86e-02	1.78	2.08e-01	1.39

$$\text{SOLUTION 4: } u(x, y) = \frac{x^3}{18} \left(3 \log(x^2) - 11 \right) + \left(y - \frac{1}{2} \right)^{\frac{8}{3}} \sqrt[5]{\left| x + \frac{1}{5} \right|^5}$$

METHOD 1

TEST 2.4		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.47e-03	—	5.46e-02	—	1.79e-01	—	6.00e-03	—
20	9.52e-02	5.47e-04	2.33	2.20e-02	1.41	5.42e-02	1.85	2.46e-03	1.38
40	4.88e-02	1.52e-04	1.91	8.94e-03	1.34	1.51e-02	1.91	9.51e-04	1.42
60	3.28e-02	7.58e-05	1.75	5.22e-03	1.36	7.01e-03	1.94	5.51e-04	1.37
80	2.47e-02	4.72e-05	1.67	3.58e-03	1.33	4.04e-03	1.95	3.71e-04	1.40
100	1.98e-02	3.33e-05	1.58	2.67e-03	1.33	2.63e-03	1.95	2.75e-04	1.35
120	1.65e-02	2.53e-05	1.52	2.10e-03	1.34	1.85e-03	1.95	2.17e-04	1.32
140	1.42e-02	2.01e-05	1.52	1.71e-03	1.33	1.37e-03	1.95	1.77e-04	1.33

METHOD 2 USING BOUNDARY CONDITION 1

TEST 2.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.08e-02	—	5.78e-01	—	1.33e-01	—	1.74e-01	—
20	9.52e-02	9.28e-03	1.85	3.33e-01	0.85	6.80e-02	1.04	1.13e-01	0.67
40	4.88e-02	2.61e-03	1.89	1.97e-01	0.78	3.27e-02	1.09	6.86e-02	0.75
60	3.28e-02	1.23e-03	1.90	1.38e-01	0.90	2.15e-02	1.05	4.48e-02	1.07
80	2.47e-02	7.10e-04	1.93	1.07e-01	0.87	1.59e-02	1.08	3.60e-02	0.77
100	1.98e-02	4.67e-04	1.89	8.72e-02	0.95	1.27e-02	1.02	2.86e-02	1.05
120	1.65e-02	3.29e-04	1.94	7.38e-02	0.92	1.05e-02	1.06	2.49e-02	0.76
140	1.42e-02	2.47e-04	1.87	6.38e-02	0.96	8.96e-03	1.01	2.13e-02	1.03

METHOD 2 USING BOUNDARY CONDITION 2

TEST 2.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.23e-02	—	2.66e-01	—	1.13e-01	—	2.90e-02	—
20	9.52e-02	3.37e-03	2.00	1.22e-01	1.21	4.66e-02	1.37	1.43e-02	1.09
40	4.88e-02	1.09e-03	1.69	5.36e-02	1.23	1.76e-02	1.46	6.03e-03	1.29
60	3.28e-02	5.25e-04	1.84	3.08e-02	1.40	9.74e-03	1.48	3.31e-03	1.51
80	2.47e-02	3.26e-04	1.68	2.09e-02	1.37	6.37e-03	1.50	2.17e-03	1.49
100	1.98e-02	2.19e-04	1.81	1.51e-02	1.45	4.57e-03	1.50	1.54e-03	1.54
120	1.65e-02	1.62e-04	1.66	1.17e-02	1.43	3.48e-03	1.50	1.17e-03	1.51
140	1.42e-02	1.23e-04	1.79	9.34e-03	1.47	2.77e-03	1.51	9.27e-04	1.55

METHOD 3

TEST 2.4		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.28e-03	—	5.26e-02	—	9.52e-02	—	6.18e-03	—
20	9.52e-02	5.32e-04	2.25	2.18e-02	1.36	3.05e-02	1.76	2.53e-03	1.38
40	4.88e-02	1.55e-04	1.84	8.94e-03	1.34	8.86e-03	1.85	9.74e-04	1.42
60	3.28e-02	7.90e-05	1.70	5.23e-03	1.35	4.18e-03	1.89	5.62e-04	1.38
80	2.47e-02	4.97e-05	1.63	3.59e-03	1.33	2.44e-03	1.90	3.77e-04	1.41
100	1.98e-02	3.52e-05	1.56	2.68e-03	1.33	1.60e-03	1.91	2.79e-04	1.36
120	1.65e-02	2.68e-05	1.52	2.10e-03	1.34	1.13e-03	1.90	2.20e-04	1.33
140	1.42e-02	2.12e-05	1.52	1.71e-03	1.33	8.49e-04	1.90	1.79e-04	1.34

METHOD 4

TEST 2.4		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.16e-03	—	5.13e-02	—	1.37e-01	—	6.73e-03	—
20	9.52e-02	5.42e-04	2.14	2.19e-02	1.32	4.34e-02	1.78	2.67e-03	1.43
40	4.88e-02	1.64e-04	1.79	8.98e-03	1.33	1.23e-02	1.88	1.01e-03	1.45
60	3.28e-02	8.45e-05	1.67	5.25e-03	1.35	5.76e-03	1.92	5.80e-04	1.41
80	2.47e-02	5.33e-05	1.62	3.60e-03	1.33	3.33e-03	1.93	3.87e-04	1.43
100	1.98e-02	3.78e-05	1.56	2.69e-03	1.33	2.18e-03	1.93	2.85e-04	1.38
120	1.65e-02	2.87e-05	1.53	2.11e-03	1.34	1.53e-03	1.93	2.24e-04	1.34
140	1.42e-02	2.27e-05	1.53	1.72e-03	1.33	1.14e-03	1.93	1.82e-04	1.36

METHOD 5 USING BOUNDARY CONDITION 1

TEST 2.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.52e-01	—	1.61e+00	—	4.29e-01	—	8.34e-01	—
20	9.52e-02	5.80e-02	1.49	1.37e+00	0.25	2.68e-01	0.73	6.43e-01	0.40
40	4.88e-02	1.88e-02	1.69	1.20e+00	0.20	1.71e-01	0.67	6.31e-01	0.03
60	3.28e-02	9.31e-03	1.76	1.08e+00	0.25	1.33e-01	0.63	6.10e-01	0.09
80	2.47e-02	5.60e-03	1.79	9.97e-01	0.29	1.10e-01	0.66	5.83e-01	0.16
100	1.98e-02	3.77e-03	1.79	9.31e-01	0.31	9.51e-02	0.68	5.58e-01	0.19
120	1.65e-02	2.72e-03	1.80	8.77e-01	0.33	8.37e-02	0.70	5.37e-01	0.22
140	1.42e-02	2.06e-03	1.81	8.33e-01	0.34	7.49e-02	0.73	5.17e-01	0.24

METHOD 5 USING BOUNDARY CONDITION 2

TEST 2.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.17e-01	—	9.37e-01	—	4.15e-01	—	6.51e-01	—
20	9.52e-02	6.70e-02	1.82	4.34e-01	1.19	1.70e-01	1.38	2.28e-01	1.63
40	4.88e-02	1.84e-02	1.93	2.11e-01	1.08	6.29e-02	1.49	7.87e-02	1.59
60	3.28e-02	8.38e-03	1.98	1.49e-01	0.87	3.62e-02	1.39	4.43e-02	1.44
80	2.47e-02	4.76e-03	1.99	1.19e-01	0.80	2.51e-02	1.30	3.01e-02	1.36
100	1.98e-02	3.09e-03	1.97	9.98e-02	0.79	1.91e-02	1.25	2.26e-02	1.31
120	1.65e-02	2.16e-03	1.98	8.62e-02	0.81	1.52e-02	1.24	1.79e-02	1.28
140	1.42e-02	1.59e-03	1.99	7.59e-02	0.83	1.26e-02	1.25	1.47e-02	1.28

