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Bianchi modular forms are a generalization of classical modular forms to imaginary quadratic fields. The study of computational aspects of Bianchi modular forms started in the 1980s by Elstrodt, Grunewald, and Mennicke. John Cremona and several of his students made notable contributions to developing theory for computing Bianchi modular forms. This thesis extends their work by providing algorithms for computing Bianchi modular forms over imaginary quadratic fields with general class groups. We also provide results, including dimension tables, of the implementation for the imaginary quadratic field $\mathbb{Q}(\sqrt{-17})$.

COMPUTATIONAL ASPECTS OF BIANCHI MODULAR FORMS

by

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To all my fellow graduate students

APPROVAL PAGE

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Chapter 1: Introduction

Modular forms play a central role in number theory and many other branches of mathematics. Modular forms were first introduced by Jacobi and Eisenstein in the nineteenth century through the theory of elliptic functions. The discriminant function and j-function were modular forms studied in their work. The study of modular forms was given a new life when Wiles [37] proved the Taniyama-Shimura conjecture for a large class of elliptic curves. As a corollary, we have an elegant proof of Fermat's last theorem. Since this development, researchers have been interested in studying if such conjectures hold for other generalizations of classical modular forms. One such generalization gives rise to Bianchi modular forms, which are modular forms over imaginary quadratic fields.

The main goal of this thesis is to analyze Bianchi modular forms from a computational perspective. To this end, this thesis describes an algorithm and its implementation to compute Bianchi modular forms as Hecke eigensystems. In particular, we compute Bianchi modular forms over an imaginary quadratic field with order 4 class group extending the computations done by Cremona and several of Cremona's students [6, 12, 14, 28, 36].

The study of computations of Bianchi modular forms started in 1980 by Grunewald, Mennicke, and others [18, 20]. They compute Bianchi modular forms for $F = \mathbb{Q}(\sqrt{-d})$ where $d = 1, 2, 3$ using modular symbols techniques. In [7], Cremona extended these computations to all five Euclidean fields. In the years that followed, some of Cremona's students worked on extending these computations. Whitley [36] in her thesis worked on extending modular symbol techniques to class number 1 fields. Bygott [6] in his thesis developed techniques for computing Bianchi modular forms over an imaginary quadratic field with class number 2. He computed explicit examples over the field $\mathbb{Q}(\sqrt{-5})$. Lingham [28] in his thesis worked on the odd class number case and computed explicit examples for the fields $\mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\sqrt{-31})$. Aranes [1] extended the M-symbol techniques over $\mathbb{Q}$ to number fields.

Similar to the work of Cremona et. al., we exploit the connection between Bianchi modular forms and the homology of certain quotients of the hyperbolic 3-space $\mathbb{H}_3$ for our computations. To compute homology we require tessellations of the hyperbolic 3-space $\mathbb{H}_3$ with an action of the congruence subgroups. Cremona and his students utilize an algorithm coming from the work of Swan [33] to compute such a tessellation. In this thesis, we use the work of Ash [3] and Koecher [25] coming from the theory of perfect Hermitian forms. Further, for Hecke operator computations, we use the reduction theory introduced by Gunnells [21]. Conveniently, we can use the implementations of these techniques by Yasaki [39].

In Chapter 2, we discuss the classical modular forms and some computational techniques. In Chapter 3, we introduce homological modular forms and discuss techniques for computing them. This section includes a brief exposition of the Voronoi theory, modular symbol, and M-symbol techniques for imaginary quadratic fields.

In Chapter 4, we introduce the notion of Bianchi modular forms and how to view them as Hecke eigensystems. This approach allows us to understand how to use homological modular forms from Chapter 3 to compute the Hecke eigensystem attached to a Bianchi modular form. In Chapter 5, we introduce the notion of a homological eigenform and explain how to use them to compute Hecke eigensystems. In this chapter, we provide algorithms for computing Bianchi modular forms over imaginary quadratic fields.

Finally in Chapter 6, we discuss the results from the implementation to the imaginary quadratic field $\mathbb{Q}(\sqrt{-17})$. We provide various types of examples observed within the scope of the computation. We also provide dimension tables and tables of Hecke eigensystems for certain levels.

Chapter 2: Classical Theory

In this chapter, we provide a discussion of the classical theory of modular forms. The goal of this chapter is to help the reader see Bianchi modular forms are a natural generalization of the classical case.

2.1 Classical Modular Forms

In this section, we state some facts regarding classical modular forms. We use generalizations of some of these facts to Bianchi modular forms for our computations. For more details, we refer the reader to [9, 16, 32].

Let $\mathbb{H}_2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ denote the upper-half plane, and let $\mathbb{H}_2^* = \mathbb{H}_2 \cup \mathbb{Q} \cup \{i\infty\}$ denote the extended upper-half plane obtained by including the cusps $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}_2$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}. \tag{2.1}$$

We can extend the action of the group $\mathrm{SL}_2(\mathbb{Z})$ to cusps $\mathbb{Q} \cup \{i\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{p}{q} = \frac{ap+bq}{cp+dq}. \tag{2.2}$$

We define modular forms as complex-valued functions on $\mathbb{H}_2$ satisfying symmetry with respect to this action by congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In particular, we consider congruence subgroups of the form

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\} \tag{2.3}$$

where $N \in \mathbb{Z}^{\geq 0}$.

Definition 2.1. A *classical modular form* of weight k and level N is a complex-valued function $f : \mathbb{H}_2 \to \mathbb{C}$ that satisfies the following conditions.

1. f is holomorphic on $\mathbb{H}_2$
2. For each $\gamma \in \Gamma_0(N)$, we have $f[\gamma]_k = f$, where $f[\gamma] = j(\gamma, \tau)^{-k} f(\gamma\tau)$ and $j(\gamma, \tau) = c\tau + d$.
3. $f[\gamma]_k$ is holomorphic at $i\infty$ where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\infty) = \alpha \in \mathbb{Q} \cup \{i\infty\}$.

The third condition can also be stated as a growth condition. Explicitly, we want $|f[\gamma]_k(z)|$ to be bounded as $\mathrm{Im}(z) \to \infty$.

These conditions imply that a modular form f has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i z},$$

for any $z \in \mathbb{H}_2$. Further, the space of modular forms $M_k(N)$ of weight k and level N is a finite-dimensional $\mathbb{C}$ vector space, and we have an explicit formula for the dimension of space of modular forms [16, Theorem 3.5.1].

If we consider the subspace of modular form $f \in M_k(N)$ that vanishes on cusps, we get the space of cuspforms denoted $S_k(N)$. Our main focus will be to compute the space of cuspforms as they are related to objects like elliptic curves.

Now we look at the interaction between levels. If $M \mid N$, then for any $g \in S_k(M)$ and for any divisor d of N/M, the form $f(z) = d^{k-1} g(dz)$ is in $S_k(N)$. We call the subspace $S_k(N)^{old}$

of such forms the *oldforms at level* N. The space $S_k(N)^{new}$ that cannot be constructed from oldforms is called the *the space of newforms at level* N.

From a computational standpoint, computing $S_k(N)^{new}$ is important as these are the forms that truly are level N. In practice, we do this by computing $S_k(N)$ systematically and accounting for oldforms at levels $M \mid N$. Since every form in $S_k(M)$ shows up in $S_k(N)$ with multiplicity equal to the number of divisors of N/M, we can recognize oldforms by looking at multiplicities.

2.2 Hecke Operators

In this section, we introduce a collection of operators on the space of cuspforms $S_k(N)$ called Hecke operators. These operators are diagonalizable and they commute. Therefore, we can obtain a basis for $S_k(N)$ consisting of simultaneous eigenforms of Hecke operators away from N. By "computing", we mean computing these Fourier coefficients of eigenforms using Hecke operators.

For $\beta \in \mathrm{GL}_2^+(\mathbb{Z})$, we can extend the action on $S_k(\Gamma)$ by:

$$f[\beta]_k = \det(\beta)^{k-1} j(\beta, \tau)^{-k} f(\gamma\beta).$$

Here $\mathrm{GL}_2^+(\mathbb{Z})$ is the subgroup of matrices in $\mathrm{GL}_2(\mathbb{Z})$ with positive determinant.

Definition 2.2. We define an operator $T_p : S_k(\Gamma) \to S_k(\Gamma)$ for any prime p by

$$f[\alpha]_k = \sum_{i=1}^{k} f[\beta_i]_k,$$

with $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma = \bigcup_{i=1}^{k} \Gamma\beta_i$ where $\beta_i = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \gamma_i$ and $\{\gamma_i\}_{i=1}^{k}$ are orbit representatives of the coset $\Gamma\backslash\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma$.

Remark 2.1. We can define Hecke operators more generally by looking at the double coset $\alpha^{-1}\Gamma\alpha \cap \Gamma$ for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$.

We set $T_1 = 1$, the identity operator. We can define an operator T_{p^r} inductively by

$$T_{p^r} = T_p T_{p^{r-1}} + p^{k-1} T_{p^{r-2}}$$

for $r > 1$ and T_n as

$$T_n = \prod_i T_{p_i^{r_i}}, \quad \text{where } n = \prod_i p_i^{r_i}.$$

Now we can show that the Hecke operators have the following properties:

Theorem 2.3. *For any $m, n \in \mathbb{Z}^+$,*

1. $T_n T_m = T_m T_n$.
2. $T_{mn} = T_m T_n$ *if* $(m, n) = 1$

Proof. These follow from the definition of T_n and [16, Proposition 5.2.4]. □

Theorem 2.4. *Hecke operators T_n for $(n, N) = 1$ are simultaneously diagonalizable.*

Proof. By [16, Theorem 5.3.3], any Hecke operator T_n with $(n, N) = 1$ on the space of cuspform $S_k(N)$ is normal with respect to the Peterson inner product given in [16, Definition 5.4.1]. Therefore, T_n is diagonalizable. Since Hecke operators commute, they are simultaneously diagonalizable. □

Definition 2.5 ([16, Definition 5.8.1])**.** A nonzero modular forms $f \in S_k(N)$ that is an eigenform for all Hecke operators T_n is an *eigenform.* An eigenform $f(z) = \sum_{n=1}^{\infty} a_n(f) q^n$, and $q = e^{2\pi i z}$ is *normalized* if $a_1(f) = 1$. A normalized eigenform in $S_k(N)^{new}$ is called a *newform.*

Now we have the following theorem about newforms.

Theorem 2.6 ([16, Theorem 5.8.2]). *The set of newforms of level N gives an orthogonal basis of $S_k(N)^{new}$ with respect to the Petersson inner product. Further, each newform f satisfies $T_n(f) = a_n(f)f$ for each $n \in \mathbb{Z}^+$.*

This means, by computing eigenvalues of the Hecke operator T_n, we can recover the Fourier coefficient $a_n(f)$ of any newform f. As $S_k(N)^{new}$ has a basis consisting of newforms, we have a complete description of the space.

2.3 Lattices

Now we look at an alternative definition for modular forms. The definition of Bianchi modular form in Section 4 is a generalization of this. For details, we refer to [16, Section 1.3 and Section 1.5]. A *lattice* in $\mathbb{C}$ is a discrete set of the form $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\{\omega_1, \omega_2\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$.

Definition 2.7. An *enhanced elliptic curve over* $\Gamma_0(N)$ is a pair (E, C), where E is a complex elliptic curve and C is a cyclic subgroup of E of order N.

Alternatively, we can view an enhanced elliptic curve (E, C) as a pair of lattice (L, L') with $L \subset L'$ where $E(\mathbb{C}) = \mathbb{C}/L$ and $L/L' \simeq \mathbb{Z}/N\mathbb{Z}$ [24].

We say two enhanced elliptic curves (E, C) and (E', C') are equivalent if there is an isomorphism $E \to E'$ that sends C to C'. The set of equivalence classes of enhanced elliptic curves is denoted by $S_0(N)$.

Functions on $S_0(N)$ satisfying the transformation

$$F(\mathbb{C}/mL, mC) = m^{-k} F(\mathbb{C}/L, C) \tag{2.4}$$

for any $m \in \mathbb{C}^\times$, can be viewed as modular forms of level N and weight k [16].

In this approach, there is a natural notion of Hecke operators which can be thought of as "averaging" operators. These operators are compatible with the Hecke operators defined in Section 2.2.

We use a generalization of this to imaginary quadratic fields in Section 4.1 to define Bianchi modular forms. This approach gives an intuitive understanding of the subtleties that arise due to the class group.

2.4 Homology

Now we restrict our attention to computational techniques for weight 2 classical modular forms.

The quotient $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}_2^*$ is a compact Riemann surface [16, Chapter 2]. Now we consider the pairing

$$\langle\ ,\ \rangle : S_2(N) \times H_1(X_0(N); \mathbb{Z}) \to \mathbb{C},$$

by

$$\langle f, \gamma\rangle = 2\pi i \int_\gamma f(z)dz$$

for any path γ in $X_0(N)$.

This pairing is non-degenerate and Hecke equivariant [32, Theorem 3.4]. That is, for any Hecke operator T_n, we have $\langle T_n f, \gamma\rangle = \langle f, T_n\gamma\rangle$. Here the action of T_n on $H_1(X_0(N); \mathbb{Z})$ is as described in [32, Chapter 3]. This means we can use the homology $H_1(X_0(N); \mathbb{Z})$ to compute the space of cuspforms as a Hecke module. See [32, chapter 3] for more details.

The homology $H_1(X_0(N); \mathbb{Z})$ can be computed by taking a tessellation of $X_0(N)$ with vertices on cusps. One such tessellation can be obtained using the theory of perfect forms.

Let us consider the vector space V of 2×2 real symmetric matrices with the positive definite inner product given by $\langle x, y\rangle = \mathrm{Tr}(xy)$. By C we denote the space of positive definite symmetric matrices of V. This is an open convex cone that is self-adjoint. That is,

$$C = C^* = \{y \in V \mid \langle x, y\rangle > 0, \text{ for all } x \in \bar{C}\backslash\{0\}\},$$

where $\bar{C} = \{y \in V \mid \langle x, y\rangle \geq 0, \text{ for all } x \in V\}$ is the closure of C. The boundary of the cone $\partial C = \bar{C}\backslash C$ is the collection of positive semidefinite symmetric matrices in V.

The space C can also be viewed as a space of positive definite quadratic forms with a given Gram matrix. This allows us to view the space of quadratic forms as an inner product space, where evaluating a quadratic form on a vector can be viewed as an inner product. Explicitly, suppose Q is a positive definite quadratic form with the Gram matrix A. Then by the Cholesky decomposition [27] of positive definite matrices, we have that $A = gg^t$ for some $g \in \mathrm{GL}_2(\mathbb{R})$. Then the inner product between A and the semidefinite form $vv^t \in \partial C$ can be viewed as evaluating the quadratic form Q at the vector v as follows:

$$Q[v] = \mathrm{Tr}(v^t A v) = \mathrm{Tr}(v^t g g^t v) = \mathrm{Tr}(gg^t \cdot vv^t) = \langle gg^t, vv^t\rangle = \langle A, vv^t\rangle.$$

Under this interpretation, ∂C corresponds to the space of positive semidefinite quadratic forms. Further, this gives a way of identifying positive semidefinite forms using linear conditions. Explicitly, $A \in \partial C$ is the same as saying that the orthogonal complement of A will intersect ∂C non-trivially. That is, if $A \in \partial C$, then there must exist $u \in \mathbb{R}^2\backslash\{0\}$ such that $\langle A, uu^t\rangle = 0$. This is the same as saying that uu^t is a non-zero vector in the orthogonal complement of Q.

The group $G = \mathrm{GL}_2(\mathbb{R})$ acts transitively on C from the left by $g \cdot x \mapsto gxg^t$. The stabilizer of the identity is $K = O(2)$. Thus, we have the identification,

$$C/\mathbb{R}^{>0} \longrightarrow \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^{>0}O(2) \simeq \mathbb{H}_2.$$

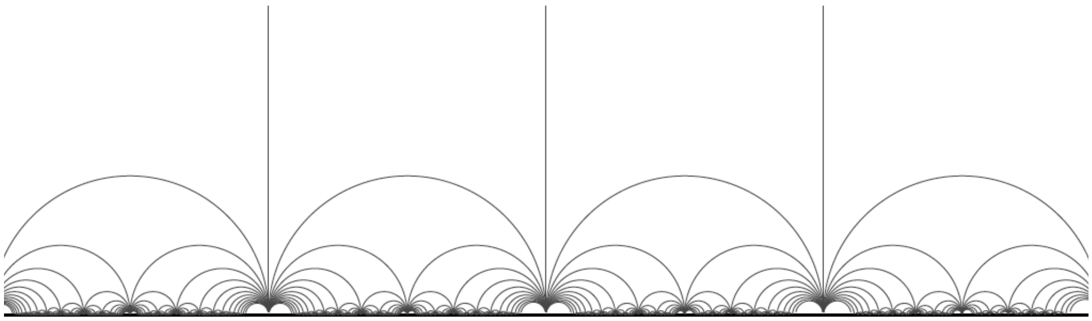


Figure 2.1. Farey tessellation of the hyperbolic plane

Define the map $q : \mathbb{Z}^2 \to \partial C$ by $v \mapsto vv^t$, and put $\Theta = q(\mathbb{Z}^2 \backslash \{0\})$. Now we define the Voronoi polytope as follows:

Definition 2.8. The *Voronoi polyhedron* Π is defined to be the convex hull of Θ.

The Voronoi polyhedron has a nice combinatorial structure. In particular, all the facets of Π are triangles. Also, each element in $q(v) \in \Theta$ can be identified with the cusps $\frac{a}{b} \in \mathbb{P}^1(\mathbb{Q})$ where $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2$. Up to homotheties, the action of $\mathrm{SL}_2(\mathbb{Z})$ is well-defined for this identification. That is, for any $\zeta > 0$, since v and ζv corresponds to the same element in $\mathbb{P}^1(\mathbb{Q})$, $g \cdot q(v) = g \cdot q(\zeta v)$ in $C/\mathbb{R}^{>0}$. This means after modding out by homotheties, we can identify the vertices of Π with elements in $\mathbb{P}^1(\mathbb{Q})$. Then up to homotheties, the facets of Π under this becomes ideal triangles in $\mathbb{H}_2^* = \mathbb{H}_2 \cup \mathbb{P}^1(\mathbb{Q})$. Thus, we get a triangulation of $\mathbb{H}_2^*$ with vertices on cusps. Further, we have a natural action by $\mathrm{SL}_2(\mathbb{Z})$ on this triangulation which is induced by the action of $\mathrm{SL}_2(\mathbb{Z})$ on C.

In practice, we can also compute this tessellation using perfect forms as introduced by Voronoi [35].

Definition 2.9. We define the *minimum* of a quadratic form Q with Gram matrix A, denoted $m(A)$, to be the minimum value of $Q[x] = \langle A, xx^t \rangle$ for all $x \in \mathbb{Z}^2 \backslash \{(0,0)\}$. We define the set of minimal vectors of A,

$$M(A) = \{x \in \mathbb{Z}^2 \backslash \{(0,0)\} \mid Q[x] = m(A)\}.$$

With this notion of a minimum, we say a quadratic form is *perfect* if it is uniquely determined by $m(A)$ and $M(A)$.

In [35], Voronoi proved that there are only finitely many perfect forms up to the action of $\mathrm{SL}_2(\mathbb{Z})$. He also proved that each facet of Π can be identified with a perfect form where vertices of the facet up to homotheties are given by $q(v)$ for $v \in M(A)$. This shows that there are only finitely many facets of Π up to the action of $\mathrm{SL}_2(\mathbb{Z})$. Further, Voronoi also provided an algorithm to enumerate these perfect forms.

From this algorithm, we can

1. find the cell σ in Π containing a form $A \in C$;

2. find a path along the edges of Π between $A, B \in C$.

These two tasks are extremely useful in computing Hecke operators.

2.5 Modular Symbols and M-symbols

Modular symbols and M-symbols provide a concrete way of writing generators and relations that describes the relative homology group $H_1(X_0(N), \partial X_0(N); \mathbb{C})$, where $\partial X_0(N)$ is the set of cusps modulo the action of $\Gamma_0(N)$.

Definition 2.10. For $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ the space of modular symbols of weight 2, denoted $\mathbb{M}_2(\Gamma)$, is a free Abelian group generated by pairs of cusps of the form $\{\alpha, \beta\}$ modulo the relations

$$\begin{aligned}
\{\alpha, \beta\} + \{\beta, \alpha\} &= 0 \\
\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} &= 0 \\
g\{\alpha, \beta\} - \{\alpha, \beta\} &= 0 \text{ for each } g \in \Gamma.
\end{aligned}$$

Theorem 2.11 ([29])**.** *The space of weight 2 modular symbols $\mathbb{M}_2(\Gamma_0(N))$ is isomorphic to the homology group $H_1(X_0(N), \partial X_0(N); \mathbb{C})$.*

Remark 2.2. We can define an action by Hecke operators on both sides compatibly. Thus, this is an isomorphism of Hecke modules.

Now we introduce the so-called Manin's trick, to show that space of modular symbols is computable.

Suppose $\{g_i\}_{i=1}^r$ is a set of right coset representatives of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ so that $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_i \Gamma_0(N) g_i$. Then for each modular symbol $\{\alpha, \beta\}$, the list $\{g_i\{\alpha, \beta\}\}_{i=1}^r$ is the complete list of distinct $\Gamma_0(N)$-translates of the symbol $\{\alpha, \beta\}$ in $\mathbb{M}_2(\Gamma_0(N))$.

Theorem 2.12 ([29])**.** *Let N be a positive integer, and let $\{g_i\}_{i=1}^r$ be a set of right coset representatives of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Then any $\{\alpha, \beta\} \in \mathbb{M}_2(\Gamma_0(N))$ can be written as*

$$\{\alpha, \beta\} = \sum_i a_i g_i \{0, i\infty\},$$

for some $a_i \in \mathbb{Z}$.

This shows that the collection $\{g_i\{0, i\infty\}\}_{i=1}^r$ gives a complete list of generators for $M_2(\Gamma_0(N))$. We call such modular symbols *unimodular.*

By $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, we denote the set of pairs $(c, d) \in \mathbb{Z}^2$ such that $\gcd(c, d, N) = 1$ modulo the relation

$$(c_1, d_2) \sim (c_2, d_2) \leftrightarrow c_1 d_2 \equiv c_2 d_1 \mod N.$$

We denote the equivalence class of a pair (c, d) by $(c : d)$. Then by [10, Proposition 2.2.2], we have $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{R}) \simeq \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. This means we can use $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ to identify coset representatives g_i.

Definition 2.13. A *Manin symbol* or *M-symbol* $(c : d)$ is the class in $H_1(X_0(N), \partial X_0(N); \mathbb{C})$ of the modular symbol $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \{0, \infty\}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

M-symbols are especially convenient for computations, but the action of Hecke operators does not preserve the space of unimodular symbols. However, we have techniques that use continued fractions like Heilbronn matrices to compute Hecke operators. More details of this process are available in [10, Chapter II].

Remark 2.3. In generalizing to imaginary quadratic fields, we will see that not every symbol can be represented by a translation of the symbol $\{0, i\infty\}$. Thus, we require to keep track of the edge "type" as well as the coset representative.

Chapter 3: Homological Modular Forms

In Section 2.4, we saw that the homology $H_1(X_0(N);\mathbb{C})$ could be used to compute classical modular forms. Here $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}_2^*$ and $\mathbb{H}_2^* = \mathbb{H}_2 \cup \mathbb{P}^1(\mathbb{Q})$, the upper half-plane $\mathbb{H}_2$ with cusps. In this section, we discuss how to generalize this to imaginary quadratic fields.

For the purpose of this thesis, we restrict ourselves to weight 2 modular forms. Treatment of higher weight modular forms over Euclidean imaginary quadratic fields is available in [15].

Let F be an imaginary quadratic field with a ring of integers $\mathcal{O}_F$. In the classical case, we considered the action of $\mathrm{SL}_2(\mathbb{Z})$ on the hyperbolic plane $\mathbb{H}_2$ by fractional linear transformations. In the Bianchi case, we consider the general linear group $\mathrm{GL}_2(\mathcal{O}_F)$ on the hyperbolic 3-space $\mathbb{H}_3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} | t > 0\}$ by

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} (z,t) = (z', t'),$$

where

$$z' = \frac{(pz+q)(\overline{rz+s}) + (pt)(\overline{st})}{|rz+s|^2 + |r|^2t^2} \quad \text{and} \quad t' = \frac{|ps-qr|t}{|rz+s|^2 + |r|^2t^2}.$$

Similar to the classical case, we can extend this action to $\mathbb{H}_3^* = \mathbb{H}_3 \cup \mathbb{P}^1(F)$, the extended hyperbolic 3-space, by defining an action on cusps $\mathbb{P}^1(F) = F \cup \{\infty\}$ as

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot (c : d) = (cp + qd : rc + sd).$$

Now for any ideal $\mathfrak{n}$ of $\mathcal{O}_F$, we can define a congruence subgroup

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \;\middle|\; c \in \mathfrak{n} \right\}.$$

This is similar to the congruence subgroups given in (2.3) for the classical case.

If $\Gamma_0(\mathfrak{n})$ is torsion-free, then the quotient $X_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n})\backslash\mathbb{H}_3^*$ will be a compact differentiable manifold, and the homology $H_1(X_0(\mathfrak{n}); \mathbb{C})$ is a generalization of the classical homology from above.

Definition 3.1. We define a *homological modular form of level* $\mathfrak{n}$ to be a class in the homology $H_1(X_0(\mathfrak{n}); \mathbb{C})$.

Remark 3.1. It is not obvious that this homology has a connection to modular forms. Fortunately, the work of Kurcanov [26] establishes a duality between the homology $H_1(X_0(\mathfrak{n}); \mathbb{C})$ and the "principal" part of Bianchi modular forms defined in Chapter 4 for imaginary quadratic fields. A more general result for other number fields is available in [19] by Franke.

Remark 3.2. In practice, we compute the relative homology group $H_1(X_0(\mathfrak{n}), \partial X_0(\mathfrak{n}); \mathbb{C})$ where $\partial X_0(\mathfrak{n})$ denotes the set of cusps up to action of $\Gamma_0(\mathfrak{n})$. This is more convenient because any path between cusps in $X_0(\mathfrak{n})$ denotes a class $H_1(X_0(\mathfrak{n}), \partial X_0(\mathfrak{n}); \mathbb{C})$. To obtain $H_1(X_0(\mathfrak{n}); \mathbb{C})$, we need to compute $\partial X_0(\mathfrak{n})$. This can be done using methods introduced in Section 4.5.

In the remainder of this section, we summarize techniques for computing homological modular forms. In Section 3.1, we discuss a generalization of Voronoi theory by Ash and Koecher. This allows us to obtain a tessellation of the hyperbolic 3-space with an action by a congruence subgroup. In Section 3.2, we describe a generalization of modular symbols, M-symbols, and a reduction theory to compute the action of principal Hecke operators on homology.

