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There is ample literature documenting that, for many decades, high school students view algebra as difficult and do not demonstrate understanding of algebraic concepts. Algebraic reasoning in elementary school aims at meaningfully introducing algebra to elementary school students in preparation for higher-level mathematics. While there is research on elementary school students' algebraic reasoning, there is a scarcity of research on how elementary school teachers implement algebraic reasoning curriculum and how their practices support algebraic reasoning. The purpose of this study therefore was to discover practices that promote algebraic reasoning in elementary classrooms by studying elementary school teachers' practices and algebraic reasoning that the practices co-constructed. Specifically, the questions that guided the study included (a) what were the teachers' routines of practice, and (b) in what ways did the routines of practice support algebraic reasoning.

I sampled On Track Learn Math project and worked with six teachers to explore their routines of practice and students' algebraic reasoning. As a participant observer, I analyzed video data of the classroom activities, memos, field notes, students' written transcripts and interview data using constructivist grounded theory approach and descriptive statistics. Member checking, data triangulation, and data coding by multiple raters ensured consistency and trustworthiness of the results.

Descriptive analysis of students' written generalizations showed that about 74% of the generalizations were explicit and about 55% of the generalizations included names

of variables indicating that students were learning how to reason algebraically. Data analysis also revealed five routines of practice. These routines are; (a) maintaining open-endedness of the tasks, (b) nurturing co-construction of ideas, (c) fostering understanding of variable, (d) creating a context for mathematical connections and (e) promoting understanding of generalizations. Teachers maintained open-endedness by giving minimal instructions when launching the tasks and providing students with workspaces. They nurtured co-construction of ideas by creating opportunities for students to collaborate, fostering collaboration, and balancing the support of discourse and content. They fostered understanding of variable as a changing quantity and as a relationship. Teachers created a context for mathematical connections between On Track tasks and students' everyday experiences, between student strategies, between different tasks, between On Track tasks and other curriculum ideas, and between different representations. Teachers promoted understanding of generalizations by encouraging students to justify their conjectures, to apply and evaluate peers' generalizations among other practices. These practices were dependent and informed each other.

CONCEPTUALIZING ROUTINES OF PRACTICE THAT SUPPORT

ALGEBRAIC REASONING IN ELEMENTARY SCHOOLS:

A CONSTRUCTIVIST GROUNDED THEORY

by

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Approved by

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Committee Co-Chair

To my mum and dad

Dorothy Catherine Ligomeka and Rex Chimombo Keara

To my husband Davie and children

Tamanda and Tadala Catherine

APPROVAL PAGE

This dissertation has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I

INTRODUCTION

Statement of the Problem

Algebra is considered difficult and a source of failure for many students. Even high performing students find algebra instruction very frustrating. For example, an accelerated seventh grade student wrote, “Algebra is quite hard, and although very educational it is very frustrating ninety percent of the time. It means hours of instruction that you don’t even come close to understanding” (House, 1988, p. 1). This view of algebra has persisted since the introduction of algebra into the United States (US) school curriculum in 1820s (Kilpatrick & Izsak, 2008; Radford & Puig, 2007). Quantitative studies have also shown that algebra has remained difficult to most students over the decades. For example, National Assessment of Educational Progress (2008) showed that from 1978 to 2008, number of 17-year-old students in US schools that demonstrated understanding of algebra skills has never been above 8%. At least two perspectives explained students’ difficulties with algebra.

One perspective attributed students’ difficulties with algebra to the time and the ways of introducing it in schools. With this perspective, most students found algebra difficult because of introducing it late and abruptly in high schools (Schifter, Bastable, Russel, Seyferth, & Riddle, 2008). Another perspective explains that, algebra is difficult because of viewing it and teaching it as meaningless manipulation of abstract systems of symbols (Arcavi, 2008; Saul, 2008).

As such, researchers (e.g., Kaput, 1999; Smith, 2008; Stump, 2011) recommended fostering algebraic reasoning in elementary schools to ease transition into formal algebra classes. Similarly, national curriculum organizations restructured mathematics curricula to focus on algebraic reasoning especially in elementary schools. Examples are Australian Education Council (1994), the Department for Education and Skills (2001) of Great Britain, and National Council of Teachers of Mathematics ([NCTM], 2000) and the Common Core State Standards Initiative (2010) in the United States. Algebraic reasoning is described as making mathematical generalizations (Blanton & Kaput, 2005; Kaput, 1999; Mason, 2008). Generalizations are a description of systematic variations or relationships in pattern-finding activities (Lins & Kaput, 2004; Mason, 2008).

Including algebraic reasoning in the curriculum is very important but is just one aspect of supporting students' algebraic reasoning. It does not ascertain that classroom practices will support algebraic reasoning. For instance, mathematical tasks aimed at promoting algebraic reasoning might simply be taught as arithmetic exercises if teachers are not well versed with ways to promote algebraic reasoning (Earnest & Balti, 2008). The actual development of algebraic reasoning depends on students' access to instructional practices that nurture such development (Blanton & Kaput, 2011; Kieran, 2011). Therefore studying productive pedagogy of algebraic reasoning is at the core of understanding how teachers support students' algebraic reasoning.

Past research shows the need for research on instructional practices that support algebraic reasoning. Researchers who study students' algebraic reasoning report that, although children are capable of making and justifying generalizations about mathematical patterns, there are some common challenges they encounter. Elementary school students generally express less sophisticated generalizations that use term-to-term change of one variable but struggle to make more sophisticated generalizations that relate independent and dependent variables (Carraher, Martinez, & Schliemann, 2008; Cooper & Warren, 2011; Lannin, Barker, & Townsend, 2006b; Moss & McNab, 2011). According to Lannin et al. (2006b), traditional teaching practices that only nurture computational fluency contribute to students' difficulties in mathematical reasoning. As such, the present focus on algebraic reasoning in the school curriculum poses challenges to mathematics educators to identify instructional practices that support algebraic thinking. Furthermore, Blanton and Kaput (2008) expressed another challenge regarding how "elementary teachers who have been schooled in a way of doing mathematics defined largely by the memorization of facts and procedures emerge from the constraints of practice that this creates . . ." (p. 361) and transition to creating classroom contexts that promote students' development of algebraic reasoning. These issues show the need for this study.

To inform teaching of algebraic reasoning, research has focused on potential instructional tasks and teaching practices of teacher educators. Beige (2011), Stump (2011), Robichaux and Rodrigue (2011) and Russel, Schifter, and Bastable (2011) wrote that patterning tasks richly support algebraic reasoning. Ellis (2011) described her teaching practices that promoted six middle school students' generalizations. Similarly, Warren (2008) wrote about her practices that supported second grade students' generalizations. Mayansky (2007) documented the teaching practices of Dr. Robert B. Davis, a distinguished mathematics education professor, which supported middle school students' generalizations of solutions to algebra problems. With such a focus in literature, more remains to be learned about teaching practices of elementary school classroom teachers that promote algebraic reasoning. Therefore, further research is necessary (Bastable & Schifter, 2008; Kieran, 2011).

Purpose and Research Questions

The primary purpose of this constructivist grounded theory study is to discover teaching practices that support algebraic thinking by studying practices of elementary classroom teachers. A constructivist grounded theory is a research methodology that aims at developing theories grounded in the data (Charmaz, 2011). Franke, Kazemi, and Battey's (2007) recommend that the core of research on teaching should be routines of practice and how they support student understanding. This dual perspective informs this study. Routines of practice are a set of practices that are regularly at the core of classroom activities intended to support understanding of different mathematical domains. The secondary purpose of this study is to understand the meanings teachers attach to their routines of practice. Meanings are teachers' description and rationale for their practice. As such, the central research question is what are the routines of practice that support algebraic reasoning in elementary school mathematics classrooms? The following are sub questions

1. What are the routines of On Track classroom practices?
2. What meanings do On Track teachers attach to their routines of practice?
3. How are the routines of practice related?
4. In what ways do On Track routines of practice appear to support student algebraic thinking?

On Track is a research project that aims at developing students' mathematical reasoning through after school enrichment lessons. On Track research group adapted Lannin et al.'s (2006b) classification of generalizations and classifies algebraic reasoning into different seven levels. Lannin et al. classify generalizations as recursive or explicit. Recursive generalizations describe the term-to-term change of a variable. For example, students might describe variation in the consecutive values of the dependent variable in an input/output table. A recursive rule might also describe changes observed when adding a unit to a geometric representation in pattern finding activities. A defining feature of recursive rules is that, the value of the $(n-1)^{th}$ term needs to be known to find the value of the n^{th} term. An explicit rule describes a general relationship between the input values and their corresponding output values. Explicit generalizations can be used to find the nth term without necessarily knowing the $(n-1)^{th}$ value. As such, explicit rules can be more powerful than recursive rules.

Based on Lannin et al.'s recursive and explicit classification of generalizations, On Track research team developed levels of generalizations represented in Table 1. From these levels of generalizations, this study's operational definition of supporting algebraic reasoning is nurturing students' tendency to express correct explicit generalizations and supporting students' progress towards expressing correct explicit generalizations.

Table 1

Levels of Generalizations

Level	Names of Generalizations	Description
1	No generalization	A student did not express any generalization
2	Incorrect recursive	A recursive generalization expressed does not work for one or more terms.
3	Transitional recursive	A correct recursive generalization that does not mention the name of a variable for which the generalization works.
4	Correct recursive	A correct recursive generalization that mentions the name of a variable for which the generalization works.
5	Incorrect explicit	An explicit generalization that does not work for one or more terms.
6	Transitional explicit	A correct explicit generalization that does not mention the name of a variable for which the generalization works.
7	Correct explicit	A correct explicit generalization that mentions the name of the variables.

Significance of the Study

The significance of studying teaching practices that support algebraic reasoning inseparably connects to the importance of students' fluency in algebraic reasoning and formal algebra classes. Making and justifying generalizations is a tool for developing children's mathematical proficiency because it promotes conceptual understanding and procedural fluency (Blanton et al., 2007; Carraher, Martinez, et al., 2008; Kaput, 1999).

Tasks that require students to generalize may demand an examination of underlying mathematical structures and their relationship thereby promoting conceptual understanding. Additionally, generalizations that relate independent and dependent variables, and the global context of the mathematical task may facilitate solving for the n^{th} term of that task. Such generalizations may facilitate solving other related mathematical tasks.

Additionally, Lee, Ng, Bull, Pen, and Ho's (2011) quantitative study with 151 students provided empirical evidence that proficiency in generalizing patterns predicts proficiency in algebra. Others (Stump, 2011) have also argued that, proficiency in generalizing patterns provides an entry to precalculus and calculus classes. Moreover, there is a strong correlation (r =7.16) between algebra 1 end of course scores and performance in Scholastic Aptitude Test Mathematics, which is an entrance exam to higher-level education (Michael, Berenson, & Store, 2009). Therefore, identifying instructional practices that support students' algebraic thinking is highly significant to students' current and future success.

This study is significant to reform policies. For example, it is significant to NCTM's (2000) algebra-for-all policy that regards knowledge of algebra as a civil right. Results of this study will inform practices that allow students' access to the study of algebra and their future success. This study is also significant to the research body on algebraic reasoning and teaching practices by exploring an area that currently remains largely unexplored.

CHAPTER II

LITERATURE REVIEW

Overview of the Chapter

In the previous chapter, I argued for the need to do research on teaching that supports development of algebraic reasoning. I stated that, historically, most students conceive algebra as difficult and that teaching algebraic reasoning is a challenge. In this chapter, I review literature that gives a historical background of algebra and school algebra and discuss the current views on algebraic thinking. I also review research that reports children's capacity for algebraic reasoning and studies on teaching practices that have been found to support students' reasoning. I conclude the reviewed literature with what previous studies have focused on and the gaps in the literature as they relate to my research questions.

Historical Background of Algebra and Algebra in School Curriculum

Rationale for Doing a Historical Background for Algebra and School Algebra

History of algebra may inform teaching that supports students' understanding (Radford, 1996). Bednarz, Kieran, and Lee (2008) explain that historical analysis of algebra and school algebra enables an understanding of students' current difficulties as they relate to teaching approaches. It gives an overview of goals for and conceptions of algebra in the school curriculum and therefore the teaching approaches that may have aligned with such goals and conceptions. It may also reveal students' capacities for algebraic reasoning that were associated with such teaching approaches. In overall, it puts the current practice of teaching and learning algebraic reasoning in an informative context by situating teaching practices in their intellectual heritage (Good, 2010).

History of Development of Algebraic Thought

Sfard (1995) argues that accounts of history of development of algebraic thought may differ because of how different authors define algebra. Defining algebra as a "science of generalized computations" (p. 18), Sfard characterizes the development of algebraic thought into three stages. These stages are rhetoric and syncopated algebra, algebra as a science of universal computations, and algebra as a science of abstract structures. These stages show an increase in complexity of the computational processes. However, seeking general ways for solving families of mathematical problems remained a constant characteristic of algebra.

The first stage in the history of algebraic thought was syncopated and rhetoric algebra which was most evident until the sixteenth century. In this era, algebraic notations were not evident. Rhetoric algebra verbally expresses computation processes. Syncopated algebra used symbols (not algebraic notations). For example, Greek geometric algebra used symbols. Squares represented area and cubes represented volume. On one hand, algebraic rules were proved using sequences of geometric transformations. On the other hand, algebra was useful in solving geometric problems. This was possible through analysis of mathematical relations that lead to equations (Charbonneau, 1996; Radford, 1996). Hence, equality was of prime importance in early algebraic thought. Representing algebra using geometry was limiting in that, it did not encompass addition of powers, or representing powers greater than three. However, representing algebraic thought using symbols or geometry helped in solving complex mathematical problems and laid a foundation for the concept of variable.

After centuries of using rhetoric and syncopated algebra in which the symbols represented unknowns, algebraic thought progressed into a science of universal computations. At this stage, the idea of symbols as ‘given knowns’ was first introduced by François Viete in 1504-1603. Symbolism for mathematical operations and relations followed. Using symbols for mathematical operations and symbols for the known eased making generalizations that algebraists were seeking. It also helped solving families of equations and using parameters. For example, using a letter as a given known pushes for functional thinking that considers an entire set of numbers and not just simply a specific entity. This view represents algebra as generalized arithmetic and a variable as a generalized number. The notions of variable changed by the mid-nineteenth century to an understanding that a variable is a thing on its own that has no meaning outside the given context. Conceiving mathematical symbols abstractly was possible, for instance, the idea that minus times minus equals plus was conceived. Thus, use of algebraic notations for the given knowns enabled a transition from dealing with concrete mathematics only to more abstract mathematics like analytical geometry and calculus.

Sfard (1995) describes the last stage of the history of development of algebraic thought as a science of abstract structures. With the foundations laid with science as universal computations, consistent axioms were proofs for existence of abstract entities. Algebraic thought did not necessarily have to connect to the physical world but had to satisfy logical analysis. As the passion for abstractness grew, newer branches of algebra developed. Group theory, theory of matrices and invariants are a few instances of algebraic thought developed at this stage.

This brief historical overview of algebraic thought shows that algebra goes beyond both geometry and arithmetic to being a superior tool for solving mathematical problems (Rojano, 1996). Algebra was used to solve geometric problems (Charbonneau, 1996; Radford, 1996). Algebra was also used to solve arithmetic problems and other theoretical problems beyond arithmetic. Rojano noted that since algebraic thought was for solving mathematical problems, symbolic algebra was developed as mathematical language that facilitates problem solving. History of development of algebraic thought shows that expressing algebraic thoughts using rhetorical language and symbols preceded understanding of abstraction. It shows that it took a long time of struggle with mathematical ideas for algebraists to express generality abstractly.

Historical Background for Algebra in the US School Curriculum

Historically, supporting schoolchildren's algebraic thinking was not in a hierarchical order resembling the three stages of history of algebraic thought. This could be because of the purposes for and the conceptions of algebra in the curriculum, which inextricably relate to teaching approaches (Usiskin, 1988).

Kilpatrick and Izsak (2008) wrote that algebra was introduced into US high schools in 1827. At the time of its introduction, the goal was to serve the needs of the workforce. During this time, algebra in the curriculum was conceived as generalized arithmetic. This view persisted into the twentieth century. In the twentieth century, the purpose for algebra in the curriculum changed to academic preparation, as it became a required course for college intending students. Châteauneuf (1929) wrote that the textbooks published until 1928 emphasized skills in factoring, manipulating algebraic expressions, and other fundamental operations. On the other hand, there was no emphasis on functional thinking in the textbooks. Much less than 1% of exercises in the textbooks required functional thinking. Teaching and learning of algebra was characterized as solving meaningless puzzles unrelated to real life experiences, which emphasized on abstract and meaningless manipulation of equations and complex radicals (Kilpatrick & Izsak, 2008). With this approach, Kilpatrick and Izsak report that many high school students failed algebra (by 1910), and opted not study it (by 1950, only 25% of high school students were enrolled in algebra).

During the period of mid 1950s to 1970s, pedagogical problems included challenges in addressing students' struggles to see logic behind algebraic theorems and teachers' struggles to convince students of the importance to prove algebraic theorems. In addition, although instruction placed emphasis on manipulation of symbols, students had difficulties developing meanings for the algebraic symbols. In addition to the stated problems, research between 1970 and 1980 showed children's lack of an understanding of the meaning of the equal sign and difficulties with functional thinking. Abrupt introduction of algebra in high schools was the blame for these difficulties. Teaching of algebra as an isolated mathematics discipline was another factor cited for students' difficulties (Kilpatrick & Izsak, 2008).

In response to the difficulties children had with algebra, NCTM in its 1988 yearbook, *The Ideas of Algebra, K-12* (Coxford & Shulte, 1988) focused on research ideas to enhance algebraic thinking from kindergarten to twelfth grade and make algebra accessible to all students. In 1989 NCTM's *Curriculum and Evaluation Standards for School Mathematics* incorporated algebraic reasoning from grade 5 through 12 in *Principles and Standards for School Mathematics*. Algebra was no longer reserved for only high school college-intending students; it was advocated that all students study algebra before graduating from high schools (Chazan, 2008). With this advocate, the question of when to introduce algebra in school arose. NCTM (2000) and the Common Core State Standards for Mathematics (2010) incorporated algebraic reasoning in prekindergarten through grade 12 standards. Kaput (2008) discussed strands of algebraic reasoning and the current (2000) NCTM standards show current conceptions of algebraic reasoning as will be discussed in the following section.

Current Perspectives on Algebraic Reasoning in the US Curriculum

The three strands of algebraic reasoning outlined by Kaput (2008) describe the current conceptions of algebra in the US school curriculum. The first strand considers early algebra as generalized arithmetic and quantitative reasoning. That is, it considers the inherent algebra in arithmetic by generalizing the arithmetic operations and generalizations of number properties. For example, to teach generalized arithmetic and quantitative reasoning, mathematical tasks may aim at making explicit the general operation properties of zero, or make generalization about summation of odd numbers.

The second strand defines early algebra as expressing generalizations about functional relationships. This translates to expressing generalizations as a description of systematic variations in sets of data or relationships across domains of data sets. Mathematical activities addressing this strand of algebraic reasoning may include generalizations about patterns generated geometrically, patterns observed from graphs or input-output tables. They may also include various kinds of change, linearity, and rate.

The third strand is an application of cluster of modeling languages. Algebraic reasoning as a cluster of modeling languages involves using unknowns, variables, or parameters to express generalizations about mathematical problems. Mathematical tasks within this strand employ syntax of algebra. Examples of mathematical tasks addressing this strand include arithmetic problems that require solving through equation with unknowns or expressing generalizations using variables.

Common to the three strands of algebraic reasoning is the idea that algebra seeks to generalize mathematical facts and relationships. The current standards for mathematics teaching (e.g., *Principles and Standards for School Mathematics* [NCTM, 2000] and the *Core Curriculum State Standards for Mathematics*, 2010) portray this view too. Table 2 shows these algebraic reasoning strands as advocated in the current standards. This table shows the conceptions of algebra in the K-12 curriculum and the activities for grades three through five to support these conceptions. The current perspective on algebra broadens the narrow view of algebra that has persisted in the school curriculum. It aligns very well with Sfard's (1995) definition of algebra in history which was "any kind of mathematical endeavor concerned with generalized computational processes, whatever the tools used to convey this generality” (p. 18).

Table 2

Algebra Standards for Grades 3 through 5

Goals	Objectives
Understand and generalize mathematical structures	Make and apply generalizations about patterns in arithmetic and properties of operations.
Understand patterns and generalization about patterns	Generate, explain, analyze patterns, and make generalizations.
Use mathematical models to represent and understand generalizations	Represent a variable as an unknown quantity, and express mathematical relationships using equations. Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.
Understand relations and functions	Make generalizations about how a change in one variable relates to a change in a second variable and analyze how corresponding terms from two variables relate.

Summary of Historical Development of Algebra and Algebra in the School Curriculum

Words and symbols, which did not resemble the current algebraic notations, expressed early algebraic reasoning. Algebraic reasoning increased in sophistication after several years to solving family of equations that were related to the physical world and then later to becoming more abstract. School algebra did not necessarily follow the historical stages of algebra itself. Although the purpose for and conceptions of algebra in the curriculum changed, students for ages were abruptly introduced to algebra as a science of abstract structures. This approach saw most students either failing or deciding not study algebra. To address this problem, NCTM advocated that all students study algebra from elementary schools. Currently, algebra perspective in elementary school is about generalization of patterns and arithmetic operations.

Children’s Algebraic Reasoning

Rationale for Studying Children’s Algebraic Reasoning

Focusing on students’ thinking can be effective in supporting students’ algebraic thinking (Maher, 2008; Mueller, 2009). Reviewing literature on students’ algebraic reasoning may reveal students’ approaches. These may inform pedagogical practices that are more likely to support students’ development of algebraic reasoning.

Children’s Capacity for Generalizing Mathematical Patterns

According to Goldenberg and Shteingold (2008) and Mason (2008), introducing early algebra in elementary schools is embedded in the belief that children have natural abilities to see patterns in nature and make generalizations. Generalizations that children make about patterns in nature are not necessarily mathematical. However, such natural abilities are a foundation for students to develop fluency in expressing and justifying mathematical generalizations.

It is well documented that children are capable of successfully engaging in algebraic reasoning (Carraher, Martinez, et al., 2008; Dougherty, 2008; Lannin, 2003; Warren & Cooper, 2008). Tierney and Monk (2008) observed and interviewed students aged 8 to 10 working on mathematical tasks that required making generalizations about change over time. Qualitative data analysis showed that the students were able to notice patterns in number sequences and make generalizations about those patterns and showed an understanding of symbolic representations. Schifter, Russel, and Bastable (2009) reported second and third graders' generalizations about number operations. In this study, students were able to make generalizations that may translate in to the form; "For any numbers x, y, and z, if $xy = z$, then $(2x)y = 2z$" (p. 234) and "For any integers x and y, $2x$, $2y$, and $2(x + y)$ are even numbers, and $2x + 2y = 2(x + y)$" (p. 233).

Warren and Cooper (2008) also wrote that elementary school students in their study were able to make generalizations about repeating patterns and growing geometric patterns. Similarly, Moss, Beaty, Barkin and Shilolo (2008) stated that fourth graders successfully generalized geometric and numeric patterns independent of teachers' help. Young students engaged in expressing generalizations about combinatorial tasks showed similar algebraic activities. Maher and Muter (2010), Tarlow and Uptegrove (2010), and Muter and Uptegrove (2010) found that students who participated in a 13-year longitudinal study were able to make generalizations about tasks and across tasks on combinatorics.

Research on students' abilities also revealed children's difficulties in making explicit generalizations. Lannin's (2005) study with sixth graders showed students focus on recursive patterns in algebraic tasks hindered their chances of explicitly generalizing. Carraher, Schlieman, and Schwartz (2008) and Carraher, Martinez, et al. (2008) found similar results in a third-grade class working on functional relationship tasks in which some students focused on the relationship of the output values without referring to the input values. This approach leads to recursive generalization (Lannin, Barker, & Townsend, 2006a). In these authors' observation, the difficulties in making explicit generalization are not due to the reason that students are not capable of seeing patterns in pattern finding tasks, it is just that the patterns children see are those do not help them make explicit generalization.

Other researchers explained that converting a visual pattern to a table of values increased the processing load and was associated with finding relationships along the sequence instead of between the pairs of numbers or variables. For example, Richardson, Berenson, and Staley (2009) found that students who focused on numeric relationships in the input-output tables did not successfully generalize or justify relationships in an algebraic pattern task. On the other hand, students who shifted from using the numeric tables to focusing on the geometric nature of the tasks were relatively much more successful in their explicit generalizations.

Contrary to these results, Warren and Cooper (2008) reported that young students (average age of 8 ½) are capable of not only relating two data sets but also of making abstract generalizations from such data. These results were from a teaching experiment with 45 students. Martinez and Brizuela (2006) reported a case study with a third-grade student who successfully expressed generalizations about relationship of numbers in input-output tables. Becker and Rivera (2009) found similar results in a three-year longitudinal study with middle school students. In this study, practice and use of multiple representations including input-output tables increased students' tendency to make explicit generalizations. Becker and Rivera (2009) explained that, much like Richardson et al. (2009) although students can make generalizations through correspondence reasoning in input-output tables, they are more successful when generalizing and justifying with geometric representations.

From this discussion, there is ample evidence in the literature that shows students' ability to make generalizations about algebraic tasks. Students make recursive generalizations with relatively more ease than explicit generalizations. While some studies report that it is a challenge for students to make generalizations about data in input-output tables, others report that students can reason with data in input-output tables. However, generalizing about geometric tasks is relatively easier than about input-output tables. It is important to note that, students' ability to express recursive generalizations is important as some tasks in the school curriculum may require recursive generalizations. Recursive generalizations are also important as they lay a foundation for students' understanding and expression of generalizations that relate how change in one variable relates to the other or how values in one domain correspond to the co-domain (Bezuzska & Kenney, 2008).

Summary of Children's Algebraic Reasoning

Literature shows that students are capable of making generalizations and justifying mathematical generalizations. However, making explicit generalizations is a challenge for many elementary students. Similarly, most students use authoritarian and empirical justification schemes as opposed to analytic schemes, which use deductive reasoning. Therefore, Jones (2000) argue that a "key issue for mathematics education is how children can be supported in shifting from 'because it looks right' or 'because it works in these cases' to convincing arguments which work in general" (p. 55).

Pedagogy of Algebraic Reasoning

Rationale for Reviewing Literature on Pedagogy of Algebraic Reasoning

Reviewing literature on pedagogy of algebraic reasoning helps in understanding where the field currently is and how research can progress it forward. It gives insight to what practices may or not support students reasoning in different contexts. I reviewed literature on teaching algebraic reasoning with an orientation that, teaching algebraic reasoning is situated in mathematics teaching practices. Therefore, whenever necessary, I also reviewed literature on general teaching practices of mathematical reasoning. The literature reviewed was selected with an understanding that teaching and learning happens in an interaction system in which one interacts with oneself, with others and with the mathematical tasks. The literature showed that the following pedagogical practices may influence children’s development of algebraic reasoning.

