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In our everyday experiences, we have developed a concept of dimension, neatly expressed as integers, i.e. a point, line, square and cube as 0-, 1-, 2-, and 3-dimensional, respectively. Less intuitive are dimensions of sets such as the Koch Curve and Cantor Set. The formal definition of toplogical dimension in a metric space conforms to our intuitive concept of dimension, but it is inadequate to describe the dimension of fractals. The purpose of this thesis is to develop notions of fractal dimension and in particular, to explore Hausdorff dimension with regard to self-similar tiles in detail. Methods of calculation of Hausdorff dimension will be discussed.

ON THE HAUSDORFF DIMENSION OF THE BOUNDARY OF A SELF-SIMILAR TILE

by

Anne E. Tally

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Approved by

Paul F. Duvall

Committee Chair

To Andrew,
my muse.

And David,
my hero.

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

Committee Chair________________________________

Committee Members________________________________

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Date of Acceptance by Committee

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CHAPTER I
INTRODUCTION

The notion of dimension has been the subject of much study in recent history. There have been many attempts to determine what it means, leading to multiple definitions, including topological, fractal, and box-counting, to name only a few. Additionally, there seems to be no consensus on the definition of "fractal dimension". This thesis will discuss three definitions of fractal dimension, explain methods for their calculation with examples from current literature, and apply these methods to other fractals. In order to understand dimension, it is necessary first to define the structure of the sets we will be discussing.

Definition 1 *A* metric space $(\mathbf{X},d)$ *is a set* $\mathbf{X}$*, together with a real-valued function,* $d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$*, which measures the distance between pairs of points,* x *and* y *in* $\mathbf{X}$*, satisfying the following axioms:*

1. $d(x, x) = 0$ *for all* $x \in \mathbf{X}$*;*
2. $d(x, y) = 0$ *iff* $x = y$*;*
3. $d(x, y) = d(y, x)$ *for all* $x, y \in \mathbf{X}$*;*
4. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in \mathbf{X}$.

The familiar Euclidean metric on the space $\mathbb{R}^2$, for example, would be defined by $d(x, y) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

A fractal is generally thought of as "a rough or fragmented geometric shape that can be subdivided into parts, each of which is (at least approximately) a reduced size copy of the whole" [10], the term itself having been coined by Mandlebrot. From a geometric perspective, a fractal exhibits several characteristics, among them: its irregularity makes it difficult to describe in Euclidean terms; it has a recursive definition; and it is often *self-similar*. A set A is *self-similar* if it is the non-overlapping union (i.e. the interiors are disjoint) of arbitrarily small copies of A. One method for fractal construction that we will be using extensively is the iteration of a set of contraction mappings.

Definition 2 *In a metric space X with metric d, if f maps X into X and if there is a number $c < 1$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$, then f is said to be a* contraction mapping *of X into X.*

Definition 3 *An* iterated function system *or* IFS *consists of a complete metric space (X, d) together with a finite set of contraction mappings $F : X \to X$, where $F = \{f_1, f_2, \ldots, f_N\}$ with respective contractivity factors c_n, for $n = 1, 2, ..., N$.*

A specific IFS can be defined by the notation, $\{X; f_n, n = 1, 2, ..., N\}$. The classical Cantor set, for example, is described as the attractor of the contractivity mappings, $\{X; \frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\}$. For each IFS, there is an associated unique geometric set called the *attractor*, which is the invariant set resulting from the recursive application of its contraction mappings. We shall discuss contraction mappings and attractors further in Chapter III.

The term "fractal" itself is difficult to define. In this thesis, we shall use Mandelbrot's concise definition, i.e. that a fractal is "a set for which the Hausdorff dimension strictly exceeds the topological dimension. Every set with a noninteger [dimension] is a fractal" [10]. Thus, we will begin with an overview of topological dimension and then lead into a more detailed discussion of three versions of fractal dimension. Since most definitions yield values between the Hausdorff and box-counting dimensions, we will define these carefully and focus on these calculations and the conditions that assure their equality. We will then return to the idea of self-similarity and continue with fractal generation and the tilings of a plane. Finally, we examine methods for the calculation of the Hausdorff dimension of the boundaries of certain self-similar sets.

CHAPTER II

DIMENSION

The formal definition of topological dimension in a metric space conforms to our intuitive concept of dimension, but as will be shown, it is inadequate to describe the dimension of fractals. All definitions of dimension, however, are dependent upon the understanding of open covers and compactness.

Definition 4 *By an* open cover *of a set K in a metric space $\mathbf{X}$, we mean a collection $\{G_i\}$ of open subsets of $\mathbf{X}$ such that $K \subset \bigcup G_i$.*

We refer to the *refinement* of a cover C of K as a covering in which every set B in the refinement C' is contained in some set A in C, the idea being that the refinement is "smaller" and a more "precise" covering of K.

Definition 5 *A subset K of a metric space $\mathbf{X}$ is said to be* compact *if every open cover of K contains a* finite *subcover. That is, if $\{G_i\}$ is an open cover of K then there are finitely many indices $i_1, \ldots, i_n$ such that $K \subset G_{i_1} \cup \cdots \cup G_{i_n}$.*

Our discussion of fractals and dimension will be limited to compact subsets of $\mathbb{R}^n$, and we shall refer to this set of all non-empty, compact subsets as $\mathcal{H}(\mathbb{R}^n)$. Since we will be investigating their relationships with each other within the space, we need to define what is meant by the distance between two sets of points.

Definition 6 *Let $(\mathbf{X}, d)$ be a metric space, with $A, B \in \mathcal{H}(\mathbf{X})$. The distance from the set A to the set B is defined to be*

$$d(A,B) = max\{d(x,B) | x \in A, \text{ where } d(x,B) = min\{d(x,y)|y \in B\}\}.$$

Definition 7 *Let* $(\mathbf{X}, d)$ *be a metric space. Then the* Hausdorff distance *between points* A *and* B *in* $\mathcal{H}(\mathbf{X})$ *is defined by* $h(A, B) = max\{d(A, B), d(B, A)\}$.

Essentially, the Hausdorff distance is the maximum of the minimum distances between any two points in A and B. If ϵ is the greatest distance from any point in B to the nearest point in A, then there is an ϵ-neighborhood, A_ϵ surrounding A that includes all points in B. Similarly, for the greatest distance, δ, from any point in A to the nearest point in B, there is a δ-neighborhood around B, B_δ, that includes all points in A. Then the Hausdorff distance is the larger of the lengths, ϵ and δ. For example, the distance between a point, A, and the unit segment, B, not on the segment, must be measured as the *greatest* distance from B to any one point on A, as in the case illustrated below. On the other hand, the distance between 2 parallel segments, C and D, one unit apart is the length of the perpendicular segment between them, since this measures the minimum distance from any point on one segment to the nearest point on the other. In this way, the Hausdorff distance gives a sense of similarity between two sets.

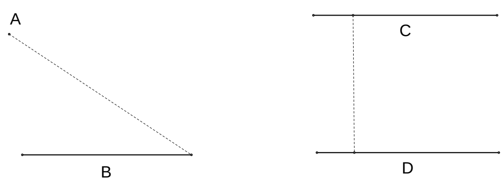


Figure 1

Definition 8 *We say a compact metric space* $\mathbf{X}$ *has* topological dimension m *if every covering* G *of* $\mathbf{X}$ *has a refinement* G' *in which every point of* $\mathbf{X}$ *occurs in at most* $m + 1$ *sets in* G' *and* m *is the smallest such integer.*

It is important to note that topological dimension is always given as an integer. Thus, it is clear that the prior definition yields the expected dimensions for a point, a line, a square, a cube, etc. By this definition, the classical Cantor set has dimension 0.

