

## Positive solutions for a class of superlinear semipositone systems on exterior domains

By: [Abraham Abebe](#), [Maya Chhetri](#), Lakshmi Sankar, and [R Shivaji](#)

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### Abstract:

We study the existence of a positive radial solution to the nonlinear eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda K_1(|x|)f(v) & -\Delta v &= \lambda K_2(|x|)g(u) & \text{in } \Omega_\epsilon \\ u(x) &= v(x) = 0 & |x| &= r_0 & u(x) \rightarrow 0 \quad v(x) \rightarrow 0 & |x| \rightarrow \infty & \lambda > 0 \\ & \text{if } & & & & \text{as } & \text{where } & \text{is a} \\ & \Delta u = \operatorname{div}(\nabla u) & \Omega_\epsilon &= \{x \in \mathbb{R}^n \mid |x| > r_0, n > 2\} \\ & K_i \in C^1([r_0, \infty), (0, \infty)) \quad i = 1, 2 & K_i(|x|) &\rightarrow 0 & |x| \rightarrow \infty \\ & \text{and } & & \text{are such that } & \text{as } \\ & f, g : [0, \infty) \rightarrow \mathbb{R} \end{aligned}$$

Here  $f, g$  are  $C^1$  functions such that they are negative at the origin (semipositone) and superlinear at infinity. We establish the existence of a positive solution for  $\lambda$  small via degree theory and rescaling arguments. We also discuss a non-existence result for  $\lambda \gg 1$  for the single equations case.

**Keywords:** superlinear | semipositone | positive solutions | existence | non-existence | exterior domains

### Article:

**\*\*\*Note: Full text of article below**

RESEARCH

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# Positive solutions for a class of superlinear semipositone systems on exterior domains

Abraham Abebe<sup>1</sup>, Maya Chhetri<sup>1\*</sup>, Lakshmi Sankar<sup>2</sup> and R Shivaji<sup>1</sup>

\*Correspondence: maya@uncg.edu

<sup>1</sup>Department of Mathematics and Statistics, The University of North Carolina at Greensboro, Greensboro, USA

Full list of author information is available at the end of the article

## Abstract

We study the existence of a positive radial solution to the nonlinear eigenvalue problem  $-\Delta u = \lambda K_1(|x|)f(v)$  in  $\Omega_e$ ,  $-\Delta v = \lambda K_2(|x|)g(u)$  in  $\Omega_e$ ,  $u(x) = v(x) = 0$  if  $|x| = r_0$  ( $> 0$ ),  $u(x) \rightarrow 0$ ,  $v(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , where  $\lambda > 0$  is a parameter,  $\Delta u = \operatorname{div}(\nabla u)$  is the Laplace operator,  $\Omega_e = \{x \in \mathbb{R}^n \mid |x| > r_0, n > 2\}$ , and  $K_i \in C^1([r_0, \infty), (0, \infty))$ ;  $i = 1, 2$  are such that  $K_i(|x|) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Here  $f, g : [0, \infty) \rightarrow \mathbb{R}$  are  $C^1$  functions such that they are negative at the origin (semipositone) and superlinear at infinity. We establish the existence of a positive solution for  $\lambda$  small via degree theory and rescaling arguments. We also discuss a non-existence result for  $\lambda \gg 1$  for the single equations case.

**MSC:** 34B16; 34B18

**Keywords:** superlinear; semipositone; positive solutions; existence; non-existence; exterior domains

## 1 Introduction

We consider the nonlinear elliptic boundary value problem

$$\left. \begin{aligned} -\Delta u &= \lambda K_1(|x|)f(v) && \text{in } \Omega_e, \\ -\Delta v &= \lambda K_2(|x|)g(u) && \text{in } \Omega_e, \\ u(x) &= v(x) = 0 && \text{if } |x| = r_0 (> 0), \\ u(x) &\rightarrow 0, v(x) \rightarrow 0 && \text{as } |x| \rightarrow \infty, \end{aligned} \right\} \quad (1.1)$$

where  $\lambda > 0$  is a parameter,  $\Delta u = \operatorname{div}(\nabla u)$  is the Laplace operator, and  $\Omega_e = \{x \in \mathbb{R}^n \mid |x| > r_0, n > 2\}$  is an exterior domain. Here the nonlinearities  $f, g : [0, \infty) \rightarrow \mathbb{R}$  are  $C^1$  functions which satisfy:

