The main goal of this dissertation is to formulate and analyze a dual wind discontinuous Galerkin method for approximating solutions to elliptic variational inequalities. A discontinuous Galerkin (DG) finite-element interior calculus is used as a common framework to describe various DG approximation methods for second-order elliptic problems. The dual-wind discontinuous Galerkin method is formulated for the obstacle problem with Dirichlet boundary conditions, \(-\Delta u \geq f\) on \(\Omega\) with \(u = g\) on \(\partial \Omega\), \(u \geq \psi\) on \(\Omega\), and \((\Delta u - f)(u - \psi) = 0\) on \(\Omega\). A complete convergence analysis is developed and numerical experiments are recorded that verify these results.

A secondary goal of this dissertation is to explore the effect of the penalty parameter on the error of the dual-wind discontinuous Galerkin method’s approximation to an elliptic partial differential equation. The dual-wind discontinuous Galerkin method is applied to the Poisson problem in two dimensions. The dual-wind discontinuous Galerkin approximation to the Poisson problem is constructed using various penalty parameters and the error is recorded for each approximation across various initial meshes and their refinements.
SYMMETRIC DUAL-WIND DISCONTINUOUS GALERKIN METHODS FOR
ELLIPTIC VARIATIONAL INEQUALITIES

by

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I would like to dedicate this dissertation to everyone who has supported me on my journey to this point. Their love and support has made this possible.
This dissertation written by Aaron Frost Rapp has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I
INTRODUCTION

The goal of this dissertation is to formulate and analyze a dual wind discontinuous Galerkin method for approximating solutions to elliptic variational inequalities. In this section, we will introduce some of the background information for the mathematical problem that will be considered in this dissertation. We start with an introduction to variational inequalities (VIs) and discuss where they arise. Next we will present the specific elliptic variational inequality problem that will be approximated, as well as give a basic breakdown of a classical continuous Galerkin method called the finite element method. Finally, we will discuss some of the benefits of discontinuous Galerkin methods over standard continuous Galerkin methods.

I.1. Elliptic Variational Inequalities

There are many problems in elasticity that can be modeled by a minimization problem. Specifically, let $U$ be a closed convex subset of a Hilbert space $V$, and let $J : V \rightarrow \mathbb{R}$ be defined by

$$J(v) := \frac{1}{2}a(v, v) - f(v),$$

where $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is a symmetric and continuous bilinear form, and $f : V \rightarrow \mathbb{R}$ is a continuous linear form. The unknown solution $u$ satisfies

$$u \in U \quad \text{and} \quad J(u) = \min_{v \in U} J(v).$$

In a physical setting, $U$ is the set of admissible displacements, the solution $u$ is the displacement of the mechanical system, and $J$ is the energy of the system.
We will continue in this subsection by stating an abstract minimization problem and presenting a theorem for the existence and uniqueness of the solution. From there, we will present another theorem stating an equivalent formulation of the minimization problem, called the variational formulation. We first introduce some necessary terminology.

**Definition I.1.** Let $(V, \| \cdot \|)$ be a normed vector space, and $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bilinear form. We say $a(\cdot, \cdot)$ is $V$-elliptic if there exist an $\alpha > 0$ such that, for all $v \in V$,

$$\alpha \|v\|^2 \leq a(v, v).$$

Note that the formulation of the abstract minimization problem below has slightly different assumptions then what was stated above.

**Definition I.2.** ([Cia02]) Let $V$ be a normed vector space with norm $\| \cdot \|$, and let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a symmetric continuous bilinear form that is $V$-elliptic. Let $f : V \to \mathbb{R}$ be a continuous linear form, and let $U$ be a non-empty subset of $V$. The abstract minimization problem is to find an element $u \in U$ such that

$$J(u) = \inf_{v \in U} J(v), \tag{I.1}$$

where the functional $J : V \to \mathbb{R}$ is defined by $J(v) := \frac{1}{2}a(v, v) - f(v)$.

**Theorem I.3.** ([Cia02]) In addition to the assumptions made to create the abstract minimization problem (I.1), further assume that $V$ is complete and $U \subset V$ is a closed, convex subset of $V$. Then the abstract minimization problem (I.1) has a unique solution.

Though the original statement of this problem is as a minimization problem, we wish to have an equivalent formulation that will allow us to more naturally pose
a discrete formulation, and thus apply our approximation method. Note that the
symmetry of $a(\cdot, \cdot)$ is not generally required for variational inequalities; however, it is
typically assumed as a way to simplify the proof of Theorem I.3.

**Theorem I.4.** ([Cia02]) An element $u$ is the solution to the abstract minimization
problem (I.1) if and only if it satisfies the following relations

$$u \in U \quad \text{and} \quad a(u, v - u) \geq f(v - u) \quad \forall \ v \in U$$

(I.2)

in the general case, or

$$u \in U \quad \text{and} \quad a(u, v - u) \geq f(v - u) \quad \forall \ v \in U \quad \text{and} \quad a(u, u) = f(u)$$

(I.3)

if $U$ is a closed convex cone with vertex 0 or

$$u \in U \quad \text{and} \quad a(u, v) = f(v) \quad \forall \ v \in U$$

(I.4)

if $U$ is a closed subspace.

One interesting consequence of this formulation is that if we take $U = V$ and
do not assume that the bilinear form $a(\cdot, \cdot)$ is symmetric, then Theorem I.3 becomes
the Lax-Milgram lemma [Cia02]. Using various Green’s formulas in Sobolev spaces, it
has been formally shown that solving an abstract minimization problem, as described
in Theorem I.4, is equivalent to solving an elliptic boundary value problem posed in
the classical way [Cia02].

**I.2. Some Applications of Variational Inequalities**

There are many areas in which variational inequalities naturally arise such as
fluid filtration in porous media, constrained heating, optimal control, and financial
mathematics [Caf98]. Other problems such as the elastic-plastic torsion problem and
the cavitation problem in the theory of lubrication can be regarded as obstacle-type problems [WHC10a]. We will introduce a few of these in more detail in this subsection, leaving the problem that we study for this thesis to be covered in Section I.3. More examples and applications of VIs can be found in the monographs [DL76, KS80, GLT81, Fri10, Glo84, Rod87].

Figure I.1. Simple Free Boundary Hydraulics Problem.

The first example of a variational inequality that naturally arises from a physical problem that we will cover is fluid filtration in a porous media, specifically for hydraulics. A simple example of this is stated in [BS76] in which Brezzi and Sacchi showed the convergence of the finite element method. Suppose that there is a dam of an isotropic homogeneous porous medium separating two water reservoirs of levels $y_1$ and $y_2$, with $y_1 > y_2$, on a common horizontal impervious base and with parallel vertical walls. An example of such a medium separating two water reservoirs is earth. The water flows through the dam from the higher reservoir to the lower reservoir uniformly. With this physical system, we wish to solve for the position and the form
of the free surface in the dam. A depiction of this physical system can be found in Figure I.1 which can be found in [BS76].

Keeping in mind the information depicted in Figure I.1, we will state the model for this problem. Let $D$ represent the rectangular dam with vertices $ABEF$, let $\Omega$ be the flow region in the dam, and let $\varphi$ be the free surface. The problem is to find a function $y = \varphi(x)$ that is defined and smooth on $[0, a]$ with $\varphi(0) = y_1$ and $\varphi(a) \geq y_2$ such that there exist a function $u(x, y)$ that is defined and smooth in the closure of $\Omega = \{(x, y) : 0 < x < a, \ 0 < y < \varphi(x)\}$ such that

$$\Delta u = 0; \quad \Omega,$$

$$u = y_1 \text{ on } [A, F],$$

$$u = y_2 \text{ on } [B, C],$$

$$u = y \text{ on } [C, C_{\varphi}],$$

$$u = y \text{ on } FC_{\varphi},$$

$$u_y = 0 \text{ on } [B, C],$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } FC_{\varphi},$$

where $FC_{\varphi}$ is the arc created by the function $\varphi$ from the point $F$ to a point on the line $[C, E]$. This problem can be reformulated as a VI using a method by C. Baiocchi [Bai72b,Bai72a] that involves restating the problem as a non-linear problem on the whole domain $D = (0, a) \times (0, y_1)$. To this end, the solution $u(x, y)$ needs to be extended to the whole domain by

$$\tilde{u}(x, y) = \begin{cases} 
    u(x, y) & \text{for } (x, y) \in \Omega, \\
    y & \text{for } (x, y) \in D \setminus \Omega.
\end{cases}$$
Furthermore, we define the function \( w(x, y) = \int_y^{y_1} \bar{u}(x, t) - t \, dt \) and the function \( g : \partial D \to \mathbb{R} \) by

\[
\begin{cases}
  g = \frac{y_2^2}{2} + \frac{y_2^2 - y_1^2}{2a} x & \text{on } (A, B), \\
  g = \frac{1}{2}(y_2 - y)^2 & \text{on } [B, C], \\
  g = \frac{1}{2}(y_1 - y)^2 & \text{on } [A, F], \\
  g = 0 & \text{elsewhere}.
\end{cases}
\]

Given the closed convex set \( K = \{ v \in H^1(D) : v|_{\partial D} = g, \ v \geq 0 \ \text{a.e. on} \ D \} \), \( w \) is the solution to the problem: find \( w \in K \) such that

\[
\int_D \nabla w \cdot \nabla (v - w) \, dx dy \geq \int_D -(v - w) \, dx dy \ \forall \ v \in K.
\]

In this form, we can now identify this free boundary problem as an elliptic variational inequality.

Another example of variational inequalities can be found in economics by studying quantity models of a commodity. Note that here, quantity is referring to the supply and demand of a certain commodity. The model that we will describe in this dissertation is a simple model. However, it can be generalized if one relaxes some of the assumptions that are made. This and more examples of variational inequalities in economics can be found in [Nag13].

Let there be \( m \) supply markets and \( n \) demand markets, where each supply market is denoted by \( 1 \leq i \leq m \) and each demand market is denoted by \( 1 \leq j \leq n \). Let \( s \in \mathbb{R}^m \) be a column vector defined by \( (s)_i = s_i \), where \( s_i \) is the supply of a commodity associated with market \( i \). Let \( \pi \in \mathbb{R}^m \) be a row vector defined by \( (\pi)_i = \pi_i \), where \( \pi_i \) is the supply price of a commodity associated with market \( i \). Let \( d \in \mathbb{R}^n \) be a column vector defined by \( (d)_j = d_j \), where \( d_j \) is the demand of a commodity associated with market \( j \). Let \( \rho \in \mathbb{R}^n \) be a row vector defined by \( (\rho)_j = \rho_j \), where \( \rho_j \) is the demand price of a commodity associated with market \( j \). Let \( Q \in \mathbb{R}^{m \times n} \) be defined
by \((Q)_{ij} = Q_{ij}\) where \(Q_{ij}\) is the non-negative commodity shipment between the supply and demand market pair \((i, j)\). Lastly, let \(c \in \mathbb{R}^{m \times n}\) defined by \((c)_{ij} = c_{ij}\), where \(c_{ij}\) is the non-negative unit transaction cost associated with trading the commodity between the market pair \((i, j)\).

We assume the following market equilibrium condition where perfect competition takes place. That is, for all \((i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}\)

\[
\pi_i + c_{ij} = \rho_j \quad \text{if} \quad Q_{ij}^* > 0, \quad (I.5a)
\]

\[
\pi_i + c_{ij} \geq \rho_j \quad \text{if} \quad Q_{ij}^* = 0. \quad (I.5b)
\]

The market equilibrium condition (I.5) can be interpreted as the supply price for market \(i\) plus the unit transaction cost between between markets \(i\) and \(j\) must be equal to the demand price in market \(j\). Otherwise, there will be no shipment between markets \(i\) and \(j\), because the supply price and transactional cost exceed the demand price at the equilibrium.

The market equilibrium condition (I.5) is what we wish to model. It is restricted by the feasibility conditions,

\[
s_i = \sum_{j=1}^{n} Q_{ij}, \quad d_j = \sum_{i=1}^{m} Q_{ij}. \quad (I.6)
\]

These feasibility conditions assume that there is an equal amount of supply and demand in the markets, as well the supply markets are the same as the commodity flows to every demand market, and the demand market is equal to all of the commodity shipments from every supply market. Define the convex set

\[
K := \{(s, Q, d) : (I.6) \text{ holds}\}.
\]
We will make further simplifying assumptions on this model that create a special case. First, restate the functions $\pi$, $\rho$, and $c$ as

$$\pi = \pi(s), \quad \rho = \rho(d), \quad c = c(Q).$$  \hspace{1cm} (I.7)

That is, in general, the supply price, demand price, and transaction cost depend upon the supply, demand, and shipments of the commodity, respectively. These functions will be smooth functions which are known in the markets. Thus, to find the optimal values for the functions $\pi$, $\rho$, and $c$, we only need to find the values for $s$, $d$, and $Q$ that optimize those functions. Note that these optimal values, which we will denote as $(s^*, d^*, Q^*)$ are in our convex set $K$.

The next assumptions are that the number of supply markets $m$ is the same as the number of demand markets $n$, the transactional cost functions are fixed, and the supply and demand price functions are symmetric. In the context of this model, symmetric implies that $\frac{\partial \pi_i}{\partial s_j} = \frac{\partial \pi_j}{\partial s_i}$ and $\frac{\partial \rho_j}{\partial d_i} = \frac{\partial \rho_i}{\partial d_j}$ for all $i, j = 1, 2, \ldots, n$. Under these assumptions, including (I.7), the market equilibrium condition collapses to a class of single commodity models which is equivalent to a variational inequality that seeks an equilibrium $(s^*, Q^*, d^*) \in K$ such that

$$(\pi(s^*), s - s^*) + (c(Q^*), Q - Q^*) - (\rho(d^*), d - d^*) \geq 0 \quad \forall (s, Q, d) \in K,$$

where $(v, w) = v \cdot w$ (see Theorem 3.1 in [Nag13]). In other words, the variational inequality is stating that there is a single, most efficient route of trading between two markets [Nag13].

I.3. The Obstacle Problem

We next define the obstacle problem, which we introduce as a specific example of the membrane problem. Approximating solutions to the obstacle problem will be
the primary focus of this dissertation. The goal of the obstacle problem is to find the equilibrium position of an elastic membrane, with tension $\tau$, which passes through the boundary $\Gamma$ of an open set $\Omega \subset \mathbb{R}^2$, that is subjected to a force of density $F = \tau f$, and lies over an obstacle which is represented by a function $\psi : \overline{\Omega} \to \mathbb{R}$. Figure I.2 is a visualization of the obstacle problem in one dimension.

\[ \psi \]

\[ \Omega \]

\[ u \]

Figure I.2. Visualization of the Obstacle Problem in 1D. In this visualization $u$ is the membrane and $\psi$ is the obstacle.

Define the set

\[ H^1(\Omega) := \left\{ v \in L^2(\Omega) : \partial_x v \in L^2(\Omega) \text{ for } 1 \leq i \leq 2 \right\}. \]

The data for the obstacle problem is

\[ \Omega \subset \mathbb{R}^2 \quad V := H^1(\Omega), \]

\[ K_g := \left\{ v \in V : v \geq \psi \text{ a.e. in } \Omega, v = g \text{ on } \partial \Omega \right\}, \quad (I.8) \]

\[ a(v, w) := \int_{\Omega} \nabla v \cdot \nabla w \, dx, \quad f(v) := \int_{\Omega} f v \, dx. \]

With this data, we have by Theorem I.3 that there exist a unique solution $u \in K_g$ such that

\[ J(u) = \min_{v \in K_g} J(v), \quad (I.9) \]
where \( J(v) = \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v \, dx - \int_{\Omega} f v \, dx \). By Theorem I.4, the unique solution to the minimization problem is also the unique solution to the variational inequality

\[
a(u, v - u) \geq f(v - u) \quad \forall \, v \in K_g. \tag{I.10}
\]

It is well-known that (I.10) has a unique solution \( u \in H^2(\Omega) \) provided \( \psi \in H^2(\Omega) \) and \( g \in H^2(\Omega) \). Furthermore, \( u \in W^{s,p}(\Omega) \) for \( 1 < p < \infty \) and \( s < 2 + 1/p \) provided that \( \Omega \) is sufficiently smooth, \( f \in L^\infty(\Omega) \cap BV(\Omega) \), and \( g, \psi \in C^3(\overline{\Omega}) \) (see [Bre71, KS80, Rod87]). These regularity results guarantee that the strong form of the obstacle problem

\[
\begin{align*}
-\Delta u &\geq f \quad \text{in } \Omega, \tag{I.11a} \\
u &= g \quad \text{on } \partial \Omega, \tag{I.11b} \\
u &\geq \psi \quad \text{in } \Omega, \tag{I.11c} \\
(-\Delta u - f)(u - \psi) &= 0 \quad \text{in } \overline{\Omega}, \tag{I.11d}
\end{align*}
\]

is well-posed and consistent with (I.9). The difficulty of approximating the obstacle problem comes from the free boundary and compatibility condition I.11d. With the condition I.11d, the problem requires \( -\Delta u = f \) or \( u = \psi \) throughout the entire domain. The free boundary, where \( u \) transitions between one of these two parts of I.11d, is unknown. This creates difficulty when finding a solution to I.11. This difficulty arises in finding the approximation of the solution to I.11 as well.

I.4. Brief Introduction to the Finite Element Method

The finite element method (FEM) is a classical numerical technique approximating solutions to partial differential equations using finite dimensional subspaces. Boris Galerkin developed a numerical technique in 1915 called the Galerkin method that
approximates the solution of a boundary value problem [Esl14]. Galerkin’s method, which gives rise to the FEM, uses a discrete space generated by basis functions across a discretized mesh of the domain. This method yields the Galerkin orthogonality condition in which the error from the approximation is orthogonal to the discrete space with respect to the bilinear form associated with the variational form of the boundary value problem. We state this precisely later in Theorem I.10. The Galerkin orthogonality condition guarantees that the discrete approximation to the boundary value problem is the closest approximation to the solution in the discrete space with regards to the norm induced by the bilinear form.

![Figure I.3. Simple Triangularization with Labeled Triangles and Edges.](image)

The mesh itself can be made of different polygons, though using triangles is standard practice. Figure I.3 is an example of a triangularization of a square domain with a labeling of the triangles and edges. The finite element method solves the discrete problem across each triangle in a mesh, then glues each piece together to create a
continuous solution. However, the number of calculations needed to create such approximations was considered too large until the introduction of modern computers. In the 1950’s and 1960’s, a plethora of research began which lead to formulations of finite element methods for different problems. In 1967, Zienkiewicz wrote the first volume of a book on finite element methods combining his and others works [Esl14].

We consider the formulation of the finite element method for a model second order linear boundary value problem in two dimensions: find a function $u$ such that

\begin{align*}
-\Delta u &= f \quad \text{on } \Omega \subset \mathbb{R}^2, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{align*}

where $\Omega$ is a bounded domain with a smooth or polygonal boundary, and $f$ is a given function. To generate the finite element method, we will need the variational formulation of the problem stated above. To do so, define the test function space

$$V_0 := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega\}.$$  

multiply each side of (I.12a) by $v \in V_0$, and apply Green’s formula. This gives us the following,

$$
\int_{\Omega} f v \, dx = -\int_{\Omega} (\Delta u) v \, dx \\
= \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial \Omega} (\nabla u \cdot n) v \, dx \\
= \int_{\Omega} \nabla u \cdot \nabla v \, dx,
$$

which results in the variational formulation of (I.12) given by

$$
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in V_0.
$$

(I.13)
Next, we will cover our domain $\Omega$ with a mesh. For simplicity, we will assume that $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain. A triangulation, or mesh, is a set of triangles $T_h = \{K\}$ such that $\bigcup_{K \in T_h} K = \overline{\Omega}$, and that each intersection of two triangles is either an edge, corner, or empty. We construct the mesh $T_h$ such that no corner of a triangle lies on the edge of another. We let $\mathcal{E}_h = \{e\}$ denote the set of triangle edges. Two subsets of $\mathcal{E}_h$ that will often arise are the set of interior triangle edges $\mathcal{E}_h^I$ in the domain $\Omega$, and the set of boundary triangle edges $\mathcal{E}_h^B$ that are on the boundary of the domain $\partial \Omega$. We also define the set of nodes $\mathcal{N} = \{N\}$, which is the set of all corners of every triangle in $T_h$, and lastly, we will define $n_p$ to be the cardinality of the set $\mathcal{N}$, $h_K$ to be the length of the longest edge for a triangle $K \in T_h$, and $h = \max_{K \in T_h} h_K$. Two mesh properties that will be referenced in the analysis are quasi-uniformity and shape-regularity.

**Definition I.5.** A mesh $T_h$ is quasi-uniform if there exist a constant $\rho > 0$ such that

$$\rho < h_K / h_{K'} < \rho^{-1} \quad \forall \, K, K' \in T_h.$$

**Definition I.6.** Let $d_K$ represent the diameter of the inscribed circle of a triangle $K$. A mesh $T_h$ is shape-regular if there is a constant $C > 0$ such that

$$\max_{K \in T_h} \frac{h_K}{d_K} \leq C.$$

Given the mesh, the next step is to define the space of piecewise polynomials over the mesh $T_h$. For simplicity, we will focus on linear polynomials. We start by first considering a single triangle $K \in T_h$. Define the set of all linear polynomials over the triangle $K$ as

$$\mathbb{P}_1(K) := \{ v : v = c_0 + c_1 x_1 + c_2 x_2, \ (x_1, x_2) \in K, \ c_0, c_1, c_2 \in \mathbb{R} \}.$$
It is important to note that each linear function over a triangle \( K \) can be uniquely determined by the values at the three nodes of \( K \). Exploiting this property, we will define the discrete space

\[
V_h := \{ v : v \in C^0(\Omega), \ v|_K \in \mathbb{P}_1(K) \ \forall K \in \mathcal{T}_h \},
\]

where \( C^0(\Omega) \) is the set of all continuous functions over the domain \( \Omega \).

Since each function in \( V_h \) is uniquely determined by its nodal values, we can define the nodal basis functions \( \{ \varphi_i \}_{i=1}^{n_p} \subset V_h \) such that

\[
\varphi_j(N_i) = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j
\end{cases}
\]

for all \( i, j = 1, 2, \ldots, n_p \). Each nodal basis function is continuous, piecewise linear, and has support only on the set of triangles that share a node [LB13]. These nodal basis functions are also called hat functions because of their shape (see Figure I.4).

![Figure I.4. Visualization of a Hat Function in \( \mathbb{R}^2 \) from [LB13].](image)

With the nodal basis functions, we can write any function \( v \in V_h \) as \( v = \sum_{i=1}^{n_p} c_i \varphi_i \), where \( \{c_i\}_{i=0}^{n_p} \subset \mathbb{R} \) and \( c_i = v(N_i) \) for \( i = 1, 2, \ldots, n_p \). Lastly, for the
problem (I.12), define the subspace \( V_{h,0} := \{ v \in V_h : v|_e = 0 \ \forall e \in \mathcal{E}_h \} \). We note that \( V_h \subset V \) and \( V_{h,0} \subset V_0 \). Due to the linear nature of the subspace \( V_{h,0} \), it is spanned by the nodal basis functions \( \{ \phi_j \}_{j=1}^{n_i} \), where \( n_i \) is the number of interior nodes [LB13].

By replacing the larger test function space \( V_0 \) with \( V_{h,0} \) in (I.13), we will obtain the discrete variational problem: find \( u_h \in V_{h,0} \) such that

\[
\int_{\Omega} \nabla u_h \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in V_{h,0},
\]

(I.15)

where \( u_h \) is the finite element method’s approximation to \( u \). With (I.15) in mind, we define the energy norm \( \| \cdot \| \) on the space \( V_0 \) by \( \| v \| := \int_{\Omega} \nabla v \cdot \nabla v \, dx \). The energy norm is essential for the error analysis.

Since each \( v \in V_{h,0} \) is a linear combination of functions in \( \{ \phi_j \}_{j=1}^{n_i} \), the problem (I.15) is equivalent to finding \( u_h \in V_{h,0} \) such that

\[
\int_{\Omega} \nabla u_h \cdot \nabla \phi_j \, dx = \int_{\Omega} f \phi_j \, dx \quad j = 1, 2, \ldots, n_i.
\]

(I.16)

Furthermore, since \( u_h \in V_{h,0} \), we can write \( u_h = \sum_{i=0}^{n_i} c_i \phi_i \). Plugging this into (I.16), we obtain

\[
\int_{\Omega} f \phi_j \, dx = \int_{\Omega} u_h \phi_j \, dx \\
= \int_{\Omega} \sum_{i=0}^{n_i} c_i \phi_i \phi_j \, dx \\
= \sum_{i=0}^{n_i} c_i \int_{\Omega} \phi_i \phi_j \, dx
\]

for all \( i, j = 1, 2, \ldots, n_i \). Next, define the matrix \( A \in \mathbb{R}^{(n_i) \times (n_i)} \) by

\[
A_{ij} = \int_{\Omega} \phi_i \phi_j \, dx,
\]
and define the two column vectors $\vec{f} \in \mathbb{R}^n$ and $\vec{c} \in \mathbb{R}^n$ by $\vec{f}_j = \int_\Omega f \varphi_j \, dx$ and $(\vec{c})_j = c_j$.

Using the linearity of the PDE, the discrete variational formulation can be written in matrix form as

$$A\vec{c} = \vec{f}.$$  

For a boundary value problem of this type, the matrix $A$ is commonly referred to as the stiffness matrix and the vector $\vec{f}$ is commonly referred to as the load vector. Since the nodal basis functions are known and $u_h = \sum_{i=0}^n c_i \varphi_i$, the finite element method solution $u_h$ is then found by solving for the coefficient vector $\vec{c}$.

Since the finite element solution $u_h$ is an approximation to the solution of (I.12), estimates of the error are needed to judge how well $u_h$ approximates $u$. The overall goal is to minimize the error as much as possible, which can be done in two steps. The first is choosing a finite dimensional subspace $V_{h,0}$ that is a good approximation of the space $V_0$. The second is to minimize the error of the approximation by finding the best approximation in the finite dimensional subspace $V_{h,0}$. Since we are choosing $V_{h,0}$ to be the set of $C^0(\Omega)$ functions that are linear polynomials over every triangle, we can get an idea of how well this space is at approximating $V_0$ by looking at how well a piecewise linear interpolation function $v_I$ approximates $v \in V_0$.

We will define $\Pi_K : V_0 \rightarrow V_{h,0}$ to be the local linear interpolation operator that we will use to build our global linear interpolant $v_I$. The rest of the definitions, lemma, and theorems in this section are restatements of those found in [BS07,Riv08,LB13] to better fit the model problem and notation given in this dissertation. First define the set $H^2(\Omega) := \{v \in L^2(\Omega) : \partial_x, \partial_z, v \in L^2(\Omega), i, j = 1, 2\}$. 