We omit proofs in this section to keep this chapter concise. The details in this section can be found in [3, 21, 25]. For an exposition, we refer the reader to a set of lecture notes by

Gunnells [22].

3.1 Voronoi Theory

In this section we discuss techniques for computing a tessellation of $\mathbb{H}_3$ from the work of Ash [3] and Koecher [25]. This is a generalization of the classical Voronoi theory introduced in Section 2.4. We state most of the results over imaginary quadratic fields, although we can work more generally on other self-adjoint homogeneous cones.

Let F be an imaginary quadratic field, and let $\mathcal{O}_F$ be the ring of integers as before. We consider the algebraic group $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, where $\mathrm{Res}_{F/\mathbb{Q}}$ denotes the Weil restriction of scalars. Let $G = \mathbf{G}(\mathbb{R})$ denote the group real points in $\mathbf{G}$.

The group G acts the vector space $V = \mathrm{Herm}_2(\mathbb{C})$ of Hermitian matrices with complex coefficients by

$$g \cdot A \mapsto gAg^*,$$

where g^* represents the conjugate transpose of the matrix g.

Let $C \subset V$ denote the space of positive definite matrices. Then from results in [2], we can show that C is a self-adjoint cone with respect to the inner product given by $\langle x, y\rangle = \mathrm{Tr}(xy^*)$ on V. The cone C is also homogeneous as the action by G is transitive. Details of a proof of both these facts for imaginary quadratic fields can be found in [23, Proposition 4.1.1].

Let $q : \mathbb{C}^2 \to \bar{C}$ be the map defined by $v \mapsto vv^*$. Here by $\bar{C}$, we denote the union of C with the boundary ∂C, which consists of positive semidefinite Hermitian forms. If we fix an embedding $F \hookrightarrow \mathbb{C}$, we can identify vectors in $\mathcal{O}_F^2$ as a discrete set in $\mathbb{C}^2$. This allows us to introduce a notion of minimum and minimal vectors for a Hermitian matrix.

Definition 3.2. We define the *minimum* of $A \in C$ by

$$m(A) = \inf\{\langle q(x), A\rangle \mid x \in \mathcal{O}_F^2\backslash\{0\}\},$$

and the set of *minimal vectors* by

$$M(A) = \{x \in \mathcal{O}_F^2 \backslash \{0\} \mid \langle q(x), A \rangle = m(A)\}.$$

Note that since $\mathcal{O}_F^2$ is a discrete set in $\mathbb{C}^2$, the set of minimal vectors $M(A)$ is a finite set. We can see this by looking at the inner product $\langle\ ,\ \rangle$ as defining a metric and noticing that the ball of radius $m(A)$ about A can only have a finite intersection with the discrete set $q(\mathcal{O}_F^2)$.

We say a form $A \in C$ is *perfect* if A is completely determined by $m(A)$ and $M(A)$. In [25], Koecher proved the existence and the finiteness of perfect forms up to the action of $\mathrm{GL}_2(\mathcal{O}_F)$.

Definition 3.3. A *perfect cone* $\sigma(A)$ attached to a perfect form A is given by

$$\sigma(A) = \left\{ \sum_{v \in M(A)} \lambda_v q(v) \;\middle|\; \lambda_v \geq 0 \right\}$$

Let Σ denote the collection of perfect cones and their proper faces.

Theorem 3.4 ([25]). *The set Σ satisfies the following properties:*

1. *Any compact set in C meets finitely many perfect cones in Σ.*

2. *Any perfect cone $\sigma \in \Sigma$ meets finitely many other perfect cones σ' such that $\sigma \cap \sigma'$ contains an element in C.*

3. *Let $\sigma, \sigma' \in \Sigma$ be different perfect cones.*

   (a) σ, σ' do not share any interior points, that is $\mathrm{Int}(\sigma) \cap \mathrm{Int}(\sigma') = \emptyset$.

   (b) $\sigma \cap \sigma'$ is a common face of both σ and σ'.

4. *If τ is a facet of a perfect cone σ that meets C, then there must exist another perfect cone σ' such that $\tau = \sigma \cap \sigma'$.*

5. $C = \bigcup_{\sigma \in \Sigma} \sigma \cap C$

This implies that Σ gives us a polyhedral decomposition of C. Further, we can show that the action of the group $\mathrm{GL}_2(\mathcal{O}_F)$ on $\mathcal{O}_F^2$ induces an action on Σ with the following properties:

Theorem 3.5. *The set Σ satisfies the following properties:*

1. *There are only finitely many $\mathrm{GL}_2(\mathcal{O}_F)$ orbits in Σ*

2. *Any $y \in C$ is contained in the interior of a unique cone in Σ*

3. *Given any cone $\sigma \in \Sigma$ with a point in C, σ has finite stabilizer.*

In particular, this means that for any congruence subgroup Γ of $\mathrm{GL}_2(\mathcal{O}_F)$, we can choose a finite list of cones in Σ which are Γ orbit representatives. We can compute this list by enumerating perfect forms. For an explicit algorithm for this, we refer the reader to [31].

We have the following identification between the cone C and the hyperbolic 3-space $\mathbb{H}_3$.

Theorem 3.6. *Let C be the cone of positive definite Hermitian matrices in V, then*

$$C/\mathbb{R}^{>0} \simeq \mathbb{H}_3.$$

Proof. First, we know that $\mathbb{H}_3$ is the symmetric space attached to $\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ and we have that $\mathbb{H}_3 \simeq G/KA_G$ where $G = \mathbb{G}(\mathbb{R}) = \mathrm{GL}_2(\mathbb{C})$ and $K = U(2)$ is the maximal compact subgroup of G and A_G is the set of positive scalar matrices.

On the other, recall that G acts on C by $g \cdot A \mapsto gAg^*$. By [2, Chapter II], we have that $C \simeq G/K$. Since A can be identified with $\mathbb{R}^{>0}$, the result follows. $\square$

Thus, the decomposition of C into polyhedral cones up to homotheties gives us a decomposition of $\mathbb{H}_3$ into ideal polytopes with an action by $\mathrm{GL}_2(\mathcal{O}_F)$. We call this the *Voronoi* *tessellation* of $\mathbb{H}_3$ for $\mathrm{GL}_2(\mathcal{O}_F)$. From the properties in Theorem 3.5, each polytope in the Voronoi tessellation will have a finite orbit and finite stabilizers.

Remark 3.3. If $v \in M(A)$ for some perfect form A, then for any $\zeta \in \mathcal{O}_F^\times$, ζv is also a minimal vector. Then the ray through $q(v)$ coincides with the ray through $q(\zeta v)$ in the perfect cone $\sigma(A)$. Thus the polytope in $\mathbb{H}_3$ induced by $\sigma(A)$ has a strictly smaller number of vertices than the number of minimal vectors of A. Further, there might also be cases where two different minimal vectors correspond to the same cusp in $\mathbb{H}_3^*$. For example, let $F = \mathbb{Q}(\sqrt{-91})$ and $\omega = \frac{1+\sqrt{-91}}{2}$. The vectors $v = \begin{pmatrix} \omega+1 \\ -\omega+4 \end{pmatrix}$ and $u = \begin{pmatrix} 5 \\ -\omega-3 \end{pmatrix}$ corresponds to the same cusp $\frac{\omega-4}{7}$ in $\mathbb{H}_3^*$.

3.2 Modular Symbols and M-symbols

In this section, we define modular symbols and M-symbols, which gives us a concrete way to write generators and relations that describe the homology group $H_1(X_0(\mathfrak{n}); \mathbb{C})$. The parallel of this material for the classical case is given in Section 2.5. We also introduce a reduction theory for modular symbols coming from the work of Gunnells [21]. This reduction theory will be helpful in computing principal Hecke operators introduced in Section 4.2.

Definition 3.7. A *modular symbol* $[u, v]$ is defined as the class in $H_1(X_0(\mathfrak{n}), \partial X_0(\mathfrak{n}); \mathbb{C})$ of a directed path from u to v, where $u, v \in \mathbb{P}^1(F)$.

We say a modular symbol $[u, v]$ is *Voronoi reduced* if the class is induced by an edge in the Voronoi tessellation.

Theorem 3.8 ([21])**.** *The set of Voronoi reduced modular symbols spans* $H_1(X_0(\mathfrak{n}), \partial X_0(\mathfrak{n}); \mathbb{C})$.

Proof. This follows from a specialization of Proposition 5 in [21] to imaginary quadratic fields. ∎

Now, we have a nice geometric algorithm for expressing an arbitrary symbol $[u, v]$ as a sum of Voronoi reduced symbols:

Theorem 3.9 ([21, Theorem 4])**.** *Given a modular symbol* $[u, v]$*, there exists a set of points* $\{x_i : 1 \leq i \leq n\}$ *with the properties:*

1. $q(u) \in R(x_1)$ *and* $q(v) \in R(x_n)$

2. *For any* $1 \leq i \leq n-1$*, there is a ray* $q(r_i)$ *in* $R(x_i) \cap R(x_{i+1})$ *such that*

$$[u, v] = [u, r_1] + [r_1, r_2] + ... + [r_n, v],$$

where $R(x)$ *denotes the rays in the cone* C *containing* $q(x)$*.*

M-symbols are a convenient way to compute modular symbols. Over Euclidean number fields, the translates of the modular symbol $\{0, i\infty\}$ give us a collection of all symbols. Thus, as described in Section 2.5, M-symbols will simply be elements $(c : d) \in \mathbb{P}^1(\mathfrak{n})$. On the other hand, non-euclidean fields have modular symbols that are not translates of the symbol $\{0, i\infty\}$. Therefore, we require more than one orbit representative.

Example 3.10. Let $F = \mathbb{Q}(\sqrt{-17})$ with the ring of integers $\mathcal{O}_F = \mathbb{Z}[\omega]$, where $\omega = \sqrt{-17}$. Consider the edges $e_1 = \{0, 1\}$, $e_2 = \{0, 3/(\omega+2)\}$. We claim that e_1 and e_2 are not equivalent. We can see this easily by looking at the cusp classes of the vertices. Since the class group of F is cyclic of order 4, we have the lists of cusps

$$c_0 = (1, \omega), c_1 = (3, \omega + 2), c_2 = (9, \omega + 8), c_3 = (27, \omega + 8).$$

Both vertices of the edge e_1 have cusps in the class c_0. The edge e_2 has one vertex of class c_0 and one of c_1. Since the action $\mathrm{GL}_2(\mathcal{O}_F)$ preserves the class of a vertex, e_1 and e_2 are not $\mathrm{GL}_2(\mathcal{O}_F)$ equivalent.

Moreover, two edges with the same types of cusps are not guaranteed to be equivalent under $\mathrm{GL}_2(\mathcal{O}_F)$. For example, the edge $e_3 = \{0, 3/(\omega+3)\}$ has the same cusp class as e_1 but they are not equivalent under the action of $\mathrm{GL}_2(\mathcal{O}_F)$. A full list of cusps and edges for this case is available in Section 6.1.

This leads to the following definition:

Definition 3.11. A *M-symbol* is a pair $\{(c:d), e\}$ where e is an oriented edge representative of a $\mathrm{GL}_2(\mathcal{O}_F)$-equivalence class of edges in the Voronoi tessellation and $(c:d) \in \mathbb{P}^1(\mathcal{O}_F/\mathfrak{n})$.

Note that we can conveniently go back and forth between modular symbols and M-symbols in the following way. If e is an edge with vertices on cusps u and v, then the M-symbol $\{(c:d), e\}$ can be identified with the modular symbol $[gu, gv]$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$.

In practice, we select edge representative e_i to be the edges in the Voronoi tessellation. Therefore, it is possible that certain edges have non-trivial stabilizer groups. For example, the edge e_3 from Example 3.10 is stabilized by the matrix

$$h = \begin{pmatrix} \omega+3 & -3 \\ 2\omega-3 & -\omega-3 \end{pmatrix} : e_3 \mapsto -e_3.$$

To account for this, we can further take the quotient of $\mathbb{P}^1(\mathcal{O}_F/\mathfrak{n})$ by the stabilizer group of edge representatives from the right. Explicitly, suppose e is an edge representative of a certain edge type, and suppose $H \subset \mathrm{GL}_2(\mathcal{O}_F)$ stabilizes the edge. Then $\mathrm{GL}_2(\mathcal{O}_F)/H$ parameterize the $\mathrm{GL}_2(\mathcal{O}_F)$ orbit of the edge e. Then for a congruence subgroup $\Gamma_0(\mathfrak{n})$, the double coset space $\Gamma_0(\mathfrak{n})\backslash \mathrm{GL}_2(\mathcal{O}_F)/H$ parameterizes $\Gamma_0(\mathfrak{n})$ orbits of the edge e. Here the right action of H on $\mathbb{P}^1(\mathfrak{n})$ is given by

$$(c:d) \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix} = (cp + dr : cq + ds).$$

Example 3.12. Consider the level $\mathfrak{p}_{2.1} = \langle 2, \omega+1\rangle$ and $\Gamma = \Gamma_0(\mathfrak{p}_{2.1})$. The points

$$\mathbb{P}^1(\mathfrak{p}_{2.1}) = \{(1:0), (0:1), (1:1)\},$$

corresponds to the coset representatives

$$\{\Gamma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Gamma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Gamma \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\} \in \Gamma\backslash \operatorname{GL}_2(\mathcal{O}_F).$$

The orbits of the edge $e_1 = \{0, 3/\omega+3\}$ contain the following edges:

$$e_1 = \{(0:1), e_1\} = \{0, 3/\omega+3\},$$
$$e_2 = \{(1:0), e_1\} = \{\infty, \omega+3/3\},$$
$$e_3 = \{(1:1), e_1\} = \{-1, (-\omega-3)/(\omega+6).\}$$

However, the edge e_1 is stabilized by the matrix

$$h = \begin{pmatrix} \omega+3 & -3 \\ 2\omega-3 & -\omega-3 \end{pmatrix} : e_1 \mapsto -e_1.$$

Let $t = \begin{pmatrix} -3 & -\omega-3 \\ -\omega-3 & -2\omega+3 \end{pmatrix} \in \Gamma$. Then, $t \cdot e_2 = -e_1$. Thus, e_1 and e_2 are in the same orbit of Γ. We can see this also by looking at the action of the matrix h on $\mathbb{P}^1(\mathfrak{p}_{2.1})$,

$$(0:1)h = (2\omega-3 : -\omega-3) = (1:0).$$

Since h swaps $(1:0)$ and $(0:1)$, we only need to consider representative e_1 and e_3 to span the orbit of e_1 in the homology.

From the above discussion, we see that M-symbols give a convenient way of writing down a list of generators for homology. Now we give an example of how to express face relations

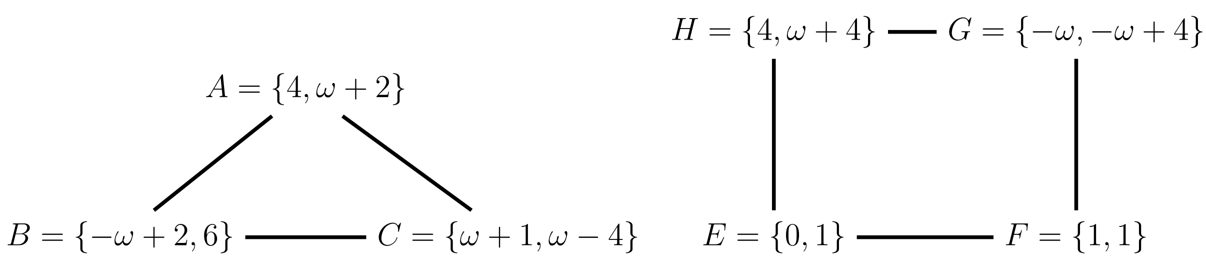


Figure 3.1. Faces in the Voronoi tessellation for $\mathrm{GL}_2(\mathcal{O}_F)$

using M-symbols.

Example 3.13. Two faces in the Voronoi tessellation for $F = \mathbb{Q}(\sqrt{-17})$ and level $\mathfrak{p}_{2.1}$ are given in Figure 3.1.

From the triangular face, we get the relation,

$$\{(0:1), AB\} + \{(0:1), BC\} + \{(0:1), CA\} = 0.$$

For the matrix $g = \begin{pmatrix} 2\omega + 9 & \omega - 9 \\ 4\omega - 4 & -2\omega - 11 \end{pmatrix} \in \Gamma_0(\mathfrak{p})$, we have $g(AB) = -(BC)$. Thus the relation of the face becomes

$$\{(0:1), AB\} - \{(0:1), (AB)\} - \{(0:1), AC\} = 0, \text{ which implies } \{(0:1), AC\} = 0.$$

The rectangular face gives us the relation

$$\{(0:1), AB\} + \{(0:1), BC\} + \{(0:1), CD\} + \{(0:1), DA\} = 0.$$

Since the edges satisfy

$$(DC) = \begin{pmatrix} \omega - 4 & 4 \\ -8 & \omega + 4 \end{pmatrix} (0:1)(AB) = (0:1)(AB)$$

and

$$(BC) = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} (0:1)(AD) = (0:1)(AD),$$

this relation will become trivial.

Remark 3.4. Principal Hecke operators introduced in Section 4.2 act on the homology $H_1(X_0(\mathfrak{n}); \mathbb{C})$. Let $\mathfrak{p}$ be a principal prime ideal, and let m_i be the Hecke matrices given in Theorem 4.20 for the Hecke operator $T_\mathfrak{p}$. Then the action of the Hecke operator $T_\mathfrak{p}$ on an M-symbol $\{(c:d), \{\alpha, \beta\}\}$ is given by

$$T_\mathfrak{p}((c:d), \{\alpha, \beta\}) = \sum_i m_i \left\{ \frac{a\alpha + b}{c\alpha + d}, \frac{a\beta + b}{c\beta + d} \right\},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$. We can apply the reduction theory for modular symbols from the work of Gunnells [21] for each modular symbol in the sum to express it as a finite sum of M-symbols.

The structure of the Voronoi tessellation determines the number of generators and relations required to identify the space of homological modular forms. Thus, understanding the number and the combinatorial types of polytopes in the tessellation is helpful in determining the difficulty of computing homological modular forms. Motivated by this, we computed and studied tessellations for a range of imaginary quadratic fields in [30].

The two figures below summarise the trends observed from the computations. From Figure 3.2, we see that the number of perfect forms increases with the discriminant of the number field. This means that the number of polytopes in the tessellation also increases with the discriminant. Thus, we expect the homology computations to get harder with the discriminant since the number of generators and relations depends on the tessellation.

Figure 3.3 shows that tetrahedra are the mostly common polytope, especially for large discriminants.

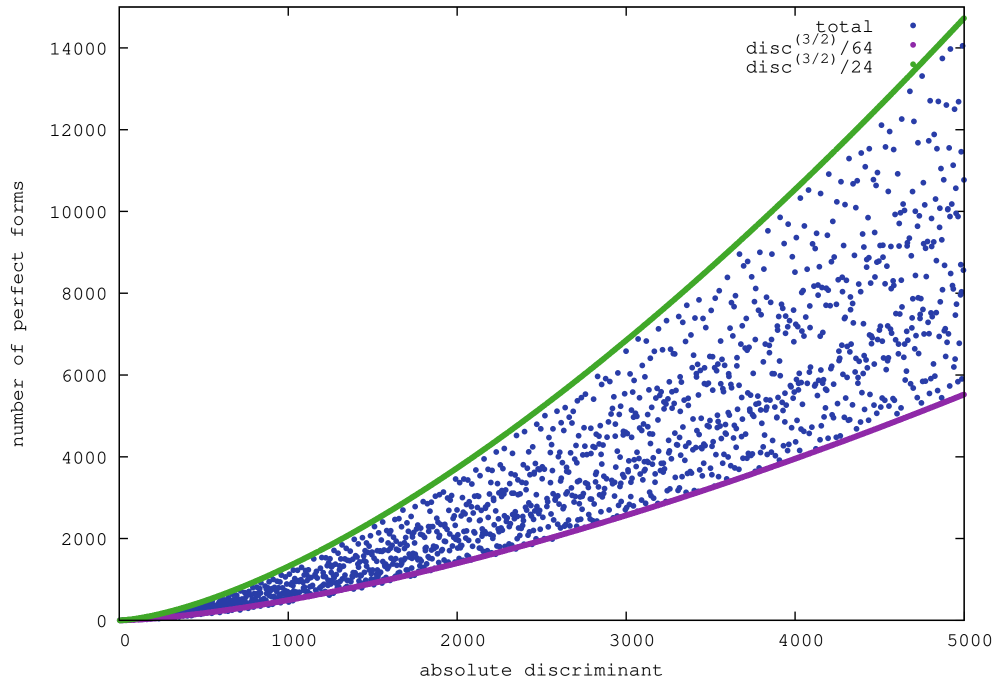


Figure 3.2. Number of perfect forms $N_{\text{perf}}(F)$, indexed by absolute discriminant of F

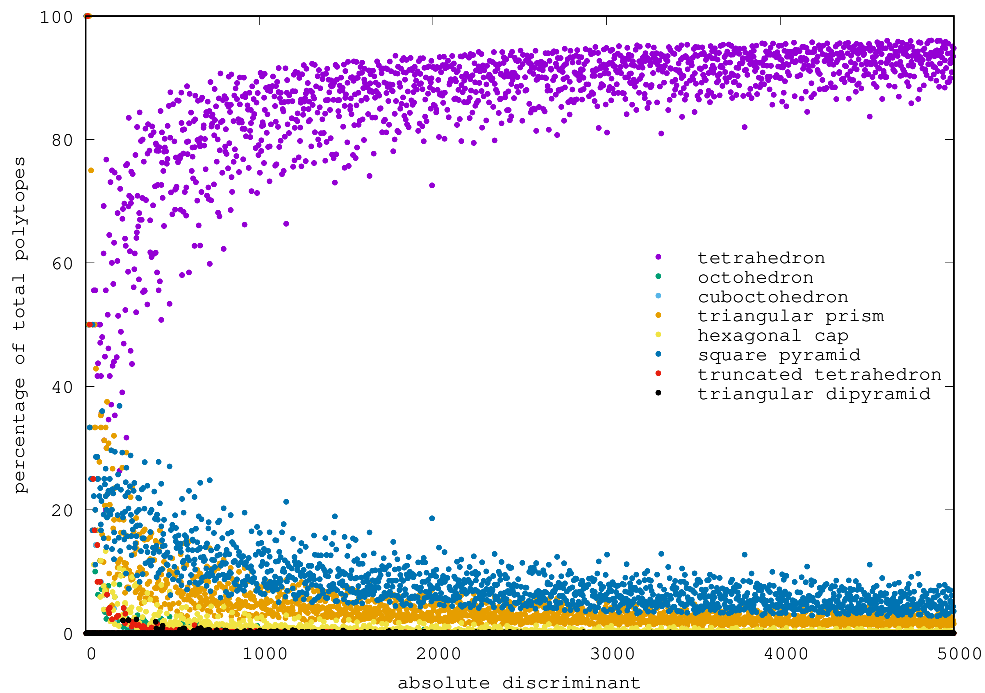


Figure 3.3. Percentage of polytope types indexed by the absolute discriminant of F.

Note that the number of perfect forms $N_{\mathrm{perf}}(F)$ is the number of polytopes in the tessellation. We also obtained a lower bound for $N_{\mathrm{perf}}(F)$.

Theorem 3.14 ([30], Theorem 4.7)**.** *Let F be an imaginary quadratic field of discriminant Δ_F. Then*

$$N_{\mathrm{perf}}(F) \geq \left\lceil \frac{|\Delta_F|^{3/2}\zeta_F(2)}{C} \right\rceil,$$

where the constant $C \sim 1200$.

Chapter 4: Bianchi Modular Forms

This section follows a preprint by John Cremona extending the work of his students Aranes, Bygott, and Lingham [1, 6, 28].

In Section 4.1, we define lattices and modular points over imaginary quadratic fields. Then we define a Bianchi modular form as a function on modular points. This approach allows us to understand the role of the class group. This generalizes the classical theory of viewing modular forms as functions on enhanced elliptic curves discussed in Section 2.3.

In Section 4.2, we define Hecke operators and study their properties. Similar to the classical case, Hecke operators commute and are diagonalizable. Thus, we have a basis for the space consisting of simultaneous eigenforms. We discuss this perspective of viewing a modular form as a Hecke eigensystem in Section 4.3.

In Section 4.4, we introduce the notion of matrices of type $(\mathfrak{a}, \mathfrak{b})$ which is used to identify Hecke matrices for principal Hecke operators. Finally, in Section 4.5, we discuss an application of matrices of type $(\mathfrak{a}, \mathfrak{b})$ to identify cusp equivalence.

4.1 Bianchi Modular Forms

We start by defining $\mathcal{O}_F$-lattices which are a generalization of classical lattices.

Definition 4.1. An $\mathcal{O}_F$*-lattice* L is $\mathcal{O}_F$-submodule of $\mathbb{C}^2$ that satisfies the following:

1. $\mathbb{C}L = \mathbb{C}^2$, that is L contains a basis for $\mathbb{C}^2$.

2. L is a finitely generated free module so that FL is a 2-dimensional F vector space.

From the structure theorem of $\mathcal{O}_F$-modules, we have the following:

Theorem 4.2 ([34, Theorem 9.3.6])**.** *For any lattice L, there exists x_1, x_2 such that $L = \mathfrak{a}_1 x_1 \oplus \mathfrak{a}_2 x_2$ for some fractional ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$.*

The class of the product $\mathfrak{a}_1\mathfrak{a}_2$ in the class group Cl_F is independent of the choice of x_1 and x_2. This invariant is called the *Steinitz class* or simply the *class* of a lattice. We use the notation $cl(L)$ to denote this.

Now we define modular points for $\Gamma_0(\mathfrak{n})$, which is the generalization of the enhanced elliptic curves for the classical case.

Definition 4.3. A *modular point for* $\Gamma_0(\mathfrak{n})$ is a pair (L, L') of lattices in $\mathcal{O}_F^2$, where $L \subseteq L'$ and $L'/L \simeq \mathcal{O}_F/\mathfrak{n}$. We call $\mathfrak{n}$ the *index* of the sublattice L in L' and denote it by $[L' : L]$. Here L is called the *underlying lattice* of (L, L'). The *class* of a modular point is the Stenitz class $cl(L)$ of the underlying lattice L. The set of modular points is denoted by $M_0(\mathfrak{n})$.

Now we define Bianchi modular forms.

Definition 4.4. A *Bianchi modular form* of weight 2 for $\Gamma_0(\mathfrak{n})$ is a function

$$f : M_0(\mathfrak{n}) \to \mathbb{C}^3,$$

which satisfies

$$f(Pzk) = f(P)\rho(zk)$$

for each $z \in \mathcal{Z}$ and $k \in K$. Here ρ is an 3-dimensional representation of $\mathcal{Z}K$, where $\mathcal{Z}$ is the center of $\mathrm{GL}_2(\mathbb{C})$ and K is the maximal compact subgroup $U(2)$.