(H<sub>1</sub>)  $f(0) < 0$  and  $g(0) < 0$  (semipositone).

(H<sub>2</sub>) For  $i = 1, 2$  there exist  $b_i > 0$  and  $q_i > 1$  such that  $\lim_{s \rightarrow \infty} \frac{f(s)}{s^{q_1}} = b_1$ , and  $\lim_{s \rightarrow \infty} \frac{g(s)}{s^{q_2}} = b_2$ .

Further, for  $i = 1, 2$ , the weight functions  $K_i \in C^1([r_0, \infty), (0, \infty))$  are such that  $K_i(|x|) \rightarrow 0$  as  $|x| \rightarrow \infty$ . In particular, we are interested in the challenging case, where  $K_i$  do not decay too fast. Namely, we assume

(H<sub>3</sub>) There exist  $\tilde{d}_1 > 0$ ,  $\tilde{d}_2 > 0$ ,  $\rho \in (0, n - 2)$  such that for  $i = 1, 2$

$$\frac{\tilde{d}_1}{|x|^{n+\rho}} \leq K_i(|x|) \leq \frac{\tilde{d}_2}{|x|^{n+\rho}} \quad \text{for } |x| \gg 1.$$

We then establish the following.

**Theorem 1.1** *Let  $(H_1)$ – $(H_3)$  hold. Then (1.1) has a positive radial solution  $(u, v)$  ( $u > 0$ ,  $v > 0$  in  $\Omega_e$ ) when  $\lambda$  is small, and  $\|u\|_\infty \rightarrow \infty$ ,  $\|v\|_\infty \rightarrow \infty$  as  $\lambda \rightarrow 0$ .*

We prove this result via the Leray-Schauder degree theory, by arguments similar to those used in [1] and [2]. The study of such eigenvalue problems with semipositone structure has been documented to be mathematically challenging (see [3, 4]), yet a rich history is developing starting from the 1980s (see [5–7]) until recently (see [8–12]). In [1, 2] the authors studied such superlinear semipositone problems on bounded domains. In particular, in [12] the authors studied the system

$$\left. \begin{aligned} -\Delta u &= \lambda f(v) && \text{in } \Omega, \\ -\Delta v &= \lambda g(u) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned} \right\}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 1$ , and establish an existence result when  $\lambda$  is small. The main motivation of this paper is to extend this study in the case of exterior domains (see Theorem 1.1).

We also discuss a non-existence result for the single equation model:

$$\left. \begin{aligned} -\Delta u &= \lambda K_1(|x|)\tilde{f}(u) && \text{in } \Omega_e, \\ u(x) &= 0 && \text{if } |x| = r_0 (> 0), \\ u(x) &\rightarrow 0 && \text{as } |x| \rightarrow \infty, \end{aligned} \right\} \quad (1.2)$$

for large values of  $\lambda$ , when  $\tilde{f}$ ,  $K_1$  satisfy the following hypotheses:

(H<sub>4</sub>)  $\tilde{f} \in C^1([0, \infty), \mathbb{R})$ ,  $\tilde{f}'(z) > 0$  for all  $z > 0$ ,  $\tilde{f}(0) < 0$ , and there exists  $m_0 > 0$  such that  $\lim_{z \rightarrow \infty} \frac{\tilde{f}(z)}{z} \geq m_0$ .