8.3. Test 3 Results

$$\text{SOLUTION 1: } u(x, y) = \sin\left(\frac{\pi}{2}(x + y)^2\right)$$

METHOD 1

TEST 3.1		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.60e-02	—	2.82e-01	—	3.42e-01	—	2.69e-01	—
20	9.52e-02	9.62e-03	2.04	1.24e-01	1.27	1.45e-01	1.32	1.19e-01	1.26
40	4.88e-02	2.51e-03	2.01	4.36e-02	1.56	5.53e-02	1.44	4.22e-02	1.55
60	3.28e-02	1.13e-03	2.01	2.19e-02	1.74	2.84e-02	1.68	2.11e-02	1.74
80	2.47e-02	6.42e-04	2.00	1.29e-02	1.86	1.71e-02	1.78	1.27e-02	1.80
100	1.98e-02	4.12e-04	2.00	8.59e-03	1.85	1.14e-02	1.83	8.45e-03	1.84
120	1.65e-02	2.87e-04	2.00	6.12e-03	1.88	8.18e-03	1.86	6.01e-03	1.88
140	1.42e-02	2.11e-04	2.00	4.57e-03	1.90	6.13e-03	1.88	4.48e-03	1.92

METHOD 2 USING BOUNDARY CONDITION 1

TEST 3.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _{\tilde{2}}$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.85e-01	—	1.31e+00	—	1.61e+00	—	4.92e-01	—
20	9.52e-02	4.93e-02	2.05	5.50e-01	1.35	6.35e-01	1.44	3.38e-01	0.58
40	4.88e-02	1.29e-02	2.01	1.80e-01	1.67	1.96e-01	1.76	1.44e-01	1.28
60	3.28e-02	5.80e-03	2.00	9.43e-02	1.63	9.35e-02	1.87	7.98e-02	1.48
80	2.47e-02	3.29e-03	2.00	5.71e-02	1.77	5.44e-02	1.91	4.97e-02	1.67
100	1.98e-02	2.11e-03	2.00	3.86e-02	1.77	3.57e-02	1.91	3.37e-02	1.76
120	1.65e-02	1.47e-03	2.00	2.82e-02	1.73	2.52e-02	1.92	2.46e-02	1.74
140	1.42e-02	1.09e-03	2.00	2.16e-02	1.75	1.89e-02	1.90	1.87e-02	1.80

METHOD 2 USING BOUNDARY CONDITION 2

TEST 3.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.31e-01	—	1.24e+00	—	1.60e+00	—	1.22e+00	—
20	9.52e-02	3.60e-02	2.00	4.33e-01	1.63	6.48e-01	1.39	4.20e-01	1.65
40	4.88e-02	9.62e-03	1.97	1.71e-01	1.39	2.31e-01	1.54	1.68e-01	1.36
60	3.28e-02	4.39e-03	1.98	8.84e-02	1.66	1.17e-01	1.71	8.74e-02	1.65
80	2.47e-02	2.51e-03	1.98	5.27e-02	1.82	7.04e-02	1.80	5.24e-02	1.80
100	1.98e-02	1.62e-03	1.98	3.51e-02	1.85	4.69e-02	1.84	3.53e-02	1.80
120	1.65e-02	1.13e-03	1.98	2.50e-02	1.87	3.34e-02	1.87	2.50e-02	1.90
140	1.42e-02	8.34e-04	1.99	1.87e-02	1.91	2.50e-02	1.89	1.86e-02	1.93

METHOD 3

TEST 3.1		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.67e-02	—	3.12e-01	—	2.25e+00	—	2.97e-01	—
20	9.52e-02	9.81e-03	2.04	1.31e-01	1.34	8.79e-01	1.45	1.24e-01	1.35
40	4.88e-02	2.56e-03	2.01	4.59e-02	1.57	2.72e-01	1.75	4.45e-02	1.53
60	3.28e-02	1.16e-03	2.01	2.31e-02	1.72	1.29e-01	1.87	2.23e-02	1.73
80	2.47e-02	6.55e-04	2.00	1.36e-02	1.88	7.52e-02	1.91	1.35e-02	1.79
100	1.98e-02	4.21e-04	2.00	9.06e-03	1.83	4.91e-02	1.93	8.98e-03	1.83
120	1.65e-02	2.93e-04	2.00	6.44e-03	1.88	3.45e-02	1.95	6.40e-03	1.88
140	1.42e-02	2.16e-04	2.01	4.78e-03	1.95	2.56e-02	1.95	4.75e-03	1.94

METHOD 4

TEST 3.1		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \delta_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.36e-02	—	2.85e-01	—	3.00e+00	—	2.66e-01	—
20	9.52e-02	8.94e-03	2.05	1.25e-01	1.28	1.16e+00	1.47	1.14e-01	1.31
40	4.88e-02	2.33e-03	2.01	4.24e-02	1.61	3.56e-01	1.77	4.08e-02	1.53
60	3.28e-02	1.05e-03	2.00	2.13e-02	1.73	1.69e-01	1.88	2.04e-02	1.75
80	2.47e-02	5.96e-04	2.00	1.25e-02	1.89	9.79e-02	1.92	1.22e-02	1.79
100	1.98e-02	3.83e-04	2.00	8.33e-03	1.83	6.38e-02	1.94	8.14e-03	1.85
120	1.65e-02	2.67e-04	2.00	5.93e-03	1.89	4.49e-02	1.95	5.80e-03	1.88
140	1.42e-02	1.96e-04	2.01	4.41e-03	1.93	3.33e-02	1.96	4.32e-03	1.92

METHOD 5 USING BOUNDARY CONDITION 1

TEST 3.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.58e-01	—	8.11e+00	—	5.12e+00	—	4.92e+00	—
20	9.52e-02	2.11e-01	1.20	6.39e+00	0.37	3.39e+00	0.64	3.83e+00	0.39
40	4.88e-02	7.32e-02	1.58	4.23e+00	0.61	1.79e+00	0.95	2.54e+00	0.61
60	3.28e-02	3.65e-02	1.75	3.19e+00	0.72	1.10e+00	1.23	1.83e+00	0.82
80	2.47e-02	2.17e-02	1.83	2.61e+00	0.71	7.48e-01	1.36	1.42e+00	0.91
100	1.98e-02	1.44e-02	1.85	2.25e+00	0.66	5.45e-01	1.43	1.17e+00	0.87
120	1.65e-02	1.02e-02	1.89	2.01e+00	0.63	4.20e-01	1.44	1.01e+00	0.82
140	1.42e-02	7.65e-03	1.91	1.83e+00	0.60	3.38e-01	1.43	8.98e-01	0.77

METHOD 5 USING BOUNDARY CONDITION 2

TEST 3.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.77e-01	—	7.76e+00	—	5.16e+00	—	7.66e+00	—
20	9.52e-02	2.14e-01	1.24	4.88e+00	0.72	3.22e+00	0.73	4.77e+00	0.73
40	4.88e-02	7.39e-02	1.59	2.34e+00	1.10	1.47e+00	1.17	2.27e+00	1.11
60	3.28e-02	3.67e-02	1.76	1.33e+00	1.42	8.16e-01	1.48	1.28e+00	1.44
80	2.47e-02	2.18e-02	1.84	8.51e-01	1.58	5.16e-01	1.62	8.30e-01	1.53
100	1.98e-02	1.45e-02	1.86	5.93e-01	1.64	3.55e-01	1.69	5.82e-01	1.61
120	1.65e-02	1.03e-02	1.89	4.38e-01	1.67	2.60e-01	1.73	4.34e-01	1.62
140	1.42e-02	7.67e-03	1.91	3.39e-01	1.68	1.98e-01	1.77	3.39e-01	1.62

$$\text{SOLUTION 2: } u(x, y) = x^2 + 3xy + \frac{1}{2}y^2 + 3$$

METHOD 1

TEST 3.2		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.60e-15	—	6.79e-14	—	3.47e-14	—	7.15e-14	—
20	9.52e-02	4.59e-15	-0.88	2.14e-13	-1.78	1.13e-13	-1.82	2.06e-13	-1.64
40	4.88e-02	8.39e-15	-0.90	7.91e-13	-1.95	4.50e-13	-2.07	8.06e-13	-2.04
60	3.28e-02	2.42e-14	-2.67	1.81e-12	-2.09	1.01e-12	-2.03	1.77e-12	-1.97
80	2.47e-02	7.63e-15	4.07	3.24e-12	-2.05	1.77e-12	-1.99	3.10e-12	-1.98
100	1.98e-02	1.53e-14	-3.15	4.98e-12	-1.95	2.69e-12	-1.91	4.89e-12	-2.08
120	1.65e-02	1.65e-14	-0.43	7.27e-12	-2.10	4.01e-12	-2.19	7.06e-12	-2.02
140	1.42e-02	2.99e-14	-3.89	9.65e-12	-1.85	5.37e-12	-1.92	9.56e-12	-1.98