Using Tasks that Support Generalizations

Several studies focused on the type of tasks that foster students’ generalizations. Robichaux and Rodrigue (2011) and Beigie (2011) found that geometric patterns support students making and expressing algebraic generalizations of middle school students. In another study (Moss & McNab, 2011), second grade students successfully reasoned algebraically with a sequence of patterning activities that started with geometric patterns, numerical patterns (function machines) and then an integration of geometrical and numerical pattern. Moss and McNab argue that, when working on geometric patterns second grade students tended to see the relationship between independent and dependent variables while as when working on the function machines, students tended to identify recursive rules rather than explicit rules.

Francisco and Maher (2005) observed mathematical reasoning of 80 students in first grade to college using video data collected over a period of three to eight years. Interviews and questionnaires were used to collect data on experiences that students thought were relevant to their mathematical reasoning. Use of isomorphic tasks was one of the conditions that supported students’ expression and justification of generalizations. Students continued to use, with modifications, the reasoning that was afforded with isomorphic tasks in elementary grades as a basis for their reasoning in college- level concepts. In another study with seventh-grade students, Steele (2007) purposefully chose and administered isomorphic tasks. Students successfully solved the algebraic tasks. Steele recommended that teachers need to encourage students to relate algebraic tasks as a way to support their reasoning.

Pre-service teachers have also shown the same gains from isomorphic tasks. In a teaching experiment, Richardson et al. (2009) studied the growth of pre-service teachers’ algebraic reasoning when given isomorphic tasks over a period of 3 weeks. The students were asked to make generalizations on the perimeter of a chain of n n-gon tables. In the second and third week, the students made generalizations based on their understanding of the first week’s tasks. As stated by the authors, the isomorphic nature of the tasks was a factor in students’ growth of understanding. Thus implementing isomorphic tasks was a pedagogical practice that supported algebraic reasoning.

Supporting Collaboration

Many studies have shown that creating collaborative learning environments is a practice that supports algebraic thinking. Ellis (2011) studied her practice in teaching experiment with six middle school students to identify practices that promoted students generalizations. She identified classroom collaboration where students shared and justified their ideas in small and whole group discussions as productive. Carraher, Schlieman, et al. (2008) observed similar occurrences in second to fourth grade students working on several generalization tasks. In Carraher, Schlieman, et al.'s study, sharing of ideas nurtured in depth reasoning by pushing students to reflect on the generalizations and providing opportunity for productive mathematical argumentation. Richardson et al. (2009) observed that when pre-service teachers were working on pattern finding activities in dyads, students who were having difficulties in completing their tasks would ask for ideas from peers either in their dyads or across groups. Interaction between and among students played an important role in enhancing students' ability to generalize and justify their rules. Rittle-Johnson and Star's (2007) quantitative study with 70 seventh-grade students had findings similar to these qualitative studies. They discovered that, an experimental group of students that compared same and alternative solution methods had greater gains in their procedural and conceptual knowledge of solving equations. From this, one can deduce that teachers' support of collaboration supports algebraic reasoning.

These studies show that engagement in collaborative learning enhances students' understanding of mathematical concepts. For this type of collaboration to be possible, there is a need to reflect on classroom practices and interventions that allow students to learn from each other. In Martino and Maher (1999) ten-year longitudinal study, teachers would sometimes group students with contradictory answers to explain and justify their answers to each other. Teachers should pay attention to students' perceptions of their cognitive abilities. Artzt and Armour-Thomas (1997) explored metacognitive behaviors of 27 students while working on problem solving tasks. The study consisted of six small groups that were randomly sampled from three classes. The groups were diverse in terms of abilities. In addition to finding out that mixing students of different abilities is more productive, they also report that, students' perceptions of their contribution and their peers' contribution played an important role in facilitating productive interactions. Students who perceived themselves as mathematically strong, and that the other students cannot contribute much to their learning either tended to dominate the group discussions or did not pay attention to other students' reasoning. On the other hand, students who perceived themselves as mathematically strong, but still believed the other students' reasoning would be beneficial tended to listen and be open to other students reasoning.

Questioning

Warren and Cooper (2008) used a more explicit approach of questioning to elicit generalization about mathematical patterns in a teaching experiment with five elementary classrooms. Working with data from input-output tables, teachers used questions to explicitly point to data sets students needed to consider in their search for patterns. For example, students were asked to look for patterns down the table (i.e. relate either input values only or output values only). Students were also asked to look for patterns across the table (i.e. relate input and output values). In this study, students successfully expressed both recursive and explicit generalizations. Moss and McNab (2011) were also explicit in their questioning regarding what they wanted students to focus on as they worked on patterning tasks. In their teaching experiment with six schools, they asked second graders to observe what was changing and what was static. Students successfully made generalizations in this study.

Martino and Maher (1999) observed that when students were working in groups making generalizations about combinatorial tasks, they did not tend to question each other to justify their solutions but they sometimes asked each other low-level questions. Similarly, students did not tend to seek justifications to their answers without teachers probing them to do so. Thus, teachers need to ask questions that elicit justifications. Examples of these types of questions as adapted from Martino and Maher (1999) are *how do you know your answer is right?* Or *how can you convince others that your answer is right?* Questions that facilitate generalizations include *what if* questions that put students in positions to relate the patterns they notice to a hypothetical related context. The other types of questions Maher and Martino discussed are those that probe students to identify relationships between isomorphic tasks, and between their reasoning and other students' reasoning. These questions may include *do you see any similarities*?

Herbal-Eisenmann and Breyfogle (2005) draws our attention to the patterns of teachers' questioning. They argue that with some patterns of questioning, teachers might end up funneling students' responses into a predetermined path. As a result, students might solve the problems as the teacher would, a situation that can hinder students' development of algebraic reasoning. Similarly, Towers (2002) recommended that teachers should allow students to struggle with mathematical tasks before intervening through questioning that aims at helping the students to see the answer in question.

Promoting Sense Making

Kaput (1999) recommends that teaching algebraic reasoning should have emphasis on sense making and conceptual understanding. Conceptual explanations are those that give reference to the context of the problem and not rely on the computational orientations only (Clark, Moore, & Clarke, 2008). Lannin et al. (2006a) explain that teachers need to encourage students to link their conjectures to the general context of the tasks. Cobb, Stephan, McClain, and Gravemeijer (2001), and Clark et al. (2008) stated that encouraging students to relate their mathematical argumentation to the tasks' context supports their understanding more as opposed to simply relating their mathematical argumentation to algorithms.

Summary of Pedagogical Practices that Support Reasoning

The use of tasks that are isomorphic, can be solved using multiple strategies, and can be represented in multiple ways support students' reasoning (Richardson, Carter, & Berenson, 2010). Use of isomorphic and open-ended tasks in collaborative learning contexts is preferred (Mueller & Maher, 2009). In such contexts, there is a need to pay attention to students' cognitive abilities, perceptions of their abilities and valuations of other students' contributions (Artzt & Armour-Thomas, 1997). Teachers are also encouraged to ask questions that ask for explanations that relate to the context of the mathematical tasks and to allow students to make sense of mathematical tasks by creating opportunities for students to justify their conjectures.

Gaps in Literature

Although there is ample evidence of students' capacity for and routes to algebraic reasoning, there is scarcity of research that focuses on teaching practices that support algebraic thinking (Kieran, 2007). Many studies focus their reports on students thinking and simply list teaching practices that researchers believe co-constructed students algebraic thinking without showing any link between change in students' thinking and the instructional practices. Moss and McNab (2011) and Blanton and Kaput (2011) are a few such examples. These practices are normally listed in the concluding paragraphs or in the description of the studies' context. This approach does not give a lot of detail about the pedagogical practices and does not give evidence of effectiveness of pedagogical practices. Even scarcer are detailed studies that report productive teaching practices that support elementary school students. In fact, *The Second Handbook of Research on Mathematics Teaching and Learning* which is the most recent summary of research in mathematics education, has a chapter on teaching of algebraic reasoning to middle and high school students, but it does not have any for teaching algebraic reasoning in elementary school. Lack of coherent research on teaching algebraic reasoning in elementary school may explain this. Both these issues leave a significant research gap.

It is notable that most research on algebraic thinking was conducted with middle school students (e.g., Ellis, 2011; Koellner, Jacobs, Borko, Roberts, & Schneider, 2011; Lannin, 2005; Rittle-Johnson & Star, 2007; Rivera & Becker, 2011; Steele, 2007, Stylinaou, 2011). Generally, teacher educators taught during teaching experiments. In a review of literature from 1968 through 2008, Good (2010) noted that, normative classroom practices of classroom teachers are different from normative practices of teacher educators. Therefore, there is a need to explore classroom teachers' instructional practices that aim at supporting algebraic thinking. Additionally, studies in literature have generally not incorporated teachers' thinking about their algebraic reasoning practices. Teaching is a thinking practice (Lampert, 1998). As such, studies on elementary school teacher practices should incorporate teachers' interpretation of or rationale for their practices.

From the literature, collaborative learning environments, choice of good tasks, questions that show what students need to focus on to generate generalizations and questions that ask students to justify support students' algebraic thinking. However, these practices are situated in normative practices of teaching mathematics in general. In schools, normative practices of classroom teachers are teacher centered (Good, 2010). Furthermore, Stein, Engle, Smith and Hughes (2008), and Smith, Hughes, Engle, and Stein (2009) noted that teachers face challenges beyond identifying tasks that potentially support algebraic reasoning, beyond setting students in the classroom to share their ideas, and beyond asking 'appropriate' questions. It is a challenge for teachers to balance the support of classroom discourse and mathematical content (Sherin, 2002). Thus, there is still need for research to identify teaching practices that support algebraic thinking when teachers have rich tasks and a set of good questions.

These gaps in literature inform purpose of this study. To recap, the primary purpose of this study is to discover elementary school teachers' routines of practices that support algebraic reasoning. The secondary purpose is to understand meanings teachers attach to their routines of practice. Chapter III will discuss the methodology I will use to serve this purpose. Chapters IV and V report findings of this study and chapter six discusses the findings in relation to literature.

CHAPTER III

METHODOLOGY

Overview of the Chapter

In the previous chapters, I discussed that for reform activities regarding algebraic reasoning to be realized, there is a need to research routines of practice that support development of algebraic reasoning. This chapter discusses how I conducted this study and describes participants, instructional tasks, and how the data were collected and analyzed. This follows a discussion of my theoretical perspective and research design.

Theoretical Perspective

I acknowledge that my world-view on knowing and knowledge affected how I conducted this study and consequently its outcome. I am informed by Greeno's (2006) situativity of knowing. Greeno and The Middle School Mathematics Through Applications Project Group (1998) argued for a perspective that synthesized behaviorist, cognitive and situated perspectives to fully inform teaching and learning of mathematics. According to Greeno, a defining feature of behaviorist assumption is that educational practice should involve learning simpler basic skills and then combine those basic skills for more complex understanding. For example, rote memorization of multiplication facts can be characterized as behaviorist. Cognitive perspective, like Piaget's perspective, focuses on cognitive structures that support understanding and reasoning. A focus on mathematics conceptual understanding may be characterized as a cognitive perspective.

Finally, a situated perspective, like Lave and Wenger's (1991), focuses on ways in which students participate in communities of practice. As opposed to considering these perspectives as opposing and mutually independent, Greeno's situative perspective argues that these perspectives complement each other in a school setting.

A situative perspective views acquiring skills as a tool for and a product of participation in (mathematical) practices of a community (Cobb & Bowers, 1999). That is, memorized multiplication facts are tools that facilitate participation in social systems. Misconceptions of mathematical ideas reflect practices of a community and participation of community members. Situative perspective also considers individual's cognitive structures as aspects of a social system. For example, Greeno & The Middle School Mathematics Through Applications Project Group (1998) explains that, as students and teachers are engaged in an inquiry practice, individuals' cognitive representations and explanations influence other classroom members' understanding. As such, one's reasoning is not just affected by the available tools in that community, but it is also part of, and affects the community too.

Greeno & The Middle School Mathematics Through Applications Project Group (1998) agrees with Lave and Wenger that knowledge is always situated and learners are practitioners. For example, school learning is situated in a social system with books, teachers, learners and others. Even for individuals who seem to be learning independently by reading a book, they are using a tool (book) made through a social system. Individuals who seem to be thinking independently use meanings and tools that were developed in a social system. With situative perspective, knowledge can be transferred from one community of practice to another (Sawyer & Greeno, 2009).

The defining feature of a situative perspective is that, “everything that people do is both social and individual” (Greeno, 1997, p. 9). Whereas there is emphasis on social systems that define learning environments, there is also emphasis “in the learning associated with individual participants, as well as the learning that corresponds to the transformations in the entire activity systems” (Sawyer & Greeno, 2009, p. 361). Greeno and The Middle School Mathematics Through Applications Group (1998) advocates for learning environments that support development of students’ perceived and constructed cognitive structures. Such learning environments, Greeno argues, support students learning of basic skills, conceptual reasoning, and participation in discursive practices. Another defining feature of Greeno’s situative perspective is that it pays attention to both cognition and relations. Greeno & The Middle School Mathematics Through Applications Group(1998) explained that situative perspective focuses on “behaving cognitive agents interacting with each other and with other subsytems in the environment” (p. 5).

Greeno explains that, fluency in basic, conceptual, and reasoning skills work together to support students’ learning. This delegitimizes the question of whether computational skills should be regarded more than conceptual understanding, or whether the use of calculators in mathematics classrooms should be approved or not. With the situative perspective, the focus becomes how teachers can meaningfully incorporate all these skills for students’ better understanding and their ability to use their understanding in other social systems (Sawyer & Greeno, 2009). This attention to affordances of an activity system requires teachers and researchers to ask: are the qualities of the social system in the classroom creating enough opportunities for all students to participate? Researchers need to pay attention to teachers and students' regular patterns of behavior to explore how students are being attuned to productive mathematical practices. In addition, researchers with a situative perspective of knowing have a fundamental view that "a community's activity can be understood best by achieving understanding from that community's perspective" (Greeno & The Middle School Mathematics Through Applications Project Group, 1998, p. 23). This perspective implies that students' algebraic reasoning intricately connects to the classroom activities they engage in. It also implies that teachers' meanings of the practices should have privilege over researchers' meanings.

Research Design

As stated earlier, the purpose of this study is to understand better pedagogical practices that support algebraic thinking. Literature reviewed shows a need for studying elementary school teachers' routines of practice that support algebraic reasoning because the existing theories were developed through studying teacher educators (e.g., Ellis, 2011), or with middle school students (e.g., Koellner et al., 2011). Therefore, a grounded theory design is the most appropriate for this study (Creswell, 2007).

Specifically, this is a constructivist grounded theory study. According to Creswell (2007), grounded theory is a systematic research that aims at developing a general explanation of a practice, action, or interactions. Charmaz (2011) discusses defining features of constructivist grounded theory as abductive reasoning, memoing, and constant comparison. Abductive reasoning involves developing tentative concepts from data, using the developed categories to inform further data collection, and considering all possible explanations for a surprising finding until a plausible explanation is found in empirical data. With grounded theory studies, researchers write memos of their reflections during data collection and analysis. Memos become part of data and are used in constant comparison between categories or concepts and phenomena (Maxwell, 2005).

The methodological approach of constructivist grounded theory roots from Glaser and Strauss (1967) grounded theory. Glaser and Strauss recommend using open, axial, and selective coding during data analysis. Open coding is the initial stage of data analysis that may involve line to line coding of text data. Axial coding involves relating codes to identify phenomena, causal conditions, action strategies and their consequences. Selective coding is the selection of a main category, relating the main category to other categories and assessing relationships within categories. At this stage, a diagrammatic representation of the relationships is recommended. Constructive grounded theorists treat this methodological approach as flexible and adaptable to different contexts as opposed to following them as rigid rules. Additionally, unlike Glaser and Strauss (1967), constructivist grounded theory view their theories as situated in the context they were generated and not fully conclusive.

Research Context

For this study, I conveniently sampled the On Track project. Convenience sampling, according to Creswell (2007), is a strategy whereby participants are sampled because they can “inform an understanding of the research problem and central phenomenon in the study” (p.125) and are accessible. The primary goal of the On Track project is to support elementary school students’ mathematical reasoning. This goal was developed with an understanding that all students are capable of reasoning mathematically if they engage in activities that nurture development of their mathematical reasoning. On Track supports students’ mathematical reasoning through providing after-school enrichment programs and professional development with elementary school teachers.

Participating Schools

At the time this study was conducted (2011), six schools from two school districts were participating in the On Track project. Only five schools participated in this study. Two of the participating schools, according to North Carolina report card, were Title 1 schools. For these Title 1 schools, percentage of students for each school at or above grade-level in mathematics during the 2010/2011school year was 85.7 and 86. About 76.67% and 76.77% of the students were classified as needy. The On Track non-Title 1 schools that participated in this study had 80.4%, 86.5% and 80.4% of students at or above grade level in mathematics. On average, 58.06% of the students were classified as needy.

Participating Students

Student participation in On Track program was voluntary. A brochure was sent to parents of students in grade three through five describing the program (see Appendix A). Students whose parents indicated interest in participating enrolled on first come first serve basis. Selected participants signed assent forms and their parents or legal guardians signed consent forms approved by Internal Review Board (see Appendix B). These consent and assent forms explained what the program is about, students' expected participation and their right to stop participation at any time. From spring to winter 2011 semester, 132 students participated in the program. Only 115 participated in this study. On Track research group did not collect data on students' ethnicity or age. Each class had up to 19 students in grades 3 through 5.

On Track Afterschool Sessions

On Track lessons were held after school and divided into sessions. Each session comprised of 10 classes, normally meeting 2 days a week. Each class started with an outdoor physical activity designed to take up to 30 minutes. Physical activities were followed by a function machine tasks whereby students were asked to find mathematical patterns and make generalizations. Function machine tasks were followed by 'main' pattern finding tasks (for a full description of these tasks, see section on instructional tasks). Although each of these tasks was designed to take 30 minutes, teachers were advised to be flexible and have their pace guided by students' thinking. Students could participate in more than one session. In 2011 when this study was conducted, there were three on Track sessions. Session 1 was conducted in the winter, session 2 was in spring and session 3 was in the fall semesters.

On Track Teachers

Teachers from the participating schools who participated in On Track professional development taught all the sessions. On Track principal investigators collaborated with the school principals to identify and ask teachers who would be willing to participate in the program. Seven female teachers participated in the 2011 spring and winter sessions (On Track sessions 1 and 2). For personal reasons, one teacher from session one and 2 could not participate in session 3. At this time, On Track research team had not collected data on her classroom activities. Consequently, she and her school did not participate in this study. Thus, this study's participants were the six teachers who taught three On Track sessions in 2011. All participating teachers signed consent forms in appendix B. From hence forth, On Track teachers refer only to those who participated in this study.

Elementary school teaching experience for participating teachers ranged from six to 25. Four of these teachers had a masters' degree in elementary school education. At graduate level, three teachers took one mathematics teaching methods course (3 or 4 credit hours). Only two teachers took one or two graduate mathematics courses. All teachers reported taking undergraduate mathematics courses. Two teachers indicated that they took 15 credit hours of undergraduate mathematics and the rest indicated taking three credit hours of undergraduate mathematics. Participating teachers had at most six credit hours of mathematics teaching methods. These data on teachers' background were self-reported. These teachers were the On Track lead teachers and were primarily responsible for teaching function machine tasks and the 'main' pattern finding tasks in sessions 1 through 3 of On Track in 2011.

For the duration of this study, the teachers underwent professional development spread over time. Teachers met with the professional developers at one of the participating schools. The meetings were once a week (90 minutes per meeting) before and in between On Track teaching sessions. The principal investigators conducted the professional development, with exception of a few times when other mathematics teacher educators were invited. During the meetings, teachers worked on the same or similar tasks as the instructional tasks. They discussed the mathematical ideas in the tasks and the connections of ideas between tasks. They watched videos of students working on the tasks. They used ideas from the videos and their classroom experiences to discuss different strategies students may use. They also discussed students' tendencies when working on these tasks and different pedagogical approaches. Teachers could give opinions on how they believed On Track teaching sessions could be improved. Altogether, teachers had 18 contact hours of professional development spread over a year for the duration of this study.

During the spring 2011 semester for On Track, each teacher had an assistant. The teacher assistants were either undergraduate elementary school pre-service teachers or mathematics education doctoral students. In the fall semester, all teaching assistants were from the participating schools, except for one school that continued to have an undergraduate student as an assistant. Teachers were given instructional materials to use with their students. These instructional materials included lesson plans and manipulatives. All teachers had incentives for participating in On Track program.

Although in this report I have referred to professional development as the meetings between the professional development providers and the participating teachers, and activities conducted during those meetings, it is important to note that the after school teaching sessions were also regarded as professional development spaces. On

Track principal investigators anticipated that the teachers would develop tendencies that reflected the expected teaching practices over time.

On Track Instructional Materials

Lesson plans. As stated earlier, On Track teachers were provided with worksheets for students and lesson plans written by the On Track's Principal investigator Sarah Berenson. To situate On Track teachers' practices better, I conducted content analysis of the lesson plans and students work sheets that teachers had access to. All On Track lesson plans stated the objective(s) of each task and discussed the underlying mathematics. For example, the objective of the square table task was for students to develop strategies for finding patterns, generalize rules with two operations and to learn to explain their thinking. The following themes regarding routines of practice were identified: facilitating whole group and small group discussions, encouraging justification of generalizations, promoting understanding of generalizations and encouraging multiple strategies. I will now discuss these themes in detail as they appear in the instructional materials. While reading the themes from the instructional materials, it is important to note that, the instructional materials broadly described expected pedagogical practices and teachers had to make pedagogical decisions based on their assessment of students' reasoning.

Facilitate whole group and small group discussions. The instructional materials advised teachers to have students work in small groups and have whole group discussions. Although in general there were no suggestions on how teachers should facilitate small group discussions, there were some suggestions for whole group discussions. Teachers were expected to call for whole class discussions after students explore the outputs for large inputs (e.g., input of 10 and 100). During whole class discussions, teachers were urged to accept both recursive and explicit rules as important patterns but privilege explicit generalizations by asking students to explain such generalizations in detail.

Encourage justifications. The lesson plans required teachers to foster understanding of generalization by asking students to justify their conjectures. Teachers were expected to ask “why and why not questions” and ask students to convince them that their generalizations work for 100 and any inputs. Lesson plans also advocated asking students to evaluate peer’s responses and question each other. Furthermore, students’ conflicting answers were to be opportunities for students to justify to each other. For train table tasks, teachers were supposed to support students’ analytic justifications that refer to the general context of the mathematical problem by asking students to explain how parts of their rules connect to the train table problem. They were discouraged from being judges of validity of responses. Rather, teachers needed to ask students to convince each other of validity of their ideas.

Creating a context for developing multiple strategies. Whereas some tasks had variable names for input and outputs written down on the t-charts, others did not. Teachers were to facilitate discussions about the different labels students could use on the input-output table. Lesson plans asked teachers to develop the habit calling for different strategies.

Instructional Tasks

On Track students engaged in several mathematical reasoning tasks. Usually, a one-hour lesson had two mathematical tasks. The first task of the lesson, called a function machine, aimed at supporting algebraic reasoning For example, students explored the following toy function machine:

> If I put $1 into the function machine, 5 toys come out. If I put $2 into the function machine, 10 toys come out. If I put $3 into the function machine, 15 toys come out. What is the rule for this machine?

Mathematical tasks that required students' combinatorial reasoning or generalizing about mathematical patterns followed function machine tasks. For this study, the focus is on the function machine tasks and other algebraic reasoning tasks. Tasks in appendix 3 were sampled from the pool of On Track tasks. The ATV function machine, toy function machine, the perimeter and the square table tasks were sampled from session 1 of On Track. The square number task, pentagon table task and the function machine without real life context were sampled from session 2. All these tasks were used again in session 3.

I delimited the study of On Track teaching practices to the routines of practice as teachers enacted the tasks in Appendix C and toy function machine task. Two criteria guided sampling of these tasks. The purpose of the study was to understand routines of practice and algebraic reasoning co-constructed. As such, it was important to sample tasks for which data on teaching practices were collected. The sampled tasks were the algebraic reasoning tasks with video data of the classroom practices. A second criterion used was the representativeness of the types of On Track algebraic reasoning tasks. On Track algebraic tasks required a quadratic rule (e.g., square number task), a one-operation rule (e.g., ATV function machine) or a two operations rule (e.g., square table tasks). These tasks were sampled because they are representative of the types of algebraic tasks On Track students engaged in.

Rationale for Sampling On Track

I was a research assistant for On Track during the period of this research project. With permission of the principal investigators, I had access to On Track research data. With that access, it was possible for me to have a deeper understanding of my research context by being a participant observer during profession development and afterschool enrichment program. I was a participant observer from January 2011 to November 2011. The access that allowed me to gain familiarity with the context was necessary for conducting constructivist grounded theory. Beyond access, On Track classes were successful in supporting students' algebraic reasoning as will be seen from overview of students' reasoning in the results chapter. Therefore, On Track was suitable for a constructivist grounded theory of routines that support algebraic reasoning.

Data Collection and Analysis

As mentioned earlier, classroom practices of the sampled lessons were video recorded. Normally, each classroom had one video recorder. The video recorder focused on the whole class to capture whole class activities. It also focused on groups of students to capture small group activities. When teachers walked around the classroom, the video camera was moved around to capture the teacher's interaction with the students. All students' written artifacts were collected at the end of each lesson.

Data analysis was in six phases as shown in Figure 1. At the initial phase of data analysis to discover On Track teachers' routines of practices, pedagogical practices of five teachers from sessions 1 and 2 were the unit of analysis. These teachers were sampled because they were the only teachers whose practices were video recorded at the time data analysis began. At this phase, video recordings and transcripts of 15 lessons were the data pool.

Tentative categories from phase 1 data analysis informed development of research questions that aimed at understanding the meanings teachers attached to their practice. I conducted one-to-one interviews with each teacher. Each interview was audio recorded and took about 15 to 30 minutes. The interview transcripts were analyzed using NVivo 9 (QSR International, 2011). Findings from the interview analysis informed refinement of the categories from phase 1 analysis.

Transcripts from the teachers' videos of classroom practices of sessions 1 and 2 were then entered into NVivo 9. The refined categories informed line by line coding of the teacher practices. At this phase, using NVivo helped in showing how robust the categories were and helped in further refining the categories for routines of practice. Data from session 3 were used for constant comparison to further refine the categories. I will now discuss in more detail the analysis for each research question (see Figure 1).