Consider the set known as the Koch Curve. Its construction can be achieved by beginning with the middle third of the unit segment, then rotating this segment twice - clockwise and counterclockwise, respectively - by $\frac{\pi}{3}$, to form two sides of an equilateral triangle. This process is repeated on each subsequently created segment (Figure 2). The topological dimension is clearly 1, and this seems consistent with our intuitive sense as it is topologically equivalent to the unit segment.

Another example, Sierpinski's Triangle, however, poses a difficulty. This set is most easily seen as an equilateral triangle reduced in size by one-half, leaving a copy in the lower left vertex of the original, then translating a copy of that one to the lower right and upper vertices of the triangle. We can conclude that its topological dimension is also 1, but we sense that this is inadequate in describing how it occupies space, especially when compared with the Koch curve.

Figure 2. Approximations to Koch Curve and Sierpinski Triangle

A more refined definition for dimension is needed in order to get a sense of a fractal's size and how it compares to other fractals. As noted earlier, there are several types of "fractal dimension", but here we develop the following for the general definition from Barnsley [3], which is referred to by Peitgen as "self-similarity dimension" [11]. Let $(\mathbf{X}, d)$ be a metric space and $A \in \mathcal{H}(\mathbf{X})$ a non-empty subset of $\mathbf{X}$. Then for $\epsilon > 0$, define $\mathcal{N}_\epsilon(A)$ to be the least number of closed balls of radius ϵ needed to cover A. Intuitively, we say that A has dimension, D if

$$\mathcal{N}_\epsilon(A) \approx C\epsilon^{-D}$$

for some constant $C > 0$. Solving for D, we see that

$$D \approx \frac{\ln \mathcal{N}_\epsilon(A) - \ln C}{\ln(1/\epsilon)}.$$

Since as $\epsilon \to 0$, $\frac{\ln C}{\ln(1/\epsilon)}$ approaches 0, and we therefore arrive at the following definition.

Definition 9 *Let $A \in \mathcal{H}(\mathbf{X})$ where $(\mathbf{X}, d)$ is a metric space. For each $\epsilon > 0$ let $\mathcal{N}_\epsilon(A)$ denote the smallest number of closed balls of radius $\epsilon > 0$ needed to cover A. If*

$$D = \lim_{\epsilon \to 0} \frac{\ln \mathcal{N}_\epsilon(A)}{\ln(1/\epsilon)}$$

exists, then D is called the fractal dimension *of A.*

By this definition then, the unit line segment retains its dimensional value of 1, so while it is a self-similar set, it is not a fractal. By contrast, for the classical Cantor set, C, the minimum number of ϵ-balls with $\epsilon = (1/3)^n$ is approximately $\mathcal{N}_\epsilon(C) = 2^n$. Then $\ln(\epsilon) = \ln(1/3^n) = -n\ln(3)$ and $\ln \mathcal{N}_\epsilon(C) = n\ln(2)$. The ratio of these two is then independent of ϵ: $-\frac{\ln \mathcal{N}(\epsilon)}{\ln(\epsilon)} = \frac{\ln(2)}{\ln(3)} \approx 0.63$. The classical Cantor set is therefore a fractal.

Applying this formula to the Koch Curve, with 4^n ϵ-balls needed when $\epsilon = (1/3)^n$, we see that it has dimension, $D = \frac{\ln(4)}{\ln(3)} \approx 1.26$. The Sierpinski Triangle then has dimension, $D = \frac{\ln(3)}{\ln(2)} \approx 1.58$, which satisfies our intuition that it is somehow "larger" in dimension than the former.

The above definition and subsequent examples rely upon visual inspection of the fractal to determine the number and size of the balls for the resulting ratio. This definition of fractal dimension is therefore useful and appropriate for comparing the "density" of self-similar sets, but for sets that are not strictly self-similar, we need a modified approach. If we begin by covering a set $A \subset \mathbb{R}^m$ with a grid, and increasingly refine the mesh of box side length ϵ and note which "boxes" contain members of the set, we can plot a log-log diagram of this data and calculate the slope, which yields the *box-counting* dimension (sometimes referred to as the *Minkowski* dimension) of the set.

Definition 10 . *Define $\mathcal{N}_\epsilon(A)$ to be the number of boxes in the mesh covering A with side length ϵ. Beginning with a grid of four boxes, and then refining these with each iteration to be of side length $(1/\epsilon)^n$, we say that if*

$$D = \lim_{n \to \infty} \frac{\ln \mathcal{N}_\epsilon(A)}{-\ln(\epsilon^n)}$$

exists, then A has box-counting *dimension D.*

Since the number of boxes that intersect A indicate how irregular the set is at each value of ϵ, the dimension of the set is an indicator of how rapidly the irregularities increase as $\epsilon \to 0$. For sets that are strictly self-similar, this dimension generally equals the self-similarity dimension, as in the case of the Sierpinski Triangle. It is easily seen that the number of boxes, beginning with $N_1 = 3, N_2 = 9, N_3 = 27...$, results in $N_n = 3^n$ for n=1, 2, 3... so that the dimension given by the above limit is once again $D = \frac{\ln(3^n)}{\ln(2^n)} \approx 1.58$.

The advantage of the box-counting dimension is its relative ease of use for experimental data. One notable example is in the recent work of Dr. Richard Taylor, who has applied this technique to analyze the drip paintings of Jackson Pollock [14]. Using high-resolution photographic prints of seventeen paintings, Dr. Taylor, et al., have demonstrated the fractal nature of Pollock's work of this style, and having compared their results to others from art known not be Pollock's, they have hypothesized that his paintings alone exhibit fractal characteristics. Their method is being used as one tool in the authentication process of recently discovered pieces.

The box-counting dimension is then useful in practical applications when we wish to calculate the fractal dimensions of physically measurable sets. A more complex version of this definition is the Hausdorff dimension, which we can use to compare the "sizes" of sets with identical fractal dimension.

We begin with the diameter of a set, which is the maximum distance between any two points in the set:

Definition 11 $\text{diam}(A) = \sup\{d(x, y)|x, y \in A\}$

Next we sum the diameters raised to the power p for a p-dimensional set. Then we define the measure $\mu \geq 0$ of a set as the maximum of the minima of all the diameters in A, which is to optimize the cover of A with sets of diameter less than ϵ.

Definition 12 *Let* $0 < \epsilon < \infty$ *and* $0 \leq p < \infty$. *Let* U *be a sequence of subsets*

$A_i \subset A$ such that $A = \bigcup_{i=1}^{\infty} A_i$. Define

$$\mu(A,p,\epsilon) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^p | \{A_i\} \in U, \textit{and } \operatorname{diam}(A_i) \leq \epsilon \textit{ for i=1,2,3,...}\right\}.$$

Then the Hausdorff p-dimensional measure of A is given by

$$\mu(A,p) = \sup\{\mu(A,p,\epsilon)|\epsilon \geq 0, \text{ for all } p \in [0,\infty), \mu(A,p,\epsilon) \in [0,\infty)\}.$$

This number takes on one of three values for any given set: zero, a positive real number, or infinity. For $m \in \mathbb{R}$, the *Hausdorff Dimension,* $D_H \in [0,m]$, represents the unique, finite real number below which the p-dimensional measure is zero and above which it is infinity.

Taking d to be the diameter of the Sierpinski Triangle, for example, the set can be covered with 3^n triangles of diameter $d \cdot \frac{1}{2^n}$.

As $n \to \infty$ for $p = 1$, $d \cdot \frac{3^n}{2^{np}} \to \infty$, so that $\mu(A,1) = \infty$.

For $p = 2$, $d \cdot \frac{3^n}{2^{np}} = d \cdot \frac{3^n}{4^n} \to 0$, so that $\mu(A,2) = 0$.