We have the decomposition of $M_0(\mathfrak{n}) = \bigsqcup_{c\in \mathrm{Cl}_F} M_0^{(c)}(\mathfrak{n})$ where $M_0^{(c)}(\mathfrak{n})$ is the collection of modular points of class $c \in \mathrm{Cl}_F$. Therefore, we can view Bianchi modular forms as a tuple of functions (f_c), where $f_c : M_0^{(c)}(\mathfrak{n}) \to \mathbb{C}^3$.

We can understand the connection between each component in the tuple (F_c) by selecting special representatives for the classes in Cl_F as follows.

Let $\mathfrak{p}_i$ for $1 \leq i \leq k$ be a set of prime ideals such that $\{[\mathfrak{p}_i] : 1 \leq i \leq k\}$ is a list of cosets in $\mathrm{Cl}_F / \mathrm{Cl}_F^2$. Let $\mathfrak{q}_j$ for $1 \leq j \leq h/k$ be ideals such that the ideal classes $[\mathfrak{q}_j^2]$ comprise Cl_F^2. We fix $\mathfrak{p}_1 = \mathfrak{q}_1 = \mathcal{O}_F$. Then each class of the class group can be written as $c_{ij} = [\mathfrak{p}_i {\mathfrak{q}_\mathrm{j}}^2]$.

Definition 4.5. The *standard lattice of class* c_{ij} is given by $L_{ij} = \mathfrak{p}_i\mathfrak{q}_j \oplus \mathfrak{q}_j$. Further, a *standard modular point of class* c_{ij} is $P_{ij} = (L_{ij}, L'_{ij})$ where $L'_{ij} = \mathfrak{p}_i\mathfrak{q}_j \oplus \mathfrak{q}_j\mathfrak{n}^{-1}$. Here $c_{ij} \in \mathrm{Cl}_F$ is a representative described above.

Every modular point in the $\mathrm{GL}_2(\mathbb{C})$-orbit of standard modular points of the same class.

Proposition 4.6 ([11])**.** *Let P be a modular point for $\Gamma_0(\mathfrak{n})$ in class c_{ij}. Then there exists a matrix $U \in \mathrm{GL}_2(\mathbb{C})$ such that $P = P_{ij}U$.*

Corollary 4.7 ([11])**.** *We have the following identification:*

$$M_0^{(c_{ij})}(\mathfrak{n}) \simeq \Gamma_0^{\mathfrak{p}_i}(\mathfrak{n}) \backslash \mathrm{GL}_2(\mathbb{C}), \tag{4.1}$$

where $\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \;\middle|\; b \in \mathfrak{p}_i^{-1}, c \in \mathfrak{n}\mathfrak{p}_i \right\}$.

Now we can write $M_0(\mathfrak{n}) = \bigsqcup_{c_{ij} \in \mathrm{Cl}_F} M_0^{(c_{ij})}(\mathfrak{n})$ and view a Bianchi modular form f as a tuple (f_{ij}), where $f_{ij} : M_0^{(c_{ij})}(\mathfrak{n}) \to \mathbb{C}^3$. We can also view a Bianchi modular form as a h-tuple of functions $g = (g_{ij}) : \mathbb{H}_3 \to \mathbb{C}^3$. Details are available in [6].

Note that the group $\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})$ is a congruence subgroup of $\mathrm{GL}_2(\mathcal{O}_F)$ only if $i = 1$. Therefore we can compute $f_{1j} : \Gamma_0(\mathfrak{n}) \backslash \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}^3$ by relating it to the homology group $H_1(X_0(\mathfrak{n}); \mathbb{C})$ from the work of Kurcanov [26].

Remark 4.1. If the class number is odd, then $\mathrm{Cl}_F^2 = \mathrm{Cl}_F$. This means each tuple of f has the form f_{1j}. Thus, we can compute each component using the homology group $H_1(X_0(\mathfrak{n}); \mathbb{C})$. On the other hand, if the class number is even $\mathrm{Cl}_F^2 \neq \mathrm{Cl}_F$, thus the homology group $H_1(X_0(\mathfrak{n}); \mathbb{C})$ only captures certain components of the Bianchi modular form.

4.2 Hecke operators

We define Hecke operators as "averaging" operators on the space of modular points. We use the notation $N(\mathfrak{a})$ to represent the ideal norm of $\mathfrak{a}$ of $\mathcal{O}_F$.

Definition 4.8. For each integral ideal $\mathfrak{a}$ coprime to $\mathfrak{n}$, we define

$$T_{\mathfrak{a}}(L, L') = N(\mathfrak{a}) \sum_{\substack{M \subseteq L \\ [L:M]=\mathfrak{a} \\ (M,M') \in M_0(\mathfrak{n})}} (M, M')$$

for each $(L, L') \in \mathcal{M}_0(\mathfrak{n})$ where M is a sublattice of L of index $\mathfrak{a}$ and $M' = M + \mathfrak{a}L'$.

Definition 4.9. For each fractional ideal $\mathfrak{b}$ coprime to $\mathfrak{n}$, we define

$$T_{\mathfrak{b},\mathfrak{b}}(L, L') = N(\mathfrak{b})^2(\mathfrak{b}L, \mathfrak{b}L').$$

Define the action of a Hecke operator given above on Bianchi modular form by

$$T(f) = f \circ T.$$

Cremona [11] defines the Atkin Lehner operator for a special case using sublattices as follows:

Definition 4.10. Let $\mathfrak{p}$ be a prime divisor of the level $\mathfrak{n}$ with $(\mathfrak{p}, \mathfrak{n}\mathfrak{p}^{-1}) = 1$. The *Atkin Lehner operator* is

$$W_{\mathfrak{p}}(L, L') = N(\mathfrak{p})(\mathfrak{p}L + \mathfrak{n}L', L + \mathfrak{p}L').$$

Remark 4.2. Cremona defines a more general class of Atkin Lehner operators in [11].

Now we state some properties of Hecke operators.

Theorem 4.11. *The Hecke operators $T_{\mathfrak{a}}$ and $T_{\mathfrak{b},\mathfrak{b}}$ satisfy the following properties:*

1. If $\mathfrak{a}$ and $\mathfrak{b}$ are coprime then

$$T_{\mathfrak{a}}T_{\mathfrak{b}} = T_{\mathfrak{a}\mathfrak{b}} = T_{\mathfrak{b}}T_{\mathfrak{a}}.$$

2. If $\mathfrak{a}$ and $\mathfrak{b}$ are coprime and $\mathfrak{b}$ coprime to the level $\mathfrak{n}$ then

$$T_{\mathfrak{a}}T_{\mathfrak{b},\mathfrak{b}} = T_{\mathfrak{b},\mathfrak{b}}T_{\mathfrak{a}}.$$

3. If $\mathfrak{p}$ does not divide the level $\mathfrak{n}$ then

$$T_{\mathfrak{p}^{n+1}} = T_{\mathfrak{p}^n}T_{\mathfrak{p}} - N(\mathfrak{p})T_{\mathfrak{p}^{n-1}}T_{\mathfrak{p},\mathfrak{p}},$$

for all $n \geq 1$.

Proof. Details are and proofs are given in [11]. □

Let $\mathbb{T}$ denote the algebra of endomorphisms generated by operators of the form $T_{\mathfrak{a}}T_{\mathfrak{b},\mathfrak{b}}$. We say a Hecke operator T has *class* $c \in \mathrm{Cl}_F$ if $T(M) \in M_0^{(cc')}(\mathfrak{n})$ for any $M \in M_0^{(c')}(\mathfrak{n})$. Similar to the space of modular points, we also have a decomposition of the Hecke algebra $\mathbb{T}$ into a disjoint union of Hecke operators $\mathbb{T}^{(c)}$ of class c.

Remark 4.3. Note that Hecke operators of non-principal classes permute the components of a tuple $f = (f_c)$. In particular, this means non-principal Hecke operators do not act on the homology group $H_1(X_0(\mathfrak{n}); \mathbb{C})$.

4.3 Hecke Eigensystems

From the previous section, we saw that Hecke operators are commutative. These operators are also diagonalizable and therefore are simultaneously diagonalizable. Thus, there exists an eigenbasis for the space of modular forms consisting of Hecke eigenforms. Each such eigenform can be viewed as a Hecke eigensystem.

This perspective is especially helpful in computations over imaginary quadratic fields with non-trivial class groups. Although we cannot compute non-principal Hecke operators directly, considering eigenforms allows us to use properties of Hecke operators to extract the eigenvalues of non-principal ideals. See Chapter 5 for more details.

Definition 4.12. Let f be a Bianchi modular form that is a simultaneous eigenvector for all Hecke operators in the Hecke algebra $\mathbb{T}$. We call a function $\lambda_f : \mathbb{T} \to \mathbb{C}$ a *Hecke eigensystem* if

$$T(f) = \lambda_f(T)f,$$

for each $T \in \mathbb{T}$. Alternatively, we can identify f with the pair (λ_f, χ_f) where $\lambda_f(\mathfrak{a}) = \lambda_f(T_\mathfrak{a})$ and $\chi_f(\mathfrak{b}) = \lambda_f(T_{\mathfrak{b},\mathfrak{b}})$. We call χ the *character* of the Hecke eigensystem.

The identification of f with a pair (λ, χ) is used more commonly in this document than as a function $\lambda_f : \mathbb{T} \to \mathbb{C}$.

Now we can restate the properties of the Hecke operators in terms of Hecke eigensystems.

Theorem 4.13. *The eigenvalues of a Hecke eigensystem (λ, χ) satisfy the following properties:*

- *If $\mathfrak{a}$ and $\mathfrak{b}$ are coprime, then*

$$\lambda(\mathfrak{a})\lambda(\mathfrak{b}) = \lambda(\mathfrak{a}\mathfrak{b}).$$

- *If $\mathfrak{p}$ does not divide the level $\mathfrak{n}$ then*

$$\lambda(\mathfrak{p}^n)\lambda(\mathfrak{p}) = \lambda(\mathfrak{p}^{n+1}) + N(\mathfrak{p})\lambda(\mathfrak{p}^{\mathfrak{n}-1})\chi(\mathfrak{p}),$$

  for all $n \geq 1$.

Given a Hecke eigensystem, we can twist by any character of the class group to obtain another Hecke eigensystem.

Definition 4.14. Let (λ, χ) be a Hecke eigensystem and let ψ be a character of the class group. We call the system $(\lambda\psi, \chi\psi^2)$ a *twist of* (λ, χ) *by* ψ.

Theorem 4.15. *If* (λ', χ') *is a twist of* (λ, χ) *by* ψ *then* $\chi = \chi'$ *if and only if* ψ *is a quadratic character.*

Proof. If (λ', χ') is a twist of (λ, χ), by definition there is a character ψ such that, for any ideal $\mathfrak{a}$, we have $\chi'(\mathfrak{a}) = \chi(\mathfrak{a})\psi^2(\mathfrak{a})$. Then $\chi' = \chi$ if and only if $\psi^2(\mathfrak{a}) = 1$ for all $\mathfrak{a}$. Thus, the character ψ is a quadratic character of the class group. $\square$

Definition 4.16. Let (λ, χ) be a Hecke eigensystem. We say the system has *an inner twist* if for some be a character ψ of the class group, we have $(\lambda, \chi) = (\lambda\psi, \chi\psi^2)$.

We call the set of Hecke eigensystems obtained by twisting a system (λ, χ) by the characters of the class group, *the twist orbit* of (λ, χ). The size of the twist orbit is at most the class number, and it is strictly less than the class number if and only if (λ, χ) has inner twists. In Chapter 5, we show that computing twist orbit representatives of Hecke eigensystems is sufficient to obtain the space of Bianchi modular forms.

From Theorem 4.15, if (λ, χ) has an inner twist by ψ then ψ must be a quadratic character. In particular, if the class group is odd, quadratic characters are not present which makes it the "easy" case.

Note that, if (λ, χ) has an inner twist by a character ψ then for each $\mathfrak{a}$,

$$\lambda(\mathfrak{a})(\psi(\mathfrak{a}) - 1) = 0. \tag{4.2}$$

Then if $\psi(\mathfrak{a}) = -1$, we have $\lambda(\mathfrak{p}) = 0$. Therefore λ is identically zero on all classes with $\psi(\mathfrak{p}) = -1$. We use this in Section 5.6 to identify Hecke eigensystems with inner twists.

4.4 Hecke Matrices

In this section, we discuss techniques for computing principal Hecke operators using explicit Hecke matrices.

Lemma 4.17 ([28, Lemma 1.2.5]). *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $\mathcal{O}_F$ such that $\mathfrak{a}\mathfrak{b} = \langle g \rangle$ is principal. Then there exists a matrix $M \in \mathrm{GL}_2(\mathcal{O}_F)$ with determinant g with the lower left entry in $\mathfrak{n}$ which induces an isomorphism: $\mathcal{O}_F \oplus \mathcal{O}_F \to \mathfrak{a} \oplus \mathfrak{b}$.*

Proof. Write $\mathfrak{a} = \alpha_1 \mathcal{O}_F + \alpha_2 \mathcal{O}_F$, where $\alpha_2 \in \mathfrak{n}$. Since $\mathfrak{a}\mathfrak{b} = \langle g \rangle$ is principal there exists some $\beta_1, \beta_2 \in \mathfrak{b}$ such that $g = \alpha_1\beta_2 - \alpha_2\beta_1$.

Now the isomorphism is given by right multiplication by the matrix $M = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ on $\mathcal{O}_F \oplus \mathcal{O}_F$. Here elements in $\mathcal{O}_F \oplus \mathcal{O}_F$ and $\mathfrak{a} \oplus \mathfrak{b}$ are represented as row vectors with the action matrix multiplication on the right. That is if $(x, y) \in \mathcal{O}_F \oplus \mathcal{O}_F$,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} x\alpha_1 + y\alpha_2 & x\beta_1 + y\beta_2 \end{pmatrix} \in \mathfrak{a} \oplus \mathfrak{b}.$$

□

We call a matrix that gives such an isomorphism a matrix an $(\mathfrak{a}, \mathfrak{b})$-*matrix* of level $\mathfrak{n}$.

There are two applications of $(\mathfrak{a}, \mathfrak{b})$-matrices of level $\mathfrak{n}$ for our computations. In this section, we discuss how to use them to find Hecke matrices for certain types of principal Hecke operators. We can also use them to identify cusp equivalence. This is discussed in Section 4.5.

Proposition 4.18 ([12, Proposition 3]). *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals. Then for any $\gamma \in \mathrm{GL}_2(F)$, we have $(\mathfrak{a} \oplus \mathfrak{b})\gamma = (\mathfrak{a} \oplus \mathfrak{b})$ if and only if $\gamma \in \Delta(\mathfrak{a}, \mathfrak{b})$, where*

$$\Delta(\mathfrak{a}, \mathfrak{b}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \,\middle|\, x, w \in \mathcal{O}_F, y \in \mathfrak{a}^{-1}\mathfrak{b}, z \in \mathfrak{a}\mathfrak{b}^{-1}, xw - yz \in \mathcal{O}_F^\times \right\}.$$

Now we will use $(\mathfrak{a}, \mathfrak{b})$-matrices of level $\mathfrak{n}$ to parameterize subspaces of the principal lattice $\mathcal{O}_F \oplus \mathcal{O}_F$.

Theorem 4.19 ([28])**.** *The collection of submodules of $\mathcal{O}_F \oplus \mathcal{O}_F$ of principal index $\mathfrak{m}$ can be parameterized by the set*

$$\{(\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{b}}g \mid g \in \Gamma_0(\mathfrak{a}\mathfrak{b}^{-1})\backslash \mathrm{GL}_2(\mathcal{O}_F)\},$$

where $\mathfrak{a}$, $\mathfrak{b}$ runs over ideals satisfying $\mathfrak{m} = \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b} \mid \mathfrak{a}$.

Proof. Let $M = (\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{b}}g$ for some $(\mathfrak{a}, \mathfrak{b})$-matrix $M_{\mathfrak{a},\mathfrak{b}}$ and $g \in \mathrm{GL}_2(\mathcal{O}_F)$. By definition, we have $(\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{b}} = \mathfrak{a} \oplus \mathfrak{b}$. Thus the lattice M is isomorphic to $\mathfrak{a} \oplus \mathfrak{b}$. Further, since $\mathfrak{a}\mathfrak{b}$ is principal, we have $M_{\mathfrak{a},\mathfrak{b}}g \in \mathrm{GL}_2(\mathcal{O}_F)$. Thus, the lattice M is a sublattice of L isomorphic to $\mathfrak{a} \oplus \mathfrak{b}$.

Now we determine when two lattices $M = (\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{b}}g$ and $M' = (\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{b}}g'$ give us the same sublattice of L.

Suppose $M = M'$. Then $(\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{b}}g = (\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{b}}g'$. This means

$$(\mathfrak{a} \oplus \mathfrak{b})g = (\mathfrak{a} \oplus \mathfrak{b})g'.$$

Since $\mathfrak{b} \mid \mathfrak{a}$, we have $g(g')^{-1} \in \mathrm{GL}_2(\mathcal{O}_F)$. Further, by Proposition 4.18, we have $g(g')^{-1}$ is in $\Delta(\mathfrak{a}, \mathfrak{b})$. Then the lower left entry of $g(g')^{-1}$ is in $\mathfrak{a}\mathfrak{b}^{-1}$. Thus, we have $\Gamma_0(\mathfrak{a}\mathfrak{b}^{-1})g = \Gamma_0(\mathfrak{a}\mathfrak{b}^{-1})g'$. This means the coset space $\Gamma_0(\mathfrak{a}\mathfrak{b}^{-1})\backslash \mathrm{GL}_2(\mathcal{O}_F)$ parameterizes the sublattices of L isomorphic to $\mathfrak{a} \oplus \mathfrak{b}$.

□

Now we consider the relation between standard principal modular point $(\mathcal{O}_F \oplus \mathcal{O}_F, \mathcal{O}_F \oplus \mathfrak{n}^{-1})$ and other principal modular points (L, L'). If (L, L') is a principal modular point, then

L is a principal lattice. Then by Proposition 4.6, there is $U \in \mathrm{GL}_2(\mathbb{C})$ such that

$$(L, L') = (\mathcal{O}_F \oplus \mathcal{O}_F, \mathcal{O}_F \oplus \mathfrak{n}^{-1})U.$$

Consider $M = (\mathcal{O}_F \oplus \mathcal{O}_F)gU$ and $M' = (\mathcal{O}_F \oplus \mathfrak{n}^{-1})gU$, where g is an $(\mathfrak{a}, \mathfrak{b})$-matrix of level $\mathfrak{n}$ such $\mathfrak{m} = \mathfrak{a}\mathfrak{b}$ is principal and $\mathfrak{b} \mid \mathfrak{a}$. Then M is a sublattice of L of index $\mathfrak{a}\mathfrak{b}$. Further, the lattice M' is a superlattice of M such that $M'/M \simeq \mathcal{O}_F/\mathfrak{n}$. Thus, the pair (M, M') is in the sum defining the Hecke operator $T_{\mathfrak{m}}$ in Definition 4.8. See [11] for additional details.

Lemma 4.20 ([28]). *For any ideal $\mathfrak{m}$ in a principal class we can compute the Hecke operator $T_{\mathfrak{m}}$ on principal lattices by*

$$T_{\mathfrak{m}}(L, L') = N(\mathfrak{m}) \sum_{\substack{\mathfrak{a}|\mathfrak{b},\\ \mathfrak{a}\mathfrak{b}=\mathfrak{m}}} \sum_{g \in \Gamma_0(\mathfrak{a}\mathfrak{b}^{-1})\backslash\Gamma} (\mathcal{O}_F \oplus \mathcal{O}_F, \mathcal{O}_F \oplus \mathfrak{n}^{-1})M_{\mathfrak{a},\mathfrak{b}}gU,$$

where $\Gamma = \mathrm{GL}_2(\mathcal{O}_F)$ and $M_{\mathfrak{a},\mathfrak{b}}$ is an $(\mathfrak{a}, \mathfrak{b})$-matrix of level $\mathfrak{n}$.

We can also compute the Hecke operator $T_{\mathfrak{a},\mathfrak{a}}$ when $\mathfrak{a}^2$ is principal using matrices as follows:

Theorem 4.21. *Let $\mathfrak{a}$ be an ideal coprime to $\mathfrak{n}$ such that $\mathfrak{a}^2$ is principal. Then*

$$T_{\mathfrak{a},\mathfrak{a}}(L, L') = N(\mathfrak{a})^2(\mathcal{O}_F \oplus \mathcal{O}_F, \mathcal{O}_F \oplus \mathfrak{n}^{-1})M_{\mathfrak{a},\mathfrak{a}}U,$$

where $M_{\mathfrak{a},\mathfrak{a}}$ is a matrix of type $(\mathfrak{a}, \mathfrak{a})$ of level $\mathfrak{n}$ and $L = (\mathcal{O}_F \oplus \mathcal{O}_F)U$.

Proof. By Definition 4.9, we have

$$T_{\mathfrak{a},\mathfrak{a}}(L, L') = N(\mathfrak{a})^2(\mathfrak{a}L, \mathfrak{a}L').$$

Suppose $L = (\mathcal{O}_F \oplus \mathcal{O}_F)U$ for some $U \in \mathrm{GL}_2(\mathbb{C})$. Then $\mathfrak{a}L$ is a sublattices of L such that

$L/\mathfrak{a}L \simeq \mathcal{O}_F/\mathfrak{a} \oplus \mathcal{O}_F/\mathfrak{a}$. Therefore, we have

$$\mathfrak{a}L = (\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a},\mathfrak{a}}U$$

for some $(\mathfrak{a}, \mathfrak{a})$ matrix $M_{\mathfrak{a},\mathfrak{a}}$. Further if $M_{\mathfrak{a},\mathfrak{a}}$ has type $\mathfrak{n}$, then we have

$$T_{\mathfrak{a},\mathfrak{a}}(L, L') = N(\mathfrak{a})^2(\mathfrak{a}L, \mathfrak{a}L') = N(\mathfrak{a})^2(\mathcal{O}_F \oplus \mathcal{O}_F, \mathcal{O}_F \oplus \mathfrak{n}^{-1})M_{\mathfrak{a},\mathfrak{a}}U.$$

□

Now we look at other types of principal Hecke operators that will be useful for computations in Chapter 5.

Theorem 4.22 (Hecke Matrices for the principal operator $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}}$)**.** *For any prime ideal $\mathfrak{p}$ and prime ideal $\mathfrak{a}$ such that $\mathfrak{p}\mathfrak{a}^2$ is principal, the following matrices can be used to compute the Hecke operator $T_{\mathfrak{p}}T_{\mathfrak{a},\mathfrak{a}}$:*

$$\{M_{\mathfrak{p}\mathfrak{a},\mathfrak{a}}V \mid V \in \Gamma_0(\mathfrak{p})\backslash \operatorname{GL}_2(\mathcal{O}_F)\},$$

where $M_{\mathfrak{p}\mathfrak{a},\mathfrak{a}}$ is a matrix of type $\mathfrak{n}$.

That is,

$$\begin{aligned} T_{\mathfrak{p}}T_{\mathfrak{a},\mathfrak{a}}(L, L') &= T_{\mathfrak{p}}T_{\mathfrak{a},\mathfrak{a}}(\mathcal{O}_F \oplus \mathcal{O}_F, \mathcal{O}_F \oplus \mathfrak{n}^{-1})U \\ &= N(\mathfrak{p}\mathfrak{a}^2) \sum_{V \in \Gamma_0(\mathfrak{p})\backslash \operatorname{GL}_2(\mathcal{O}_F)} (\mathcal{O}_F \oplus \mathcal{O}_F, \mathcal{O}_F \oplus \mathfrak{n}^{-1})M_{\mathfrak{p}\mathfrak{a},\mathfrak{a}}VU. \end{aligned}$$

Proof. By definition,

$$\begin{aligned} T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}}(L,L') &= T_{\mathfrak{a},\mathfrak{a}}(T_{\mathfrak{p}}(L,L')) \\ &= N(\mathfrak{a}^2\mathfrak{p}) \sum_{\substack{M\subset L,\\ [L:M]=\mathfrak{p}\\ (M,M')\in M_0(\mathfrak{n})}} (\mathfrak{a}M, \mathfrak{a}M'). \end{aligned}$$

If $\mathfrak{a}$ is an integral ideal, then $\mathfrak{a}M$ is also an $\mathcal{O}_F$-submodule of L, and we have $\mathfrak{a}M \subset M \subset L$. Then we can compute the index of $\mathfrak{a}M$ in L as follows:

$$cl([L:\mathfrak{a}M]) = cl(\mathfrak{a}M)/cl(L) = cl(\mathfrak{a}^2)cl(M)/cl(L) = [\mathfrak{a}^2\mathfrak{p}].$$

Therefore, the lattice $\mathfrak{a}M$ is a principal index submodule of L such that M is an $\mathcal{O}_F$-submodule of L of index $\mathfrak{p}$. This means $\mathfrak{a}M \simeq \mathfrak{a}\mathfrak{p} \oplus \mathfrak{a}$. Thus, we have $\mathfrak{a}M = L \cdot M_{\mathfrak{a}\mathfrak{p},\mathfrak{a}} \cdot V$, where $V \in \Gamma_0(\mathfrak{p})\backslash \mathrm{GL}_2(\mathcal{O}_F)$. Since we can select $M_{\mathfrak{a}\mathfrak{p},\mathfrak{a}}$ to have level $\mathfrak{n}$, the result follows. $\square$

Theorem 4.23. *We can compute the principal Hecke operator $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}^2}$ using the following matrices:*

$$\{M_{\mathfrak{a}\mathfrak{p},\mathfrak{a}\mathfrak{p}}\} \cup \{M_{\mathfrak{a}\mathfrak{p}^2,\mathfrak{a}} \cdot V \mid V \in \Gamma_0(\mathfrak{p}^2)\backslash \mathrm{GL}_2(\mathcal{O}_F)\}.$$

Proof. By definition,

$$\begin{aligned} T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}^2}(L,L') &= T_{\mathfrak{a},\mathfrak{a}}(T_{\mathfrak{p}^2}(L,L')) \\ &= N(\mathfrak{a}^2\mathfrak{p}^2) \sum_{\substack{M\subset L,\\ [L:M]=\mathfrak{p}^2\\ (M,M'\in M_0(\mathfrak{n}))}} (\mathfrak{a}M, \mathfrak{a}M') \end{aligned}$$

If $\mathfrak{a}$ is an integral ideal, then $\mathfrak{a}M$ is also an $\mathcal{O}_F$-submodule of L, and we have $\mathfrak{a}M \subset M \subset L$.