METHOD 2 USING BOUNDARY CONDITION 1

TEST 3.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.93e-02	—	6.76e-02	—	3.71e-02	—	2.18e-02	—
20	9.52e-02	8.19e-03	1.97	3.68e-02	0.94	1.91e-02	1.03	1.37e-02	0.72
40	4.88e-02	2.17e-03	1.98	1.92e-02	0.97	9.61e-03	1.02	7.08e-03	0.99
60	3.28e-02	9.83e-04	1.99	1.34e-02	0.91	6.26e-03	1.08	4.88e-03	0.94
80	2.47e-02	5.58e-04	2.00	9.99e-03	1.03	4.68e-03	1.03	3.76e-03	0.92
100	1.98e-02	3.59e-04	2.00	8.03e-03	0.99	3.72e-03	1.03	3.05e-03	0.95
120	1.65e-02	2.50e-04	2.00	6.68e-03	1.02	3.08e-03	1.05	2.52e-03	1.06
140	1.42e-02	1.84e-04	2.00	5.73e-03	1.00	2.64e-03	1.01	2.17e-03	0.99

METHOD 2 USING BOUNDARY CONDITION 2

TEST 3.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.11e-15	—	1.55e-14	—	9.43e-15	—	1.81e-14	—
20	9.52e-02	1.85e-15	-0.79	6.54e-14	-2.23	3.44e-14	-2.00	6.34e-14	-1.94
40	4.88e-02	3.77e-15	-1.06	2.21e-13	-1.82	1.39e-13	-2.08	2.31e-13	-1.93
60	3.28e-02	4.16e-15	-0.25	4.92e-13	-2.02	3.30e-13	-2.19	4.99e-13	-1.94
80	2.47e-02	6.41e-15	-1.52	8.66e-13	-1.99	5.67e-13	-1.91	8.61e-13	-1.93
100	1.98e-02	5.83e-15	0.43	1.37e-12	-2.09	9.02e-13	-2.10	1.33e-12	-1.97
120	1.65e-02	9.63e-15	-2.78	1.91e-12	-1.82	1.28e-12	-1.94	1.87e-12	-1.88
140	1.42e-02	9.07e-15	0.39	2.55e-12	-1.91	1.72e-12	-1.91	2.47e-12	-1.82

METHOD 3

TEST 3.2		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.16e-15	—	7.16e-14	—	1.82e-14	—	7.61e-14	—
20	9.52e-02	2.26e-15	-0.07	2.07e-13	-1.64	5.38e-14	-1.68	2.07e-13	-1.55
40	4.88e-02	5.47e-15	-1.32	7.64e-13	-1.95	2.05e-13	-2.00	7.75e-13	-1.97
60	3.28e-02	6.84e-15	-0.56	1.72e-12	-2.04	4.69e-13	-2.09	1.68e-12	-1.95
80	2.47e-02	1.50e-14	-2.77	3.15e-12	-2.14	8.03e-13	-1.90	2.98e-12	-2.03
100	1.98e-02	1.02e-14	1.76	4.92e-12	-2.02	1.32e-12	-2.25	4.75e-12	-2.11
120	1.65e-02	8.49e-15	1.00	6.99e-12	-1.95	1.88e-12	-1.95	6.76e-12	-1.95
140	1.42e-02	1.62e-14	-4.24	9.56e-12	-2.05	2.56e-12	-2.02	9.28e-12	-2.07

METHOD 4

TEST 3.2		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.74e-15	—	4.60e-14	—	2.65e-14	—	5.17e-14	—
20	9.52e-02	3.62e-15	-1.13	2.02e-13	-2.29	1.16e-13	-2.29	2.09e-13	-2.16
40	4.88e-02	5.81e-15	-0.71	8.19e-13	-2.09	4.41e-13	-2.00	7.80e-13	-1.97
60	3.28e-02	1.01e-14	-1.39	1.80e-12	-1.99	9.78e-13	-2.00	1.78e-12	-2.07
80	2.47e-02	7.52e-15	1.04	3.12e-12	-1.93	1.73e-12	-2.01	3.06e-12	-1.92
100	1.98e-02	1.04e-14	-1.47	4.86e-12	-2.01	2.66e-12	-1.94	4.67e-12	-1.92
120	1.65e-02	1.26e-14	-1.05	7.13e-12	-2.12	3.97e-12	-2.22	7.04e-12	-2.27
140	1.42e-02	1.35e-14	-0.47	9.70e-12	-2.01	5.31e-12	-1.89	9.65e-12	-2.06

METHOD 5 USING BOUNDARY CONDITION 1

TEST 3.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	9.12e-02	—	8.26e-01	—	1.54e-01	—	2.92e-01	—
20	9.52e-02	3.07e-02	1.69	6.52e-01	0.36	8.31e-02	0.95	2.63e-01	0.16
40	4.88e-02	8.82e-03	1.86	4.85e-01	0.44	4.36e-02	0.96	2.04e-01	0.38
60	3.28e-02	4.15e-03	1.90	4.04e-01	0.46	3.19e-02	0.79	1.73e-01	0.41
80	2.47e-02	2.44e-03	1.88	3.55e-01	0.46	2.68e-02	0.61	1.54e-01	0.41
100	1.98e-02	1.60e-03	1.90	3.20e-01	0.47	2.35e-02	0.59	1.41e-01	0.42
120	1.65e-02	1.13e-03	1.91	2.94e-01	0.47	2.09e-02	0.65	1.30e-01	0.43
140	1.42e-02	8.48e-04	1.90	2.73e-01	0.47	1.86e-02	0.75	1.22e-01	0.43

METHOD 5 USING BOUNDARY CONDITION 2

TEST 3.2		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.21e-14	—	1.52e-13	—	1.10e-13	—	1.83e-13	—
20	9.52e-02	2.68e-13	-2.86	7.57e-13	-2.49	6.05e-13	-2.64	7.81e-13	-2.25
40	4.88e-02	1.10e-12	-2.11	4.56e-12	-2.68	2.82e-12	-2.30	4.16e-12	-2.50
60	3.28e-02	3.17e-12	-2.66	1.21e-11	-2.45	8.39e-12	-2.75	1.07e-11	-2.37
80	2.47e-02	4.73e-12	-1.41	1.70e-11	-1.21	1.27e-11	-1.45	1.58e-11	-1.37
100	1.98e-02	4.47e-12	0.26	1.81e-11	-0.30	1.30e-11	-0.13	1.78e-11	-0.55
120	1.65e-02	2.30e-13	16.42	1.24e-11	2.10	6.99e-12	3.44	1.23e-11	2.04
140	1.42e-02	4.55e-12	-19.52	2.41e-11	-4.32	1.62e-11	-5.49	2.32e-11	-4.15

$$\text{SOLUTION 3: } u(x, y) = e^{xy+2y}$$

METHOD 1

TEST 3.3		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.14e-02	—	3.55e-01	—	3.84e-01	—	2.78e-01	—
20	9.52e-02	1.17e-02	1.96	1.23e-01	1.64	1.45e-01	1.50	1.03e-01	1.54
40	4.88e-02	3.05e-03	2.01	3.82e-02	1.75	4.52e-02	1.75	3.03e-02	1.83
60	3.28e-02	1.39e-03	1.99	1.85e-02	1.82	2.18e-02	1.83	1.43e-02	1.90
80	2.47e-02	7.89e-04	1.99	1.07e-02	1.94	1.28e-02	1.87	8.32e-03	1.91
100	1.98e-02	5.06e-04	2.01	7.05e-03	1.88	8.43e-03	1.91	5.43e-03	1.93
120	1.65e-02	3.53e-04	1.99	4.98e-03	1.92	5.96e-03	1.92	3.83e-03	1.94
140	1.42e-02	2.60e-04	2.00	3.70e-03	1.95	4.43e-03	1.94	2.85e-03	1.93