The p-dimensional measure of A will therefore be finite and non-zero when $3^n = 2^{np}$, or when $p = \frac{\ln 3}{\ln 2} \approx 1.58$. Thus, the Hausdorff Dimension in this case agrees with the similarity and box-counting dimensions. In fact, for most sets, this will be the case.

Consider, however, the set $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}...\}$. Because the set A consists of discrete values, the box-counting method is not ideal for calculating its dimension. Let $\epsilon > 0$. Since the distance from a point in the set, $1/n$, to its nearest neighbor is $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, we choose ϵ so that $\frac{1}{n(n-1)} > \epsilon \geq \frac{1}{n(n+1)}$. Then there are at least n "boxes", or in this case, segments, of length ϵ required to cover A, and thus

$$\frac{\ln \mathcal{N}_\epsilon(A)}{-\ln \epsilon} \geq \frac{\ln n}{\ln n(n-1)}.$$

Letting $\epsilon \to 0$, we get $\dim_B(A) \geq 1/2$.

From the other direction, if we choose n so that $\frac{1}{n(n-1)} > \epsilon \geq \frac{1}{n(n+1)}$, then we can cover $[0, \frac{1}{n}]$ with $n+1$ segments of length ϵ, and the remaining $n-1$ points of A with $n-1$ segments. Thus, $\mathcal{N}_\epsilon(A) = 2n$, and

$$\frac{\ln \mathcal{N}_\epsilon(A)}{-\ln \epsilon} \leq \frac{\ln 2n}{\ln n(n-1)},$$

giving $\dim_B(A) \leq 1/2$.

Thus, the box-counting dimension is $1/2$, whereas the Hausdorff Dimension is 0, indicating that a non-integer box-counting dimension does not always guarantee fractal properties of a set.

As noted earlier, the box-counting dimension is more feasible for use with experimental data. But even though the Hausdorff dimension may be more difficult to calculate, it better lends itself for use in theoretical situations where we wish to compare the "sizes" of two sets with the same fractional dimension. It is important to understand, however, that the two calculations are related in the following way.

If we cover a set A with $\mathcal{N}_d(A)$ sets of diameter d, then by Definition 12, $\mu(A, p, \epsilon) \leq \mathcal{N}_p(A) \cdot d^p$. If $\mu(A, p, \epsilon) > 1$, then for d arbitrarily small,

$$\ln(\mathcal{N}_p(A)) + p \cdot \ln(d) > 0.$$

Therefore,

$$p \leq \lim_{d \to 0} \frac{\ln(\mathcal{N}_d(A))}{-\ln(d)}$$

so that $D_H(A) \leq D_B(A)$. Note that

$$\mathcal{N}_p(A) \cdot d^p = \inf \left\{ \sum_{i=1}^{\infty} d^p | \{A_i\} \text{ is a } p\text{-cover of} A \right\}$$

and

$$D_H(A) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{diam}(A_i)^p | \{A_i\} \text{ is a } p\text{-cover of } A\right\}.$$

In the latter, the $\mathrm{diam}(A_i)^p$ are different weights assigned to the A_i set covers. In the former, the mesh diameters, d^p, are of the same weight for each set cover, making the calculation of the box-counting dimension a simpler process in general [7].

To summarize, we state the following theorem.

Theorem 1 *Let m be a positive integer and let A be a subset of the metric space $(\mathbb{R}^m,$ Euclidean$)$. Let $D_B(A)$ denote the box-counting dimension of A and let $D_H(A)$ denote the Hausdorff dimension of A. Then $0 \leq D_H(A) \leq D_B(A) \leq m$.*

See [7] for a formal proof of this theorem. Because the Hausdorff dimension is somewhat non-intuitive in nature and is difficult to apply, we shall use this theorem and others to be discussed that facilitate the calculation of this number.

We now further explore self-similarity and how fractals can be generated.

CHAPTER III
CONTRACTION MAPPINGS AND SELF-SIMILARITY

At the heart of fractal behavior is the contraction mapping, as defined in the Introduction. Recall from our discussion of dimension that if X is a complete metric space, with the Hausdorff metric, denoted h, we refer to the set of all non-empty compact subsets of (X, h) as $\mathcal{H}(X)$. Now let $f_1, \ldots, f_N$ be N contractions on $\mathcal{H}(X)$ and $A \subset \mathcal{H}(X)$. Then

$$F(A) = f_1(A) \cup \cdots \cup f_N(A) \text{ is a contraction mapping on } A. \text{ [9]}$$

We illustrate the basic idea of Hutchinson's proof with the example of two mappings. Begin with two contraction mappings, f_1 and f_2, and their respective contractivity factors, $c_1, c_2 < 1$. We need to show that for any two compact sets, A and B, the Hausdorff distance between $F(A) = f_1(A) \cup f_2(A)$ and $F(B) = f_1(B) \cup f_2(B)$, denoted $h(F(A), F(B))$ is less than $h(A, B)$ reduced by the greater of the two contractivity factors.

If $d = h(A, B)$, then any point in B is in the d-neighborhood of A, A_d, and any point in A is in the d-neighborhood of B, B_d. Thus $B \subset A_d$ and $A \subset B_d$. Let $c = max\{c_1, c_2\}$. Because f_1 and f_2 are contraction mappings,

$$f_1(B) \subset f_1(A_d) \subset (c \cdot d)\text{-neighborhood of } f_1(A), \text{ and}$$

$$f_2(B) \subset f_2(A_d) \subset (c \cdot d)\text{-neighborhood of } f_2(A).$$

Then $f_1(B), f_2(B) \subset (c \cdot d)$-neighborhood of $f_1(A) \cup f_2(A)$.

Similarly, $f_1(A), f_2(A) \subset (c \cdot d)$-neighborhood of $f_1(B) \cup f_2(B)$.

Therefore, $h(F(A), F(B)) < c \cdot h(A, B)$.

So for a subset A of $\mathcal{H}(X)$, the attractor, K, is the unique fixed point of such contraction mappings, F, given by

$$K = \lim_{n\to\infty} F^n(A).$$

Note that we often use the term "point" to refer to this set of points, which we call the *attractor* of the IFS. We also refer to the set, K, as invariant; i.e. $F(K) = K$. As the Banach Contraction Mapping Principle states:

Theorem 2 *If X is a complete metric space, and if F is a contraction of X into X, then there exists one and only one $x \in X$ such that $F(x) = x$. Moreover, x is given by $x = \lim_{n\to\infty} F^n(y)$ for any point $y \in X$.*

In addition to proving the existence and uniqueness of the attractor, an important result of this principle is that it reveals the rate of convergence toward the invariant set. Because we know the distance from the initial set, A_0, to A_1, we can estimate the distance from A_n to A_∞. Let f be a contraction mapping with contractivity factor, c, and $a_0, a_1, \ldots$ be a sequence in a complete metric space, X, such that $a_{n+1} = f(a_n)$. Then,

$$d(f(a_0), a_\infty) = d(f(a_0), f(a_\infty)) \le c \cdot d(a_0, a_\infty)$$

By the triangle inequality,

$$d(a_0, a_\infty) \le d(a_0, f(a_0)) + d(f(a_0), a_\infty) \le d(a_0, f(a_0)) + c \cdot d(a_0, a_\infty)$$

Then,

$$d(a_0, a_\infty) \le \frac{d(a_0, f(a_0))}{1-c} \text{ and } d(a_n, a_\infty) \le \frac{d(a_{n-1}, a_{n+1})}{1-c}$$

for all $n = 1, 2, \ldots$ Also,

$$d(a_n, a_{n+1}) \leq c \cdot d(a_{n-1}, a_n) \leq c^2 \cdot d(a_{n-2}, a_{n-1}) \leq \cdots \leq c^n \cdot d(a_0, a_1)$$

Thus,

$$d(a_n, a_\infty) \leq c^n \cdot d(a_0, a_1)/(1 - c). \text{ [11]}$$

Many self-similar fractals share yet another property that allow the calculation of their dimensions.