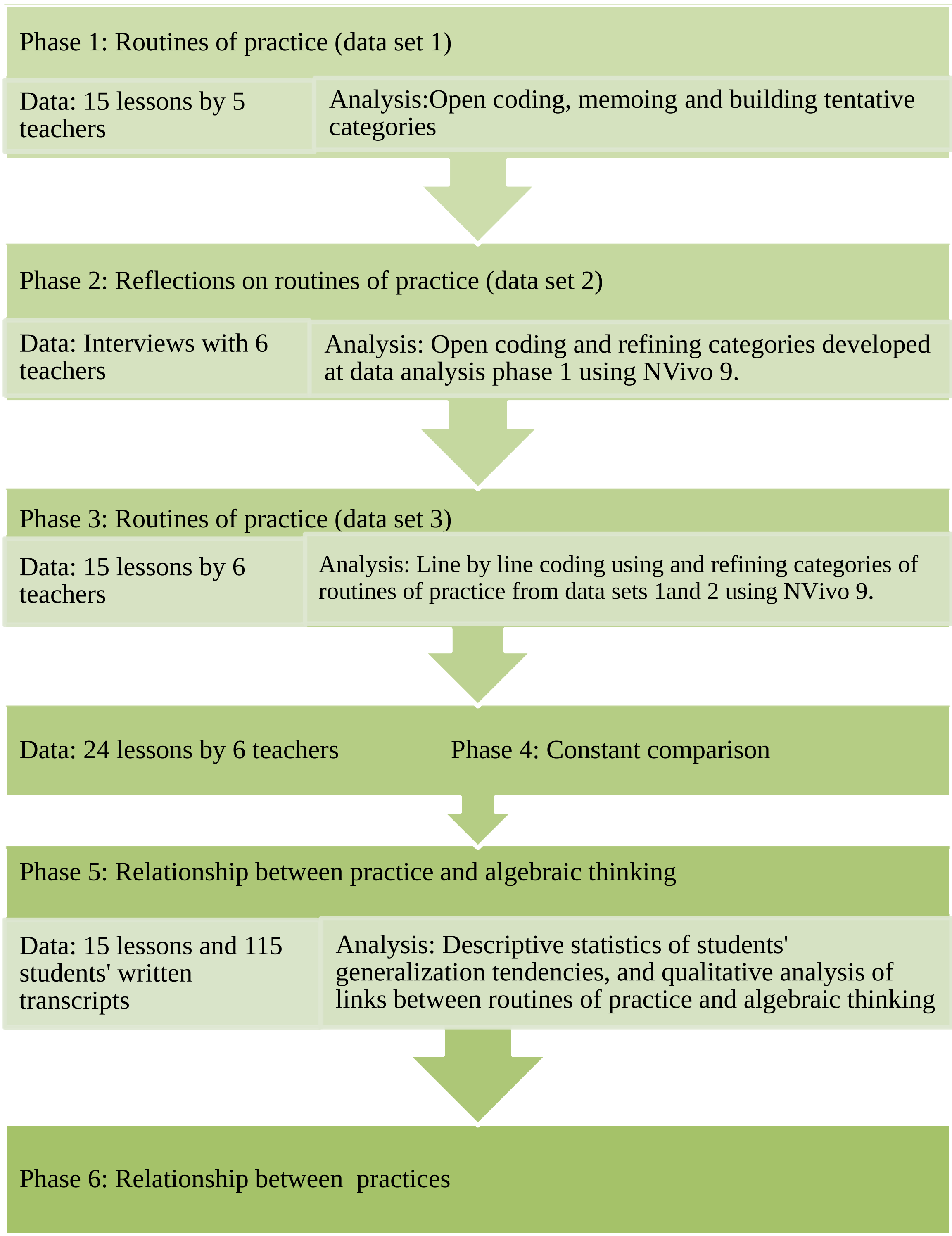


Figure 1. Data Analysis Process

Data Analysis: What were Teachers' Routines of Practice?

Data analysis methods of instructional practices treated On Track teachers as a single case and developed a constructivist grounded theory of practices from across teachers. To make data analysis more manageable, practices associated with each task were analyzed separately. For each task, I started with line-by-line coding of video transcripts of the classroom activities to get sensitivity of the data and where possible label with active verbs. Being a participant observer for a year in On Track contexts informed my line-by-line coding especially when describing the classroom activities. Secondly, I synthesized the codes to make categories of teaching practices. I used memoing to write narratives to describe properties and dimensions of each category. I created a matrix of categories or practices against tasks to identify common practices across tasks and those particular to some tasks.

Memoing was ongoing and part of the data analysis process. Memoing helped in describing the categories in detail and record my reflections of the data. I conducted data analysis of routines of practice in phases (see Figure 1 to continuously refine, confirm and disconfirm my initial findings. I used tentative categories to analyze the data from which the categories were developed and apply the categories to a new set of data. This process showed how robust the categories were and helped in reducing initial 40 codes to five categories. Furthermore, NVivo 9 showed how robust the categories were by revealing the percent coverage of the data by each category. Descriptions of the categories were then given to one person who worked as an On Track teacher assistant throughout the period of this study and was familiar with On Track context to use in coding a subset of the data. This purpose for this was checking if the dimensions in the categories were meaningful and to check for consistency in the coding.

Data Analysis: What Meanings do Teachers Attach to their Routines of Practice?

Data from interviews with teachers were sources of data for this inquiry. Teacher interviews focused on teacher actions that seemed critical to student understanding, teachers' meanings of their practices, and what teachers normally did to promote algebraic thinking in On Track classrooms. The tentative routines of practice served as a framework for developing questions for interview protocol. Teachers were asked to talk about their On Track instructional practices and their rationale for using different strategies. I also asked the teachers to talk about the instructional practices that I observed in their classrooms but they did not list them as their practices. Relevant episodes from each teacher's class were used to facilitate the interviews. Appendix D contains questions that flexibly guided the interviews. I interviewed all the teachers in session 3.

Again, to analyze interview data, all On Track teachers were one case. Flexibly guided by the tentative categories developed from teachers' observed practices, I did line by line coding of all data to further refine categories of routines of teacher practices. Coding and building themes for teachers' rationale for the different instructional practices followed. Reported instructional practices were matched to observed routines of practice. The reported practices and the rationale for the practices were the meanings teachers attached to their practices. In line with constructivist grounded theory approach, these meanings were privileged over the meanings I attached to teachers' practices. That is, whenever appropriate, teachers' meanings became the names and the descriptions of the

On Track routines of practice bringing together analysis of observed and reported practices. Consequently, data analysis of teachers' meanings was with the same rigor as described in data analysis for routines of practice.

Data from session 3 were primarily for constant comparison to confirm, disconfirm, or refine the categories developed from sessions 1 and 2 data. Unlike recorded data from sessions 1 and 2, not all data from session 3 were transcribed. Instead, I observed the classroom activities and watched the videos of the classroom practices to see if the practices were typical of the session 1 and 2 data. Tentative routine of practices guided analysis of session 3 data. This analysis further characterized the categories. When outlier cases of practices were identified, their explanation was sought from the data and if necessary tentative categories were re-characterized. This process continued until saturation occurred (Strauss & Corbin, 1998).

Data Analysis: How Routines of Practice Supported Algebraic Reasoning

Having identified routines of practice, I analyzed algebraic activities that the practices co-constructed. Supporting algebraic thinking was interpreted at three levels. These are: supporting students' tendency to make transitional explicit and correct explicit generalizations, students' tendency to use variable names, and supporting students' progress towards expressing correct explicit generalizations. Descriptive statistics assessed students' tendencies in making generalizations.

Identifying how the routines of practice supported students' progress towards expressing correct explicit generalizations were analyzed using grounded theory methods. After identifying the main categories of practices and themes within those practices, I looked for instances that the practices supported algebraic thinking. These instances were from a pool of students' written work that was collected at the end of each lesson (lessons from which routines of practices emerged) and from transcripts of classroom activities that had evidence of student thinking. Students' work samples were chosen based on clarity in terms of how the thinking or strategies changed due to a students' experience of a teaching practice. I considered one category/practice at a time, treating it as an intervening condition for students' algebraic thinking whenever possible. Ellis (2011) methodological approach to studying generalizing-promoting actions informed analysis of consequences of teaching practice. Three criteria were used.

1. An observed progression in student generalization directly related to a teaching action or practice.
2. A progression in generalization reflected a new idea introduced by a teaching practice.
3. A students' idea or strategy linked to teaching practice over time.

A hypothetical presentation of how instructional practices supported students' algebraic reasoning is presented in Table 3.

Students and students' work samples used in this analysis were not randomly sampled but were sampled based on clarity of the algebraic activities. It was therefore necessary to supplement this analysis with teachers' assessment of how their practices supported algebraic reasoning. Themes developed from analyzing student work samples, and themes developed from teachers' assessment of how their practices supported algebraic reasoning were matched as qualitative evidence of how routines of practice supported algebraic reasoning. This reduced bias from sampling strategy and gave a more holistic understanding.

Table 3

A Hypothetical Example of Routines of Practice that Supported Algebraic Reasoning

Element	Description
Context	Mathematical task and activities a student is engaged in.
Phenomena	Student's initial reasoning e.g., a student makes a transitional recursive generalization.
Intervening condition	Instructional practice e.g., showing inefficiency of this strategy by asking how a recursive generalization will help solve for the 100^{th} term.
Consequence	Revising the initial reasoning e.g., attempting to generalize explicitly.

An understanding of how the routines of practice supported algebraic reasoning was further supported by analyzing all students' written generalizations. Students written generalizations were coded using the seven levels of generalizations presented in Table 1. Four raters coded a subset of the generalizations. Kendall's W statistic was used to analyze agreement of the four coders. For the tasks analyzed, Kendall's W was approximately .7 (at least) for the first round of coding. After a discussion of the codes, Kendall's W was at least .91 during the second round of coding. Discussions followed this analysis until there was a total agreement on the codes for students' generalizations. Another research team member and I then coded all student generalizations. Whenever there was a disagreement about our codes, we discussed why each one of us coded the way we did. We then revisited definitions of each generalization level to come to an agreement. Consequently, there was a total agreement of the analysis of student's generalizations. I ran descriptive statistical analysis of student generalizations to find out the students' tendency to make explicit generalizations and their tendency to use variable names.

Altogether, data for this study included 24 lessons and interviews with six teachers. This meets at least 20 data points recommended for grounded theory research (Charmaz, 2011; Creswell, 2007). Table 4 illustrates the type of data that informed each research question and a brief description of the analysis.

Table 4

Research Crosswalk

Research questions	Connections to research problem	Data sources	Data analysis
What are the routines of practice?	To identify elementary school teachers' routines of practice	Video recording of classroom activities and memos.	1. Code teaching practices of each task. 2. Identify categories and subcategories. Find relationships between and within categories of each task and between tasks.
What meanings do teachers attach to their routines of practice?	To deepen understanding of and validate practices as routines	Teacher interview	1. Identify themes and categories of teachers' meanings. 2. Relate categories to observed practices.

Table 4 (cont.)

Research questions	Connections to research problem	Data sources	Data analysis
How do routines of practice support student algebraic thinking?	To identify productive routines of practice	Students' written artifacts, video transcripts, interview transcripts and field notes.	1. Analyze teachers' assessment of how their routines support algebraic reasoning. 2. Find an instance of routine and identify students' algebraic activity or change in thinking associated with each routine. 3. Run descriptive statistical analysis of students' levels of generalization

Validity Issues and Rationale for this Methodological Approach

On Track teachers whose practices were being studied have different backgrounds. Their teaching experiences, educational backgrounds and their other histories are different. These possibly affected their instructional practices. It is not in the interest of this study to find out the effects of these historical factors. These historical factors were treated as background, not a focus of this study. It was also not within the scope of this study to compare what practices were associated with different teachers or to explain why those differences occured. As mentioned before, the unit of analysis is the practice across cases. Just as in many studies that explored practices of different teachers, the differences in teachers did not affect the validity of this study. Examples of studies with teachers of different backgrounds included Carlone, Haun-Frank, and Kimmel (2010) ethnographic case with 13 teachers and Wilson's (2009) study with 35 teachers.

Consistency and Trustworthiness

Consistency and trustworthiness was built into the study through triangulation of data sources. Data sources included students written transcripts, video recordings, field notes and interviews with teachers that aimed at understanding their routines of practice. I acknowledge that my worldview affected my interpretation of the data. To minimize how this may affect the trustworthiness of this study, member checking (Creswell, 2007) was conducted. Triangulating teachers' meanings of their practices with my interpretation served as member checking. Additionally, the findings of this study were presented to On Track teachers. Analysis by On Track teacher assistant also built in consistency in this study. Furthermore, investigator triangulation built in trustworthiness of interpretation.

Summary of Chapter III

This chapter discussed my theoretical orientation as situative perspective of knowing which calls for a focus on individuals and group activities, and transfer of skills from one context to another. With that theoretical perspective, I have argued that constructivist grounded theory was appropriate for this study because it enabled me to address the need for a theory grounded in data and elementary school teachers' meanings on practices that support algebraic reasoning. This research design informed data analysis of 24 lessons, 6 interviews, and 115 students' written transcripts. The results of this analysis are in Chapters IV and V. A literature based discussion of these results and implications for practice and future research are in Chapter VI.

CHAPTER IV

RESULTS

Chapter Overview

As discussed in the previous chapters, the primary purpose of this study was to discover practices that supported algebraic reasoning in On Track classrooms. To do this, I explored On Track students' algebraic reasoning and routines of practice that co-constructed student's reasoning. Four research questions guided this exploration. These were: (a) what are the routines of On Track classroom practices, (b) What meanings do teachers attach to their routines of practice, (c) How are the routines of practice related, and (d) in what ways do On Track routines of practice support student algebraic reasoning? Using a constructivist grounded theory approach, I explored patterns across the teachers that emerged from analysis of their practices, reflections of their practices, content analysis of students' artifacts, and teachers' assessment of student understanding. All names in this report are pseudonyms and "teachers" in this chapter refer to On Track teachers that participated in this study.

The organizational structure of this chapter reflects an acknowledgement that teachers' routines of practice, meanings they attach to those routines, and students' learning mesh in practice. I first present an overview of students' algebraic reasoning based on their written work. This is followed by categories of routines of practice that the teachers reported and that were observable in their classrooms. These categories represent routines of practice common to all the teachers. In line with constructivist grounded theory, atypical cases are also discussed. As I report each routine of practice, I also report how the practices co-constructed students algebraic activities based on my observations, students' written work, and teachers' assessment. I conclude with a summary of this chapter and an overview of Chapter V.

Overview of Students' Algebraic Reasoning

In the previous sections, I defined algebraic reasoning as making generalizations about mathematical patterns. I also discussed that in this study, algebraic reasoning is assessed at six levels. These are (a) giving no response to a generalization task, (b) making an incorrect recursive generalization, (c) making a transitional recursive generalization, (d) making an incorrect explicit generalization, (e) making a transitional generalization, and (f) making a correct explicit generalization. Examples of students' generalization at each of these levels are in Table 4. As it may be noticed, generalizations that mentioned variable names (correct recursive and correct explicit generalizations) are regarded as higher reasoning levels than those that do not mention variable names (transitional generalizations). With that orientation, I report on how students reasoned algebraically in On Track classrooms by giving an overview of the mathematical generalizations they made followed by their tendency to use variable names for both geometric and function machine tasks.

Across tasks with geometric representations, Figure 2 shows that students tended to relate independent and dependent variables and correctly write explicit generalizations. That is, 54.5% (n = 55), 60.8% (n = 74) and 97.14% (n = 35) and 70.97% (n = 31) of student responses for square tables, perimeter, square number and pentagon table tasks respectively were explicit generalizations. Students also tended to make explicit generalizations when working on function machine tasks (see Figure 3). There were 74.28% (*n* = 70), 94.62% (*n* = 42) and 81.24% (*n* = 48) explicit generalizations for Toys, ATV and function machine task with no real life context respectively. In overall, 74.08% of all the generalizations were explicit generalizations that corresponded input and output variables.

Table 4

Examples of Student Generalizations for the Pentagon Tasks

Generalizations	**Example of Student Generalizations**
Correct Explicit	(Px3)+2 = N for input P and output N
Transitional Explicit	X3+2
Incorrect Explicit	tables x 3 because that is the pattern of the output values.
Correct Recursive	People plus 3 each time gets next person

Table 4 (cont.)

Transitional Recursive	You take away 1 and then add 4
Incorrect Recursive	Starting with 5 add two people each time

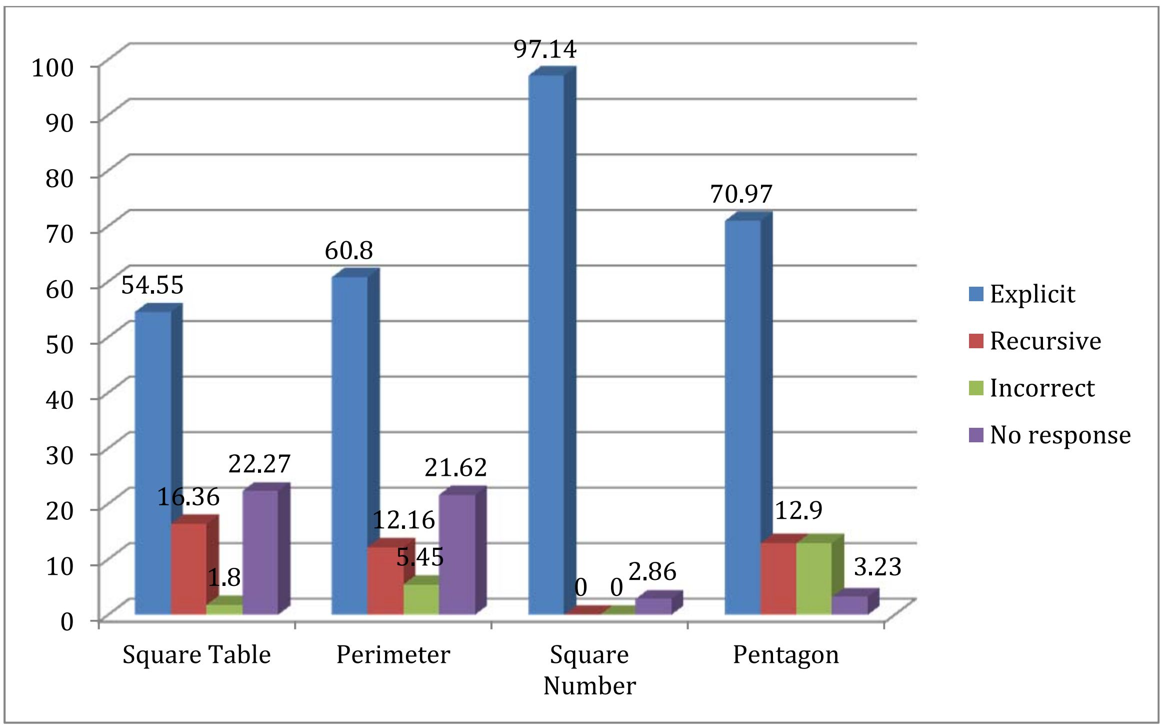


Figure 2. Percentages of Generalizations on Geometric Pattern-finding Tasks

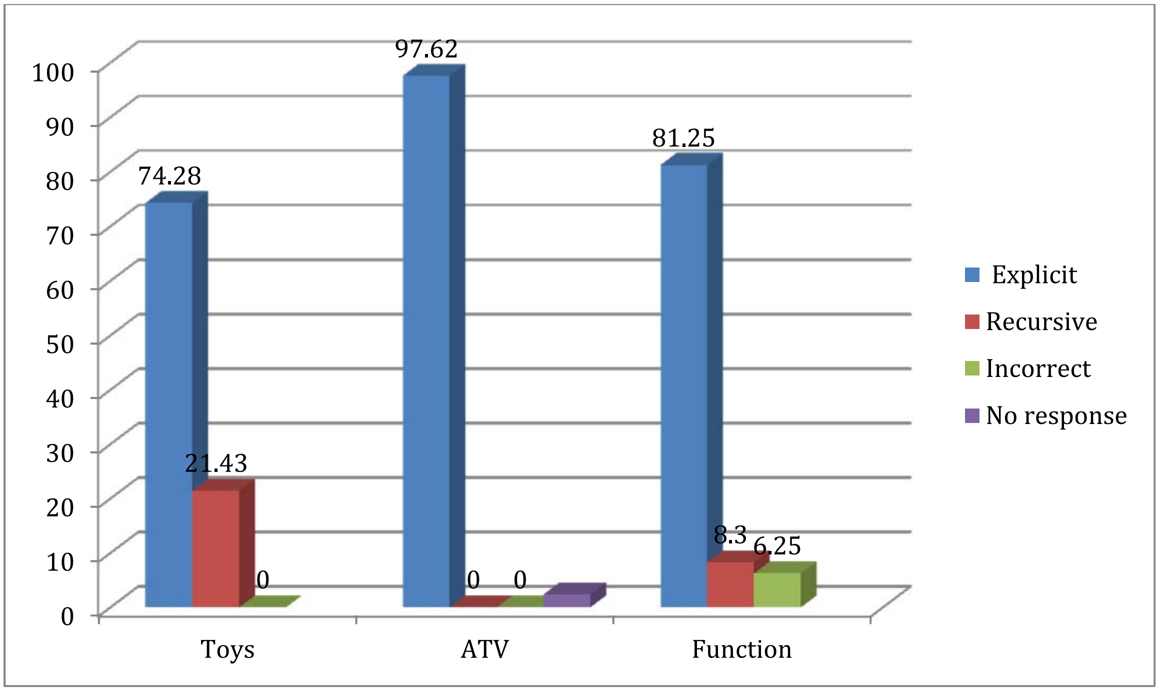


Figure 3. Percentage of Students' Generalizations on Function Machine

Content analysis of students' written work was also conducted to assess levels' of students' algebraic generalizations. Descriptive statistics were used to explore tendencies of reasoning in terms of different generalization levels. Figures 4 and 5 show percentages of students' written generalizations to geometric and function machine tasks respectively, while participating in these classroom activities. Of the explicit generalizations students wrote for tasks with a geometric representation, more students used variable names for the square table and pentagon table tasks (see Figure 4). Similarly, there were more recursive rules with variable names than without. A slightly different pattern was observed on use of variable names with the function machine task. More students used variable names for their explicit generalizations on two out of three function machines (ATV function machine and function machine with no rela life context). Only 28.57% of the generalizations on the toys function machine mentioned variable names when expressing explicit generalizations.

For recursive generalizations, a different pattern on use of variable names is observed. There were more recursive generalizations that mentioned variable names than transitional recursive generalizations for all the geometric tasks. However,there were more transitional recursive generalizations than correct recursive generalization for all function machine tasks. Altogether, approximately 55.26% of all the correct written generalizations (*n* = 304) mentioned variable names whereas approximately 45% of the correct responses did not mention variable names.

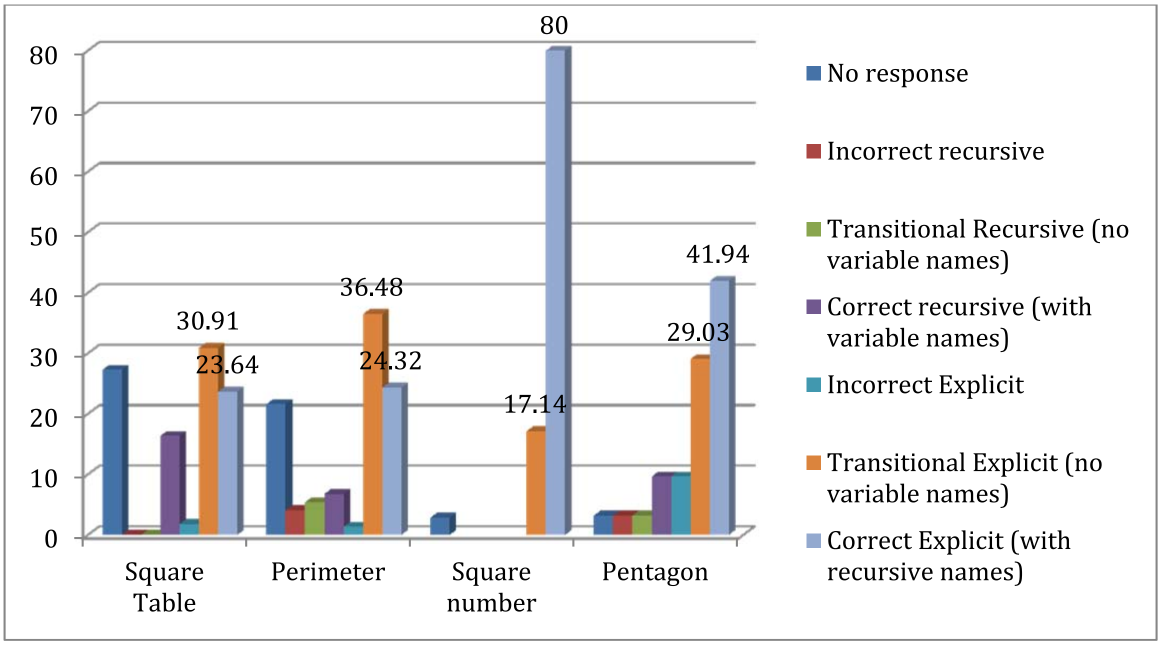


Figure 4. Percentage of Generalizations on Geometric Tasks Comparing Use of Variable Names

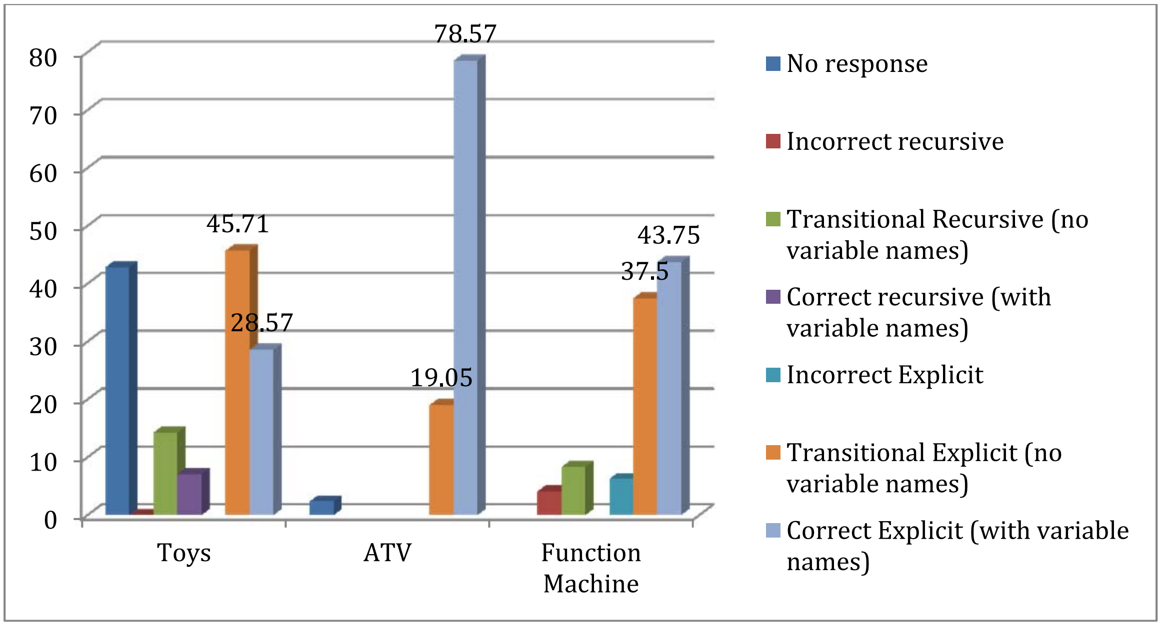


Figure 5. Percentage of Generalizations on Function Machine Tasks Comparing Use of Variable Names

In summary, analysis of On Track students written work shows a tendency to make generalizations that relate the independent and dependent variables. X It also shows students' tendencies to use variable names in their generalizations, which is a practice regarded as a higher thinking level in this study. To understand routines of practice that co-constructed students' algebraic reasoning, I explored teachers' routines of practice. The results of this exploration in reported in the following section.