METHOD 2 USING BOUNDARY CONDITION 1

TEST 3.3.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _{\hat{2}}$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.32e-01	—	3.05e+00	—	1.41e+00	—	6.20e-01	—
20	9.52e-02	1.00e-01	1.85	1.44e+00	1.16	7.29e-01	1.02	4.22e-01	0.60
40	4.88e-02	2.82e-02	1.90	1.10e+00	0.40	3.75e-01	0.99	2.27e-01	0.92
60	3.28e-02	1.31e-02	1.93	8.42e-01	0.68	2.81e-01	0.73	2.32e-01	-0.05
80	2.47e-02	7.52e-03	1.96	6.38e-01	0.98	2.12e-01	0.99	1.99e-01	0.55
100	1.98e-02	4.85e-03	1.99	5.15e-01	0.97	1.68e-01	1.07	1.68e-01	0.75
120	1.65e-02	3.38e-03	1.99	4.31e-01	0.98	1.39e-01	1.02	1.45e-01	0.83
140	1.42e-02	2.48e-03	2.02	3.71e-01	0.98	1.19e-01	1.03	1.26e-01	0.90

METHOD 2 USING BOUNDARY CONDITION 2

TEST 3.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.22e-01	—	1.03e+00	—	1.06e+00	—	8.26e-01	—
20	9.52e-02	4.04e-02	1.71	4.54e-01	1.27	4.11e-01	1.47	3.88e-01	1.17
40	4.88e-02	1.12e-02	1.92	1.61e-01	1.55	1.30e-01	1.72	1.23e-01	1.72
60	3.28e-02	5.24e-03	1.92	8.12e-02	1.73	6.41e-02	1.77	5.74e-02	1.91
80	2.47e-02	3.02e-03	1.94	4.86e-02	1.81	3.78e-02	1.87	3.42e-02	1.83
100	1.98e-02	1.95e-03	1.97	3.25e-02	1.82	2.49e-02	1.89	2.27e-02	1.85
120	1.65e-02	1.37e-03	1.96	2.33e-02	1.85	1.77e-02	1.91	1.62e-02	1.87
140	1.42e-02	1.01e-03	1.97	1.74e-02	1.89	1.32e-02	1.93	1.22e-02	1.88

METHOD 3

TEST 3.3		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.08e-02	—	3.54e-01	—	8.72e-01	—	2.78e-01	—
20	9.52e-02	1.15e-02	1.96	1.23e-01	1.63	3.04e-01	1.63	1.03e-01	1.54
40	4.88e-02	3.01e-03	2.01	3.81e-02	1.75	9.16e-02	1.80	3.03e-02	1.83
60	3.28e-02	1.37e-03	1.98	1.85e-02	1.82	4.35e-02	1.88	1.43e-02	1.90
80	2.47e-02	7.79e-04	1.99	1.06e-02	1.94	2.53e-02	1.91	8.32e-03	1.91
100	1.98e-02	5.00e-04	2.01	7.03e-03	1.88	1.65e-02	1.93	5.43e-03	1.93
120	1.65e-02	3.49e-04	1.99	4.97e-03	1.92	1.16e-02	1.94	3.83e-03	1.94
140	1.42e-02	2.57e-04	2.00	3.69e-03	1.95	8.63e-03	1.95	2.85e-03	1.93

METHOD 4

TEST 3.3		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \delta_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.95e-02	—	3.54e-01	—	1.15e+00	—	2.80e-01	—
20	9.52e-02	1.12e-02	1.95	1.23e-01	1.64	3.39e-01	1.88	1.03e-01	1.54
40	4.88e-02	2.91e-03	2.01	3.80e-02	1.75	1.03e-01	1.78	3.05e-02	1.83
60	3.28e-02	1.33e-03	1.98	1.84e-02	1.82	4.99e-02	1.83	1.44e-02	1.90
80	2.47e-02	7.54e-04	1.99	1.06e-02	1.94	2.90e-02	1.92	8.35e-03	1.91
100	1.98e-02	4.84e-04	2.02	7.02e-03	1.88	1.92e-02	1.87	5.46e-03	1.93
120	1.65e-02	3.37e-04	1.99	4.96e-03	1.92	1.35e-02	1.96	3.84e-03	1.94
140	1.42e-02	2.49e-04	2.00	3.68e-03	1.95	9.85e-03	2.04	2.86e-03	1.93

METHOD 5 USING BOUNDARY CONDITION 1

TEST 3.3.1		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.37e-01	—	1.90e+00	—	1.46e+00	—	6.89e-01	—
20	9.52e-02	1.33e-01	0.89	2.17e+00	-0.21	1.37e+00	0.09	1.05e+00	-0.65
40	4.88e-02	5.53e-02	1.31	1.90e+00	0.20	1.30e+00	0.09	1.32e+00	-0.35
60	3.28e-02	2.97e-02	1.57	1.46e+00	0.66	1.06e+00	0.50	1.12e+00	0.40
80	2.47e-02	1.97e-02	1.45	1.16e+00	0.81	9.18e-01	0.52	9.07e-01	0.76
100	1.98e-02	1.27e-02	2.00	9.55e-01	0.87	7.72e-01	0.79	7.27e-01	1.00
120	1.65e-02	9.34e-03	1.68	8.27e-01	0.79	6.76e-01	0.74	6.13e-01	0.95
140	1.42e-02	7.28e-03	1.63	7.40e-01	0.73	6.22e-01	0.54	5.45e-01	0.77

METHOD 5 USING BOUNDARY CONDITION 2

TEST 3.3		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	5.41e-01	—	2.18e+00	—	1.17e+00	—	3.71e+00	—
20	9.52e-02	2.01e-01	1.53	1.16e+00	0.97	6.17e-01	0.99	2.71e+00	0.49
40	4.88e-02	6.40e-02	1.71	4.91e-01	1.29	2.93e-01	1.11	1.59e+00	0.80
60	3.28e-02	3.14e-02	1.79	2.74e-01	1.46	1.74e-01	1.31	1.06e+00	1.02
80	2.47e-02	1.86e-02	1.85	1.77e-01	1.55	1.17e-01	1.41	7.89e-01	1.03
100	1.98e-02	1.24e-02	1.82	1.22e-01	1.67	8.59e-02	1.39	6.00e-01	1.24
120	1.65e-02	8.88e-03	1.87	8.99e-02	1.71	6.59e-02	1.47	4.89e-01	1.13
140	1.42e-02	6.65e-03	1.89	6.89e-02	1.74	5.28e-02	1.45	4.13e-01	1.11

$$\text{SOLUTION 4: } u(x, y) = \frac{x^3}{18} \left(3 \log(x^2) - 11 \right) + \left(y - \frac{1}{2} \right)^{\frac{8}{3}} \sqrt[5]{\left| x + \frac{1}{5} \right|^5}$$