Then we can compute the index of $\mathfrak{a}M$ in L as follows:

$$cl([L:\mathfrak{a}M]) = cl(\mathfrak{a}M)/cl(L) = cl(\mathfrak{a}^2)cl(M)/cl(L) = [\mathfrak{a}^2\mathfrak{p}^2]$$

Therefore, the lattice $\mathfrak{a}M$ is a principal index submodule of L. This means $\mathfrak{a}M \simeq \mathfrak{a}\mathfrak{p}^2 \oplus \mathfrak{a}$ or $\mathfrak{a}M \simeq \mathfrak{a}\mathfrak{p} \oplus \mathfrak{a}\mathfrak{p}$.

If $L = (\mathcal{O}_F \oplus \mathcal{O}_F)U$ for some $U \in \mathrm{GL}_2(\mathbb{C})$, then $\mathfrak{a}M = (\mathcal{O}_F \oplus \mathcal{O}_F) \cdot M_{\mathfrak{a}\mathfrak{p}^2,\mathfrak{a}}V \cdot U$ where $V \in \Gamma_0(\mathfrak{p}^2)\backslash \mathrm{GL}_2(\mathcal{O}_F)$, or $\mathfrak{a}M = (\mathcal{O}_F \oplus \mathcal{O}_F)M_{\mathfrak{a}\mathfrak{p},\mathfrak{a}\mathfrak{p}} \cdot U$. Therefore, the Hecke matrices required are

$$\{M_{\mathfrak{a}\mathfrak{p},\mathfrak{a}\mathfrak{p}}\} \cup \{M_{\mathfrak{a}\mathfrak{p}^2,\mathfrak{a}} \cdot V \mid V \in \Gamma_0(\mathfrak{p}^2)\backslash \mathrm{GL}_2(\mathcal{O}_F)\}.$$

□

Theorem 4.24. *If $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals such that $(\mathfrak{p}\mathfrak{q}\mathfrak{a}, \mathfrak{n}) = 1$, then we can compute the principal Hecke operator $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}\mathfrak{q}}$ using the following matrices:*

$$\{M_{\mathfrak{a}\mathfrak{p}\mathfrak{q},\mathfrak{a}} \cdot V \mid V \in \Gamma_0(\mathfrak{p}\mathfrak{q})\backslash \mathrm{GL}_2(\mathcal{O}_F)\}.$$

Proof. By definition,

$$\begin{aligned} T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}\mathfrak{q}}(L, L') &= T_{\mathfrak{a},\mathfrak{a}}(T_{\mathfrak{p}\mathfrak{q}}(L, L')) \\ &= T_{\mathfrak{a},\mathfrak{a}}N(\mathfrak{p}\mathfrak{q}) \sum_{\substack{M\subset L,\\ [L:M]=\mathfrak{p}\mathfrak{q}\\ (M,M')\in M_0(\mathfrak{n})}} (M, M') \\ &= N(\mathfrak{a}^2\mathfrak{p}\mathfrak{q}) \sum_{\substack{M\subset L,\\ [L:M]=\mathfrak{p}\mathfrak{q}\\ (M,M')\in M_0(\mathfrak{n})}} (\mathfrak{a}M, \mathfrak{a}M'). \end{aligned}$$

If $\mathfrak{a}$ is an integral ideal then $\mathfrak{a}M$ is also an $\mathcal{O}_F$-submodule of L, and we have $\mathfrak{a}M \subset M \subset L$.

Then we can compute the index of $\mathfrak{a}M$ in L,

$$cl([L : \mathfrak{a}M]) = cl(\mathfrak{a}M)/cl(L) = cl(\mathfrak{a}^2)cl(M)/cl(L) = [\mathfrak{a}^2\mathfrak{p}\mathfrak{q}].$$

Therefore, the lattice $\mathfrak{a}M$ is a sublattice of L of principal index $\mathfrak{p}\mathfrak{q}\mathfrak{a}^2$. By the structure theorem of $\mathcal{O}_F$-modules, we have $M \simeq \mathfrak{p}\mathfrak{q}\mathfrak{a} \oplus \mathfrak{a}$. Such sublattices are parameterized by matrices of the form $M_{\mathfrak{p}\mathfrak{q}\mathfrak{a},\mathfrak{a}}V$, where $\Gamma_0(\mathfrak{p}\mathfrak{q})V$ is a coset representatives of $\Gamma_0(\mathfrak{p}\mathfrak{q})\backslash \mathrm{GL}_2(\mathcal{O}_F)$. □

Cremona [11] identifies Hecke matrices for the Atkin Lehner operator $W_\mathfrak{q}$ where prime $\mathfrak{q} \mid\mid \mathfrak{n}$ with $(\mathfrak{m}\mathfrak{q}, \mathfrak{m})$-matrices whose transposes are also $(\mathfrak{m}\mathfrak{q}, \mathfrak{m})$-matrices. For this thesis, we use the following proposition from his work:

Proposition 4.25 (Hecke matrices for Atkin Lehner operators $T_\mathfrak{p}W_\mathfrak{q}$)**.** *Suppose $\mathfrak{p}$ is a prime not dividing the level $\mathfrak{n}$ and $\mathfrak{q}$ is prime with $\mathfrak{q} \mid\mid \mathfrak{n}$ such that $\mathfrak{p}\mathfrak{q}$ is principal. Then the operator $T_\mathfrak{p}W_\mathfrak{q}$ is described by the matrices M that satisfy the following:*

1. $\mathfrak{p}\mathfrak{q} = \langle \det(M) \rangle$,
2. $M \in \begin{pmatrix} \mathfrak{q} & \mathcal{O}_F \\ \mathfrak{n} & \mathfrak{q} \end{pmatrix}$,
3. $(\mathcal{O}_F \oplus \mathcal{O}_F)M \subset L$,

where L runs through lattices of index $\mathfrak{p}$.

4.5 Cusp Equivalence

To compute $H_1(X_0(\mathfrak{n}); \mathbb{C})$, we need to understand the set $\partial X_0(\mathfrak{n})$, which is the set of cusps $\mathbb{P}^1(F) = F \cup \{\infty\}$ modulo the action of $\Gamma_0(\mathfrak{n})$. In this section, we state an algorithm from [12] that allows us to use $(\mathfrak{a}, \mathfrak{b})$-matrices introduced in Lemma 4.17 to enumerate cusps.

The group $\mathrm{GL}_2(\mathcal{O}_F)$ acts on cusps by left multiplication if we identify elements in $\mathbb{P}^1(F)$ with the column vector in $\mathcal{O}_F^2\backslash\{0\}$.

Theorem 4.26. *The set of cusp modulo the action of* $\mathrm{GL}_2(\mathcal{O}_F)$ *is parameterized by the class group. That is,*

$$\mathrm{GL}_2(\mathcal{O}_F)\backslash\mathbb{P}^1(F) \simeq \mathrm{Cl}_F,$$

where the isomorphism is given by mapping $\mathrm{GL}_2(\mathcal{O}_F)\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{P}^1(F) \mapsto [\langle a, b\rangle] \in \mathrm{Cl}_F$

We can see from the theorem above that if F has a non-trivial class group, then there are multiple cusp classes. We also have more cusp classes for congruence subgroups of $\mathrm{GL}_2(\mathcal{O}_F)$.

Example 4.27. For the number field $F = \mathbb{Q}(\sqrt{-17})$, there are 4 cusps up to the action of $\mathrm{GL}_2(\mathcal{O}_F)$. Namely,

$$c_0 = (1, \omega), \qquad c_1 = (3, \omega + 2), \qquad c_2 = (9, \omega + 8), \qquad c_3 = (27, \omega + 8).$$

The list of cusps up to the action of the congruence subgroup $\Gamma_0(\mathfrak{p}_{2.1})$ is contained in the set

$$\begin{array}{llll} c_0 = (1, \omega), & c_1 = (3, \omega + 2), & c_2 = (9, \omega + 8), & c_3 = (27, \omega + 8) \\ d_0 = (1, 0), & d_1 = (-\omega - 2, \omega + 5), & d_2 = (-\omega + 10, 2\omega - 2), & d_3 = (2\omega - 11, -2\omega - 16). \end{array}$$

Therefore, we have at most twice as many cusp classes because $\Gamma_0(\mathfrak{p}_{2.1})$ is an index 2 subgroup of $\mathrm{GL}_2(\mathcal{O}_F)$.

More generally, the double coset $\Gamma_0(\mathfrak{n})\backslash \mathrm{GL}_2(\mathcal{O}_F)/\Gamma_\alpha$ parameterizes the $\Gamma_0(\mathfrak{n})$-orbit of a cusp α, where Γ_α is the stabilizer in $\mathrm{GL}_2(\mathcal{O}_F)$ of α. A systematic way to parameterize the $\Gamma_0(\mathfrak{n})$-orbit of a cusp α is to first obtain a list of coset representative for $\Gamma_0(\mathfrak{n})\backslash \mathrm{GL}_2(\mathcal{O}_F)$ to get a total list of cusps. Then check their equivalence using techniques given by Aranes and Cremona [12].

We first state two necessary conditions for cusp equivalence, which provide two quick exits.

Theorem 4.28 ([12]). *If α and α' are two cusps that are $\Gamma_0(\mathfrak{n})$-equivalent then*

1. *$[\alpha] = [\alpha']$ where $[\alpha]$ denotes the class of a cusp.*

2. *$\mathfrak{d}_\mathfrak{n}(\alpha) = \mathfrak{d}_\mathfrak{n}(\alpha')$ where $\mathfrak{d}(\alpha)_\mathfrak{n} = \mathfrak{d}(\alpha) + \mathfrak{n}$ and $\mathfrak{d}(\alpha)$ is the denominator ideal of a cusp $\alpha = \begin{pmatrix} a \\ b \end{pmatrix}$ given by $\mathfrak{d}(\alpha) = \langle b \rangle / \langle a, b \rangle$.*

The following theorem provides a technique to check $\Gamma_0(\mathfrak{n})$-equivalence of cusps.

Theorem 4.29 ([12, Theorem 6]). *Suppose $\alpha = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\alpha' = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}$ are cusps that satisfy the necessary conditions above. Let $\mathfrak{a} = \langle a_1, a_2 \rangle = \langle a_1', a_2' \rangle$ Then the following are equivalent:*

1. *There exists $\gamma \in \Gamma_0(\mathfrak{n})$ such that $\gamma(\alpha) = \alpha'$.*

2. *There exist $(\mathfrak{a}, \mathfrak{b})$-matrices $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ and $M' = \begin{pmatrix} a_1' & b_1' \\ a_2' & b_2' \end{pmatrix}$, where $\mathfrak{b} = \mathfrak{a}^{-1}$ such that for some $\gamma \in \mathrm{GL}_2(\mathcal{O}_F)$ we have $\gamma M = M'$.*

3. *There exist $u \in \mathcal{O}_F$ coprime to $\mathfrak{n}$ and $v \in \mathcal{O}_F^\times$ such that*

   (a) $a_2' = u a_2 \mod (\mathfrak{a}\mathfrak{n})$,

   (b) $u a_1' = v a_1 \mod (\mathfrak{d}\mathfrak{a})$.

Chapter 5: Algorithm

In Section 4.1, we introduced Bianchi modular forms and remarked on the deficiency of the homology group $H_1(X_0(\mathfrak{n};\mathbb{C}))$ when the class group of the field is non-trivial. Further, we commented in Section 4.3, that by considering eigenforms and eigensystems, one can extract Hecke eigenvalues of non-principal Hecke operators. In this chapter, we describe an explicit algorithm to compute Bianchi modular forms as Hecke eigensystems using principal Hecke operators. Parts of this section are closely related to the preprint by John Cremona. We cite and remark on these connections throughout the chapter.

In Chapter 3, we described techniques to compute the homology group $H_1(X_0(\mathfrak{n};\mathbb{C}))$, which is a finite-dimensional complex vector space with an action by principal Hecke operators. Since Hecke operators are diagonalizable and simultaneously diagonalizable, we can construct a basis for this space in terms of eigenforms. We defined these as *homological eigenforms* in Section 5.1 and explain how to compute them. Further, we define what it means for a homological eigenform to match an eigensystem. In Section 5.2, we show that Hecke eigensystems that match the same homological eigenform are in the same twist orbit. The main goal is to compute a representative of the Hecke eigensystem orbit for each homological eigenform.

In Section 5.3, we classify the characters of Hecke eigensystems that match a homological eigenform. We call this the character orbit. Then we explain how to compute the eigenvalues of other Hecke operators if the character is fixed. In Section 5.4, we show that for any prime $\mathfrak{p}$, the Hecke eigenvalue $\lambda(\mathfrak{p})$ is a root of a certain quadratic polynomial. If $[\mathfrak{p}] \in \mathrm{Cl}_F^2$, we show that this root is known exactly. On the other hand, if $[\mathfrak{p}] \notin \mathrm{Cl}_F^2$, each choice of root occurs in the twist orbit. Then we explain how to make choices of roots to obtain a single Hecke eigensystem in the twist orbit. In Sections 5.5 and 5.6, we explain how to carry out computations for Hecke eigensystems with and without inner twists. Finally, we conclude this chapter by providing an algorithm that utilizes these techniques.

5.1 Homological Eigenforms

Definition 5.1. We say a class f in the homology $H_1(X_0(\mathfrak{n}); \mathbb{C})$ is a *homological eigenform* if

1. $T_{\mathfrak{a}} f = a_f(\mathfrak{a}) f, \quad$ for each principal ideal $\mathfrak{a}$;

2. $T_{\mathfrak{b},\mathfrak{b}} f = a_f(\mathfrak{b},\mathfrak{b}) f, \quad$ for each ideal $\mathfrak{b} \in \mathrm{Cl}_F[2]$.

In particular, this means f is a simultaneous eigenvector for all Hecke operator $T_{\mathfrak{a}} T_{\mathfrak{b},\mathfrak{b}}$ where $\mathfrak{a}\mathfrak{b}^2$ is principal.

Remark 5.1. From the work of Kurcanov [26], we know that the eigenvalues of a homological eigenform lie in a finite extension of $\mathbb{Q}$. However, we do not know this field until we compute Hecke eigensystems.

Now we explain how we compute homological eigenforms over an imaginary quadratic field F of a given level $\mathfrak{n}$.

First, we compute the homology $H_1(X_0(\mathfrak{n}); \mathbb{C})$ using M-symbol techniques introduced in Section 3.2. Then up to a certain bound B, we compute:

1. Hecke operators $T_{\mathfrak{p}}$ on $H_1(X_0(\mathfrak{n}); \mathbb{C})$ for all principal prime ideals $\mathfrak{p}$ up to norm B using Hecke matrices in Theorem 4.20;

2. Hecke operators $T_{\mathfrak{p}} T_{\mathfrak{a},\mathfrak{a}}$ on $H_1(X_0(\mathfrak{n}); \mathbb{C})$ for all non-principal prime ideals $\mathfrak{p} \in \mathrm{Cl}_F^2$ up to norm B using the Hecke matrices from Theorem 4.22. Here $\mathfrak{a}$ is any prime ideal such that $\mathfrak{a}^2 \in [\mathfrak{p}^{-1}]$.

Finally, we compute the simultaneous diagonalization of these matrices using standard techniques from linear algebra to obtain a list of homological eigenforms.

Remark 5.2. It is possible for the bound B to not be sufficient to obtain a complete diagonalization. In theory, the bound B might depend on the discriminant of the number field and the degree of the Hecke field. In our implementation for $F = \mathbb{Q}(\sqrt{-17})$ a bound $B = 100$ was sufficient for the scope of our computation.

We denote the subgroup of order 2 elements in the class group Cl_F by $\mathrm{Cl}_F[2]$. This means that $[\mathfrak{p}] \in \mathrm{Cl}_F[2]$ if and only if $\mathfrak{p}^2$ is a principal ideal.

Definition 5.2. We say a homological eigenform f *matches* a Hecke eigensystem (λ, χ) if the following are satisfied:

1. For each principal ideal $\mathfrak{a}$, we have $\lambda(\mathfrak{a}) = a_f(\mathfrak{a})$, where $T_{\mathfrak{a}} f = a_f(\mathfrak{a}) f$.
2. For each ideal $\mathfrak{b}$ coprime to the level $\mathfrak{n}$ in $\mathrm{Cl}_F[2]$, that is $\mathfrak{b}^2$ is principal, we have $\chi(\mathfrak{b}) = a_f(\mathfrak{b}, \mathfrak{b})$, where $T_{\mathfrak{b},\mathfrak{b}} f = a_f(\mathfrak{b}, \mathfrak{b}) f$.

In [11], Cremona characterizes the Hecke eigensystems with the same restriction to the principal component, i.e., those Hecke eigensystems that match the same homological eigenform.

In the remainder of this chapter, we explain how to compute Hecke eigensystems using homological eigenforms. First, we show that computing one Hecke eigensystems representative per homological eigenform is sufficient to compute all Hecke eigensystems attached to Bianchi modular forms. Next, we explain how to pick a character for the representative eigensystem using the action of $T_{\mathfrak{a},\mathfrak{a}}$ for $[\mathfrak{a}] \in \mathrm{Cl}_F[2]$. Finally, we explain how to compute eigenvalues $\lambda(\mathfrak{p})$ for various primes $\mathfrak{p}$. By definition, if $\mathfrak{p}$ is principal then we can compute $\lambda(\mathfrak{p})$ from the action of $T_{\mathfrak{p}}$ on the homological eigenform. If the character is fixed, we show how to compute $\lambda(\mathfrak{p})$ for primes $\mathfrak{p}$ with $[\mathfrak{p}] \in \mathrm{Cl}_F^2$. Computing $\lambda(\mathfrak{p})$ with $[\mathfrak{p}] \notin \mathrm{Cl}_F^2$ is more technical. We explain how to do this in Section 5.5 for eigensystems without inner twists and in Section 5.6 for eigensystems with inner twists.

Remark 5.3. Homological eigenforms are sufficient to compute newforms as their restrictions to the principal component are non-trivial. However, there are certain oldforms with trivial restriction to the principal component. We give an example of this in Section 6.3.

5.2 Twist Orbit of Hecke eigensystems

In this section, we discuss the connection between Hecke eigensystems that match the same homological eigenform. We claim that two different Hecke eigensystems can match the same homological eigenform if and only if they are in the same twist orbit. In particular, this means if we can identify a Hecke eigensystem (λ, χ) that matches a homological eigenform f, then we can obtain all the systems that match f by twisting (λ, χ).

The converse of this claim is not hard to see because the characters of the class group are trivial on the principal component. Proving the forward direction requires the bulk of work. The results in this section will follow a preprint by Cremona [11] closely.

The main group theoretical result needed for the proofs in this section is the ability to extend a quadratic character of a subgroup of a 2-group to the full group. We state this result for our context.

Theorem 5.3. *Let H be a subgroup of Cl_F such that $\mathrm{Cl}_F^2 \subseteq H$. Then any quadratic character $\psi : H \to \{1, -1\}$ can be extended to a quadratic character $\psi' : \mathrm{Cl}_F \to \{1, -1\}$ such that $\psi'|_H = \psi$.*

Proof. Since $\mathrm{Cl}_F^2 \subseteq H$ and quadratic characters are trivial on Cl_F^2, we can view ψ as a group homomorphism of H/Cl_F^2. From the third isomorphism theorem, we have $H/\mathrm{Cl}_F^2 \trianglelefteq \mathrm{Cl}_F/\mathrm{Cl}_F^2$, thus H/Cl_F^2 is also a 2-group. Therefore, we can view H/Cl_F^2 as a $\mathbb{F}_2$ vector space with a basis $\{[\mathfrak{a}_i]\mathrm{Cl}_F^2 \mid 1 \leq i \leq l\}$ and ψ as a linear transformation. Now, we can extend the basis above to a basis $\{[\mathfrak{a}_i]\,\mathrm{Cl}_F^2 \mid 1 \leq i \leq k\}$ of $\mathrm{Cl}_F / \mathrm{Cl}_F^2$ and extend ψ to $\psi' colon\, \mathrm{Cl}_F / \mathrm{Cl}_F^2 \to \{1, -1\}$, by $\psi'(\mathfrak{a}_i) = \epsilon_i$, where $\epsilon_i = \psi([\mathfrak{a}_i]\,\mathrm{Cl}_F^2)$ for $i \leq l$ and $\epsilon_i(\mathfrak{a}_i) = 1$ for $i > l$.

Since $\{[\mathfrak{a}_i]\,\mathrm{Cl}_F^2 \mid 1 \leq i \leq k\}$ is a basis for $\mathrm{Cl}_F / \mathrm{Cl}_F^2$ and ψ is a linear transformation defined on a basis, it extends to $\mathrm{Cl}_F / \mathrm{Cl}_F^2$ by linearity. The composition of ψ with the projection map from Cl_F to $\mathrm{Cl}_F / \mathrm{Cl}_F^2$ determines a quadratic character ψ' of the class group. Now by construction, we have $\psi'|_H = \psi$. □

We look at the relation between the eigenvalues of two Hecke eigensystems with the same character that matches the same homological eigenform.

Theorem 5.4 ([11])**.** *Let (λ, χ) and (λ', χ) be two Hecke eigensystems that match a homological eigenform f. Then either $\lambda(T) = \lambda'(T)$ or $\lambda(T) = -\lambda'(T)$ for any Hecke operator T.*

Proof. Suppose T is a Hecke operator of class $[\mathfrak{a}]$. Let $\mathfrak{b} \in [\mathfrak{a}^{-1}]$ be a prime ideal coprime to the level. Then the Hecke operator $T^2 T_{\mathfrak{b},\mathfrak{b}}$ is a principal Hecke operator. Since both (λ, χ) and (λ', χ) match f, we have

$$\lambda(T^2 T_{\mathfrak{b},\mathfrak{b}}) = a_f(T^2 T_{\mathfrak{b},\mathfrak{b}}) = \lambda'(T^2 T_{\mathfrak{b},\mathfrak{b}}).$$

Since the two Hecke eigensystems have the same character, we have $\lambda(T^2) = \lambda'(T^2)$. From the multiplicative property of λ and λ', we get $\lambda(T)^2 = \lambda'(T)^2$. Therefore either $\lambda(T) = \lambda'(T)$ or $\lambda(T) = -\lambda'(T)$, as desired. □

Note that the above theorem gives a result only at the level of a single Hecke operator. To compare two eigensystems, we need to consider their connection in the full Hecke algebra.

For a large collection of Hecke operators, namely Hecke operators with classes in Cl_F^2, we can prove more.

Theorem 5.5 ([28])**.** *Let (λ, χ) and (λ', χ) be two Hecke eigensystems that match a homological eigenform f. Then $\lambda(T) = \lambda'(T)$ for any operator T in the Hecke algebra $\mathbb{T}$ of class $[\mathfrak{a}] \in \mathrm{Cl}_F^2$.*

Proof. Let $[\mathfrak{a}] \in \mathrm{Cl}_F^2$. If the ideal $\mathfrak{b} \in [\mathfrak{a}^{-1}]$ then $\mathfrak{a}\mathfrak{b}^2$ is principal. Therefore, for any $T \in \mathbb{T}$ of class $[\mathfrak{a}] \in \mathrm{Cl}_F^2$, the Hecke operator $TT_{\mathfrak{b},\mathfrak{b}}$ is a principal. Since both Hecke eigensystems match with the same eigenform f, we have $\lambda(TT_{\mathfrak{b},\mathfrak{b}}) = a_f(TT_{\mathfrak{b},\mathfrak{b}}) = \lambda'(TT_{\mathfrak{b},\mathfrak{b}})$. Since both systems have the same character, $\lambda(T) = \lambda'(T)$. $\square$

Now with these ingredients, we can show the following:

Theorem 5.6 ([11])**.** *Let f be a homological eigenform that matches two different eigensystems (λ, χ) and (λ', χ). Then there exists a quadratic character ψ such that $\lambda = \psi\lambda'$.*

Proof. Let $H \subseteq \mathrm{Cl}_F$ be the subset where λ is not identically zero on Hecke operators with classes in H. Since $\lambda(T) = \pm\lambda'(T)$, Theorem 5.4 implies that λ' is also not identically zero on Hecke operators with classes in H. That is, for every class $c \in H$, there exists a Hecke operator $T \in \mathbb{T}$ of class c where $\lambda(T) \neq 0$ and $\lambda'(T) \neq 0$. Note that H is a subgroup because λ is multiplicative, that is $\lambda(TT') = \lambda(T)\lambda(T')$. Also, we have $\mathrm{Cl}_F^2 \subseteq H$ because $T_{\mathfrak{a},\mathfrak{a}}$ is not identically zero for any class $[\mathfrak{a}]$ in Cl_F.

Let ψ be a map on H given by $\psi([\mathfrak{a}]) = \dfrac{\lambda(T)}{\lambda'(T)}$ where $T \in \mathbb{T}$ of class $[\mathfrak{a}]$ such that $\lambda'(T) \neq 0$. Note that since $[\mathfrak{a}] \in H$, we can always find such a Hecke operator. From Corollary 5.4, we know that $\psi([\mathfrak{a}]) \in \{1, -1\}$.

Now we show that ψ is well-defined. That is, the value of ψ does not depend on the choice of the Hecke operator $T \in \mathbb{T}$ of class $[\mathfrak{a}]$. Suppose T and T' in $\mathbb{T}$ of class $[\mathfrak{a}]$ such that $\lambda'(T) \neq 0$ and $\lambda'(T') \neq 0$. Then TT' is a Hecke operator of class $[\mathfrak{a}^2]$. Theorem 5.5 implies that $\lambda(TT') = \lambda'(TT')$. This means $\frac{\lambda(T)}{\lambda'(T)} \cdot \frac{\lambda(T')}{\lambda'(T')} = 1$. Since $\frac{\lambda(T)}{\lambda'(T)}$ and $\frac{\lambda(T')}{\lambda'(T')}$ are ± 1, we have $\frac{\lambda(T)}{\lambda'(T)} = \frac{\lambda(T')}{\lambda'(T')}$.

Now we show that ψ is a group homomorphism on H. That is for any $[\mathfrak{a}_1], [\mathfrak{a}_2] \in H$, we need to show that $\psi([\mathfrak{a}_1\mathfrak{a}_2]) = \psi([\mathfrak{a}_1])\psi([\mathfrak{a}_2])$. Suppose $T_1, T_2 \in \mathbb{T}$ are Hecke operators of class $[\mathfrak{a}_1]$ and $[\mathfrak{a}_2]$, respectively. Then the operator $T = T_1T_2$ has class $[\mathfrak{a}_1\mathfrak{a}_2]$ and $\lambda(T) \neq 0$ because

λ is multiplicative. By definition

$$\psi([\mathfrak{a}_1\mathfrak{a}_2]) = \frac{\lambda(T)}{\lambda'(T)} = \frac{\lambda(T_1T_2)}{\lambda'(T_1T_2)} = \frac{\lambda(T_1)\lambda(T_2)}{\lambda'(T_1)\lambda'(T_2)} = \psi([\mathfrak{a}_1])\psi([\mathfrak{a}_2]).$$

This means ψ is a quadratic character of H. Now by Theorem 5.3, we can find a quadratic character ψ' of the class group that restricts to ψ on H. Thus we have $\lambda = \psi\lambda'$.