(H<sub>5</sub>) The weight function  $K_1 \in C^1([r_0, \infty), (0, \infty))$  is such that  $s^{-\frac{2(n-1)}{n-2}} K_1(r_0 s^{\frac{1}{2-n}})$  is decreasing for  $s \in (0, 1]$ .

We establish the following.

**Theorem 1.2** *Let  $(H_3)$ – $(H_5)$  hold. Then (1.2) has no nonnegative radial solution for  $\lambda \gg 1$ .*

We establish Theorem 1.2 by recalling various useful properties of solutions established in [13], where the authors prove a uniqueness result for  $\lambda \gg 1$  for such an equation in the case when  $\tilde{f}$  is sublinear at  $\infty$ . However, the properties we recall from [13] are independent of the growth behavior of  $\tilde{f}$  at  $\infty$ . Non-existence results for such superlinear semipositone problems on bounded domain also have a considerable history starting from the work in the 1980s in [14] leading to the recent work in [15]. Here we discuss such a result for the first time on exterior domains.

Finally, we note that the study of radial solutions  $(u(r), v(r))$  (with  $r = |x|$ ) of (1.1) corresponds to studying

$$\left. \begin{aligned} -(r^{n-1}u'(r))' &= \lambda r^{n-1}K_1(r)f(v(r)) & \text{for } r > r_0, \\ -(r^{n-1}v'(r))' &= \lambda r^{n-1}K_2(r)g(u(r)) & \text{for } r > r_0, \\ u(r) = v(r) &= 0 & \text{if } r = r_0 (> 0), \\ u(r) \rightarrow 0, v(r) &\rightarrow 0 & \text{as } r \rightarrow \infty, \end{aligned} \right\}$$

which can be reduced to the study of solutions  $(u(s), v(s)); s \in [0, 1]$  to the singular system:

$$\left. \begin{aligned} -u''(s) &= \lambda h_1(s)f(v(s)), & 0 < s < 1, \\ -v''(s) &= \lambda h_2(s)g(u(s)), & 0 < s < 1, \\ u(0) = u(1) &= 0, v(0) = v(1) = 0, \end{aligned} \right\} \quad (1.3)$$

via the Kelvin transformation  $s = (\frac{r}{r_0})^{2-n}$ , where  $h_i(s) = \frac{r_0^2}{(n-2)^2} s^{\frac{-2(n-1)}{(n-2)}} K_i(r_0 s^{\frac{1}{2-n}})$ ,  $i = 1, 2$  (see [16]).

**Remark 1.3** The assumption  $(H_3)$  implies that  $\lim_{s \rightarrow 0^+} h_i(s) = \infty$ , for  $i = 1, 2$ ,  $\hat{h} = \inf_{t \in (0,1)} \{h_1(t), h_2(t)\} > 0$ , and there exist  $d > 0, \eta \in (0, 1)$  such that  $h_i(s) \leq \frac{d}{s^\eta}$  for  $s \in (0, 1]$ , and for  $i = 1, 2$ . When in addition  $(H_5)$  is satisfied,  $h_1$  is decreasing in  $(0, 1]$ .

We will prove Theorem 1.1 in Section 2 by studying the singular system (1.3), and Theorem 1.2 in Section 3 by studying the corresponding single equation

$$\left. \begin{aligned} -u''(s) &= \lambda h_1(s)\tilde{f}(u(s)), & 0 < s < 1, \\ u(0) &= u(1) = 0. \end{aligned} \right\} \quad (1.4)$$

## 2 Existence result

We first establish some useful results for solutions to the system

$$\left. \begin{aligned} -u''(s) &= b_1 h_1(s)|v(s) + l|^{q_1}, & 0 < s < 1, \\ -v''(s) &= b_2 h_2(s)|u(s) + l|^{q_2}, & 0 < s < 1, \\ u(0) = u(1) &= 0, v(0) = v(1) = 0, \end{aligned} \right\} \quad (2.1)$$

where  $l \geq 0$  is a parameter. (Clearly, any solution  $(u_l, v_l)$  of (2.1) for  $l > 0$  must satisfy  $u_l(s) > 0, v_l(s) > 0$  for  $s \in (0, 1)$ . This is also true for any nontrivial solution when  $l = 0$ .) We prove the following.