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**Definition I.7.** Let \( v \in H^2(\Omega) \), and let \( \mathcal{T}_h \) be a shape-regular mesh of \( \Omega \). On a triangle \( K \in \mathcal{T}_h \) with nodes \( \{N_i\}_{i=1}^3 \) and the nodal basis functions \( \{\varphi_i\}_{i=1}^3 \), the local interpolant \( \Pi_Kv \) is defined by

\[
\Pi_Kv := \sum_{i=1}^3 v(N_i)\varphi_i|_K
\]

The global linear interpolant follows from combining the local linear interpolants together.

**Definition I.8.** Let \( v \in H^2(\Omega) \) and let \( \mathcal{T}_h \) be a shape-regular mesh of \( \Omega \). The global linear interpolant \( v_I \) of \( v \) is defined by

\[
v_I|_K := \Pi_Kv \quad \forall K \in \mathcal{T}_h
\]

Since \( v_I \in V_h \), we can use it to understand how well the space \( V_h \) approximates \( V \).

**Lemma I.9.** Let \( v \in H^2(\Omega) \), and let \( \mathcal{T}_h \) be a shape-regular mesh of \( \Omega \). Then

\[
||v - v_I|| \leq Ch|v|_{H^2(\Omega)}.
\]

Consequently, \( v_I \to v \) as \( h \to 0 \). Therefore, the space \( V_{h,0} \) is a good approximation to \( V_0 \) in the sense that we can find an approximation to any function \( v \in V_0 \) that will converge to \( v \) as the mesh is refined. We now state the Galerkin orthogonality condition for our simple FEM and use Lemma I.9 to bound the error.

**Theorem I.10.** Let \( u \) be the solution to (I.13). The finite element solution \( u_h \) in (I.15) satisfies the Galerkin orthogonality condition

\[
\int_{\mathcal{T}_h} \nabla(u - u_h)\nabla v \, dx = 0 \quad \forall v \in V_{h,0}.
\]
With the Galerkin orthogonality condition, we can show that the finite element approximation is the best approximation to $u$ in the space $V_{h,0}$ with respect to $\|\cdot\|$. 

**Theorem I.11.** Let $u \in H^2(\Omega)$ be the solution to (I.13). The finite element solution $u_h$ for (I.15) satisfies the best approximation result

$$\|u - u_h\| \leq \|u - v\| \quad \forall v \in V_{h,0}.$$ 

From Lemma I.9 and Theorem I.11, we can use the two inequalities to conclude that

$$\|u - u_h\| \leq \|u - u_I\| \leq Ch|u|_{H^2(\Omega)}.$$ 

Therefore, as $h \to 0$, the finite element approximation from (I.15) will converge to the solution from (I.13). This also implies that as we refine our mesh for (I.15), we will be getting a better approximation to $u$ from (I.13). This observation inherently makes sense because, as we gather more information about the true solution, we should be able to form a better approximation.

Standard finite element methods for the obstacle problem (I.9)/(I.10) were investigated in [Fal74, BHR77, Wan02], where the optimal *a priori* error estimates have been derived with respect to the regularity of the exact solution. Similar results have been extended to other methods including nonconforming finite element methods in [Wan03, CK17a], virtual element methods in [WW20], and various discontinuous Galerkin methods in [WHC10b]. More numerical methods for the obstacle problem (I.9)/(I.10) and for other types of variational inequalities were also investigated in [BHR78, GLT81, Glo84, HHN96, BSZ12, BSZZ12, ABdVV13, GP15, GSV17, CZ19] and the references therein.
I.5. Brief History of Discontinuous Galerkin Methods

The first discontinuous Galerkin (DG) method was introduced by Reed and Hill [RH73] for hyperbolic partial differential equations (PDEs). What made this method different from other finite element approximations is that the spaces used in the numerical method consist of totally discontinuous piecewise polynomials with respect to the underlying triangulation. Consequently, $V_h \not\subset V$.

Convection problems started the endeavor into DG methods for two main reasons. First, the exact solution to a purely (nonlinear) convective problem will develop discontinuities in finite time. Secondly, the exact solution can display complex behavior near these discontinuities. Methods such as the continuous Galerkin finite element method cannot capture discontinuities in the approximation, and they may smooth out the complex behavior around these discontinuities. Therefore, an approximation method to this type of problem must be able to guarantee the discontinuities of the approximation are physically relevant, and do not create oscillations that will reduce the accuracy of the approximation around these discontinuities. Since DG methods assume a discontinuous solution to the problem in which they are applied, DG methods can naturally capture the physically relevant discontinuities without producing spurious oscillations [CKS12].

Other methods, such as high-resolution finite difference methods and finite volume methods, are also able to handle the discontinuities from convection problems. However, DG methods have still have some advantages. One advantage is that the actual order of accuracy of a DG method is solely dependent upon the exact solution to the problem. That is, DG methods of high order accuracy can be obtained by simply increasing the degree of the approximating polynomials. Another advantage is that the mass matrix is block diagonal and each block’s size is dependent only on the
corresponding element since each element of the solution is discontinuous. Further, each block can be inverted individually and only needs to be inverted once.

A third advantage for DG methods is that they can easily handle complicated geometries as well as maintain a high order of accuracy with a simple treatment of the boundary conditions of the problem. Lastly, since DG methods do not have continuity restrictions for the approximate solution, they can easily handle adaptive strategies for meshes. This is a consequence from the block nature of the mass matrix. This is especially important for hyperbolic problems which can have complexities around discontinuities, where a refinement of a mesh may be necessary to fully capture the behavior of the solution. The DG approximation to the solution can also adapt by changing degrees from element to element, which provides another way to resolve complexities [CKS12].

Since Reed and Hill introduced the first DG method for a hyperbolic problem, DG methods have been applied to other types of PDEs including convection-diffusion equations [CCSS01, PS02], Navier-Stokes equations [BR97, CKS04], Hamilton-Jacobi equations [HS99, KLHS00], the radiative transfer equation [HHE10], and many others. A historical record of the methods listed above, and others, can be found in [CKS12]. DG methods for elliptic equations were independently proposed in the 1970s, where many variants were introduced and studied [WHC10a]. These original methods are usually referred to as interior penalty methods.

For elliptic problems, there are historically two classes of DG methods. The first class is the class of interior penalty (IP) discontinuous Galerkin methods which are created by adding penalty terms in the primal formulations. There are multiple interior penalty methods, but the first one introduced is called the symmetric interior penalty method. This method is stable if the penalization of jumps across elements
is large enough. Other interior penalty methods include the non-symmetric interior penalty Galerkin method and the incomplete interior penalty Galerkin method [Riv08].

The second class of DG methods uses the mixed formulation of a PDE and involves picking appropriate numerical fluxes to ensure well-posedness of the numerical method. Some examples of these types of methods are the local discontinuous Galerkin method (LDG) [CCPS00,CS98], the method of Bassi et al. [BRM+97], the method of Bassi and Rebay [BR97], and the minimal dissipation local discontinuous Galerkin method (MD-LDG) [CCSS01,CD07]. The LDG method is stable for any positive penalty parameter, and the MD-LDG method is stable provided the boundary penalty parameter is positive and all interior penalty parameters are non-negative. Interior penalty and LDG methods for elliptic problems were compared in [ABCM00,ABCM02], specifically showing that penalizing across element boundaries can be obtained as special cases of choosing appropriate numerical fluxes.

For a comprehensive survey of DG methods, we refer the readers to [ABCM02] and the references therein. We note that most DG methods for elliptic problems require positive penalty parameters to guarantee the stability of the schemes with the exception of the non-symmetric interior penalty methods proposed in [OBB98,LN04a,LN04b] and the minimal-dissipation DG method for advection-diffusion problems [CD07]. This is important to this thesis since we wish to prove convergence of a symmetric DG method without penalty. Many of the standard DG methods were studied for the obstacle problem in [WHC10a].
CHAPTER II
DUAL-WIND DISCONTINUOUS GALERKIN METHODS

II.1. Introduction and Definitions

Recently, an interior DG differential calculus was established in [FLN16], where certain one-sided discrete partial derivatives for piecewise weakly differentiable functions are defined and various calculus identities are shown. Using these one-sided derivatives as building blocks, known DG methods are reformulated, and entirely new DG methods for a variety of linear and nonlinear PDEs are developed. In particular, using both the up-wind gradient operator and the down-wind gradient operator as independent auxiliary variables in the mixed formulation, the symmetric dual-wind DG (DWDG) methods were developed for linear elliptic problems in [LN14,FLW15].

DWDG methods are natural extensions of finite difference methods when written in primal form. They naturally enforce both Dirichlet and/or Neumann boundary conditions, and DWDG methods are stable without the introduction of interior or boundary penalization. A complete description of the discrete derivatives and additional analytic properties of these operators as well as implementation aspects can be found in [FLN16].

The DWDG method belongs to the second family of DG methods, but is typically written in primal form like IPDG methods. The new methods also have a strong stability property in that they do not require the introduction of a penalization term. The DWDG method is even stable for some negative penalty parameters when formulated for elliptic second order and fourth order PDEs. The DWDG method is the first symmetric DG method that does not require either boundary or interior
penalization on a general class of quasi-uniform triangulations. To start, we first provide the notation and the definitions of the one-sided discrete derivatives that will be used in our formulation of DWDG methods for (I.10).

We choose to introduce the discrete one-sided derivatives for a general dimension \( d \geq 1 \), but note that the analysis for the obstacle problem will be performed in 2D. Throughout the paper we use \( W^{m,p}(\Omega) \) to denote set of all \( L^p(\Omega) \) functions whose distributional derivatives up to order \( m \) are in \( L^p(\Omega) \), and \( W^{m,p}_0(\Omega) \) to denote the set of \( W^{m,p}(\Omega) \) functions whose traces vanish up to order \( m-1 \) on \( \partial\Omega \). The special case \( p = 2 \) is denoted by \( H^m := W^{m,2} \) and \( H^m_0 := W^{m,2}_0 \). The corresponding vector-valued Sobolev spaces are indicated in bold-face format, e.g., \( W^{m,p}(\Omega) := [W^{m,p}(\Omega)]^d \), \( H^m(\Omega) := [H^m(\Omega)]^d \), etc.

We define the piecewise vector spaces with respect to the triangulation

\[
W^{m,p}(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} W^{m,p}(K), \quad W^{m,p}(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} W^{m,p}(K), \quad \text{etc},
\]

and the special cases \( \mathcal{V}_h := W^{1,1}(\mathcal{T}_h) \cap C^0(\mathcal{T}_h) \) and \( \mathcal{V}_h := [\mathcal{V}_h]^d \). The piecewise \( L^2 \) inner product with respect to the triangulation is given by

\[
(v, w)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \int_K vw \, dx,
\]

and the piecewise \( L^2 \) inner product over a subset \( \mathcal{S}_h \subset \mathcal{E}_h \) is given by

\[
\langle v, w \rangle_{\mathcal{S}_h} := \sum_{e \in \mathcal{S}_h} \int_e vw \, ds.
\]

For an integer \( r \geq 0 \), we let \( V^r_h \) denote the space of piecewise polynomials of degree less than or equal to \( r \) with respect to the triangulation \( \mathcal{T}_h \); that is, \( V^r_h = \prod_{K \in \mathcal{T}_h} \mathbb{P}_r(K) \), where \( \mathbb{P}_r(K) \) denotes the space of polynomials with domain \( K \) and
degree not exceeding \( r \). The vector-valued DG space is defined as \( V^r_h := [V^r_h]^d \). We note the inclusions \( V^r_h \subset V_h \) and \( V^r_h \subset \mathcal{V}_h \).

Let \( K^+, K^- \in \mathcal{T}_h \) and \( e = \partial K^- \cap \partial K^+ \). Without loss of generality we assume that the global labeling number of \( K^+ \) is lower than that of \( K^- \). We then define the jumps and averages across the \((d - 1)\)-dimensional simplex \( e \) as

\[
[v]_e := v_+ - v_- \quad \{v\}_e := \frac{1}{2}(v_+ + v_-) \quad \forall v \in \mathcal{V}_h,
\]

where \( v_\pm := v|_{K^\pm} \). For a boundary \((d - 1)\)-dimensional simplex \( e = \partial K^+ \cap \partial \Omega \) we set \([v]_e := v_+\) and \( \{v\}_e := v_+ \). Set \((n^{(1)}_e, n^{(2)}_e, \ldots, n^{(d)}_e)^t = n_e := n_{K^+}|e = -n_{K^-}|e\) to be the unit normal on \( e \). For \( i \in \{1, 2, \ldots, d\} \) and \( v \in \mathcal{V}_h \), we define the following three trace operators on \( e \in \mathcal{E}^f_h \) in the direction \( x_i \):

\[
Q^\pm_i(v) := \{v\} \pm \frac{1}{2} \text{sgn}(n_e^{(i)}) [v], \quad \text{where} \quad \text{sgn}(n_e^{(i)}) = \begin{cases} 
1 & \text{if } n_e^{(i)} > 0, \\
-1 & \text{if } n_e^{(i)} < 0, \\
0 & \text{if } n_e^{(i)} = 0,
\end{cases}
\]

\[
\overline{Q}_i(v) := \frac{1}{2} \left( Q^-_i(v) + Q^+_i(v) \right) = \{v\}.
\]

The operators \( Q^-_i(v) \) and \( Q^+_i(v) \) can be regarded respectively as the “backward” and “forward” limit of \( v \) in the \( x_i \) direction on \( e \in \mathcal{E}^f_h \). On a boundary simplex \( e \in \mathcal{E}^b_h \), we simply take \( Q^\pm_i(v) = \overline{Q}_i(v) = v \).

With the trace operators \( Q^-_i, Q^+_i \) and \( \overline{Q}_i \) defined, we are ready to introduce our discrete partial derivative operators \( \partial^-_{h,x_i}, \partial^+_{h,x_i}, \overline{\partial}_{h,x_i} : \mathcal{V}_h \to V^r_h \).
Definition II.1. Let $v \in \mathcal{V}_h$, $g \in L^1(\partial \Omega)$, and $i \in \{1, 2, \ldots, d\}$.

(i) The discrete partial derivatives $\partial_{h,x_i}^{\pm}, \overline{\partial}_{h,x_i} : \mathcal{V}_h \to \mathcal{V}_h^{r}$ are defined by:

$$
(\partial_{h,x_i}^{\pm} v, \varphi)_h := \langle Q_i^{\pm}(v)n^{(i)}_i, \varphi \rangle_{E_h} - (v, \partial_{x_i} \varphi)_h \quad \forall \varphi \in \mathcal{V}_h^{r},
$$

$$
\overline{\partial}_{h,x_i} v := \frac{1}{2} \left( \partial_{h,x_i}^- v + \partial_{h,x_i}^+ v \right).
$$

Here, $\partial_{x_i}$ denotes the usual (weak) partial derivative operator in the direction $x_i$, and $n^{(i)}_i$ is the piecewise constant vector-valued function satisfying $n^{(i)}_i|_e = n^{(i)}_e$.

(ii) The discrete partial derivatives with given boundary data $\partial_{h,x_i}^{\pm,g}, \overline{\partial}_{h,x_i}^{g} : \mathcal{V}_h \to \mathcal{V}_h^{r}$ are defined by

$$
(\partial_{h,x_i}^{\pm,g} v, \varphi)_h := (\partial_{h,x_i}^{\pm} v, \varphi)_h + \langle (g - v)n^{(i)}_i, \varphi \rangle_{E_h} \quad \forall \varphi \in \mathcal{V}_h^{r},
$$

$$
\overline{\partial}_{h,x_i}^{g} v := \frac{1}{2} \left( \partial_{h,x_i}^{-,g} v + \partial_{h,x_i}^{+,g} v \right).
$$

(iii) The discrete gradient operators $\nabla_{h}^{\pm}, \nabla_{h}, \nabla_{h,g}^{\pm}, \nabla_{h,g} : \mathcal{V}_h \to \mathcal{V}_h^{r}$ are defined as

$$
(\nabla_{h}^{\pm} v)_i := \partial_{h,x_i}^{\pm} v, \quad (\nabla_{h} v)_i := \overline{\partial}_{h,x_i} v, \quad (\nabla_{h,g}^{\pm} v)_i := \partial_{h,x_i}^{\pm,g} v, \quad (\nabla_{h,g} v)_i := \overline{\partial}_{h,x_i}^{g} v.
$$

(iv) The discrete divergence operators $\text{div}_{h}^{\pm}, \overline{\text{div}}_{h} : \mathcal{V}_h \to \mathcal{V}_h^{r}$ are defined as

$$
\text{div}_{h}^{\pm} v := \sum_{i=1}^{d} \partial_{h,x_i}^{\pm} v^{(i)}, \quad \overline{\text{div}}_{h} v = \frac{1}{2} \left( \text{div}_{h}^{+} v + \text{div}_{h}^{-} v \right).
$$

Remark. Equivalently, the discrete gradients are defined as the unique functions in $\mathcal{V}_h^{r}$ satisfying the following conditions:
\[
(\nabla_h^\pm v, \varphi_h)_{\mathcal{T}_h} = (\nabla v, \varphi_h)_{\mathcal{T}_h} - \left\langle [v], \{\varphi_h \cdot n\}\right\rangle_{\mathcal{E}_h^l}
\]
\[
\pm 1/2 \sum_{i=1}^{d} \left\langle [v]|n^{(i)}|, [\varphi_h^{(i)}]\right\rangle_{\mathcal{E}_h^l} \quad \forall \varphi_h \in \mathbf{V}_h^r,
\]
\[
(\nabla_{h,g}^\pm v, \varphi_h)_{\mathcal{T}_h} = (\nabla v, \varphi_h)_{\mathcal{T}_h} - \left\langle [v], \{\varphi_h \cdot n\}\right\rangle_{\mathcal{E}_h}
\]
\[
+ \left\langle g, \varphi_h \cdot n\right\rangle_{\mathcal{E}_h^b} \pm 1/2 \sum_{i=1}^{d} \left\langle [v]|n^{(i)}|, [\varphi_h^{(i)}]\right\rangle_{\mathcal{E}_h^l} \quad \forall \varphi_h \in \mathbf{V}_h^r.
\]

These two identities will be used frequently in the subsequent analysis.

**Remark.** From Definition II.1 (ii), we have the following relationship between the operators \(\nabla_{h,g}^\pm\) and \(\nabla_{h,0}^\pm\):

\[
(\nabla_{h,g}^\pm v, \varphi_h)_{\mathcal{T}_h} = (\nabla_{h,0}^\pm v, \varphi_h)_{\mathcal{T}_h} + \left\langle g, \varphi_h \cdot n\right\rangle_{\mathcal{E}_h} \quad \forall \varphi_h \in \mathbf{V}_h^r.
\]

A direct result of Definition II.1 is that the operators \(\nabla_{h,0}^\pm\) and \(\nabla_{h,0}^\pm\) are linear on \(\mathcal{V}_h\).

We also have for any \(v, w \in \mathcal{V}_h\)

\[
\left\langle \nabla_{h,0}^\pm (v - w), \varphi_h \right\rangle_{\mathcal{T}_h} = (\nabla_{h,0}^\pm v - \nabla_{h,0}^\pm w, \varphi_h)_{\mathcal{T}_h} \quad \forall \varphi_h \in \mathbf{V}_h^r.
\]

Furthermore, the following discrete analog of integration by parts holds \((i = 1, 2, ..., d)\) (see [FLN16])

\[
(\partial_{h,x}^\pm v_h, w_h)_{\mathcal{T}_h} = -(v_h, \partial_{h,x}^\mp w_h)_{\mathcal{T}_h} + \left\langle v_h, w_h n^{(i)}\right\rangle_{\mathcal{E}_h^B} \quad \forall v_h, w_h \in \mathbf{V}_h^r.
\]

yielding

\[
(\nabla_h^\pm v_h, \varphi_h)_{\mathcal{T}_h} = -(v_h, \text{div}^\pm_h \varphi_h)_{\mathcal{T}_h} + \left\langle v_h, \varphi_h \cdot n\right\rangle_{\mathcal{E}_h^B} \quad \forall v_h \in \mathbf{V}_h^r, \varphi_h \in \mathbf{V}_h^r,
\]
\[
(\nabla_{h,0}^\pm v_h, \varphi_h)_{\mathcal{T}_h} = -(v_h, \text{div}_{h,0}^\pm \varphi_h)_{\mathcal{T}_h} \quad \forall v_h \in \mathbf{V}_h^r, \varphi_h \in \mathbf{V}_h^r.
\]
II.2. The Dual-Wind Discontinuous Galerkin Methods for the Obstacle Problem

The numerical method that we consider for approximating the solution to the obstacle problem (I.11) replaces the Laplacian operator by a discrete version that was first proposed using the dual-wind discontinuous Galerkin method (DWDG) method for Poisson’s equation in [LN14]. First, we consider the special case $K_g = H^1_g(\Omega)$, where

$$H^1_g(\Omega) := \{ v \in H^1(\Omega) : v = g \text{ on } \partial\Omega \}.$$ (II.8)

Thus, the primal form of (I.10) reduces to a prototypical second order elliptic PDE

$$-\Delta u = f \quad \text{in } \Omega,$$ (II.9a)
$$u = g \quad \text{on } \partial\Omega.$$ (II.9b)

Now we define the discrete Laplace operator $\Delta_{h,g} : V^r_h \to V^r_h$ by

$$\Delta_{h,g} v_h := \frac{\text{div}_h^+ \nabla_{h,g}^- v_h + \text{div}_h^- \nabla_{h,g}^+ v_h}{2} \quad \forall v_h \in V^r_h,$$ (II.10)

which involves an up-wind gradient operator and a down-wind gradient operator utilized in a symmetric way. Consider the approximation of (II.9): Find $u_h \in V^r_h$ such that

$$-\Delta_{h,g} u_h + j_{h,g}(u_h) = P_h f,$$ (II.11)

where $P_h f \in V^r_h$ is the $L^2$ projection of $f$ onto $V^r_h$ defined by $(P_h f, v_h)_{\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}$ for all $v_h \in V^r_h$ and $j_{h,g} : V_h \to V^r_h$ satisfies

$$(j_{h,g}(v), w_h)_{\mathcal{T}_h} = \left( \frac{\gamma_e}{h_e} [v], [w_h] \right)_{\mathcal{E}_h} - \left( \frac{\gamma_e}{h_e} g, w_h \right)_{\mathcal{E}_h} \quad \forall w_h \in V^r_h.$$ (II.12)
The piecewise constant $\gamma_e$ is a "penalty" parameter on $e \in \mathcal{E}_h$. We say "penalty" because we will not assume $\gamma_e > 0$. Using (II.7) and (II.3), a direct calculation shows that problem (II.11) is equivalent to finding $u_h \in V_r^r$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_r^r,$$

where

$$B_h(v_h, w_h) := \frac{1}{2} \left( (\nabla^+_{h,0} v_h, \nabla^+_{h,0} w_h)_{T_h} + (\nabla^-_{h,0} v_h, \nabla^-_{h,0} w_h)_{T_h} \right) + \left\langle \frac{\gamma_e}{h_e} \lVert [v_h], [w_h] \rVert_{\mathcal{E}_h}^2 \right\rangle,$$

$$F_h(v) := (f, v)_{T_h} + \left\langle \frac{\gamma_e}{h_e} g, v \right\rangle_{\mathcal{E}_h^B} - \left\langle g, \nabla_{h,0} v \cdot \mathbf{n} \right\rangle_{\mathcal{E}_h^B} \quad \forall v \in \mathcal{V}_h.$$  

Next, we define the approximations of the constraint set $K_g$ defined in I.8 to be

$$K^1_h := \left\{ v_h \in V^1_h : v_h(p) \geq \psi(p) \quad \forall p \in V_K, \quad \forall K \in \mathcal{T}_h \right\},$$

$$K^2_h := \left\{ v_h \in V^2_h : v_h(m) \geq \psi(m) \quad \forall m \in M_K, \quad \forall K \in \mathcal{T}_h \right\},$$

where $\mathcal{N}_K$ is the set of nodes/vertices of $K \in \mathcal{T}_h$ and $M_K$ is the set of the midpoints of the edges of $K \in \mathcal{T}_h$. Motivated by the weak form of the VI (I.10), our dual-wind DG methods for the approximation of (I.9)/(I.10) are to seek $u_h \in K^r_h$ ($r = 1, 2$) such that

$$B_h(u_h, v_h - u_h) \geq F_h(v_h - u_h) \quad \forall v_h \in K^r_h.$$  

Along with the discrete formulation of the obstacle problem, we introduce the notation

$$\lVert v_h \rVert^2_{1,h} := \frac{1}{2} \left( \lVert \nabla^+_{h,0} v_h \rVert_{L^2(T_h)}^2 + \lVert \nabla^-_{h,0} v_h \rVert_{L^2(T_h)}^2 \right) + \sum_{e \in \mathcal{E}_h} \frac{\gamma_e}{h_e} \lVert [v_h] \rVert_{L^2(e)}^2,$$

$$\lVert v_h \rVert^2_{1,h} := \frac{1}{2} \left( \lVert \nabla^+_{h,0} v_h \rVert_{L^2(T_h)}^2 + \lVert \nabla^-_{h,0} v_h \rVert_{L^2(T_h)}^2 \right).$$  

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Notationally, we define $\gamma_{\text{min}} = \min_{e \in \mathcal{E}_h} \gamma_e$. In Section III.2, we will introduce properties that guarantee both $\|\cdot\|_{1,h}$ and $\|\cdot\|_{1,h}$ are norms over the space $V_h^r$. We show the well-posedness of the DWDG discrete problem (II.18) in Section IV.
CHAPTER III

ANALYTIC PROPERTIES FOR DISCONTINUOUS GALERKIN METHODS

In this chapter, we will record several inequalities that will be useful for the convergence analysis. In the first section we will list some standard analytic properties, and in the second section we will state proven properties of the DWDG method.

III.1. Standard Analytic Properties

The following are classical analytic properties that are used in the convergence analysis. The results are listed in no particular order.

Theorem III.1. (Cauchy-Schwarz inequality) Let $V$ be a vector space with the inner product $(\cdot, \cdot): V \times V \to V$ and let $\| \cdot \|$ be the corresponding induced norm. Then, for any $u, v \in V$,

$$(u, v) \leq \|u\|\|v\|.$$ 

Theorem III.2. (Young’s inequality) Let $a$ and $b$ be non-negative real numbers, let $p$ and $q$ be real numbers greater than 1 such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $\epsilon > 0$. Then,

$$ab \leq \frac{a^p \epsilon^p}{\epsilon p} + \frac{\epsilon b^q}{q}.$$ 

Theorem III.3. (Trace inequality) [Riv08] Let $K \subset \mathbb{R}^2$ be a shape-regular triangle, and let $|K|$ represent the area of $K$. For any $v \in H^1(K)$ there is a constant $C$ that is independent of $h_K$ and $v$ such that,

$$\forall e \in \partial K, \quad \|v\|_{L^2(e)} \leq Ch_e^{1/2}|K|^{-1/2} \left( \|v\|_{L^2(K)} + h_K\|\nabla v\|_{L^2(K)} \right). \quad (III.1)$$
Note that the shape regularity of $T_h$ is important here since the ratio $\frac{h_K}{|K|}$ appears on the right hand side of the trace inequality. The shape regularity ensures that this ratio on right-hand side of (III.1) is bounded above by a constant times some form of $h_K$, making the trace inequality useful for error analysis.