Definition 13 *If there exists an open set $V \subset \mathbb{R}^n$ such that*

$$\bigcup_{i=1}^{m} f_i(V) \subseteq V \textit{ and } f_i(V) \cap f_j(V) = \emptyset \textit{ for } i \neq j,$$

then the f_i are said to satisfy the open set condition (OSC).

In [2], Bandt, et al., explain the OSC analytically. They define a "potential neighbor set," $h(A)$, to be the mapping, $f_i^{-1} f_j(A)$, which transforms the small sets, $f_i(A)$ and $f_j(A)$, into A and $h(A)$. Formally,

Definition 14 *Let $S^* = \bigcup_{n \geq 1} S^n$. Define* neighbor maps *in A as* $\mathcal{N} = \{h(A) = f_i^{-1} f_j | i, j \in S^*, i_1 \neq j_1\}$.

Rewriting the second condition of the OSC gives $V \cap f_i^{-1} f_j(V) = \emptyset$. The existence of such an open set V dictates that $h(A)$ cannot be near the identity map, $i(A)$.

Letting the norm for a mapping f on $\mathbb{R}^n$ be the usual $||f|| = sup_{|x| \leq 1} |f(x)|$, they give the algebraic formulation for the OSC as follows: *There is a constant $\kappa > 0$ such that $||h - i(A)|| > \kappa$ for all neighbor maps h.*

Thus, the open set condition ensures minimal overlap of the sets A_i. Additionally, it implies that there exists a positive integer n such that at most n pieces

A_j of size $\geq \epsilon$ can intersect the ϵ-neighborhood of a piece A_i of diameter ϵ [2]. From a geometric perspective, the condition holds that the sets A_i and A_j cannot be arbitrarily close to each other relative to their size. Or, in the terminology of Barnsley, a fractal that satisfies the open set condition is "just-touching", rather than overlapping. The OSC is a precondition for theorems that will allow us to find the dimension of certain self-similar sets.

As discussed previously, we can construct fractals using iterated function systems, which consist of contraction mappings that generate self-similar sets. An IFS, therefore, is comprised of a scaling factor plus linear transformations which may include translations, reflections and/or rotations. A mapping of this type is called a *similitude*. Using Barnsley's notation, we can express the contractions with rotations or reflections in a matrix and any linear translations in a column vector. Thus, an affine similitude in the Euclidean plane can be written as

$$f(x) = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

Combining multiple mappings defined in this way, we can create many familiar fractals. The IFS for the Sierpinski Triangle, for example, is given by $\{f_1, f_2, f_3\}$ where

$$\begin{aligned} f_1 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f_2 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ f_3 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \end{aligned}$$

This has the effect of contracting a triangle by one-half, leaving a reduced copy in the lower left vertex of the original, then translating a copy of that one to the lower right and upper vertices of the triangle.

Theorem 2 also provides us with a way to derive experimentally an IFS from the image of its attractor. By determining the number of copies in the initial iteration and measuring their distances and directions from a point in the original image, we can deduce a set of mappings that describes this attractor. By inspecting the image of the Koch Curve, for example, it can be deduced that the center one-third of a segment is replaced by two sides of an equilateral triangle, which is created by rotating each side clockwise and counterclockise, respectively, by $\frac{\pi}{3}$. This process is repeated on each subsequently created segment, which can be described by the following IFS:

$$\begin{aligned}
f_1 &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
f_2 &= \begin{pmatrix} \frac{1}{3}\cos\frac{\pi}{3} & -\frac{1}{3}\sin\frac{\pi}{3} \\ \frac{1}{3}\sin\frac{\pi}{3} & \frac{1}{3}\cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \\
f_3 &= \begin{pmatrix} \frac{1}{3}\cos\frac{\pi}{3} & \frac{1}{3}\sin\frac{\pi}{3} \\ -\frac{1}{3}\sin\frac{\pi}{3} & \frac{1}{3}\cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3}\cos\frac{\pi}{3} \end{pmatrix} \\
f_4 &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}
\end{aligned}$$

We now quote the following theorem of Hutchinson that allows the calculation of the fractal dimension of a non-overlapping fractal using its IFS of similitudes.

Theorem 3 *Let A be the attractor for a non-overlapping IFS of similitudes given by $f_1, f_2, \cdots, f_N$ where c_n is the contractivity factor for each f_n. Then A has dimension $D(A)$ given by the unique solution of $\sum_{n=1}^{N} |c_n|^{D(A)} = 1$.*

Barnsley's outline of a proof covers the case of the totally disconnected IFS [3]. The proof of the general theorem can be found in [9].

Outline of Proof. Let c_i be the non-zero scaling factor for each similitude $f_i \neq 0$, $i \in 1, 2, ..., N$, and let $\epsilon > 0$. We note that each similitude maps closed balls onto closed balls, i.e.

$$f_i(B(x, \epsilon)) = B(f_i(x), |c_i|\epsilon).$$

Since $c_i \neq 0$, f_i is invertible,

$$f_i^{-1}(B(x, \epsilon)) = B(f_i^{-1}(x), |c_i|^{-1}\epsilon).$$

Thus,

$$\mathcal{N}(A, \epsilon) = \mathcal{N}(f_i(A), |c_i|\epsilon). \ (1)$$

Note also that the attractor of the IFS is the disjoint union of the sets obtained by applying the similitudes, i.e. $A = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A)$. Then for ϵ arbitrarily small, some point $x \in X$, and some $i \in 1, 2, ..., N$, we have $B(x, \epsilon) \cap f_i(A) \neq \emptyset$ and $B(x, \epsilon) \cap f_j(A) = \emptyset$ for all $j \in 1, 2, ..., N, j \neq i$. Therefore,

$$\mathcal{N}(A, \epsilon) = \mathcal{N}(f_1(A), \epsilon) + \mathcal{N}(f_2(A), \epsilon) + \cdots + \mathcal{N}(f_N(A, \epsilon)).$$

Substituting (1) into this last equation, we see that

$$\mathcal{N}(A, \epsilon) = \mathcal{N}(A, |c_1|^{-1}\epsilon) + \mathcal{N}(A, |c_2|^{-1}\epsilon) + \cdots + \mathcal{N}(A, |c_N|^{-1}\epsilon). \ (2)$$

Substituting the earlier definition of $\mathcal{N}(A, \epsilon) \approx C\epsilon^{-D}$ into (2),

$$C\epsilon^{-D} \approx C|c_1|^D\epsilon^{-D} + C|c_2|^D\epsilon^{-D} + \cdots + C|c_N|^D\epsilon^{-D}.$$

Therefore, $1 = |c_1|^D + |c_2|^D + \cdots + |c_N|^D = \sum_{n=1}^{N} |c_n|^{D(A)}.\square$

A set of mappings that fulfills the open set condition is enough to guarantee that the dimension of the set will be positive and is given by this formula. Moreover, self-similar sets have equal Hausdorff and box-counting dimensions, though this dimension may be less than D if the OSC does not hold.[7]

As an example, we refer to the IFS for the classical Cantor set given earlier as $\{X; \frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\}$, yielding $(1/3)^D + (1/3)^D = 1$. Then $D = \dfrac{\ln 2}{\ln 3} \approx 0.63$, which agrees with our earlier result. From the IFS for the Koch Curve above, we see that there are four contractions of $1/3$ each, giving $4(1/3)^D = 1$, and thus, $D = \dfrac{\ln 4}{\ln 3} \approx 1.26$. Additionally, the Sierpinski Triangle is described by three contractions of $1/2$ each, so that $3(1/2)^D = 1$, and its dimension is easily given by $D = \dfrac{\ln 3}{\ln 2} \approx 1.58$. Because all these sets fulfill the OSC, the calculations reflect both the box-counting and Hausdorff dimensions.