Routines of Practice

Five categories, that are not mutually exclusive, emerged from the data as teachers' routines of practice. Teachers tended to keep algebraic tasks open through different teaching practices. They fostered co-construction of algebraic reasoning in multiple ways. Teachers supported understanding of variables. They also supported opportunities for students to connect mathematical ideas. Another routine of practice that emerged from the data is supporting understanding of generalizations. I will now discuss these practices and how they co-constructed student algebraic reasoning in detail. Since this is a constructivist grounded theory, wherever appropriate, teachers' meanings of their practices were privileged over my interpretation. That is, the practices are discussed using teachers' descriptions unless otherwise stated.

Routine of Practice: Keeping Algebraic Tasks Open

Mathematical tasks used in On Track classrooms were cognitively demanding, as they required students to observe relationships and develop generalizations or formulas that can be used to predict outcomes. I described keeping algebraic tasks open as a practice that maintained the open-endedness and therefore the cognitive demands of these tasks. An open-ended task can be solved using multiple strategies and may have more than one correct answer. Two themes emerged in this category as practices that maintained open-endedness of the tasks. These are teachers' style when launching the tasks and providing students with workspaces. After discussing these practices, I will discuss how these practices supported students' algebraic reasoning.

Launching the tasks. On Track teaching style that kept algebraic tasks open was observed as the teachers launched the tasks. For both the function machine tasks and the tasks with a geometric representation, teachers launched the tasks by discussing with the students what the input and output values were for input of one through three. The purpose for launching tasks in this way was to make sure students understood how to represent the tasks (e.g., how to build the models) and in general for students to understand what the task was. For example, when working on the train table tasks, teachers asked students to build the models for input of 1, 2, and 3 and asked them to count how many people would sit around each train of tables. Teachers went around the classroom to check if students were representing this task correctly through the models they were building. Those students who represented the tables of the first three stages as disjoint were helped to build them as trains for each stage. After launching the tasks by discussing how to build the models and data for the first three stages, students explored the mathematical patterns with little assistance from teachers.

According to the teachers, they aimed at creating contexts whereby students explored and discovered mathematical concepts and relationships. Their view of discovery method of teaching reasoning had an emphasis on exploring mathematical relationships other than a focus on trying to find an existing specific algebraic generalization as teachers explained in episode 1 and 2. Discovery in On Track classrooms was supported by "holding back" and not giving many instructions that funnel students thinking. Teachers described this approach as relaxed and a flexibly structured teaching style whereby you give students a task with an attitude of "here it is, see what you can do with it. And just ask them few questions and not trying to lead them, just seeing where they go."

> Episode 1
>
> For On Track, my goal is to make it discovery based. My style is to let the students work through it, not feel like there is an answer that they have to achieve as their goal, but help them to understand that there might be, maybe there is an answer, but there is more than one way to get to that answer. And sometimes there is more than one answer.

> Episode 2
>
> My On Track teaching style is exploratory, I mean, letting the students see what they can discover and not trying to give them too much direction. So, some days in here it seems to get a little bit loud, but they are all actually on task and very involved in what they are doing. But I like seeing that they are trying to solve it on their own.

Providing workspaces. Teachers supported student engagement through providing them with "work spaces." In On Track classrooms, workspaces were "opportunities for students to use their own critical thinking." One way students had workspaces is by having opportunities to think by "getting a lot of time to explore." As one teacher explained, "I am not having to teach to a test so I feel like I can just kind of give as much time as I can to the task. And allowing them (students) a lot more time to think it through rather than giving them the direct answer." Students' thinking and progress guided the pace of the lessons. Since students normally progressed at different paces, students or groups of students who figured out mathematical relationships faster were asked to give space for their peers to explore and think about the tasks they were working on. For example, during one of the classes, a teacher asked students: "put a thumb up on your desk when you get it, . . . until everybody has had an equal chance to figure it out."

Teachers also created workspaces by providing a lot of blank space on the paper for students to use as they thought through the tasks. In contrast to more structured worksheets that provide little spaces for students to explore and try out their mathematical conjectures, On Track teachers preferred more workspace. At their recommendation, the tasks for students were in as few words as possible at the top of a page (See Appendix E for On Track student worksheet). The rest of the page was students' workspace. Students had as much paper as they required exploring the mathematical relationships between independent and dependent variable. This type of workspace nurtured explorations and supported algebraic thinking because as stated in episode 3, it helped in making the mathematical tasks less intimidating. As observed from students' worksheets, workspaces accommodated divergent thinking and strategies. It appears that, workspaces supported ownership of ideas by accommodating unique and divergent thinking.

> Episode 3
>
> . . . less intimidating to the kids, it is not so much on there (the paper) . . . And when they have all that space, they know that they can use it. I mean, I have so many students flipping over, front and back, every single time. They are just drawing and they are writing out these patterns and I feel like it (workspace) is encouraging them to do that.

Algebraic thinking and keeping tasks open. As noted by the teachers, when students had workspaces, they engaged in explorations to make their generalizations. Figure 6 is an example of students' work that shows explorations in search for algebraic generalization. As may be noted, this student tried out several rules to come up with explicit rules. The student had enough workspace to try several rules and test them.

Additionally, based on teachers' assessment, these practices supported students' algebraic thinking in multiple ways. As Episode 3 shows, workspaces helped in making the mathematical tasks less intimidating and encouraged explorations. Explorations were important because students "need (ed) to explore the different strategies to be able to come up with the algebraic equations without being told, ‘this is how you do it’.” Similarly, teachers reported that, when they “hold back, not guide them (students) as much, and let them use their own critical thinking skills rather than leading them,” students exhibited more divergent thinking. Such learning “is more meaningful and . . . they (students) are going to learn it (algebraic thinking) that way and remember it more than if you try to tell them.”

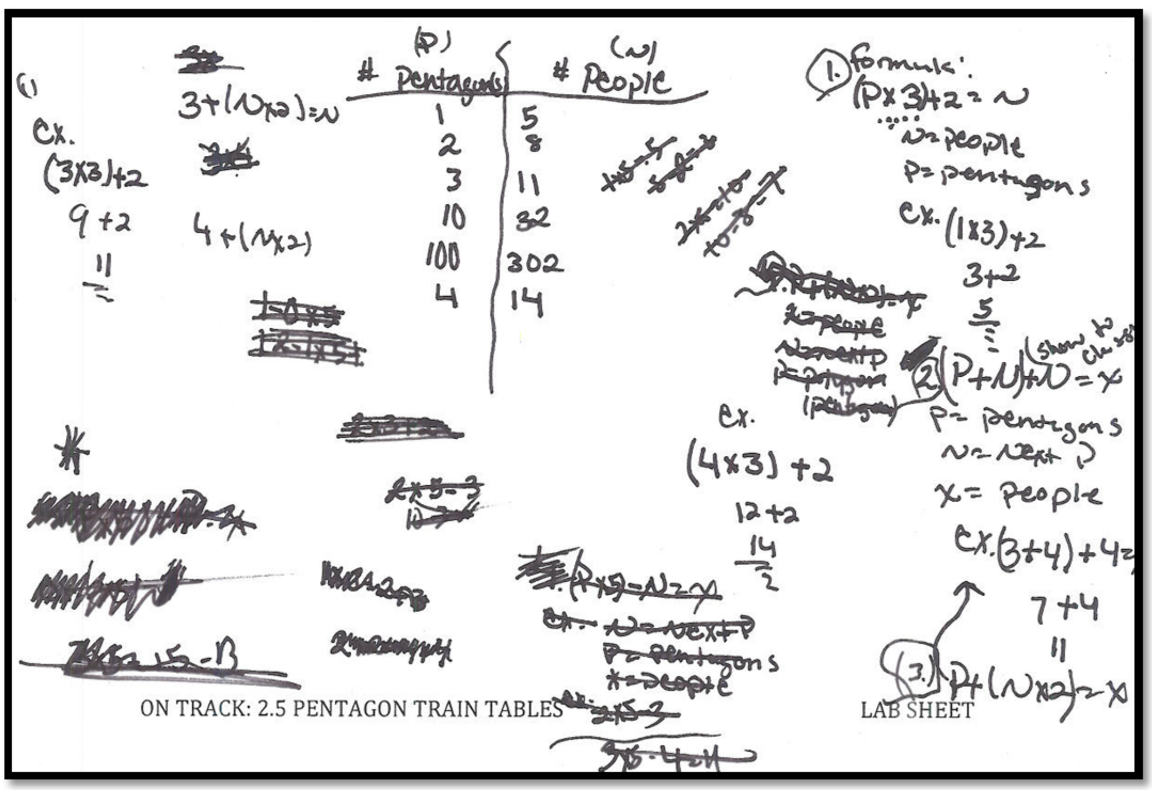


Figure 6. A Student’s Explorations of the Pentagon Table Task

Summary on keeping algebraic tasks open. Cognitive demands and open-endedness of algebraic tasks were maintained during implementation by flexible, “laid back” teaching styles that made the lessons discovery based. Providing students workspaces through allowing them enough time and physical space to explore mathematical patterns contributed to maintaining the cognitive demands of the algebraic tasks. The practices made algebraic tasks less intimidating to the students, allowed students to try out different strategies, and made students' learning more meaningful. These practices of keeping algebraic tasks open supported students' algebraic reasoning. I will now discuss supporting co-construction of algebraic ideas as another practice that emerged from the data.

Routine of Practice: Nurturing Co-construction of Algebraic Reasoning

Nurturing co-construction of ideas was one of the evident routines of practice of this study's participating teachers. I defined nurturing co-construction as teachers' practice that created or took up opportunities for students to develop their reasoning through participating in collaborative mathematical practices. Data analysis revealed six themes as practices for nurturing co-construction of ideas. These practices are (a) communicating to the students that they were expected to collaborate, (b) giving students reasons for collaboration, (c) creating opportunities for collaboration to occur, (d) fostering collaboration, (e) assessing if students saw the need for collaboration, and (f) balancing the support of discourse and algebraic reasoning. In this category, I will discuss these themes and the patterns of fostering collaboration that emerged from the data. I will also discuss how fostering collaboration supported students' algebraic thinking.

Communicating expectations and giving reasons for collaboration. Teachers communicated to the students expectations to work with a partner as a team. Although this expectation was communicated to the students throughout the lesson, most emphasis was made at the beginning of the activities. Teachers gave reasons to students about why it was important for them to collaborate with partners. These reasons included telling students how collaboration may help them in their mathematical reasoning. For example, in one class, a teacher told the students "We are here to broaden our minds, use good thinking, and when we learn from each other we get so much more." According to another teacher,

> … you always have to give kids a reason for what they are doing. Just to make sense, I think as a teacher my job is to help them make sense out of their world. I know because as a student, I always asked: why am I doing this? What purpose does this have? And nobody ever answered those questions. And when I look at them learning, they want to know why they have to do this, what does this mean to them. And especially with math . . .

Creating opportunities for students to collaborate. Teachers created opportunities for students to collaborate as they explored algebraic tasks. These opportunities included fostering collaboration in small group and whole class activities. The following sections report how teachers fostered collaboration in small and whole group activities and how they made these collaboration activities support algebraic reasoning more.

Creating opportunities for and fostering collaboration in small groups. Teachers created opportunities for students to co-construct mathematical ideas with peers in small groups. Students were encouraged to "work as a team with a partner or partners."

Students worked in groups of two or three as explained in Episode 4. Teachers considered several factors when grouping students. They assessed each student's engagement with the tasks during small group activities to see ways of pairing students that may encourage engagement (see episode 4). Students' assessed ability in algebraic reasoning was also used in making decisions on how to group students. Students with seemingly like abilities tended to be grouped together to avoid tendencies to disengage and depend on peers' reasoning by those who may perceive themselves as less capable (Episode 5).

> Episode 4
>
> Most of them are just in groups of two, so they have a buddy that they work with. A couple of them I have might have three in a group if there's an odd man out, or whatever. And I did have to change a couple of groups that were not staying on task and staying focused and they were doing a little bit too much playing around . . . And so a couple of them, they don't work very well together. Like, they were fussing over the blocks or whatever. But the fourth and fifth grade students are awesome working together and they really get into it, and they work really well.
>
> Episode 5
>
> You usually are going to have one (student) that is a little bit quicker to be able to understand what is going on, and then you will have one that is kind of copying or you know, just saying, okay. . . . So I have like the third graders working with third graders, fourth graders with fourth graders, fifth graders with fifth graders, so they are the same age group.

As the students worked in small groups, teachers fostered collaboration by encouraging dialogue between students. Students were expected to explain their thinking and engage in mathematical argumentation until each member of their group made sense of their ideas. Normally, students were encouraged to discuss their ideas before writing them down. In cases where some students wrote down their ideas before discussing them, partners were encouraged to ask questions like “why did you do that, or why is that on your paper?” especially if a partner “does not see what (mathematical ideas) the other one sees.” For example, when students were discussing their thinking about the square number task, teachers went around the groups to monitor students’ work, they asked members of the groups to explain their thinking. When a student used “I” in explaining an idea, some teachers normally asked such a student, “did you work with a partner?” In response to such a question, students changed to using “we” in their explanations, acknowledging that their ideas were co-constructed. Teachers also encouraged collaboration by questions like; “What did you talk about with your partner?”

Almost all the time On Track students worked with a partner. However, exceptions existed. Teachers allowed students who would rather work alone to do so (Episode 6). An exception also existed when a teacher chose to have one student work without a partner (Episode 7) to give all students opportunities to explore mathematical relationships. For students who worked without partners, teachers created opportunities for them to share their ideas in small groups. They asked such students to explain as clearly as possible to convince their peers about the observed mathematical relationships and conclusions.

> Episode 6
>
> I give them a choice. When we started this I said okay, every time we do this you can either work directly with your partner and you and your partner can try to solve this together, or you can work individually and then compare. I would rather they have the choice. And they all choose differently. Some of them really work together, like Jane and Joe always want to work it together and talk it through.

> And then I have others that really want to work it on their own. Nate needs to focus on: let me figure this out in my own head, and then he wants to look at what the other person does and kind of figure out and compare. That gives them the opportunity to do it in a way that they are comfortable doing it.
>
> Episode 7
>
> She will normally be the first one to figure it out. I tend to put her by herself, because I noticed that, when she is in a group with someone, I could not really see what they were doing, because she could immediately catch on. So what I have tried to do, I have one really strong (student) who can do that. Then I have two more, because they have done it (algebraic activities) before, that tend to remember. So now I have also started separating or pairing those two. So if those two are not working together then they are not working with anyone else either. So I can kind of see. And then what I have been doing is, once they get it, I have been seeing how they can take what they know, and go to the third graders, who are not as strong in multiplication, and guide them without telling them. So, when they get over there (to the third graders), and they can't tell them, they are like, 'I don't know.' I tell them, sure you can. What thought process did you go through? Now see if they can get there. That has worked, I like the way that works.

Creating opportunities for and fostering collaboration in whole class discussions. Teachers fostered collaboration through whole class discussions. After students worked in pairs, they presented their ideas to the whole group. In most On Track classrooms, students were expected to present their ideas together with their partners. As they presented, they shared responsibilities. In Figure 7, students collaborated to share with the class their observations on the square number task. At times, one student was responsible for displaying their work while another explained their reasoning. Although often times, only one partner presented their ideas. Like in small groups, students who worked with partners were encouraged to use "we" to show that the ideas were co-owned. Throughout whole group discussions, teachers encouraged students to be good listeners, telling them, "Because if you are almost there and you have not quite gotten it and somebody else may do something to trigger it (your understanding) so that is why you have to be a good listener."

Figure 7. Students Collaborate in Sharing Their Ideas during Whole Class Discussions

Assessing if students find collaboration important. Teachers asked students if they found collaboration helpful. They encouraged students to reflect on how sharing ideas deepened their understanding. They asked students if their peers helped them in algebraic thinking. Teachers also checked if students found collaboration important by inquiring about their feelings when others listened to them as they were "trying to share their best thinking." It should be noted that in On Track classes, listening to partners as they shared their thinking was always praised.

Balancing the support of discourse and algebraic reasoning. In addition to the discussed practices that supported collaboration, teachers purposefully selected and sequenced responses during whole class discussions to balance their support of discourse and development of algebraic ideas. Selecting and sequencing responses were the purposeful decisions about what responses should be discussed during whole class discussion and in what order. Teachers used three criteria to select responses for public display during whole group discussions. Teachers made sure that “everybody gets a chance to share their work.” Each student may get a chance to discuss their work with the whole class on each class day or overtime (Episode 9). One strategy used at one school to achieve this goal was by writing student names on Popsicle sticks and putting the sticks in a box. Once a student shared his or her work to the whole class, the Popsicle stick with that student name was transferred to another side of the box. That way teachers tracked students’ chances to participate. Teachers combined this selection criterion with other strategies so that those students whose names were called to discuss their ideas from the Popsicle sticks did not disengage from the whole class discussions after their presentation.

The other criteria depended on going around the class during small groups and trying to understand student reasoning as they work out the mathematical tasks (Episode 8). Selection of responses were guided by the level of sophistication of student responses. Responses selected included both low (recursive generalizations) and high (explicit generalizations) level of sophistication. Low levels of sophistication were selected to commend everybody’s thinking (Episode 8).

Episode 8

> Usually it is someone I have talked to. I try to talk to every one before whole group discussion. If I have seen their work I try to pick some of the easier strategies and the more obvious patterns like the recursive ones so those kids feel: oh yeah I found something.

The third criterion for selecting responses was the perceived difference in student responses. Teachers selected responses that they believed were different from other responses presented during whole class discussion as explained in episode 9. Teachers also considered students' view on whether their responses were different from the others. They were often asked: "if you have a different generalization or strategy, come up and share it."

Episode 9

> I look to see who has a little different slant on it (strategies to particular tasks). You know, to where the other kids can say: oh well, I did not look at it that way. And then get them to thinking divergently more. You know. And when we do not have time for all of them to share in one day, I have been rotating because most of them want to share and explain their thinking.

Selected responses were purposefully sequenced. Generally, less sophisticated responses were presented first (Episode 10). The more sophisticated responses or the "ones that show the highest level of thinking are saved for last." Two reasons were common as teachers' rationale for this sequence. As in Episode 11, teachers aimed at commending all thinking levels and motivating students to share during whole class discussions. Some teachers saved sophisticated responses for last because although they wanted all students to feel good about their effort and their thinking about particular tasks, teachers also wanted students to see that some strategies are limited in the types of ways they can be justified. One teacher explained, "I wanted the others (with less sophisticated generalizations) to present theirs and realize when I ask them to "prove it" that they had no way to do so. I was hoping that when she (a student with an explicit generalization) then presented that some of the other student might start to go deeper in developing their generalizations." Teachers believed students do not notice this sequence during whole group discussion. However, an exception may have occurred at one school. During class the teacher said, one student "always gets mad at me, because I pick her last." In this class, this student noted that she is usually the last one to present her idea, however, these data do not show if the student realized that she was picked last because her responses were often more sophisticated.

> Episode 10
>
> Like sometimes, not surprisingly, it will be the younger kids (grade 3 students) that will have just the recursive generalization and the more simplistic thinking. I let them go first and then let the older ones that have thought it out more deeply present after. So it is kind of a sequence of the easiest to more complicated strategies.
>
> Episode 11
>
> If you pick a kid that has figured out what the function is first, then everyone else after that is like, oh mine is not as good. Mine is like the small one, the not so smart one. I try to pick responses so it is like a progression of easier to harder. I don't think anyone would ever notice that is what we do. But we try to do it that way to keep everyone's answer justified. But to go backwards, I think it would make few people want to share.

Patterns of practices that fostered co-construction of ideas. During data analysis, a pattern of practices emerged (see Figure 8). Recalling the definition of routines of practice as a collection of related practices, this pattern shows the relationship between those practices and further informs the routine of fostering co-construction of ideas. Teachers generally started by communicating to the students that they needed to collaborate. They created opportunities for students to create through whole group and small group discussions. After creating these opportunities, teachers used multiple ways to encourage collaboration in both whole group and small group discussions. Teachers then checked if students found collaboration helpful with their algebraic activities.

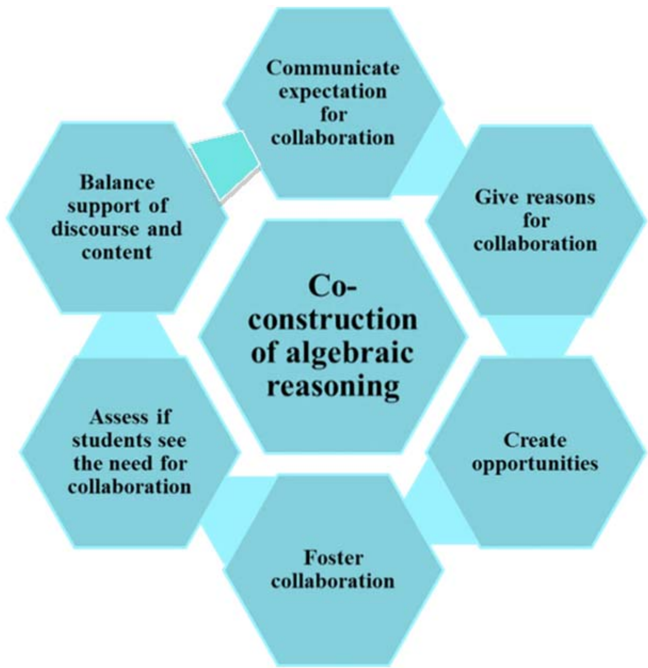


Figure 8. Routine of Practices that Nurtured Co-construction of Algebraic Reasoning

It is very important to note that these five practices for supporting collaboration (communicating expectation for collaboration, giving reasons, creating opportunities, fostering collaboration, and assessing if students found collaboration important) were not mutually exclusive. Although in these data encouraging collaboration mainly took the form of reminding students to collaborate and acknowledge that their peers contributed to their mathematical thinking, teachers sometimes encouraged collaboration by constantly giving reasons for collaboration. It is also important to note that, these practices although not necessarily linear depended on each other. For example, students had to have opportunities for collaboration to be able to assess if collaboration was helpful to them. Teachers also sometimes gave reasons for collaboration after assessing students' view of the role collaboration plays in their algebraic reasoning. Construction of ideas was further supported by practices regarding collaboration were further supported by practices that aimed at balancing the support of discourse and content. Teachers encouraged students to share their ideas. They purposefully selected and sequenced responses in ways that they believed brought out the mathematical content in the talk and allowed students to access other students' thinking to construct their own understanding. Thus, these practices worked together to support students' co-construction of ideas.

How co-construction of ideas supported algebraic reasoning. Collaboration supported students' algebraic thinking in different ways. For example, as a teacher explained, "We did have one time when somebody thought that there was a second generalization that could have worked, but then as they were sharing it, they realized that doesn't work anymore." Episode 12 is a case where a student realized during whole class discussions that his rule does not work. Through collaboration, students evaluated the validity of their algebraic generalizations. They were able to see divergent ways of approaching algebraic tasks (Episode 13). Collaboration also helped students to reason beyond what they could on their own, as a teacher noted in episode 14. Building on ideas of others occurred in both small groups and whole class discussions (see Episode 15).

Episode 12

Student: The number S times 5 minus the next number equals

Teacher: And S is what
Student: S is the number of pentagons, like the input. And N is the next input. For example you could have 3 times 5 minus n equals; so will be minus 4 equals; 3 times 5 equals 15 minus 4 would equal 11

Teacher: And the 4. Explain the 4 again. So 3 is the input and you multiply that by 5

Student: N is one more than the number you currently have

Teacher: You have multiplied every number by 5 and then subtracting the next input. Everybody see that? I just want you guys to look at one of your input numbers, multiply it by 5. Let us do 4 together. Ok? So we have 4, multiply that by 5. What is 4 times 5?

Class: 20

Teacher: And then we have to subtract the next input from 4 which is 5 and we get what

Class: 15

Teacher: So 4 x 5 is 20, minus 5 is

Class: 15

Teacher: Is that what you got for the input of 4?

Student: I think I put something wrong . . . It won't for (inputs of) 1 and 2 . . . Wait oh I messed them up

Episode 13

They (students) are able to see different approaches to the problem, or different generalizations, and how this student may have come up with the same generalization, but approached it differently, or worked it out differently. So that communication is building their thinking. It is expanding their reasoning skills I think. Because they are able to hear what someone else says, and acknowledge it and accept it. And I think it is just building that acceptance, like, ok, there is not just this one way to get it. There are several ways. So I think that in general, when we do whole group discussions they build that.

Episode 14

> It is interesting how it works out; one of them will usually say something and it kind of triggers some thinking with the others. I think collaboration is really important because they can hear what the other one is thinking and then they can kind of put their thinking together and it really builds on that way . . .

Episode 15

> At first it is making (mathematical) connections with each other in their pairs, and they do not always work with the same partner. So I think they are able to make more connections each time when they are not working with the same person, or they are able to understand other people's thinking, or listen to other people's understanding of different concepts. But you know, first it is their connections, it is making connections with each other, and then I talk about that and I ask; what do you think? Or, do you agree? Or, did you find something out that is different within the pairs? And then when they get up to share (during whole class discussion), I ask the same questions as with the groups: who came up with something different, or who came up with something similar to what has been presented? So it starts off in pairs and then it ends with connections to the whole group.

Summary of practices that nurtured co-construction of algebraic reasoning. Teachers created opportunities for students to share their ideas and learn from others in small group and whole class discussion. Students who preferred to work alone during small group discussion were asked to explain their ideas to others. During collaboration, students built on other people's reasoning and were able to assess the validity of their own algebraic rules. Purposeful selection and sequencing of responses during whole class discussions further supported students' algebraic thinking. Students also made connections between their reasoning and other students' strategies.

Routine of Practice: Fostering Understanding of Variables

Understanding of the concept of variable in On Track classrooms entailed (a) being able to identify independent and dependent variables from a mathematical problem's context, and (b) being able to associate variables with their corresponding values. Based on their assessment and teaching experiences, teachers reported that students lacked an understanding of "what it means to have a variable and what it means to have values of a variable." Therefore, to support students' algebraic thinking, their routines included practices that fostered students' understanding of the idea of inputs and outputs, identifying and naming variables and corresponding independent and dependent variable.

Fostering understanding of the idea of input and output. A snack machine was a common representation used to discuss concept of input and output (see Figure 9). Teachers used it to discuss that the input is what has to go into the snack machine to produce the output (Episode 16). A snack machine was also preferred because students were familiar with it. Some teachers used "in" and "out" instead of input and output until the teachers believed (based on their observations) that the students were ready to move to these terms.