Furthermore, if $v$ is a polynomial and $T_h$ is a quasi-uniform mesh, we can take advantage of norm equivalencies in a finite-dimensional space as well as the area of each triangle being bounded above by the same constant to simplify the result of the trace inequality. That is, let $K \in T_h$, let $v \in \mathbb{P}_k(K)$, and let $e \in \partial K$. Then,

\[
\|v\|_{L^2(e)} \leq C h_K^{-1/2} \|v\|_{L^2(K)},
\]

(III.2)

\[
\|\nabla v \cdot n\|_{L^2(e)} \leq C h_K^{-1/2} \|\nabla v\|_{L^2(K)},
\]

(III.3)

where $C$ is a constant that is independent of $h_K$, but dependent upon the degree of the polynomial, and $n$ is the outward normal vector.

The following inverse inequality will be used in the convergence analysis.

**Theorem III.4.** (Inverse Inequalities) [Riv08] Let $K$ be a bounded domain in $\mathbb{R}^d$ with diameter $h_K$. Then, there is a constant $C$ independent of $h_K$ such that for any polynomial function $v$ of degree $k$ defined on $K$ we have

\[
\|\nabla^j v\|_{L^2(K)} \leq C h_K^{-j} \|v\|_{L^2(K)} \quad \forall \quad 0 \leq j \leq k.
\]

(III.4)

Another result that will be necessary is the error estimate for the standard polynomial interpolates $v_I$ of a function $v \in H^s(\Omega)$, with $s \geq 2$ in $\|\cdot\|_{1,h}$, where $v_I(x_i) = v(x_i)$ for all nodes $x_i \in K$ for all $K \in T_h$. 


Theorem III.5. ([Riv08]) Let $\mathcal{T}_h$ be a shape-regular triangulation of a polygonal domain $\Omega$. Let $v \in H^s(\Omega)$ for $s \geq 2$. Let $k \geq 0$ be an integer. There exist a constant independent of $v$ and $h$ and a function $v_I \in \mathbb{P}_k(K)$ for $K \in \mathcal{T}_h$, such that

$$\forall 0 \leq q \leq s, \quad \|v - v_I\|_{H^q(K)} \leq Ch_k^{\min(k+1,s)-q}|v|_{H^s(K)}.$$ 

Though the result of Theorem III.5 is local, it easily extends to a global error bound over a shape-regular mesh $\mathcal{T}_h$. Thus, if $v \in H^2(\Omega)$ and $v_I$ represents the standard linear interpolant of $v$, then

$$\|v - v_I\|_{L^2(\Omega)} \leq Ch^2|v|_{H^2(\Omega)}.$$ (III.5)


The following are a few necessary results from papers in which the Dual-Wind Discontinuous Galerkin methods were applied to the Poisson problem as well as the fourth order biharmonic equation with clamped boundaries [FLN16,LN14,FLW15]. A property of the discrete derivatives from Definition II.1 of a function $v$ is that they reduce to the $L^2$ projection of the derivative of $v$ provided $v \in H^1(\Omega)$.

Lemma III.6 ([FLN16]). For any $v \in H^1(\Omega)$, both $\partial_{h,x_i}^\pm v$ and $\overline{\partial}_{h,x_i} v$ are the $L^2$ projections of $\partial_{x_i} v$ onto $V^r_h$; that is, $(\partial_{h,x_i}^\pm v, w_h)_{\mathcal{T}_h} = (\overline{\partial}_{h,x_i} v, w_h)_{\mathcal{T}_h} = (\partial_{x_i} v, w_h)_{\mathcal{T}_h} \forall w_h \in V^r_h$. Moreover, if $v \in H^1(\Omega)$ satisfies $v|_{\partial \Omega} = g$ for some $g \in L^2(\Omega)$, then both $\partial_{h,x_i}^\pm v$ and $\overline{\partial}_{h,x_i} v$ are the $L^2$ projection of $\partial_{x_i} v$ onto $V^r_h$.

From Lemma III.6 it is possible to derive a few more useful identities and properties of the discrete differential operators. In particular, one result determines a relationship between $\Delta$ and the discrete operators $\nabla_{h,g}$ and $\nabla_{h,0}$ over the triangulation $\mathcal{T}_h$. 
Lemma III.7. ([LN14]) Let $\varphi \in H^2(\Omega)$ with $\varphi = g$ on the boundary $\partial\Omega$. Then there holds

$$-(\Delta \varphi; v_h)_{\mathcal{T}_h} = (\nabla_h g \varphi, \nabla_h g v_h)_{\mathcal{T}_h} + \langle \{ \nabla_h g \varphi - \nabla \varphi \} \cdot n \| v_h \| \rangle_{\mathcal{E}_h} \quad \forall v_h \in V_h^r. \quad (III.6)$$

A key component of the analysis of the DWDG method (II.18) and a key to removing the penalization term is the following result.

Theorem III.8. ([LN14]) If $\gamma_{\text{min}} > 0$, then

$$\gamma_{\text{min}} \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|_{L^2(e)}^2 \leq \| v_h \|_{1,h}^2 \quad \forall v_h \in V_h^r. \quad (III.7)$$

Moreover, if the triangulation is quasi-uniform, and if each simplex in the triangulation has at most one boundary face/edge, then there exists a constant $C_* > 0$ independent of $h$ and $\gamma_e \forall e \in \mathcal{E}_h$ such that

$$(C_* + \gamma_{\text{min}}) \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|_{L^2(e)}^2 \leq \| v_h \|_{1,h}^2 \quad \forall v_h \in V_h^r. \quad (III.8)$$

The derivation of (III.8) is quite technical and can be found in [LN14]. This result in very important because it shows that the penalized jumps that are usually necessary for stability of DG methods can be bounded above by the upwind/downwind DG derivatives. This is what allows us to zero out the penalty parameter for the DWDG method and still obtain optimal convergence. For the rest of this paper, $C_*$ will always denote the constant from Theorem III.8. As a corollary to (III.8), we also have

$$C_* \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|_{L^2(e)}^2 \leq \| v_h \|_{1,h}^2 \quad \forall v_h \in V_h^r. \quad (III.9)$$
CHAPTER IV
A PRIORI ANALYSIS

In this chapter we will establish the \textit{a priori} error analysis of DWDG methods for the obstacle problem (I.9)/(I.10), extending the analysis in [LN14] for boundary value problems to variational inequalities. The main results of Chapter IV are published in [LRZ20]. To the best of our knowledge, this is the first analysis of symmetric DG methods without penalty for the obstacle problem. A condition that we will assume for the remainder of this paper is that \( \Omega \subset \mathbb{R}^2 \) is a convex polyhedral.

IV.1. Well Posedness

We first show existence and uniqueness of the DWDG approximation for the obstacle problem. We consider two cases based on whether \( \gamma_{\min} > 0 \) or not. Suppose \( \gamma_{\min} > 0 \). In this case, \( \|v_h\|_{1,h}^2 = 0 \) implies that \( v_h \in V_h^r \cap H^1_0(\Omega) \) and \( \nabla v_h = P_h(\nabla v_h) = \nabla_{h,0}^\pm v_h = 0 \), which then implies \( v_h = 0 \). Therefore, \( \|\cdot\|_{1,h} \) is a norm on \( V_h^r \), and the stability of the bilinear form \( B_h(\cdot, \cdot) \) with respect to \( \|\cdot\|_{1,h} \) holds. We also have the boundedness of \( B_h(\cdot, \cdot) \) in the sense that

\[
B_h(v, w) \leq \|v\|_{1,h} \|w\|_{1,h} \quad \forall v, w \in V_h^r + H^2(\Omega). \tag{IV.1}
\]

Note that (IV.1) also holds for \( \gamma_{\min} = 0 \).

Whenever \( \gamma_{\min} < 0 \), the second term on the right-hand side of (II.19) may be negative. Therefore \( \|\cdot\|_{1,h} \) may not necessarily be a norm on \( V_h^r \). To address this, we assume the triangulation \( \mathcal{T}_h \) is quasi-uniform and each simplex in the triangulation has at most one boundary edge.
Thus, (III.8) guarantees that $\|\cdot\|_{1,h}$ is a norm on $V_h^r$ and $B_h(\cdot, \cdot)$ is stable. However, the boundedness of $B_h(\cdot, \cdot)$ only holds on $V_h^r \times V_h^r$:

$$B_h(v_h, w_h) \leq \|v_h\|_{1,h} \|w_h\|_{1,h} \quad \forall v_h, w_h \in V_h^r$$

(IV.2)

when $-C_* < \gamma_{\text{min}} < 0$. Specifically, Theorem III.8 shows us that $\|v\|_{1,h} = 0$ only if $\nabla_h^+ v = 0$ and $[v] = 0$ across all edges. In the discontinuous space $V_h^r$, the only function that satisfies these conditions is the zero function. However, if we extend our function space to include $H^1(\Omega)$ functions, the null space for $\|\cdot\|_{1,h}$ is no longer trivial.

In Theorem IV.7 and Theorem IV.8 we are considering the energy norm of a function in a space extended from $V_h^r$. For example, in Theorem IV.7, we will be considering the energy norm of a function in the extended space $V_h^r + H^2(\Omega)$. Since $H^2(\Omega) \subset C^0(\Omega)$ in two dimensions, there are no jumps in the function across all interior edges. This allows for functions in the extended space that are non-zero to exist in the null space of $\|\cdot\|_{1,h}$. Thus we are only guaranteed that $\|\cdot\|_{1,h}$ is a semi-norm on the extended space $V_h^r + H^2(\Omega)$ when $\gamma_{\text{min}} \geq 0$. To verify this, we will present two examples of non-zero functions that are in the null space of $\|\cdot\|_{1,h}$.

Let $\Omega = [-1, 1]$ and consider the two functions

$$v(x) = \begin{cases} \frac{3}{2}x^2 + 2x + \frac{1}{2} & \text{if } x \leq 0, \\ \frac{3}{2}x^2 - 2x + \frac{1}{2} & \text{if } x > 0 \end{cases}$$

and

$$w(x) = \begin{cases} x^2 + x & \text{if } x \leq 0, \\ -x^2 + x & \text{if } x > 0. \end{cases}$$

Since each function is continuous at $x = 0$, we can find the weak derivatives. The weak derivatives are
\[ v'(x) = \begin{cases} 
3x + 2 & \text{if } x \leq 0, \\
3x - 2 & \text{if } x > 0
\end{cases} \]

and

\[ w'(x) = \begin{cases} 
2x + 1 & \text{if } x \leq 0, \\
-2x + 1 & \text{if } x > 0.
\end{cases} \]

This implies that \( v \in H^1_0(\Omega) \) and \( w \in H^2(\Omega) \cap H^1_0(\Omega) \). We will construct the DG derivatives \( \nabla^\pm_h \) in the discrete space \( V^1_h = \text{span}\{1, x\} \). Thus, \( \mathcal{T}_h \) is comprised of one interval \([-1, 1]\).

Since \( v, w \in H^1(\Omega) \), then, by Lemma III.6,

\[ (\nabla^\pm_h v, \phi_h)_{\mathcal{T}_h} = (\nabla_h v, \phi_h)_{\mathcal{T}_h} \quad \forall v \in V^1_h, \]

which implies \( \nabla^\pm_h v, \nabla^\pm_h w \in V^1_h \) and are in the span of \( \{1, x\} \). We only need to calculate the following integrations:

\[
\int_{-1}^{1} v' \cdot 1 \, dx = \int_{-1}^{0} 3x + 2 \, dx + \int_{0}^{1} 3x - 2 \, dx \\
= \left[ \frac{3}{2} x^2 + 2x \right]_0^0 + \left[ \frac{3}{2} x^2 - 2x \right]_0^1 \\
= 0,
\]

\[
\int_{-1}^{1} v' \cdot x \, dx = \int_{-1}^{0} 3x^2 + 2x \, dx + \int_{0}^{1} 3x^2 - 2x \, dx \\
= \left[ x^3 + x^2 \right]_{-1}^0 + \left[ x^3 - x^2 \right]_0^1 \\
= 0,
\]
\[
\int_{-1}^{1} w' \cdot 1 \, dx = \int_{-1}^{0} 2x + 1 \, dx + \int_{0}^{1} -2x + 1 \, dx \\
= [x^2 + x]_{-1}^{0} + [-x^2 + x]_{0}^{1} \\
= 0,
\]

\[
\int_{-1}^{1} w' \cdot x \, dx = \int_{-1}^{0} 2x^2 + x \, dx + \int_{0}^{1} -2x^2 + x \, dx \\
= \left[ \frac{2}{3}x^3 + \frac{1}{2}x^2 \right]_{-1}^{0} + \left[ -\frac{2}{3}x^3 + \frac{1}{2}x^2 \right]_{0}^{1} \\
= 0.
\]

Therefore, we can conclude that \( \| \nabla_{\pm} v \|_{L^2(\Omega)} = \| \nabla_{\pm} w \|_{L^2(\Omega)} = 0 \). Furthermore, since \( v \) and \( w \) are continuous on \([-1, 1]\) and zero on the boundary, then

\[
\sum_{e \in E_h} \gamma_e \| [v]_{e} \|_{L^2(e)}^2 = \sum_{e \in E_h} \gamma_e \| [w]_{e} \|_{L^2(e)}^2 = 0.
\]

Thus \( \| v \|_{1,h} = \| w \|_{1,h} = 0 \). We address this issue in Section IV.2 with the introduction of Corollary IV.9.

We now show that the DWDG method has a unique solution to II.18. Let \( \psi_I \) be the standard nodal interpolation of the obstacle function \( \psi \) on \( V^r_h \). Since \( \psi_I \in K^r_h \) \((r = 1, 2)\), \( K^r_h \) is a nonempty closed convex subset of \( H^1(\Omega) \). This, together with \( F_h \) being a continuous linear operator and the stability and boundedness of \( B_h(\cdot, \cdot) \) over \( V^r_h \times V^r_h \), imply the existence and uniqueness of the solution to (II.18) (see [KS80, Glo84, Rod87]). We summarize the result in the following theorem.
Theorem IV.1. There exists a unique solution to (II.18) provided \( \gamma_{\text{min}} > 0 \). Suppose the triangulation is quasi-uniform and each simplex in the triangulation has at most one boundary edge. Then there is a unique solution to (II.18) provided \( \gamma_{\text{min}} > -C_* \), where \( C_* \) is the constant in (III.9).

For the rest of this paper, unless stated otherwise, we will always assume the necessary conditions on \( T_h \) and \( \gamma_{\text{min}} \) that guarantee the well-posedness of (II.18).

IV.2. Analytic Lemmas

For the error analysis of this method for the obstacle problem, we will need a few more inequalities which we will state as Lemmas, as well as an error bound for a polynomial interpolant in the \( \|\cdot\|_{1,h} \) norm. The first result is a relationship between the \( \|\cdot\|_{1,h} \) and \( \|\cdot\|_{1,h} \) norms.

Lemma IV.2. For \( \gamma_{\text{min}} \geq 0 \), we have

\[
\| v_h \|_{1,h} \leq \| v_h \|_{1,h} \quad \forall \ v_h \in V_h^r. \tag{IV.3}
\]

If the mesh \( T_h \) is quasi-uniform and \( \gamma_{\text{min}} > -C_* \) for \( C_* > 0 \) then

\[
\| v_h \|_{1,h} \leq \left( 1 + \frac{|\gamma_{\text{min}}|}{C_* + \gamma_{\text{min}}} \right)^{\frac{1}{2}} \| v_h \|_{1,h} \quad \forall \ v_h \in V_h^r. \tag{IV.4}
\]

Proof. If \( \gamma_{\text{min}} \geq 0 \), then the proof follows from the definitions of the \( \|\cdot\|_{1,h} \) and \( \|\cdot\|_{1,h} \) norms. Assume that the mesh \( T_h \) is quasi-uniform and that \( \gamma_{\text{min}} > -C_* \). Let \( v_h \in V_h \), and assume that \( \gamma_{\text{min}} < 0 \). Thus \( \gamma_{\text{min}} = -|\gamma_{\text{min}}| \), and

\[
\| v_h \|_{1,h}^2 - \| v_h \|_{1,h}^2 = \sum_{e \in \mathcal{E}_h} \frac{\gamma_e}{h_e} \| [v_h] \|_{L^2(e)}^2 \geq \gamma_{\text{min}} \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|_{L^2(e)}^2 = -|\gamma_{\text{min}}| \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|_{L^2(e)}^2.
\]
Multiply and divide the last line above by the constant \( C_\ast + \gamma_{\min} \) from (III.8). Then by Theorem III.8, we have

\[
-|\gamma_{\min}| \sum_{e \in E_h} h_e^{-1} \|v_h\|_{L^2(e)}^2 = \frac{-|\gamma_{\min}|}{C_\ast + \gamma_{\min}} (C_\ast + \gamma_{\min}) \sum_{e \in E_h} h_e^{-1} \|v_h\|_{L^2(e)}^2
\]

\[
\geq \frac{-|\gamma_{\min}|}{C_\ast + \gamma_{\min}} \|v_h\|_{1,h}^2.
\]

Thus,

\[
\|v_h\|_{1,h}^2 - \|v_h\|_{1,h}^2 \geq \frac{-|\gamma_{\min}|}{C_\ast + \gamma_{\min}} \|v_h\|_{1,h}^2,
\]

and the result follows. \( \square \)

The following is another relationship between the dual-wind derivative operators in the \( L^2 \) norm and the \( \| \cdot \|_{1,h} \) norm, which also gives us a relationship to the \( \| \cdot \|_{1,h} \) norm because of Lemma IV.2.

**Lemma IV.3.** Let \( v_h \in V_h^r \). Then

\[
\| \nabla_{h,0} v_h \|_{L^2(\Omega_h)}^2 \leq \| v_h \|_{1,h}^2.
\]

**Proof.** Let \( v_h \in V_h^r \). Then, by the Triangle Inequality and the algebraic identity \( 2ab \leq a^2 + b^2 \),

\[
\| \nabla_{h,0} v_h \|_{L^2(\Omega_h)}^2 = \frac{1}{4} \| \nabla_{h,0}^+ v_h + \nabla_{h,0}^- v_h \|_{L^2(\Omega_h)}^2
\]

\[
\leq \frac{1}{4} \left( \| \nabla_{h,0}^+ v_h \|_{L^2(\Omega_h)} + \| \nabla_{h,0}^- v_h \|_{L^2(\Omega_h)} \right)^2
\]

\[
\leq \frac{1}{4} \left( 2\| \nabla_{h,0}^+ v_h \|_{L^2(\Omega_h)}^2 + 2\| \nabla_{h,0}^- v_h \|_{L^2(\Omega_h)}^2 \right)
\]

\[
= \frac{1}{2} \| \nabla_{h,0}^+ v_h \|_{L^2(\Omega_h)}^2 + \frac{1}{2} \| \nabla_{h,0}^- v_h \|_{L^2(\Omega_h)}^2
\]

\[
= \| v_h \|_{1,h}^2.
\]

\( \square \)
By the definition of \( \| \cdot \|_{1,h} \), we also have the following relationship for all functions in \( V_h^r \)

\[
\| \nabla_{h,0}^\pm v_h \|_{L^2(T_h)}^2 \leq 2 \| v_h \|_{1,h}^2 \quad \forall v_h \in V_h^r.
\] (IV.5)

**Remark.** Here and throughout the paper, the letter \( C \) with or without subscripts will denote a generic positive constant, independent of \( h \), that may take different values at different occurrences.

Another necessary relationship to address is between \( | \cdot |_{H^1(T_h)} \) and \( ||| \cdot |||_{1,h} \). This lemma will be necessary in the convergence analysis.

**Lemma IV.4.** Let \( \gamma_{\text{min}} > 0 \). Then, for all \( v_h \in V_h^r \), there holds

\[
\| \nabla v_h \|_{L^2(T_h)}^2 \leq C \left( 1 + \frac{1}{\gamma_{\text{min}}} \right) \| v_h \|_{1,h}^2.
\] (IV.6)

Let \( \gamma_{\text{min}} > -C_* \). Then, if \( T_h \) is quasi-uniform and each triangle has at most one boundary edge, there holds

\[
\| \nabla v_h \|_{L^2(T_h)}^2 \leq C \left( 1 + \frac{1 + |\gamma_{\text{min}}|}{C_* + \gamma_{\text{min}}} \right) \| v_h \|_{1,h}^2 \quad \forall v_h \in V_h^r
\] (IV.7)

**Proof.** Let \( v_h \in V_h^r \). By (II.2b) we have that

\[
\| \nabla_{h,0}^\pm v_h \|_{L^2(T_h)}^2 = (\nabla_{h,0}^\pm v_h, \nabla_{h,0}^\pm v_h)_{T_h}
\]

\[
= (\nabla v_h, \nabla_{h,0}^\pm v_h)_{T_h} - \langle [v_h]_{\partial T}, \{ \nabla_{h,0}^\pm v_h \} \cdot n \rangle_{E_h} - \frac{1}{2} \sum_{i=1}^d \langle [v_h]_{\partial T}, \langle [\nabla_{h,0}^\pm v_h]^{(i)} \rangle_{E_h} \rangle_{E_h} \pm \frac{1}{2} \sum_{i=1}^d \langle [v_h]_{\partial T}, \langle \nabla v_h \rangle^{(i)}_{E_h} \rangle_{E_h} \pm \frac{1}{2} \sum_{i=1}^d \langle [v_h]_{\partial T}, \langle \nabla v_h \rangle^{(i)}_{E_h} \rangle_{E_h}.
\]
Using the representation of $\|\nabla_h^\pm v_h\|_{L^2(\Omega_h)}^2$ found above in the definition of $\| \cdot \|_{1,h}^2$, we have

$$
\|v_h\|_{1,h}^2 = \frac{1}{2} \|\nabla_h^+ v_h\|_{L^2(\Omega_h)}^2 + \frac{1}{2} \|\nabla_h^- v_h\|_{L^2(\Omega_h)}^2
- \frac{1}{2} \langle [v_h], \{\nabla_h^+ v_h\} \cdot \mathbf{n} \rangle_{E_h} - \frac{1}{2} \langle [v_h], \{\nabla_h^- v_h\} \cdot \mathbf{n} \rangle_{E_h}
- \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^+ v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l
- \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^- v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l
= \langle \nabla v_h, \nabla v_h \rangle_{\mathcal{H}} - \langle [v_h], \{\nabla_h^+ v_h\} \cdot \mathbf{n} \rangle_{E_h} - \langle [v_h], \{\nabla_h^- v_h\} \cdot \mathbf{n} \rangle_{E_h}
+ \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^+ v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l
+ \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^- v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l,
$$

or, after rearranging the terms,

$$
\|\nabla v_h\|_{L^2(\Omega_h)}^2 = \|v_h\|_{1,h}^2 + \langle [v_h], \{\nabla_h^+ v_h\} \cdot \mathbf{n} \rangle_{E_h} + \langle [v_h], \{\nabla_h^- v_h\} \cdot \mathbf{n} \rangle_{E_h}
- \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^+ v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l
+ \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^- v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l.
$$

(IV.8)

Let $\gamma_{\min} > 0$. By Lemma IV.2, we have

$$
\|\nabla v_h\|_{L^2(\Omega_h)}^2 \leq \|v_h\|_{1,h}^2 + \langle [v_h], \{\nabla_h^+ v_h\} \cdot \mathbf{n} \rangle_{E_h} + \langle [v_h], \{\nabla_h^- v_h\} \cdot \mathbf{n} \rangle_{E_h}
- \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^+ v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l
+ \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^- v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l
\leq \|v_h\|_{1,h}^2 + \left| \langle [v_h], \{\nabla_h^+ v_h\} \cdot \mathbf{n} \rangle_{E_h} \right| + \left| \langle [v_h], \{\nabla_h^- v_h\} \cdot \mathbf{n} \rangle_{E_h} \right|
+ \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^+ v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l
+ \frac{1}{4} \sum_{i=1}^d \langle [v_h]|n^{(i)}|, \langle \nabla_h^- v_h \rangle_{E_h}^{(i)} \rangle_{E_h}^l.
$$

(IV.9)
Let $\beta > 0$. By the Cauchy-Schwartz Inequality and Young’s Inequality, we obtain the following

\[
\left| \langle \|v_h\|, \{\nabla v_h\} \cdot \mathbf{n} \rangle \right|_{E_h} \leq \sum_{e \in E_h} \frac{1}{\sqrt{h_e}} \|\|v_h\||_{L^2(e)} \sqrt{h_e} \|\{\nabla v_h\}\|_{L^2(e)}
\]

\[
\leq \sum_{e \in E_h} \frac{1}{2\beta h_e} \|\|v_h\||^2_{L^2(e)} + \sum_{e \in E_h} \frac{\beta h_e}{2} \|\{\nabla v_h\}\|_{L^2(e)}
\]

\[
\leq \sum_{e \in E_h} \frac{1}{2\beta h_e} \|\|v_h\||^2_{L^2(e)} + \sum_{e \in E_h} \frac{\beta h_e}{8} \|\nabla v_h\|_{K_e^-} + \|\nabla v_h\|_{K_e^+}^2_{L^2(e)}
\]

\[
\leq \sum_{e \in E_h} \frac{1}{2\beta h_e} \|\|v_h\||^2_{L^2(e)} + \sum_{e \in E_h} \frac{\beta h_e}{4} \|\nabla v_h\|_{K_e^-}^2_{L^2(e)} + \sum_{e \in E_h} \frac{\beta h_e}{4} \|\nabla v_h\|_{K_e^+}^2_{L^2(e)}
\]

where $K_e^- \cap K_e^+ = e$. By the trace theorem with scaling, this can be further estimated by

\[
\left| \langle \|v_h\|, \{\nabla v_h\} \cdot \mathbf{n} \rangle \right|_{E_h} \leq \sum_{e \in E_h} \frac{1}{\sqrt{h_e}} \|\|v_h\||_{L^2(e)} \sqrt{h_e} \|\{\nabla v_h\}\|_{L^2(e)} + \frac{C}{4} \sum_{K \in T_h} \|\nabla v_h\|_{L^2(K)}^2.
\] (IV.10)

Letting $\beta > 0$ be sufficiently small, we have

\[
\frac{C}{4} \sum_{e \in E_h} \sum_{K \in T_{h,e}} \|\nabla v_h\|_{L^2(K)}^2 \leq \frac{1}{2} \|\nabla v_h\|_{L^2(T_h)}^2.
\] (IV.11)