On the other hand, this theorem may also be used to compute the upper limit of an overlapping IFS such as Barnsley's Wreath [3], which is constructed from six functions with two different contraction factors and on which the OSC is not fulfilled. Its upper dimension is then calculated from the equation, $3(1/2)^D + 3(1/4)^D \leq 1$, the result of which is, therefore, $D \leq \dfrac{\ln(3+\sqrt{21})}{\ln(2)} - 1 \approx 1.92$. This higher dimension satisfies our sense that an overlapping set is somewhat "denser" than those that are "just-touching".

In [8], Falconer also studied subsets of the union of self-similar-sets and called them *sub-self-similar*; i.e. the closed set E is sub-self-similar under contracting similitudes $F = \{f_1, f_2, \ldots, f_n\}$ if $E \subset \displaystyle\bigcup_{i=1}^{n} f_i(E)$. Our interest in sub-self-similar sets is in the example of the *boundary* of a self-similar set, notated by ∂E, by which we mean that any open set containing points in ∂E also necessarily contains points in E and the complement of E. Falconer shows that ∂E is sub-self-similar by noting that if $x \in \partial E$, then $x \in E$, and therefore, $x \in f_i(E)$, for some i. Since

$x \in \partial E$, every neighborhood of x contains points in the complement of $f_i(E)$, and thus $x \in \partial F(E) = f_i(\partial E)$. In other words, ∂E is sub-self-similar since $\mathrm{int}(E)$ is mapped into itself by the open mappings, F. We shall use this principle in the remaining chapters for our investigations into the boundary dimensions of some specific fractals.

CHAPTER IV
THE BOUNDARY OF A SELF-SIMILAR TILE

By our earlier definition, we can think of a self-similar set as being somewhat "self-contained" in that the similarity is repeated at increasingly smaller scales within the set. By translating copies of these sets in a particular way, we can cover the Euclidean plane without overlaps in a *tiling*.

Definition 15 *A* tiling *is a collection, τ, of non-empty compact subsets, called* tiles*, of* $\mathbb{R}^2$ *such that:*

1. *each tile is the closure of its interior;*
2. *the union of the tiles in τ is $\mathbb{R}^2$;*
3. *distinct tiles are non-overlapping.*

Well-known examples of tilings can be seen in Islamic architecture and in the work of M. C. Escher. For our purposes, we define a subclass of tiles called *self-affine tiles*. Let T be the attractor of the contraction mapping, $F = \{f_1, \ldots, f_N\}$ on $\mathcal{H}(X)$. For the following definition, we use the language and notation of Vince [15].

Definition 16 *A tile, T, is* self-affine *if there is an expansive matrix A and a collection of vectors $\mathcal{D}$, called the* digit set*, such that*

$$A(T) = T + \mathcal{D} = \bigcup_{d \in \mathcal{D}} (T + d).$$

From this, we can see that the tile, T, is a self-affine set given by $T = \bigcup(A^{-1}T + A^{-1}d)$.

For purposes of clarification, we emphasize here that an IFS is *self-similar* if it consists entirely of contractive similitudes of the same magnitude and *self-affine* if it is composed of affine functions. Affine transformations may contain different contraction factors and varying directions. If the matrix A, representing the functions $f_1, ..., f_N$, for a self-affine tile is a similitude, then T is a *self-similar tile*, which is characterized by the following:

1. each f_i is a similitude with the same contraction factor $1/c$, $c > 1$;

2. T is the closure of its interior;

3. $int(f_i(T_0)) \cap int(f_j(T_0)) = \emptyset$ for any $i \neq j$.

Also note that an *expansive* matrix is one for which the moduli of its eigenvalues are all greater than one, i.e. $|\lambda_i| > 1$. With a further restriction that the digit set be composed of lattice points, i.e. those with integral coordinates, the translates become a *lattice tiling*, and $\mathcal{D}$ is a set of coset representatives of $\mathbb{Z}^d/A(Z^d)$ with $0 \in \mathcal{D}$. In this case, the determinant of the matrix A, $m = |\det(A)|$, determines the number of elements in the digit set that constitutes the tile. This can be verified by observing that in the above definition, the expression $\bigcup_{d\in\mathcal{D}}(T + d)$ enlarges the area of T by a factor of the number of digit sets, and $A(T)$ increases it by a factor of $|\det(A)|$.

A straight-forward example is the unit square, where $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, translated to the set of points, $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, will produce a checkerboard pattern that can tile the plane. This tile is an example of a digit set with integral entries. The digit set is then a complete set of coset representatives for the quotient group $\mathbb{Z}^2/A\mathbb{Z}^2$.

In general, our matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an expansive mapping, so that A^{-1} is a contraction mapping, and the functions $F = \bigcup f_i$ approximate the attractor regardless of the intial value of x_0. Note that for A as described above, $|\det(A)| = |ab - cd| = m$ gives the area of the parallelogram, P, spanned by the vectors $v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$. The digit set is formed by those vectors with integer coordinates having a vertex at the origin, and the lattice L is generated by application of the mapping, g. Bandt refers to these vectors as a *complete residue system* for A, because if g is an expansive mapping, the lattice, L, given by $L = \bigcup\{y_i + g(L) | i = 1, ..., m, \text{ where } y_i \neq 0\}$, forms a subgroup $g(L)$. [1]

All linear combinations of the column vectors of A comprise the vertices of P and a subset of L, forming a grid of parallelograms congruent to P and all with n points of L. Each point $y_i \in L$ lies within a copy of P and is an element of a coset of $L/g(L)$. Thus, the y_i's form a complete residue system for A, and each vector in the system determines the location of each tile. Different bases can generate the same lattice, but $|\det(A)|$ is uniquely determined. The following definition and theorem of Bandt summarize.

Definition 17 *A closed set, $A_1 \in \mathbb{R}^n$ with a non-empty interior is called an* m-rep tile *if there are sets $A_2, \ldots, A_m$ congruent to A_1, such that* $\text{int}(A_i) \cap \text{int}(A_j) = \emptyset$ *for $i \neq j$ and $A_1 \cup \cdots \cup A_m = g(A_1)$, where g is a similitude.*

Theorem 4 *Let g be a linear expansive map on $\mathbb{R}^n$ with integer matrix and $\{y_1, \ldots, y_m\}$ a residue system of g. Then there is a unique m-rep tile A_i such that*

$$g(A_i) = A_1 \cup \cdots \cup A_m \text{ with } A_i = y_i + A_1. \text{ [1]}$$

Recall the prior example of the square that is a digit tile. Note that in this case, $\lim_{n\to\infty} \partial T_n = T$ and that this limit is space filling. Here, we are concerned primarily with those sets for which $\lim_{n\to\infty} \partial T_n = \partial T$. This calculation is dependent upon the expansion matrix, A, and the digit set, $\mathcal{D}$.

As an example, let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, so that $m = |\det(A)| = 2$, and $\mathcal{D} = \{(0,0),(1,0)\}$. A lattice point y is determined by the unique solution (x,d) of the equation $Ax+d=y$, where $x \in \mathbb{Z}^d$ and $d \in \mathcal{D}$. The lattice is completed by repeated application of the algorithm,

$$L \leftarrow L \cup \{x \in \mathbb{Z}^d | Ax + d = y, \text{ for some } d \in \mathcal{D} \text{ and } y \in \mathcal{D} + L\},$$

until the sets are equal. For example,

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

yields the solution $(-1,1)$. Continuing in this manner, we find that

$$L = \{(0,0),(0,1),(1,0),(1,-1),(0,-1),(-1,0),(-1,1)\},$$

resulting in an approximation of the Twin Dragon [5]. This process can be shown to produce finite points based on the fact that a lattice point generated by any iteration of the mapping must be an integer that is bounded by a maximum distance from the origin that is dependent upon the contraction factor of the system.