Figure 9. A Gumball Machine Used to Foster Understanding of Input and Output Variables

> Episode 16
>
> The gumball machine just came to me as I was thinking about how to present function machines in a way in which the students could relate. It was something I knew they would all know. It was colorful to catch their attention; it is a fun thing in and of itself . . . and it seemed to cement the concept of putting something in while getting something different out. I used it until they all said “oh no, not the gumball machine again!”

After students got familiar with inputs and outputs, they identified an output corresponding to a given input from a mathematical story. Asking students to identify and name both the dependent and independent variables followed. Students were asked questions like “how did you determine what your input is?” or “why did you choose this as your output?” At all levels of understanding variables, teachers asked students to enter or generate data for input and output variables.

Fostering understanding of a variable as a changing quantity. As students explored patterns in the algebraic reasoning tasks, teachers fostered understanding of a variable as a changing quantity. For example, when students were working on tasks with geometric representations and building models for those tasks, teachers asked students to pay attention to what was happening with each construction as the representations moved from one stage to another. For example, as students were working on train table tasks, teachers asked students to observe how the number of people (output variable) was changing. In response to such practices, students described the change of variables by describing recursive rules. Thus, teachers supported understanding of a variable as a quantity that changes.

Encouraging students to correspond independent and dependent variables. As students looked for mathematical patterns in their data sets, they tended to focus on one variable at a time and consequently developing recursive generalizations. In such cases, teachers encouraged them to look for patterns that showed correspondence of values of the dependent and independent variables. Normally students looked for a relationship between two corresponding values and tried out if that relationship held for all other corresponding values (see Episode 17).

Episode 17

> Mathematically how does this number (input values) become this number (output value)? And then that is what they keep looking for. They say, well I can add this, and I say, okay then go to the next number on the list, if you add that again is it going to work? If the answer is no, I ask: then mathematically how did it happen? It had to happen some way mathematically.

When students had difficulties finding relationship between independent and dependent variables, they tended to focus on one variable (normally the output) and develop recursive generalizations (see Episode 18). Teachers accepted recursive generalizations and encouraged students to continue looking for different patterns. In On Track classrooms, students were constantly reminded to look for patterns that related dependent and independent variables.

Episode 18

> And when they get stumped, they still want to go to that recursive pattern, but they know they are going to have to look for another pattern. So I like knowing that they realize that there is more to this than just looking at these outputs and

> seeing how the outputs are related. I like seeing that they understand I have got to look for a relationship between the input and the output.

How fostering understanding of variable supported algebraic thinking.

Supporting students to understand independent and dependent variables helped them as they were writing rules. Students focused on finding the mathematical operations that needed to be carried out on the input to be able to get the output. Knowing independent and dependent variables, and the exercises of naming variables were important when students were asked to write explicit correct rules that referred to the variable names. Additionally, accepting recursive pattern as correct and then asking students to look for patterns that corresponded independent and dependent variables acknowledged students' different reasoning levels. It also created opportunities for students to search for multiple strategies for approaching the tasks and for students to engage in mathematical conversations. For example, in episode 18, after a student found a recursive rule for the square table task, the teacher encouraged them to discuss their ideas with partners and look for explicit generalizations. This student later announced that he found an explicit rule.

> Episode 18
>
> Teacher: And we were talking about looking at patterns on the table. And a lot of people saw a pattern this way. What pattern did you guys see this way?
>
> Student: adding two.
>
> Teacher: You are adding two. Okay, 'If you look this way (points at the output values) you can see that they are adding two each time, right?
>
> Students: Yes

> Teacher: Yes. When you look at the vertical pattern. The only thing about doing that was? What did you come up with doing it that way?
>
> Student: You have to go all the way to 100 (when you want to find the output for input of 100).
>
> Teacher: Yes, yes. Which, sometimes you don't want to do that. You don't want to go all the way to 100. So we were looking for a pattern between these two numbers (input and output values). You see how is one related to four? So you need to look for different patterns. So I want you guys to start looking not only for this pattern the vertical pattern, but I want you to look horizontally for a pattern as well. Okay? Play with your blocks with partners and see if you can figure that out.
>
> Students: (Inaudible talk)

Summary on supporting understanding of variables and algebraic thinking. In this category, teachers promoted students' understanding of variables by using representations that explained ideas of inputs and outputs. Students were encouraged to identify independent and dependent variable. Although both recursive and explicit rules were accepted, teachers encouraged students to find rules that related the different variables. These practices supported students' understanding by encouraging them to explore different strategies to solve the tasks and to think of the variable names as they made their generalizations. Another routine of practice that worked in conjunction with these practices is connection of mathematical ideas.

Routine of Practice: Creating a Context for Mathematical Connections

As a routine of practice, supporting mathematical connections is a practice where by teachers make attempts for students to see how their mathematical ideas connect to ideas from different contexts. These contexts are; students' everyday experiences, different mathematical tasks, different representations of ideas in tasks, different curriculum ideas and different strategies of exploring the same task. This section discusses how teachers supported students' activities in connecting mathematical ideas within and across these contexts and how this practice supported students' algebraic reasoning.

Connecting algebraic tasks and everyday experiences. On Track teachers' routines of practice were characterized by creating or taking up opportunities for students to make different connections. From this data, teachers made connections between the classroom activities and students' everyday life in ways that supported reasoning. An example of this connection is the use of snack machine that was discussed as a representation for input variable and output variable. In a different example, before students worked on the perimeter task to find patterns between stage numbers and the perimeter of the shapes, they went out to run around a basketball court to discuss what perimeter is. In a classroom Episode 19 and Figure 10, the teacher connected the square table task to a real life experience of inviting people for dinner. One teacher's rationale for connecting algebraic activities to everyday life is in Episode 20. Connections were made between these algebraic activities and everyday life so that students could be aware that thinking exercised during On Track is useful in all their activities. In a classroom Episode 21, a teacher explained to students that the problem solving process they go through as they engage in On Track activities is similar to problem solving process in everyday life. Additionally teachers made connections with every day activities because when students make such connections, the concepts "become more real to them. And they can think about it more concretely when they have a connection that they can make with something that they have experienced." Discussions on such connections were not limited to On Track activities. Teachers discussed with the students how mathematical activities are part of everyday life (see Episode 22).

> Episode 19
>
> Teacher: We are going to go back to our square tables. So everybody put a square table in front of them just put one. If you have a square table in front of you and one person can sit on each side of the table, how many people can sit at this one table?
>
> Student: 4.
>
> Teacher: So I have a table and if I am inviting my friends over I can only invite enough friends that can sit around the table. How many friends can I invite for dinner to sit around my square table?
>
> Student: 3. You can only invite 3 so you can have a seat yourself too.
>
> Teacher: Ahh that was a trick question and you absolutely got it. I can only invite 3 because don't I want to hang out with my friends? Now put 2 tables together. How many guests including myself can sit around my table?
>
> Students: 6
>
> Teacher: I can sit 6. Let us check if that is right (draws smiley faces around the table). I have lots of happy friends.
>
> Students: (laughter)
>
> Teacher: Here are my 6 people at my table and you are right, I can invite 5 people because I would want to hang out with my friends. So now, you go ahead and make your square table

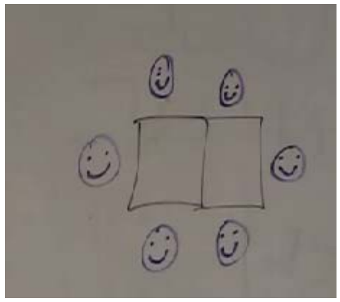


Figure 10. A Representation Connecting the Square Table Task to Real Life Experience

Episode 20

I want the students to see the relevance of the work they have been doing and to think about how it can help them in their everyday learning versus in isolation in this math "club." I also believe that this class will help them with all problem solving activities and just wanted to help them see that it is relevant to "school."

Episode 21

I hope this (On Track activities) has helped you in more than just our time together. I hope this helps you in your math class, in school, and just helping you figure out problem solving. Remember there is always more than one way to look at things. So if you figure out something and say, oh I think I got it, and you don't get it, well the fact is you thought about it. And now you just need to think about it differently. And that is problem solving in life. Think about it, if it doesn't work out; think about it differently. And you will find a huge, huge success that way.

Episode 22

Everything is math. They try to challenge me and they are like reading is not math. I ask them, how many pages did you read last night? I can turn about anything into math. And just help them realize that their world really is mathematical. Our world runs on money, which is math. Money and time are the two things our world runs on. And that is all math.

Connecting ideas from different tasks. Connections were also made between mathematical ideas from different tasks. Teachers made these connections by referring back to previous activities as they launched new activities as explained in Episode 23. This way of connecting aimed at assessing students' previous understanding and for teachers to position themselves to better attend to student thinking. The tasks themselves also had connected mathematical ideas. For example, the square table tasks and the pentagon task could both be solved by observing that number of people that can sit around each train is dependent on number of seats each table on the train is contributing to the train. Teachers called on students to think about mathematical ideas they discovered previously and see how those ideas related to tasks they were working on. Episode 24 is a classroom episode whereby a teacher was encouraging students to make connections between different mathematical tasks.

> Episode 23
>
> I like to refer back to previous lessons as a way to start the thinking process and get the students involved in the new day. I think that starting off where we left off helps to build on the learning as well as helping me understand if the students first, understand what we have learned in order to move forward or if I need to reteach anything. And second, to see if having a few days to absorb the learning has helped them advance their thinking without further teaching. This helps me know how much I can challenge them. Either way it helps me to launch into whatever the lesson is at the point of knowledge where the students are.
>
> Episode 24
>
> Let me show you the next activity. It is about something we have already done and something we have not. So try to put your input-output table together and make some connections. So you should be making connections between the different patterns you have done with the blocks and see if you can see a pattern between them . . . You can see that it is

some kind of growing generalization or some pattern that they have in common.

Connecting ideas across the curriculum. Teachers encouraged connections between On Track activities and other ideas in the curriculum. Developing fluency in multiplication was commonly incorporated into On Track activities. Figure 11 shows teachers connecting On Track activities with multiplication strategy for 10 and 100. In another class, students connected the square number task to ideas on exponents. In Episode 25, students were teaching their peers about this connection. Teachers also took many opportunities to connect the tasks across the curricular by using terms in the activities. For example, discussion on geometric shapes arose when students were exploring the relationship between pentagon trains and the stage number of pentagon table task.

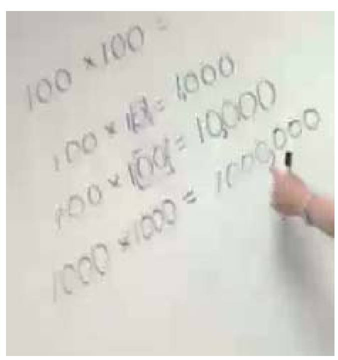


Figure 11. A Teacher Taking up Opportunity to Teach Multiplication Skills

Episode 25

Teacher: Ok girls, would you be able to explain the exponent that you all came up with (one girl goes to the document camera). You go and help her explain it (talks to the girl’s partner). I want you to listen because there is different way to do this task called square numbers. There is such a thing called squaring numbers . . . So explain to us how it relates to this particular input-output task.

Student: yeah exponents are like the number say like 10, (writes on “exponents” on active board) so exponents are like let us say you take a number 10 you write a little 2 (writes 10^2) . . .

Teacher: (to the partners presenting) How do you read 10 with a little 2?

Student: 10 to the power 2

Teacher: Or you can say 10 squared right? 10 to the second power.

Student: so (10 x 10) x 10) will be 10 cubed.

Teacher: That will be 10 to the third power right?

Student: Yes it would. Emelyn you can do this (asks partner Emelyn to explain).

Emelyn: So if you have a 2 right here (points to power 2) . . .

Teacher: Which equals what?

Emelyn: Which equals 100 (writes 100 on the board)

Teacher: So what if you had 10 to the power 3 what will that be?

Emelyn: (writes 10^3 and (10 x 10 x 10) on board) 10 to the power 3 will be 3 tens so 10 x 10 x 10.

Teacher: So how does exponent relate to this task with square numbers? Can you show the relationship? Write the generalization using exponents. (Emelyn gestures her partner to explain)

Student: Like this (writes 2 x 2). You can do the exponent (crosses out x 2 and writes power 2 and continues to write the output values as squares of the input values while explaining this idea)

Using and connecting multiple representations. In On Track classrooms, students were encouraged to use different representations. More often than not, students used physical manipulatives (see Figure 12). Students used pattern blocks to explore patterns when working on square table, pentagon table, perimeter, and square number tasks. Students were encouraged to draw the geometric representation of the tasks (see Episode 26). According to the teachers, drawing geometric representations of the tasks helped students to identify what is varying and what is constant, specifically with the table tasks. Some teachers noted that, some students “use the manipulatives more and they build more with the manipulatives, rather than drawing it on their papers. Whereas others do not want to use the manipulatives, and they just wanted to draw it.” In such cases, students’ preferences were considered but teachers still helped students to see how they might benefit from using both geometric representations and using manipulatives to represent the problems. For example, in one class in Episode 27, the teacher helped the students see that because they did not represent the task using manipulatives, it was difficult for them to collect their data correctly.

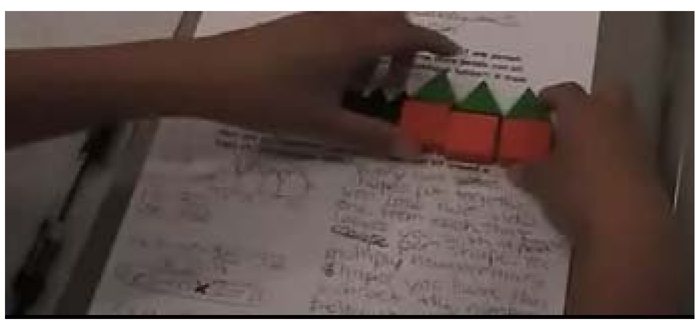


Figure 12. A Student Models the Pentagon Table Task Using Physical Manipulatives

Episode 26

Some just want to look at the mathematical part of things with the (input-output) tables, but generally they had to draw them. They had to draw the tables at the beginning (of the task), but then after like four tables it was

> like, I don't need to waste my time, I can see what is mathematically happening. But in actuality they needed to be looking at the pictures, to see what was actually changing. Yes, there was a generalization, but what in the actual manipulative representation, what was that showing?
>
> Episode 27
>
> Ok so when I first gave you this activity today I said, ok go ahead and build your tables and not a single person did that. And now you started making your input output tables and you didn't count around your tables. And I came around and said, ok how did you get your number? And you showed to me and you realized your number were often times wrong. That may be something you want to think about.

Students were expected to collect data from their explorations and systematically organize them in input-output charts. As explained in Episode 25, students started exploring patterns with manipulatives and pictorial representations and moved to exploring patterns using input-output tables. Figure 13 is an example of a student's input-output table for the pentagon table task. Using t-charts made it easier for students to identify patterns because it "isolates pertinent information from everything else in the word problem." Since students tended to look for patterns in one variable and developing recursive generalizations, teachers used t-charts to encourage students to look for patterns between the input and output variables and come up with algebraic equations. In Figure 13, a student used an input–output table to explore relationships between independent and dependent variable. The exploration led to a conclusion that, for the pentagon task, multiplying the number of pentagons by three, and adding two to the product gives the number of people that can sit around a train of pentagon tables.

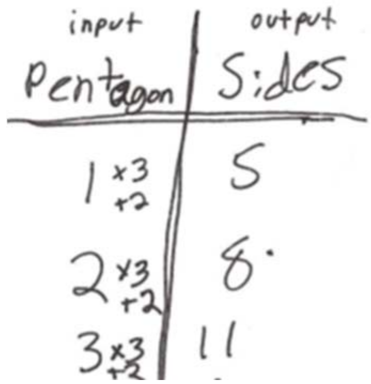


Figure 13. Student's Input-Output of the Pentagon Table Task

Connecting mathematical ideas from different student strategies. As reported earlier, whole class and small group discussions created opportunities for students to connect their reasoning to other students' reasoning. Teachers discussed how students' ideas were related to one another after using a variety of colors to write different student ideas on the whiteboard (see Figure 14). For example, in one class while working on the relationship between time and distance of a function machine task, one student referred to figure 15 and explained her strategy as "the number you have in minutes is going to be the same number in feet with a five at the end." To connect this strategy to the one whereby students multiplied minutes by 10 and then added 5 to get number of feet, the teacher asked, "why is this generalization of writing a 5 in front working? Why is it giving the same number as multiplying by 10 and adding 5?" These questions led to a discussion about place value that by writing a 5 in front of the number, the value for minutes gets promoted to a tens place therefore the strategies are mathematically the same.

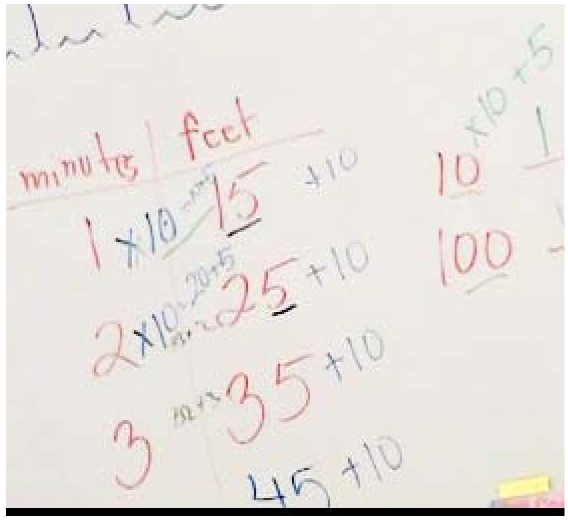


Figure 14. A Display of Student Strategies Used to Approach the Function Machine Task

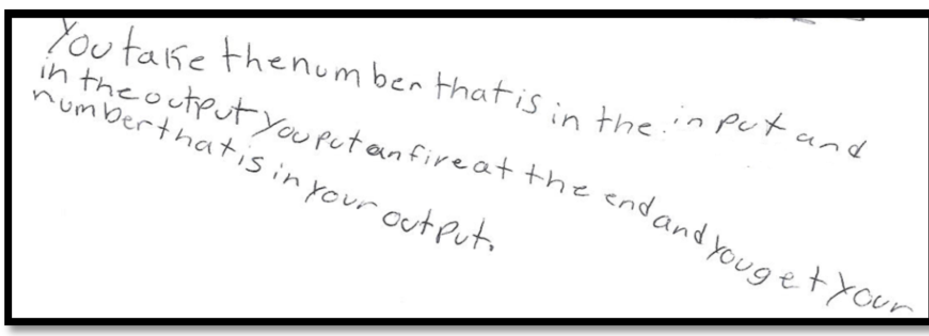


Figure 15. Student's Generalization about the Relationship between Time and Distance Based on T-chart in Figure 10

How connecting mathematical ideas supported algebraic thinking. Most of the rules for On Track tasks used multiplication. For example; $2x + 2 = y$ for square table task, $3x + 2 = y$ for pentagon task, $x^2 = y$ for square number task, and $4x = y$ for perimeter task where x is the independent variable (input) and y is the dependent variable (output). As noted, these generalizations required some understanding of multiplication.

By also focusing on developing other skills in the school mathematics curriculum, teachers positioned students to be able to make and understand these generalizations.

Teachers also found geometric representations to be very helpful with students' algebraic activities, especially with understanding what is constant and what is changing in the geometric patterns (Episode 28). In Episode 28, a teacher explained that using pictorial representations helped students to discover that on the square table tasks, there is a 2 that is constant on each train, and the other number of seats is two times number of tables. This discovery helped students understand the rules that they found using the t-charts more.

> Episode 28
>
> I think it (a pictorial representation) helps them to be able to visualize, for example with the tables that we have been having with people sitting around (table tasks) . . . it was just easier for them to visualize it when you know, you had the two on the side and you take the top number and the bottom number . . . but without that picture it may not have made sense to them.

Table 5 is a series of a student's work that agrees with the teacher's observation in Episode 28. After a student explored patterns with geometric representations of the square table task, he wrote a recursive rule. When he continued his explorations with a t-chart, he related the independent and dependent variable and wrote a correct explicit rule. This student went back to the geometric representation to explain that his explicit rule is valid because the number of seats would be equal to two times the length of the train table plus the two ends. This example shows student's algebraic reasoning through connecting ideas from different representations.

Table 6

A Student's Series of Reasoning between Different Representations

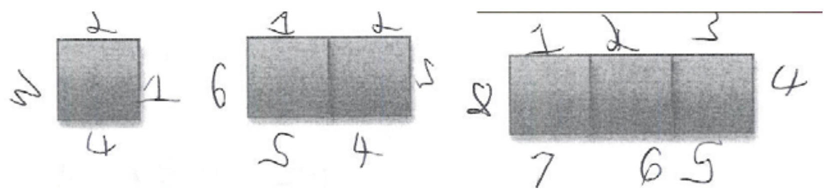
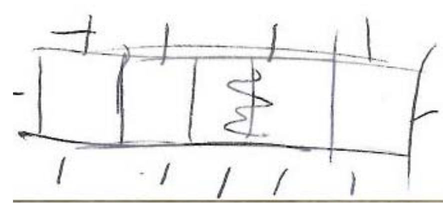
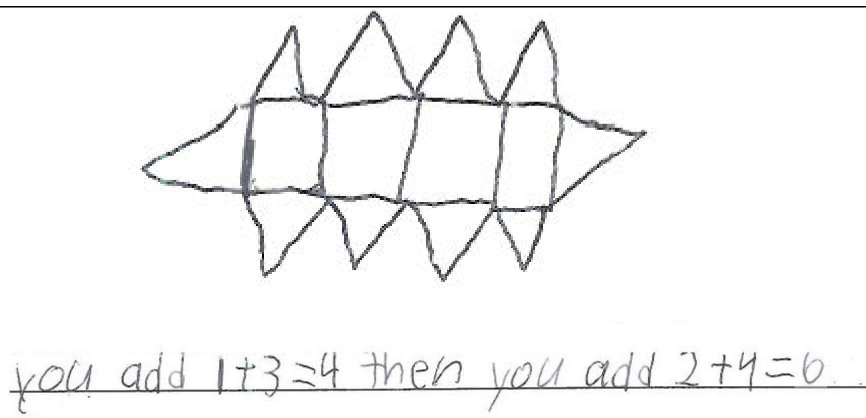
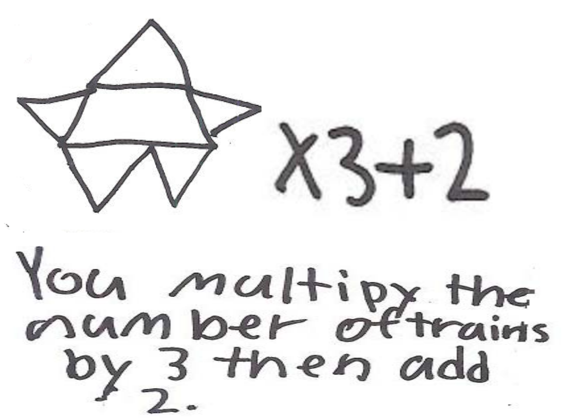
Explorations with geometric representations	
Pattern observed after geometric explorations	add 1 table to the group you add 2 chairs
Explorations with t-chart	table | chairs 1x2+2 = 4 2x2+2= 6 3x2+2= 8 4x2+2=10
Pattern observed after explorations with t-chart	~~add 1 table~~ the input x2 +2 = out put
Geometric representation used to explain pattern observed from t-chart	

Table 6 is another example of how creating mathematical connections supported algebraic reasoning. In this example, a student made connections across tasks and different student strategies to reason about the pentagon table task.

When this student was working on the square table task, she seemed to have observed a pattern between input and output values. She showed that for input of 1, the output is 1 + 3 and for 2 square tables, the output is 2 + 4. Although this reasoning could lead to a generalization of x + (x + 2) = y for x tables, she did not take it further beyond using input of 1 and 2 and could not explain it. From her drawing, it also seems she reasoned by considering available seats on each train. It appears she was on track to generating an explicit rule although she did not.

Table 7

Student's Connection of Mathematical Ideas

Task	Student's Representations and Generalizations
Square table task	
Pentagon table task	

During whole class discussion, another student described his explicit rule as 2n + 2 = p (where n is number of tables and people is number of people) and explained that this rule works because each square table contributes 2 seats to the train and the train had 2 other seats on its ends. More than a month later, this student was exploring the pentagon table task. As seen from her representation, she used reasoning from previous task (square table task) to reason about the pentagon table task that 3n + 2 = p where n is the number of pentagon tables and p is the number of people that can sit around it.

Connecting mathematical ideas across contexts. These five contexts as represented in Figure 16 together supported students' algebraic thinking. As discussed, connections within one context (e.g., student strategies) also led to connecting mathematical ideas with other contexts (e.g., curriculum and tasks). The discussed example of a students' work in Table 6 illustrates the connections across different tasks and different strategies in response to teachers' routines of practice. It is therefore important to note that there were connections within and between contexts.

Summary on connecting mathematical ideas and algebraic reasoning. Teachers situated learning for students to make connections of mathematical ideas from different contexts. These connections were between the tasks and (a) students' real life experiences, (b) between students' strategies, (c) different representations, and (d) different ideas in the curriculum and across different tasks. Teachers positioned students to make these connections by asking them to think about how the tasks they were working on were similar to other tasks, asking students to identify underlying mathematical ideas of different strategies and asking students to share their ideas during whole class discussions. Connecting mathematical ideas helped students' algebraic thinking in coming up with the generalizations and checking if those generalizations were valid.