Theorem 2 along with (IV.11) gives us

\[
\left| \langle \|v_h\|, \{\nabla v_h\} \cdot \mathbf{n} \rangle \right|_{E_h} \leq \sum_{e \in E_h} \frac{1}{2\beta h_e} \|\|v_h\||^2_{L^2(e)} + \frac{1}{2} \|\nabla v_h\|_{L^2(T_h)}^2
\]

\[
\leq \frac{1}{2\beta \gamma_{\min}} \|\|v_h\||^2_{L^2(e)} + \frac{1}{2} \|\nabla v_h\|_{L^2(T_h)}^2.
\] (IV.12)

Analogously, we obtain the same bound in (IV.12) replacing $\nabla_{h,0} v_h$ with $\nabla v_h$. Furthermore, by Lemma IV.3, and Lemma IV.2 we obtain
\[
\left\langle \llbracket v_h \rrbracket, \{ \nabla_{h, 0} v_h \} \cdot n \right\rangle_{\mathcal{E}_h} \leq \frac{1}{2 \beta \gamma_{\text{min}}} \| v_h \|_{1,h}^2 + \frac{1}{2} \| \nabla_{h, 0} v_h \|_{L^2(\mathcal{T}_h)}^2 \\
\leq \frac{1}{2 \beta \gamma_{\text{min}}} \| v_h \|_{1,h}^2 + \frac{1}{2} \| v_h \|_{1,h}^2 \\
\leq \frac{1}{2 \beta \gamma_{\text{min}}} \| v_h \|_{1,h}^2 + \frac{1}{2} \| v_h \|_{1,h}^2 \\
= \left( \frac{1}{2} + \frac{1}{2 \beta \gamma_{\text{min}}} \right) \| v_h \|_{1,h}^2.
\]

Next, let \((|n|)_{(i)} = |n_{(i)}|\). Then,
\[
\left| \sum_{i=1}^{d} \left\langle \llbracket v_h \rrbracket | n_{(i)} \|, \llbracket (\nabla_{h, 0} v_h)_{(i)} \rrbracket \right\rangle_{\mathcal{E}_h} \right| = \left| \left\langle \llbracket v_h \rrbracket, \llbracket \nabla_{h, 0} v_h \rrbracket \cdot n \right\rangle_{\mathcal{E}_h} \right|,
\]
which allows us to obtain a result similar to (IV.12). Specifically,
\[
\left| \left\langle \llbracket v_h \rrbracket, \llbracket \nabla_{h, 0} v_h \rrbracket \cdot n \right\rangle_{\mathcal{E}_h} \right| \leq \frac{1}{2 \beta \gamma_{\text{min}}} \| v_h \|_{1,h}^2 + \frac{1}{2} \| \nabla_{h, 0} v_h \|_{L^2(\mathcal{T}_h)}^2.
\]

Then, by (IV.5) as well as Lemma IV.2 we have,
\[
\left| \sum_{i=1}^{d} \left\langle \llbracket v_h \rrbracket | n_{(i)} \|, \llbracket (\nabla_{h, 0} v_h)_{(i)} \rrbracket \right\rangle_{\mathcal{E}_h} \right| \leq \frac{1}{2 \beta \gamma_{\text{min}}} \| v_h \|_{1,h}^2 + \frac{1}{2} \| v_h \|_{1,h}^2 \\
\leq \frac{1}{2 \beta \gamma_{\text{min}}} \| v_h \|_{1,h}^2 + \frac{1}{2} \| v_h \|_{1,h}^2 \\
= \left( 1 + \frac{1}{2 \beta \gamma_{\text{min}}} \right) \| v_h \|_{1,h}^2.
\]

Using the bounds found in (IV.12), (IV.13), and (IV.14) in (IV.9), we obtain the following
\[ \| \nabla v_h \|_{L^2(\mathcal{T}_h)}^2 \leq \| v_h \|_{1,h}^2 + \left| \left[ \left[ v_h \right], \{ \nabla_h, 0 v_h \} \cdot n \right] \right| \varepsilon_h + \left| \left[ \left[ v_h \right], \{ \nabla v_h \} \cdot n \right] \right| \varepsilon_h \]

\[ + \frac{1}{4} \sum_{i=1}^{d} \left\langle \| v_h \|_{\mathcal{E}_h} \| n^{(i)} \|, \left\| \left( \nabla_h + v_h \right)^{(i)} \right\| \varepsilon_h \right\rangle \]

\[ \leq \left( 2 + \frac{3}{4 \beta \gamma_{\min}} \right) \| v_h \|_{1,h}^2 + \left| \left[ \left[ v_h \right], \{ \nabla v_h \} \cdot n \right] \right| \varepsilon_h \]

\[ \leq \left( 2 + \frac{3}{4 \beta \gamma_{\min}} \right) \| v_h \|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{2 \beta h_e} \| \left[ \left[ v_h \right] \right] \|_{L^2(e)}^2 + \frac{C \beta}{4} \sum_{K \in \mathcal{T}_h} \| \nabla v_h \|_{L^2(K)}^2 \]

\[ \leq \left( 2 + \frac{3}{4 \beta \gamma_{\min}} \right) \| v_h \|_{1,h}^2 + \frac{1}{4 \beta \gamma_{\min}} \| v_h \|_{1,h}^2 + \frac{C \beta}{2} \sum_{K \in \mathcal{T}_h} \| \nabla v_h \|_{L^2(K)}^2 \]

\[ \leq \left( 2 + \frac{5}{4 \beta \gamma_{\min}} \right) \| v_h \|_{1,h}^2 + \frac{C \beta}{4} \sum_{K \in \mathcal{T}_h} \| \nabla v_h \|_{L^2(K)}^2. \]

By (IV.11), with \( \beta > 0 \) sufficiently small, we obtain

\[ \| \nabla v_h \|_{L^2(\mathcal{T}_h)}^2 \leq \left( 2 + \frac{5}{4 \beta \gamma_{\min}} \right) \| v_h \|_{1,h}^2 + \frac{1}{2} \| \nabla v_h \|_{L^2(\mathcal{T}_h)}^2. \]

Thus,

\[ \| \nabla v_h \|_{L^2(\mathcal{T}_h)}^2 \leq C \left( 1 + \frac{1}{\gamma_{\min}} \right) \| v_h \|_{1,h}^2. \]

Next, let \( \mathcal{T}_h \) be quasi-uniform and \( C_* > 0 \) such that \( \gamma_{\min} > -C_* \). Continuing from (IV.8), by Lemma IV.2, we now have

\[ \| \nabla v_h \|_{L^2(\mathcal{T}_h)}^2 \leq \left( 1 + \frac{|\gamma_{\min}|}{C_* + \gamma_{\min}} \right) \| v_h \|_{1,h}^2 + \left| \left[ \left[ v_h \right], \{ \nabla_h, 0 v_h \} \cdot n \right] \right| \varepsilon_h \]

\[ + \left| \left[ \left[ v_h \right], \{ \nabla v_h \} \cdot n \right] \right| \varepsilon_h \]

\[ + \frac{1}{4} \sum_{i=1}^{d} \left| \left[ \left[ v_h \right], \{ \nabla v_h \} \cdot n^{(i)} \right] \right| \varepsilon_h \]

\[ + \frac{1}{4} \sum_{i=1}^{d} \left| \left[ \left[ v_h \right], \{ \nabla v_h \} \cdot n^{(i)} \right] \right| \varepsilon_h \]

\[ + \frac{1}{4} \sum_{i=1}^{d} \left| \left[ \left[ v_h \right], \left[ \left[ \nabla_h, 0 v_h \right] \cdot n^{(i)} \right] \right] \right| \varepsilon_h \]
Let $\beta > 0$. From Theorem III.8 and (IV.11), with $\beta$ sufficiently small, we obtain the following

$$
\left| \left\langle [v_h], \{\nabla v_h\} \cdot \mathbf{n} \right\rangle_{E_h} \right| \leq \sum_{e \in E_h} \frac{1}{2\beta h_e} \|v_h\|_{L^2(e)}^2 + \frac{1}{2} \|\nabla v_h\|_{L^2(T_h)}^2
$$

$$
\leq \frac{1}{2\beta (C_* + \gamma_{\min})} \|v_h\|_{1,h}^2 + \frac{1}{2} \|\nabla v_h\|_{L^2(T_h)}^2. \quad \text{(IV.16)}
$$

Analogously, we obtain the same bound in (IV.16) for $\nabla_{h,0} v_h$. Furthermore, by Lemma IV.3 and Lemma IV.2,

$$
\left| \left\langle [v_h], \{\nabla_{h,0} v_h\} \cdot \mathbf{n} \right\rangle_{E_h} \right| \leq \frac{1}{2\beta (C_* + \gamma_{\min})} \|v_h\|_{1,h}^2 + \frac{1}{2} \|\nabla_{h,0} v_h\|_{L^2(T_h)}^2
$$

$$
\leq \frac{1}{2\beta (C_* + \gamma_{\min})} \|v_h\|_{1,h}^2 + \frac{1}{2} \|v_h\|_{1,h}^2
$$

$$
\leq \frac{1}{2\beta (C_* + \gamma_{\min})} \|v_h\|_{1,h}^2 + \frac{1}{2} \left(1 + \frac{|\gamma_{\min}|}{C_* + \gamma_{\min}}\right) \|v_h\|_{1,h}^2
$$

$$
\leq C \left(1 + \frac{1 + |\gamma_{\min}|}{C_* + \gamma_{\min}}\right) \|v_h\|_{1,h}^2. \quad \text{(IV.17)}
$$

Similarly to (IV.14), with Lemma IV.2 we obtain,

$$
\sum_{i=1}^{d} \left| \left\langle [v_h]|n^{(i)}|, [(\nabla_{h,0} v_h)^{(i)}] \right\rangle_{E_h^i} \right| \leq C \left(1 + \frac{1 + |\gamma_{\min}|}{C_* + \gamma_{\min}}\right) \|v_h\|_{1,h}^2. \quad \text{(IV.18)}
$$
After applying the bounds found in (IV.16), (IV.17), (IV.18) and (IV.11) in (IV.15), we have

\[
\| \nabla v_h \|_{L^2(T_h)}^2 \leq \left( 1 + \frac{|\gamma_{\text{min}}|}{C_* + \gamma_{\text{min}}} \right) (\| v_h \|_{1,h}^2 + \left| \left\langle [v_h], \{ \nabla_{h,0} v_h \} \cdot n \right\rangle_{E_h} \right| + \frac{1}{4} \left| \sum_{i=1}^d \left\langle [v_h], [\nabla^+ v_h]^{(i)} \right\rangle_{E_h} \right| \\
+ \frac{1}{4} \left| \sum_{i=1}^d \left\langle [v_h], [\nabla^- v_h]^{(i)} \right\rangle_{E_h} \right| \\
+ \frac{1}{4} \sum_{i=1}^d \left\langle [v_h], [\nabla h, 0 v_h] \cdot n \right\rangle_{E_h} \right| \\
\leq C \left( 1 + \frac{1 + |\gamma_{\text{min}}|}{C_* + \gamma_{\text{min}}} \right) (\| v_h \|_{1,h}^2 + \left| \left\langle [v_h], \{ \nabla_{h,0} v_h \} \cdot n \right\rangle_{E_h} \right| + \frac{C\beta}{2} \sum_{K \in T_h} \| \nabla v_h \|_{L^2(K)}^2 ) \\
\leq C \left( 1 + \frac{1 + |\gamma_{\text{min}}|}{C_* + \gamma_{\text{min}}} \right) (\| v_h \|_{1,h}^2 + \frac{1}{2} \| \nabla v_h \|_{L^2(T_h)}^2 ) .
\]

After subtracting \( \frac{1}{2} \| \nabla v_h \|_{L^2(T_h)}^2 \) from both sides and multiplying by 2 we have

\[
\| \nabla v_h \|_{L^2(T_h)}^2 \leq C \left( 1 + \frac{1 + |\gamma_{\text{min}}|}{C_* + \gamma_{\text{min}}} \right) \| v_h \|_{1,h}^2 .
\]

\[ \square \]

IV.3. Polynomial Interpolation Error

Let \( v \in H^{s+1}(\Omega) \) for \( s \geq 1 \) and \( v_I \) be the standard nodal interpolation of \( v \) to \( V_h^r \) for \( r \geq s \). The following standard interpolation error estimates [Cia02,BS07] will be used frequently throughout the paper:

\[
\| v - v_I \|_{L^2(K)} + h_K \| v - v_I \|_{H^1(K)} \leq C h_K^{s+1} \| v \|_{H^{s+1}(K)} . \tag{IV.19}
\]

The global version of (IV.19) is

\[
\| v - v_I \|_{L^2(\Omega)} + h \| v - v_I \|_{H^1(\Omega)} \leq C h^{s+1} \| v \|_{H^{s+1}(\Omega)} \quad \forall v \in H^{s+1}(\Omega) . \tag{IV.20}
\]
Next, we show that the interpolation error estimate holds in the $\| \cdot \|_{1,h}$ energy norm as well.

**Theorem IV.5.** Let $v \in H^{s+1}(\Omega)$ for $s \geq 1$ with $v|_{\partial \Omega} = g \in H^1(\Omega)$, and let $v_I$ be the standard polynomial interpolation of $v$ with degree $r \geq s$. Then, there exists a positive constant $C$ independent of $h$ such that

$$\|v - v_I\|_{1,h}^2 \leq C (1 + |\gamma|_{\text{max}}^s) h^{2s} |v|_{H^{s+1}(\Omega)}^2. \quad (IV.21)$$

**Proof.** From (II.4) and applying the triangle inequality, we have

$$\|\nabla_{h,0}^\pm (v - v_I)\|_{L^2(\tau_h)} = \|\nabla_{h,g}^\pm v - \nabla_{h,g}^\pm v_I\|_{L^2(\tau_h)} \leq \|\nabla_{h,g}^\pm v - \nabla v\|_{L^2(\tau_h)} + \|\nabla v - \nabla v_I\|_{L^2(\tau_h)} + \|\nabla v_I - \nabla_{h,g}^\pm v_I\|_{L^2(\tau_h)}. \quad (IV.22)$$

It follows from Lemma III.6 and (IV.20) that

$$\|\nabla_{h,g}^\pm v - \nabla v\|_{L^2(\tau_h)} + \|\nabla v - \nabla v_I\|_{L^2(\tau_h)} \leq Ch^s |v|_{H^{s+1}(\Omega)}. \quad (IV.23)$$

It then suffices to estimate the third term on the right-hand side of (IV.22).

Taking $\varphi_h = \nabla v_I - \nabla_{h,g}^\pm v_I \in V_h^r$ in (II.2b), applying the Cauchy-Schwarz inequality, the trace theorem with scaling, Lemma III.6, and (IV.19), we obtain
\[ \| \nabla v_I - \nabla_{h,g}^\pm v_I \|^2_{L^2(T_h)} = \langle g - v_I, (\nabla v_I - \nabla_{h,g}^\pm v_I) \cdot n \rangle_{E^B_h} \]  
(IV.24)

\[ \leq \left( \sum_{e \in \mathcal{E}^B_h} h_e^{-1} \| g - v_I \|^2_{L^2(e)} \right)^\frac{1}{2} \left( \sum_{e \in \mathcal{E}^B_h} h_e \| \nabla v_I - \nabla_{h,g}^\pm v_I \|^2_{L^2(e)} \right)^\frac{1}{2} \]

\[ \leq C \left( \sum_{K \in \mathcal{T}_h} h_K^{-2} \| v - v_I \|^2_{L^2(K)} + \| \nabla (v - v_I) \|^2_{L^2(K)} \right)^\frac{1}{2} \times \| \nabla v_I - \nabla_{h,g}^\pm v_I \|_{L^2(T_h)} \]

\[ \leq Ch^s |v|_{H^{s+1}(\Omega)} \| \nabla v_I - \nabla_{h,g}^\pm v_I \|_{L^2(T_h)}, \]

Combining (IV.22)-(IV.24), we have

\[ \| \nabla_{h,0}^\pm (v - v_I) \|_{L^2(T_h)} \leq Ch^s |v|_{H^{s+1}(\Omega)}. \]  
(IV.25)

Next, we estimate the jump term in the \( \| \cdot \|_{1,h} \) norm. For \( e \in \mathcal{E}^B_h \), define \( K_e \in \mathcal{T}_h \) as the unique triangle in \( \mathcal{T}_h \) such that \( K_e \cap \partial \Omega = e \). By the fact that \( \| v - v_I \|_e = 0 \) for all \( e \in \mathcal{E}^I_h \), the trace theorem with scaling, and (IV.19),

\[ \sum_{e \in \mathcal{E}_h} \frac{\gamma_e}{h_e} \| [v - v_I] \|^2_{L^2(e)} = \sum_{e \in \mathcal{E}^B_h} \frac{\gamma_e}{h_e} \| [v - v_I] \|^2_{L^2(e)} \]

\[ \leq \sum_{e \in \mathcal{E}^B_h} \frac{\gamma_e}{h_e^2} \| v - v_I \|^2_{L^2(K_e)} \]

\[ \leq C |\gamma|_{\max} h^2 |v|_{H^{s+1}(\Omega)}^2. \]  
(IV.26)

The estimate (IV.21) follows from (IV.25) and (IV.26).

**IV.4. Convergence Analysis**

In this section, we prove optimal order error estimates in the \( \| \cdot \|_{1,h} \) norm for both linear and quadratic DWDG methods when the solution is sufficiently regular. Note that the convexity of \( \Omega \) implies the full regularity \( H^2(\Omega) \) for the exact solution \( u \) to the obstacle problem (I.9)/(I.10).
First, we derive an abstract discrete energy norm error estimate of $\|u_I - u_h\|_{1,h}$ that holds for both linear and quadratic DWDG methods. We will bound the direct comparisons $\|\nabla u - \nabla u_h\|_{L^2(\mathcal{T}_h)}$ and $\|\nabla u - \nabla_{h,g}^\pm u_h\|_{L^2(\mathcal{T}_h)}$ in Corollary IV.9.

**Lemma IV.6.** Let $u \in H^2(\Omega)$ and $u_h \in K^r_h$ ($r = 1, 2$) be the solutions to (I.10) and (II.18), respectively. There holds

$$
\|u_I - u_h\|^2_{1,h} \leq B_h(u_I - u, u_I - u_h) + (-\Delta u + f, u_I - u_h)_{\mathcal{T}_h} \tag{IV.27}
- \langle \{\nabla_{h,g} u - \nabla u\} \cdot n, [u_I - u_h] \rangle_{E_h}.
$$

*Proof.* By the definition of $\|\cdot\|_{1,h}$, we have

$$
\|u_I - u_h\|^2_{1,h} = B_h(u_I - u_h, u_I - u_h) \tag{IV.28}
= B_h(u_I - u, u_I - u_h) + B_h(u, u_I - u_h) - B_h(u_h, u_I - u_h).
$$

Since $u_I \in K^r_h$ ($r = 1, 2$), we apply (II.18) to get

$$
B_h(u, u_I - u_h) \geq F_h(u_I - u_h) = (f, u_I - u_h)_{\mathcal{T}_h} + \left\langle \frac{\gamma_e}{h_e} g, u_I - u_h \right\rangle_{E_h} \tag{IV.29}
- \langle g, \nabla_{h,0} (u_I - u_h) \cdot n \rangle_{E_h}.
$$

The regularity $u \in H^2(\Omega)$ implies that $[u] = 0$ on any interior edge and thus $\langle [u], [u_I - u_h] \rangle_{E_h} = 0$. Then we have

$$
B_h(u, u_I - u_h) = \frac{1}{2} (\nabla_{h,0}^+ u, \nabla_{h,0}^+ (u_I - u_h))_{\mathcal{T}_h} + \frac{1}{2} (\nabla_{h,0}^- u, \nabla_{h,0}^- (u_I - u_h))_{\mathcal{T}_h} \tag{IV.30}
+ \left\langle \frac{\gamma_e}{h_e} g, u_I - u_h \right\rangle_{E_h}.
$$
By focusing on the first two terms on the right-hand side, and using (II.3), Lemma III.7, and Lemma III.6, there holds

\[
\frac{1}{2} \langle \nabla^+_{h,0} u, \nabla^+_{h,0} (u_I - u_h) \rangle_{T_h} + \frac{1}{2} \langle \nabla^-_{h,0} u, \nabla^-_{h,0} (u_I - u_h) \rangle_{T_h} = (\nabla^+_{h} u, \nabla^-_{h,0} (u_I - u_h))_{T_h}
\]

(IV.31)

\[
\frac{1}{2} \langle \nabla^-_{h} u, \nabla^+_{h,0} (u_I - u) \rangle_{T_h} + \frac{1}{2} \langle \nabla^-_{h,0} u, \nabla^-_{h,0} (u_I - u_h) \rangle_{T_h} - \langle g, \nabla^-_{h,0} (u_I - u_h) \cdot n \rangle_{E_h}^B
\]

\[
- \langle g, \nabla^-_{h,0} (u_I - u_h) \cdot n \rangle_{E_h}^B
\]

Therefore, from (IV.28)-(IV.31) and (II.15), we obtain

\[
\| u_I - u_h \|^2_{1,h} = B_h(u_I - u, u_I - u_h) + B_h(u, u_I - u_h) - B_h(u_h, u_I - u_h)
\]

\[
\leq B_h(u_I - u, u_I - u_h) + (-\Delta u, u_I - u_h)_{T_h}
\]

\[
- \langle \{ \nabla^-_{h} u - \nabla u \} \cdot n, [u_I - u_h] \rangle_{\mathcal{E}_h} \]

\[
+ \langle \frac{\gamma_e}{h_e} g, u_I - u_h \rangle_{E_h}^B
\]

\[
- \langle g, \nabla^-_{h,0} (u_I - u_h) \cdot n \rangle_{E_h}^B - \langle f, u_I - u_h \rangle_{T_h}
\]

\[
- \langle \frac{\gamma_e}{h_e} g, u_I - u_h \rangle_{E_h}^B + \langle g, \nabla^-_{h,0} (u_I - u_h) \cdot n \rangle_{E_h}^B
\]

\[
= B_h(u_I - u, u_I - u_h) + (\Delta u + f, u_I - u_h)_{T_h}
\]

\[
- \langle \{ \nabla^-_{h} u - \nabla u \} \cdot n, [u_I - u_h] \rangle_{\mathcal{E}_h}.
\]

\[
\square
\]

In order to obtain the concrete error estimates, we will split the main results into Theorem IV.7 for the linear DWDG method and Theorem IV.8 for the quadratic DWDG method, respectively.
Theorem IV.7. Let \( u \in K_g \) and \( u_h \in K_h^1 \) be the unique solutions to (I.10) and (II.18), respectively. Assume \( u \in H^2(\Omega) \) and \( \psi \in H^2(\Omega) \). There exists a positive constant \( C \) independent of \( h \) such that

\[
||| u - u_h |||_{1,h} \leq Ch
\]

(IV.32)

provided \( \gamma_{\text{min}} > 0 \). Furthermore, (IV.32) holds whenever \( \gamma_{\text{min}} > -C_* \) if the triangulation is quasi-uniform and each triangle has no more than one boundary edge.

Proof. By the triangle inequality and Theorem IV.5 we have

\[
||| u - u_h |||_{1,h} \leq ||| u - u_I |||_{1,h} + ||| u_I - u_h |||_{1,h}
\]

(IV.33)

\[
\leq Ch|u|_{H^2(\Omega)} + ||| u_I - u_h |||_{1,h}.
\]

From Lemma IV.6, the key to bound \( ||| u_I - u_h |||_{1,h} \) is to bound the three terms on the right-hand side of (IV.27). That is,

\[
||| u_I - u_h |||_{1,h} \leq B_h(u_I - u, u_I - u_h) + ((\Delta u + f), u_I - u_h)_{\Omega_h} \\
- \langle \{ \nabla_{h,g} u - \nabla u \} \cdot n, ||| u_I - u_h |||_{E_h} \rangle.
\]

Since the estimate of the second term is independent of the choice of \( \gamma_{\text{min}} \), we discuss this first. Note that since \( u_h \) and \( \psi_I \) are piecewise linear, \( u_h \geq \psi \) at the vertices of every triangle which implies \( u_h \geq \psi_I \) on \( \Omega \). Using this fact, the Cauchy-Schwarz inequality, (I.11a), (I.11d), and (IV.20), we have
\((-\Delta u + f), u_I - u_h\)_{\mathcal{T}_h} = (-\Delta u + f), u_I - u\)_{\mathcal{T}_h} + (-\Delta u + f), u - \psi\)_{\mathcal{T}_h} \tag{IV.34}

\leq (-\Delta u + f), u_I - u\)_{\mathcal{T}_h} + (-\Delta u + f), \psi - \psi_I\)_{\mathcal{T}_h}

\leq \|\Delta u + f\|_{L^2(\Omega)} (\|u - u_I\|_{L^2(\Omega)} + \|\psi - \psi_I\|_{L^2(\Omega)})

\leq C\|\Delta u + f\|_{L^2(\Omega)} (|u|_{H^2(\Omega)} + |\psi|_{H^2(\Omega)}) h^2.

Next, we turn to the estimate of \(B_h(u_I - u, u_I - u_h)\). In the case of \(\gamma_{\text{min}} \geq 0\), it follows from (IV.1), (IV.20), and Young's inequality that

\[B_h(u_I - u, u_I - u_h) \leq \|u_I - u\|_{1,h} \|u_I - u_h\|_{1,h}\]

\[\leq \|u_I - u\|_{1,h}^2 + \frac{1}{4} \|u_I - u_h\|_{1,h}^2\]

\[\leq C h^2 |u|_{H^2(\Omega)}^2 + \frac{1}{4} \|u_I - u_h\|_{1,h}^2.

Whenever \(\gamma_{\text{min}} > -C_*\), using Theorem IV.5, (IV.2), (IV.4), (III.8), (IV.20), the trace theorem with scaling, the Cauchy-Schwarz inequality, and Young's inequality, we obtain

\[B_h(u_I - u, u_I - u_h) \leq \|u_I - u\|_{1,h} \|u_I - u_h\|_{1,h} + \sum_{e \in \mathcal{E}_h} \frac{|\gamma_e|}{h_e} \|u_I - u\|_{L^2(e)} \|u_I - u_h\|_{L^2(e)}\]

\[\leq C \|u_I - u\|_{1,h} \|u_I - u_h\|_{1,h}\]

\[+ \left(\sum_{e \in \mathcal{E}_h} \frac{|\gamma_e|}{(\gamma_{\text{min}} + C_*) h_e} \|u_I - u\|_{L^2(e)}^2\right)^{1/2} \left(\sum_{e \in \mathcal{E}_h} \frac{\gamma_{\text{min}} + C_*}{h_e} \|u_I - u_h\|_{L^2(e)}^2\right)^{1/2}\]

\[\leq C \left[\|u_I - u\|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^{-2} \|u_I - u\|_{L^2(K)}^2 + \|\nabla (u_I - u)\|_{L^2(\Omega)}^2\right] + \frac{1}{4} \|u_I - u_h\|_{1,h}^2\]

\[\leq C h^2 |u|_{H^2(\Omega)}^2 + \frac{1}{4} \|u_I - u_h\|_{1,h}^2.