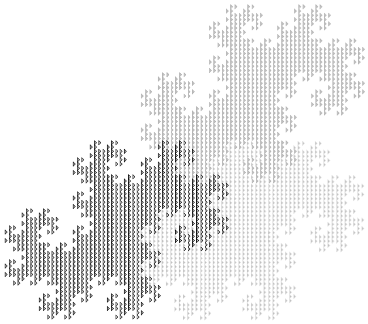


Figure 3. Lattice Translates of the Twin Dragon

Theorem 5 *Let T be a self-similar digit tile constructed from matrix A, with expansion factor c, and digit set $\mathcal{D}$, and let T_0 be the unit square with vertices at $(0,0),(1,0),(0,1)$ and $(1,1)$. If $T_n = F^n(T_0)$ are approximating tiles, then:*

1. $\lim_{n\to\infty} \partial T_n = \partial T$;
2. $\lim_{n\to\infty} \partial T_n$ *is not space-filling;*
3. $m(T) = 1$;
4. $\{T + x | x \in \mathbb{Z}^d\}$ *is a tiling of* $\mathbb{R}^d$.

Moreover, these conditions are equivalent, and if satisfied, then there is a constant a such that

$$h(\partial T, \partial T_n) < a/c^n,$$

where h denotes the Hausdorff metric.

It is this last assertion that leads to the main result of [5], so its proof is included here.

Proof. If $T_n = F^n(T_0)$, then $h(T, T_n) \leq a/c^n$ for some constant, a, by the Contraction Mapping Principle. We need to show that any $x \in \partial T_n$ is within the distance a/c^n of a point in ∂T. Then x must satisfy one of three cases:

1. If $x \in \partial T$, then clearly, the inequality holds;

2. If $x \notin T$, there exists $y \in T$ such that $h(x, y) \leq a/c^n$, as shown above. Then there is a point z on the segment xy that is also on ∂T, so that $h(x, z) \leq h(x, y) \leq a/c^n$;

3. If $x \in int(T)$, consider the possible tilings of $\mathbb{R}^n$:

$$\mathcal{T}_n = \{p + T_n | p \in \mathbb{Z}^d\}$$

$$\mathcal{T} = \{p + T | p \in \mathbb{Z}^d\}.$$

$\mathcal{D}$ is a set of coset representatives of $\mathbb{Z}^d/A(\mathbb{Z}^d)$, so it follows that $\mathcal{T}_n$ is a tiling of $\mathbb{R}^d$. Since $x \in \partial T_n$, then there is a copy $y + T_n$ in $\mathcal{T}_n$ such that $x \in \partial(y + T_n)$, but $x \notin y + T$. There is a point $z \in y + T$ such that $h(x, z) \leq a/c^n$, and thus a point $w \in \partial T$ on the segment zx such that $w \in \partial(y + T)$ and $h(w, x) \leq h(z, x) \leq a/c^n$. Similarly, $h(\partial T, \partial T_n) \leq a/c^n$.$\Box$[5]

In order to compute the dimension of the boundary of a tiling, information about the behavior of the boundary must be derived from a matrix constructed from the digit set and lattice. The entries of this *contact matrix* are determined by the number of occurrences of an element in the digit set from which a point in the lattice is generated. In other words, an element of the contact matrix is found by counting the d's of the digit set that produce each y of the lattice; i.e.,

$$C_{xy} = |\{d \in \mathcal{D} | x_d = y\}|, \text{ for } x, y \in L \backslash \{0\}.$$

In the Twindragon example, the contact matrix is, therefore,

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which the characteristic polynomial is then determined to be

$$\det(C - \lambda I) = \lambda^4(1-\lambda)^2 - 4 = (\lambda+1)(\lambda^2 - 2\lambda + 2)(\lambda^3 - \lambda^2 - 2).$$

The modulus of the largest eigenvalue of this matrix is then used in the calculation of the Hausdorff dimension of the boundary of this set, as will be shown below.

Theorem 6 *Let $T = T(A, \mathcal{D})$ be a self-similar digit tile where A has expansion factor c and the contact matrix C has largest eigenvalue λ. Under any of the well-behaved boundary conditions of the previous theorem,*

$$\dim_H(\partial T) = \frac{\ln \lambda}{\ln c}.$$

Proof. Let T_0 be the unit cube centered at (0,0) with edges parallel to the axes and let $T_n = F^n(T_0)$ be the nth approximation to the self-similar digit tile T. As above, the IFS, as given in Definition 18, is then

$$T_n = \bigcup \{T_0 + d_0 + Ad_1 + Ad_2 + \cdots + A^{n-1}d_{n-1} | d_i \in \mathcal{D}\},$$

and $T = \lim_{n\to\infty} T_n$. Thus T_n is the non-overlapping union of copies of $A^{-n}(T_0)$, each copy being a cube of edge length $1/c^n$. Under the mapping, A^n, there is a bijection between this set of cubes of T_n and the set of lattice points $\mathcal{D}_n$. For the given pair, $(A, \mathcal{D})$, let $L' = L(A, \mathcal{D})\backslash\{0\}$. For any matrix M, let $|M|$ denote the sum of the entries of M. Then $|C^n|$ denotes the number of triples (x, y, d) that are solutions to the equation, $d + x \in A^n y + \mathcal{D}_n$ where $x, y \in L'$ and $d \in \mathcal{D}$. Let B_n denote the set of $d \in \mathcal{D}_n$ such that $d + x \in A^n y + \mathcal{D}_n$ for some $x, y \in L'$, and let β_n denote the cardinality of B_n. Thus

$$\beta_n \leq |C^n| \leq (k-1)^2 \beta_n,$$

where k is the cardinality of L. Under the bijection described above, B_n also corresponds to a certain set of cubes in T_n. For simplification, we refer to this set of cubes as B_n. By Lemma 1 in [5] and straightforward induction, $\mathcal{D}+L \subset A^n L+\mathcal{D}_n$, for $n = 1, 2, \ldots$, and $\{\pm e_1, \ldots, \pm e_d\} \in L$. Let $b = b(A, \mathcal{D})$ denote the greatest Euclidean distance from the origin to any point in the neighborhood $L(A, \mathcal{D})$. The center of each cube in B_n has distance at most $(a + b)/c^n$ from a center of such a cube. Consider the following tiling of $\mathbb{R}^d$ by cubes of edge length $1/c^n$:

$$\{x + A^{-n}(T_0) | x \in A^{-n}(\mathbb{Z}^d)\}.$$

The number of such tiles of edge length $1/c^n$ within distance of $(a + b)/c^n$ of, say, the origin, is bounded by a constant h that depends only on the dimension of d, not on n. Let α_n be the smallest number of tiles of edge length $1/c^n$ whose union covers $\partial(T)$. Thus $\beta_n \leq h\alpha_n$ and $\alpha_n \leq h\beta_n$. Moreover, there are positive constants α' and β' such that

$$\alpha'|C^n| \leq \alpha_n \leq \beta'|C^n|.$$

By a standard result for non-negative matrices, we have $\lim_{n\to\infty} (|C^n|)^{1/n} = \lambda$, which implies that

$$\lim_{n\to\infty} \ln |C^n| = \ln \lambda.$$

As previously demonstrated, ∂T is a *sub-self-similar set*, and the Hausdorff dimension coincides with the box-counting dimension for ∂T. Then

$$\dim_H \partial T = \dim_B \partial T = \lim_{n\to\infty} \frac{\ln \alpha_n}{\ln c^n}.$$