□

Remark 5.4. Note that the quadratic character ψ from the above theorem might not be unique. However, this only happens if (λ, χ) has an inner twist. For example, suppose the eigensystems λ and λ' have eigenvalue zero for all ideals in some class $[\mathfrak{q}]$. This means that (λ, χ) has an inner twist by a quadratic character non-trivial on $\mathfrak{q}$ and trivial everywhere else.

Up to now, we have only compared systems with the same character. The following corollary gives the more general result for systems with different characters that match the same homological eigenform:

Corollary 5.7. *Let f be a homological eigenform. Both Hecke eigensystems (λ, χ) and (λ', χ') match f if and only if (λ', χ') is a twist of (λ, χ).*

Proof. Suppose (λ, χ) and (λ', χ') are two Hecke eigensystems that match the homological eigenform f. Then for any $\mathfrak{a} \in \mathrm{Cl}_F[2]$, we have that $\chi(\mathfrak{a}) = \chi'(\mathfrak{a})$. This means the character $\chi(\chi')^{-1}$ is a character that is trivial on $\mathrm{Cl}_F[2]$. Therefore $\chi'(\chi)^{-1}$ is a square. That is $\chi' = \chi\psi^2$ for some character ψ. Now the twist of (λ, χ) by ψ is $(\lambda\psi, \chi')$.

Using Theorem 5.6, there exists a quadratic character ψ' such that the twist of $(\lambda\psi, \chi')$ by ψ' is (λ', χ'). We can twist (λ, χ) by $\psi\psi'$ to obtain (λ', χ'). Thus, they are in the same twist orbit. □

Corollary 5.8. *Let f be a homological eigenform. There are at most $h = |\,\mathrm{Cl}_F\,|$ Hecke eigensystems (λ, χ) that match f.*

Proof. We have shown in Corollary 5.7 that any two eigensystems matching the same homological eigenform must be in the same twist orbit. Since the size of the twist orbit is at most h, the result follows. $\square$

From Corollary 5.7, we know that all Hecke eigensystems matching a homological eigenform are in the same twist orbit. This means computing a single Hecke eigensystem will be sufficient to compute all eigensystems that match it.

5.3 Computing the Character

In this section, we explain how to identify the characters of Hecke eigensystems matching a given homological eigenform f using principal Hecke operators.

Note that twisting a system (λ, χ) by a character only changes χ by a square. This means the number of distinct characters in the same twist orbit of a Hecke eigensystem is $|\operatorname{Cl}_F^2|$. In particular, if the order of the class group is even then $\operatorname{Cl}_F^2 \neq \operatorname{Cl}_F$. Thus, a homological modular form f can only match Hecke eigensystems with certain characters. Since this collection of characters only depends on the homological eigenform, we call it the *character orbit of a homological eigenform* or simply the *character orbit*.

Theorem 5.9. *Let f be a homological eigenform that matches the Hecke eigensystem (λ, χ). Then we can compute the characters χ up to twist using principal Hecke operators on f.*

Proof. Using Theorem 4.21, we can compute the Hecke operator $T_{\mathfrak{a},\mathfrak{a}}$ on f for each ideal $\mathfrak{a}$ such that $\mathfrak{a}^2$ is principal. Thus $\chi(\mathfrak{a})$ can be computed for each $\mathfrak{a} \in \operatorname{Cl}_F[2]$. Also, we know that $\chi(\mathfrak{a}) \in \{1, -1\}$ because $\chi(\mathfrak{a}^2) = \chi(\mathfrak{a})^2 = 1$ for any $\mathfrak{a} \in \operatorname{Cl}_F[2]$. Let χ' be a character of the class group such that $\chi(\mathfrak{a}) = \chi'(\mathfrak{a})$ for each $\mathfrak{a} \in \operatorname{Cl}_F[2]$. Then $\chi\chi'^{-1}(\mathfrak{a})$ must be trivial on $\operatorname{Cl}_F[2]$. Since the characters that are trivial on order two elements are squares, we have $\chi = \chi'\psi^2$ for some ψ. Thus χ' determines the character χ up to a twist. $\square$

Remark 5.5. In practice, for a list of ideal representative $\mathfrak{a}_i$ from classes in $\mathrm{Cl}_F[2]$, we compute $T_{\mathfrak{a}_i,\mathfrak{a}_i}$ and pick a character of the class group that agrees with the eigenvalues of $T_{\mathfrak{a}_i,\mathfrak{a}_i}$. As stated before, since we only need to identify one eigensystem, we do not miss anything by fixing a character from the character orbit.

5.4 Hecke Eigenvalues

Once we fix a character from the character orbit, we can obtain certain information about the eigenvalues of any Hecke eigensystem matching f. Recall that we can view a Hecke eigensystem as a function on the Hecke algebra $\lambda : \mathbb{T} \to \mathbb{C}$ or as a pair (λ, χ). We write $\lambda(\mathfrak{a})$ for $\lambda(T_\mathfrak{a})$ and $\chi(\mathfrak{a})$ for $\lambda(T_{\mathfrak{a},\mathfrak{a}})$.

Corollary 5.10. *Let f be a homological eigenform. If (λ, χ) is any Hecke eigensystem that matches f with a known character χ, then $\lambda(\mathfrak{p})$ can be computed using principal Hecke operators for any prime $\mathfrak{p}$ where $[\mathfrak{p}] \in \mathrm{Cl}_F^2$. In particular, at a fixed prime $\mathfrak{p}$ with $[\mathfrak{p}] \in \mathrm{Cl}_F^2$, all eigensystems with the same character that match f have the same eigenvalue.*

Proof. Consider the Hecke operators $T_\mathfrak{p}$. Since $[\mathfrak{p}] \in \mathrm{Cl}_F^2$, there exists an ideal $\mathfrak{a}$ such that $T_\mathfrak{p} T_{\mathfrak{a},\mathfrak{a}}$ is principal. Using Hecke matrices from Theorem 4.22, we can compute

$$a_f(T_\mathfrak{p} T_{\mathfrak{a},\mathfrak{a}}) = \lambda(T_\mathfrak{p} T_{\mathfrak{a},\mathfrak{a}}) = \lambda(\mathfrak{p})\chi(\mathfrak{a}).$$

Since χ is known, we know $\lambda(\mathfrak{p})$ is computable. The second statement follow from Theorem 5.5 for $T = T_\mathfrak{p}$. $\square$

We define the quantity

$$t(\mathfrak{p}) = a_f(T_{\mathfrak{p}^2} T_{\mathfrak{a},\mathfrak{a}}) + N(\mathfrak{p}) \tag{5.1}$$

for any $\mathfrak{a} \in [\mathfrak{p}^{-1}]$. Note that $t(\mathfrak{p})$ only depends on the homological eigenform f and the principal Hecke operator $T_{\mathfrak{p}^2} T_{\mathfrak{a},\mathfrak{a}}$.

Now we study the Hecke eigenvalues for Hecke operators of the form $T_{\mathfrak{p}}$, where $\mathfrak{p}$ is prime.

Theorem 5.11. *Let f be a homological eigenform. If (λ, χ) is any Hecke eigensystem matching f, then for any prime ideal $\mathfrak{p}$ the value of $\lambda(\mathfrak{p})$ is a root of the polynomial $x^2 - t(\mathfrak{p})\chi(\mathfrak{p})$.*

Proof. Since $\mathfrak{a} \in [\mathfrak{p}^{-1}]$, the ideal $\mathfrak{p}^2\mathfrak{a}^2$ is principal. By Definition 5.2, we have $\lambda(T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}}) = a_f(T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}})$, where $T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}}f = a_f(T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}})f$. From the properties of Hecke eigenvalues, we get

$$\lambda(\mathfrak{p}^2) = \frac{\lambda(T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}})}{\chi(\mathfrak{a})} = \chi^{-1}(\mathfrak{a})a_f(T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}}) = \chi(\mathfrak{p})a_f(T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}}),$$

and

$$\lambda(\mathfrak{p})^2 = \lambda(\mathfrak{p}^2) + \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p}) = \chi(\mathfrak{p})a_f(T_{\mathfrak{p}^2}T_{\mathfrak{a},\mathfrak{a}}) + \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p}) = \chi(\mathfrak{p})t(\mathfrak{p}).$$

Therefore $\lambda(\mathfrak{p})$ is a root of the polynomial $x^2 - t(\mathfrak{p})\chi(\mathfrak{p})$. □

Corollary 5.12. *Let f be a homological eigenform. Any Hecke eigensystem (λ, χ) that matches f satisfies $\lambda(\mathfrak{p}) \in \{\sqrt{\chi(\mathfrak{p})t(\mathfrak{p})}, -\sqrt{\chi(\mathfrak{p})t(\mathfrak{p})}\}$.*

Proof. This follows easily from Theorem 5.11. □

We now make several observations. First, if $t(\mathfrak{p}) = 0$, then $\lambda(\mathfrak{p}) = 0$ in any Hecke eigensystem. Otherwise, if $[\mathfrak{p}] \notin \mathrm{Cl}_F^2$ both values will occur in the twist orbit. That is, for a fixed prime ideal $\mathfrak{p} \notin \mathrm{Cl}_F^2$, there exists systems (λ, χ) and (λ', χ) that matches f, where $\lambda'(\mathfrak{p}) = \sqrt{\chi(\mathfrak{p})t(\mathfrak{p})}$ and $\lambda(\mathfrak{p}) = -\sqrt{\chi(\mathfrak{p})t(\mathfrak{p})}$.

Since our goal is to compute a single eigensystem in its character orbit, we must make certain choices for primes $[\mathfrak{p}] \notin \mathrm{Cl}_F^2$. That is, if for a certain prime $\mathfrak{p}$ we have $t(\mathfrak{p}) \neq 0$, then we must decide whether to compute the system with the eigenvalue $\sqrt{\chi(\mathfrak{p})t(\mathfrak{p})}$ or $-\sqrt{\chi(\mathfrak{p})t(\mathfrak{p})}$. Also, there is a limit on the number of choices because the size of the twist orbit is bounded by the class number. Further, after making a choice for an ideal $\mathfrak{p}$ where $t(\mathfrak{p}) \neq 0$, then any other ideal $\mathfrak{q}$ with $[\mathfrak{q}] \in [\mathfrak{p}]\,\mathrm{Cl}_F^2$ is known. For example, suppose $\mathfrak{p} \notin \mathrm{Cl}_F^2$ is a prime ideal and

(λ, χ) is the Hecke eigensystem with $\lambda(\mathfrak{p}) = \sqrt{\chi(\mathfrak{p})t(\mathfrak{p})}$. Let $\mathfrak{q}$ be a prime ideal such that $\mathfrak{pq}$ is principal. Then we can compute the eigenvalue $\lambda(\mathfrak{pq})$ exactly using the principal Hecke operator $T_{\mathfrak{pq}}$. Then we have $\lambda(\mathfrak{q}) = \frac{\lambda(\mathfrak{pq})}{\sqrt{t(\mathfrak{p})}}$.

It is possible for $t(\mathfrak{p}) = 0$ for all ideals in certain classes of $\mathrm{Cl}_F \setminus \mathrm{Cl}_F^2$. This only happens if the Hecke eigensystem has inner twists. We discuss how to compute such systems in Section 5.6.

5.5 Eigensystems without Inner Twists

Recall from Definition 4.16, we say a Hecke eigensystem (λ, χ) has inner twist if for some character ψ of the class group $(\lambda, \chi) = (\lambda\psi, \lambda, \chi\psi^2)$. This section discusses techniques to compute Hecke eigensystems without inner twists.

Definition 5.13. We call a set of ideals $\{\mathfrak{a}_i \mid 1 \leq i \leq k\}$ *admissible* for the Hecke eigensystem (λ, χ) if the set $\{[\mathfrak{a}_i] \mathrm{Cl}_F^2 \mid 1 \leq i \leq k\}$ is a $\mathbb{Z}$-basis for $\mathrm{Cl}_F / \mathrm{Cl}_F^2$ and $\lambda(\mathfrak{a}_i) \neq 0$ for all i.

If a set of ideals is admissible for a Hecke eigensystem then it is also admissible for any eigensystem in its twist orbit. Therefore, an admissible set only depends on the homological eigenform.

Theorem 5.14. *Let (λ, χ) be a Hecke eigensystem. The eigensystem has an admissible set if and only if (λ, χ) does not have inner twists.*

Proof. Suppose (λ, χ) is a system without inner twists. Let $\{\mathfrak{a}_i, 1 \leq i \leq k\}$ be any set such that $\{[\mathfrak{a}_i] \mid 1 \leq i \leq k\}$ is a $\mathbb{Z}$-basis for $\mathrm{Cl}_F / \mathrm{Cl}_F^2$. Suppose $\lambda(\mathfrak{a}_j) = 0$ for some j. Applying Theorem 5.3 for $H = \mathrm{Cl}_F$, we can find a quadratic character ψ defined by

$$\psi(\mathfrak{a}_i) := \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{for } i \neq j. \end{cases}$$

Since (λ, χ) does not have an inner twist by ψ, there exist a ideal $\mathfrak{b}$ such that $\lambda\psi \neq \lambda$. This could only happen if $\lambda(\mathfrak{b}) \neq 0$ and $\psi(\mathfrak{b}) = -1$. By the definition of a $\mathbb{Z}$-basis, we have $[\mathfrak{b}] \operatorname{Cl}_F^2 = [\mathfrak{a}_i^{l_i}] \operatorname{Cl}_F^2$ for some exponents $l_i \in \{0, 1\}$. Since ψ is trivial on Cl_F^2, we have that $\psi(\mathfrak{b}) = \prod_{i=1}^{k} \psi(\mathfrak{a}_j)^{l_i} = -1$. Therefore, for some j, we have $l_j = 1$ and $\psi(\mathfrak{a}_i) = -1$. This means, we can replace $\mathfrak{a}_j$ by $\mathfrak{b}$ to obtain a new set where $\lambda(\mathfrak{a}_j) \neq 0$. We can repeat this process iteratively to obtain an admissible set. $\square$

Remark 5.6. In practice, we have the freedom to pick a set $\{\mathfrak{a}_i\}$ that is a $\mathbb{Z}$-basis of $\operatorname{Cl}_F / \operatorname{Cl}_F^2$. However, for computational running time reasons, there might be a better basis where the norms of the ideals $\mathfrak{a}_i$ are smaller. The Example 5.15 demonstrates this. See Remark 5.7 for additional details.

Example 5.15. Suppose $F = \mathbb{Q}(\sqrt{-5190})$ where $\omega = \sqrt{-5190}$. Then $\operatorname{Cl}_F = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle \simeq C_2 \times C_4 \times C_8$. Thus $\operatorname{Cl}_F^2 = \langle g_2^2 \rangle \times \langle g_3^2 \rangle \simeq C_2 \times C_4$ and $\operatorname{Cl}_F / \operatorname{Cl}_F^2 = \langle g_1 \operatorname{Cl}_F^2 \rangle \times \langle g_2 \operatorname{Cl}_F^2 \rangle \times \langle g_3 \operatorname{Cl}_F^2 \rangle \simeq C_2^3$. Thus, we can pick $\mathfrak{a}_1 = \langle 2, \omega + 1 \rangle \in g_1$, $\mathfrak{a}_2 = \langle 499, \omega + 171 \rangle \in g_2$ and $\mathfrak{a}_3 = \langle 7, \omega + 5 \rangle \in g_3$ to be a $\mathbb{Z}$-basis. Alternatively, we can replace $\mathfrak{a}_2$ by $\mathfrak{a}_2' = \langle 43, \omega + 20 \rangle \in g_2 g_3$ to get another $\mathbb{Z}$-basis. The second collection might be preferable because the norm of $\mathfrak{a}_2'$ is significantly smaller than the norm of $\mathfrak{a}_2$.

Theorem 5.16. *Let f be a homological eigenform without inner twists. For any admissible set $\{\mathfrak{a}_i \mid 1 \leq i \leq k\}$, there is a Hecke eigensystem (λ, χ) that matches f such that for all i, we have $\lambda(\mathfrak{a}_i) = \sqrt{\chi(\mathfrak{a}_i) t(\mathfrak{a}_i)}$.*

Proof. Let $\{\mathfrak{a}_i\}_{i=1}^{k}$ be an admissible set for f. Let (λ, χ) be the Hecke eigensystem that matches f. Suppose $(\epsilon_i) \in \{1, -1\}^k$, where $\lambda(\mathfrak{a}_i) = \epsilon_i \sqrt{\chi(\mathfrak{a}_i) t(\mathfrak{a}_i)}$. Let ψ be the unique quadratic character with $\psi(\mathfrak{a}_i) = \epsilon_i$ obtained from Theorem 5.3 for $H = \operatorname{Cl}_F$. Then the statement holds for the twist of the system (λ, χ) by ψ. $\square$

Now we show that the knowledge of the eigenvalues on an admissible set is sufficient for computing the Hecke eigenvalues of other ideals using principal Hecke operators.

Theorem 5.17. *Let f be a homological eigenform without inner twists, and let $\{\mathfrak{a}_i\}$ be an admissible set. Suppose (λ, χ) is the Hecke eigensystem where $\lambda(\mathfrak{a}_i) = \sqrt{\chi(\mathfrak{a}_i)t(\mathfrak{a}_i)}$. Then for any ideal $\mathfrak{p}$, we can compute the Hecke eigenvalue $\lambda(\mathfrak{p})$ from the action of principal Hecke operators.*

Proof. Let (λ, χ) be a Hecke eigensystem that matches f with $\lambda(\mathfrak{a}_i) = \sqrt{\chi(\mathfrak{a}_i)t(\mathfrak{a}_i)}$. Note that since $\{\mathfrak{a}_i\}_{i=1}^k$ is an admissible set such a Hecke eigensystem exists by Theorem 5.16. Now for any $\mathfrak{p}$, there exists exponents $l_i \in \{0, 1\}$ such that $[\mathfrak{p}] \operatorname{Cl}_F^2 = [\prod_{i=1}^k \mathfrak{a}_i^{l_i}] \operatorname{Cl}_F^2$. Let $\mathcal{I} = \{i \mid l_i \neq 0\}$, and set $\mathfrak{b} = \mathfrak{p} \prod_{i \in \mathcal{I}} \mathfrak{a}_i$. Note that $[\mathfrak{b}] \in \operatorname{Cl}_F^2$, therefore, we can compute the eigenvalue $\lambda(\mathfrak{b})$ using the action of the principal Hecke operator $T_{\mathfrak{b}} T_{\mathfrak{a},\mathfrak{a}}$ of f for some $\mathfrak{a}$ such that $\mathfrak{a}^2\mathfrak{b}$ is principal. Now using properties of Hecke eigenvalues, we have

$$\lambda(\mathfrak{b}) = \lambda(\mathfrak{p})\lambda(\prod_{i \in \mathcal{I}} \mathfrak{a}_i) = \lambda(\mathfrak{p}) \prod_{i \in \mathcal{I}} \lambda(\mathfrak{a}_i).$$

Therefore,

$$\lambda(\mathfrak{p}) = \frac{\lambda(\mathfrak{b})}{\prod_{i \in \mathcal{I}} \sqrt{t(\mathfrak{a}_i)\chi(\mathfrak{a}_i)}}.$$

□

The main takeaway of these results is as follows. If f is a homological modular form without inner twist with a fixed character χ from the character orbit, then the pairs $(\mathfrak{a}_i, \lambda(\mathfrak{a}_i))$ for an admissible set are sufficient to compute all eigenvalues of the system (λ, χ) that matches f. Thus, the datum $\{f, \chi, \{(\mathfrak{a}_i, \lambda(\mathfrak{a}_i))\}\}$ identifies a unique Hecke eigensystem that matches f.

Remark 5.7. The speed of computing a Hecke operator depends on the number of Hecke matrices used. To compute $\lambda(\mathfrak{p})$ using the theorem above, we need about

$$N(\mathfrak{p}) \prod_{i=1}^k N(\mathfrak{a}_i) + \sum_{i=1}^k N(\mathfrak{a}_i)^2 + 2k$$

matrices, where $k = |\operatorname{Cl}_F / \operatorname{Cl}_F^2|$. Alternatively, we can first compute eigenvalues for a complete list of small norm representatives for Cl_F and use the representative of the inverse class of $\mathfrak{p}$ to compute $\lambda(\mathfrak{p})$. This requires about

$$N(\mathfrak{p})N(\mathfrak{a}_j) + \sum_{i=1}^{h} N(\mathfrak{a}_i)^2 + 2k$$

matrices. If the norm of the ideal $\mathfrak{p}$ is large, this method might be faster.

5.6 Eigensystems with Inner Twists

In this section, we discuss some aspects of computing Hecke eigensystems with inner twists. The existence of an inner twists makes the computations over imaginary quadratic fields with even-order class groups interesting and challenging. One aspect that makes computations difficult is the non-existence of admissible sets. Therefore, we require more information to identify a unique Hecke eigensystem that matches a homological eigenform.

First, we discuss a strategy that we use to identify systems with inner twists. This strategy is the main idea behind Algorithm 5.22. The theory behind the statement was explained through personal communications with J. Cremona [13].

Suppose (λ, χ) is a Hecke eigensystem that matches the homological eigenform f of level $\mathfrak{n}$. Suppose f has an inner twist by a quadratic character ψ. Then the contribution of f to the oldspace at level $\mathfrak{m}$ is equal to $\#\{\mathfrak{d} \mid \mathfrak{d}|(\mathfrak{m}/\mathfrak{n}), \psi(\mathfrak{d}) = 1\}$. This means that for a set of prime ideals $\{\mathfrak{a}_i \mid 1 \leq i \leq k\}$ where $\{[\mathfrak{a}_i] \mid 1 \leq i \leq k\}$ is a $\mathbb{Z}$-basis for $\operatorname{Cl}_F / \operatorname{Cl}_F^2$, we can compare the oldform contribution of f to level $\mathfrak{n}\mathfrak{a}_i$ for each $\mathfrak{a}_i$. If the contribution of f to the oldspace is one dimensional, then $\psi(\mathfrak{a}_i) = -1$ and if it is 2-dimensional then $\psi(\mathfrak{a}_i) = 1$. Therefore, by Theorem 5.3, we can identify ψ uniquely.

For example, suppose the set of prime ideals $\{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3\}$ is a basis for $\operatorname{Cl}_F / \operatorname{Cl}_F^2$. The table 5.1 describes three quadratic characters of Cl_F.

Table 5.1. Character values

	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_3$
ψ_1	-1	1	1
ψ_2	1	-1	1
ψ_3	-1	-1	1

If the system has inner twist by ψ_1, the old space contribution of f to $\mathfrak{n}\mathfrak{a}_1$, $\mathfrak{n}\mathfrak{a}_2$, $\mathfrak{n}\mathfrak{a}_3$ would be 1, 2 and 2, respectively. If the system has inner twist by ψ_2, the old space contribution of f to $\mathfrak{n}\mathfrak{a}_1$, $\mathfrak{n}\mathfrak{a}_2$, $\mathfrak{n}\mathfrak{a}_3$ would be 2, 1 and 2, respectively. Finally, if the system has inner twist by ψ_3, the old space contribution of f to $\mathfrak{n}\mathfrak{a}_1$, $\mathfrak{n}\mathfrak{a}_2$, $\mathfrak{n}\mathfrak{a}_3$ would be 1, 1 and 2, respectively. This means we can identify the inner twist character uniquely by looking at the oldspace contribution to $\mathfrak{n}\mathfrak{a}_1$, $\mathfrak{n}\mathfrak{a}_2$, $\mathfrak{n}\mathfrak{a}_3$.

Now we discuss how to identify a Hecke eigensystem that matches a homological eigenform f if the inner twist character ψ is known. By Theorem 5.7, if we can find a Hecke eigensystem that matches f, then its orbit gives all the eigensystems matching f. However, unlike the non-twist case, the Hecke eigensystem orbit has strictly fewer than $h = \#\operatorname{Cl}_F$ elements.

Theorem 5.18. *Let f be a homological eigenform and let (λ, χ) be a Hecke eigensystem that matches f with an inner twist by a quadratic character ψ. Let H be the subgroup of classes of Hecke operators in Cl_F where λ is not identically zero. Then there exists a set of ideals $\{\mathfrak{b}_i\}$ such that $H/\operatorname{Cl}_F^2 = \langle [\mathfrak{b}_i] \operatorname{Cl}_F^2 \rangle$ and $\lambda(\mathfrak{b}_i) \neq 0$.*

Proof. Since the Hecke eigensystem has a twist by ψ, this means λ cannot be identically zero on the classes in H. Therefore, we can find some ideal $\mathfrak{b}_i$ with $\lambda(\mathfrak{b}_i) \neq 0$. $\square$

Now we show the existence of a Hecke eigensystem with known eigenvalues on a certain set of ideals satisfying the condition in the Theorem 5.18.

Theorem 5.19. *Let f be a homological eigenform with an inner twist by ψ. Let H be the subgroup of classes $[\mathfrak{p}]$ of ideals with $\psi(\mathfrak{p}) = 1$. Let $\{\mathfrak{b}_i\}$ such that $H/\operatorname{Cl}_F^2 = \langle \operatorname{Cl}_F^2[\mathfrak{b}_i] \rangle$ and*

$t(\mathfrak{b}_i) \neq 0$. *Then there is a Hecke eigensystem* (λ, χ) *that matches* f *such that for all* i*, we have* $\lambda(\mathfrak{b}_i) = \sqrt{\chi(\mathfrak{b}_i)t(\mathfrak{b}_i)}$.

Proof. The proof is similar to that of Theorem 5.16. We define ϵ_i to be $\lambda(\mathfrak{b}_i) = \epsilon_i\sqrt{\chi(\mathfrak{b}_i)t(\mathfrak{b}_i)}$ for a basis $\{\mathfrak{b}_i\}$ of H/Cl_F^2 that satisfies $t(\mathfrak{b}_i) \neq 0$ for all i. From Theorem 5.3, we can find a quadratic character ψ such that the twist of (λ, χ) by ψ desired the property. □

Remark 5.8. In contrast to the non-twist case, the quadratic character ψ from Theorem 5.6 is not unique.

Now we show the uniqueness of the eigensystem from Theorem 5.19. In particular, we show that once eigenvalues for a set of ideals $\{\mathfrak{b}_i\}$ are computed, then the eigenvalues of any other ideal are known.