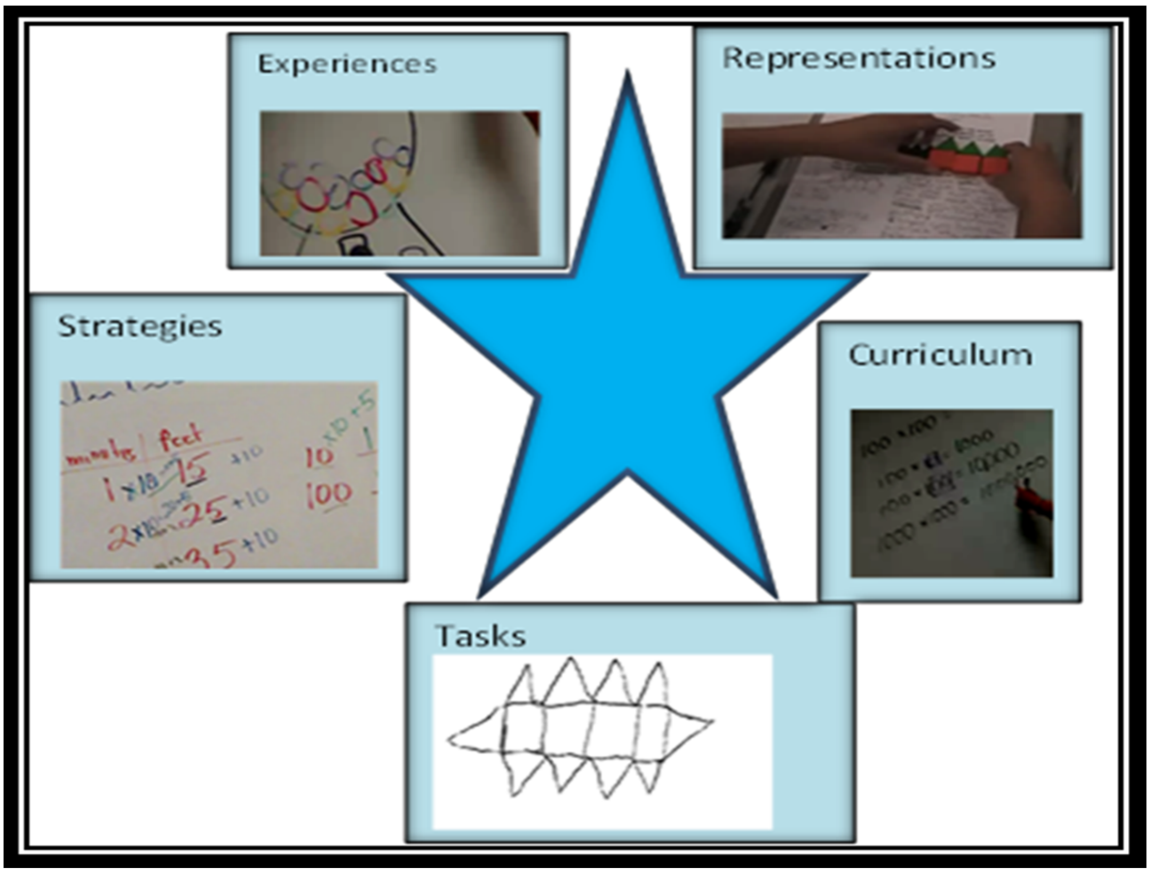


Figure 16. Five Contexts Used to Connect Mathematical Ideas

Routine of Practice: Promoting Understanding of Generalizations

Teachers reported that a successful algebraic reasoning class is the one in which students are not only able to come up with the generalizations but also to understand their own and their peer's generalizations. Based on the data, I defined understanding generalizations as being able to explain the thinking involved in the generalization and being able to apply the generalizations to solve related problems. Within this routine of promoting understanding of generalizations, teachers encouraged students to use variable names when reporting their generalizations, asked students to justify their conjectures, encouraged verbalization of rules and re-voiced students' reasoning. Students were also encouraged to apply and discuss efficiency of different rules.

Encouraging use of variable names. Teachers promoted students' understanding of their peers' generalizations (e.g., Episode 30). Often teachers made the students assume responsibility of making others understand their generalization. In cases where students wrote explicit generalizations that did not mention the names of variables to which mathematical operations can be carried out and what the output variable will be, teachers asked the authors if their peers can understand such generalizations. They asked students to put in a lot of detail as they write their generalizations to make it clear for others to understand. Questions to get students to think of including input variable names in their generalizations included "what are you multiplying?" or "what are you adding?" depending on the mathematical operations students had in their generalizations. To get students to think of including output variable names, teacher asked questions like, "what are you trying to get" when performing the mathematical operations in your generalization?

Episode 30

Teacher: Yeah, well just pretend that well there is somebody that is not very good in math, and they don't understand it. They don't understand what the input

> relationship is to the output. They want you to explain it to them, so that they would understand.

Using justifications as a tool for understanding generalizations. Teachers promoted students' understanding of generalizations made by themselves or their peers. Understanding of generalization was promoted through asking students to evaluate their mathematical conjectures. Teachers encouraged students to feel confident about their inferences. This was normally done during small group discussions when teachers often asked students "how do you know your rule works?," "test your rule," "convince me your rule works" or "does that work? Prove it." These practices aimed at promoting understanding of students' own generalizations.

In response to such justification promoting probes, students verified their generalizations by comparing if the output values they get by using their generalizations are the same as the values they get from the problem statement into the input-output table. For example, in Figure 17, a student argued that "an input value multiplied by itself gives the output" for the square number task is a valid generalization because the output values from this generalization is the same as the values from the t-chart. In cases where students may build or draw a geometric model, for example the table tasks, students verified their generalizations by comparing output values from their generalizations and those from their models. Teachers accepted and encouraged these types of justifications.

Teachers also tended to ask students to think of multiple ways of verifying their conjectures by asking questions like "is there another way you can convince me your rule works?" This practice was often associated with students' tendency to look for a different pattern or express their patterns in different ways. Justifications were then provided by checking if the different generalizations gave the same output values. For instance, a student's justification in Figure 18 used an explicit generalization (3 x number of pentagons + 2 = number of people) and recursive patterns (the output values increase by 3) to check that these two strategies gave the same output values and conclude that both generalizations were valid for the pentagon table task.

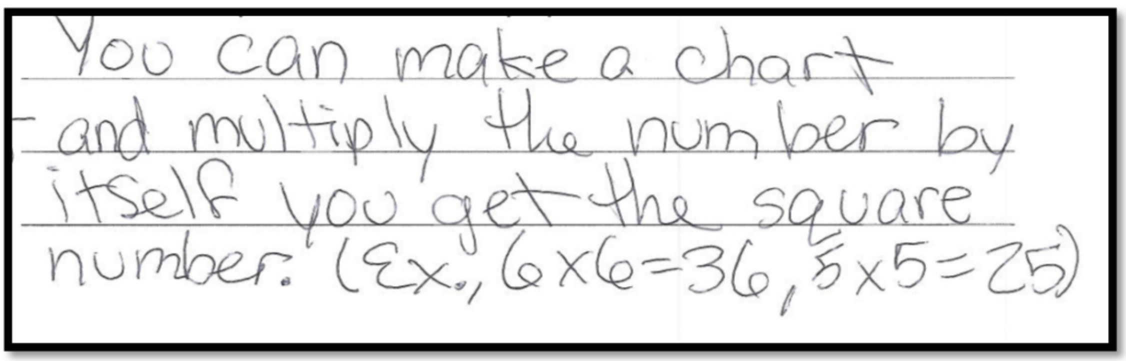


Figure 17. Justifying by Comparing Output from a Generalization to Output from a T-chart

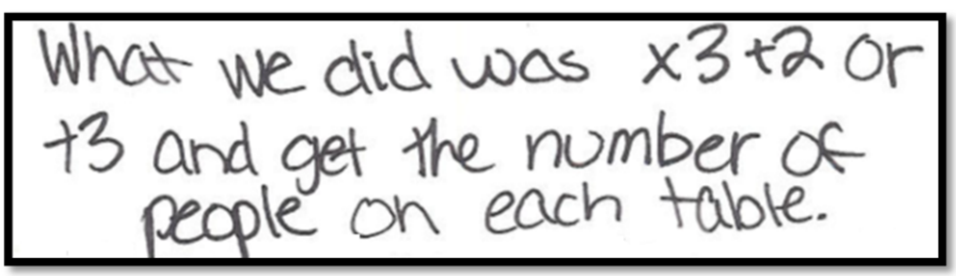


Figure 18. Justifying by Comparing Output Values from 2 Generalizations

Teachers also promoted student understanding of their peers' generalizations during whole class discussion. During presentations of generalizations, students were expected to convince the teacher and their peers that their generalizations were valid (Episode 30). Students evaluated their peers reasoning to see if it made sense to them. Teachers asked students to apply their peer's generalizations to see if they gave the same output values as their own generalizations.

> Episode 30
>
> Teacher: Okay, how many people think this is going to work? Do you believe it?
>
> Students: Yes.
>
> Teacher: I don't. She has to prove it to me. You have to prove it (talks to the author of the generalization). As she is proving it, take a look at the work on your paper and think about what she is saying. Listen to what she is saying, and see if it works for you.

In general, student justification practices aligned with their understanding of what it meant and what it took to convince someone. To the students, convincing others meant showing others how a generalization worked (see Episode 31). In addition, students' justification practices aligned with teachers' routines in asking students to justify their generalizations. Questions as how do you know it works or convince me it works were normally responded to with example-based justifications. Additionally, teachers promoted example-based justifications by asking students to try their rules out and see if they worked. Additionally teachers accepted and promoted empirical justifications whereby output values from students' generalizations were compared to values found by counting around geometric models (e.g., Episode 32). Thus, On Track teachers' routines of practice generally promoted understanding of generalizations through promoting empirical justification practices.

Episode 31

Teacher: What does it mean to convince me? What does it mean to convince me of something?

Students: showing you

Teacher: You can convince me that something is right by showing me. Ok what else can you do? Give me another word.

Student: Persuade

Teacher: Ok you want to persuade me so you want to use really vivid language. (Include) lots of detail about how you would convince me that your rule works. How would you convince me that your rule is the best way?

Episode 32

Teacher: Now I want you to prove that this works. Now this is how I want you to do it. I want you to figure out for input of 4 using the formula. Then I want you to build it and see if it works.

Student: Ok 4 times 3 is 12, plus 2 is 14.

Teacher: Put that down. Now prove it. Prove it by building a 4 pentagon table.

An exception in routine of practice aimed at promoting understanding of generalizations was when justifying generalizations about the square and pentagon train table tasks. Generalizations for these tasks are $2x + 2 = y$ and $3x + 2 = y$ respectively, where x is number of tables in a train and y is number of people that can sit around it. Teachers asked students to explain the relationship between the generalizations and other representations by explaining where the 2 x, 3 x and +2 come from. In response, students explained that, the 2 x and the 3 x come from the number of sides each table contributes to the train and the +2 come from the 2 sides at the ends of the train. This was an exception because these practices encouraged students to justify beyond trying out a few examples and required students to analyze the general context of the problem.

Encouraging verbalization of student reasoning. On Track teachers encouraged students to verbalize their thinking for two purposes. They believed that students "sometimes get a concept by verbalizing it." That is, by verbalizing their thinking or their generalizations, students got to understand their algebraic generalizations. When students were having difficulties verbalizing their thinking, teachers used their questioning techniques to encourage them to talk (Episode 33). They encouraged students to explain their thinking or their generalizations in more than one way. As noted in episode 33, teachers also encouraged students to verbalize their thinking because they used students' verbal explanations as formative assessment of students' understanding of the algebraic activities. At times, when a student was having difficulties verbalizing their written generalization, their partners helped in explaining their thinking. Episode 34 is an example of a student helping a partner to explain his reasoning presented in Figure 19. Teachers accepted this practice.

Episode 33

Just to keep dialogue, keep asking: tell me more or how do you know that or explain it a different way. Use different words to explain what you mean. Because if they can explain it two or three different ways then they pretty much understand what they have done. But just to keep asking them, well, how do you know? Why? Well, tell me more, or I don't understand, or something like that.

Episode 34

Student 1: We have used both the input and the output so we can do mmh I found out we can have three to this one (points at input column) so we can go all the way, we can go down. We can add this (input of 1) plus 4

Partner: Ok I know what you are doing. Let me just explain it for a minute. You do 1 + 4 = 5 so 2 + 6 = 8 so 3 + 8 = 11 (shows middle column of her t-chart). It is like 4, 6, 8.

Student 1: 10 and then you can go to 12.

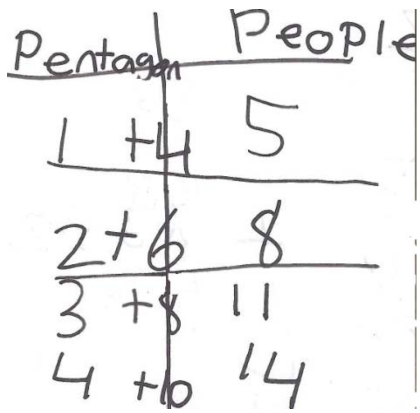


Figure 19. A Student's T-chart Showing the Middle Column Used to Explain a Partner's Reasoning about the Pentagon Task

Teachers also noticed that, students had difficulties writing down justifications for their generalizations. As such, they asked them to talk about the patterns they observed to come up with the generalizations. Teachers asked students to talk about how they checked that their generalizations worked. Students were then asked to put their verbal explanations down in writing (see Episode 35).

Episode 35

The hardest thing I think for all of the kids is writing down how they know that the rule is going to work for all (inputs) . . . So sometimes I have to get them to

> articulate it first and then I say now write that down in words. Because they can get the pattern, they can write the equation or but expressing how do you know them expressing that is hard for elementary kids, but they can tell you, and so it's just a matter of getting them to tell me and then saying okay, now, write down what you just told me, that is your thinking of how you know that what you have done is the answer.

Re-voicing. When students did not understand what the peers' reasoning was, teachers re-voiced student's reasoning. For example, in Episode 36, one student was explaining his reasoning about how many people would sit around a train of 100 square tables. Other students reported that they did not understand their peer's reasoning. The teacher re-voiced the student's reasoning by illustrating the student's reasoning with a picture and asking the student if that picture was a correct representation of his reasoning. As explained in Episode 37, teachers re-voiced student reasoning to express it more clearly for others to understand the algebraic reasoning involved. Additionally, teachers re-voiced students' ideas to show them that their reasoning can be understood by others.

> Episode 36
>
> Teacher: Why are you doing 100 plus 100?
>
> Student: Because there would be 100 seats on each side.
>
> Teacher: Okay.
>
> Student: That would make 200. 100 plus 100 would make 200. And you add, two plus 200 and that will be 200 and two.
>
> Teacher: Why do you add 200 plus two?
>
> Student: Because, there would be two seats on the ends. Because there are two seats, one seat on each side.
>
> Teacher: Okay, does everybody understand what he is saying?

> Students: No.
>
> Teacher: Okay, I am going to explain it to you. You can talk and I am going to illustrate. He is saying if he has 100 tables, let us just pretend that these are 100 tables; okay (draws a representation of 100 tables like this l l on the board). There is a seat above each of those tables, right. And then there is a seat below each one of those tables. Is that an illustration of what you are saying?
>
> Student: Yes
>
> Teacher: Okay, and then finish that for us.
>
> Episode 37
>
> Sometimes I feel like the kids just don't talk loud enough, and I want to make sure everyone hears what they are saying. I kind of like to reiterate the group what was their reasoning. And in a way it is justifying everyone's answer because sometimes I think the other kids are like, I don't get it. And I feel like, if I restate it, then they are saying, okay, she gets what I am saying. Sometimes I just state it in a more obvious way like when they dance around the issue. Whereas if I restate it, like I know what they mean, and I can just state it in a clearer way, so the other kids can understand.

Applying and discussing efficiency of different generalizations. Teachers asked students to apply their rules to find outputs for bigger input values. They asked students to find the output for an input of 100. Teachers took this exercise as an opportunity to discuss efficiency of different generalizations. While discussing with students that, recursive generalizations can correctly be used to find values for such big numbers, teachers helped students realize inefficiency of such rules. In one class, the teacher used the idea of a big giant paper as necessary to find output for big input values when using recursive rules (see Episode 38). Such discussions helped students realize the need to find a generalization that worked for any input value.

> Episode 38
>
> It is very easy for them . . . to give them an input of 1 to 3, and say now go to 10. And right away they do 4, 5, 6, 7, 8 and 9 because it is easy to do. But it is funny to watch their faces and say, okay, now your input is 100, go. And they just kind of get paralyzed and look at you like, I can’t do that. And they can. They can go to 11 and 12 and 13 and 14. But my question to them is always, do you want to do that? And the answer is always no. I don’t want to do that. And I tell them, in order to do that you would need a big giant piece of paper because you would have to do every number between 10 and 100. So I think that helps them focus on, okay I can’t just rotely do numbers here I have to figure out how I can make this work for any number.

How promoting understanding of generalizations supported algebraic reasoning. Promoting students’ understanding of generalizations supported students’ algebraic reasoning in multiple ways. As students justified their rules, they were able to identify mistakes in their generalizations. From Episode 38, discussing efficiency of different generalizations and asking students to apply their rules to find outputs for large input pushed students to think of explicit generalizations. Justifications like in episode 36 became a tool for algebraic reasoning as in the case of a student whose work is Table 6. In addition, as mentioned, re-voicing made reasoning more accessible to others.

Summary on promoting understanding of generalizations and algebraic thinking. On Track teachers promoted students’ understanding of generalizations. They re-voiced students’ reasoning and encouraged students to use variable names to make it more accessible to all students. On Track classes discussed efficiency of different generalizations and asked students to apply their own and other students’ generalizations. Teachers also encouraged students to verbalize their reasoning as a way of supporting other students' understanding of generalizations. These practices worked together as shown in Figure 20 to promote understanding of generalizations.

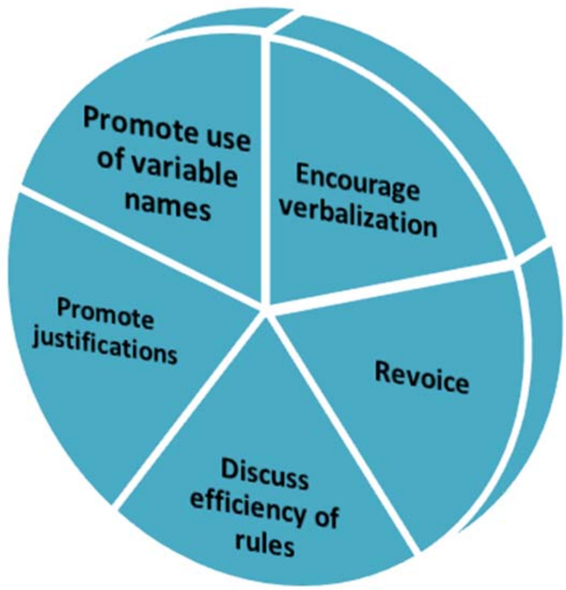


Figure 20. Practices that Promoted Understanding of Variables

Summary of Results

The analysis results of this study show that many On Track students successfully made generalizations when working on pattern finding tasks. Most students made explicit generalizations and used variable names in their generalizations. Five routines of practice helped co-construct students' algebraic reasoning. Teachers maintained open-endedness of algebraic reasoning tasks by giving few instructions when launching tasks and by creating workspaces for students. Teachers nurtured co-construction of ideas by supporting collaboration and balancing the support of discourse and content. Teachers fostered understanding of variables. They promoted understanding of generalizations and created a context for making mathematical connections.

Chapter V discusses how these practices relate to each other. Chapter VI discusses the study’s findings in reference to ideas and empirical evidence from literature. I will also discuss the significance of these findings and the implications they pose.

CHAPTER V

RELATIONSHIPS AMONG ON TRACK TEACHERS' ROUTINES OF PRACTICE

Overview of Chapter V

Chapter III explained that my theoretical orientation is the situativity of knowing and that this is a constructivist grounded theory study. Implications for research from this theoretical view and research design include the requirement to study relations among resources or practices that promote algebraic reasoning (Charmaz, 2011; Greeno & The Middle School Mathematics Through Applications Project Group, 1998). This chapter discusses the relationships among the On Track teachers' routines of practice using Figure 21. A discussion of these relationships reflects a mesh between teaching and learning. In this chapter, "teachers" refer to On Track teachers who participated in this study.

Five major themes emerged from the data as routines of practice that supported algebraic reasoning. These routines of practice were—keeping tasks open, nurturing co-construction of ideas, creating a context for mathematical connections, fostering understanding of the concept of variable, and promoting understanding of generalizations. Further data analysis revealed relationships between and among these routines of practice and their elements. Figure 21 is a model of this relationship. Within each routine of practice, teachers maintained open-endedness of the tasks and nurtured co-construction of ideas. In the same way, creating a context for mathematical connections supported fostering understanding of variable and promoting understanding of generalizations. As may be noted, exploring how the routines related to each other gave a richer background for successful implementation of these routines in the classroom and beyond. The following sections discuss in more depth these routines of practice.

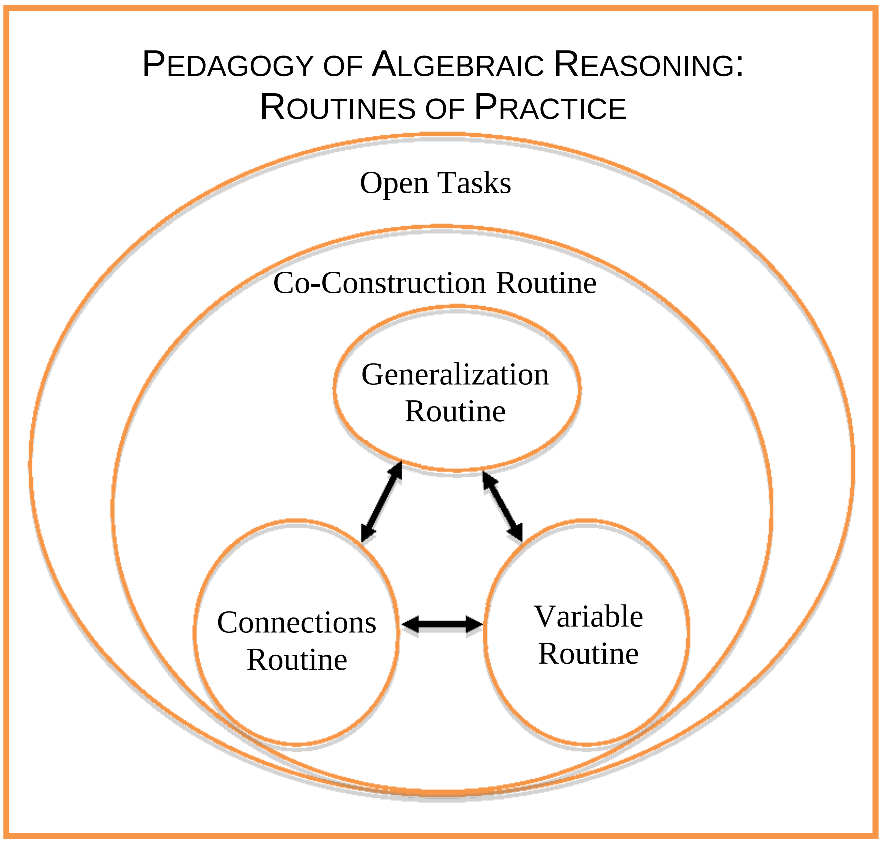


Figure 21. Routines of Practice

Relationship between Keeping Tasks Open and Nurturing Co-construction of Ideas

Keeping tasks open was a teaching practice that maintained open-endedness of algebraic reasoning tasks through minimal directions at launching stages. Teachers also maintained open-endedness of the tasks by providing students ample time to work on the tasks and by providing a lot of writing space as students' workspaces. Maintaining open-endedness of the tasks was important because it created fertile grounds for the other routines of practice to flourish.

As a routine of practice, keeping algebraic tasks open created space for co-construction of ideas. To recall, creating a context for co-construction of ideas was a routine of practice that fostered collaboration among students, and balanced the support of discourse and content to promote student understanding. To foster collaboration, time as a workspace was important for students to work in small groups and engage in whole group discussions. Open-ended tasks allowed multiple student strategies. Consequently, maintaining open-endedness of the tasks throughout the classroom activities appeared to move collaboration activities beyond 'showing and telling the right answer' to evaluating ideas and strategies to inform and build on one's thinking.

Nurturing Co-construction and Other Routines of Practice

As explained before, nurturing co-construction of algebraic reasoning supported mathematical talk in the classroom and engineered the mathematical talk to support students' algebraic reasoning. To nurture co-construction of ideas, teachers created opportunities for students to work with peers with seemingly like abilities. These opportunities included student engagement in small and whole group discussions. Teachers also selected responses and strategies that were different and purposefully sequenced them for presentation during whole class discussions. These practices were observable throughout classroom activities. As such, routines that promoted understanding of generalizations, fostered understanding of variables and created a context for mathematical connections were part of routines that nurtured co-construction of ideas.

For example, students made mathematical connections between different strategies during small group and whole group discussions. Thus, nurturing co-construction of ideas created a context for mathematical connections. Moreover, through purposeful selection and sequencing of ideas for whole class discussions, teachers supported understanding of variables by sequencing generalizations from recursive to explicit. Thus, supporting co-construction of ideas also supported mathematical connections, understanding of variables and generalizations. Figure 21 illustrates this relationship.

Whereas large group and small group discussions with peers enhanced overall understanding across a spectrum of ideas presented in the classroom, as Figure 21 suggests, not all practices regarding co-construction of ideas were specific to supporting mathematical connections, and understanding of generalizations and variables. For example, practices like instituting didactical contracts for students to try to understand their peers reasoning and decisions on how to group students and practices on encouraging engagement by all students were a productive part of nurturing co-construction of ideas as discussed in Chapter IV. However, these practices were not specific to fostering understanding of variables, promoting understanding of generalizations or creating a context for making mathematical connections.

Fostering Understanding of Variable and Promoting Understanding of Generalizations

On Track teachers fostered understanding of variable through supporting understanding of independent and dependent variables by asking students to identify inputs and outputs. They also fostered understanding of variable by encouraging students to find mathematical relationships between input and output variables. Practices that promoted understanding of generalizations were (a) use of variable names, (b) students' verbalization of reasoning, (c) re-voicing, (d) encouraging students to apply and evaluate efficiency of rules, and (e) justifying.

Practices for fostering understanding of variables and for promoting understanding of generalizations were dependent. To promote peers' understanding of generalizations, On Track teachers asked students to use variable names when writing generalizations, justifying conjectures, and when verbalizing their reasoning. To use variable names required students to identify their variables. Therefore, promoting understanding of generalizations depended on fostering understanding of variables. Moreover, as teachers fostered understanding of variables, they positioned themselves to promote understanding of generalizations through practices that required students' understanding of variable as a changing quantity and as a relationship. For example, to evaluate efficiency of generalizations in solving outputs for large inputs, students applied understanding of ideas of relationship between inputs and outputs. That is, application and evaluation of generalizations depended on understanding of variables. This shows the inter-dependence between practices for nurturing understanding of variables and promoting understanding of generalizations. Figure 22 illustrates this inter-dependence.