Thus, this limit exists, and with the previous results,

$$\dim_H \partial T = \lim_{n\to\infty} \frac{\ln \alpha_n}{n \ln c} = \lim_{n\to\infty} \frac{\ln |C^n|}{n \ln c} = \frac{\ln \lambda}{\ln c}. \ \Box[5]$$

Applying this theorem to the earlier example of the Twindragon, the Hausdorff dimension of its boundary is shown to be $\dim_H(\partial T) \approx \frac{\ln 1.69962}{\ln \sqrt{2}} = 1.523627\ldots$

CHAPTER V
THE BOUNDARY OF THE LÉVY DRAGON

The Twin Dragon lends itself to the methods described because of the simplicity of its IFS. Compare this with the IFS for a self-similar set first studied by Lévy in 1938, now known as the Lévy Dragon. As with the Twin Dragon, we begin with a triangle, T_0, and contraction factor of root two. The first rotation remains the same, while the and another is added with a translation to the other vertex of the original triangle. Its IFS is then

$$f_1(x,y) = \left(\tfrac{x-y}{2}, \tfrac{x+y}{2}\right)$$
$$f_2(x,y) = \left(\tfrac{x+y+1}{2}, \tfrac{-x+y+1}{2}\right).$$

It is this second function adding a different rotation that complicates the calculation of its boundary dimension. Since this dragon is not a self-similar digit tile, its dimension cannot be computed using the methods previously described. One approach [4] is to use Falconer's results on sub-self-similar sets in the following manner.

Recall that for the attractor, K, of a set of contracting similitudes, $F = \{f_1, f_2, \ldots, f_N\}$, the boundary of K, ∂K, is a sub-self-similar set because $\mathrm{int}(K)$ is mapped onto itself by the open mappings, F. Thus, F satisfies the open set condition on $\mathrm{int}(K)$. In [8], Falconer went on to prove that this condition is sufficient to show that the set has positive s-dimensional Hausdorff measure, given by the unique nonnegative s satisfying $\tau(s) = 1$ where

$$\tau(s) = \lim_{k\to\infty} \left(\sum_{i \in A_k} c_i^s \right)^{1/k}.$$

Duvall and Keesling [4], use this formula to derive an equivalent one that is easier to use for computing $\dim_H \partial K$. In their notation, the infinite set of sequences generating the not-covered triangles in the boundary of K, $A_k = \{I \in \Omega_k | f_{I_k}(K) \cap \partial K \neq \emptyset\}$, where Ω_k denotes the set of sequences of length k and f_{I_k} denotes the composition of mappings $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}$. Note also that c_{I_k} is meant to be the product of the c_i's. Then for convenience, they assume that all the c_i's have the same value, c, so that the sum can be written as $|A_k|c^{ks}$, and thus

$$\tau(s) = \lim_{k\to\infty} \left(\sum_{I_k \in A_k} c_{I_k}^s \right)^{1/k} = \lim_{k\to\infty} (|A_k|c^{ks})^{1/k} = \alpha c^s.$$

The Hausdorff dimension of the boundary of the set K, $\dim_H \partial K$, is then the solution to $\alpha c^s = 1$, or

$$\dim_H \partial K = -\frac{\ln \alpha}{\ln c}.$$

In the case of the Lévy Dragon, let T_0 be the initial triangle in the dragon and $F = \{f_1, f_2\}$ be the contraction mappings. Then the attractor is defined to be $K = \lim\limits_{k\to\infty} F^k(T_0)$. Since f_1 and f_2 are open mappings of $\mathrm{int}(K) \to \mathrm{int}(K)$, ∂K is a sub-self-similar set. Thus, F satisfies the open set condition on $\mathrm{int}(K)$.

Part of the difficulty in computing the dimension of the boundary is in determining a value for $|A_k|$. To address this issue, the authors instead define $B_k = \{I_k \in \Omega_k | f_{i_k}(T_0) \cap \partial F^k(T_0) \neq \emptyset\}$, the set of finite sequences, F^k, of functions on the initial set, T_0. It is necessary to examine the dragon's construction in order to calculate its dimension using the sets, B_k.

Any triangle intersecting the boundary of the initial triangle, T_0, is referred to as a *neighboring* triangle, T', so that $F^1(T_0)$ has 14 neighbors, and subsequently, each $T \in \mathcal{T}_k$, the triangulation of $\mathbb{R}^n$ at the kth iteration, also has 14 neighbors. Number the neighboring triangles 1-15, denoted $N(T)[1] - N(T)[15]$, as follows: 1, the initial triangle, T; 2, the triangle that shares its hypotenuse with that of T; and 3-15 clockwise around T from there. For any $T \in \mathcal{T}_k$, the neighborhood of T, $N(T) = \{T' \in \mathcal{T}_k | T \cap T' \neq \emptyset\}$. Then let $\rho(T) = \{\rho_1, \rho_2, \ldots, \rho_{15}\}$ be defined as the *neighborhood type* where

$$\rho_i = \begin{cases} 1, & \text{if } N(T)[i] \in F^k(T_0); \\ 0, & \text{otherwise.} \end{cases}$$

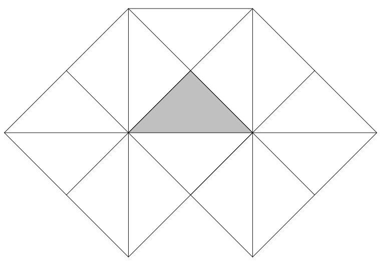
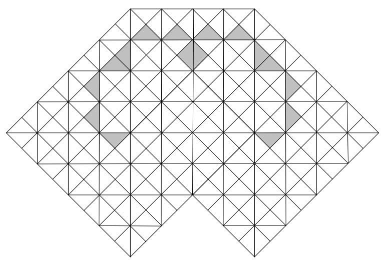


Figure 4. T_0 in N_0



$F^4(T_0)$ in N_4

If the ρ_i's are all ones, the triangle, $T \in \mathcal{T}_k$, is said to be *covered*, meaning that $T \in \mathcal{T}_k$ and all its 14 neighbors are contained in $F^k(T_0)$. Not surprisingly, it takes many iterations before covered triangles become apparent. Through computer experimentation, the authors discovered the first occurrence at the 14th iteration, and then only 8 of the 2^{14} triangles were covered. It can be seen, however, that in the subsequent iteration, any T that is covered yields 2 covered triangles, and therefore, $T \subset F^{k+m}(T_0)$ for all $m > 0$, so that $T \subset \text{int } K$. To summarize this, along with other properties of the neighborhoods:

1. Let $N_0 = N(T_0)$. Then $F(N_0) \subset N_0$, and thus, $K \subset N_0$.

2. If $T \in \mathcal{T}_k$ and $T' \in \mathcal{T}_{k+1}$, then $N(T') \subset N(T)$.

3. If $T \in \mathcal{T}_k$ is covered and $T' \subset T$ for some $T' \in \mathcal{T}_{k+1}$, then T' is covered.

4. If $T \in \mathcal{T}_k$ is covered, then $T \subset \mathrm{int}K$.

5. If v is a vertex of any triangle $f_{I_k}(T_0) \subset F^k(T_0)$, it is also a vertex of some triangle $f_{I_{k+1}}(T_0) \subset F^{k+1}$, and therefore, $v \in K$.

As previously noted, finding a value for $|A_k|$ is more difficult than computing one for $|B_k|$. The authors show that $B_k \subset A_k$ as follows, and they use this to prove the theorem stating the formula needed to calculate the boundary dimension.