### Lemma 2.1

- (i) *There exists  $l_0 > 0$  such that 2.1 has no solution if  $l \geq l_0$ .*
- (ii) *For each  $l \in [0, l_0)$ , there exists  $M > 0$  (independent of  $l$ ) such that if  $(u_l, v_l)$  is a solution of (2.1), then  $\max\{\|u_l\|_\infty, \|v_l\|_\infty\} \leq M$ .*

*Proof of (i)* Let  $\lambda_1 := \pi^2$ ,  $\phi_1 := \sin(\pi s)$ . Here  $\lambda_1$  is the principal eigenvalue and  $\phi_1$  a corresponding eigenfunction of  $-\phi''(s) = \lambda \phi(s)$  in  $(0, 1)$  with  $\phi(0) = 0 = \phi(1)$ . Let  $a > \frac{\lambda_1}{\sqrt{b_1 b_2} \hat{h}}$ ,  $c > 0$  be such that  $(s + l)^{q_i} \geq as - c$  for all  $s \geq 0$  and for  $i = 1, 2$ . Now let  $(u_l, v_l)$  be a solution of

(2.1). Multiplying (2.1) by  $\phi_1$  and integrating, we obtain

$$\lambda_1 \int_0^1 u_l \phi_1 ds = b_1 \int_0^1 h_1(s)(v_l + l)^{q_1} \phi_1 ds \geq b_1 \int_0^1 h_1(s)(av_l - c) \phi_1 ds$$

and

$$\lambda_1 \int_0^1 v_l \phi_1 ds = b_2 \int_0^1 h_2(s)(u_l + l)^{q_2} \phi_1 ds \geq b_2 \int_0^1 h_2(s)(au_l - c) \phi_1 ds.$$

By Remark 1.3,  $\hat{h} = \inf_{t \in (0,1)} \{h_1(t), h_2(t)\} > 0$ , and  $\|h_i\|_1 := \int_0^1 h_i(s) ds < \infty$  for  $i = 1, 2$ . Then from the above inequalities we obtain

$$\int_0^1 v_l \phi_1 ds \leq \frac{1}{ab_1 \hat{h}} \left( \lambda_1 \int_0^1 u_l \phi_1 ds + b_1 c \|h_1\|_1 \right)$$

and

$$\int_0^1 u_l \phi_1 ds \leq \frac{1}{ab_2 \hat{h}} \left( \lambda_1 \int_0^1 v_l \phi_1 ds + b_2 c \|h_2\|_1 \right).$$

Hence we deduce that

$$\int_0^1 u_l \phi_1 ds \leq \frac{m_1}{m} := m_2,$$

where  $m := (ab_2 \hat{h} - \frac{\lambda_1^2}{ab_1 \hat{h}})$ , and  $m_1 := \frac{\lambda_1 c \|h_1\|_1}{a \hat{h}} + b_2 c \|h_2\|_1$ . This implies

$$\int_0^1 (v_l + l)^{q_1} \phi_1 ds \leq \frac{\lambda_1 m_2}{b_1 \hat{h}} := m_3.$$

In particular, this implies  $\int_{\frac{1}{4}}^{\frac{3}{4}} l^{q_1} ds \leq \frac{m_3}{\inf_{[\frac{1}{4}, \frac{3}{4}]} \phi_1}$ . Since  $m_3$  is independent of  $l$ , clearly this is a contradiction for  $l \gg 1$ , and hence there must exist an  $l_0 > 0$  such that for  $l \geq l_0$ , (2.1) has no solution.