METHOD 1

TEST 3.4		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.68e-03	—	2.62e-02	—	1.79e-01	—	3.13e-02	—
20	9.52e-02	1.37e-03	1.53	1.19e-02	1.21	5.39e-02	1.85	1.47e-02	1.17
40	4.88e-02	5.17e-04	1.45	5.65e-03	1.12	1.51e-02	1.91	6.47e-03	1.23
60	3.28e-02	2.92e-04	1.44	3.42e-03	1.26	7.03e-03	1.92	3.75e-03	1.37
80	2.47e-02	1.95e-04	1.43	2.35e-03	1.33	4.10e-03	1.90	2.64e-03	1.25
100	1.98e-02	1.41e-04	1.46	1.77e-03	1.29	2.70e-03	1.89	1.92e-03	1.43
120	1.65e-02	1.09e-04	1.45	1.40e-03	1.29	1.93e-03	1.87	1.53e-03	1.27
140	1.42e-02	8.70e-05	1.45	1.14e-03	1.34	1.45e-03	1.84	1.23e-03	1.41

METHOD 2 USING BOUNDARY CONDITION 1

TEST 3.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.97e-02	—	2.62e-01	—	1.50e-01	—	8.25e-02	—
20	9.52e-02	1.22e-02	2.17	1.56e-01	0.80	8.29e-02	0.92	5.38e-02	0.66
40	4.88e-02	3.41e-03	1.91	8.04e-02	0.99	4.25e-02	1.00	3.07e-02	0.84
60	3.28e-02	1.58e-03	1.94	5.92e-02	0.77	2.90e-02	0.96	2.38e-02	0.64
80	2.47e-02	9.30e-04	1.87	4.41e-02	1.04	2.13e-02	1.08	1.79e-02	1.00
100	1.98e-02	6.06e-04	1.94	3.54e-02	1.00	1.70e-02	1.03	1.46e-02	0.93
120	1.65e-02	4.34e-04	1.85	2.95e-02	1.01	1.39e-02	1.10	1.21e-02	1.03
140	1.42e-02	3.23e-04	1.94	2.53e-02	1.00	1.19e-02	1.02	1.04e-02	1.00

METHOD 2 USING BOUNDARY CONDITION 2

TEST 3.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.70e-02	—	1.28e-01	—	1.14e-01	—	1.10e-01	—
20	9.52e-02	3.64e-03	2.39	4.19e-02	1.73	4.83e-02	1.32	5.10e-02	1.18
40	4.88e-02	9.53e-04	2.00	2.05e-02	1.06	1.71e-02	1.55	1.89e-02	1.49
60	3.28e-02	4.47e-04	1.91	1.18e-02	1.40	9.71e-03	1.43	1.02e-02	1.55
80	2.47e-02	2.71e-04	1.76	7.77e-03	1.47	6.39e-03	1.48	6.71e-03	1.47
100	1.98e-02	1.85e-04	1.72	5.55e-03	1.52	4.62e-03	1.47	4.84e-03	1.48
120	1.65e-02	1.38e-04	1.64	4.23e-03	1.51	3.53e-03	1.49	3.73e-03	1.44
140	1.42e-02	1.07e-04	1.64	3.36e-03	1.50	2.80e-03	1.52	2.94e-03	1.57

METHOD 3

TEST 3.4		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.65e-03	—	2.58e-02	—	9.49e-02	—	3.03e-02	—
20	9.52e-02	1.35e-03	1.54	1.19e-02	1.20	3.01e-02	1.78	1.43e-02	1.16
40	4.88e-02	5.12e-04	1.45	5.62e-03	1.12	8.78e-03	1.84	6.35e-03	1.21
60	3.28e-02	2.89e-04	1.44	3.40e-03	1.26	4.24e-03	1.83	3.70e-03	1.36
80	2.47e-02	1.93e-04	1.42	2.34e-03	1.32	2.56e-03	1.78	2.61e-03	1.24
100	1.98e-02	1.40e-04	1.45	1.76e-03	1.29	1.73e-03	1.77	1.90e-03	1.43
120	1.65e-02	1.08e-04	1.44	1.39e-03	1.29	1.27e-03	1.72	1.51e-03	1.26
140	1.42e-02	8.64e-05	1.45	1.14e-03	1.34	9.82e-04	1.67	1.22e-03	1.40

METHOD 4

TEST 3.4		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \check{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.37e-03	—	2.52e-02	—	1.60e-01	—	3.24e-02	—
20	9.52e-02	1.58e-03	1.58	1.17e-02	1.18	4.86e-02	1.85	1.50e-02	1.19
40	4.88e-02	5.72e-04	1.52	5.61e-03	1.10	1.37e-02	1.90	6.55e-03	1.24
60	3.28e-02	3.16e-04	1.49	3.41e-03	1.26	6.41e-03	1.90	3.79e-03	1.38
80	2.47e-02	2.08e-04	1.47	2.34e-03	1.32	3.77e-03	1.88	2.65e-03	1.25
100	1.98e-02	1.50e-04	1.49	1.76e-03	1.29	2.49e-03	1.88	1.93e-03	1.44
120	1.65e-02	1.15e-04	1.48	1.40e-03	1.29	1.78e-03	1.85	1.54e-03	1.28
140	1.42e-02	9.15e-05	1.48	1.14e-03	1.34	1.35e-03	1.83	1.24e-03	1.41

METHOD 5 USING BOUNDARY CONDITION 1

TEST 3.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.58e-01	—	1.40e+00	—	3.76e-01	—	6.71e-01	—
20	9.52e-02	5.51e-02	1.63	1.13e+00	0.33	2.16e-01	0.85	4.32e-01	0.68
40	4.88e-02	1.60e-02	1.85	9.04e-01	0.34	1.23e-01	0.85	3.46e-01	0.33
60	3.28e-02	7.48e-03	1.91	7.80e-01	0.37	9.44e-02	0.66	3.10e-01	0.28
80	2.47e-02	4.34e-03	1.92	6.94e-01	0.42	7.91e-02	0.63	2.82e-01	0.33
100	1.98e-02	2.85e-03	1.92	6.33e-01	0.42	6.88e-02	0.63	2.63e-01	0.33
120	1.65e-02	2.00e-03	1.95	5.85e-01	0.43	6.07e-02	0.69	2.46e-01	0.36
140	1.42e-02	1.48e-03	1.96	5.45e-01	0.46	5.37e-02	0.79	2.31e-01	0.42

METHOD 5 USING BOUNDARY CONDITION 2

TEST 3.4		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.31e-01	—	8.15e-01	—	4.16e-01	—	7.35e-01	—
20	9.52e-02	7.19e-02	1.81	3.29e-01	1.40	1.61e-01	1.47	2.73e-01	1.53
40	4.88e-02	1.96e-02	1.94	1.40e-01	1.28	5.37e-02	1.64	9.80e-02	1.53
60	3.28e-02	9.03e-03	1.95	9.12e-02	1.08	2.81e-02	1.63	5.59e-02	1.41
80	2.47e-02	5.23e-03	1.92	6.79e-02	1.04	1.80e-02	1.57	3.91e-02	1.26
100	1.98e-02	3.40e-03	1.95	5.40e-02	1.04	1.26e-02	1.60	2.95e-02	1.27
120	1.65e-02	2.39e-03	1.96	4.49e-02	1.02	9.50e-03	1.58	2.38e-02	1.19
140	1.42e-02	1.77e-03	1.96	3.85e-02	1.00	7.52e-03	1.53	1.99e-02	1.16

8.4. Test 4 Results

Recall from Sections 5.1 and 5.5 the following matrix and solution:

$$\text{SOLUTION 5: } u(x, y) = x^{4/3} - y^{4/3} \quad A(x, y) = \frac{16}{9} \begin{bmatrix} x^{2/3} & -x^{1/3}y^{1/3} \\ -x^{1/3}y^{1/3} & y^{2/3} \end{bmatrix}$$