Theorem 5.20. *Let* f *be a homological eigenform with inner twists by* ψ*. Let* $\{\mathfrak{b}_i\}$ *such that* $H/\operatorname{Cl}_F^2 = \langle \operatorname{Cl}_F^2[\mathfrak{b}_i]\rangle$ *and* $t(\mathfrak{b}_i) \neq 0$*. Suppose* (λ, χ) *is the Hecke eigensystem where* $\lambda(\mathfrak{b}_i) = \sqrt{\chi(\mathfrak{b}_i)t(\mathfrak{b}_i)}$*. Then for any ideal* $\mathfrak{p}$*, we can compute the Hecke eigenvalue* $\lambda(\mathfrak{p})$ *from the action of principal Hecke operators.*

Proof. If $\psi(\mathfrak{p}) = -1$, then $\lambda(\mathfrak{p}) = 0$. On the other hand, if $\psi(\mathfrak{p}) = 1$ then $[\mathfrak{p}] \in H$. Therefore, for some $l_i \in \{0, 1\}$ we have $[\mathfrak{p}]\operatorname{Cl}_F^2 = [\mathfrak{b}_i]^{l_i}\operatorname{Cl}_F^2$. Now $\mathfrak{p}\mathfrak{b}_i^{l_i} \in \operatorname{Cl}_F^2$, so we can compute the eigenvalue using principal Hecke operators. Then

$$\lambda(\mathfrak{p}) = \frac{\lambda(\mathfrak{p}\mathfrak{b}_i^{l_i})}{\chi(\mathfrak{b}_i)\sqrt{\chi(\mathfrak{b}_i)t(\mathfrak{b}_i)}}.$$

□

The following example demonstrates this strategy:

Example 5.21. Let f be a homological eigenform over a field F, where

$$\operatorname{Cl}_F = \langle g_1, g_2\rangle \simeq C_4 \times C_4.$$

Then $\mathrm{Cl}_F / \mathrm{Cl}_F^2 = \langle g_1 \mathrm{Cl}_F^2 \rangle \times \langle g_2 \mathrm{Cl}_F^2 \rangle$. Suppose f has inner twist by the quadratic character $\psi : g_1 \mapsto -1, g_2 \mapsto -1$. Then

$$H = \{1, g_1^2, g_2^2, g_1g_2, g_1^2g_2^2, g_1g_2^3, g_2g_1^3\}.$$

Then $H/\mathrm{Cl}_F^2 = \{\mathrm{Cl}_F^2, \mathrm{Cl}_F^2(g_1g_2)\}$. Let $\mathfrak{b}$ be an ideal in $[g_1g_2]$ such that $t(\mathfrak{b}) \neq 0$. This is possible since any Hecke eigensystem that matches f is not identically zero in $[g_1g_2]$. Then there is a eigensystem (λ, χ) that matches f with $\lambda(\mathfrak{b}) = \sqrt{\chi(\mathfrak{b})t(\mathfrak{b})}$.

If $\mathfrak{p}$ is a prime ideal with $\psi(\mathfrak{p}) = -1$, then we know $\lambda(\mathfrak{p}) = 0$. On the other hand if $\psi(\mathfrak{p}) = 1$, then $[\mathfrak{p}] \in H$. Thus, either $[\mathfrak{p}] \in \mathrm{Cl}_F^2$ or $[\mathfrak{p}] \in \mathrm{Cl}_F^2[\mathfrak{b}]$. If $[\mathfrak{p}] \in \mathrm{Cl}_F^2$, we can compute $\lambda(\mathfrak{p})$ exactly. On the other hand if $[\mathfrak{p}] \in \mathrm{Cl}_F^2[\mathfrak{b}]$ then $[\mathfrak{p}\mathfrak{b}] \in \mathrm{Cl}_F^2$ and $\lambda(\mathfrak{p}) = \frac{\lambda(T_{\mathfrak{p}\mathfrak{b}})}{\sqrt{\chi(\mathfrak{b})t(\mathfrak{b})}}$.

Remark 5.9. In the non-twist case, there are 2^k systems matching a homological modular form f with the same character, where $k = |\mathrm{Cl}_F / \mathrm{Cl}_F^2|$. Therefore, there are k ideals to pick signs arbitrarily. If the system has inner twists, we only have 2^{k-1} systems matching a homological modular form f with the same character. Therefore, we only have $k-1$ ideals to pick signs arbitrarily.

If the imaginary quadratic field has a cyclic class group of even order, the case is much simpler because there is only one quadratic twist. See 6.10 for an example of such a system.

5.7 Algorithms for Computing Eigensystems

In this section, we give algorithms for computing the Hecke eigensystems attached to Bianchi modular forms with general class groups.

First, we give an algorithm to check if a homological modular form f has inner twists. If the class number is odd, there are no quadratic twists. Thus, we only check inner twists for fields with even class numbers.

Algorithm 5.22 (Checking inner twists)**.**

Input: *Homological eigenform f of level $\mathfrak{n}$ over the imaginary quadratic field F with even class number*

Output: *"True" if f has inner twist or "False" if f does not have inner twists*

1. *Select a list R of small prime ideal representatives of each class in the class group Cl_F.*

2. *Identify a subset $\{\mathfrak{a}_i\}$ of R whose classes form a $\mathbb{Z}$-basis for $\mathrm{Cl}_F / \mathrm{Cl}_F^2$.*

3. *For each $\mathfrak{a}_i$, compute the dimension contribution of f to the oldspace homological modular forms at level $\mathfrak{a}_i\mathfrak{n}$. Let $\mathcal{I}$ be the set of indices where the oldspace dimension of f in $\mathfrak{a}_i\mathfrak{n}$ is 1-dimensional.*

4. *If $\mathcal{I}$ is non-empty, output "True", $\mathcal{I}$ else return "False".*

If f is a homological eigenform without an inner twist, we use the following algorithm to identify a Hecke eigensystem that matches f.

Algorithm 5.23.

Input: *Homological eigenform f over the imaginary quadratic field F*

Output: *List R of ideal representative $\mathcal{O}_F$, character χ in the character orbit of f, set of pairs $\{[\mathfrak{a}_i, \lambda(\mathfrak{a}_i)]\}$*

1. *Select a list R of small prime ideal representatives of each class in the class group Cl_F.*

2. *For each ideal $\mathfrak{a} \in R$ where $\mathfrak{a}^2$ is principal, compute the Hecke operator $T_{\mathfrak{a},\mathfrak{a}}$ using Hecke matrices from Theorem 4.21. Select the character χ such that $T_{\mathfrak{a},\mathfrak{a}}f = \chi(\mathfrak{a})f$.*

3. *Check if inner twist using Algorithm 5.22. If no inner twists, let $H = \mathrm{Cl}_F$. Otherwise if f has inner twists by ψ, let $H = ker(\psi)$.*

4. *Identify a subset $\{\mathfrak{a}_i\}$ of R whose classes form a $\mathbb{Z}$-basis for H.*

5. *Compute* $t(\mathfrak{a}_i)$ *for each* i *where* $t(\mathfrak{p}) = a_f(T_{\mathfrak{p}^2} T_{\mathfrak{a},\mathfrak{a}}) + N(\mathfrak{p})$.

6. *If* $t(\mathfrak{a}_i) \neq 0$ *for each* i, *return* R,χ, *set of pairs* $\{[\mathfrak{a}_i, \sqrt{\chi(\mathfrak{a}_i) t(\mathfrak{a}_i)}]\}$. *Also, return* ψ *if the system has inner twists.*

7. *If* $t(\mathfrak{a}_i) = 0$ *for some* i, *select a different set in step* 3 *and repeat.*

If the Hecke eigenvalues of a full collection of representatives of the class group are known, then computing the Hecke eigenvalue of another ideal $\mathfrak{p}$ would simply amount to finding the representative $\mathfrak{q}$ for the class $[\mathfrak{p}^{-1}]$ with $\lambda(\mathfrak{q}) \neq 0$ and computing the principal Hecke operator $T_{\mathfrak{p}\mathfrak{q}}$ using Hecke matrices in Theorem 4.20. As this is a speedup especially if the norm of the prime $\mathfrak{q}$ is large, we give a grid algorithm to compute the eigenvalues of such a list.

Algorithm 5.24 (Constructing the Grid)**.**

Input: *Homological eigenform* f *over the imaginary quadratic field* F *without inner twists,* R,χ, *set of pairs* $\{[\mathfrak{a}_i, \lambda(\mathfrak{a}_i)]\}$

Output: $\{(\mathfrak{p}, \lambda(\mathfrak{p})) : \mathfrak{p} \in R\}$

For each $\mathfrak{p} \in R$, *we do the following:*

1. *Compute exponents* $l_i \in \{0, 1\}$ *such that* $[\mathfrak{p}]\,\mathrm{Cl}_F^2 = [\prod_{i=1}^k [\mathfrak{a}_i^{l_i}]\,\mathrm{Cl}_F^2$.

2. *Let* $\mathfrak{b} = \mathfrak{p} \prod_{l_i=1} \mathfrak{a}_i$. *Pick an ideal* $\mathfrak{a}$ *such that* $\mathfrak{a}^2\mathfrak{b}$ *is a principal ideal. Compute the principal Hecke operator* $T_{\mathfrak{b}} T_{\mathfrak{a},\mathfrak{a}}$.

3. *Compute* $\lambda(\mathfrak{p})$. *See proof of Corollary 5.17 for more details.*

Chapter 6: Computational Results

In this chapter, we summarize the results from the implementation of our algorithms to the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-17})$ with the LMFDB label 2.0.68.1, with the ring of integers $\mathcal{O}_F = \mathbb{Z}[\omega]$, where $\omega = \sqrt{-17}$. The class group Cl_F of F is a cyclic group of order 4, and its unit group is $\{\pm 1\}$. The scope of our computation is as follows:

1. We compute the homology $H_1(X_0(\mathfrak{n}); \mathbb{C})$ for levels $\mathfrak{n}$ with norm $N(\mathfrak{n}) < 850$.
2. For level $\mathfrak{n}$ with $N(\mathfrak{n}) \leq 300$, we computed Hecke operators for prime ideals and certain composite ideals with class in Cl_F^2 with norm less than 100.
3. The space of Bianchi modular forms for levels $\mathfrak{n}$ with norm $N(\mathfrak{n}) \leq 200$.

In this chapter, we use the notation $\mathfrak{p}_{p.n}$ to represent a prime ideal with LMFDB label $p.n$. Examples of this are given in Table 6.1.

6.1 Voronoi Tessellation

In this section, we state information about the structure of the Voronoi tessellation of the full modular group $\mathrm{GL}_2(\mathcal{O}_F)$.

There are 12 perfect forms up to $\mathrm{GL}_2(\mathcal{O}_F)$ equivalence. Table 6.2 lists these forms and the number of minimal vectors. The second column of the table contains $[a, b_1, b_2, c]$ where the matrix representation of each form is $\begin{pmatrix} a & b_1 + b_2\omega \\ b_1 - b_2\omega & c \end{pmatrix}$.

Table 6.1. Ideals of $\mathcal{O}_F$ and their generators

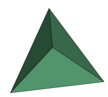
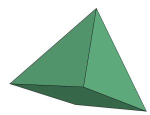
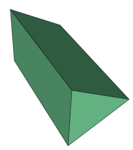
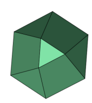
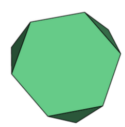
Norm	LMFDB Label	Generators	Symbol
2	2.1	$\langle 2, \omega + 1 \rangle$	$\mathfrak{p}_{2.1}$
3	3.1	$\langle 3, \omega + 4 \rangle$	$\mathfrak{p}_{3.1}$
3	3.2	$\langle 3, \omega + 2 \rangle$	$\mathfrak{p}_{3.2}$
7	7.1	$\langle 7, \omega + 4 \rangle$	$\mathfrak{p}_{7.1}$
7	7.2	$\langle 7, \omega + 7 \rangle$	$\mathfrak{p}_{7.2}$
11	11.1	$\langle 11, \omega + 4 \rangle$	$\mathfrak{p}_{11.1}$
11	11.2	$\langle 11, \omega + 7 \rangle$	$\mathfrak{p}_{11.2}$
13	13.1	$\langle 13, \omega + 3 \rangle$	$\mathfrak{p}_{13.1}$
13	13.2	$\langle 13, \omega + 10 \rangle$	$\mathfrak{p}_{13.2}$
17	17.1	$\langle 17, \omega \rangle$	$\mathfrak{p}_{17.2}$
23	23.1	$\langle 23, \omega + 11 \rangle$	$\mathfrak{p}_{23.1}$
23	23.2	$\langle 23, \omega + 12 \rangle$	$\mathfrak{p}_{23.2}$
25	25.2	$\langle 5 \rangle$	$\mathfrak{p}_{25.1}$
31	31.1	$\langle 31, \omega + 13 \rangle$	$\mathfrak{p}_{31.1}$
31	31.2	$\langle 31, \omega + 18 \rangle$	$\mathfrak{p}_{31.2}$
53	53.1	$\langle 53, \omega + 6 \rangle$	$\mathfrak{p}_{53.1}$
53	53.2	$\langle 53, \omega + 47 \rangle$	$\mathfrak{p}_{53.2}$
89	89.1	$\langle 89, \omega + 28 \rangle$	$\mathfrak{p}_{89.1}$
89	89.2	$\langle 89, \omega + 61 \rangle$	$\mathfrak{p}_{89.2}$

Table 6.2. Perfect forms up to $\mathrm{GL}_2(\mathcal{O}_F)$ equivalence for the number field $F = \mathbb{Q}(\sqrt{-17})$

	$[a, b_1, b_2, c]$	number of minimal vectors
P_1	$[5/3, -1/2, 29/102, 1]$	6
P_2	$[19/9, -5/6, 29/102, 1]$	4
P_3	$[13/3, -31/18, 223/306, 25/9]$	4
P_4	$[24/5, -39/20, 137/170, 31/10]$	5
P_5	$[25/12, -7/8, 14/51, 1]$	5
P_6	$[5/3, -11/18, 83/306, 1]$	5
P_7	$[25/12, -41/48, 227/816, 1]$	4
P_8	$[35/8, -85/48, 601/816, 17/6]$	4
P_9	$[7/4, -7/8, 4/17, 1]$	6
P_{10}	$[13/8, -5/8, 9/34, 1]$	4
P_{11}	$[5/4, -5/8, 15/68, 1]$	9
P_{12}	$[1, -1/2, 7/34, 1]$	12

Table 6.3 gives the combinatorial type, the face vector of each polytope, and the number of distinct polytopes of each type observed in the tessellation. The face vector is a tuple of integers $[v, e, f]$, where v, e, f are the number of vertices, edges, and faces of each polytope type respectively.

Table 6.3. Combinatorial types of polytopes in the Voronoi tessellation for $F = \mathbb{Q}(\sqrt{-17})$

Combinatorial type	polytope	Face Vector	Number of polytopes
Tetrahedron		$[4, 6, 4]$	5
Square Pyramid		$[5, 8, 5]$	3
Triangular Prism		$[6, 9, 5]$	2
Hexagonal Cap		$[9, 15, 8]$	1
Truncated Tetrahedron		$[12, 18, 8]$	1

We list below the 27 vertices of the 12 polytopes:

$C_1 = [0, 1]$	$C_2 = [1, 0]$	$C_3 = [1, 1]$	$C_4 = [3, w + 1]$
$C_5 = [3, w + 2]$	$C_6 = [3, w + 3]$	$C_7 = [4, w + 2]$	$C_8 = [4, w + 3]$
$C_9 = [4, w + 4]$	$C_{10} = [5, w + 2]$	$C_{11} = [5, w + 3]$	$C_{12} = [-w, -w + 4]$
$C_{13} = [-w, -w + 5]$	$C_{14} = [-w + 2, 4]$	$C_{15} = [-w + 2, 5]$	$C_{16} = [-w + 2, 6]$
$C_{17} = [-w + 3, 5]$	$C_{18} = [-w + 3, 6]$	$C_{19} = [-w + 6, w + 8]$	$C_{20} = [w + 1, w - 4]$
$C_{21} = [w + 1, w - 3]$	$C_{22} = [w + 2, w - 4]$	$C_{23} = [w + 2, w - 3]$	$C_{24} = [w + 2, w - 2]$
$C_{25} = [w + 3, w - 2]$	$C_{26} = [w + 5, 2w - 1]$	$C_{27} = [2w + 3, 2w - 7]$	

There are 18 types of edges up to $\mathrm{GL}_2(\mathcal{O}_F)$ equivalence. Thus, these are the generators of the homology for the full group $\mathrm{GL}_2(\mathcal{O}_F)$.

$E_1 = C_1C_5$	$E_2 = C_5C_{16}$	$E_3 = C_7C_{16}$	$E_4 = C_1C_7$	$E_5 = C_6C_{20}$	$E_6 = C_1C_6$
$E_7 = C_1C_{20}$	$E_8 = C_{19}C_{20}$	$E_9 = C_7C_{20}$	$E_{10} = C_{20}C_{26}$	$E_{11} = C_8C_{26}$	$E_{12} = C_8C_{19}$
$E_{13} = C_8C_{27}$	$E_{14} = C_1C_9$	$E_{15} = C_8C_9$	$E_{16} = C_1C_8$	$E_{17} = C_1C_3$	$E_{18} = C_8C_{21}$

There are 29 types of faces. These faces contribute relations in homology.

$F_1 = C_8C_{26}C_{27}$	$F_2 = C_1C_4C_5$	$F_3 = C_1C_2C_3$	$F_4 = C_5C_6C_{20}$
$F_5 = C_1C_6C_{20}$	$F_6 = C_1C_5C_{20}$	$F_7 = C_1C_5C_6$	$F_8 = C_{16}C_{19}C_{20}$
$F_9 = C_7C_{19}C_{20}$	$F_{10} = C_7C_{16}C_{20}$	$F_{11} = C_7C_{16}C_{19}$	$F_{12} = C_8C_{18}C_{21}$
$F_{13} = C_8C_{20}C_{26}$	$F_{14} = C_8C_{19}C_{20}$	$F_{15} = C_1C_7C_8$	$F_{16} = C_8C_{19}C_{27}$
$F_{17} = C_1C_8C_9$	$F_{18} = C_7C_8C_{20}$	$F_{19} = C_1C_6C_8$	$F_{20} = C_8C_9C_{13}$
$F_{21} = C_1C_7C_{20}$	$F_{22} = C_1C_8C_{20}$	$F_{23} = C_1C_5C_7C_{16}$	$F_{24} = C_1C_3C_9C_{12}$
$F_{25} = C_1C_3C_8C_{21}$	$F_{26} = C_1C_6C_9C_{13}$	$F_{27} = C_4C_5C_{16}C_{22}$	$F_{28} = C_{19}C_{20}C_{26}C_{27}$
$F_{29} = C_1C_3C_7C_{11}C_{23}C_{24}$			

6.2 Hecke Eigensystems

In this section, we discuss and provide a detailed example of how to compute Hecke eigensystems matching a homological eigenform f over the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-17})$.

The class group Cl_F of F is a cyclic grop of order 4. Let g be a generator of Cl_F such that $\mathfrak{p}_{3.1} \in g$, where $\mathfrak{p}_{3.1} = \langle 3, 4+\omega \rangle$. The character group is also a cyclic group of order 4. Let χ_1 be the character defined by $\chi_1(g) = i = \sqrt{-1}$, and let χ_k denote χ_1^k. For concreteness, the character values are given in Table 6.4.

By $\mathrm{Cl}_F[2]$ we denote the order 2 elements in the class group Cl_F. From Theorem 5.9 and Remark 5.5, we know computing $T_{\mathfrak{a},\mathfrak{a}}$ for all $\mathfrak{a} \in \mathrm{Cl}_F[2]$ is sufficient to identify the character orbit. For the field F, since $\mathrm{Cl}_F[2] = \{1, g^2\}$, we can use one ideal $\mathfrak{p} \in g^2$ to compute the

Table 6.4. Characters of the class group of $\mathbb{Q}(\sqrt{-17})$

	1	g	g^2	g^3
χ_0	1	1	1	1
χ_1	1	i	1	$-i$
χ_2	1	-1	1	-1
χ_3	1	$-i$	1	i

character orbit. If $\lambda(T_{\mathfrak{p},\mathfrak{p}}) = 1$ for some Hecke operators $T_{\mathfrak{p},\mathfrak{p}}$ with $\mathfrak{p} \in g^2$ then χ must be a character that is trivial on $\mathrm{Cl}_F[2]$. Thus, $\chi = \chi_0$ or $\chi = \chi_2$. On the other hand, if the eigenvalue for $T_{\mathfrak{p},\mathfrak{p}}$ for some $\mathfrak{p} \in g^2$ is -1, then $\chi = \chi_1$ or χ_3. To make our computations simpler, if $\lambda(T_{\mathfrak{p},\mathfrak{p}}) = 1$, we pick $\chi = \chi_0$ and otherwise pick $\chi = \chi_1$.

Because $\mathrm{Cl}_F / \mathrm{Cl}_F^2$ is generated by $g\,\mathrm{Cl}_F^2$, the set $\{\mathfrak{q}\}$ where $\mathfrak{q} \in g$ and $\lambda(\mathfrak{q}) \neq 0$ is an admissible set. Therefore, by computing the Hecke eigenvalue $\lambda(\mathfrak{q})$ for some such ideal $\mathfrak{q} \in g$, we can uniquely identify a Hecke eigensystem.

Example 6.1. At level $\mathfrak{p}_{2.1} = \langle 2, 1+\omega \rangle$, there is one homological eigenform f_2. Since the homology is one-dimensional, the principal eigenvalues of f_2 are rational.

Let $\mathfrak{p} = \mathfrak{p}_{13.1} = \langle 13, 3+\omega \rangle$. Then $\mathfrak{p} \in \mathrm{Cl}_F^2$, and the eigenvalue of the principal Hecke operator $T_{\mathfrak{p},\mathfrak{p}}$ is 1. Therefore, the character orbit of f_2 is $\{\chi_0, \chi_2\}$. Now, we can fix the character to be $\chi = \chi_0$ to identify a Hecke eigensystem.

The smallest non-principal prime ideals are the two primes above 3, $\mathfrak{p}_{3.1} = \langle 3, 4+\omega \rangle \in g$ and $\mathfrak{p}_{3.2} = \langle 3, 2+\omega \rangle \in g^3$. The Hecke eigenvalue of the principal Hecke operator $T_{\mathfrak{p}_{3.1}\mathfrak{p}_{3.2}}$ is non-zero. This means that the set $\{\mathfrak{p}_{3.1}\}$ is an admissible $\mathbb{Z}$-basis for $\mathrm{Cl}_F / \mathrm{Cl}_F^2$. Now we compute $t(\mathfrak{p}_{3.1})$ with Theorem 5.11 and picking $\lambda(\mathfrak{p}_{3.1}) = \sqrt{t(\mathfrak{p}_{3.1})}$ for some $\mathfrak{a} \in [\mathfrak{p}^{-1}]$. Note that $\chi(\mathfrak{p}_{3.1}) = 1$.

For this example, picking $\mathfrak{a} = \mathfrak{p}_{3.2}$ and using matrices in Theorem 4.23, we get that $\lambda(T_{\mathfrak{p}_{3.1}^2} T_{\mathfrak{p}_{3.2},\mathfrak{p}_{3.2}}) = 3$. Then we get $\lambda(\mathfrak{p}_{3.1}) = 2\sqrt{2}$.

Table 6.5. Hecke eigensystem representative at level $\mathfrak{p}_{2.1}$

prime	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$
class	2	3	1	3	1	3	2	2	0	1	3
f_2	2α	-2α	0	0	-2α	2α	-2	-2	-6	-4α	4α

Now, we can compute all the eigenvalues in this particular system using principal Hecke operators. For example, for the eigenvalue of $\mathfrak{p}_{3.2}$, we only need the principal Hecke operator $T_{\mathfrak{p}_{3.1}\mathfrak{p}_{3.2}}$ to compute $\lambda(\mathfrak{p}_{3.1}\mathfrak{p}_{3.2}) = -8$. Then,

$$\lambda(\mathfrak{p}_{3.2}) = \frac{\lambda(\mathfrak{p}_{3.1}\mathfrak{p}_{3.2})}{\lambda(\mathfrak{p}_{3.2})} = \frac{-8}{2\sqrt{2}} = -2\sqrt{2}.$$

With these choices, we get the Hecke eigensystem given the Table 6.5, where $\alpha = \sqrt{2}$.

Now we discuss how to obtain the full space of Bianchi modular forms using homological eigenforms. Recall from Chapter 5 that we can compute a complete collection of all Hecke eigensystems that match a homological eigenform f_2 by twisting one Hecke eigensystem (λ, χ) that matches with f_2 by the characters of the class group. For the field $F = \mathbb{Q}(\sqrt{-17})$, the characters in consideration are given in Table 6.4. If the homological eigenform f matches (λ, χ), then the twist orbit

$$\{(\lambda, \chi), (\lambda\chi_1, \chi\chi_2), (\lambda\chi_2, \chi), (\lambda\chi_3, \chi\chi_2)\}$$

gives us all eigensystems matching f_2.

Example 6.2. Table 6.5 describes a Hecke eigensystem matching the unique homological eigenform at level $\mathfrak{p}_{2.1}$. This Hecke eigensystem does not have inner twists since we have an admissible basis for $\mathrm{Cl}_F / \mathrm{Cl}_F^2$. Now we can twist this system by the character of the class group to obtain the Hecke eigensystems given in Table 6.6, where $\alpha = \sqrt{2}$ and $\beta = \sqrt{-2}$.

Table 6.6. Hecke eigensystems at level $\mathfrak{p}_{2.1}$

prime	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$	$\mathfrak{p}_{25.1}$
class	3	1	1	3	3	1	2	2	0	1	3	0
f_2	2α	-2α	0	0	-2α	2α	-2	-2	-6	-4α	4α	2
$f_2 \otimes \chi_1$	2β	2β	0	0	-2β	-2β	2	2	-6	4β	4β	2
$f_2 \otimes \chi_2$	-2α	2α	0	0	2α	-2α	-2	-2	-6	4α	-4α	2
$f_2 \otimes \chi_3$	-2β	-2β	0	0	2β	2β	2	2	-6	-4β	-4β	2

Note that the twist orbit decomposes into two Galois orbits with the following Galois alignments:

$$\sigma_1(f_2) = f_2 \otimes \chi_2 \text{ and } \sigma_2(f_2 \otimes \chi_1) = f_2 \otimes \chi_3,$$

where $\sigma_1 : \alpha \mapsto -\alpha$ and $\sigma_2 : \beta \mapsto -\beta$.

Now we give an example of a case where the Galois orbits and twists orbits align exactly.

Example 6.3. At level $\mathfrak{p}_{3.1}^2 = \langle 9, -1-\omega \rangle$, we observe one homological eigenform. As before, the principal Hecke operators are rational. Let $\mathfrak{p} = \mathfrak{p}_{13.1}$. The eigenvalue of the principal Hecke operator $T_{\mathfrak{p},\mathfrak{p}}$ is -1. Therefore, the character orbit of the homological eigenform is $\{\chi_1, \chi_3\}$. Table 6.7 gives a Hecke eigensystem that matches the homological eigenform where α is a root of $x^4 + 1$.

In this case, we recover all the systems using the action of the Galois group of $x^4 + 1$. Table 6.8 shows these four systems.