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For the estimate of the third term on the right-hand side of (IV.27), we will show that it can be bounded by

$$\left| \langle \{ \nabla h,g u - \nabla u \} \cdot n, \| u_I - u_h \| \rangle_{\mathcal{E}_h} \right| \leq C h^2 |u|_{H^2(\Omega)}^2 + \frac{1}{4} \| u_I - u_h \|_{1,h}^2. \quad \text{(IV.37)}$$

If $\gamma_{\text{min}} > 0$, from the Cauchy-Schwartz inequality, the trace theorem with scaling, and (III.7), we have

$$\left| \langle \{ \nabla h,g u - \nabla u \} \cdot n, \| u_I - u_h \| \rangle_{\mathcal{E}_h} \right| \leq \left( \sum_{e \in \mathcal{E}_h} \frac{h_e}{\gamma_e} \| \{ \nabla h,g u - \nabla u \} \|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} \frac{\gamma_e}{h_e} \| u_I - u_h \|_{L^2(e)}^2 \right)^{1/2} \quad \text{(IV.38)}$$

$$\leq C \left( \sum_{K \in \mathcal{T}_h} \| \nabla h,g u - \nabla u \|_{L^2(K)}^2 + h_K^2 \| \nabla (\nabla h,g u - \nabla u) \|_{L^2(K)}^2 \right)^{1/2} \| u_I - u_h \|. \quad \text{(IV.39)}$$

Since $\nabla h,g u$ is the $L^2$ projection of $\nabla u$ into $V_h^1$, we have that

$$\| \nabla h,g u - \nabla u \|_{L^2(K_h)}^2 \leq C h^2 |u|_{H^2(\Omega)}^2. \quad \text{(IV.39)}$$

Moreover, letting $(\nabla u)_I \in V_h^r$ be the standard linear interpolation of $\nabla u$, we have by the inverse inequality, (IV.20), and (IV.39) that

$$\sum_{K \in \mathcal{T}_h} h_K^2 \| \nabla (\nabla h,g u - \nabla u) \|_{L^2(K)}^2 \leq C \sum_{K \in \mathcal{T}_h} h_K^2 \| \nabla (\nabla h,g u - \nabla u) \|_{L^2(K)}^2 \quad \text{(IV.40)}$$

$$\leq C \sum_{K \in \mathcal{T}_h} \| \nabla h,g u - (\nabla u)_I \|_{L^2(K)}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 \| \nabla u - (\nabla u)_I \|_{L^2(K)}^2 \leq C \| \nabla h,g u - (\nabla u)_I \|_{L^2(K)}^2 + C h^2 |u|_{H^2(\Omega)}^2 \leq C h^2 |u|_{H^2(\Omega)}^2. \quad \text{(IV.40)}$$

Therefore, the estimate (IV.37) is proved by combining (IV.38)-(IV.40) and applying Young's inequality when $\gamma_{\text{min}} > 0$. A slight modification of (IV.38) by changing from $\gamma_e$ to $\gamma_{\text{min}} + C_*$ also proves (IV.37) in the case of $\gamma_{\text{min}} > -C_*$. 

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From the upper bounds (IV.34), (IV.35)/(IV.36), and (IV.37) found above, we have

\[
\|u_I - u_h\|_{1,h}^2 \leq B_h(u_I - u, u_I - u_h) + (-(\Delta u + f), u_I - u_h)_{\Gamma_h} \\
- \left\langle \left\{ \nabla_{h,g} u - \nabla u \right\} \cdot n, [u_I - u_h] \right\rangle_{E_h} \\
\leq Ch^2|u|_{H^2(\Omega)}^2 + \frac{1}{4}\|u_I - u_h\|_{1,h}^2 \\
+ C\|\Delta u + f\|_{L^2(\Omega)} \left( |u|_{H^2(\Omega)} + |\psi|_{H^2(\Omega)} \right) h^2 \\
+ Ch^2|u|_{H^2(\Omega)}^2 + \frac{1}{4}\|u_I - u_h\|_{1,h}^2 \\
\leq Ch^2 + \frac{1}{2}\|u_I - u_h\|_{1,h}^2.
\]

After subtracting \(\frac{1}{2}\|u_I - u_h\|_{1,h}^2\), multiplying by 2, and taking the square root of both sides, we obtain

\[
\|u_I - u_h\|_{1,h} \leq Ch.
\] (IV.41)

Finally, by combining (IV.33) and (IV.41), we complete the proof. \(\square\)

Now we will prove our main convergence result for the quadratic DWDG method if the exact solution \(u\) and data have more regularity. The idea of the proof follows from convergence analysis of a finite element method on the obstacle problem in [Wan02]. However, several nuances of the DWDG method must be addressed.

**Theorem IV.8.** Let \(u\) and \(u_h \in K_h^2\) be the unique solutions to (I.10) and (II.18), respectively. Assume \(u \in W^{s,p}(\Omega)\) for \(1 < p < \infty\), \(s < 2 + \frac{1}{p}\), \(\psi \in H^3(\Omega)\), and \(f \in L^\infty(\Omega) \cap H^1(\Omega)\). There exists a positive constant \(C\) independent of \(h\) such that

\[
\|u - u_h\|_{1,h} \leq Ch^{3/2-\epsilon}
\] (IV.42)
for any $\epsilon > 0$. Note that (IV.42) holds for $\gamma_{\min} > 0$ with shape-regular triangulations and for $\gamma_{\min} > -C_*$ with quasi-uniform triangulations where no boundary element has more than one edge along $\partial \Omega$.

Proof. Let $\epsilon > 0$. Similar to the proof of Theorem IV.7, the following estimates can be obtained easily by utilizing the regularity $u \in H^{2.5, -\epsilon}(\Omega)$ and the corresponding interpolation and $L^2$ projection error estimates:

\[
B_h(u_I - u, u_I - u_h) \leq C h^{3 - 2\epsilon} |u|_{H^{2.5, -\epsilon}(\Omega)}^2 + \frac{1}{4} ||u_I - u_h||_{1,h}^2, \quad \text{(IV.43)}
\]

\[
\left| \left\langle \{\nabla_h g - \nabla u\} \cdot n, \|u_I - u_h\|_{E_h} \right\rangle \right| \leq C h^{3 - 2\epsilon} |u|_{H^{2.5, -\epsilon}(\Omega)}^2 + \frac{1}{4} ||u_I - u_h||_{1,h}^2. \quad \text{(IV.44)}
\]

Now it suffices to estimate $(-\Delta u + f, u_I - u_h)_{\mathcal{T}_h}$. For simplicity, we focus on the case of $\gamma_{\min} > 0$ as the other case is similar. To this aim, we denote $w := -(-\Delta u + f)$ and define the following subsets of $\Omega$:

\[
\Omega^+ := \{x \in \Omega : u(x) > \psi(x)\},
\]

\[
\Omega^0 := \{x \in \Omega : u(x) = \psi(x)\},
\]

\[
\Omega^+_h := \{K \in \mathcal{T}_h : K \subset \Omega^+\},
\]

\[
\Omega^0_h := \{K \in \mathcal{T}_h : K \cap \Omega^0 \neq \emptyset\}.
\]

Note that we have $\Omega = \Omega^+ \cup \Omega^0 = \Omega^+_h \cup \Omega^0_h$. From (I.11d), we see $w = 0$ in $\Omega^+_h \subset \Omega$ and, therefore,

\[
(-\Delta u + f, u_I - u_h)_{\mathcal{T}_h} = (w, u_I - u_h)_{\mathcal{T}_h} = \sum_{K \in \Omega^0_h} (w, u_I - u_h)_K + \sum_{K \in \Omega^0_h} (w, \psi I - u_h)_K, \quad \text{(IV.45)}
\]
where we used the $(\cdot,\cdot)_K$ notation to represent the inner product in $L^2(K)$. We will now bound the two terms on the right-hand side.

We will first bound \( \sum_{K \in \Omega_h^0} (w, (u - \psi)_I - (u - \psi))_K \). Let the constants \( p, q, \) and \( s \) be defined by

\[
p = \frac{1}{2\epsilon - \epsilon_1}, \\
q = \frac{p}{p-1} = 1 + \frac{2\epsilon - \epsilon_1}{1 - 2\epsilon + \epsilon_1}, \\
s = 2 + \frac{1}{q} - \epsilon_1 = 3 - 2\epsilon,
\]

for \( 0 < \epsilon_1 < 2\epsilon \). Then, by the Cauchy-Schwartz inequality, Hölder's inequality, and the interpolation estimates \([Cia02,BS07]\),

\[
\sum_{K \in \Omega_h^0} (w, (u - \psi)_I - (u - \psi))_K \leq \sum_{K \in \Omega_h^0} \|w\|_{L^p(K)} \|(u - \psi)_I - (u - \psi)\|_{L^q(K)} \quad (IV.46)
\]

\[
\leq \left( \sum_{K \in \Omega_h^0} \|w\|_{L^p(K)}^p \right)^{\frac{1}{p}} \times \left( \sum_{K \in \Omega_h^0} \|(u - \psi)_I - (u - \psi)\|_{L^q(K)}^q \right)^{\frac{1}{q}}
\]

\[
\leq \left( \sum_{K \in \Omega_h^0} \|w\|_{L^p(K)}^p \right)^{\frac{1}{p}} \left( C \sum_{K \in \Omega_h^0} h_K^{s}\|u - \psi\|_{W^{s,q}(K)}^q \right)^{\frac{1}{q}}
\]

\[
\leq C \|w\|_{L^p(\Omega)} \|u - \psi\|_{W^{s,q}(h)} h^{3-2\epsilon}.
\]

Next, we will bound \( \sum_{K \in \Omega_h^0} (w, \psi_I - u_h)_K \). Let \( P : L^2(\Omega) \to P_0(T_h) := \oplus_{K \in T_h} P_0(K) \) be defined by \( P|_K := P_K \) for all \( K \in T_h \), where

\[
P_K(v) = \frac{1}{|K|} \int_K v \, dx \quad \forall \, v \in L^2(K).
\]
Recall that $w \geq 0$ a.e. in $\Omega$ from (I.11a), and $u_h(m_i) \geq \psi(m_i)$ for the three midpoints $m_i \in M_K$ ($i = 1, 2, 3$). Thus,

$$
\int_K (\psi_I - u_h) \, dx = |K| \sum_{i=1}^{3} (\psi_I - u_h)(m_i) \leq 0,
$$

(IV.47)

and it follows that

\[
\sum_{K \in \Omega_h^o} (w, \psi_I - u_h)_K \leq \sum_{K \in \Omega_h^o} (w - P_K(w), \psi_I - u_h)_K \tag{IV.48}
\]

\[
= \sum_{K \in \Omega_h^o} \left( w - P_K(w), (\psi_I - u_h) - P_K(\psi_I - u_h) \right)_K \\
= \sum_{K \in \Omega_h^o} \left( w - P_K(w), (\psi_I - \psi) - P_K(\psi_I - \psi) \right)_K \\
+ \sum_{K \in \Omega_h^o} \left( w - P_K(w), (\psi - u) - P_K(\psi - u) \right)_K \\
+ \sum_{K \in \Omega_h^o} \left( w - P_K(w), (u - u_h) - P_K(u - u_h) \right)_K \\
:= B_1 + B_2 + B_3.
\]

By the Cauchy-Schwartz inequality and interpolation error estimates we have

\[
B_1 \leq \sum_{K \in \Omega_h^o} \| w - P_K(w) \|_{L^2(K)} \| (\psi_I - \psi) - P_K(\psi_I - \psi) \|_{L^2(K)} \tag{IV.49}
\]

\[
\leq C \sum_{K \in \Omega_h^o} \| w \|_{L^2(K)} h_K \| \psi_I - \psi \|_{H^1(K)} \\
\leq C \sum_{K \in \Omega_h^o} \| w \|_{L^2(K)} h_K^3 \| \psi \|_{H^3(K)} \\
\leq C h^3 \| w \|_{L^2(\Omega)} \| \psi \|_{H^3(\Omega)}.
\]

We next bound $B_2$. Note that $\psi - u \in W^{2+\frac{1}{t} - \epsilon_2, t}(\Omega)$ for $1 < t < \infty$ and $\epsilon_2 > 0$ and $\nabla (\psi - u) \in W^{1+\frac{1}{t} - \epsilon_2, t}(\Omega) \hookrightarrow C^{0,1-\frac{1}{t} - \epsilon_2}(\Omega)$ by the Sobolev embedding [AF03].
Also, for any $K \in \Omega^n_h$, there exists a point $x_K \in K$ such that $\nabla (\psi - u)(x_K) = 0$. Therefore, for all $x \in K$ in $\Omega^n_h$,

$$
|\nabla (\psi - u)(x)| = |\nabla (\psi - u)(x) - \nabla (\psi - u)(x_K)| \\
\leq Ch_K^{1-\frac{\epsilon_1}{t} - \epsilon_2} \|\psi - u\|_{W^{2+1/t-\epsilon_2, t}(\Omega)},
$$

which implies

$$
|\psi - u|_{H^1(K)} = \left( \int_K |\nabla (\psi - u)|^2 \, dx \right)^{\frac{1}{2}} \leq Ch_K^{2-\frac{\epsilon_1}{t} - \epsilon_2} \|\psi - u\|_{W^{2+1/t-\epsilon_2, t}(\Omega)}. \tag{IV.50}
$$

Since $w \in W^\frac{1}{p'-\epsilon_1}(\Omega)$ for $1 < p' < \infty$ and $\epsilon_1 > 0$, applying (IV.50) and Hölder’s inequality ($\frac{1}{p'} + \frac{1}{q} = 1$), we have

$$B_2 \leq \sum_{K \in \Omega^n_h} \left\| w - P_K(w) \right\|_{L^2(K)} \| (\psi - u) - P_K(\psi - u) \|_{L^2(K)} \tag{IV.51}
\leq C \sum_{K \in \Omega^n_h} h_K^{2-\frac{1}{p'} - \epsilon_1} \| w \|_{W^{1/p'-\epsilon_1, q'}(K)} |\psi - u|_{H^1(K)} \\
\leq C \sum_{K \in \Omega^n_h} h_K^{4-\frac{1}{p'} - \frac{1}{t} - \epsilon_1 - \epsilon_2 - \frac{2}{q}} \| w \|_{W^{1/p'-\epsilon_1, q'}(K)} \|\psi - u\|_{W^{2+1/t-\epsilon_2, t}(\Omega)} \\
\leq Ch_K^{4-\frac{1}{p'} - \frac{1}{t} - \epsilon_1 - \epsilon_2 - \frac{2}{q}} \sum_{K \in \Omega^n_h} h_K^{\frac{1}{2}} \| w \|_{W^{1/p'-\epsilon_1, q'}(K)} \|\psi - u\|_{W^{2+1/t-\epsilon_2, t}(\Omega)} \\
\leq Ch_K^{4-\frac{1}{p'} - \frac{1}{t} - \epsilon_1 - \epsilon_2 - \frac{2}{q}} \left( \sum_{K \in \Omega^n_h} h_K^2 \| w \|_{W^{1/p'-\epsilon_1, q'}(\Omega)} \right)^{\frac{1}{2q}} \|\psi - u\|_{W^{2+1/t-\epsilon_2, t}(\Omega)} \\
\leq Ch_K^{4-\frac{1}{p'} - \frac{1}{t} - \epsilon_1 - \epsilon_2 - \frac{2}{q}} \| w \|_{W^{1/p'-\epsilon_1, q'}(\Omega)} \|\psi - u\|_{W^{2+1/t-\epsilon_2, t}(\Omega)}.
$$

Now we can choose $p' = 1 + \epsilon_1$, $q' = \frac{p'}{p'-1}$, and $t = (2\epsilon + \frac{1}{p'} - \epsilon_1 - \epsilon_2 - 1)^{-1}$ such that $4 - \frac{1}{p'} - \frac{1}{t} - \epsilon_1 - \epsilon_2 - \frac{2}{q'} = 3 - 2\epsilon$, which implies that

$$B_2 \leq Ch^{3-2\epsilon} \| w \|_{W^{1/p'-\epsilon_1, q'}(\Omega)} \|\psi - u\|_{W^{2+1/t-\epsilon_2, t}(\Omega)}. \tag{IV.52}
$$

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Similar to how $B_1$ was bound, we apply the Cauchy Schwartz inequality, interpolation error estimates, Lemma IV.4, and Young’s inequality to estimate $B_3$:

\[
B_3 \leq \sum_{K \in \Omega_h^o} \| w - P_K(w) \|_{L^2(K)} \| (u - u_h) - P_K(u - u_h) \|_{L^2(K)} 
\]

\[
\leq C \sum_{K \in \Omega_h^o} h^{3/2-\epsilon} \| w \|_{H^{1/2-\epsilon}(K)} \| u - u_h \|_{H^1(K)} 
\]

\[
\leq C \sum_{K \in \Omega_h^o} h^{3/2-\epsilon} \| w \|_{H^{1/2-\epsilon}(K)} \left( \| u - u_I \|_{H^1(K)} + \| u_I - u_h \|_{H^1(K)} \right) 
\]

\[
\leq Ch^{3/2-\epsilon} \| w \|_{H^{1/2-\epsilon}(\Omega)} \left( \sum_{K \in \Omega_h^o} | u - u_I |_{H^1(K)}^2 + \sum_{K \in \Omega_h^o} | u_I - u_h |_{H^1(K)}^2 \right)^{1/2} 
\]

\[
\leq Ch^{3-2\epsilon} \| w \|_{H^{1/2-\epsilon}(\Omega)} | u |_{H^{2.5-\epsilon}(\Omega)} + Ch^{3/2-\epsilon} \| u_I - u_h \|_{1,h} 
\]

\[
\leq Ch^{3-2\epsilon} \| w \|_{H^{1/2-\epsilon}(\Omega)} | u |_{H^{2.5-\epsilon}(\Omega)} + Ch^{3-2\epsilon} + \frac{1}{4} \| u_I - u_h \|_{1,h}^2. 
\]

Combining (IV.45), (IV.46), (IV.48), (IV.49), (IV.52), and (IV.53), we obtain

\[
(-\Delta u + f), u_I - u_h)_{\mathcal{T}_h} \leq Ch^{3-2\epsilon} + \frac{1}{4} \| u_I - u_h \|_{1,h}^2. \quad (IV.54)
\]

By Lemma IV.6, (IV.43), (IV.44), (IV.54), we then have obtained

\[
\| u_I - u_h \|_{1,h} \leq C h^{3/2-\epsilon}. \quad (IV.55)
\]

Finally, by Theorem IV.5, (IV.55), and the triangle inequality we obtain our result. □

As stated in Section IV.1, the energy norm $\| \cdot \|_{1,h}$ is a norm on the space $V_h^r$, but not necessarily on an extended space. Therefore, we are not directly guaranteed from Theorem IV.7 and Theorem IV.8 that our DWDG approximation converges to the true solution. However, as a consequence of Theorem IV.7 and Theorem IV.8, we can also derive similar error estimates for $\| \nabla u - \nabla u_h \|_{L^2(\mathcal{T}_h)}$ and $\| \nabla u - \nabla_{h,g} u_h \|_{L^2(\mathcal{T}_h)}$. 59
This corollary, along with the fact that our boundary information is enforced either naturally or weakly (depending on the choice of $\gamma$), will guarantee convergence of our approximation to the true solution.

**Corollary IV.9.** Let $u$ and $u_h \in K_1^h$ be the unique solutions to (I.10) and the linear DWDG method (II.18), respectively. Under the assumptions in Theorem IV.7, we have

$$\|\nabla u - \nabla u_h\|_{L^2(\mathcal{T}_h)} + \|\nabla u - \nabla_{h,g}^{\pm} u_h\|_{L^2(\mathcal{T}_h)} \leq Ch. \quad (IV.56)$$

Furthermore, if $u_h \in K_2^h$ is the unique solution to the quadratic DWDG method (II.18), under the assumptions in Theorem IV.8, we have for any $\epsilon > 0$,

$$\|\nabla u - \nabla u_h\|_{L^2(\mathcal{T}_h)} + \|\nabla u - \nabla_{h,g}^{\pm} u_h\|_{L^2(\mathcal{T}_h)} \leq Ch^{3/2-\epsilon}. \quad (IV.57)$$

**Proof.** First, we assume $u_h$ is the solution to the linear DWDG method. By the triangle inequality and Lemma IV.4, we have

$$\|\nabla u - \nabla u_h\|_{L^2(\mathcal{T}_h)} \leq \|\nabla (u - u_I)\|_{L^2(\mathcal{T}_h)} + \|\nabla (u_I - u_h)\|_{L^2(\mathcal{T}_h)}$$

$$\leq |u - u_I|_{H^1(\Omega)} + C\|u_I - u_h\|_{1,h}. \quad (IV.58)$$

From (II.4) and Lemma IV.4, we are also able to obtain a bound for the second term in (IV.56)
\[
\|\nabla u - \nabla_{h,g}^{\pm} u_h\|_{L^2(T_h)} \leq \|\nabla u - \nabla_{h,g}^{\pm} u\|_{L^2(T_h)} + \|\nabla_{h,g}^{\pm} u - \nabla_{h,g}^{\pm} u_I\|_{L^2(T_h)} \\
+ \|\nabla_{h,g}^{\pm} u_I - \nabla_{h,g}^{\pm} u_h\|_{L^2(T_h)} \\
= \|\nabla u - \nabla_{h,g}^{\pm} u\|_{L^2(T_h)} + \|\nabla_{h,0}^{\pm}(u - u_I)\|_{L^2(T_h)} \\
+ \|\nabla_{h,0}^{\pm}(u_I - u_h)\|_{L^2(T_h)} \\
\leq \|\nabla u - \nabla_{h,g}^{\pm} u\|_{L^2(T_h)} + \|\nabla_{h,0}^{\pm}(u - u_I)\|_{L^2(T_h)} \\
+ \|u_I - u_h\|_{1,h}.
\] (IV.59)

By applying (IV.20), (IV.41), Lemma III.6, and (IV.25) to (IV.58) and (IV.59), we obtain

\[
\|\nabla u - \nabla u_h\|_{L^2(T_h)} + \|\nabla u - \nabla_{h,g}^{\pm} u_h\|_{L^2(T_h)} \leq |u - u_I|_{H^1(\Omega)} + C\|u - u_h\|_{1,h} \\
+ \|\nabla u - \nabla_{h,g}^{\pm} u\|_{L^2(T_h)} \\
+ \|\nabla_{h,0}^{\pm}(u - u_I)\|_{L^2(T_h)} \\
+ \|u_I - u_h\|_{1,h} \\
\leq Ch.
\]

Similarly, we can prove (IV.57) for the quadratic DWDG method by using (IV.55). \(\square\)
In this section, we apply the DWDG methods to four examples of the obstacle problem (I.9)/(I.10). The goal of these experiments is to validate Theorem IV.7, Theorem IV.8, and Corollary IV.9. The nonlinear discrete problems are solved by using the primal-dual active set strategy that can be found in [HIK02]. We give a brief description of this solver in Section V.1. The initial meshes were refined by a uniform bisection method that can be found in the iFEM MATLAB package [Che09]. Note that for all experiments we use a constant penalty parameter across all edges, including the boundary edges. We consider the penalty parameters $\gamma \in \{-1, 0, 1, 100\}$ for each problem. We will record the absolute errors

- $\| \nabla u - \nabla u_h \|_{L^2(T_h)}$,
- $\| \nabla u - \nabla_{h,g} u_h \|_{L^2(T_h)}$,
- $\| u - u_h \|_{1,h}$,

as well as the corresponding rates for each error measurement.

In general, the lower bound $-C_*$ of $\gamma$ depends on the shape regularity of the mesh, which is difficult to determine in practice. Therefore, the zero-penalty method can be crucial for practical applications. We chose the following four experiments to confirm that our method converges for a few choices of $\gamma$, including $\gamma$ being zero across all edges. The first two experiments are examples of the obstacle problem with known solution. The solutions match the necessary regularity results that are needed to observe optimal convergence rates.
The third experiment also has a known solution and was chosen to see how the DWDG method would perform on a non-convex mesh with a corner singularity. The analysis for any problem with a corner singularity can be complicated. Thus, verification that the DWDG method achieves optimal rates is preferred before attempting the analysis. The solution for this problem is $H^{5/3-\tau}(\Omega)$ for all $\tau > 0$. Thus we would expect to see a rate of approximately $\frac{2}{3}$. Further, this test will give indication as to whether or not we can eliminate the convex domain condition for the method.

The forth experiment is an example without a known solution. Thus, we measure relative errors and the resulting rates. We still expect to achieve optimal regularity because the obstacle function is smooth enough, and we will not lose regularity because of the solution coming in contact with the obstacle. Further examination of rates and errors in each experiment are discussed in the following sections.

V.1. Primal-Dual Active Set Strategy

Identifying a solver that can handle a discrete constrained problem such as (II.18) is necessary before we can run any experiments to verify the prospective results stated in Section IV.2. The primal-dual active set (PDAS) strategy is a semismooth Newton solver that uses Lagrange multipliers to solve problems of the form

\begin{align}
Ay + \lambda &= f, \\
\lambda - \max\{0, \lambda + c(\psi - y)\} &= 0
\end{align}

for all $c > 0$, where $A$ is an $n \times n$ P-matrix and $f, \psi \in \mathbb{R}^n$. Note that the max operator is assumed to be component-wise. The assumption that $A$ is a P-matrix is necessary for the existence and uniqueness of the solution [HIK02]. Since the DWDG bilinear form defined in (II.14) is symmetric positive definite (SPD), then the resulting
matrix operator will be SPD as well. Since SPD matrices are a subset of P-matrices [HIK02], then the matrix system formed from (II.18) satisfies the conditions necessary for PDAS.