METHOD 1

TEST 4.5		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.94e-02	—	1.12e+00	—	2.12e-01	—	1.12e+00	—
20	9.52e-02	2.21e-02	0.44	1.07e+00	0.08	2.52e-01	-0.26	1.07e+00	0.08
40	4.88e-02	1.62e-02	0.46	9.90e-01	0.11	2.53e-01	-0.01	9.90e-01	0.11
60	3.28e-02	1.36e-02	0.44	9.39e-01	0.13	2.48e-01	0.05	9.39e-01	0.13
80	2.47e-02	1.21e-02	0.43	9.03e-01	0.14	2.42e-01	0.08	9.03e-01	0.14
100	1.98e-02	1.10e-02	0.42	8.75e-01	0.14	2.37e-01	0.11	8.75e-01	0.14
120	1.65e-02	1.02e-02	0.41	8.52e-01	0.15	2.31e-01	0.12	8.52e-01	0.15
140	1.42e-02	9.59e-03	0.41	8.33e-01	0.15	2.27e-01	0.14	8.33e-01	0.15

METHOD 2 USING BOUNDARY CONDITION 1

TEST 4.5		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.27e-02	—	9.63e-01	—	1.59e-01	—	8.53e-01	—
20	9.52e-02	2.32e-02	0.53	9.50e-01	0.02	2.14e-01	-0.46	8.80e-01	-0.05
40	4.88e-02	1.72e-02	0.45	9.04e-01	0.08	2.44e-01	-0.19	8.57e-01	0.04
60	3.28e-02	1.43e-02	0.45	8.68e-01	0.10	2.49e-01	-0.06	8.32e-01	0.08
80	2.47e-02	1.26e-02	0.45	8.41e-01	0.11	2.48e-01	0.02	8.10e-01	0.09
100	1.98e-02	1.15e-02	0.44	8.20e-01	0.12	2.45e-01	0.06	7.92e-01	0.10
120	1.65e-02	1.06e-02	0.43	8.01e-01	0.12	2.41e-01	0.09	7.76e-01	0.11
140	1.42e-02	9.93e-03	0.43	7.86e-01	0.13	2.36e-01	0.12	7.63e-01	0.11

METHOD 2 USING BOUNDARY CONDITION 2

TEST 4.5		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.74e-02	—	9.88e-01	—	1.54e-01	—	9.88e-01	—
20	9.52e-02	2.29e-02	0.28	9.53e-01	0.05	2.07e-01	-0.46	9.53e-01	0.05
40	4.88e-02	1.71e-02	0.44	9.04e-01	0.08	2.39e-01	-0.21	9.04e-01	0.08
60	3.28e-02	1.43e-02	0.45	8.68e-01	0.10	2.46e-01	-0.07	8.68e-01	0.10
80	2.47e-02	1.26e-02	0.44	8.41e-01	0.11	2.45e-01	0.00	8.41e-01	0.11
100	1.98e-02	1.15e-02	0.44	8.19e-01	0.12	2.43e-01	0.05	8.19e-01	0.12
120	1.65e-02	1.06e-02	0.43	8.01e-01	0.12	2.39e-01	0.09	8.01e-01	0.12
140	1.42e-02	9.92e-03	0.43	7.86e-01	0.13	2.35e-01	0.11	7.86e-01	0.13

METHOD 3

TEST 4.5		$\ U - u\ _2$		$\ \hat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \hat{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \hat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.86e-02	—	1.12e+00	—	1.66e-01	—	1.12e+00	—
20	9.52e-02	2.18e-02	0.42	1.07e+00	0.08	2.11e-01	-0.37	1.07e+00	0.08
40	4.88e-02	1.62e-02	0.45	9.87e-01	0.11	2.31e-01	-0.13	9.87e-01	0.11
60	3.28e-02	1.36e-02	0.44	9.37e-01	0.13	2.33e-01	-0.02	9.37e-01	0.13
80	2.47e-02	1.20e-02	0.43	9.01e-01	0.14	2.31e-01	0.03	9.01e-01	0.14
100	1.98e-02	1.10e-02	0.42	8.73e-01	0.14	2.28e-01	0.06	8.73e-01	0.14
120	1.65e-02	1.02e-02	0.41	8.51e-01	0.14	2.25e-01	0.09	8.51e-01	0.14
140	1.42e-02	9.58e-03	0.41	8.32e-01	0.15	2.21e-01	0.11	8.32e-01	0.15

METHOD 4

TEST 4.5		$\ U - u\ _2$		$\ \widehat{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \delta_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \widehat{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	2.73e-02	—	1.14e+00	—	1.58e-01	—	1.14e+00	—
20	9.52e-02	2.12e-02	0.39	1.08e+00	0.08	1.94e-01	-0.31	1.08e+00	0.08
40	4.88e-02	1.59e-02	0.43	1.00e+00	0.12	2.12e-01	-0.13	1.00e+00	0.12
60	3.28e-02	1.34e-02	0.43	9.49e-01	0.14	2.15e-01	-0.04	9.49e-01	0.14
80	2.47e-02	1.19e-02	0.42	9.11e-01	0.14	2.15e-01	0.01	9.11e-01	0.14
100	1.98e-02	1.09e-02	0.41	8.82e-01	0.15	2.13e-01	0.04	8.82e-01	0.15
120	1.65e-02	1.01e-02	0.40	8.58e-01	0.15	2.11e-01	0.06	8.58e-01	0.15
140	1.42e-02	9.52e-03	0.40	8.38e-01	0.15	2.08e-01	0.08	8.38e-01	0.15

METHOD 5 USING BOUNDARY CONDITION 1

TEST 4.5		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	5.74e-02	—	1.00e+00	—	3.30e-01	—	9.05e-01	—
20	9.52e-02	4.56e-02	0.36	1.08e+00	-0.11	4.18e-01	-0.37	1.02e+00	-0.18
40	4.88e-02	3.30e-02	0.48	1.21e+00	-0.18	4.47e-01	-0.10	1.18e+00	-0.22
60	3.28e-02	2.72e-02	0.49	1.33e+00	-0.23	4.48e-01	-0.00	1.31e+00	-0.26
80	2.47e-02	2.38e-02	0.46	1.43e+00	-0.26	4.44e-01	0.03	1.41e+00	-0.27
100	1.98e-02	2.16e-02	0.44	1.52e+00	-0.27	4.38e-01	0.06	1.50e+00	-0.28
120	1.65e-02	2.00e-02	0.42	1.59e+00	-0.27	4.33e-01	0.07	1.58e+00	-0.28
140	1.42e-02	1.88e-02	0.41	1.66e+00	-0.28	4.28e-01	0.08	1.65e+00	-0.29

METHOD 5 USING BOUNDARY CONDITION 2

TEST 4.5		$\ U - u\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$		$\ \bar{\delta}_{x,y;h}^2 U - u_{xy}\ _2$		$\ \bar{\delta}_{y,h_y}^2 U - u_{yy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	6.31e-02	—	1.01e+00	—	3.50e-01	—	1.01e+00	—
20	9.52e-02	4.67e-02	0.46	1.07e+00	-0.08	4.23e-01	-0.29	1.07e+00	-0.08
40	4.88e-02	3.32e-02	0.51	1.21e+00	-0.19	4.50e-01	-0.09	1.21e+00	-0.19
60	3.28e-02	2.73e-02	0.50	1.33e+00	-0.24	4.50e-01	-0.00	1.33e+00	-0.24
80	2.47e-02	2.39e-02	0.47	1.43e+00	-0.26	4.46e-01	0.04	1.43e+00	-0.26
100	1.98e-02	2.16e-02	0.44	1.52e+00	-0.27	4.40e-01	0.06	1.52e+00	-0.27
120	1.65e-02	2.00e-02	0.43	1.59e+00	-0.27	4.34e-01	0.07	1.59e+00	-0.27
140	1.42e-02	1.88e-02	0.41	1.66e+00	-0.28	4.29e-01	0.08	1.66e+00	-0.28