If all the systems in the twist orbit are distinct, as it is in the examples above, the dimension of the space of Bianchi modular forms is 4 times the dimension of the space of homological eigenforms. We have non-distinct eigensystems in the twist orbit only when the

Table 6.7. Hecke eigensystems at level $\mathfrak{p}_{3.1}^2$

prime	$\mathfrak{p}_{2.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$	$\mathfrak{p}_{25.1}$
class	2	1	1	3	3	1	2	2	0	1	3	0
f_9	$-\alpha^2$	$2\alpha^3$	$-4\alpha^3$	2α	-3α	$-6\alpha^3$	$-2\alpha^2$	$2\alpha^2$	0	$4\alpha^3$	2α	4

Table 6.8. All Hecke eigensystems at level $\mathfrak{p}_{3.1}^2$

prime	$\mathfrak{p}_{2.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$
class	2	1	1	3	3	1	2	2	0	1	3
f_9	$-\alpha^2$	$2\alpha^3$	$-4\alpha^3$	2α	-3α	$-6\alpha^3$	$-2\alpha^2$	$2\alpha^2$	0	$4\alpha^3$	2α
$f_9 \otimes \chi_1$	α^2	2α	-4α	$2\alpha^3$	$-3\alpha^3$	-6α	$2\alpha^2$	$-2\alpha^2$	0	4α	$2\alpha^3$
$f_9 \otimes \chi_2$	$-\alpha^2$	$-2\alpha^3$	$4\alpha^3$	-2α	3α	$6\alpha^3$	$-2\alpha^2$	$2\alpha^2$	0	$-4\alpha^3$	-2α
$f_9 \otimes \chi_3$	α^2	-2α	4α	$-2\alpha^3$	$3\alpha^3$	6α	$2\alpha^2$	$-2\alpha^2$	0	-4α	$-2\alpha^3$

homological eigenform has inner twists. In that case, the dimension of the twist orbit will be 2 times the dimension of the homological eigenform. Up to norm 200, the only inner twist cases observed are at level $\mathfrak{p}_{2.1}^6$. Thus, we can accurately compute the dimension of the space of Bianchi modular forms up to level norm 200.

6.3 Oldforms and Newforms

From the last section, we saw how to compute the dimension of the space of Bianchi modular forms. Now we discuss how to identify the space of newforms.

Similar to the classical case, if $\mathfrak{m}$ divides $\mathfrak{n}$ then $S_2(\mathfrak{m}) \subset S_2(\mathfrak{n})$. For each divisor $\mathfrak{d}$ of $\mathfrak{n}/\mathfrak{m}$,

Table 6.9. Hecke eigensystem at level $\mathfrak{p}_{2.1}^2$

prime	$\mathfrak{p}_{2.1}$	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$	multiplicity
class	2	3	1	1	3	3	1	2	2	0	1	3	
f_4	1	α	$-\alpha$	3α	-3α	$-\alpha$	α	4	4	6	α	$-\alpha$	1
f_2	1	2α	-2α	0	0	-2α	2α	-2	-2	-6	-4α	4α	2

there is a map $S_2(\mathfrak{m}) \to S_2(\mathfrak{n})$. Therefore, each Hecke eigensystem of level $\mathfrak{m}$ contributes to the old space of $S_2(\mathfrak{n})$ with the multiplicity of the number of divisors of $\mathfrak{n}/\mathfrak{m}$. We can see this contribution at the homology level as well. However, if the Hecke eigensystem has inner twists, the dimension of the old space in homology might be lower. See Section 6.8 for more details.

Example 6.4. At level $\mathfrak{p}_{2.1}^2$, we observe 2 homological eigenforms, one with multiplicity 1 and the other with multiplicity 2. Table 6.9 represents one Hecke eigensystem representative that matches each homological modular form, where $\alpha = \sqrt{2}$.

Up to the scope of the computation, f_2 matches the Hecke eigensystem from level $\mathfrak{p}_{2.1}$ as expected from Atkin Lehner theory.

At level $\mathfrak{p}_{2.1}^3$, the 7 homological eigenforms are given in Table 6.10, where $\alpha = \sqrt{2}$.

The Hecke eigensystem at level $\mathfrak{p}_{2.1}$ shows up with multiplicity 3, and the new Hecke eigensystem for level $\mathfrak{p}_{2.1}^2$ shows up with multiplicity 3 as expected.

The oldform contribution in the homology of the inner twist systems is more subtle. Suppose $f \in S_2(\mathfrak{n})$ with inner twists by ψ. Then the contribution of f to the homology at level $\mathfrak{n}$ is the number of divisor $\mathfrak{d}|(\mathfrak{m}/\mathfrak{n})$ with $\psi(\mathfrak{d}) = 1$. From communications with Cremona, it was explained that this is due to the image of the adelic analog of the map "multiplication by $\mathfrak{d}$" for $\psi(\mathfrak{d}) = -1$. If the system has an inner twist by ψ, then these images will only

Table 6.10. Hecke eigensystems at level $\mathfrak{p}_{2.1}^3$

prime	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$	multiplicity
class	3	1	1	3	3	1	2	2	0	1	3	
$f_{8,1}$	0	0	-2α	2α	-4α	4α	-2	-2	-6	2α	-2α	1
$f_{8,2}$	2	-2	-2	2	2	-2	2	2	2	6	-6	1
f_4	α	$-\alpha$	3α	-3α	$-\alpha$	α	4	4	6	α	$-\alpha$	2
f_2	2α	-2α	0	0	-2α	2α	-2	-2	-6	-4α	4α	3

be supported on non-principal classes. Hence, we cannot observe them in homology. The following example demonstrates this phenomenon:

Example 6.5. At the levels $\mathfrak{p}_{2.1}^2\mathfrak{p}_{3.1}$ and $\mathfrak{p}_{2.1}^2\mathfrak{p}_{3.2}$, there are 80 homological eigenforms. All homological eigenforms, except the two from level $\mathfrak{p}_{2.1}^6$, do not have any inner twist. Therefore, the size of the twist orbit of each of these 78 homological eigenform is 4. The two homological eigenforms from level $\mathfrak{p}_{2.1}^6$ each only have an orbit of size 2. Thus, the total number of Hecke eigensystems coming from the twist orbits of homological eigenforms is 316.

Now, we look at the oldspace contributions to level $\mathfrak{p}_{2.1}^6\mathfrak{p}_{3.1}$ from the Bianchi modular forms. This is given in the table below: From this we can see that the total dimension of the space of Bianchi modular forms is 320. This means that the homology is missing 4 Hecke eigensystems. These four "missing" systems are the ones that are supported on non-principal components. These are coming from the image of the map "multiplication by $\mathfrak{p}_{3.1}$" on the four inner twist eigensystems at level $\mathfrak{p}_{2.1}^6$.

Table 6.11. Oldspace dimensions at level $\mathfrak{p}_{2.1}^6\mathfrak{p}_{3.1}$

level	d	d_{new}	Multiplicity	χ_0	χ_1	χ_2	χ_3
$\mathfrak{p}_{2.1}$	4	4	12	2	0	2	0
$\mathfrak{p}_{2.1}^2$	12	4	10	2	0	2	0
$\mathfrak{p}_{2.1}^3$	28	8	8	4	0	4	0
$\mathfrak{p}_{2.1}^4$	60	16	6	0	8	0	8
$\mathfrak{p}_{2.1}^6$	144	20	2	5	5	5	5
$\mathfrak{p}_{2.1}^2\mathfrak{p}_{3.1}$	28	4	5	0	2	0	2
$\mathfrak{p}_{2.1}^4\mathfrak{p}_{3.1}$	136	4	3	2	0	2	0
$\mathfrak{p}_{2.1}^6\mathfrak{p}_{3.1}$				92	68	92	68

6.4 Dimension Tables

In this section, we provide tables of dimensions of the space of Bianchi modular forms and spaces of newforms up to level norm 200.

The columns of Tables 6.13 and Table 6.14 give the following information. The first column of these tables represents the LMFDB label of ideal $\mathfrak{n}$. The second column is the total dimension d of the cuspidal space. The third column specifies the dimension of new space d_{new}. The fourth column is a pair $[t, n]$ where t represents the number of Hecke eigensystems with character orbit $\{\chi_0, \chi_2\}$ and n represents the number of Hecke eigensystems with character $\{\chi_1, \chi_3\}$. We indicate the level with Hecke eigensystems with an inner twist with an asterisk.

For example, the column corresponding to level $\mathfrak{p}_{16.1}$ is given below. The dimension of the Bianchi modular forms space at this level is 60 with 16 eigensystems that are new. Since $[t, n] = [0, 16]$, all 16 Hecke eigensystems had character either χ_1 or χ_3.

Table 6.12. Excerpt of Table 6.13

$\mathfrak{n}$	d	d_{new}	$[t, n]$
16.1	60	16	$[0, 16]$

Remark 6.1. From the independent work of Cremona, dimension tables are available for the field $\mathbb{Q}(\sqrt{-17})$ in LMFDB up to level norm bound 1000. With a careful analysis, we confirmed these agreed with our computations up to level norm 200. For all levels in this range, the newspace dimensions from the LMFBD tables are always half the number t in the pair $[n, t]$ on Tables 6.13 through Table 6.17. For example, from our tables the dimension of the total cuspidal space at level $\mathfrak{p}_{3.1}^3$, the pair $[t, n] = [4, 16]$. This means the newspace at this level has 4 Hecke eigensystems with character χ_0 or χ_2. Therefore, the new space dimension in LMFDB tables for this level is 2. Since our table does not explicitly state character details for oldspaces, direct comparisons are not possible. However, since oldform contributions are well understood, one can compute and verify the data in these tables.

6.5 Newform Tables

In this section, we provide tables with information about levels containing newforms.

The first four columns have the same format as in the dimension Tables 6.13 and 6.14. Note that on each row $t + n = d_{new}$. The fifth column gives information about the restriction of Hecke eigensystems to the principal component. The sixth column gives information about the Hecke field of the Hecke eigensystem. The entries in both columns are sequences of pairs of integers. The first entry of a pair corresponds to the degree of the Hecke field and the second to the total number of systems over the Hecke field. The number of such pairs gives the number of distinct Hecke fields observed at each level.

For example, the row corresponding to the level $\mathfrak{p}_{16.1}$ is given below.

Table 6.13. Dimension tables for levels $0 \le N(\mathfrak{n}) \le 200$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
1.1	0	0	$[0,0]$
2.1	4	4	$[4,0]$
3.2	0	0	$[0,0]$
3.1	0	0	$[0,0]$
4.1	12	4	$[4,0]$
6.2	8	0	$[0,0]$
6.1	8	0	$[0,0]$
7.1	4	4	$[4,0]$
7.2	4	4	$[4,0]$
8.1	28	8	$[8,0]$
9.1	4	4	$[0,4]$
9.3	4	4	$[0,4]$
9.2	8	8	$[8,0]$
11.2	0	0	$[0,0]$
11.1	0	0	$[0,0]$
12.1	28	4	$[0,4]$
12.2	28	4	$[0,4]$
13.2	0	0	$[0,0]$
13.1	0	0	$[0,0]$
14.2	16	0	$[0,0]$
14.1	16	0	$[0,0]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
16.1	60	16	$[0,16]$
17.1	4	4	$[4,0]$
18.3	20	0	$[0,0]$
18.1	20	0	$[0,0]$
18.2	36	4	$[4,0]$
21.1	12	4	$[4,0]$
21.4	12	4	$[4,0]$
21.3	8	0	$[0,0]$
21.2	8	0	$[0,0]$
22.2	8	0	$[0,0]$
22.1	8	0	$[0,0]$
23.1	0	0	$[0,0]$
23.2	0	0	$[0,0]$
24.1	64	0	$[0,0]$
24.2	64	0	$[0,0]$
25.1	12	12	$[12,0]$
26.2	12	4	$[0,4]$
26.1	12	4	$[0,4]$
27.4	8	0	$[0,0]$
27.2	44	20	$[4,16]$
27.1	8	0	$[0,0]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
27.3	44	20	$[4,16]$
28.2	36	0	$[0,0]$
28.1	36	0	$[0,0]$
31.2	0	0	$[0,0]$
31.1	0	0	$[0,0]$
32.1	92	0	$[0,0]$
33.2	0	0	$[0,0]$
33.1	0	0	$[0,0]$
33.3	0	0	$[0,0]$
33.4	0	0	$[0,0]$
34.1	20	4	$[4,0]$
36.3	60	4	$[4,0]$
36.2	100	4	$[4,0]$
36.1	60	4	$[4,0]$
39.1	0	0	$[0,0]$
39.4	0	0	$[0,0]$
39.3	0	0	$[0,0]$
39.2	0	0	$[0,0]$
42.1	40	0	$[0,0]$
42.4	40	0	$[0,0]$
42.3	40	8	$[4,4]$

Table 6.14. Dimension tables for levels $0 \leq N(\mathfrak{n}) \leq 200$ (continued)

$\mathfrak{n}$	d	d_{new}	$[t,n]$
42.2	40	8	$[4,4]$
44.1	24	0	$[0,0]$
44.2	24	0	$[0,0]$
46.2	8	0	$[0,0]$
46.1	8	0	$[0,0]$
48.1	136	4	$[4,0]$
48.2	136	4	$[4,0]$
49.3	12	4	$[4,0]$
49.2	36	20	$[20,0]$
49.1	12	4	$[4,0]$
50.1	44	12	$[12,0]$
51.1	8	0	$[0,0]$
51.2	8	0	$[0,0]$
52.1	32	0	$[0,0]$
52.2	32	0	$[0,0]$
53.1	0	0	$[0,0]$
53.2	0	0	$[0,0]$
54.4	32	0	$[0,0]$
54.2	120	0	$[0,0]$
54.3	120	0	$[0,0]$
54.1	32	0	$[0,0]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
56.2	72	0	$[0,0]$
56.1	72	0	$[0,0]$
62.2	12	4	$[4,0]$
62.1	12	4	$[4,0]$
63.6	36	8	$[0,8]$
63.3	40	0	$[0,0]$
63.2	20	0	$[0,0]$
63.1	36	8	$[0,8]$
63.5	20	0	$[0,0]$
63.4	40	0	$[0,0]$
64.1*	144	20	$[10,10]$
66.1	20	4	$[4,0]$
66.2	28	12	$[0,12]$
66.4	20	4	$[4,0]$
66.3	28	12	$[0,12]$
68.1	52	8	$[8,0]$
69.1	0	0	$[0,0]$
69.2	0	0	$[0,0]$
69.4	0	0	$[0,0]$
69.3	0	0	$[0,0]$
71.1	0	0	$[0,0]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
71.2	0	0	$[0,0]$
72.3	124	0	$[0,0]$
72.1	124	0	$[0,0]$
72.2	220	24	$[24,0]$
75.2	24	0	$[0,0]$
75.1	24	0	$[0,0]$
77.1	8	0	$[0,0]$
77.3	8	0	$[0,0]$
77.4	8	0	$[0,0]$
77.2	8	0	$[0,0]$
78.1	24	0	$[0,0]$
78.3	44	20	$[4,16]$
78.4	24	0	$[0,0]$
78.2	44	20	$[4,16]$
79.1	0	0	$[0,0]$
79.2	0	0	$[0,0]$
81.3	164	28	$[12,16]$
81.1	12	0	$[0,0]$
81.4	80	0	$[0,0]$
81.2	80	0	$[0,0]$
81.5	12	0	$[0,0]$

Table 6.15. Dimension tables for levels $0 \leq N(\mathfrak{n}) \leq 200$ (continued)

$\mathfrak{n}$	d	d_{new}	$[t,n]$
84.4	92	0	$[0,0]$
84.1	92	0	$[0,0]$
84.3	96	0	$[0,0]$
84.2	96	0	$[0,0]$
88.2	56	0	$[0,0]$
88.1	56	0	$[0,0]$
89.1	0	0	$[0,0]$
89.2	0	0	$[0,0]$
91.3	8	0	$[0,0]$
91.2	8	0	$[0,0]$
91.4	8	0	$[0,0]$
91.1	8	0	$[0,0]$
92.2	24	0	$[0,0]$
92.1	24	0	$[0,0]$
93.1	0	0	$[0,0]$
93.3	8	8	$[4,4]$
93.2	8	8	$[4,4]$
93.4	0	0	$[0,0]$
96.1	208	0	$[0,0]$
96.2	208	0	$[0,0]$
98.2	116	28	$[28,0]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
98.3	40	4	$[4,0]$
98.1	40	4	$[4,0]$
99.2	8	0	$[0,0]$
99.5	8	0	$[0,0]$
99.4	32	16	$[0,16]$
99.3	32	16	$[0,16]$
99.1	8	0	$[0,0]$
99.6	8	0	$[0,0]$
100.1	100	16	$[16,0]$
101.2	0	0	$[0,0]$
101.1	0	0	$[0,0]$
102.1	40	0	$[0,0]$
102.2	40	0	$[0,0]$
104.2	72	4	$[4,0]$
104.1	72	4	$[4,0]$
106.1	12	4	$[0,4]$
106.2	12	4	$[0,4]$
107.1	0	0	$[0,0]$
107.2	0	0	$[0,0]$
108.2	280	16	$[12,4]$
108.3	280	16	$[12,4]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
108.4	92	0	$[0,0]$
108.1	92	0	$[0,0]$
112.1	144	4	$[0,4]$
112.2	144	4	$[0,4]$
117.5	8	0	$[0,0]$
117.2	8	0	$[0,0]$
117.4	16	0	$[0,0]$
117.1	8	0	$[0,0]$
117.3	16	0	$[0,0]$
117.6	8	0	$[0,0]$
119.2	16	0	$[0,0]$
119.1	16	0	$[0,0]$
121.2	36	36	$[36,0]$
121.3	4	4	$[0,4]$
121.1	4	4	$[0,4]$
124.2	32	0	$[0,0]$
124.1	32	0	$[0,0]$
126.4	136	0	$[0,0]$
126.2	80	0	$[0,0]$
126.3	136	0	$[0,0]$
126.5	80	0	$[0,0]$

Table 6.16. Dimension tables for levels $0 \leq N(\mathfrak{n}) \leq 200$ (continued)

$\mathfrak{n}$	d	d_{new}	$[t,n]$	$\mathfrak{n}$	d	d_{new}	$[t,n]$	$\mathfrak{n}$	d	d_{new}	$[t,n]$
126.1	108	12	$[4,8]$	143.4	0	0	$[0,0]$	154.3	32	0	$[0,0]$
126.6	108	12	$[4,8]$	143.3	0	0	$[0,0]$	154.1	32	0	$[0,0]$
128.1	196	0	$[0,0]$	143.2	0	0	$[0,0]$	156.3	112	0	$[0,0]$
131.2	0	0	$[0,0]$	144.3	252	8	$[4,4]$	156.1	72	0	$[0,0]$
131.1	0	0	$[0,0]$	144.2	460	40	$[0,40]$	156.2	112	0	$[0,0]$
132.1	64	0	$[0,0]$	144.1	252	8	$[4,4]$	156.4	72	0	$[0,0]$
132.3	80	0	$[0,0]$	147.6	40	8	$[8,0]$	157.2	0	0	$[0,0]$
132.4	64	0	$[0,0]$	147.5	80	0	$[0,0]$	157.1	0	0	$[0,0]$
132.2	80	0	$[0,0]$	147.2	80	0	$[0,0]$	158.1	8	0	$[0,0]$
136.1	140	40	$[24,16]$	147.4	24	0	$[0,0]$	158.2	8	0	$[0,0]$
137.2	0	0	$[0,0]$	147.1	40	8	$[8,0]$	159.3	4	4	$[4,0]$
137.1	0	0	$[0,0]$	147.3	24	0	$[0,0]$	159.4	0	0	$[0,0]$
138.3	20	4	$[0,4]$	149.2	0	0	$[0,0]$	159.1	0	0	$[0,0]$
138.4	16	0	$[0,0]$	149.1	0	0	$[0,0]$	159.2	4	4	$[4,0]$
138.1	16	0	$[0,0]$	150.1	92	4	$[4,0]$	161.4	8	0	$[0,0]$
138.2	20	4	$[0,4]$	150.2	92	4	$[4,0]$	161.3	12	4	$[4,0]$
139.1	0	0	$[0,0]$	153.1	20	0	$[0,0]$	161.1	8	0	$[0,0]$
139.2	0	0	$[0,0]$	153.3	20	0	$[0,0]$	161.2	12	4	$[4,0]$
142.1	8	0	$[0,0]$	153.2	44	12	$[12,0]$	162.5	60	16	$[8,8]$
142.2	8	0	$[0,0]$	154.2	32	0	$[0,0]$	162.2	212	8	$[4,4]$
143.1	0	0	$[0,0]$	154.4	32	0	$[0,0]$	162.1	60	16	$[8,8]$

Table 6.17. Dimension tables for levels $0 \leq N(\mathfrak{n}) \leq 200$ (continued)

$\mathfrak{n}$	d	d_{new}	$[t,n]$
162.4	212	8	$[4,4]$
162.3	396	16	$[16,0]$
163.1	0	0	$[0,0]$
163.2	0	0	$[0,0]$
167.2	0	0	$[0,0]$
167.1	0	0	$[0,0]$
168.4	184	8	$[4,4]$
168.1	184	8	$[4,4]$
168.2	184	0	$[0,0]$
168.3	184	0	$[0,0]$
169.2	36	36	$[36,0]$
169.1	0	0	$[0,0]$
169.3	0	0	$[0,0]$
175.2	32	0	$[0,0]$
175.1	32	0	$[0,0]$
176.2	120	0	$[0,0]$
176.1	120	0	$[0,0]$
178.2	24	16	$[4,12]$
178.1	24	16	$[4,12]$
182.4	40	0	$[0,0]$
182.1	40	0	$[0,0]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
182.3	40	0	$[0,0]$
182.2	40	0	$[0,0]$
184.1	56	0	$[0,0]$
184.2	56	0	$[0,0]$
186.1	28	4	$[0,4]$
186.3	40	0	$[0,0]$
186.4	28	4	$[0,4]$
186.2	40	0	$[0,0]$
187.2	8	0	$[0,0]$
187.1	8	0	$[0,0]$
189.4	132	8	$[0,8]$
189.6	144	0	$[0,0]$
189.1	60	0	$[0,0]$
189.8	60	0	$[0,0]$
189.5	132	8	$[0,8]$
189.7	32	0	$[0,0]$
189.2	32	0	$[0,0]$
189.3	144	0	$[0,0]$
192.1*	320	0	$[0,0]$
192.2*	320	0	$[0,0]$
196.3	84	4	$[0,4]$

$\mathfrak{n}$	d	d_{new}	$[t,n]$
196.2	228	16	$[16,0]$
196.1	84	4	$[0,4]$
198.4	144	8	$[0,8]$
198.6	48	0	$[0,0]$
198.1	48	0	$[0,0]$
198.3	144	8	$[0,8]$
198.2	84	20	$[12,8]$
198.5	84	20	$[12,8]$
199.1	0	0	$[0,0]$
199.2	0	0	$[0,0]$
200.1	244	72	$[40,32]$

Table 6.18. Excerpt of Table 6.19

$\mathfrak{n}$	d	d_{new}	$[t,n]$	principal	full
16.1	60	16	$[0,16]$	$[1,16]$	$[2,8],[4,4],[4,4]$

This means all of the 16 Hecke eigensystems are rational when restricted to the principal component. Hence, we only observe one new homological eigenform at this level. The full eigensystems come from two different Hecke fields. There are 8 Hecke eigensystem systems over the degree two fields $\mathbb{Q}(i)$. There are four Hecke eigensystem with Hecke field $\mathbb{Q}(\zeta_8)$ where ζ_8 is a primitive 8^{th} root of unity (4.0.256.1), and the other four are over the field $\mathbb{Q}(\zeta_{12})$, where ζ_{12} is a primitive 12^{th} root of unity (4.0.144.1). Note that this means the Galois conjugates Hecke eigensystem over the degree 2 field and their twists are all different eigensystems. However, the Galois conjugate and twists of the two eigensystems align with each other.

6.6 Base Change

According to [8], a Hecke eigensystem over an imaginary quadratic field F is base change of a classical modular form $f \in S_2(N,\psi)$ if the eigenvalues satisfy the following properties:

1. If p is split and $p = \mathfrak{p}\overline{\mathfrak{p}}$, then $\lambda(\mathfrak{p}) = \lambda(\overline{\mathfrak{p}}) = a_p$;
2. If p is inert, then $\lambda(\mathfrak{p}) = a_p^2 - 2\psi(p)p$;
3. If p is ramified and if $(p) = \mathfrak{p}^2$, then $\lambda(\mathfrak{p}) = a_p$,

where a_p is the Hecke eigenvalue of f at p.