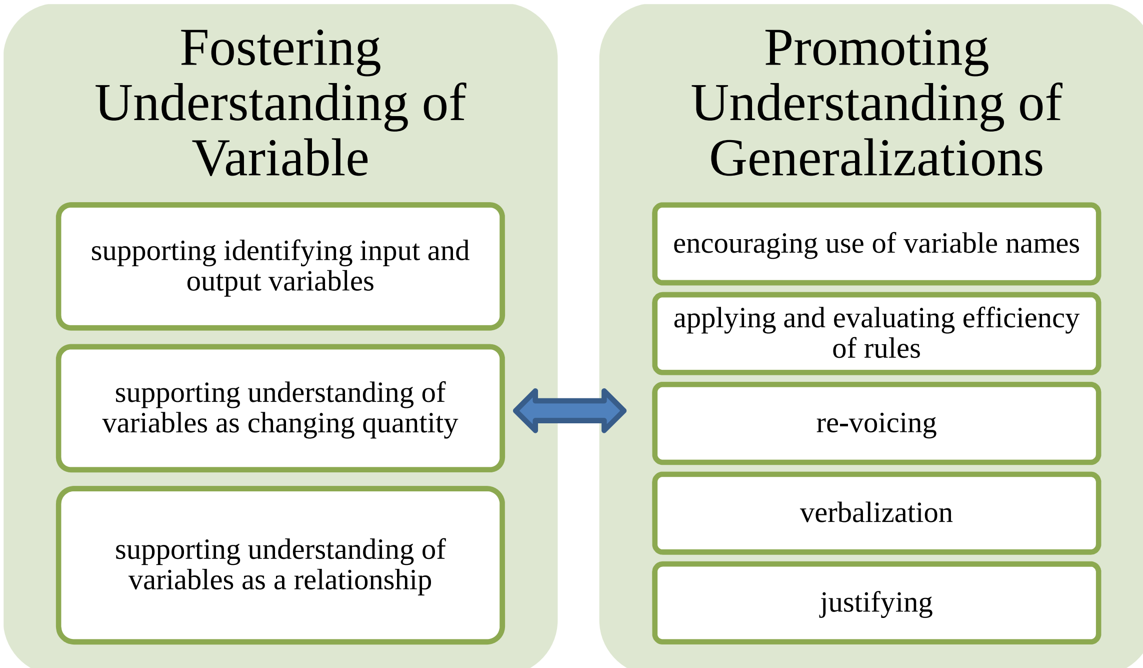


Figure 22. Relationship between Supporting Understanding of Variables and of Generalizations

Promoting Understanding of Generalizations and Mathematical Connections

To recall, teachers created a context for connecting mathematical ideas from five more localized contexts. These are students' everyday experience, different representations, student strategies, different curriculum ideas, and algebraic reasoning tasks. Promoting understanding of generalizations and creating a context for mathematical connections were different routines of practice. Creating a context for mathematical connections was observable throughout implementation of algebraic task. Teachers used the representation of snack machine and invitation of friends to dinner for the square table tasks (Figures 9 and 10 respectively) when launching the tasks. Connecting ideas from different strategies during whole group discussions was towards the end of the lessons. On the other hand, promoting understanding of generalizations was mainly observable after students expressed their generalizations.

Despite these differences, there was a relationship between creating a context for mathematical connections and promoting understanding of generalizations. Everyday experiences were used when naming variables to make generalizations easily understood by others (e.g., Episode 19). Additionally, teachers encouraged students to connect ideas from multiple representations and to connect different strategies to evaluate and justify generalizations. Similarly, On Track teachers supported connections between curricular ideas to support justifying, applying, and evaluating efficiency of generalizations. For example, when students had difficulties with multiplication strategies, teachers discussed multiplication strategies so that students were able to apply and evaluate generalizations when given large input values. Table 6 is another example of a student's written work that made connections of mathematical ideas from other students' verbalized justifications. Similarly, applying and evaluating efficiency of rules supported connections with students' everyday experiences. For instance, students who reported recursive generalizations needed a big giant paper when the inputs were large. Therefore, supporting mathematical connections and supporting understanding of generalizations appeared to be interdependent. Figure 23 illustrates this interdependence.

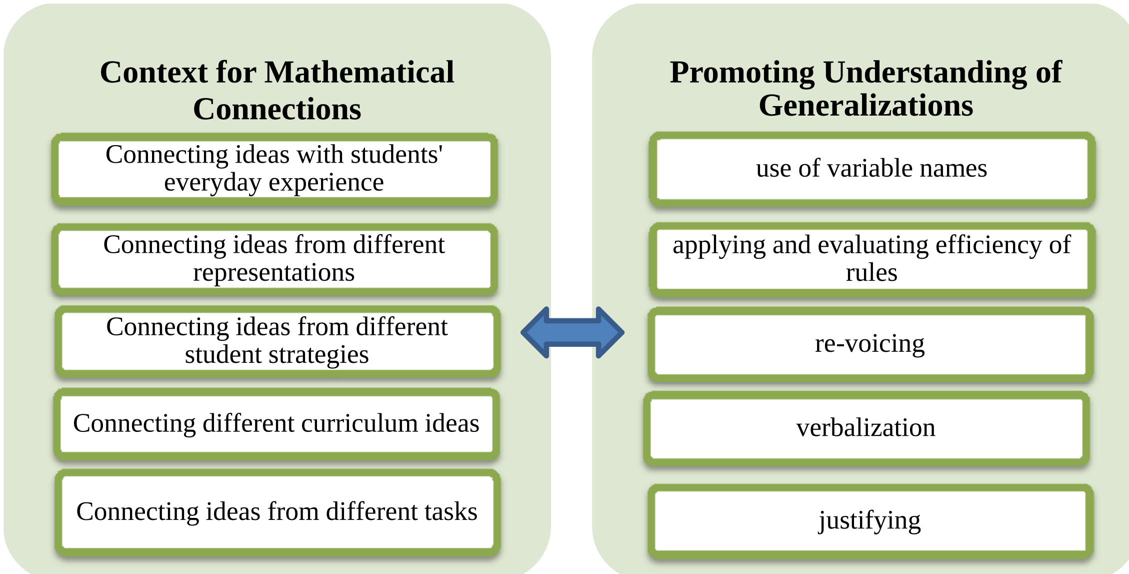


Figure 23. Relationship between Creating a Context for Mathematical Connections and Promoting Understanding of Generalizations

Fostering Understanding of Variables and Mathematical Connections

The main difference between creating a context for mathematical connections and supporting understanding of variables is that, supporting mathematical connections applies to a broader spectrum of classroom activities than promoting understanding of variables. Fostering students' identification of input and output variables depended on connections with everyday experiences (e.g., the case of snack machine representation). As students built geometric models of the tasks, teachers asked them to identify how the variables changed with each build. This practice supported students' understanding that the input and output variables change their quantities. By connecting different strategies for recursive generalizations, teachers also fostered understanding of variable as a changing quantity. Thus, connecting ideas from different representations fostered understanding of variable as a changing quantity. Supporting understanding of variable as a changing quantity depended on using and connecting ideas from multiple representations.

Practices that created a context for mathematical connections depended on students' understanding of variable and discussions of what was changing and what was constant (e.g., the case of one student's reasoning in Table 6). In the same way, context for connections of ideas in the curriculum depended on supporting understanding of variable as a relationship. For example, connecting multiplication strategies and On Track algebraic reasoning tasks was observable mostly when teachers gave students large input values to push students to find the relationship between independent and dependent variables. Figure 24 represents interdependence of fostering understanding of variables and mathematical connections.

This chapter reports the relationships of routines of practice from the outer to the inner parts of the model in Figure 21. The only reason for this pattern in the report is to clarify the discussion of relationships. It does not mean the relationships among the practices are linear. To illustrate this nonlinearity, consider a relationship between nurturing co-construction and keeping open-ended tasks open. As discussed, keeping open-ended tasks nurtured co-construction of ideas. In the same way, sharing of ideas demonstrated that On Track tasks could be approached using different strategies thereby instituting the open-endedness of the tasks in the classrooms.

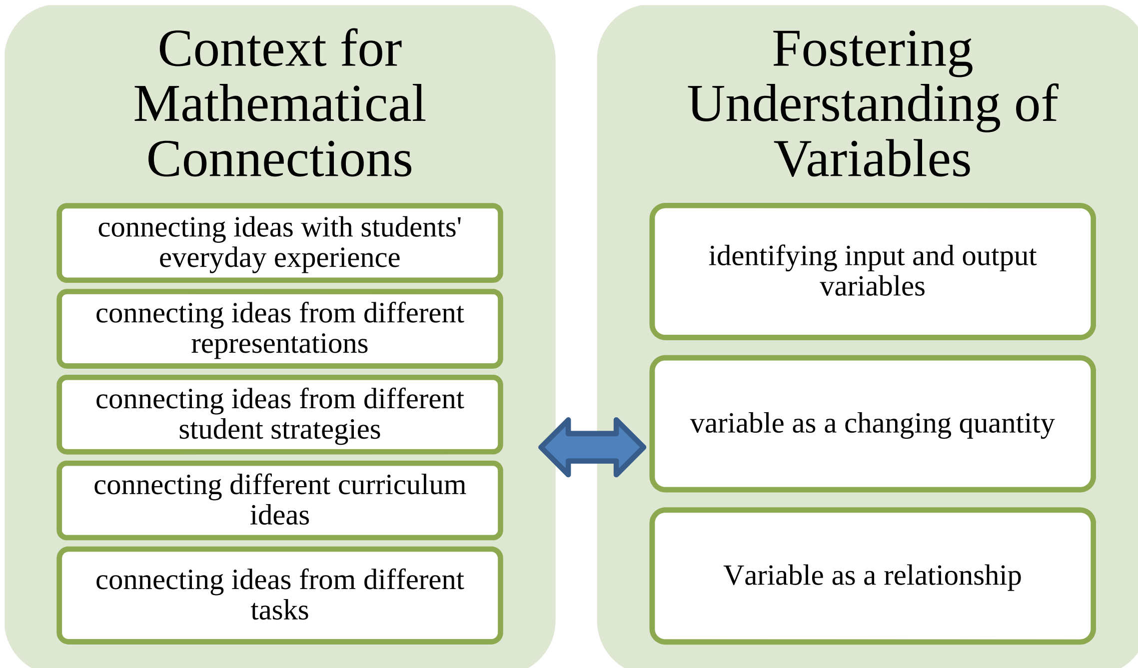


Figure 24. Relationship between Creating a Context for Mathematical Connections and Fostering Understanding of Variables

Essence of this Grounded Theory of Productive Routines of Practice

One of the central ideas in this theory is the relationships and dependency of the different routines of practice. To create or nurture a productive classroom context, all these routines need to be present to support each other in contributing to richer classroom experiences. Several authors, as will be discussed in Chapter VI, have discussed some components of this theory in more isolated ways than data in this study have suggested. This theory however, emphasizes interaction and dependency of the elements within each routine and between different routines of practice. Because of this dependency, each element (e.g., connecting algebraic reasoning tasks with students' every day experiences) and each routine (e.g., maintaining open-endedness of the tasks) is essential in supporting students' reasoning.

Summary of Chapter V

This chapter reported interdependence and relationships between On Track teachers' routines of practice. Supporting understanding of variables, generalizations and connecting mathematical ideas were interdependent. Each of these practices created a context for the other practices to be meaningful. Within each of these three practices, teachers supported co-construction of ideas and maintained open-endedness of algebraic tasks. However, not all practices aimed at supporting co-construction of ideas were specific to supporting mathematical connections, understanding of variables, and understanding of generalizations. Similarly, teachers maintained open-endedness of the tasks throughout the classroom activities and the other routines of practice worked together to maintain the open-endedness of On Track algebraic reasoning tasks. Providing students with workspace and launching by giving minimal directions added to keeping tasks open. Chapter VI discusses these practices in relation to literature and discusses implications from relationships of these practices.

CHAPTER VI

DISCUSSION

Overview of the Chapter

The purpose of this study was to discover teaching practices that support algebraic reasoning in On Track classes. Using a constructivist grounded theory approach; five practices emerged from the data collected from On Track research project. These practices are maintaining open-endedness of algebraic tasks, supporting co-construction of algebraic ideas, promoting understanding of variables and generalizations, and connecting mathematical ideas. Analysis of generalizations written by 115 students in grades three through five showed that 74% of all generalizations made were explicit and approximately 55.26% of all the correct written generalizations mentioned variable names. In this chapter, I will discuss these finding in relation to related literature, the research context and my experiences as an educator. I will conclude by discussing implications for practice and research.

Discussion of Results

Overview of Students' Reasoning

Students' difficulties in generalizing about pattern finding activities are well documented. Warren and Cooper (2008)[b] reported that of the 45 students with an average age of 8 years and six months, about half could not observe patterns and extend them to the next term before an intervention through a teaching experiment. That is, they could not make recursive generalizations. In another study with 70 students in grades 2 through 5, only a third of the students were able to represent their generalizations as an algebraic expression after engaging in algebraic reasoning intervention for one and a half years (Brizuela & Schlieman, 2004). Without undermining the fact that elementary school students are capable of algebraic reasoning, these studies show to what extent algebraic reasoning may be a challenge to elementary school students. They also show that, On Track teaching practices were relatively very successful in supporting generalization activities during the 10-lesson sessions.

Maintaining Open-endedness of Algebraic Tasks

Earnest and Balti (2008) wrote that algebraic reasoning tasks may be turned into arithmetic tasks if students simply use recursive generalizations to find the next term in a pattern. By discussing outputs for the first three terms of On Track algebraic reasoning tasks and asking students to find outputs for large inputs, On Track teachers maintained algebraic characteristics of the tasks as Earnest and Balti may argue. As this study shows, workspaces are important in maintaining characteristics of algebraic reasoning task. However, in classrooms, students rarely have workspaces. From my experiences as a student, as a teacher and as someone who has worked with teachers for many years, students are given worksheets that are full of words and with only a little space for quick calculation and for writing a solution to a mathematical problem. On Track teacher practice shows the need to reduce words on the worksheet and provide students with a lot of workspace. In addition, as explained in this study, the teachers were able to give students workspace by allocating a lot of time for mathematical explorations. This was possible because On Track was a low-stake teaching context with students' progress driven teaching as opposed to test driven teaching. This requires educators and school administrators to reflect on factors that hinder productive pedagogy. It requires education stakeholders to promote practices that balance students' thinking and mathematical content to be drivers of education pace. Without such a balance, school and classroom practices may hinder students' reasoning.

Nurturing Co-construction of Ideas

Several studies explain that co-construction of ideas support mathematical reasoning. As such, teachers are encouraged to give students opportunities to work in small groups and engage in whole group discussions (Carraher et al., 2008; Cobb et al., 2001; Mueller, 2009). While attention has been paid to how teachers can make whole group discussions more productive (e.g., Stein et al., 2008), little attention has been paid to how teachers can support co-construction in small group discussions. In this study, and other studies (e.g., Chapin, O'Connor, & Anderson, 2003), it is learned that, grouping students does not simply lead to students working together. Students might just sit together and not share ideas or they might work independently and just show each other their solutions to mathematical problems. To make group discussions more productive, Martino and Maher (1999) reported that grouping students with contradictory responses support mathematical reasoning. Additionally, Artzt and Armour-Thomas (1997) reported that grouping students with different cognitive abilities support productive small group discussions. On Track teachers grouped students differently. Teachers grouped students with like abilities together. This created a context for nurturing co-construction of ideas and not simply a peer-teaching context where students that are more knowledgeable teach students perceived as less knowledgeable. This context was also supported by teachers explicitly and implicitly communicating to students' the expectation to understand peers thinking and to make others understand their thinking.

This study also showed that, to support algebraic reasoning, it is important to engage all students in the class discussions. Since past research (Lannin et al., 2006b) reports that students develop recursive generalizations with relative ease, selecting recursive generalizations for whole class discussions is one way to include responses from more students during whole class discussions. To balance the support of discourse and algebraic reasoning requires that teachers also use other criteria to select responses for public display. From the results, selecting strategies that have different mathematical ideas and those that have same mathematical ideas but are expressed differently supports algebraic reasoning. Such a selection may contribute to building socio-mathematical norms that there are multiple acceptable strategies for solving algebraic reasoning tasks—a practice that is a key to fostering algebraic reasoning (Rathouz, 2009). Contrary to reports (e.g., Depaepe, De Corte & Verschaffel, 2007) that it is a challenge for teachers to teach mathematics in ways that show that there are different ways of solving and representing mathematical ideas, On Track teachers communicated to the students and had practices that reflected this idea.

Fostering Understanding of Variables

As discussed before, meaningless manipulation of algebra variables contributes to students' lack of understanding of high school algebra concepts. By supporting students' conceptual understanding of independent and dependent variables in functional relationships as On Track teachers did, students are better prepared to develop meanings to algebraic symbols (Blanton & Kaput, 2011). Moreover, English and Warren (1998) and Ellis (2011) explain that students' understanding of what an input and an output variable is required for algebraic reasoning. Similar to the Connected Mathematics Project curriculum, On Track classroom practices did not constitute variables as placeholders for unknown quantities but as quantities that change or vary used to represent relationship—a practice which supports students algebraic reasoning in middle school years (Cai, Moyer, Wang, & Nie, 2011). Teaching understanding of variables in this way prepares students for meaningful manipulation of symbols and potentially supports students' understanding of higher-level algebraic concepts. Teaching practices of identifying variables in algebraic reasoning tasks is a foundation to move students from focusing on recursive pattern to finding relationship between two variables. As reported, elementary school students often require probing to look for relationships between variables. Like On Track teachers, Warren and Cooper (2008)[b] also reported a questioning strategy that focuses students on independent and dependent variables. Researchers (e.g., Blanton & Kaput, 2011) who feel that an emphasis on only recursive patterns in elementary classrooms may hinder students' algebraic reasoning may promote this practice.

Creating a Context for Mathematical Connections

On Track teachers encouraged connections of mathematical ideas with real world contexts that are familiar to students. Mathematical standards for teaching mathematics (e.g. NCTM, 2000) encourage this practice. Examples in On Track classrooms were from real life experiences based on the teachers' belief that students were familiar with those examples. This is an important aspect in supporting mathematical reasoning as Warren, Cooper and Lamb (2006) found that "contexts are only useful when students understand and are familiar with them" (p. 219). When students are not familiar with the real world context, it may become a hindrance to students' reasoning. The focus then should be on connecting mathematical ideas with students' everyday experiences as opposed to simply making connections with real world context.

Connecting ideas across tasks support algebraic reasoning. Richardson et al. (2009) reported that connecting ideas from isomorphic tasks supports algebraic reasoning. Tasks do not necessarily have to be isomorphic to make mathematical connections across tasks. This fosters an understanding that mathematical concepts are not distinct but connected in that an understanding in one area might foster an understanding in another area as observed in On Track classrooms. Similarly, connecting ideas from different representations and different students' ideas support mathematical reasoning (Stein et al., 2008). Connecting ideas from different students and different representations creates opportunities for conceptual discussions as noted in this study and as reported in other studies (e.g., Piccolo, Harbaugh, Carter, Capraro, & Capraro, 2008).

Promoting Understanding of Generalizations

Lannin et al.'s (2006b) study with middle grades students found that "few students naturally developed a deep understanding of the explicit rules they generated . . ." (p. 316). It is therefore important for teachers to support students' understanding of generalizations. On Track teachers, just like elementary school teachers in Hernon's (2004) research found that verbalizing is an important teaching strategy that supports understanding of generalizations. Chapin, O'Connor and Anderson (2003) concurred with this.

Lannin (2005) and Ellis (2007) reported that when students justify their generalizations, they refine their reasoning and understand the generalizations better. Justifications support algebraic reasoning. Lannin (2005) advocated for supporting students to go beyond example-based justifications and use justifications that regarded the general context of the problem as a better support of algebraic reasoning. In this study, teachers encouraged students to justify their conjectures. For all tasks, teachers situated example based justifications as acceptable. For train table tasks however, teachers asked for justifications that required students to identify the relationships between the generalizations and the square table task. Perceived justification types inherent in different algebraic reasoning tasks may explain this change in practice across tasks. This calls for attention to instructional tasks and the justification schemes different tasks may afford when researchers report or discuss students' justification activities when working on algebraic reasoning tasks.

Earnest and Balti (2008) reported that asking students to apply their generalizations to find output values for large input support students' activities in generalizing. This study supports Earnest and Balti's finding. It has shown that, in addition to applying their own generalizations, encouraging students to apply other students' generalization support algebraic reasoning. This practice links to Cobb et al.'s

(2001) research findings that when students have authority to evaluate different strategies and solutions, their mathematical reasoning is nurtured. Evaluation of strategies encompasses evaluating efficiency of different types of generalizations as was done in this study.

Others (e.g., Johnson, 1990) argue that re-voicing has potential to take authorship away from students and does not encourage students to pay attention when their peers are presenting during whole class discussion. This could be due to how re-voicing is done. On Track teachers re-voiced when it was necessary to make a student's reasoning accessible to others thereby making whole group discussions more productive. When re-voicing, teachers acknowledged authorship of the ideas by acknowledging proponents of those ideas, mentioning student names and using phrases like "his idea" or "her idea." This also brings attention to the importance of considering the practices as working in concert to support student algebraic reasoning. According to Gerson and Bateman (2010), students who justify their ideas and evaluate other people's ideas as On Track students did, get a sense of authority that comes by authoring or co-authoring, and authority that comes by evaluating mathematical ideas. Gerson and Bateman argue that such sense of authority is a potential mathematical empowerment tool that supports mathematical understanding and students' mathematical autonomy. Therefore, On Track teacher practices of supporting understanding of generalizations also supported students' sense of authorship.

Implications

Research with classroom teachers indicates that their normative practices are teacher centered (Good, 2010). The ON TRACK study revealed that elementary school teachers are capable of productive mathematical practices that involve students in the co-construction of ideas. On Track's professional development and low stake teaching environment may contribute to teachers' productive practices. Perhaps, teachers develop teacher-centered practices that focus on procedural skills because of lack of professional support and other school related constraining factors. This has implications for teacher education and development.

Teachers indicated that, they were becoming more comfortable with teaching for mathematical reasoning over time in the low stake environment since they did not feel like they had to teach to the test. This comfort developed as teachers practiced ideas shared and observed during professional development. This suggests that, for professional development aimed at supporting pedagogy of mathematical reasoning, creating practice spaces is recommendable. Similarly, in the preparation of teachers for mathematical reasoning, practice spaces are important.

On Track teachers were relatively very successful in supporting algebraic reasoning, about 74% of the generalizations were explicit. These results call for more focus on engaging all students in algebraic reasoning to achieve "access for all" goal in higher-level mathematics. In addition, to the support of students' meaningful algebraic activities with variable in higher algebra classes, teaching practice needs to focus on how to support development of students' understanding of variables.

Relationships among the different routines of practice have implications for practice and research. The dependency of routines on other routines shows the need for research on teaching to take an approach that allows a holistic understanding of the interactions between and among the variety of tasks, tools and students. This dependency also shows the importance of teachers becoming experts in different practices to be able to create and support mathematical reasoning in their classrooms. This requires teacher educators and professional developers to provide spaces that support in-service and pre-service teachers becoming expert teachers of mathematical reasoning.

This study is limited to understanding a group of teachers who underwent the same professional development. For a broader understanding of practices of elementary school teachers when implementing algebraic reasoning, further research is required. This study also only explored practices of teachers in a low stake environment. Further research is required to understand how teachers implement algebraic reasoning curriculum given everyday school contextual factors.

To recap, teachers indicated that as they were teaching On Track classes, they were becoming more and more attuned to teaching mathematical reasoning. The area of research on how teachers become expert teachers of mathematical reasoning is largely unexplored but required to support education of both in-service and pre-service teachers.

REFERENCES

Arcavi, A. (2008). Algebra: Purpose and empowerment. In C. E. Greenes & R. Rubestein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 37– 49). Reston, VA: National Council of teachers of Mathematics.

Artzt, A. F., & Armour-Thomas E. (1997). Mathematical problem solving in small groups: Exploring the interplay of students' metacognitive behaviors, perception and ability levels. *Journal of Mathematical Behavior, 16*(1), 63–74.

Australian Education Council. (1994). *Mathematics: A curriculum profile for Australian schools.* Carlton, VIC: Curriculum Corporation.

Bastable, V., & Schifter, D. (2008). Classroom stories: Examples of elementary students engaged in early algebra. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 165–184). New York: Lawrence Erlbaum.

Becker F. D., & Rivera J. R. (2009). Algebraic reasoning through patterns. *Mathematics Teaching in the Middle School*, *15*(14), 212–221.

Beigie, D. (2011). The leap from patterns to formulas. *Mathematics Teaching in Middle School*, *16*(6), 328–335.

Bednarz, N., Kieran, C., & Lee, L. (2008). Approaches to algebra: Perspectives for research and teaching. In N. Bednarz, C. Kieran, & L. Lee, *Approaches to algebra: Perspectives for research and teaching* (pp. 3–14). Dordrecht, Netherlands: Kluwer Academic Publishers.

Bezuzska, S., & Kenney, M. J. (2008). The three R’s: Recursive thinking, recursion, and recursive formulas. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 81–98). Reston, VA: NCTM.

Blanton, M. L., & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education, 36*(5), 412–446.

Blanton, M. L., & Kaput, J. J. (2008). Building district capacity for teacher development in algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 361–388). New York: Lawrence Erlbaum.

Blanton M. L., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5–24). Springer-Verlag: Berlin Heidelberg.

Blanton, M., Schifter, D., Inge, V., Lofgren, P., Willis, C., Davis, F., & Confrey, J. (2007). Early algebra. In V. J. Katz (Ed.), *Algebra: Gateway to technological future* (pp. 7–14). The Mathematical Association of America.

Brizuela, B., & Schliemann, A. (2004) 10-year old students solving linear equations. *For the Learning of Mathematics* 24(2), 33-40.

Cai, J., Moyer, J. C., Wang, N., & Nie, B. (2011). Standards-based and traditional curricula in the United States. In In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 161–185). Springer-Verlag: Berlin Heidelberg.

Carlone, H. B., Haun-Franke, J., & Kimmel, S. C. (2010). Tempered radicals: Elementary teachers' narratives of teaching science within and against prevailing meanings of schooling. *Cultural Studies of Science Education, 5,* 941–965.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM Mathematics Education, 40,* 3–22.

Carraher, D. W., Schliemann, A., & Schwartz, J. L. (2008). Early algebra is not algebra early. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). New York: Lawrence Erlbaum.

Chapin, S. H., O'Connor, C., & Anderson, N. C. (2003). *Classroom discussions: Using math talk to help students learn*. Sausalito, CA: Math Solutions Publications

Charbonneau, L. (1996). From Euclid to Descartes: Algebra and its relation to Geometry. In N. Berdnarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 15–38). Dordrecht, Netherlands: Kluwer Academic Publishers.

Charmaz, K. (2011). Constructivist theory analysis of losing and regaining a valued self. In F. J. Wertz, K. Charmaz, L. M. McMullen, R. Josselson, R. Anderson, & E. McSpadden (Eds.), *Five ways of doing qualitative analysis: Phenomenological psychology, grounded theory, discourse analysis, narrative research, and intuitive inquiry* (pp. 165–204). New York: Guilford Press.

Chateaeneouf, A. O. (1929). *Changes in the content of elementary algebra since the beginning of high school movement as revealed by the textbooks of the period*. (Doctoral dissertation). University of Pennsylvania.

Chazan, D. (2008). The shifting landscape of algebra in the United States. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 19–36). Reston, VA: NCTM.

Clark, P. G., Moore, K. C., & Clarke, P. M. (2008). Documenting the emergence of “speaking with meaning” as a sociomathematical norm in professional learning community discourse. *The Journal of Mathematical Behavior, 27*, 297–310.

Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspective in theory and practice. *Educational Researcher*, *28*(4), 4–15.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, J. (2001). Participating in classroom mathematical classrooms. *The Journal of the Learning Sciences, 10*(1&2), 113–163.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/the-standards

Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students’ ability to generalise: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 187–214). Springer-Verlag: Berlin Heidelberg.

Coxford, A. F., & Shulte, A. P. (1988). *The ideas of algebra, K-12: 1988 yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Creswell, J.W. (2007). *Qualitative inquiry and research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage.