8.5. Truncation Errors

We record the truncation errors for the various solutions and operators $\tilde{D}_h^2, \hat{D}_h^2, \bar{D}_h^2$.

$$\text{SOLUTION 1: } u(x, y) = \sin\left(\frac{\pi}{2}(x + y)^2\right)$$

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 1

SOLUTION 1		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \hat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.18e+00	—	6.62e-01	—	3.82e+00	—	3.20e+00	—
20	9.52e-02	4.80e+00	0.62	2.49e-01	1.51	2.49e+00	0.67	6.10e-01	1.18
40	4.88e-02	3.03e+00	0.69	7.53e-02	1.79	1.53e+00	0.72	2.27e-01	1.48
60	3.28e-02	2.39e+00	0.59	3.55e-02	1.89	1.20e+00	0.61	1.19e-01	1.64
80	2.47e-02	2.04e+00	0.55	2.06e-02	1.93	1.03e+00	0.56	7.21e-02	1.75
100	1.98e-02	1.82e+00	0.53	1.34e-02	1.94	9.11e-01	0.54	4.83e-02	1.81
120	1.65e-02	1.65e+00	0.52	9.42e-03	1.95	8.29e-01	0.53	3.46e-02	1.85
140	1.42e-02	1.53e+00	0.52	6.98e-03	1.96	7.65e-01	0.52	2.60e-02	1.88

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 2

SOLUTION 1		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \hat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.51e+00	—	6.62e-01	—	2.50e+00	—
20	9.52e-02	1.90e+00	1.34	2.49e-01	1.51	1.04e+00	1.36
40	4.88e-02	7.09e-01	1.47	7.53e-02	1.79	3.76e-01	1.52
60	3.28e-02	3.95e-01	1.47	3.55e-02	1.89	2.06e-01	1.51
80	2.47e-02	2.60e-01	1.48	2.06e-02	1.93	1.34e-01	1.51
100	1.98e-02	1.87e-01	1.48	1.34e-02	1.94	9.63e-02	1.51
120	1.65e-02	1.43e-01	1.49	9.42e-03	1.95	7.33e-02	1.51
140	1.42e-02	1.14e-01	1.49	6.98e-03	1.96	5.82e-02	1.51

APPROXIMATING u_{xy}

SOLUTION 1		$\ \tilde{\delta}_{x,y;h}^2 u - u_{xy}\ _2$		$\ \widehat{\delta}_{x,y;h}^2 u - u_{xy}\ _2$		$\ \bar{\delta}_{x,y;h}^2 u - u_{xy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	4.31e+00	—	6.62e-01	—	2.48e+00	—
20	9.52e-02	1.70e+00	1.44	2.49e-01	1.51	9.75e-01	1.45
40	4.88e-02	5.24e-01	1.76	7.53e-02	1.79	2.99e-01	1.76
60	3.28e-02	2.48e-01	1.88	3.55e-02	1.89	1.42e-01	1.88
80	2.47e-02	1.44e-01	1.92	2.06e-02	1.93	8.22e-02	1.92
100	1.98e-02	9.37e-02	1.94	1.34e-02	1.94	5.36e-02	1.94
120	1.65e-02	6.59e-02	1.95	9.42e-03	1.95	3.76e-02	1.95
140	1.42e-02	4.88e-02	1.96	6.98e-03	1.96	2.79e-02	1.96

SOLUTION 2: $u(x, y) = x^2 + 3xy + \frac{1}{2}y^2 + 3$

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 1

SOLUTION 2		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \widehat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	8.13e-01	—	2.76e-14	—	4.07e-01	—	1.33e-01	—
20	9.52e-02	6.02e-01	0.46	1.19e-13	-2.26	3.01e-01	0.46	2.42e-14	-2.65
40	4.88e-02	4.36e-01	0.48	4.77e-13	-2.07	2.18e-01	0.48	1.10e-13	-2.26
60	3.28e-02	3.59e-01	0.49	1.13e-12	-2.18	1.80e-01	0.49	2.67e-13	-2.23
80	2.47e-02	3.12e-01	0.49	1.88e-12	-1.77	1.56e-01	0.49	4.54e-13	-1.87
100	1.98e-02	2.80e-01	0.49	2.92e-12	-2.01	1.40e-01	0.49	7.04e-13	-1.99
120	1.65e-02	2.56e-01	0.50	4.42e-12	-2.29	1.28e-01	0.50	1.14e-12	-2.68
140	1.42e-02	2.37e-01	0.50	6.00e-12	-1.99	1.19e-01	0.50	1.50e-12	-1.77

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 2

SOLUTION 2		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \widehat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.72e-14	—	2.76e-14	—	1.70e-14	—
20	9.52e-02	8.35e-14	-2.44	1.19e-13	-2.26	4.77e-14	-1.60
40	4.88e-02	3.55e-13	-2.16	4.77e-13	-2.07	1.56e-13	-1.77
60	3.28e-02	8.54e-13	-2.21	1.13e-12	-2.18	3.13e-13	-1.75
80	2.47e-02	1.39e-12	-1.72	1.88e-12	-1.77	5.28e-13	-1.84
100	1.98e-02	2.21e-12	-2.10	2.92e-12	-2.01	8.12e-13	-1.95
120	1.65e-02	3.28e-12	-2.19	4.42e-12	-2.29	1.24e-12	-2.34
140	1.42e-02	4.51e-12	-2.07	6.00e-12	-1.99	1.63e-12	-1.81

APPROXIMATING u_{xy}

SOLUTION 2		$\ \tilde{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.71e-14	—	1.53e-14	—	5.41e-15	—
20	9.52e-02	7.75e-14	-2.34	7.94e-14	-2.54	2.52e-14	-2.38
40	4.88e-02	2.72e-13	-1.87	2.72e-13	-1.84	9.13e-14	-1.92
60	3.28e-02	6.54e-13	-2.21	6.54e-13	-2.20	2.11e-13	-2.11
80	2.47e-02	1.18e-12	-2.07	1.17e-12	-2.06	3.78e-13	-2.06
100	1.98e-02	1.74e-12	-1.79	1.71e-12	-1.73	5.66e-13	-1.83
120	1.65e-02	2.63e-12	-2.27	2.62e-12	-2.36	8.26e-13	-2.09
140	1.42e-02	3.62e-12	-2.09	3.63e-12	-2.13	1.12e-12	-2.01

$$\text{SOLUTION 3: } u(x, y) = e^{xy+2y}$$

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 1

SOLUTION 3		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \widehat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	7.57e-01	—	4.46e-03	—	3.77e-01	—	3.60e+00	—
20	9.52e-02	7.39e-01	0.04	1.91e-03	1.31	3.69e-01	0.03	3.51e-03	0.05
40	4.88e-02	6.09e-01	0.29	6.18e-04	1.69	3.04e-01	0.29	1.68e-03	1.10
60	3.28e-02	5.22e-01	0.39	2.99e-04	1.83	2.61e-01	0.38	9.22e-04	1.50
80	2.47e-02	4.64e-01	0.42	1.75e-04	1.88	2.32e-01	0.42	5.77e-04	1.65
100	1.98e-02	4.21e-01	0.44	1.15e-04	1.91	2.10e-01	0.44	3.93e-04	1.73
120	1.65e-02	3.88e-01	0.45	8.12e-05	1.93	1.94e-01	0.45	2.85e-04	1.78
140	1.42e-02	3.62e-01	0.46	6.03e-05	1.94	1.81e-01	0.46	2.16e-04	1.82