*Proof of (ii)* Assume the contrary. Then without loss of generality we can assume there exists  $\{l_n\} \subset (0, l_0)$  such that  $\|u_{l_n}\|_\infty \rightarrow \infty$  as  $n \rightarrow \infty$ . Clearly  $u''_{l_n}(s) < 0$ , and  $v''_{l_n}(s) < 0$  for all  $s \in (0, 1)$ . Let  $s_{1(l_n)} \in (0, 1)$ ,  $s_{2(l_n)} \in (0, 1)$  be the points at which  $u_{l_n}$  and  $v_{l_n}$  attain their maximums. Now since  $u''_{l_n}(s) < 0$  for all  $s \in (0, 1)$ , we have

$$u_{l_n}(s) \geq \begin{cases} \frac{s u_{l_n}(s_{1(l_n)})}{s_{1(l_n)}} & \text{for } s \in (0, s_{1(l_n)}), \\ \frac{(1-s) u_{l_n}(s_{1(l_n)})}{1-s_{1(l_n)}} & \text{for } s \in (s_{1(l_n)}, 1). \end{cases}$$

Hence  $u_{l_n}(s) \geq \min \left\{ \frac{s \|u_{l_n}\|_\infty}{s_{1(l_n)}}, \frac{(1-s) \|u_{l_n}\|_\infty}{1-s_{1(l_n)}} \right\}$ , and in particular, for  $s \in [\frac{1}{4}, \frac{3}{4}]$ ,

$$u_{l_n}(s) \geq \min \left\{ \frac{1}{4} \|u_{l_n}\|_\infty, \frac{1}{4} \|u_{l_n}\|_\infty \right\} = \frac{1}{4} \|u_{l_n}\|_\infty.$$

Let  $\tilde{s}_{l_n}, \bar{s}_{l_n} \in [\frac{1}{4}, \frac{3}{4}]$  be such that  $\min_{[\frac{1}{4}, \frac{3}{4}]} u_{l_n}(s) = u_{l_n}(\tilde{s}_{l_n})$ , and  $\min_{[\frac{1}{4}, \frac{3}{4}]} v_{l_n}(s) = v_{l_n}(\bar{s}_{l_n})$ . Now for  $s \in [\frac{1}{4}, \frac{3}{4}]$ ,

$$v_{l_n}(s) \geq b_2 \hat{h} \tilde{m} \int_{\frac{1}{4}}^{\frac{3}{4}} |u_{l_n}(t) + l|^{q_2} dt,$$

where  $\tilde{m} := \min_{[\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]} G(s, t) (> 0)$ , and  $G$  is the Green's function of  $-Z''$  with  $Z(0) = 0 = Z(1)$ . In particular,  $v_{l_n}(\bar{s}_{l_n}) \geq b_2 \hat{h} \frac{\tilde{m}}{2} (u_{l_n}(\tilde{s}_{l_n}))^{q_2}$ . Similarly  $u_{l_n}(\tilde{s}_{l_n}) \geq b_1 \hat{h} \frac{m}{2} (v_{l_n}(\bar{s}_{l_n}))^{q_1}$ . Hence, there exists a constant  $A > 0$  such that

$$u_{l_n}(\tilde{s}_{l_n}) \geq A (u_{l_n}(\tilde{s}_{l_n}))^{q_1 q_2}.$$

This is a contradiction since  $q_1 q_2 > 1$  and  $u_{l_n}(\tilde{s}_{l_n}) \geq \frac{1}{4} \|u_{l_n}\|_{\infty} \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus (ii) holds.  $\square$