Depaepe, F., De Corte, E., & Verschaffel, L. (2007). Unraveling the culture of the mathematics classroom: A video based study in sixth grade. *International Journal of Educational Research, 46,* 266–279.

Department for Education and Skills. (2001). *Frameworks for teaching mathematics: Years 7, 8, and 9.* London: DES Publications.

Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). New York: Lawrence Erlbaum.

Earnest, D., & Balti, A. A. (2008). Instructional strategies for teaching algebra in elementary school: Findings from a research-practice collaboration. *Teaching Children Mathematics*, *14*(9), 518–522.

Ellis, A. B. (2011). Generalization-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, *42*(4), 308–345.

English, L., & Warren, E. (1998). Introducing the variable through pattern exploration. *Mathematics Teacher,* 91, 166–170.

Francisco, J. M., & Maher, C.A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, 24, 361–372.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. K. Lester (Ed.), *Second handbook of research on* *mathematics teaching and learning* (pp. 707–762). Charlotte, NC: Information Age.

Piccolo, D.L., Harbaugh, A. P., Carter, T.A., Capraro, M.M., Capraro, R.M. (2008). Quality of Instruction: Examining Discourse in Middle School. *Mathematics Instruction Journal of Advanced Academics*, 19 (3), 376–410.

Gerson, H., & Bateman, E. (2010). Authority in an agency-centered, inquiry-based university calculus classroom. *The Journal of Mathematical Behavior*, 29(4), 195–206.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Goldenberg, E. P., & Shteingold, N. (2008). Early algebra: The math workshop perspective. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 449–478). New York: Lawrence Erlbaum.

Good, T. L. (2010). Forty years of research on teaching 1968-2008: What do we know that we didn't know then? In R. J. Marzano (Ed.), *On excellence in teaching* (pp. 31–64). Bloomington, IN: Solution Tree.

Greeno, J. (2006). Learning in activity. In K. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (pp. 79–96), New York: Cambridge University Press.

Greeno, J.G. (1997). On claims that answer the wrong questions. *Educational Researcher*, 26(1), 5–17.

Greeno J. G., & The Middle School Mathematics Through Applications Project Group. (1998). The situativity of knowing, learning, and research. *American Psychologist, 53*, 5–26.

Herbal-Eisenmann, B. A., & Breyfogle, M. L. (2005). Questioning our patterns of questioning. *Mathematics Teaching in Middle School*, *10*(9), 484–489.

Hernon, C. A. (2004). *Teacher exploration of instructional strategies to promote algebraic thinking.* (Doctoral dissertation), University of Nevada.

House, P. A. (1988). Reshaping school algebra: Why and how? In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 1–7). Reston, VA: National Council of Teachers of Mathematics.

Johnson, D. R. (1990). *Every minute counts. Making your math class work.* Palo Alto, CA: Dale Seymour Publications

Jones, K. (2000). Providing foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44,* 55–85.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Mahwah, NJ: Lawrence Erlbaum.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–18). New York: Lawrence Erlbaum.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Charlotte, NC: Information Age.

Kieran, C. (2011). Overall commentary on early algebraization: Perspectives for research and teaching. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 579–593). Springer-Verlag: Berlin Heidelberg.

Kilpatrick, J., & Izsak, A. (2008). History of algebra in the school curriculum. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 3–18). Reston, VA: National Council of teachers of Mathematics.

Koellner, K., Jacobs, J., Borko, H., Roberts, S., & Schneider, C. (2011). Professional development to support students' algebraic reasoning: An example from the problem-solving cycle model. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 429–452). Springer-Verlag: Berlin Heidelberg.

Lampert, M. (1998). Studying teaching as a thinking practice. In J. Greeno & S. G. Goldman (Eds.), *Thinking practices in mathematics and science learning* (pp. 53–78). Mahwah, NJ: Erlbaum.

Lannin, J. K. (2003). Developing algebraic reasoning through generalization. *Mathematics Teaching in the Middle School, 8*(7), 342–348.

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning: An International Journal*, *25,* 299–317.

Lannin, J. K., Barker, D. D., & Townsend B. E. (2006a). Algebraic generalization strategies: Factors influencing student strategy selection. *Mathematics Education Research Journal*, *18*(3), 3–28.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006b). Recursive and explicit rules: How can we build student algebraic understanding? *The Journal of Mathematical Behavior, 25*(4), 299–317.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York: Cambridge University Press.

Lee, K., Ng, S. W., Bull, R., Pe, M. L., & Ho, R. H. M. (2011). Are patterns important? An investigation of the relationships between proficiencies in patterns, computation, executive functioning and algebraic word problems. *Journal of Educational Psychology, 103*(2), 269–281. doi:10.1037/a0023068

Lester, F. K. (2007). *Second handbook of research on mathematics teaching and learning.* Charlotte, NC: Information Age.

Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra*: *The 12th ICMI Study* (pp. 47– 70). Dordrecht: Kluwer.

Maher, C. A. (2008). Video recordings as pedagogical tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol. 2: Tools and processes in mathematics teacher education* (pp. 65– 83). Rotterdam, The Netherlands: Sense Publishers.

Maher, C. A., & Muter, E. (2010). Responding to Ankur's challenge: Co-construction of argument leading to proof. In C. A. Maher, A. B. Powell, & E. B. Uptegrove (Eds.), *Combinatorics and reasoning: Representing, justifying and building isomorphisms* (pp. 89– 96). New York: Springer.

Martinez, M., & Brizuela, B. M. (2006). A third grader's way of thinking about linear function tables. *Journal of Mathematical Behavior, 25,* 285– 298.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior, 18*(1), 53–78.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 57– 95). New York: Lawrence Erlbaum.

Maxwell, J. A. (2005). *Qualitative research design: An interactive approach* (2nd ed.). Thousand Oaks, CA: Sage.

Mayansky, E. (2007). *An analysis of the pedagogy of Robert B. Davis: Young children working on the tower of Hanoi problem* (Doctoral dissertation, Rutgers University). New Brunswick, NJ.

Michael, J. J., Berenson, S. B., & Store, J. C. (2009). *Algebra 1: The career gatekeeper*. Conference Proceedings of the 33rd annual meeting of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.

Moss, J., Beaty, R., Barkin, S., & Shilolo, G. (2008). “What is your theory? What is your rule?” Fourth graders build an understanding of functions through patterns and generalizing problems. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 81–98). Reston, VA: NCTM.

Moss, J., & McNab, S. L. (2011). An approach to geometric and numeric patterns that fosters second grade students’ reasoning and generalizations about functions and covariation. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 277– 301). Springer-Verlag: Berlin Heidelberg.

Mueller, M., & Maher, C. (2009). Convincing and justifying through reasoning. *Mathematics Teaching in the Middle School*, *15*(2), 109–116.

Mueller, M. F. (2009). The co-construction of arguments by middle school students. *The Journal of Mathematical Behavior, 28,* 138–149.

Muter, E. M., & Uptegrove, E. B. (2010). Representations and connections. In C. A. Maher, A. B. Powell, & E. B. Uptegrove (Eds.), *Combinatorics and reasoning: Representing, justifying and building isomorphisms* (pp. 105–120). New York: Springer.

National Assessment of Educational Progress. (2008). *The Nation’s Report Card: Trends in Academic Progress in Reading and Mathematics*. Retrieved from http://nces.ed.gov/nationsreportcard/pubs/main2008/2009479.asp

National Council for Teachers of Mathematics. (1989). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

QSR International (2011). NVivo 9 (Computer software). Victoria, Australia: QSR International

Radford, L. (1996). Some reflections on teaching algebra through generalization. In N. Berdnarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 107–114). Dordrecht, Netherlands: Kluwer.

Radford, L., & Puig, L. (2007). Syntax and meaning as sensuous, visual, historical forms of algebraic thinking. *Educational Studies in Mathematics, 66*, 145–164. doi: 10.1007/s10649-006-9024-6 C

Rathouz, M. M. (2009). Support preservice teachers' reasoning and justifying. *Teaching Children Mathematics*, 16(4), 214–221.

Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers' use of representation to reason algebraically. *Journal of Mathematical Behavior, 28*(2/3*)*, 188–199.

Richardson, K., Carter, T., & Berenson, S. (2010). Connected tasks: The building blocks of reasoning and proof. *Australian Primary Mathematics Classroom, 15*(4), 17–23.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561–574.

Rivera, F. D., & Becker, J. R. (2011). Formation of pattern generalization involving linear figural patterns among middle school students: Results from a three-year study. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 323–366). Springer-Verlag: Berlin Heidelberg.

Robichaux, R. R., & Rodrigue, P. R. (2011). Dine on rich functional examples. *Mathematics Teaching in the Middle School, 16*(6), 368–374.

Rojano, T. (1996). The role of problems and problem solving in the development of algebra. In N. Berdnarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 15– 38). Dordrecht, Netherlands: Kluwer.

Russel, S. J., Schifter, D., & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 43–69). Springer-Verlag: Berlin Heidelberg.

Saul, M. (2008). Algebra: The mathematics and the pedagogy. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 63–80). Reston, VA: NCTM.

Sawyer, R. K., & Greeno, J. G. (2009). Situativity and learning. In P. Robbins & M. Ayedede (Eds.), The Cambridge handbook of situated cognition (pp. 347–367). New York: Cambridge University Press.

Schifter, D., Bastable, V., Russel, S. J., Seyferth, L., & Riddle, M. (2008). Algebra in the grades K-5 curriculum: Learning opportunities for students and teachers. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 263–273). Reston, VA: National Council of teachers of Mathematics.

Schifter, D., Russel, S. J., & Bastable, B. (2009). Early algebra to reach the range of learners. *Teaching Children Mathematics, 16*(4), 230– 237.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *The Journal of Mathematical Behavior*, *14,* 15–39.

Sherin, G. M. (2002). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, *5,* 205–233.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 133–160). New York: Taylor & Francis Group.

Smith, S. S., Hughes, E. K., Engle, R. A., & Stein, M. K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School*, *14*(9), 548–556.

Steele, D. F. (2007). Understanding students' problem solving knowledge through their writing. *Mathematics Learning in the Middle School, 13,* 102–109.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313–340.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research techniques and procedures for developing grounded theory* (2nd ed.). London: Sage.

Stump, S. L. (2011). Patterns to develop algebraic reasoning. *Teaching children Mathematics, 17*(7), 410–418.

Stylinaou, D. A. (2011). An examination of middle school students' representation practices in mathematical problem solving through the lens of expert: Towards an organizing scheme. *Educational Studies in Mathematics, 76*(3), 265–280.

Tarlow, L. D., & Uptegrove, E. B. (2010). Block towers: Co-construction of proof. In C. A. Maher, A. B. Powell, & E. B. Uptegrove (Eds.), *Combinatorics and reasoning: Representing, justifying and building isomorphisms* (pp. 97–104). New York: Springer.

Tierney, C., & Monk, S. (2008). Children's reasoning about change over time. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 185–200). New York: Lawrence Erlbaum.

Towers, J. (2002). Blocking the growth of mathematical understanding: A challenge for teaching. *Mathematics Education Journal, 14*(2) 121–132.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, k-12* (pp. 8–19). Reston, VA: NCTM.

Warren, E., & Cooper, T. J. (2008a). Generalizing the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67,* 171–185.

Warren, E., & Cooper, T. J. (2008b). Patterns that support early algebraic thinking in the elementary school. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 113–126). Reston, VA: National Council of Teachers of Mathematics.

Warren, E. A., Cooper, T. J., & Lamb, J. T. (2006). Investigating functional thinking in the elementary classroom: Foundations of early algebraic reasoning. *Journal of Mathematical Behavior , 25*, 208–223.

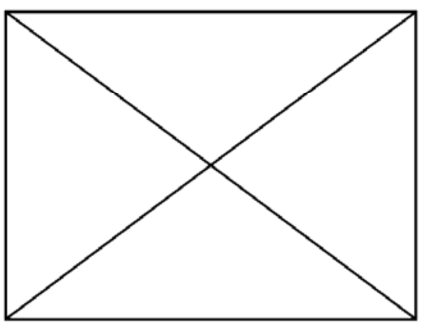
Wilson, P. H. (2009). *Teachers' uses of a learning trajectory for equipartitioning.* (Doctoral dissertation), North Carolina State University.

APPENDIX A

ON TRACK BROCHURE

ON TRACK LEARN MATH

An After-School Enrichment Program for Students in Grades 3-5 to Promote Problem Solving



Partners

[redacted]

What: Fun with math problems and puzzles
Where: [redacted]
When: [redacted]
Who: Students in grades 3-5

Please fill in this request for an application and return as soon as possible to [redacted]. Space is limited and students will be selected on the basis of the date your application for ON TRACK is received.

There are no extra charges to participate in this program.

This program is supported in part by a grant from the Fund for the Improvement of Education, US Office of Education, Grant 10-0397; 11-0156s.

I am interested in having my student participate in the ON TRACK program beginning [redacted] at [redacted]. Please mail me an application.

Print student's name:

____

My student's grade: ______

Print student's math Teacher:

____

Please mail the application to me at the following address (print):

____

(name)

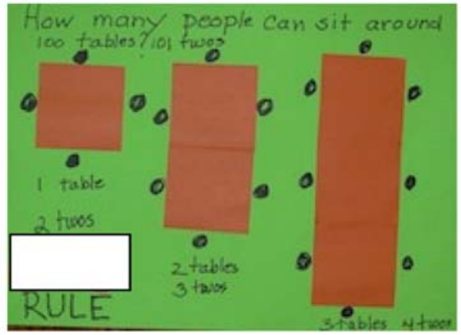
____

____

____

The goals of the ON TRACK program are:

- To learn math with sports and games.
- To learn problem solving strategies with patterns.
- To build confidence in problem solving situations.



Special Thanks to

[redacted]

for their assistance with this program.

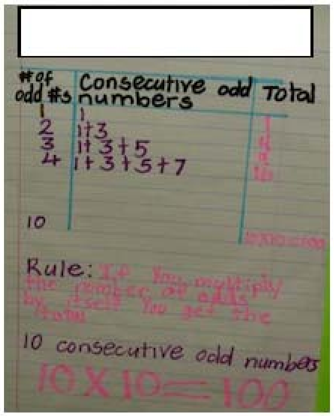
THE PROGRAM

Each afternoon begins with 30-40 minutes of Sports Math using lots of physical activity. We recognize that children need to exercise after being in school all day.

After Sports Math students will work together to solve interesting problems. These problems are designed to promote elementary students' understanding of early algebra ideas.

Do you see a pattern?
How many stars will you draw in the next row?
Can you find a rule to predict how many stars are in the 10th row? How do you know you are correct?

ON TRACK LEARN MATH



Contact for more information:

[redacted]

APPENDIX B

IRB CONSENT/ASSENT FORMS

UNIVERSITY OF NORTH CAROLINA AT GREENSBORO

CONSENT FOR A MINOR TO ACT AS A HUMAN PARTICIPANT: LONG FORM

Project Title: ON TRACK-LEARN MATH
Project Director: (name)
Participant's Name (Your child's name): ____

What is the study about?
The purpose of this project is to teach mathematical problem solving and mathematical reasoning to students in grades 3 to 5. In general, students will engage in enrichment activities that focus on problem solving, mathematical reasoning and pattern finding in after-school programs at selected schools. These concepts will be taught using a student-centered approach to problem solving whereby students will explore and share their ideas.

Why are you asking my child?
Your child goes to ____ Elementary school and is in either grade 3, 4 or 5. He/she has been accepted into the ON TRACK - LEARN MATH PROJECT taking place in the after school care program. The program will focus on teaching math through problem solving. We need your permission to include your child in this research project. We also need your permission to include your child's words and written work in our study of the participating students' learning.

What will you ask my child to do if I agree to let him or her be in the study?
Your child will be asked to participate in the learning activities that we have designed to teach most of the state mathematics objectives for grades 3 to 5. The learning activities will take place in the after school care program over 5 week sessions. Each class will take about 90 minutes. The students will meet 2 afternoons each week. All the children in this study will be audio recorded as they engage in the learning tasks. Some students will be videotaped within the classroom and whole group class activities. Your child's work will become part of the study. A few students will be interviewed on their learning experiences during this project. The interviews will take up to 45 minutes.

Is there any audio/video recording of my child?
Because your child's face will be potentially identifiable by anyone who hears and sees the tapes, the confidentiality for things your child says or does on the tape cannot be guaranteed although the researcher will try to limit access to the tape as described below. Children will be asked to adopt a research name on their desks and papers rather than

their own for the study. Video cameras will focus on children's desks rather than their faces during class and interview filming. Whole class filming will be done from the back of the room.

What are the dangers to my child?
Other than the slight risk of the breach of confidentiality, there are no anticipated dangers to your child in this project. The measures taken to minimize the risks are described in the section above.
If you have any concerns about your child's rights or how your child is being treated or if you have questions, want more information or have suggestions, please contact (name) in the Office of Research Compliance at UNCG at (phone number). Questions, concerns or complaints about this project or your child's benefits or risks associated with being in this study can be answered by (name, email, phone).

Are there any benefits to my child as a result of participation in this research study?
Students may learn more about each of the mathematics objectives that will be investigated during the project.

Are there any benefits to society as a result of my child taking part in this research?
We may be able to provide evidence that teaching problem solving increases students' mathematics achievement and better prepares them for their future education and careers. Insights from the study will also help in the preparation of mathematics teachers at UNCG.

Will my child get paid for being in the study? Will it cost me anything for my kid to be in this study?
There are no costs to you or payments made for participating in this study.

How will my child's information be kept confidential?
Audio CDs and videotapes that could identify your child will be stored in a locked file cabinet in (place). The electronic identifiable data will be stored (place). This information will be stored for 3 years after the end of the project before being destroyed as follows: Electronic data will be overwritten electronically, CDs will be shredded, and ink text files will be shredded. Transcripts and students work that maintain confidentiality will be maintained for 5 years after the data are collected. All information obtained in this study is strictly confidential unless disclosure is required by law.

What if my child wants to leave the study or I want him/her to leave the study?
You have the right to refuse to have you child participate or to withdraw at any time, without penalty. If you do withdraw your child, it will not affect him/her in any way. If you choose to withdraw your child, you may request that any of your child's data which has been collected be destroyed unless it is in a de-identifiable state.

What about new information/changes in the study?
If significant new information relating to the study becomes available which may relate to your willingness allow your child to continue to participate, this information will be provided to you.

Voluntary Consent by Participant:
By signing this consent form, you are agreeing that you have read it or it has been read to you, you fully understand the contents of this document and consent to your child taking part in this study. All of your questions concerning this study have been answered. By signing this form, you are agreeing that you are the legal parent or guardian of the child who wishes to participate in this study described above in this document.

_____________________________________ Date: ________________
Participant's Parent/Legal Guardian's Signature

_____________________________________ Date: ________________
Participant's Parent/Legal Guardian's Signature

UNIVERSITY OF NORTH CAROLINA AT GREENSBORO

Assent Form

Study Title: ON TRACK-LEARN MATH

What is this about?
We would like to teach you about problem solving in mathematics. Some of the activities include finding patterns and rules that explain the patterns. We want to learn about how you think about these problem solving ideas.

Did my parents say it was ok?
Your parent(s) said it was ok for you to be in this study and have signed a form like this one.

Why me?
We would like you to take part because you are ether in grade 3, 4 or 5 at this school. We wanted to study with 3rd, 4th, and 5th graders at this school.

What if I want to stop?
You do not have to say "yes", if you do not want to take part. We will not punish you if you say "no". Even if you say "yes" now and change your mind after you start doing this study, you can stop and no one will be mad at you. It will not affect your grade either.

What will I have to do?
We will be meeting in room ______ after school from _______ to __________. We will be meeting 2 times each week for 5 weeks. We will be working on some mathematical tasks in small groups and discuss them as a class. Video and audio recordings will be made of you as you work. We will collect your written work at the end of each day. We may have to interview you later during the project.

Will anything bad happen to me?
We do not think that anything bad will happen to you. This is just like a regular after school class with different teachers and helpers. Your work in this project does not count towards your math grade.

Will anything good happen to me?
You may learn ideas about problem solving in mathematics. This may help you during your regular math lessons and on the end of grade tests.

Do I get anything for being in this study?
You get to participate in the ON TRACK -LEARN MATH program.

What if I have questions?
You are free to ask questions at any time.

If you understand this study and want to be in it, please write your name below.

______________________________	____________________
Signature of child	Date

UNIVERSITY OF NORTH CAROLINA AT GREENSBORO

CONSENT TO ACT AS A HUMAN PARTICIPANT: LONG FORM

Project Title: ON TRACK-LEARN MATH

Project Director: (name)

Participant's Name: ____

What is the study about?
This is a research project. The purpose of this research is to study changes in your perception of teaching and learning mathematical reasoning over your participation in the project. We will study your teaching of mathematical reasoning during the ON TRACK-LEARN MATH project.

Why are you asking me?
You are being asked to participate in this study because you will be paid to teach the ON TRACK-LEARN MATH project students.

What will you ask me to do if I agree to be in the study?
Your perception will be sought during both the professional development period and the times you will be teaching the ON TRACK-LEARN MATH student participants. Your teaching practice will also be observed.

Are there any audio/video recording?
Your perceptions and teaching practice will be video and audio recorded. Because your voice will be potentially identifiable by anyone who hears the tape, your confidentiality for things you say on the tape cannot be guaranteed although the researcher will try to limit access to the tape as described below.

What are the dangers to me?
The Institutional Review Board at the University of North Carolina at Greensboro has determined that participation in this study poses minimal risk to participants.
If you have any concerns about your rights, how you are being treated or if you have questions, want more information or have suggestions, please contact (name) in the Office of Research Compliance at (place and email) . Questions, concerns or complaints about this project or benefits or risks associated with being in this study can be answered by (name) who may be contacted at (phone email email).

Are there any benefits to me for taking part in this research study?
There are no direct benefits to participants in this study.

Are there any benefits to society as a result of me taking part in this research?
Your perceptions and practice may be very valuable in designing the ON TRACK-LEARN MATH activities to improve students learning. You may also help in improving mathematics education by providing insights into issues that need to be considered in teaching mathematical reasoning.

Will I get paid for being in the study? Will it cost me anything?
There are no costs to you or payments made for participating in this study.

How will you keep my information confidential?
Audio CDs and videotapes that could identify your child will be stored in a locked file cabinet in (place). This information will be stored for 3 years after the end of the project before being destroyed as follows: Electronic data will be overwritten electronically, CDs will be shredded, and ink text files will be shredded. Transcripts and students work that maintain confidentiality will be maintained for 5 years after the data are collected. All information obtained in this study is strictly confidential unless disclosure is required by law.

What if I want to leave the study?
You have the right to refuse to participate or to withdraw at any time, without penalty. If you do withdraw, it will not affect you in any way. If you choose to withdraw, you may request that any of your data that has been collected be destroyed unless it is in a de-identifiable state.

What about new information/changes in the study?
If significant new information relating to the study becomes available which may relate to your willingness to continue to participate, this information will be provided to you.

Voluntary Consent by Participant:
By signing this consent form you are agreeing that you have read it, or that it has been read to you and you fully understand the contents of this document and are openly willing to consent to take part in this study. All of your questions concerning this study have been answered. By signing this form, you are agreeing that you are 18 years of age or older and are agreeing to participate, or have the individual specified above as a participant participate, in this study described to you by ________.

Signature: ________________________________ Date: ____________________

APPENDIX C

ON TRACK LAB SHEET

NAME____________________________SCHOOL___________________GRADE______

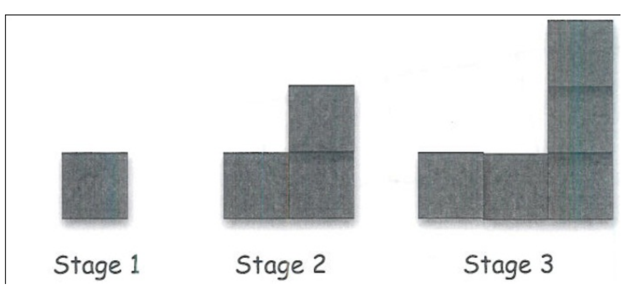
PARTNER'S NAME____________________________________

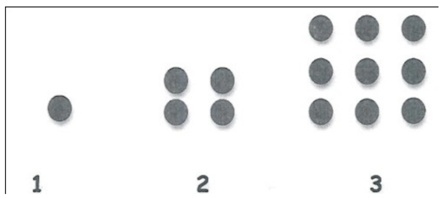
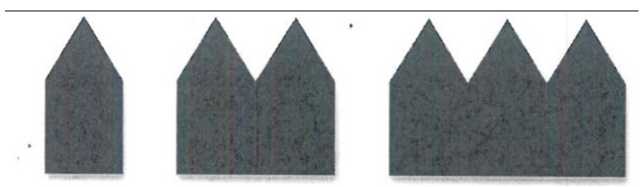
Suppose you make a train of pentagon tables. If one person can sit on one side of the pentagon, how many people can sit at 1 pentagon table? A train of two pentagon tables?] **A train of three pentagon tables?**

How many people can sit around a train of 10 pentagon tables that are put together? How many people can sit around a train of 100 pentagon tables?

APPENDIX D

INSTRUCTIONAL TASKS

Function Machine task	Guess my rule If I put 1 into the function machine, 12 comes out. If I put 2 into the function machine, 22 comes out. If I put 3 into the function machine, 32 comes out. What is the rule for this function machine? Rule:10x + 2 = y
ATV Function Machine Task	It costs $5 to rent an ATV for 1 hour. You are the owner of the ATV and need to make a table of the costs of renting the ATV. Make the table where hours are the input variable and dollars are the output variable. Is there a rule that will tell you the cost for any number of hours? Rule: 5x = y
Perimeter Task	Predict the perimeter of the figure at stages 5, 10, 100. Write your rule.  Rule: 4x = y

Square Table Task	If one person sits on each side of a square in this pattern, how many people would sit around a train of 100 squares? Write your rule.  Rule: $4x + 2 = y$
Square number task	Predict number of dots for stages 5, 10, 100. Write your rule.  Rule: $x^2 = y$
Pentagon task	How many people would sit around a train of 100 pentagon tables. Write your rule.  Rule: $3x + 2 = y$

APPENDIX E

TEACHER INTERVIEW PROTOCOL

As you know, you have been teaching On Track students to reason algebraically. From the way On Track students are making their generalizations, you On Track teachers are doing an amazing job. I would like to hear more about what you do and why you do it when you are teaching On Track sessions.

1. How would you characterize your On Track teaching style?
2. Tell me everything you do during On Track lessons to help students to reason algebraically.
    a. You have identified the following practices (*repeat the list of practices to teachers*). Is there anything else you would like to add to the list?
3. You mentioned using this practice to support students' reasoning (*mention each practice at a time*). Tell me more about that.
    a. What is your rationale for using it?
4. What do you think about this practice (*observed practice but not listed*).
    a. Do you use it during On Track lessons?
    b. Why or why not.
5. In terms of supporting students algebraic thinking, what would you differently.
    a. What instructional practices would you do or not do?
    b. What are the barriers to your engagement in this practices you wish to engage in.