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 2

SOLUTION 3		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \widehat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	9.42e-02	—	4.46e-03	—	4.67e-02	—
20	9.52e-02	5.49e-02	0.83	1.91e-03	1.31	2.73e-02	0.83
40	4.88e-02	2.50e-02	1.18	6.18e-04	1.69	1.25e-02	1.17
60	3.28e-02	1.48e-02	1.32	2.99e-04	1.83	7.38e-03	1.32
80	2.47e-02	1.00e-02	1.37	1.75e-04	1.88	5.00e-03	1.37
100	1.98e-02	7.35e-03	1.40	1.15e-04	1.91	3.67e-03	1.40
120	1.65e-02	5.69e-03	1.42	8.12e-05	1.93	2.84e-03	1.42
140	1.42e-02	4.57e-03	1.43	6.03e-05	1.94	2.28e-03	1.43

APPROXIMATING u_{xy}

SOLUTION 3		$\ \tilde{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \hat{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.31e+00	—	3.64e-01	—	8.29e-01	—
20	9.52e-02	4.50e-01	1.65	1.24e-01	1.67	2.85e-01	1.65
40	4.88e-02	1.33e-01	1.82	3.63e-02	1.83	8.42e-02	1.82
60	3.28e-02	6.27e-02	1.89	1.71e-02	1.90	3.96e-02	1.90
80	2.47e-02	3.63e-02	1.93	9.86e-03	1.93	2.29e-02	1.93
100	1.98e-02	2.37e-02	1.94	6.42e-03	1.95	1.49e-02	1.94
120	1.65e-02	1.66e-02	1.95	4.51e-03	1.96	1.05e-02	1.95
140	1.42e-02	1.23e-02	1.96	3.34e-03	1.96	7.78e-03	1.96

SOLUTION 4: $u(x, y) = \frac{x^3}{18} \left(3 \log(x^2) - 11 \right) + \left(y - \frac{1}{2} \right)^{\frac{8}{3}} \sqrt[5]{\left| x + \frac{1}{5} \right|^5}$

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 1

SOLUTION 4		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \hat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.57e+00	—	7.63e-02	—	7.93e-01	—	8.25e-02	—
20	9.52e-02	1.21e+00	0.41	3.28e-02	1.31	6.06e-01	0.42	9.27e-02	0.69
40	4.88e-02	8.93e-01	0.45	1.40e-02	1.28	4.47e-01	0.45	5.06e-02	0.91
60	3.28e-02	7.40e-01	0.47	8.63e-03	1.21	3.70e-01	0.47	3.43e-02	0.97
80	2.47e-02	6.46e-01	0.48	6.19e-03	1.17	3.23e-01	0.48	2.60e-02	0.98
100	1.98e-02	5.81e-01	0.48	4.81e-03	1.14	2.91e-01	0.48	2.09e-02	0.99
120	1.65e-02	5.32e-01	0.49	3.93e-03	1.12	2.66e-01	0.49	1.75e-02	0.99
140	1.42e-02	4.94e-01	0.49	3.32e-03	1.11	2.47e-01	0.49	1.50e-02	0.99

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 2

SOLUTION 4		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \widehat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	3.62e-01	—	7.63e-02	—	2.00e-01	—
20	9.52e-02	2.09e-01	0.85	3.28e-02	1.31	1.12e-01	0.89
40	4.88e-02	1.08e-01	0.98	1.40e-02	1.28	5.63e-02	1.03
60	3.28e-02	7.20e-02	1.03	8.63e-03	1.21	3.70e-02	1.05
80	2.47e-02	5.39e-02	1.02	6.19e-03	1.17	2.75e-02	1.04
100	1.98e-02	4.30e-02	1.02	4.81e-03	1.14	2.19e-02	1.04
120	1.65e-02	3.58e-02	1.02	3.93e-03	1.12	1.82e-02	1.03
140	1.42e-02	3.07e-02	1.01	3.32e-03	1.11	1.55e-02	1.03

APPROXIMATING u_{xy}

SOLUTION 4		$\ \tilde{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \bar{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.39e-01	—	1.79e-01	—	9.70e-02	—
20	9.52e-02	4.34e-02	1.80	5.44e-02	1.85	3.08e-02	1.77
40	4.88e-02	1.23e-02	1.88	1.51e-02	1.91	8.92e-03	1.85
60	3.28e-02	5.74e-03	1.92	7.01e-03	1.94	4.21e-03	1.89
80	2.47e-02	3.32e-03	1.93	4.03e-03	1.95	2.45e-03	1.91
100	1.98e-02	2.16e-03	1.94	2.62e-03	1.96	1.61e-03	1.92
120	1.65e-02	1.52e-03	1.95	1.84e-03	1.96	1.14e-03	1.92
140	1.42e-02	1.13e-03	1.95	1.36e-03	1.96	8.46e-04	1.93

$$\text{SOLUTION 5: } u(x, y) = x^{4/3} - y^{4/3}$$

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 1

SOLUTION 5		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \hat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 U - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.19e+00	—	1.56e+00	—	1.12e+00	—	8.53e-01	—
20	9.52e-02	1.35e+00	-0.19	1.78e+00	-0.20	1.27e+00	-0.20	1.20e+00	-0.29
40	4.88e-02	1.52e+00	-0.18	2.01e+00	-0.18	1.44e+00	-0.18	1.40e+00	-0.23
60	3.28e-02	1.63e+00	-0.17	2.16e+00	-0.18	1.54e+00	-0.18	1.51e+00	-0.20
80	2.47e-02	1.71e+00	-0.17	2.27e+00	-0.17	1.62e+00	-0.17	1.60e+00	-0.19
100	1.98e-02	1.78e+00	-0.17	2.35e+00	-0.17	1.68e+00	-0.17	1.66e+00	-0.18
120	1.65e-02	1.83e+00	-0.17	2.43e+00	-0.17	1.73e+00	-0.17	1.72e+00	-0.18
140	1.42e-02	1.88e+00	-0.17	2.49e+00	-0.17	1.78e+00	-0.17	1.77e+00	-0.18

APPROXIMATING u_{xx} USING BOUNDARY CONDITION 2

SOLUTION 5		$\ \tilde{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \hat{\delta}_{x,h_x}^2 u - u_{xx}\ _2$		$\ \bar{\delta}_{x,h_x}^2 u - u_{xx}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.18e+00	—	1.56e+00	—	1.11e+00	—
20	9.52e-02	1.34e+00	-0.20	1.78e+00	-0.20	1.27e+00	-0.20
40	4.88e-02	1.52e+00	-0.18	2.01e+00	-0.18	1.44e+00	-0.18
60	3.28e-02	1.63e+00	-0.18	2.16e+00	-0.18	1.54e+00	-0.18
80	2.47e-02	1.71e+00	-0.17	2.27e+00	-0.17	1.62e+00	-0.17
100	1.98e-02	1.78e+00	-0.17	2.35e+00	-0.17	1.68e+00	-0.17
120	1.65e-02	1.83e+00	-0.17	2.43e+00	-0.17	1.73e+00	-0.17
140	1.42e-02	1.88e+00	-0.17	2.49e+00	-0.17	1.78e+00	-0.17

APPROXIMATING u_{xy}

SOLUTION 5		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \widehat{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$		$\ \overline{\delta}_{x,y;\mathbf{h}}^2 u - u_{xy}\ _2$	
N_x	h	ERROR	RATE	ERROR	RATE	ERROR	RATE
10	1.82e-01	1.92e-15	—	2.56e-15	—	5.03e-16	—
20	9.52e-02	8.57e-15	-2.31	8.54e-15	-1.86	2.73e-15	-2.62
40	4.88e-02	3.63e-14	-2.16	3.79e-14	-2.23	9.10e-15	-1.80
60	3.28e-02	6.29e-14	-1.38	6.47e-14	-1.35	2.42e-14	-2.46
80	2.47e-02	1.26e-13	-2.45	1.29e-13	-2.44	3.68e-14	-1.48
100	1.98e-02	1.66e-13	-1.25	1.73e-13	-1.33	6.32e-14	-2.45
120	1.65e-02	2.71e-13	-2.71	2.75e-13	-2.57	7.78e-14	-1.16
140	1.42e-02	3.91e-13	-2.39	3.95e-13	-2.36	1.15e-13	-2.54

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