The PDAS strategy uses (V.1b) as a prediction strategy by finding the next primal-dual pair \((y, \lambda)\) from the next active and inactive sets given by

\[
I = \{ i : \lambda_i + c(\psi - y)_i \leq 0 \},
\]
\[
A = \{ i : \lambda_i + c(\psi - y)_i > 0 \}.
\]

This leads to the Primal-Dual Active Set Algorithm stated in [HIK02]:

**Algorithm V.1** (Primal-Dual Active Set Algorithm).

**Input:** Initial guess for \(y\) and \(\lambda\)

**Output:** \(y^{k+1}\)

1. Initialize \(y^0 = y\) and \(\lambda^0 = \lambda\). Set \(k = 0\).

2. Set the index sets

\[
I_k = \{ i : \lambda^k_i + c(\psi - y^k)_i \leq 0 \},
\]
\[
A_k = \{ i : \lambda^k_i + c(\psi - y^k)_i > 0 \}.
\]

3. Solve

\[
Ay^{k+1} + \lambda^{k+1} = f,
\]

with

\[
\begin{align*}
    y^{k+1}_i &= \psi_i & \text{if} & & i \in A_k \\
    \lambda^{k+1}_i &= 0 & \text{if} & & i \in I_k.
\end{align*}
\]
4. Stop, or set $k = k + 1$ and return to 2.

The stopping condition can be based on the relative error between consecutive primal-dual pairs, or if $\mathcal{I}_{k+1} = \mathcal{I}_k$ and $\mathcal{A}_{k+1} = \mathcal{A}_k$. For our PDAS system, we used the second of the stopping conditions.

In the solving step (Step 3), it is beneficial to state exactly which system we are solving and how we are solving it. Notice that for the indices in the active set $\mathcal{A}_k$, we will be setting $y^{k+1}_i = \psi_i$. Therefore, the active set can be interpreted as where the function is on top of the obstacle, which is a known value. On the other hand, for the indices in the inactive set $\mathcal{I}_k$ we are setting $\lambda^{k+1}_i = 0$. Therefore, the inactive set can be interpreted as where the function is above the obstacle. With this information, it is only necessary to solve for $y^{k+1}_i$ when $i \in \mathcal{I}_k$, and solve for $\lambda^{k+1}_i$ when $i \in \mathcal{A}_k$. To determine both unknowns, it is beneficial to rearrange the indices in such a way that the active and inactive indices occur in consecutive order.

To this end, reorganize the matrix $A$ as the block matrix,

$$A = \begin{pmatrix} A_{\mathcal{I}_k,\mathcal{I}_k} & A_{\mathcal{I}_k,\mathcal{A}_k} \\ A_{\mathcal{A}_k,\mathcal{I}_k} & A_{\mathcal{A}_k,\mathcal{A}_k} \end{pmatrix}$$

as well as $y = (y_{\mathcal{I}_k}, y_{\mathcal{A}_k})$, $f = (f_{\mathcal{I}_k}, f_{\mathcal{A}_k})$, and $\psi = (\psi_{\mathcal{I}_k}, \psi_{\mathcal{A}_k})$. Ignoring the known values $y^{k+1}_{\mathcal{A}_k}$ and $\lambda^{k+1}_{\mathcal{I}_k}$, we can simplify this system to solving for $y^{k+1}_{\mathcal{I}_k}$ in the new system,

$$A_{\mathcal{I}_k,\mathcal{I}_k} y^{k+1}_{\mathcal{I}_k} = f_{\mathcal{I}_k} - A_{\mathcal{I}_k,\mathcal{A}_k} \psi_{\mathcal{A}_k}.$$ 

To solve for $y^{k+1}_{\mathcal{I}_k}$, we used the installed Matlab solver. After solving for $y^{k+1}_{\mathcal{I}_k}$, we use it to update $\lambda^{k+1}_{\mathcal{A}_k}$ by the formula

$$\lambda^{k+1}_{\mathcal{A}_k} = A_{\mathcal{A}_k,\mathcal{I}_k} y^{k}_{\mathcal{I}_k} - A_{\mathcal{A}_k,\mathcal{A}_k} \psi_{\mathcal{A}_k} - f_{\mathcal{A}_k}.$$
Once the stopping condition is met, the ending vector $y$ is the vector of coefficients to the basis function that make up the DWDG approximation $u_h$.

V.2. Experiments 1-2: Known Solutions

Consider the obstacle problem on the domain $\Omega := (-1.5, 1.5)^2$ with a constant function $f \equiv -2$ and the obstacle function $\psi \equiv 0$. The Dirichlet boundary condition $g$ is given as the trace of the exact solution

$$u(r) = \begin{cases} \frac{r^2}{2} - \ln(r) - \frac{1}{2} & \text{if } r \geq 1, \\ 0 & \text{otherwise,} \end{cases} \quad (V.2)$$

where $r = \sqrt{x^2 + y^2}$.

For this experiment, we used a quasi-uniform mesh for both approximation spaces $K^1_h$ and $K^2_h$ and for all choices of $\gamma$. An example of a mesh for this domain is depicted in Figure V.2. A plot of the solution as well as an approximation can be found in Figure V.1. Tables V.1 and V.2 are the various errors and rates for the different values of $\gamma$ used in finding the DWDG solution in both $K^1_h$ and $K^2_h$. As we can see in the tables, when finding the DWDG solution in $K^1_h$ we get a rate of 1 appearing for all choices of $\gamma$ for all three error measurements. This matches the analytic results of Theorem IV.7. When $u_h \in K^2_h$, we see a rate of approximately 1.6 appearing across all choices of $\gamma$ and error measurements. This is slightly better than the analytic result of Theorem IV.8, but not so much as to cause concern.

Consider the $\| \nabla u - \nabla u_h \|_{L^2(\mathcal{T}_h)}$ and $\| \nabla u - \nabla_{h,g} u_h \|_{L^2(\mathcal{T}_h)}$ errors that can be found in Table V.1 when the approximation $u_h \in K^1_h$. In both of these columns on the finest mesh, the smallest errors are appearing when $\gamma \equiv -1$, but the errors when $\gamma \equiv 0$ are smaller than when $\gamma \equiv 1, 100$. However, on the four coarsest meshes, the error is smaller for the choices $\gamma \equiv 1, 100$. This indicates that as we refine our mesh, the choice $\gamma \equiv 0$ will produce smaller errors than a choice of $\gamma$ that is positive. This
is similar for $u_h \in K_h^2$ except that the smaller values of $\gamma$ result in smaller errors after the initial mesh.

Table V.1. Experiment 1, $u_h \in K_h^1$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$h$</th>
<th>$|\nabla u - \nabla u_h|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$|\nabla u - \nabla_{K,h} u_h|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$|u - u_h|_{1,h}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\equiv -1$</td>
<td>3/2</td>
<td>1.7159e+00</td>
<td>1.0016</td>
<td>1.7159e+00</td>
<td>0.9422</td>
<td>1.1364e+00</td>
<td>0.7809</td>
</tr>
<tr>
<td>$\equiv 0$</td>
<td>3/4</td>
<td>5.8702e-01</td>
<td>0.5589</td>
<td>5.5413e-01</td>
<td>0.6884</td>
<td>4.8911e-01</td>
<td>0.4605</td>
</tr>
<tr>
<td>$\equiv 1$</td>
<td>3/6</td>
<td>2.5960e-01</td>
<td>1.1642</td>
<td>2.4136e-01</td>
<td>1.1991</td>
<td>2.2856e-01</td>
<td>1.0976</td>
</tr>
<tr>
<td>$\equiv 100$</td>
<td>3/8</td>
<td>1.3191e-01</td>
<td>0.9767</td>
<td>1.1963e-01</td>
<td>1.0126</td>
<td>1.1828e-01</td>
<td>0.9503</td>
</tr>
<tr>
<td>$\equiv 1000$</td>
<td>3/2</td>
<td>6.4319e-02</td>
<td>1.0362</td>
<td>5.7616e-02</td>
<td>1.0541</td>
<td>5.8141e-02</td>
<td>1.0246</td>
</tr>
<tr>
<td>$\equiv 10000$</td>
<td>3/4</td>
<td>3.2009e-02</td>
<td>1.0968</td>
<td>2.8417e-02</td>
<td>1.0197</td>
<td>2.9035e-02</td>
<td>1.0033</td>
</tr>
</tbody>
</table>

Figure V.1. An Approximation and Solution for Experiment 1
Table V.2. Experiment 1, \( u_h \in K_h^2 \).

| \( h \) | \( \| \nabla u - \nabla u_h \|_{L^2(T_h)} \) | Rate | \( \| \nabla u - \nabla_{h,g} u_h \|_{L^2(T_h)} \) | Rate | \( ||| u - u_h |||_{1,h} \) | Rate |
|------|----------------|------|----------------|------|----------------|------|
| \( \gamma = -1 \) |   |                         |      |                         |      |                         |      |
| \( \gamma = 0 \) |   |                         |      |                         |      |                         |      |
| \( \gamma = 1 \) |   |                         |      |                         |      |                         |      |
| \( \gamma = 100 \) |   |                         |      |                         |      |                         |      |

Figure V.2. Triangularization of \( \Omega \) from Experiment 1 with \( h = 3/8 \)

Next, consider the obstacle problem with the domain \( \Omega \subset \mathbb{R}^2 \) a square with corners \( \{ (-1,0), (0,-1), (1,0), (0,1) \} \). Defining \( r = \sqrt{x^2 + y^2} \), let \( \psi(r) = 1 - 2r^2 \). With \( r_0 = \frac{\sqrt{2}-1}{\sqrt{2}} \), we will define the load function \( f \) as
\[ f(r) = \begin{cases} 
0 & \text{if } r < r_0, \\
\frac{4r_0}{r} & \text{if } r \geq r_0 
\end{cases} \]  
(V.3)

so that the exact solution to the obstacle problem is

\[ u(r) = \begin{cases} 
1 - 2r^2 & \text{if } r < r_0, \\
4r_0(1 - r) & \text{if } r \geq r_0 
\end{cases} \]  
(V.4)

with the boundary function \( g(r) \) being the trace of the solution \( u \) on \( \partial \Omega \). A plot of the solution as well as an approximation can be found in Figure V.3. Tables V.3 and V.4 are the outcomes for each choice of \( \gamma \) in both \( K^1_h \) and \( K^2_h \), respectively.

As we can see in the tables, when finding the DWDG solution in \( K^1_h \), we again get a rate of 1 appearing for all choices of \( \gamma \) for all three error measurements. This matches the analytic results of Theorem IV.7. When \( u_h \in K^2_h \), we see a rate of approximately 1.5 or 1.6 appearing across all choices of \( \gamma \) and error measurements. What the rate is converging to in our numerical results is not as clear as in Experiment 1, but we can still see that the rate is converging to something close to, if not slightly higher than 1.5. Thus, the results from this numerical experiment still correspond to the analytic results of Theorem IV.8.
Table V.3. Experiment 2, $u_h \in K^1_h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|\nabla u - \nabla u_h|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$|\nabla u - \nabla_{H^1} u_h|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$|u - u_h|_{1,h}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt[2]{2}/4$</td>
<td>4.1448e-01</td>
<td>0.8417</td>
<td>4.3967e-01</td>
<td>0.7566</td>
<td>4.0925e-01</td>
<td>0.8666</td>
</tr>
<tr>
<td>$\sqrt[2]{2}/8$</td>
<td>2.9159e-01</td>
<td>0.5074</td>
<td>2.8878e-01</td>
<td>0.6065</td>
<td>2.883e-01</td>
<td>0.5028</td>
</tr>
<tr>
<td>$\sqrt[2]{2}/16$</td>
<td>1.391e-01</td>
<td>1.0687</td>
<td>1.2982e-01</td>
<td>1.1534</td>
<td>1.3208e-01</td>
<td>1.1210</td>
</tr>
<tr>
<td>$\sqrt[2]{2}/32$</td>
<td>0.8157e-01</td>
<td>1.0924</td>
<td>0.8176e-01</td>
<td>1.0312</td>
<td>0.8213e-01</td>
<td>1.0061</td>
</tr>
<tr>
<td>$\sqrt[2]{2}/64$</td>
<td>0.473e-01</td>
<td>1.0992</td>
<td>0.5866e-01</td>
<td>1.0029</td>
<td>0.6447e-01</td>
<td>1.0008</td>
</tr>
</tbody>
</table>

Figure V.3. An Approximation and Solution for Experiment 2

$K^1_h$, Approximation with: $h = \sqrt[2]{2}/\gamma$, $\gamma = 0$

$\sqrt[2]{2}/2$ | 7.3762e-01 | 0.8618 | 4.2280e-01 | 0.8074 | 4.1147e-01 | 0.8532 |

$\sqrt[2]{2}/4$ | 4.034e-01 | 0.9740 | 2.885e-01 | 0.5496 | 2.927e-01 | 0.4911 |

$\sqrt[2]{2}/8$ | 1.391e-01 | 1.0722 | 1.301e-01 | 1.1501 | 1.3485e-01 | 1.1183 |

$\sqrt[2]{2}/16$ | 7.016e-02 | 0.9936 | 6.470e-02 | 1.0082 | 6.732e-02 | 1.0021 |

$\sqrt[2]{2}/32$ | 3.496e-02 | 0.9992 | 3.209e-02 | 1.0116 | 3.354e-02 | 1.0052 |

$\sqrt[2]{2}/64$ | 1.750e-02 | 0.9985 | 1.602e-02 | 1.0016 | 1.677e-02 | 1.0001 |

$\sqrt[2]{2}/128$ | 0.875e-03 | 1.0000 | 0.815e-03 | 1.0000 | 0.837e-03 | 1.0000 |

Figure V.3. An Approximation and Solution for Experiment 2

$\sqrt[2]{2}/2$ | 7.3762e-01 | 0.8618 | 4.2280e-01 | 0.8074 | 4.1147e-01 | 0.8532 |

$\sqrt[2]{2}/4$ | 4.034e-01 | 0.9740 | 2.885e-01 | 0.5496 | 2.927e-01 | 0.4911 |

$\sqrt[2]{2}/8$ | 1.391e-01 | 1.0722 | 1.301e-01 | 1.1501 | 1.3485e-01 | 1.1183 |

$\sqrt[2]{2}/16$ | 7.016e-02 | 0.9936 | 6.470e-02 | 1.0082 | 6.732e-02 | 1.0021 |

$\sqrt[2]{2}/32$ | 3.496e-02 | 0.9992 | 3.209e-02 | 1.0116 | 3.354e-02 | 1.0052 |

$\sqrt[2]{2}/64$ | 1.750e-02 | 0.9985 | 1.602e-02 | 1.0016 | 1.677e-02 | 1.0001 |

$\sqrt[2]{2}/128$ | 0.875e-03 | 1.0000 | 0.815e-03 | 1.0000 | 0.837e-03 | 1.0000 |

$\sqrt[2]{2}/256$ | 0.438e-03 | 1.0000 | 0.402e-03 | 1.0000 | 0.421e-03 | 1.0000 |

$\sqrt[2]{2}/512$ | 0.219e-03 | 1.0000 | 0.201e-03 | 1.0000 | 0.220e-03 | 1.0000 |

$\sqrt[2]{2}/1024$ | 0.109e-03 | 1.0000 | 0.100e-03 | 1.0000 | 0.110e-03 | 1.0000 |

$\sqrt[2]{2}/2048$ | 0.55e-04 | 1.0000 | 0.525e-04 | 1.0000 | 0.562e-04 | 1.0000 |

$\sqrt[2]{2}/4096$ | 0.28e-04 | 1.0000 | 0.262e-04 | 1.0000 | 0.289e-04 | 1.0000 |

$\sqrt[2]{2}/8192$ | 0.14e-04 | 1.0000 | 0.131e-04 | 1.0000 | 0.146e-04 | 1.0000 |

$\sqrt[2]{2}/16384$ | 0.71e-05 | 1.0000 | 0.685e-05 | 1.0000 | 0.736e-05 | 1.0000 |

$\sqrt[2]{2}/32768$ | 0.35e-05 | 1.0000 | 0.334e-05 | 1.0000 | 0.369e-05 | 1.0000 |

$\sqrt[2]{2}/65536$ | 0.18e-05 | 1.0000 | 0.173e-05 | 1.0000 | 0.190e-05 | 1.0000 |

$\sqrt[2]{2}/131072$ | 0.90e-06 | 1.0000 | 0.883e-06 | 1.0000 | 0.945e-06 | 1.0000 |

Figure V.3. An Approximation and Solution for Experiment 2
### Table V.4. Experiment 2, $u_h \in K^2_h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|\nabla u - \nabla u_h|_{L^2(\Omega_h)}$</th>
<th>Rate</th>
<th>$|\nabla u - \nabla_{h,g} u_h|_{L^2(\Omega_h)}$</th>
<th>Rate</th>
<th>$|u - u_h|_{1,h}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{2}$</td>
<td>6.7151e-01</td>
<td></td>
<td>6.7151e-01</td>
<td></td>
<td>6.8447e-01</td>
<td></td>
</tr>
<tr>
<td>$\sqrt{2}/2$</td>
<td>3.3885e-01</td>
<td>0.9868</td>
<td>3.0893e-01</td>
<td>2.3211</td>
<td>2.672e-02</td>
<td>2.3211</td>
</tr>
<tr>
<td>$\sqrt{2}/4$</td>
<td>6.2813e-02</td>
<td>2.4861</td>
<td>6.1821e-02</td>
<td>2.3211</td>
<td>2.672e-02</td>
<td>2.3211</td>
</tr>
<tr>
<td>$\sqrt{2}/8$</td>
<td>2.5889e-02</td>
<td>1.2642</td>
<td>2.4400e-02</td>
<td>1.3412</td>
<td>2.5124e-02</td>
<td>1.3119</td>
</tr>
<tr>
<td>$\sqrt{2}/16$</td>
<td>8.9453e-03</td>
<td>1.5044</td>
<td>8.9344e-03</td>
<td>1.4494</td>
<td>9.0816e-03</td>
<td>1.4680</td>
</tr>
<tr>
<td>$\sqrt{2}/32$</td>
<td>3.0874e-03</td>
<td>1.5044</td>
<td>2.9763e-03</td>
<td>1.5044</td>
<td>3.043e-03</td>
<td>1.5773</td>
</tr>
<tr>
<td>$\sqrt{2}/64$</td>
<td>9.9542e-04</td>
<td>1.6300</td>
<td>9.8514e-04</td>
<td>1.5907</td>
<td>1.0093e-03</td>
<td>1.5922</td>
</tr>
<tr>
<td>$\gamma = -1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>6.6925e-01</td>
<td></td>
<td>6.6925e-01</td>
<td></td>
<td>6.8447e-01</td>
<td></td>
</tr>
<tr>
<td>$\sqrt{2}/2$</td>
<td>3.3530e-01</td>
<td>0.9971</td>
<td>3.0906e-01</td>
<td>1.1147</td>
<td>3.1390e-01</td>
<td>1.1206</td>
</tr>
<tr>
<td>$\sqrt{2}/4$</td>
<td>6.2813e-02</td>
<td>2.4861</td>
<td>6.1821e-02</td>
<td>2.3211</td>
<td>2.672e-02</td>
<td>2.3211</td>
</tr>
<tr>
<td>$\sqrt{2}/8$</td>
<td>2.5889e-02</td>
<td>1.2642</td>
<td>2.4400e-02</td>
<td>1.3412</td>
<td>2.5124e-02</td>
<td>1.3119</td>
</tr>
<tr>
<td>$\sqrt{2}/16$</td>
<td>8.9453e-03</td>
<td>1.5044</td>
<td>8.9344e-03</td>
<td>1.4494</td>
<td>9.0816e-03</td>
<td>1.4680</td>
</tr>
<tr>
<td>$\sqrt{2}/32$</td>
<td>3.0874e-03</td>
<td>1.5044</td>
<td>2.9763e-03</td>
<td>1.5044</td>
<td>3.043e-03</td>
<td>1.5773</td>
</tr>
<tr>
<td>$\sqrt{2}/64$</td>
<td>9.9542e-04</td>
<td>1.6300</td>
<td>9.8514e-04</td>
<td>1.5907</td>
<td>1.0093e-03</td>
<td>1.5922</td>
</tr>
</tbody>
</table>

| $\gamma = 0$ | | | | | | |
| $\sqrt{2}$ | 6.6740e-01 | | 6.6740e-01 | | 6.8447e-01 | |
| $\sqrt{2}/2$ | 3.3269e-01 | 0.9971 | 3.0931e-01 | 1.1147 | 3.1422e-01 | 1.1155 |
| $\sqrt{2}/4$ | 6.2133e-02 | 2.4212 | 6.1737e-02 | 2.3248 | 6.2785e-02 | 2.3233 |
| $\sqrt{2}/8$ | 2.5670e-02 | 1.2748 | 2.4391e-02 | 1.3405 | 2.5110e-02 | 1.3211 |
| $\sqrt{2}/16$ | 8.8910e-03 | 1.5314 | 8.9103e-03 | 1.4528 | 9.0678e-03 | 1.4699 |
| $\sqrt{2}/32$ | 3.0741e-03 | 1.5314 | 2.9653e-03 | 1.5873 | 3.0418e-03 | 1.5758 |
| $\sqrt{2}/64$ | 9.8965e-04 | 1.6352 | 9.8077e-04 | 1.5962 | 1.0061e-03 | 1.5962 |

| $\gamma = 1$ | | | | | | |
| $\sqrt{2}$ | 6.5826e-01 | | 6.5826e-01 | | 6.8447e-01 | |
| $\sqrt{2}/2$ | 3.1723e-01 | 0.9412 | 3.1550e-01 | 1.0510 | 3.1600e-01 | 1.0547 |
| $\sqrt{2}/4$ | 6.2433e-02 | 2.3448 | 6.2344e-02 | 2.3375 | 6.2955e-02 | 2.3275 |
| $\sqrt{2}/8$ | 2.4824e-02 | 1.3309 | 2.4648e-02 | 1.3309 | 2.4856e-02 | 1.3407 |
| $\sqrt{2}/16$ | 8.6756e-03 | 1.5167 | 8.6738e-03 | 1.5060 | 8.7441e-03 | 1.5072 |
| $\sqrt{2}/32$ | 2.9786e-03 | 1.5424 | 2.9678e-03 | 1.5484 | 2.9861e-03 | 1.5509 |

![Figure V.4. Triangularization of $\Omega$ from Experiment 2 with $h = \sqrt{2}/8$](image)

**V.3. Experiment 3: Known Solution with Corner Singularity**

Since we have verified our analytic results with the experiments in Section V.2, we wish to further test the convergence of the DWDG method on other variations
of the obstacle problem. In Experiment V.3, we will look at the convergence of the DWDG method on the obstacle problem with a corner singularity. Note that this implies that Ω is no longer convex. The presence of optimal convergence rates would indicate that we may be able to relax the convex domain condition for optimal convergence of the DWDG method.

We will consider the problem with the constant obstacle \( \psi \equiv 0 \) on the L-shaped domain \( \Omega := (-1.5, 1.5)^2 \setminus (0, 1.5) \times (-1.5, 0) \). The function \( f \) is given in polar coordinates by

\[
f(r, \theta) := \begin{cases} 
-2 & \text{if } r \geq 0.75, \\
-r^{2/3} \sin(2\theta/3) (\gamma_1'(r)/r + \gamma_2''(r)) & \text{if } r < 0.75, \\
-\frac{4}{3} r^{-1/3} \gamma_1'(r) \sin(2\theta/3) - \gamma_2(r), & \text{if } r < 0.75,
\end{cases}
\]

where \((\cdot)'\) denotes the derivative with respect to the radius \( r \), \( \frac{d}{dr} \). Furthermore, with \( \tau := 2(r - 1/4) \), the functions \( \gamma_1(r) \) and \( \gamma_2(r) \) are defined as

\[
\gamma_1(r) = \begin{cases} 
1 & \text{if } \tau < 0, \\
-6\tau^5 + 15\tau^4 - 10\tau^3 + 1 & \text{if } 0 \leq \tau < 1, \\
0 & \text{if } \tau \geq 1,
\end{cases}
\]

\[
\gamma_2(r) = \begin{cases} 
0 & \text{if } r < 5/4, \\
1 & \text{otherwise}.
\end{cases}
\]

With this, the exact solution in polar coordinates is

\[
u(r, \theta) = \begin{cases} 
\frac{r^2}{2} - \ln(r) - \frac{1}{2} & \text{if } r \geq 1, \\
0 & \text{if } 0.75 \leq r < 1, \\
r^{2/3} \gamma_1(r) \sin(2\theta/3) & \text{if } r < 0.75,
\end{cases} \tag{V.5}
\]

which has a corner singularity at the origin. The solution for this problem is in \( H^{5/3-\tau}(\Omega) \) for all \( \tau > 0 \). We would expect a rate of almost 2/3 to appear when our approximation \( u_h \) is in either \( K^1_h \) or \( K^2_h \). A plot of the solution and an approximation are in Figure V.5. The error results are found in Tables V.5 and V.6.
As we can see in Table V.5, the rate that we are achieving is approximately 0.94 when $u_h \in K^1_h$. This is possibly due to the super convergence of the method. When $u_h \in K^2_h$, we see a difference in that the rate seems to be continually decreasing after $h = 3/16$. This indicates that a rate of $2/3$ may be observed after running multiple refinements of the mesh, based on the behavior of the $K^2_h$ solution.

<table>
<thead>
<tr>
<th>$\gamma \equiv -1$</th>
<th>$| \nabla u - \nabla u_h |_{L^2(T)}$</th>
<th>Rate</th>
<th>$| \nabla u - \nabla_{h,g} u_h |_{L^2(T)}$</th>
<th>Rate</th>
<th>$| u - u_h |_{1,h}$</th>
<th>Rate</th>
</tr>
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<tbody>
<tr>
<td>$\gamma \equiv -1$</td>
<td>$| \nabla u - \nabla u_h |_{L^2(T)}$</td>
<td>Rate</td>
<td>$| \nabla u - \nabla_{h,g} u_h |_{L^2(T)}$</td>
<td>Rate</td>
<td>$| u - u_h |_{1,h}$</td>
<td>Rate</td>
</tr>
<tr>
<td>$\gamma \equiv 0$</td>
<td>$| \nabla u - \nabla u_h |_{L^2(T)}$</td>
<td>Rate</td>
<td>$| \nabla u - \nabla_{h,g} u_h |_{L^2(T)}$</td>
<td>Rate</td>
<td>$| u - u_h |_{1,h}$</td>
<td>Rate</td>
</tr>
<tr>
<td>$\gamma \equiv 100$</td>
<td>$| \nabla u - \nabla u_h |_{L^2(T)}$</td>
<td>Rate</td>
<td>$| \nabla u - \nabla_{h,g} u_h |_{L^2(T)}$</td>
<td>Rate</td>
<td>$| u - u_h |_{1,h}$</td>
<td>Rate</td>
</tr>
</tbody>
</table>

Table V.5. Experiment 3, $u_h \in K^1_h$. 