Since B_k is the set of sequences that generate non-covered triangles in the kth iteration, there is a triangle T' in the neighborhood of some $T = f_{I_k}(T_0)$ that is not in $F^k(T_0)$. So this T' is in ${F_a}^k(T_a)$ for some neighborhood in that iteration. For a vertex, v, in both T and T', v is in the intersection of the two sets at that level and is therefore in the boundary of K. Since v must also be the result of functions applied to a vertex of T_0, it is in the boundary of K, and therefore, $I_k \in A_k$. It follows that $B_k \subset A_k$ and $|B_k| \leq |A_k|$.

The authors use this to establish inequality in the one direction, then find the box-counting dimension and invoke Falconer's theorem on the equality of the box-counting and Hausdorff dimensions of self-similar sets for the other.

Recall that $\dim_H \partial K = -\dfrac{\ln \alpha}{\ln c}$ where $\alpha = \lim\limits_{k \to \infty} |A_k|^{1/k}$.

Theorem 7 *Let* $\beta = \lim\limits_{k \to \infty} |B_k|^{1/k}$.

$$\dim_H \partial K = \frac{\ln \beta}{\ln \sqrt{2}}.$$

Proof. Since $\beta \leq \alpha$, the earlier result gives

$$\frac{\ln \beta}{\ln \sqrt{2}} \leq \dim_H \partial K.$$

For any T in the kth triangulation, $\mathcal{T}_k$, of N_0, if its neighborhood type is the zeros vector or the all one's vector, then T is either not in K, or is in the interior of K, respectively, and therefore, contains no points in ∂K. Thus, the boundary of K is contained in the set of triangles whose neighborhood types are not constant.

Then $\partial K \subset \bigcup\{N(T)|N(T) \text{ contains some } f_{I_k}(T_0), I_k \in B_k\}$. For each k, there is a maximum of $15|B_k|$ such $N(T)$, so ∂K is covered by a maximum of $15|B_k|$ sets of diameter $\leq 3(\sqrt{2})^{-k}$. Therefore,

$$\lim_{k\to\infty} -\frac{\ln(15|B_k|)}{\ln(3(\sqrt{2})^{-k}} = \frac{\ln\beta}{\ln\sqrt{2}}.\square[4]$$

Finding a value for β to use in the computation of the boundary dimension involves more careful analysis of the neighborhood structures. Begin by indexing all the 2^{15} possible combinations of neighborhood types and then defining $V(k)$, a vector of that length with each element representing the count of each type of neighborhood at the kth iteration; for example, $V(1)_1$ contains the number of triangles in $\mathcal{T}_1$ of the first neighborhood type in the index. Next, construct a matrix, M so that each element counts the number of one particular neighborhood type that resulted from iterating another; i.e. $M_{i,j}$ is the number of j-neighborhood types created from the iteration of an i-neighborhood type. Consequently, the nature of M is that it can be constructed from the initial triangle and its iteration into N_0 and thereafter has the property that $V(k+1) = V(k) \cdot M$. Define a column vector J, where $J_i = 1$ for i odd and less than 32767, and 0 otherwise. Then, the number of not-covered triangles, $|B_k| = V(0) \cdot M^k \cdot J$, can be calculated from the eigenvalues of M. The largest of these will give us the fewest number of ϵ-balls whose union will cover ∂K.

Clearly, the number of combinations of neighborhoods and their descendants creates a matrix that is too cumbersome to manage with the technology that was available to the authors at the time. Fortunately, they are able to determine that there are actually far fewer neighborhood types that occur. Additionally, through computational methods, they ascertain that for iterations beyond $F^{19}(T_0)$, there are no new combinations. The neighborhoods are said to be stable after this, with 752 combinations. Thus, $V(k)$ and M are redefined accordingly to represent only these combinations and thus contain 752 and 752×752 elements, respectively. Then $|B_k| = V(k) \cdot M \cdot J$, and so $|B_{19+k}| = V(19) \cdot M^k \cdot J$. By identifying triangles that do not add to the boundary, the authors reorganize the matrix based on information about these neighborhood types. Thus, M can be rewritten in block form as

$$M = \begin{pmatrix} P & Q & R \\ 0 & C & L \\ 0 & 0 & I \end{pmatrix},$$

where P is a permutation matrix, I is the 2×2 identity matrix, and C is a 734×734 matrix. Sorting the elements of the vectors V and J similarly, we get

$$|B_{19+k}| = V(k) \cdot \begin{pmatrix} P & Q \\ 0 & C \end{pmatrix}^k \cdot J.$$

C has an eigenvalue equal to its spectral radius, and $\lim_{k\to\infty} \frac{1}{\lambda^k} C^k = D$, where D is a positive matrix. Since, in this case, $\lambda > 1$, then

$$\lim_{k\to\infty} \frac{1}{\lambda^k} \begin{pmatrix} P & Q \\ 0 & C \end{pmatrix}^k = \lim_{k\to\infty} \begin{pmatrix} \frac{1}{\lambda^k} P^k & \frac{1}{\lambda^k} P^{k-1} Q \\ 0 & \frac{1}{\lambda^k} C^k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

This gives

$$\lim_{k\to\infty} \frac{|B_{k+19}|}{\lambda^{k+19}} = \frac{1}{\lambda^{k+19}} V(19) \cdot D \cdot J = q > 0.$$

Therefore,

$$\beta = \lim_{m\to\infty} |B_k|^{1/m} = \lim_{m\to\infty} \lambda \cdot q^{1/m} = \lambda, \text{ and}$$

$$\dim_H \partial K = \frac{\ln \lambda}{\ln \sqrt{2}}, \text{ by Theorem 7. [4]}$$

The value of λ, which indicates the number of ϵ-balls of size $1/\sqrt{2}^k$ needed to cover the pieces of the boundary, is calculated to be $\lambda \approx 1.954776399$, and consequently, the boundary dimension of the Lévy Dragon is computed to be $\dim_H \partial K \approx 1.934007183$.

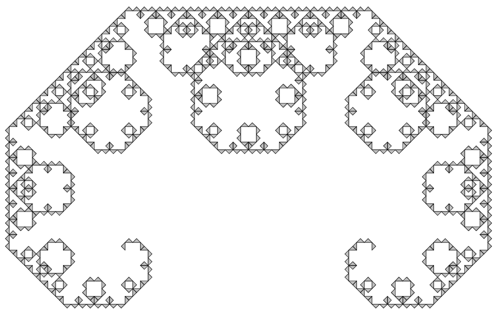


Figure 5. Approximation to the Levy Dragon

In [13], a method for computing this dimension is developed without relying on satisfaction of the open set condition. Instead, the authors, Strichartz and Wang, devise a method of constructing a contact matrix by using the translates to investigate pieces of the boundary. By examining the intersections of the attractor with its translates, as determined by the expansion matrix, A, they derive a contact matrix in much the same manner as described in the previous chapter. The computation of the boundary dimension is further simplified by elimination of the matrix entries that result from reflections of coordinates in the lattice.

Because of the computer limitations at the time [4] was written, this technique provided a far more streamlined approach to finding the boundary dimension of the Lévy Dragon. Strichartz and Wang's result is an 11×11 matrix with a 9th degree characteristic polynomial that has been shown to be a factor of the one derived by Duvall and Keesling, and thus, the spectral radius and resulting boundary dimension values agree.

It is interesting to note that, in this case, the development of sophisticated mathematical software greatly reduces the need for the application of more complex mathematical theory. Had Duvall and Keesling had access to current methods, perhaps Strichartz and Wang would not have striven to develop their alternate approach to the calculation of the Hausdorff dimension of the Lévy Dragon. Yet it has historically been these types of endeavors that have led to more elegant solutions to complex problems. On the other hand, considering the number of proofs devised for centuries for a formula as well-established as the Pythagorean Theorem, it seems likely that the quest for elegance will not be hampered by technological developments.

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