Remark 6.2. A special case of base change can happen if level $\mathfrak{n}$ is Galois stable. If f is a homological eigenform of level $\mathfrak{n}$ that matches (λ,χ) then by $\bar{f}$ we mean the conjugate

Table 6.19. Newforms space $0 \leq N(\mathfrak{n}) \leq 200$

$\mathfrak{n}$	d	n	$[t, n]$	principal	full
2.1	4	4	$[4, 0]$	$[1, 4]$	$[2, 2], [2, 2]$
4.1	12	4	$[4, 0]$	$[1, 4]$	$[2, 2], [2, 2]$
7.1	4	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
7.2	4	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
8.1	28	8	$[8, 0]$	$[1, 8]$	$[1, 2], [2, 2], [2, 2], [2, 2]$
9.1	4	4	$[0, 4]$	$[1, 4]$	$[4, 4]$
9.3	4	4	$[0, 4]$	$[1, 4]$	$[2, 4]$
9.2	8	8	$[8, 0]$	$[1, 8]$	$[1, 4], [2, 4]$
12.1	28	4	$[0, 4]$	$[1, 4]$	$[4, 4]$
12.2	28	4	$[0, 4]$	$[1, 4]$	$[4, 4]$
16.1	60	16	$[0, 16]$	$[1, 16]$	$[2, 8], [4, 4], [4, 4]$
17.1	4	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
18.2	36	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
21.1	12	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
21.4	12	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
25.1	12	12	$[12, 0]$	$[3, 12]$	$[3, 6], [6, 6]$
26.2	12	4	$[0, 4]$	$[1, 4]$	$[4, 4]$
26.1	12	4	$[0, 4]$	$[1, 4]$	$[2, 4]$
27.2	44	20	$[4, 16]$	$[1, 4], [4, 16]$	$[1, 2], [2, 2], [16, 16]$
27.3	44	20	$[4, 16]$	$[1, 4], [4, 16]$	$[1, 2], [2, 2], [16, 16]$
34.1	20	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$

Table 6.20. Newforms space $0 \leq N(\mathfrak{n}) \leq 200$ (continued)

$\mathfrak{n}$	d	n	$[t, n]$	principal	full
36.3	60	4	$[4, 0]$	$[1, 4]$	$[1, 4]$
36.2	100	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
36.1	60	4	$[4, 0]$	$[1, 4]$	$[2, 2], [2, 2]$
42.3	40	8	$[4, 4]$	$[1, 4], [2, 4]$	$[1, 2], [2, 2], [4, 4]$
42.2	40	8	$[4, 4]$	$[1, 4], [2, 4]$	$[1, 2], [2, 2], [4, 4]$
48.1	136	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
48.2	136	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
49.3	12	4	$[4, 0]$	$[1, 4]$	$[2, 2], [2, 2]$
49.2	36	20	$[20, 0]$	$[5, 10], [5, 10]$	$[5, 10], [10, 10]$
49.1	12	4	$[4, 0]$	$[1, 4]$	$[2, 2], [2, 2]$
50.1	44	12	$[12, 0]$	$[1, 12]$	$[1, 6], [2, 6]$
62.2	12	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
62.1	12	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
63.6	36	8	$[0, 8]$	$[2, 8]$	$[4, 4], [4, 4]$
63.1	36	8	$[0, 8]$	$[2, 8]$	$[4, 8]$
64.1*	144	20	$[10, 10]$	$[1, 4], [2, 16]$	$[1, 2], [2, 2], [2, 8], [4, 8]$
66.1	20	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
66.2	28	12	$[0, 12]$	$[1, 4], [2, 8]$	$[4, 4], [8, 8]$
66.4	20	4	$[4, 0]$	$[1, 4]$	$[1, 2], [2, 2]$
66.3	28	12	$[0, 12]$	$[1, 4], [2, 8]$	$[2, 4], [4, 8]$
68.1	52	8	$[8, 0]$	$[2, 8]$	$[2, 4], [4, 4]$

Table 6.21. Newforms space $0 \le N(\mathfrak{n}) \le 200$ (continued)

$\mathfrak{n}$	d	n	$[t,n]$	principal	full
72.2	220	24	$[24,0]$	$[1,4],[2,4],[2,4],[3,12]$	$[1,2],[2,2],[2,4],[3,6],[4,4],[6,6]$
78.3	44	20	$[4,16]$	$[1,4],[4,16]$	$[1,4],[8,16]$
78.2	44	20	$[4,16]$	$[1,4],[4,16]$	$[1,2],[2,2],[16,16]$
81.3	164	28	$[12,16]$	$[1,12],[2,8],[2,8]$	$[1,4],[2,2],[2,2],[2,4],[4,8],[8,8]$
93.3	8	8	$[4,4]$	$[1,4],[2,4]$	$[1,2],[2,2],[4,4]$
93.2	8	8	$[4,4]$	$[1,4],[2,4]$	$[1,2],[2,2],[4,4]$
98.2	116	28	$[28,0]$	$[1,8],[2,8],[3,12]$	$[1,6],[2,2],[2,4],[3,6],[4,4],[6,6]$
98.3	40	4	$[4,0]$	$[1,4]$	$[2,2],[2,2]$
98.1	40	4	$[4,0]$	$[1,4]$	$[2,2],[2,2]$
99.4	32	16	$[0,16]$	$[1,4],[3,12]$	$[2,4],[6,12]$
99.3	32	16	$[0,16]$	$[2,4],[6,12]$	$[4,4],[12,12]$
100.1	100	16	$[16,0]$	$[1,4],[3,6],[3,6]$	$[1,2],[2,2],[3,6],[6,6]$
104.2	72	4	$[4,0]$	$[1,4]$	$[2,2],[2,2]$
104.1	72	4	$[4,0]$	$[1,4]$	$[2,2],[2,2]$
106.1	12	4	$[0,4]$	$[2,4]$	$[4,4]$
106.2	12	4	$[0,4]$	$[2,4]$	$[4,4]$
108.2	280	16	$[12,4]$	$[1,8],[2,8]$	$[1,2],[2,2],[2,4],[4,4],[4,4]$
108.3	280	16	$[12,4]$	$[1,8],[2,8]$	$[1,2],[2,4],[2,6],[4,4]$
112.1	144	4	$[0,4]$	$[2,4]$	$[4,4]$
112.2	144	4	$[0,4]$	$[2,4]$	$[4,4]$
121.2	36	36	$[36,0]$	$[1,4],[8,32]$	$[1,2],[2,2],[8,16],[16,16]$

Table 6.22. Newforms space $0 \leq N(\mathfrak{n}) \leq 200$ (continued)

$\mathfrak{n}$	d	n	$[t,n]$	principal	full
121.3	4	4	$[0,4]$	$[2,4]$	$[4,4]$
121.1	4	4	$[0,4]$	$[2,4]$	$[4,4]$
126.1	108	12	$[4,8]$	$[1,4],[2,8]$	$[1,2],[2,2],[4,4],[4,4]$
126.6	108	12	$[4,8]$	$[1,4],[2,8]$	$[1,2],[2,2],[4,4],[4,4]$
136.1	140	40	$[24,16]$	$[1,8],[2,8],[2,8],$	$[1,4],[2,4],[2,4],[2,4],[4,4],$
				$[4,16]$	$[4,4],[8,16]$
138.3	20	4	$[0,4]$	$[2,4]$	$[4,4]$
138.2	20	4	$[0,4]$	$[2,4]$	$[4,4]$
144.3	252	8	$[4,4]$	$[1,4],[2,4]$	$[2,2],[2,2],[4,4]$
144.2	460	40	$[0,40]$	$[1,20],[2,8],[3,12]$	$[2,4],[4,4],[4,4],[4,8],[8,8],[12,12]$
144.1	252	8	$[4,4]$	$[1,8]$	$[2,2],[2,2],[4,4]$
147.6	40	8	$[8,0]$	$[1,8]$	$[1,4],[2,4]$
147.1	40	8	$[8,0]$	$[1,8]$	$[1,4],[2,4]$
150.1	92	4	$[4,0]$	$[1,4]$	$[1,2],[2,2]$
150.2	92	4	$[4,0]$	$[1,4]$	$[1,2],[2,2]$
153.2	44	12	$[12,0]$	$[1,4],[2,4],[2,4]$	$[1,2],[2,2],[2,4],[4,4]$
159.3	4	4	$[4,0]$	$[1,4]$	$[1,2],[2,2]$
159.2	4	4	$[4,0]$	$[1,4]$	$[1,2],[2,2]$
161.3	12	4	$[4,0]$	$[1,4]$	$[1,2],[2,2]$
161.2	12	4	$[4,0]$	$[1,4]$	$[1,2],[2,2]$
162.5	60	16	$[8,8]$	$[2,4],[2,4],[2,8]$	$[4,4],[4,4],[4,4],[4,4]$
162.2	212	8	$[4,4]$	$[1,8]$	$[1,2],[2,2],[4,4]$

Table 6.23. Newforms space $0 \leq N(\mathfrak{n}) \leq 200$ (continued)

$\mathfrak{n}$	d	n	$[t, n]$	principal	full
162.1	60	16	$[8, 8]$	$[2, 4], [2, 4], [2, 8]$	$[4, 4], [4, 4], [4, 4], [4, 4]$
162.4	212	8	$[4, 4]$	$[1, 8]$	$[1, 2], [2, 2], [4, 4]$
162.3	396	16	$[16, 0]$	$[1, 16]$	$[1, 2], [2, 2], [2, 6], [2, 6]$
168.4	184	8	$[4, 4]$	$[1, 4], [2, 4]$	$[1, 2], [2, 2], [4, 4]$
168.1	184	8	$[4, 4]$	$[1, 4], [2, 4]$	$[1, 2], [2, 2], [4, 4]$
169.2	36	36	$[36, 0]$	$[1, 4], [2, 8]$,	$[1, 2], [2, 2], [4, 4], [4, 4]$,
				$[3, 12], [3, 12]$	$[6, 6], [6, 6], [6, 6], [6, 6]$
178.2	24	16	$[4, 12]$	$[1, 16]$	$[1, 2], [2, 10], [4, 4]$
178.1	24	16	$[4, 12]$	$[1, 16]$	$[1, 2], [2, 14]$
186.1	28	4	$[0, 4]$	$[2, 4]$	$[4, 4]$
186.4	28	4	$[0, 4]$	$[2, 4]$	$[4, 4]$
189.4	132	8	$[0, 8]$	$[1, 8]$	$[4, 4], [4, 4]$
189.5	132	8	$[0, 8]$	$[1, 8]$	$[4, 4], [4, 4]$
196.3	84	4	$[0, 4]$	$[2, 4]$	$[4, 4]$
196.2	228	16	$[16, 0]$	$[4, 16]$	$[4, 8], [8, 8]$
196.1	84	4	$[0, 4]$	$[2, 4]$	$[4, 4]$
198.4	144	8	$[0, 8]$	$[1, 8]$	$[2, 8]$
198.3	144	8	$[0, 8]$	$[1, 8]$	$[2, 8]$
198.2	84	20	$[12, 8]$	$[1, 8], [3, 6], [3, 6]$	$[2, 8], [3, 6], [6, 6]$
198.5	84	20	$[12, 8]$	$[1, 8], [3, 6], [3, 6]$	$[2, 8], [3, 6], [6, 6]$
200.1	244	72	$[40, 32]$	$[1, 16], [3, 6], [3, 18]$,	$[1, 10], [2, 6], [4, 8], [4, 8], [6, 6]$,
				$[4, 8], [4, 8], [4, 8], [4, 8]$	$[6, 6], [6, 6], [6, 6], [8, 8], [8, 8]$

eigenform that matches a conjugate system denoted by $(\bar{\lambda}, \bar{\chi})$ satisfying

$$\bar{\lambda}(\mathfrak{p}) = \lambda(\bar{\mathfrak{p}}) \text{ and } \bar{\chi}(\mathfrak{p}) = \chi(\bar{\mathfrak{p}}).$$

Here the level of the conjugate form $\bar{f}$ is $\bar{\mathfrak{n}}$. If the level is Galois stable, that is if $\bar{\mathfrak{n}} = \mathfrak{n}$, then the forms f and $\bar{f}$ will have the same level. Further, if the eigenspace of f is one dimensional, then $f = \bar{f}$. Then there is an eigensystem that matches f with the property

$$\lambda(\mathfrak{p}) = \lambda(\bar{\mathfrak{p}}) \text{ and } \chi(\mathfrak{p}) = \chi(\bar{\mathfrak{p}}).$$

Then this system (λ, χ) is base change.

Example 6.6. The level $\mathfrak{p}_{17.1} = \langle 17, \omega \rangle$ is a Galois stable level with a one-dimensional homological eigensystem as described in the remark above. Thus, we expect a Hecke eigensystem that matches the homological eigenform to come from base change. The following table represents such a Hecke eigensystem:

Table 6.24. Hecke eigensystem at level $\mathfrak{p}_{17.1}$

prime	$\mathfrak{p}_{2.1}$	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$	$\mathfrak{p}_{25.1}$
class	2	3	1	1	3	3	1	2	2	0	1	3	0
	-1	0	0	-4	-4	0	0	-2	-2	$*$	-4	-4	-6

This system matches the point counts of the following rational elliptic curve

$$E : y^2 = x^3 - 11x + 6$$

with LMFDB label 17.a. Thus, this system is a base change of the classical newform at level

17 with trivial character (17.2.a.a).

Example 6.7. At level $\mathfrak{p}_{17.1}\mathfrak{p}_{2.1}^2$, there are two new homological eigenforms. Table 6.25 gives two Hecke eigensystems that match these eigenforms where $\alpha = \sqrt{3}$.

Table 6.25. Hecke eigensystem at level $\mathfrak{p}_{17.1}\mathfrak{p}_{2.1}^2$

prime	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{23.1}$
class	3	1	3	1	3	1	2	2	1
g_1	$\alpha+1$	$\alpha+1$	$-\alpha-1$	$-\alpha-1$	$\alpha-3$	$\alpha-3$	$2\alpha+2$	$2\alpha+2$	$\alpha-3$
g_2	$-\alpha-1$	$-\alpha-1$	$\alpha+1$	$\alpha+1$	$-\alpha+3$	$-\alpha+3$	$2\alpha+2$	$2\alpha+2$	$-\alpha+3$

Both systems are one-dimensional in homology, so we expect the systems to come from base change.

From the data available on LMFDB, we find that the base change of a newform with level 68 with trivial character (68.2.a.a) matches g_1, and the base change of the a newform with level 272 with trivial character (272.2.a.e) matches with g_2 up to the scope of our computation.

We also observe that the two systems have the following agreement:

$$g_2 = \sigma(g_1) \otimes \chi_2,$$

where $\sigma : \alpha \mapsto -\alpha$. On the classical modular forms side, we see similar symmetry. The classical modular form with LMFDB label 272.2.a.e is a twist of the form with LMFDB label 68.2.a.a by the quadratic Dirichlet character with conductor 4 and LMFBD label 4b.

We also observe Hecke eigensystem (λ, χ) with the property $\lambda(\mathfrak{p}) = \pm\lambda(\bar{\mathfrak{p}})$ on split primes. Thus, finding a character ψ might be possible such that the twisted system $(\lambda\psi, \chi\psi^2)$ is base change. We call such a system a *twisted base change*. The following is a detailed example that demonstrates this phenomenon.

Example 6.8. We observe one homological eigenform at level $\mathfrak{p}_{2.1} = \langle 2, \omega + 1 \rangle$. The system f_2 given in Table 6.5 is not base change because the eigenvalues on conjugate primes are different. However, the two twists $f_2 \otimes \chi_1$ and $f_2 \otimes \chi_3$ given in Table 6.6 agree on conjugate primes. Thus, the eigensystems $f_2 \otimes \chi_1$ and $f_2 \otimes \chi_3$ are potentially base change of a classical newform.

On split primes, the Hecke eigensystem $f_2 \otimes \chi_1$ agrees exactly with the classical newform at level 34 with a quadratic Dirichlet character ψ of conductor 34 (34.2.b.a). On inert prime, we have the following relationship.

Table 6.26. Hecke eigenvalues of the system $f_2 \otimes \chi_1$ at inert primes

p	5	19	43	59
$\psi(p)$	-1	1	1	1
a_p	$-2\sqrt{-2}$	-4	-4	12
$a_{\mathfrak{p}} = a_p^2 - 2\psi(p)p$	2	-22	-70	26

Thus the system $f_2 \otimes \chi_1$ agrees with the base change of the classical modular newform mentioned above for the scope of our computations.

Further, the L-function attached to this Hecke eigensystem f_2 matches the L-function of the Hilbert cusp form with LMFDB label 2.2.17.1-32.3-a for the scope of the computation. This Hilbert modular form matches with the isogeny classes of two the elliptic curve of conductor 32 over $\mathbb{Q}(\sqrt{17})$ (32.3a, 32.4a). These objects have the degree 4 L-function with conductor 9248 (4-9248-1.1-c1e2-0-2).

Another example of twisted base change can be found in Section 6.8.

$$\begin{tikzcd}
\text{BMF } f_2 \text{ at level } \mathfrak{p}_{2.1} \arrow[rr, leftrightarrow, "\text{L-function}"] \arrow[d, "\chi_1"'] & & \text{HMF 2.2.17.1-32.3-a} \arrow[d, "\psi"] \\
\text{BMF } f_2 \otimes \chi_1 \text{ at level } \mathfrak{p}_{2.1} \arrow[dr, leftrightarrow, "\text{base change to } \mathbb{Q}(\sqrt{-17})"'] & & \text{HMF 2.2.17.1-4.1-a} \arrow[dl, leftrightarrow, "\text{base change to } \mathbb{Q}(\sqrt{17})"] \\
& \text{CMF 34.2.b.a with character 34.b} &
\end{tikzcd}$$

Figure 6.1. Conjectural connections at level $\mathfrak{p}_{17.1}$

6.7 Elliptic Curves

From the proof of the Shimura-Taniyama-Weil conjecture by Bruiel, Conrad, Diamond, Taylor, and Wiles [4, 38], we know that rational elliptic curves are modular. Explicitly, this means that there is a rational newform f of weight 2 over the level equal to the conductor of the elliptic curve that has the same L-function. It is of interest to understand if and when there is such a relationship between elliptic curves over other number fields. By the work of Dieulefait, Guerberoff, and Pacetti [17], we have an algorithm to prove modularity for a given elliptic curve over an imaginary quadratic field. They also provide some of the first known examples of modular elliptic curves over an imaginary quadratic field that are not base change of an elliptic curve over $\mathbb{Q}$. In this section, we add to this work by proving the modularity of several elliptic curves over $F = \mathbb{Q}(\sqrt{-17})$. To our knowledge, these are the first explicit examples of modular elliptic curves over number fields with the class number 4.

Example 6.9. Here we show how we use the algorithm given in [17] to prove the modularity of the elliptic curve,

$$E : y^2 = x^3 + (-11664\omega - 23355)x + (-1714608\omega - 256662)$$

with conductor $\mathfrak{p}_{7.1} = \langle 7, 5+\omega \rangle$.

At level $\mathfrak{p}_{7.1}$, there is one homological eigenform. The following table represents one Hecke eigensystem that matches it.

Table 6.27. Hecke eigensystem at level $\mathfrak{p}_{7.1}$

	$\mathfrak{p}_{2.1}$	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{23.1}$	$\mathfrak{p}_{23.2}$	$\mathfrak{p}_{25.1}$	field
class	2	3	1	1	3	3	1	2	2	0	1	3	0	
	-1	-2	-2	0	1	-6	2	-2	6	-2	8	0	-2	$x-1$

Now we explain the details of using the algorithm [17, Algorithm 2.2] to show that the above system matches the elliptic curve.

1. First we chose primes ideal, $\mathfrak{p}_1 = \mathfrak{p}_{2.1}$ and $\mathfrak{p}_2 = \mathfrak{p}_{3.1}$. The root of characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}_i}$, $\alpha_{\mathfrak{p}_i}$ and $\beta_{\mathfrak{p}_i}$ satisfy $\alpha_{\mathfrak{p}_i} + \beta_{\mathfrak{p}_i} \neq 0$. Also, 2 in the extension $\mathbb{Q}[\alpha_{\mathfrak{p}_i}]$ has inertia degree 1 as needed. At these primes, the trace of E is equal to the Hecke Eigenvalues:

$$a(\mathfrak{p}_{2.1}) = -1 \qquad\qquad a(\mathfrak{p}_{3.1}) = -2$$

2. For this case, the modulus $\mathfrak{m}_F = \mathfrak{p}_{2.1}^5\mathfrak{p}_{7.1}\mathfrak{p}_{7.2}\mathfrak{p}_{17.1}$. and the ray class group is

$$\mathrm{Cl}_F(\mathfrak{m}_F) \simeq C_2 \times C_2 \times C_2 \times C_{24} \times C_{96}.$$

3. There are 31 index two subgroups of $\mathrm{Cl}_F(\mathfrak{m}_F)$. For each of these and the full group, we consider the corresponding quadratic extension L of F. We compute the modulus $\mathfrak{m}_L$ and the corresponding ray class group $\mathrm{Cl}_F(\mathfrak{m}_L)$. Now for each of these ray class groups,

we select a set of generators $\{\chi_j\}_{j=1}^n$ of cubic characters. For each of these generators, we pick a set of primes $\mathfrak{q}_j$ such that

$$\langle \log(\chi_1(\mathfrak{q}_j), ..., \log(\chi_n(\mathfrak{q}_j)\rangle_{j=1}^{n'} = (\mathbb{Z}/3\mathbb{Z})^n$$

for some $n' \in \mathbb{Z}$. In our example, we only used prime above 3 and 23. The eigenvalue for these primes are

$$a(\mathfrak{p}_{3.1}) = a(\mathfrak{p}_{3.2}) = -2, \qquad a(\mathfrak{p}_{23.1}) = 0, \qquad a(\mathfrak{p}_{23.2}) = 8.$$

Each of these eigenvalues was even as required. Thus, we proceeded to the next step.

4. For this step, we needed a basis of quadratic characters $\{\chi_i\}_{i=1}^n$ of $\mathrm{Cl}_F(\mathfrak{m}_F)$. And we also needed a set of prime ideal $\{\mathfrak{p}_i\}$ coprime to $\mathfrak{m}_F$, such that

$$\{\log(\chi_1(\mathfrak{q}_j), \log(\chi_2(\mathfrak{q}_j), \ldots, \log(\chi_n(\mathfrak{q}_j)\}_{j=1}^{2^n-1} = (\mathbb{Z}/2\mathbb{Z})^n \backslash \{0\}.$$

For our example, $n = 5$. Therefore, we needed to further compute the eigenvalue of primes below

$$\{11, 13, 5, 31, 53, 79, 89, 107, 131, 149, 157, 167, 257, 281,$$
$$361, 457, 593, 1721\}.$$

As the Hecke eigenvalues for those primes agreed with the trace of Frobenius of E, we moved on to the next step.

5. In the last step of our computation, we needed to check if the local L-factor of E and f are equal at primes dividing $2\mathfrak{n}(E)\mathfrak{n}(f)\overline{\mathfrak{n}(f)}\Delta(F)$, that is we needed to compute Hecke

eigenvalue of the following prime ideals:

$$\{\mathfrak{p}_{2.1}, \mathfrak{p}_{7.1}, \mathfrak{p}_{7.2}, \mathfrak{p}_{17.1}\}$$

On the side of the modular form, for the prime that divides the conductor, we can use the Atkin Lehner operators to compute the Hecke eigenvalues. These can be computed using matrices specified in Proposition 4.25.

For elliptic curves, we need to understand the type of reduction at the bad primes as mentioned in [5]. At primes, $\{\mathfrak{p}_{2.1}, \mathfrak{p}_{7.2}, \mathfrak{p}_{17.1}\}$ has good reduction. Therefore, we can compute the trace of Frobenius. At the prime, $\mathfrak{p}_{7.1}$, E has split multiplicative reduction thus the corresponding $a(\mathfrak{p}_{7.1})$ is 1 which agrees with the Hecke eigenvalue for the Atkin Lehner operator.

6.8 Inner Twists

For our imaginary quadratic field F, the character χ_2 from Table 6.4 is the only quadratic character. Thus, we can recognize inner twist forms by checking if $\lambda(\mathfrak{a}) = 0$ for each $\mathfrak{a} \in g$ or g^3 where g is a generator of the class group Cl_F.

In this case, there are only two Hecke eigensystems that match the homological eigenform f because we have the alignment

$$f = f \otimes \chi_2 \text{ and } f \otimes \chi_1 = f \otimes \chi_3.$$

We can observe this reduction in homology by looking at oldspace contributions of twisted systems. That is, if f is a homological eigenform of level $\mathfrak{n}$ with an inner twist by χ_2, then the oldspace contribution of f to level $\mathfrak{n}\mathfrak{a}$ will be the number of divisors $\mathfrak{d}$ of $\mathfrak{a}$ where $[\mathfrak{d}] \in \{0, c^2\}$. This means if $\mathfrak{a}$ has divisors such that $[\mathfrak{d}] \in \{c, c^3\}$, then the oldspace contribution of f is fewer than the number of total divisors. In particular, if $\mathfrak{a}$ is a prime such that $\chi_2(\mathfrak{a}) = -1$, then we observe a one-dimensional homological eigenform in level $\mathfrak{an}$ that is now an old homological eigenform. An explicit example is given below.

Example 6.10. At level $\mathfrak{p}_{2.1}^6$, there are 6 one-dimensional homological eigenforms. Two homological eigenforms have zero eigenvalues for the principal Hecke operators of the form $T_{\mathfrak{pq}}$, where $\mathfrak{p} \in c$ and $\mathfrak{q} \in c^3$. The two Hecke eigensystems that match these two homological eigenforms are given Table 6.28, where $i = \sqrt{-1}$.

Table 6.28. Hecke eigensystems at level $\mathfrak{p}_{2.1}^6$

	$\mathfrak{p}_{3.1}$	$\mathfrak{p}_{3.2}$	$\mathfrak{p}_{7.1}$	$\mathfrak{p}_{7.2}$	$\mathfrak{p}_{11.1}$	$\mathfrak{p}_{11.2}$	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{17.1}$	$\mathfrak{p}_{25.1}$	$\mathfrak{p}_{53.1}$	$\mathfrak{p}_{53.2}$	$\mathfrak{p}_{89.1}$	$\mathfrak{p}_{89.2}$
class	3	1	3	1	1	3	2	2	0	0	0	0	2	2
f_1	0	0	0	0	0	0	6	6	2	-6	14	14	10	10
f_2	0	0	0	0	0	0	$-6i$	$6i$	-2	-6	-14	-14	$-10i$	$10i$

The eigenvalues computed for both these systems at all primes in generator classes are 0. The contribution of two eigenform to the oldspace at level $\mathfrak{p}_{2.1}^6\mathfrak{p}_{3.1}$ is 1-dimensional in homology. Thus, this means for the unramified characters of the class group in Table 6.4, we have

$$f_1 = f_1 \otimes \chi_2 \text{ and } f_1 \otimes \chi_1 = f_1 \otimes \chi_3;$$

$$f_2 = f_2 \otimes \chi_2 \text{ and } f_2 \otimes \chi_1 = f_2 \otimes \chi_3.$$

This also means the newspace dimension of the Bianchi modular forms is 20 dimensional instead of being 4 times the dimension of homology.

We also observe that the first system matched the base change to F of the rational elliptic curve

$$E : y^2 = x^3 - x$$

with LMFDB label 32.a3, which has CM by $\mathbb{Q}(\sqrt{-1})$.

The inner twist of f_1 explains the existence of a CM elliptic curve.

A twist of the second Hecke eigensystem f_2 by a character ψ of the ray class group $\mathrm{Cl}_F(\mathfrak{m})$, where $\mathfrak{m} = \mathfrak{p}_{2.1}^2$, matches the base change of classical newform 32.2.a.a. This newform also has CM by $\mathbb{Q}(\sqrt{-1})$. Thus, $a_p = 0$ for each inert prime in $\mathbb{Q}(\sqrt{-1})$. These rational primes align with primes in the generator classes of Cl_F.

We identify a candidate for ψ by looking at principal primes and primes in the class c^2. This candidate character on the ray class group $\mathrm{Cl}_F(\mathfrak{m}) = \langle g \rangle \simeq C_8$ is defined by $\psi(g) = \zeta_8$, where ζ_8 is an 8^{th} root of unity. We summarize these eigenvalues in Table 6.29.

Table 6.29. Eigenvalues of the inner twist Hecke eigensystems at level $\mathfrak{p}_{2.1}^6\mathfrak{p}_{3.1}$

	$\mathfrak{p}_{13.2}$	$\mathfrak{p}_{13.1}$	$\mathfrak{p}_{53.2}$	$\mathfrak{p}_{53.1}$	$\mathfrak{p}_{89.1}$	$\mathfrak{p}_{89.2}$
class in $\mathrm{Cl}_F^{\mathfrak{m}}$	$2g$	$6g$	$4g$	$4g$	$6g$	$2g$
class in Cl_F	c^2	c^2	0	0	c^2	c^2
$\psi(g)$	ζ_8^2	ζ_8^6	-1	-1	ζ_8^6	ζ_8^2
f_2	$-6\zeta_8^2$	$6\zeta_8^2$	-14	-14	$-10\zeta_8^2$	$10\zeta_8^2$
$f_2 \otimes \psi(g)$	6	6	14	14	-10	-10

Thus, the system f_2 potentially is a twisted base change system.

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