As we can see in Table V.5, the rate that we are achieving is approximately 0.94 when $u_h \in K^1_h$. This is possibly due to the super convergence of the method. When $u_h \in K^2_h$, we see a difference in that the rate seems to be continually decreasing after $h = 3/16$. This indicates that a rate of $2/3$ may be observed after running multiple refinements of the mesh, based on the behavior of the $K^2_h$ solution.
Table V.6. Experiment 3, $u_h \in K_h^2$.

| $\gamma$ | $h$ | $\|\nabla u - \nabla u_h\|_{L^2(\Omega_h)}$ | Rate | $\|\nabla u - \nabla_{h,g}u_h\|_{L^2(\Omega_h)}$ | Rate | $|||u - u_h|||_{1,h}$ | Rate |
|----------|-----|----------------|-------|----------------|-------|----------------|-------|
| $\equiv -1$ | 3/2 | 1.3997e+00 | 1.274e+00 | 1.2484e+00 | 1.3984 |
| | 3/4 | 5.2762e-01 | 1.3117 | 4.3336e-01 | 1.5546 | 4.3577e-01 | 1.3984 |
| | 3/8 | 2.2693e-01 | 1.2172 | 1.7624e-01 | 1.2981 | 1.9410e-01 | 1.2868 |
| | 3/16 | 7.3776e-02 | 1.6211 | 5.4745e-02 | 1.6868 | 6.3098e-02 | 1.6232 |
| | 3/32 | 2.7080e-02 | 1.4459 | 9.9586e-02 | 1.4857 | 2.3729e-02 | 1.4089 |
| | 3/64 | 1.3571e-02 | 0.9967 | 9.9557e-03 | 0.9735 | 1.2431e-02 | 0.9326 |
| | 3/128 | 8.0865e-03 | 0.7469 | 5.9703e-03 | 0.7377 | 5.5174e-03 | 0.7258 |
| $\equiv 0$ | 3/2 | 1.3988e+00 | 1.2741e+00 | 1.2510e+00 | 1.3813 |
| | 3/4 | 5.2561e-01 | 1.3162 | 4.3543e-01 | 1.5490 | 4.8022e-01 | 1.3813 |
| | 3/8 | 2.2681e-01 | 1.2125 | 1.7917e-01 | 1.2811 | 1.9885e-01 | 1.2720 |
| | 3/16 | 7.3686e-02 | 1.6220 | 5.5533e-02 | 1.6899 | 6.4777e-02 | 1.6196 |
| | 3/32 | 2.6993e-02 | 1.4488 | 1.9513e-02 | 1.5089 | 2.4411e-02 | 1.4064 |
| | 3/64 | 1.3498e-02 | 0.9999 | 9.7776e-03 | 0.9969 | 1.2767e-02 | 0.9352 |
| | 3/128 | 8.0370e-03 | 0.7480 | 5.8336e-03 | 0.7451 | 7.7145e-03 | 0.7267 |
| $\equiv 1$ | 3/2 | 1.3986e+00 | 1.2752e+00 | 1.2534e+00 | 1.3673 |
| | 3/4 | 5.2432e-01 | 1.3190 | 4.3796e-01 | 1.5418 | 4.8582e-01 | 1.3673 |
| | 3/8 | 2.2698e-01 | 1.2079 | 1.8259e-01 | 1.2661 | 2.0288e-01 | 1.2661 |
| | 3/16 | 7.3704e-02 | 1.6228 | 5.6360e-02 | 1.6919 | 6.6154e-02 | 1.6164 |
| | 3/32 | 2.6956e-02 | 1.4512 | 1.9574e-02 | 1.5258 | 2.5064e-02 | 1.4637 |
| | 3/64 | 1.3455e-02 | 1.0924 | 9.6818e-03 | 1.0156 | 1.3063e-02 | 0.9367 |
| | 3/128 | 8.0967e-03 | 0.7489 | 5.7519e-03 | 0.7512 | 7.8902e-03 | 0.7273 |
| $\equiv 100$ | 3/2 | 1.3981e+00 | 1.2707 | 1.2994e+00 | 1.3149e+00 |
| | 3/4 | 5.4215e-01 | 1.2707 | 5.2646e-01 | 1.3035 | 5.4795e-01 | 1.2851 |
| | 3/8 | 2.4718e-01 | 1.1331 | 2.4045e-01 | 1.1306 | 2.4758e-01 | 1.1439 |
| | 3/16 | 8.3492e-02 | 1.5659 | 8.0158e-02 | 1.5848 | 8.4871e-02 | 1.5445 |
| | 3/32 | 3.2617e-02 | 1.3560 | 3.0772e-02 | 1.3812 | 3.4193e-02 | 1.3138 |
| | 3/64 | 1.7038e-02 | 0.9369 | 1.5933e-02 | 0.9496 | 1.8215e-02 | 0.9063 |
| | 3/128 | 1.0277e-02 | 0.7203 | 9.885e-03 | 0.7327 | 1.1053e-02 | 0.7206 |

Figure V.5. An Approximation and Solution for Experiment 3
V.4. Experiment 4: Unknown Solution

In this last experiment, we examine the DWDG methods on the obstacle problem with an unknown solution. We take the data for this problem to be $\Omega = [-1, 1]^2$, $f \equiv -5$, $u = 1.5$ on $\partial \Omega$, and $\psi(x_1, x_2) = \cos(3\pi x_1) + \cos(3\pi x_2)$. This problem was chosen because the obstacle is slightly more complex than the others seen in this dissertation, but it is still smooth. Since the exact solution is unknown, we will replace $u$ with $u_{h/2}$ and perform the error calculations on the mesh that generated $u_{h/2}$. A plot of two different approximations can be found in Figure V.7.

As we can see in Table V.7 and Table V.8, the rates match the results from Theorems IV.7-IV.8. The rates in Table V.7 seem to be converging to 1, and the rates in Table V.8 float around 1.5. We can again see that the errors are smallest for the $K_2^h$ approximation, ranging from half to a third of the error seen for the $K_1^h$ approximation. Similar to Experiment 1, when $u_h \in K_1^h$, we see that $\gamma \equiv -1, 0$ are producing the smaller absolute errors in the two $L^2$ norms on the finer meshes and $\gamma \equiv 1, 100$ have
the smaller errors on the coarser meshes. Again, when \( u_h \in K_h \), we see that \( \gamma \equiv 1, 100 \) are producing the smaller absolute errors for \( \| \nabla u_{h/2} - \nabla u_h \|_{L^2(T_h)} \). However, \( \gamma \equiv 0, 1 \) are producing the smaller absolute errors for \( \| \nabla_{h/2,g} u_{h/2} - \nabla_{h/2,g} u_h \|_{L^2(T_h)} \) which is different from previous experiments.

Table V.7. Experiment 4, \( u_h \in K_h \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( | \nabla u_{h/2} - \nabla u_h |_{L^2(T_h)} )</th>
<th>Rate</th>
<th>( | \nabla_{h/2,g} u_{h/2} - \nabla_{h/2,g} u_h |_{L^2(T_h)} )</th>
<th>Rate</th>
<th>( | u_{h} - u_{h/2} |_{1,h} )</th>
<th>Rate</th>
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Table V.8. Experiment 4, \( u_h \in K_h^2 \).

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<th>( \gamma \equiv -1 )</th>
<th>( h )</th>
<th>( | \nabla h - \nabla u_h |_{L^2(\mathcal{T}_h)} ) Rate</th>
<th>( | \nabla h |_{L^2(\mathcal{T}_h)} ) Rate</th>
<th>( | u_h - u_h |_1, h ) Rate</th>
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<th>( | \nabla h |_{L^2(\mathcal{T}_h)} ) Rate</th>
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Figure V.7. Two Approximations for Experiment 4
Figure V.8. Triangularization of $\Omega$ in Experiment 4 with $h = 1/4$
CHAPTER VI
INVESTIGATING THE PENALTY PARAMETER FOR THE DUAL-WIND
DISCONTINUOUS GALERKIN METHOD

Another goal of this dissertation is to understand how the choice of \( \gamma \) will effect the approximation from the DWDG method. The penalty parameter \( \gamma \) is traditionally an artificial parameter that is introduced into the formulation to guarantee stability of the method. In the context of interior-penalty methods, the penalty term naturally arises in the derivation of a discrete Friedrich’s inequality (see [Bre03, ABCM02]). In the context of LDG methods, the parameter does not arise as naturally, and can be eliminated in very special cases [Lew16, CD07]. Thus, we investigate the penalty parameter in an ad hoc manner. In experiments conducted in Section V.2, we saw the measured errors increase as \( \gamma \) increased given a fine enough mesh. As \( \gamma \to \infty \), we would recover a finite element solution. To establish a relationship between the penalty parameter \( \gamma \) and the error, we first run several numerical experiments for a simple elliptic problem before making observations for elliptic variational inequalities.

For this investigation, we will study the DWDG method for the Poisson problem. The homogeneous Poisson problem was given in (I.12). For our experiments, we will test the DWDG method on both homogeneous and non-homogeneous examples. A more general statement of the Poisson problem is as follows:

\[
-\Delta u = f; \quad \Omega = [0, 1]^2, \\
u = g; \quad \partial \Omega.
\] (VI.1)
As was discussed in Section II.2, we can obtain the Poisson problem from the obstacle problem by assuming that our solution never comes in contact with the obstacle, giving us a connection between the two problems. The DWDG method was studied for the Poisson problem in [LN14], where optimal convergence results were established. The regularity of solutions for the Poisson problem will allow us to mitigate any influential factors that may come from low-regularity solutions, allowing us to focus on the effect of the penalty parameter to the approximation. If we can establish a solid relationship between the choice of $\gamma$ and the errors for this type of problem, we then have a starting point to establish such relationships for other elliptic problems, including the obstacle problem. Lastly, we have chosen the Poisson problem because there are many examples with known solutions that we can choose from. This allows us to calculate exact errors and eliminate the possibility of a particular example influencing our results.

VI.1. The Dual-Wind Discontinuous Galerkin Approximation for the Poisson Problem

Recall from Section II.2 that the discrete DWDG Poisson problem consists of finding $u_h \in V_h^r$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall \ v_h \in V_h^r,$$  \hspace{1cm} (VI.2)

where

$$B_h(v_h, w_h) := \frac{1}{2} \left( (\nabla_{h,0}^+ v_h, \nabla_{h,0}^+ w_h)_{\partial h} + (\nabla_{h,0}^- v_h, \nabla_{h,0}^- w_h)_{\partial h} \right) + \left\langle \frac{\gamma_h}{h_e} [v_h], [w_h] \right\rangle_{E_h},$$  \hspace{1cm} (VI.3)

$$F_h(v) := (f, v)_{\partial h} + \left\langle \frac{\gamma_h}{h_e} g, v \right\rangle_{E_h} - \left\langle g, \nabla_{h,0} v \cdot n \right\rangle_{E_h} \quad \forall \ v \in V_h,$$  \hspace{1cm} (VI.4)
with $\gamma_e = \gamma$ for all $e \in \mathcal{E}_h$. In [LN14], they established error results for the Poisson problem with Dirichlet boundary conditions. Similar to the error results for the obstacle problem, the DWDG approximation required a shape-regular mesh for $\gamma > 0$ and a quasi-uniform mesh for $\gamma \geq -C_*$. In the experiments that follow, we hope to identify a possible relationship between the errors and $\gamma$.

Since the constant $C_*$ has a complex derivation, the exact value is generally unknown. We calculate the DWDG approximation to the solution of the PDE above with $\gamma \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, 100\}$. Going only as low as $\gamma \equiv -1$ should guarantee that the method is stable, and gives us information about the effect of $\gamma$ decreasing towards zero and becoming negative. We measure the error of the DWDG approximation in the the $L^2$ norm, $\|u - u_h\|_{L^2(\mathcal{T}_h)}$, on a shape-regular mesh.

To account for the various effects of the meshing strategy on the error, we ran the test on four differing initial meshes for the domain $[0, 1]^2$. By running the various Poisson problems across refinements from all four initial meshes, we can establish a relationship that may be less dependent upon the mesh. The first initial mesh (A) consists of four equilateral triangles of equal size and has a minimal angle of $45^\circ$. The second mesh (B) consists of three triangles with a minimal angle of approximately $26.6^\circ$. The third mesh (C) is a modification of the initial mesh (A) where all four triangles share a node at the point $(0.25, 0.25)$ resulting in a minimal angle of approximately $18.4^\circ$. The fourth mesh (D) consists of five triangles with a minimal angle of approximately $14^\circ$.

The initial meshes are labeled in order of largest minimal angle to smallest minimal angle. A visualization of each of these starting meshes can be found in Figure VI.1. The meshing strategy for (A) is interesting in that it leads to criss-cross meshes. LDG methods are known to require $\gamma > 0$ for criss-cross meshes [Lew16]. Thus,
the choice represents a potentially challenging meshing strategy for DWDG methods without penalization.

Figure VI.1. The Four Initial Starting Meshes That Were Used for Each of the Examples (VI.5)-(VI.8).

VI.2. Homogeneous Dirichlet Boundary Condition

The first numerical experiment that was run was on the problem

\[-\Delta u = 2x(1 - x) + 2y(1 - y); \quad \Omega = [0, 1]^2, \]

\[u = 0; \quad \partial \Omega.\]

The solution for (VI.5) is

\[u(x, y) = x(1 - x)y(1 - y).\]
In the plots of Figure VI.2, Figure VI.3, and Figure VI.4 below, we give the errors
from a coarser and finer mesh that are refined from each initial mesh for both linear,
quadratic, and cubic basis functions.

In Figure VI.2, Figure VI.3, and Figure VI.4 we see a relationship appear. Specifically, there seems to be a strictly increasing relationship between $\gamma$ and the $L^2$ error. In these figures there are two exceptions. The first is on the coarser mesh that is a refinement of initial mesh (D) when using a linear basis (see Figure VI.2). In the plot of the coarser mesh, we see a check-mark error curve with the minimal error occurring at $\gamma = -\frac{1}{2}$. However, in the finer mesh, we no longer see the check-mark relationship.

Further investigation of this case (see Figure B.10) shows that the check-mark relationship persists throughout most refinements with either $\gamma = 0$ or $\gamma = -\frac{1}{2}$ producing the smallest $L^2$ error. However, we can see that the relationship turns into a strictly increasing relationship as seen on the final refinement from initial mesh (D). This indicates that the strictly increasing relationship will occur for linear basis polynomials as long as the mesh is fine enough.

The second exception occurs when using cubic basis polynomials on the refinements from initial mesh (A) (see Figure VI.4). Further investigation of the cubic basis case (see Figure B.1) shows that the check-mark relationship persists throughout all refinements. Furthermore, there seems to be some instability of the error measurement on the finest mesh. Otherwise, the check-mark relationship only appears on the initial meshes (B), (C), (D), the first refinement of initial mesh (A) of quadratic basis polynomials (see Figure B.2), and the first refinement of initial mesh (B) of linear basis polynomials (see Figure B.4). For all other refinements of the initial meshes (B), (C), and (D), we see a strictly increasing relationship between $\gamma$ and the $L^2$ error.
Figure VI.2. Error Plots for the Linear DWDG Approximation to (VI.5) with Linear Basis Functions. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.3. Error Plots for the DWDG Approximation to (VI.5) with Quadratic Basis Functions. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.4. Error Plots for the DWDG Approximation to (VI.5) with Cubic Basis Functions. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Since the solution of (VI.5) is in $V_h^2$ and $V_h^3$, the DWDG method may be recovering the exact solution. Indeed, the errors are $O(10^{-11})$ and may have floating point rounding errors. This may be causing some influence on our results due to the level of floating point precision needed in the calculations. We ran a second experiment whose solution does not exist in any of our test function spaces $V_h^r$ to see if it has any effect on the relationship we are currently observing.

The second homogeneous numerical experiment was run on the problem

$$-\Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y); \; \Omega = [0, 1]^2,$$

$$u = 0; \; \partial \Omega.$$  \hspace{1cm} (VI.6a)

The solution for VI.6 is

$$u(x, y) = \sin(\pi x) \sin(\pi y).$$

In the plots of Figure VI.5, Figure VI.6, and Figure VI.7 below, we give the errors from a coarser and finer mesh that are refined from all initial meshes for linear, quadratic, and cubic basis functions. In these figures we see only a strictly increasing relationship between $\gamma$ and the $L^2$ error. The special case of the persisting check-mark curve that was found in Figure (VI.4) is not replicated in this example, giving evidence to our conjecture of over-approximating the solution and floating point errors.

We again see the check-mark relationship occur in the error plots for the initial meshes (B), (C), and (D), and the first refinement for linear, quadratic, and cubic basis functions. However, the relationship quickly becomes strictly increasing as the mesh is refined. Therefore, we conclude that as long as the mesh is refined enough, there is a strictly increasing relationship between the value of $\gamma$ and the $L^2$ error. All plots for the homogeneous numerical experiments can be found in Appendix B.
Figure VI.5. Error Plots for the DWDG Approximation to (VI.6) with Linear Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.6. Error Plots for the DWDG Approximation to (VI.6) with Quadratic Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Coarser Mesh

Gammas:
- Row 1 from Initial Mesh (A)
- Row 2 from initial mesh (B)
- Row 3 from initial mesh (C)
- Row 4 from initial mesh (D)

Figure VI.7. Error Plots for the DWDG Approximation to (VI.6) with Cubic Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
VI.3. Non-Homogeneous Dirichlet Boundary Condition

The third numerical experiment was run on the non-homogeneous problem

\[ \Delta u = 6xy(1 - y) - 2x^3; \ \Omega = [0, 1]^2, \]  

\[ u|_{\partial \Omega} = \begin{cases} 
    y(1 - y) & \text{if } x = 1, \\
    0 & \text{otherwise}.
\end{cases} \]

The solution for (VI.7) is

\[ u(x, y) = y(1 - y)x^3. \]

Figure VI.8, Figure VI.9, and Figure VI.10 give the errors for a coarser and finer mesh from each initial mesh.

In Figure VI.8, Figure VI.9, and Figure VI.10, we can see a relationship between \( \gamma \) and the \( L^2 \) error similar to the homogeneous cases. In these figures, we see the check-mark relationship appear on some coarser meshes and only see a strictly increasing relationship on the finer meshes with only one exception. We observe a check-mark relationship that persists through all refinements from the initial mesh (D) for this experiment when using linear basis functions (see Figure B.34). However, the \( \gamma \) that minimizes the error seems to be moving towards \( \gamma = -1 \) as the mesh is refined, and the difference between the error for \( \gamma = -\frac{1}{2} \) and \( \gamma = -1 \) decreases.

This relationship indicates that the mesh may need to be refined further to achieve the strictly increasing relationship that we see in most other plots. Note that the same behavior was seen in the DWDG approximation to the first homogeneous problem (VI.5) on the initial mesh (D) and its refinements. The plots for the \( L^2 \) error of the quadratic and cubic DWDG method all follow a strictly increasing relationship after the initial mesh (D), similar to the DWDG approximation for (VI.5). For this
example, we conclude that the $L^2$ error increases as $\gamma$ increases as long as the mesh is fine enough.

Though we saw the same strictly increasing relationship appear in this non-homogeneous example, we have to consider the fact that the solution to (VI.7) is in $V_h^3$. Therefore, there might be some influence on our results. To make sure this is not the case, we will run another non-homogeneous example whose solution is not in any of our test function spaces $V_h^r$. The fourth numerical experiment was run on the problem

$$-\Delta u = 2\pi^2 \cos(\pi x) \cos(\pi y); \quad \Omega = [0,1]^2,$$

$$u|_{\partial \Omega} = \begin{cases} 
\cos(\pi y) & \text{if } x = 0, \\
\cos(\pi x) & \text{if } y = 0, \\
-\cos(\pi y) & \text{if } x = 1, \\
-\cos(\pi x) & \text{if } y = 1.
\end{cases} \quad (\text{VI.8})$$

The solution for (VI.8) is

$$u(x,y) = \cos(\pi x) \cos(\pi y).$$

In the plots of Figure VI.11, Figure VI.12, and Figure VI.13 below, we give the errors from a coarser and finer mesh that are refined from all initial meshes for linear, quadratic, and cubic basis functions. In these figures we see the same strictly increasing relationship between $\gamma$ and the $L^2$ error that was established for the second homogeneous problem after some refinement of the initial meshes. Again, the only exception is on all refinements from the initial mesh (D) using linear basis polynomials (see Figure B.46). Notice that on the last three refinements from this initial mesh, the $\gamma$ that is achieving the lowest error is $\gamma = -\frac{1}{2}$. 

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Similar to Experiment 3, we see that the difference in error between $\gamma = -1$ and $\gamma = -\frac{1}{2}$ decreases as the mesh is refined. The relationship that we see in these plots may suggest that the check-mark relationship would disappear as the mesh is refined, but they are not conclusive. Since the check-mark relationship does not persist for a quadratic basis, and is non-existent with a cubic basis, we can conclude that this is only the case with the DWDG approximation using linear basis functions. With consideration of the one exception, we conclude that as long as the mesh is refined enough, there is a strictly increasing relationship between the value of $\gamma$ and the $L^2$ error for this experiment. More importantly, the error is increasing for $\gamma \in [0, 100)$. All plots for the non-homogeneous numerical experiments across all initial meshes (A)-(D) and their 6 refinements can be found in Appendix B.
Figure VI.8. Error Plots for the DWDG Approximation to (VI.7) with Linear Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.9. Error Plots for the DWDG Approximation to (VI.7) with Quadratic Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.10. Error Plots for the DWDG Approximation to (VI.7) with Cubic Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.11. Error Plots for the DWDG Approximation to (VI.8) with Linear Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.12. Error Plots for the DWDG Approximation to (VI.8) with Quadratic Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
Figure VI.13. Error Plots for the DWDG Approximation to (VI.8) with Cubic Basis Polynomials. Row 1 is from Initial Mesh (A), row 2 is from initial mesh (B), row 3 is from initial mesh (C), and row 4 is from initial mesh (D).
VI.4. Conclusion from Poisson Experiments

As we can see in each experiment presented, we either saw a strictly increasing relationship between $\gamma$ and the $L^2$ error, or we would see the minimum appear at a small, non-negative value of $\gamma$ creating a check-mark relationship. On any refinement where the check-mark relationship appeared, further refinement of the mesh would change it to a strictly increasing relationship for values of gamma between $-1$ and $100$, with the only exceptions occurring when our solution is in our approximation space $V^r_h$, or the DWDG approximation was constructed using linear basis polynomials. In the latter case, we have reason to believe that with further refinement of our mesh we would observe a strictly increasing relationship. Based on all of our observations, as long as the solution is smooth enough and the mesh is fine enough we experimentally see a strictly increasing relationship between $\gamma$ and the $L^2$ error.

The increasing relationship indicates that to minimize the error on a fine mesh, we would want to choose $\gamma$ to be negative. However, it would be practically impossible to calculate the optimal choice, and choosing an appropriate choice for $\gamma < 0$ is not straightforward. If we choose $\gamma$ to be too close to or below $-C_*$, the bilinear form may become unstable and our approximation will no longer converge. Therefore, a safer and convenient option to reduce the error is to set $\gamma \equiv 0$.

Furthermore, it has been established in [LN00,BQS10] that as we allow $\gamma \to \infty$, an interior penalty DG method will converge to a continuous Galerkin method in the $L^2$ norm. We would expect the same for the DWDG method since the formulation reduces to a $C^0$ FEM when restricting $V^r_h$ to contain only continuous basis functions. Thus, based on the relationships provided in the plots in Appendix B we can assume that the DWDG method will produce better errors than a continuous Galerkin method.
In this chapter, we will further investigate the DG derivatives that are used to build the DWDG methods for the Poisson problem VI.1. Specifically, we wish to study the discrete Laplacian $\Delta_{h,g}$. We will present a theorem and two conjectures for the consistency of the discrete operator $\Delta_{h,g}$ and the DWDG approximation $u_h$ to the discrete Poisson problem (II.11). This will be followed by numerical experiments that support the theorem and conjectures. The main emphasis is to further understand the consequences of eliminating the penalty terms.

VII.1. Consistency Results

For most approximation methods for the Poisson problem, the approximation comes from solving the weak discrete formulation. Therefore, it is common to only study the convergence of the approximation in either the $L^2$ norm or $H^1$ semi-norm. This is because other methods, such as $C^0$ methods and IPDG methods, are not able to create a high order approximation to $\Delta u$ since each derivative reduces the degree of the approximating space. Other DG methods that pick numerical fluxes, such as LDG methods, are able to create high order approximations to $\nabla u$. In fact, the error for the LDG gradient approximation can be either $O(h^{r+\frac{1}{2}})$ or $O(h^r)$ depending on the choice of the penalty parameter [Riv08]. Since the dual-wind derivatives in (II.1) are able to naturally construct a discrete Laplacian operator, it invites further study into the accuracy of these higher order derivative approximations that is uncommon for other FEM and DG methods. Our main result for this chapter will be the following.
Theorem VII.1. Let $u \in H^{s+3}(\Omega)$ be the solution to the Poisson problem (VI.1). Let $T_h$ be a quasi-uniform mesh such that each element has at most one boundary edge, and let $u_h \in V_h^r$ be the DWDG approximation (VI.2) with $1 \leq r \leq s$. If $\gamma_e = 0 \ \forall \ e \in \mathcal{E}_h$, then

$$\|\Delta_{h,g}u_h - \Delta u\| \leq Ch^{r+1}. \quad (VII.1)$$

Otherwise, if $\gamma_e \neq 0$ and $\gamma_e > -C_* \ \forall \ e \in \mathcal{E}_h$, then

$$\|\Delta_{h,g}u_h - \Delta u\| \leq Ch^{r-1}. \quad (VII.2)$$

Proof. From the DWDG approximation to the primal form of the Poisson problem (II.11), we have $u_h \in V_h^r$ is the solution to

$$-\Delta_{h,g}u_h + j_{h,g}(u_h) = \mathcal{P}_hf,$$

where $\mathcal{P}_hf$ is the $L^2$ projection of $f$, i.e. $(\mathcal{P}_hf, v_h)_{T_h} = (f, v_h)_{T_h} \ \forall \ v_h \in V_h^r$, and $j_{h,g} : \mathcal{V}_h \rightarrow V_h^r$ is defined by

$$(j_{h,g}(v), w_h)_{T_h} = \langle \frac{\gamma_e}{h_e} [v], [w_h] \rangle_{\mathcal{E}_h} - \langle \frac{\gamma_e}{h_e} g, w_h \rangle_{\mathcal{E}_h^B}. \quad (VII.3)$$

Let $T_h$ be a quasi-uniform mesh of $\Omega$. Since $\Omega$ is a polygonal domain, then $\Omega = \cup_{K \in \mathcal{T}_h} K$. Thus,

$$\|\Delta_{h,g}u_h - \Delta u\|_{L^2(\Omega)} = \|\Delta_{h,g}u_h + f\|_{L^2(\Omega)}$$

$$= \|j_{h,g}(u_h) - \mathcal{P}_hf + f\|_{L^2(\Omega)}$$

$$\leq \|f - \mathcal{P}_hf\|_{L^2(\Omega)} + \|j_{h,g}(u_h)\|_{L^2(\Omega)}$$
Since $\mathcal{P}_h f$ is the $L^2$ projection of $f$ into the space $V^r_h$, and $f = -\Delta u \in H^{s+1}$, we have that
\[ \| f - \mathcal{P}_h f \|_{L^2(\Omega)} \leq C h^{r+1}. \]

Therefore,
\[ \| \Delta_{h,g} u_h - \Delta u \|_{L^2(\Omega)} \leq C h^{r+1} + \| j_{h,g}(u_h) \|_{L^2(\Omega)}. \] (VII.4)

If $\gamma_e = 0 \forall e \in \mathcal{E}_h$, then $j_{h,g}(u_h) = 0$, resulting in (VII.1).

To prove (VII.2), assume that $\gamma_e \neq 0$ and $\gamma_e > -C_\ast \forall e \in \mathcal{E}_h$. By (VII.3), we have
\begin{align*}
\| j_{h,g}(u_h) \|_{L^2(\Omega)}^2 &= \left( j_{h,g}(u_h), j_{h,g}(u_h) \right)_{\mathcal{T}_h} \\
&= \left( \frac{\gamma_e}{h_e} [u_h], [j_{h,g}(u_h)] \right)_{\mathcal{E}_h} - \left( \frac{\gamma_e}{h_e} g, j_{h,g}(u_h) \right)_{\mathcal{E}_h^B}.
\end{align*}

Since $u = g$ on the $\partial \Omega = \sum_{e \in \mathcal{E}_h^B} e$ and $[u] = 0$ on all $e \in \mathcal{E}_h^I$, we have
\begin{align*}
\| j_{h,g}(u_h) \|_{L^2(\Omega)}^2 &= \left( \frac{\gamma_e}{h_e} [u_h], [j_{h,g}(u_h)] \right)_{\mathcal{E}_h} - \left( \frac{\gamma_e}{h_e} u, j_{h,g}(u_h) \right)_{\mathcal{E}_h^B} \\
&= \left( \frac{\gamma_e}{h_e} [u_h - u], [j_{h,g}(u_h)] \right)_{\mathcal{E}_h} \\
&= \left( \frac{\gamma_e}{h_e^{3/2}} [u_h - u], h_e^{1/2} [j_{h,g}(u_h)] \right)_{\mathcal{E}_h}.
\end{align*}
By the Cauchy-Schwartz inequality, the trace theorem with scaling, and the quasi-
uniformity of $T_h$, we have

$$
\| j_{h,g}(u_h) \|_{L^2(\Omega)}^2 \leq \left( \sum_{e \in \mathcal{E}_h} \frac{\gamma_e^2}{h_e^2} \| [u_h - u] \|_{L^2(e)}^2 \right)^{\frac{1}{2}} \left( \sum_{e \in \mathcal{E}_h} h_e \| [j_{h,g}(u_h)] \|_{L^2(e)}^2 \right)^{\frac{1}{2}}
\leq \left( C \sum_{K \in T_h} \frac{\gamma_K^4}{h_K^4} \| u_h - u \|_{L^2(K)}^2 \right)^{\frac{1}{2}} \left( C \sum_{K \in T_h} \| j_{h,g}(u_h) \|_{L^2(K)}^2 \right)^{\frac{1}{2}}
\leq |\gamma_{\text{max}}| \left( C \sum_{K \in T_h} h_K^{-4} \| u_h - u \|_{L^2(K)}^2 \right)^{\frac{1}{2}} \left( C \sum_{K \in T_h} \| j_{h,g}(u_h) \|_{L^2(K)}^2 \right)^{\frac{1}{2}}
\leq C |\gamma_{\text{max}}| h^{-4} \| u - u_h \|_{L^2(T_h)} \| j_{h,g}(u_h) \|_{L^2(T_h)}.
$$

Therefore,

$$
\| j_{h,g}(u_h) \|_{L^2(\Omega)} \leq C |\gamma_{\text{max}}| h^{-2} \| u - u_h \|_{L^2(T_h)}.
$$

By [LN14, Theorem 7], we have

$$
\| u - u_h \|_{L^2(T_h)} \leq Ch^{r+1}.
$$

Thus,

$$
\| j_{h,g}(u_h) \|_{L^2(\Omega)} \leq C |\gamma_{\text{max}}| h^{-2} \| u - u_h \|_{L^2(T_h)}
\leq C |\gamma_{\text{max}}| h^{-1},
$$

giving us

$$
\| \Delta_{h,g} u_h - \Delta u \|_{L^2(\Omega)} \leq Ch^{r+1} + C |\gamma_{\text{max}}| h^{-1}
\leq C |\gamma_{\text{max}}| h^{r-1}.
$$

The proof is complete. \qed
Theorem VII.1 proves that $\Delta_{h,g} u_h$ is the $L^2$ projection of $\Delta u$ into $V_h^r$ if $\gamma_e = 0$ for all $e \in \mathcal{E}_h$. If $\gamma_e \neq 0$ for any $e \in \mathcal{E}_h$ then we no longer have the $L^2$ projection property. This gives us a hint as to why $\gamma = 0$ is an important choice for the Poisson problem, and why we still can accurately approximate $u$ without penalization.

Two other comparisons that we will like to study are the consistency of the operator $\Delta_{h,g}$ with $\Delta$. This will give us a better understanding on how well the DWDG method approximates higher order derivatives from the standpoint of the DG derivatives themselves. We state the following conjecture which is informed by the numerical experiments found in Section VII.2.

**Conjecture VII.2.** Let $u \in H^{s+2}$ with $s \geq 1$, and let $\Delta_{h,g}$ be defined as in (II.10). Then, for $1 \leq r \leq s$, we have

$$\|\Delta_{h,g} u - \Delta u\|_{L^2(\mathcal{T}_h)} \leq C h^r. \quad (VII.5)$$

Lastly, another important observation to make is that the DWDG approximation is only approximately Galerkin orthogonal. Thus, it is appropriate to investigate the implications of this observation. Let $\tilde{u}_h$ be the function in $V_h^r$ that satisfies the Galerkin orthogonality condition

$$B_h(u - \tilde{u}_h, v_h) = 0 \quad \forall v_h \in V_h^r. \quad (VII.6)$$

Consider the special case of (VII.6), with $\gamma_e = 0$ for all $e \in \mathcal{E}_h$. For the rest of the chapter, $\tilde{u}_h$ will represent the function that satisfies the Galerkin orthogonality condition in this special case.
Since $\nabla_{h,0}^+ (\tilde{u}_h - u) \in V_h^r$ we can apply (II.7) to (VII.6). So, for any $v_h \in V_h^r$, we have

$$0 = B_h(u - \tilde{u}_h, v_h) = \frac{1}{2} (\nabla_{h,0}^- (\tilde{u}_h - u), \nabla_{h,0}^- v_h)_{T_h} + \frac{1}{2} (\nabla_{h,0}^+ (\tilde{u}_h - u), \nabla_{h,0}^+ v_h)_{T_h} = (\Delta_{h,0}(\tilde{u}_h - u), v_h)_{T_h} = (\Delta_{h,g}\tilde{u}_h - \Delta_{h,g} u, v_h)_{T_h},$$

which implies that

$$\Delta_{h,g}\tilde{u}_h = \Delta_{h,g} u.$$  \hfill (VII.7)

Thus, $\tilde{u}_h = (\Delta_{h,g})^{-1}(\Delta_{h,g} u)$. In contrast to (VII.7), $\Delta_{h,g} u_h = \mathcal{P}_h(\Delta u)$ when $\gamma = 0$.

We will measure how well $u_h$ approximates the Galerkin orthogonality by comparing it to the Galerkin projection of $\tilde{u}_h$. We state the following conjecture about $\tilde{u}_h$. This conjecture was informed by numerical experiments for a homogeneous Poisson problem. The results of these experiments can be found in Section VII.2

**Conjecture VII.3.** Let $u \in H^{s+1}$ be the solution to (1.12). Let $T_h$ be a quasi-uniform mesh. Let $u_h$ be the DWDG approximation from (VI.2) and $\tilde{u}_h$ be defined by (VII.7). If $1 \leq r \leq s$, then

$$\|u - \tilde{u}_h\|_{T_h} \leq C h^{r+1},$$ \hfill (VII.8)

$$\|\nabla u - \nabla \tilde{u}_h\|_{T_h} \leq C h^r.$$ \hfill (VII.9)
Furthermore, if $\gamma_e = 0$ for all $e \in \mathcal{E}_h$, then

$$\|u_h - \tilde{u}_h\|_{\mathcal{T}_h} \leq C h^{r+2},$$  \hspace{1cm} (VII.10)

$$\|\nabla_{h,g}^\pm u_h - \nabla_{h,g}^\pm \tilde{u}_h\|_{\mathcal{T}_h} \leq C h^{r+1}.$$  \hspace{1cm} (VII.11)

Otherwise, if $\gamma_e \neq 0$ and $\gamma_e \geq -C_*$ for all $e \in \mathcal{E}_h$, then

$$\|u_h - \tilde{u}_h\|_{\mathcal{T}_h} \leq C h^{r+1},$$  \hspace{1cm} (VII.12)

$$\|\nabla_{h,g}^\pm u_h - \nabla_{h,g}^\pm \tilde{u}_h\|_{\mathcal{T}_h} \leq C h^r.$$  \hspace{1cm} (VII.13)

VII.2. Numerical Experiments for Higher Order Derivative Approximations

For our numerical experiments, we will use the Poisson problems (VI.7) and (VI.8) from Chapter VI. We ran the experiments on initial mesh (A) from Chapter VI and its refinements using linear, quadratic, and cubic basis functions to build the DWDG approximation. For simplicity, we choose $\gamma_e = \gamma$ to be constant for all $e \in \mathcal{E}_h$, and let $\gamma \in \{-1, 0, 1\}$. This will allow us to verify that any non-zero penalization will result in a lesser rate as seen in Theorem VII.1, Conjecture VII.2, and Conjecture VII.3.

The first experiment is for the homogeneous Poisson problem (VI.6):

$$-\Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y); \Omega = [0, 1]^2,$$  \hspace{1cm} (VII.14a)

$$u = 0; \partial\Omega.$$  \hspace{1cm} (VII.14b)

Recall the solution for (VI.6) is $u(x, y) = \sin(\pi x) \sin(\pi y)$. The error and rates for all choices of $\gamma$ with the DWDG approximation constructed with linear, quadratic, and cubic basis polynomials can be found in Table VII.1. When $\gamma = -1, 1$ we see a rate of one less than the basis polynomial degree in each column. This includes a rate of 0.
when using linear basis polynomials. When $\gamma = 0$ we see a rate of one more than the degree of the basis polynomials in each column. The results of the error measurements mentioned in Conjecture VII.2 and Conjecture VII.3 can be found in Table VII.1.

Table VII.1. $\Delta_{h,g}u_h$ Error for a Homogeneous Poisson Experiment.

<table>
<thead>
<tr>
<th>$\gamma \equiv -1$</th>
<th>$\gamma \equiv 0$</th>
<th>$\gamma \equiv 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$|\Delta u - \Delta_{h,g}u_h|_{L^2(T_h)}$</td>
<td>Rate</td>
</tr>
<tr>
<td>1/2</td>
<td>1.0242e+00</td>
<td>-0.3858</td>
</tr>
<tr>
<td>1/4</td>
<td>1.6087e+00</td>
<td>-0.2655</td>
</tr>
<tr>
<td>1/8</td>
<td>1.6821e+00</td>
<td>-0.0644</td>
</tr>
<tr>
<td>1/16</td>
<td>1.6068e+00</td>
<td>0.0660</td>
</tr>
<tr>
<td>1/32</td>
<td>1.5300e+00</td>
<td>0.0707</td>
</tr>
<tr>
<td>1/64</td>
<td>1.4802e+00</td>
<td>0.0477</td>
</tr>
</tbody>
</table>

Table VII.2. $\Delta_{h,g}V_h$ Error for a Homogeneous Poisson Experiment. Note that $\gamma_e$ is not necessary for the construction of $\Delta_{h,g}$.

<table>
<thead>
<tr>
<th>$\gamma \equiv -1$</th>
<th>$\gamma \equiv 0$</th>
<th>$\gamma \equiv 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$|\Delta u - \Delta_{h,g}V_h|_{L^2(T_h)}$</td>
<td>Rate</td>
</tr>
<tr>
<td>1/2</td>
<td>1.0242e+00</td>
<td>-0.3858</td>
</tr>
<tr>
<td>1/4</td>
<td>1.6087e+00</td>
<td>-0.2655</td>
</tr>
<tr>
<td>1/8</td>
<td>1.6821e+00</td>
<td>-0.0644</td>
</tr>
<tr>
<td>1/16</td>
<td>1.6068e+00</td>
<td>0.0660</td>
</tr>
<tr>
<td>1/32</td>
<td>1.5300e+00</td>
<td>0.0707</td>
</tr>
<tr>
<td>1/64</td>
<td>1.4802e+00</td>
<td>0.0477</td>
</tr>
</tbody>
</table>
Table VII.3. Comparing $\tilde{u}_h$ and the Dual-Wind Discontinuous Galerkin Approximation $u_h$ for a Homogeneous Poisson Experiment. This is for Conjecture VII.3.

<table>
<thead>
<tr>
<th>$\gamma \equiv -1$</th>
<th>$|u_h - \tilde{u}<em>h|</em>{L^2(\Omega_h)}$</th>
<th>$|\nabla_{h,g} u_h - \nabla_{h,g} \tilde{u}<em>h|</em>{L^2(\Omega_h)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma \equiv 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma \equiv 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma \equiv -1$</th>
<th>LINEAR ($r = 1$)</th>
<th>QUADRATIC ($r = 2$)</th>
<th>CUBIC ($r = 3$)</th>
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<tbody>
<tr>
<td>$h$</td>
<td>Error</td>
<td>Rate</td>
<td>Error</td>
</tr>
<tr>
<td>1</td>
<td>4.5004e-02</td>
<td>—</td>
<td>2.2552e-03</td>
</tr>
<tr>
<td>1/2</td>
<td>8.2745e-03</td>
<td>2.4433</td>
<td>5.8069e-04</td>
</tr>
<tr>
<td>1/4</td>
<td>1.7109e-03</td>
<td>2.2739</td>
<td>4.9195e-05</td>
</tr>
<tr>
<td>1/8</td>
<td>3.7998e-04</td>
<td>2.1707</td>
<td>5.9015e-06</td>
</tr>
<tr>
<td>1/16</td>
<td>8.1312e-05</td>
<td>2.2244</td>
<td>7.4438e-07</td>
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</table>

<table>
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<tbody>
<tr>
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<td>Rate</td>
<td>Error</td>
</tr>
<tr>
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<td>3.6122e-01</td>
<td>—</td>
<td>1.7658e-02</td>
</tr>
<tr>
<td>1/2</td>
<td>1.4816e-01</td>
<td>1.2858</td>
<td>1.9012e-02</td>
</tr>
<tr>
<td>1/4</td>
<td>5.2471e-02</td>
<td>1.4975</td>
<td>3.1409e-03</td>
</tr>
<tr>
<td>1/8</td>
<td>2.2086e-02</td>
<td>1.2484</td>
<td>7.1783e-04</td>
</tr>
<tr>
<td>1/16</td>
<td>9.5616e-03</td>
<td>1.2078</td>
<td>1.7790e-04</td>
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</table>

<table>
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<tbody>
<tr>
<td>$h$</td>
<td>Error</td>
<td>Rate</td>
<td>Error</td>
</tr>
<tr>
<td>1</td>
<td>4.4439e-02</td>
<td>—</td>
<td>6.5320e-03</td>
</tr>
<tr>
<td>1/2</td>
<td>7.3556e-03</td>
<td>2.5949</td>
<td>3.9404e-04</td>
</tr>
<tr>
<td>1/4</td>
<td>9.2549e-02</td>
<td>1.9255</td>
<td>7.7781e-04</td>
</tr>
<tr>
<td>1/8</td>
<td>1.3472e-05</td>
<td>1.9996</td>
<td>5.9017e-05</td>
</tr>
<tr>
<td>1/16</td>
<td>2.2522e-03</td>
<td>2.0086</td>
<td>5.5066e-06</td>
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<table>
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<tr>
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<th>QUADRATIC ($r = 2$)</th>
<th>CUBIC ($r = 3$)</th>
</tr>
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<tbody>
<tr>
<td>$h$</td>
<td>Error</td>
<td>Rate</td>
<td>Error</td>
</tr>
<tr>
<td>1</td>
<td>3.5445e-02</td>
<td>—</td>
<td>5.3548e-02</td>
</tr>
<tr>
<td>1/2</td>
<td>1.3767e-01</td>
<td>1.3644</td>
<td>1.5114e-02</td>
</tr>
<tr>
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<td>3.6240e-02</td>
<td>1.9255</td>
<td>7.7781e-04</td>
</tr>
<tr>
<td>1/8</td>
<td>9.0628e-03</td>
<td>1.9996</td>
<td>5.9017e-05</td>
</tr>
<tr>
<td>1/16</td>
<td>2.2522e-03</td>
<td>2.0086</td>
<td>5.5066e-06</td>
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</table>

<table>
<thead>
<tr>
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<th>CUBIC ($r = 3$)</th>
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<tbody>
<tr>
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<td>Rate</td>
<td>Error</td>
</tr>
<tr>
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<td>4.4004e-02</td>
<td>—</td>
<td>1.0579e-02</td>
</tr>
<tr>
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<td>9.4253e-03</td>
<td>2.2230</td>
<td>4.4163e-04</td>
</tr>
<tr>
<td>1/4</td>
<td>1.7031e-03</td>
<td>2.4684</td>
<td>3.4806e-05</td>
</tr>
<tr>
<td>1/8</td>
<td>3.1684e-04</td>
<td>2.4264</td>
<td>4.7780e-06</td>
</tr>
<tr>
<td>1/16</td>
<td>6.3375e-05</td>
<td>2.3218</td>
<td>6.3247e-07</td>
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</table>

<table>
<thead>
<tr>
<th>$\gamma \equiv 0$</th>
<th>LINEAR ($r = 1$)</th>
<th>QUADRATIC ($r = 2$)</th>
<th>CUBIC ($r = 3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>Error</td>
<td>Rate</td>
<td>Error</td>
</tr>
<tr>
<td>1</td>
<td>3.9260e-01</td>
<td>—</td>
<td>9.3560e-02</td>
</tr>
<tr>
<td>1/2</td>
<td>1.5431e-01</td>
<td>1.1782</td>
<td>9.7017e-03</td>
</tr>
<tr>
<td>1/4</td>
<td>5.1523e-02</td>
<td>1.5826</td>
<td>1.8366e-03</td>
</tr>
<tr>
<td>1/8</td>
<td>1.8552e-02</td>
<td>1.4736</td>
<td>5.4646e-04</td>
</tr>
<tr>
<td>1/16</td>
<td>7.4441e-03</td>
<td>1.3174</td>
<td>1.4565e-04</td>
</tr>
</tbody>
</table>
Table VII.4. Error for $\tilde{u}_h$ Approximation for a Homogeneous Poisson Experiment. This is for Conjecture VII.3.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$V^1_h$ Error</th>
<th>Rate</th>
<th>$V^2_h$ Error</th>
<th>Rate</th>
<th>$V^3_h$ Error</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.3350e-01</td>
<td></td>
<td>3.7789e-02</td>
<td></td>
<td>3.5475e-03</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>4.2524e-02</td>
<td>1.6505</td>
<td>4.2805e-03</td>
<td>3.1421</td>
<td>4.6268e-04</td>
<td>2.9387</td>
</tr>
<tr>
<td>1/4</td>
<td>1.2804e-02</td>
<td>1.7317</td>
<td>4.8093e-04</td>
<td>3.1539</td>
<td>3.1809e-05</td>
<td>3.8625</td>
</tr>
<tr>
<td>1/8</td>
<td>3.5478e-03</td>
<td>1.8515</td>
<td>5.6442e-05</td>
<td>3.0910</td>
<td>2.0594e-06</td>
<td>3.9691</td>
</tr>
</tbody>
</table>

The second experiment is for the non-homogeneous Poisson problem (VI.8):

$$-\Delta u = 2\pi^2 \cos(\pi x) \cos(\pi y); \quad \Omega = [0,1]^2,$$

$$u|_{\partial\Omega} = \begin{cases} 
\cos(\pi y) & \text{if } x = 0, \\
\cos(\pi x) & \text{if } y = 0, \\
-\cos(\pi y) & \text{if } x = 1, \\
-\cos(\pi x) & \text{if } y = 1.
\end{cases}$$

Recall the solution for (VI.8) is $u(x, y) = \cos(\pi x) \cos(\pi y)$. The error and rates for all choices of $\gamma$ with the DWDG approximation constructed with linear, quadratic, and cubic basis polynomials can be found in Table VII.5.

When $\gamma = -1, 1$ we again see a rate of one less than the basis polynomial degree in each column. This includes a rate of 0 when using linear basis polynomials. When $\gamma = 0$ we see a rate of one more than the degree of the basis polynomials in each column. This implies that the boundary condition does not effect the rates we achieve when considering the consistency of $\Delta_{h,g}$. We did not investigate the error results for Conjecture VII.3 on the non-homogeneous example. This is due to the fact that the discrete Laplacian $\Delta_{h,g}$ is affine and not linear when $g \neq 0$. However, such comparisons are possible and may be of interest going forward.
Table VII.5. $\Delta_{h,g} u_h$ Error for a Non-homogeneous Poisson Experiment.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|\Delta u - \Delta_{h,g} u_h|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$|\Delta u - \Delta_{h,g} u_h|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$|\Delta u - \Delta_{h,g} u_h|_{L^2(T_h)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0747e+00</td>
<td>—</td>
<td>3.1573e-01</td>
<td>—</td>
<td>2.7647e-02</td>
<td>—</td>
</tr>
<tr>
<td>1/4</td>
<td>1.7426e+00</td>
<td>0.0277</td>
<td>2.0877e-01</td>
<td>1.1070</td>
<td>1.0914e-02</td>
<td>2.1592</td>
</tr>
<tr>
<td>1/16</td>
<td>1.6652e+00</td>
<td>0.0654</td>
<td>1.0234e-01</td>
<td>1.0286</td>
<td>2.2892e-03</td>
<td>2.1330</td>
</tr>
<tr>
<td>1/32</td>
<td>1.5058e+00</td>
<td>0.0627</td>
<td>5.1075e-02</td>
<td>1.0026</td>
<td>5.3014e-04</td>
<td>2.1104</td>
</tr>
<tr>
<td>1/64</td>
<td>1.4662e+00</td>
<td>0.0384</td>
<td>2.5548e-02</td>
<td>0.9994</td>
<td>1.2585e-04</td>
<td>2.0747</td>
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</table>

Table VII.6. $\Delta_{h,g}$ Error for a Non-homogeneous Poisson Experiment. Note that $\gamma_e$ is not necessary for the construction of $\Delta_{h,g}$. 

<table>
<thead>
<tr>
<th>$h$</th>
<th>$V_h^1$ $|\Delta u - \Delta_{h,g} u|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$V_h^2$ $|\Delta u - \Delta_{h,g} u|_{L^2(T_h)}$</th>
<th>Rate</th>
<th>$V_h^3$ $|\Delta u - \Delta_{h,g} u|_{L^2(T_h)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.1906e+00</td>
<td>—</td>
<td>7.9513e-01</td>
<td>—</td>
<td>5.2131e-01</td>
<td>—</td>
</tr>
<tr>
<td>1/2</td>
<td>3.3393e+00</td>
<td>0.6363</td>
<td>5.1391e-01</td>
<td>0.6297</td>
<td>9.1537e-02</td>
<td>2.5097</td>
</tr>
<tr>
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<td>1.2172e-02</td>
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<tr>
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<td>1.0228</td>
<td>5.1092e-04</td>
<td>2.1768</td>
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<td>1.0123</td>
<td>1.2098e-04</td>
<td>2.0795</td>
<td>2.8495e-06</td>
<td>3.0084</td>
</tr>
</tbody>
</table>
CHAPTER VIII
CONCLUSION

For my dissertation, we have proven that the DWDG approximation given by (II.18) for the obstacle problem (I.11) optimally converges to the solution of the obstacle problem (I.11) as we continuously refine our mesh $\mathcal{T}_h$. We were able to generate the expected theoretical rates in Experiments 1, 2, and 4, even when $\gamma = 0$. This verifies our results in Section IV. Also, this shows that when approximating solutions to the Obstacle Problem (I.11), the DWDG method can do so without the need for a penalty parameter. This is because the DWDG operators $\nabla^\pm_h$ inherently penalize the jumps in the solution eliminating the need for a standard DG penalization. Of course, this is only true if the mesh requirements are satisfied. We also proved standard rates if we penalize the jumps independent of the quasi-uniformity of the meshes.

We were also able to experimentally show that $\gamma \leq 0$ was optimal for the DWDG method for the Poisson problem. Since the barrier $-C_\star$ is hard or impossible to determine, $\gamma = 0$ is a consistent natural choice for tuning the DWDG method. In application, the choice of $\gamma = 0$ implies that only the DG derivatives need to be coded for the DWDG approximation for the Poisson problem. This eliminates the need for calculations done on the interior edges of the mesh when building the stiffness matrix. This coupled with the observation that the DWDG method on a rectangular mesh is a generalization of the finite difference method [LN14] indicates that the DWDG method is a natural high-order choice for industrial applications.
CHAPTER IX
FUTURE WORK

One interesting result of the numerical experiments is that some meshes did have triangles with two boundary edges, but we were still able to obtain the desired rates. On these triangles, the value of the function on the boundary edge was reflected throughout the entire triangle, which does cause some concern. As far as we could tell, this error did not perpetrate throughout neighboring triangles, let alone the entire mesh. Furthermore, this error seemed to be mitigated as the mesh was refined. This indicates the condition that no more than one edge of every triangle can lie on the boundary may not be a necessary one, at least for application. Tests in [LN14] also indicate the quasi-uniformity of the mesh may be nonessential when $\gamma \equiv 0$. We would like to further investigate whether the mesh restrictions can be removed analytically.

Another direction that can be taken is to apply DWDG methods to other elliptic PDEs such as the friction problem or the double obstacle problem. Some DG methods have already been studied for the friction problem in [WHC10a]. It would also be possible to apply DWDG methods to parabolic and hyperbolic problems, where DG methods are more commonly used. Lastly, as an extension of the obstacle problem, we have started investigating the use of the DWDG method on optimal control problems with elliptic partial differential equation restraints.

Since DG methods are capable of easily handling adaptive mesh techniques, we plan to attempt to implement and analyze adaptivity. Approximation error for obstacle problems usually occurs where the solution and the obstacle come in contact. Refinement around these free boundaries can decrease the errors seen in our
experiments and improve convergence with less resources. Examples of continuous Galerkin methods and discontinuous Galerkin with adaptive meshes for the obstacle problem can be found in [PP13, BC04, CK17b].
BIBLIOGRAPHY


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