*Proof of Theorem 1.1* We first extend  $f$  and  $g$  as even functions on  $\mathbb{R}$  by setting  $f(-s) = f(s)$  and  $g(-s) = g(s)$ . Then we use the rescaling,  $\lambda = \gamma^\delta$ ,  $w_1 = \gamma u$ , and  $w_2 = \gamma^\theta v$  with  $\gamma > 0$ ,  $\theta = \frac{q_2+1}{q_1+1}$ , and  $\delta = \frac{q_1 q_2 - 1}{q_1 + 1}$ . With this rescaling, (1.3) reduces to

$$\begin{cases} -w_1''(s) = F(s, \gamma, w_2), & 0 < s < 1, \\ -w_2''(s) = G(s, \gamma, w_1), & 0 < s < 1, \\ w_1(0) = w_1(1) = 0, \quad w_2(0) = w_2(1) = 0, \end{cases} \quad (2.2)$$

where

$$\begin{aligned} F(s, \gamma, w_2) &:= \gamma^{1+\delta} h_1(s) \left( f\left(\frac{w_2}{\gamma^\theta}\right) - b_1 \left| \frac{w_2}{\gamma^\theta} \right|^{q_1} \right) + b_1 |w_2|^{q_1} h_1(s), \quad \text{and} \\ G(s, \gamma, w_1) &:= \gamma^{\theta+\delta} h_2(s) \left( g\left(\frac{w_1}{\gamma}\right) - b_2 \left| \frac{w_1}{\gamma} \right|^{q_2} \right) + b_2 |w_1|^{q_2} h_2(s). \end{aligned}$$

Note that by our hypothesis (H<sub>2</sub>),  $F(s, \gamma, w_2) \rightarrow b_1 |w_2|^{q_1} h_1(s)$  and  $G(s, \gamma, w_1) \rightarrow b_2 |w_1|^{q_2} \times h_2(s)$  as  $\gamma \rightarrow 0$ . Hence we can continuously extend  $F(s, \gamma, w_2)$  and  $G(s, \gamma, w_1)$  to  $F(s, 0, w_2) = b_1 |w_2|^{q_1} h_1(s)$  and  $G(s, 0, w_1) = b_2 |w_1|^{q_2} h_2(s)$ , respectively. Note that proving (1.3) has a positive solution for  $\lambda$  small is equivalent to proving (2.2) has a solution  $(w_1, w_2)$  with  $w_1 > 0$ ,  $w_2 > 0$  in  $(0, 1)$  for small  $\gamma > 0$ . We will achieve this by establishing that the limiting equation (when  $\gamma = 0$ )

$$\begin{cases} -w_1''(s) = F(s, 0, w_2) = b_1 h_1(s) |w_2|^{q_1}, & 0 < s < 1, \\ -w_2''(s) = G(s, 0, w_1) = b_2 h_2(s) |w_1|^{q_2}, & 0 < s < 1, \\ w_1(0) = w_1(1) = 0, \quad w_2(0) = w_2(1) = 0 \end{cases} \quad (2.3)$$

(which is the same as (2.1) with  $l = 0$ ) has a positive solution  $w_1 > 0$ ,  $w_2 > 0$  in  $(0, 1)$  that persists for small  $\gamma > 0$ .

Let  $X = C_0[0, 1] \times C_0[0, 1]$  be the Banach space equipped with  $\|\underline{w}\|_X = \|(w_1, w_2)\|_X = \max\{\|w_1\|_{\infty}, \|w_2\|_{\infty}\}$ , where  $\|\cdot\|_{\infty}$  denotes the usual supremum norm in  $C_0([0, 1])$ . Then for fixed  $\gamma \geq 0$ , we define the map  $S(\gamma, \cdot) : X \rightarrow X$  by

$$S(\gamma, \underline{w}) := \underline{w} - (K(F(s, \gamma, w_2)), K(G(s, \gamma, w_1))),$$

where  $K(H(s, \gamma, Z(s))) = \int_0^1 G(t, s)H(t, \gamma, Z(t)) dt$ . Note that  $F(s, \gamma, \cdot), G(s, \gamma, \cdot) : C_0([0, 1]) \rightarrow L^1(0, 1)$  are continuous and  $K : L^1(0, 1) \rightarrow C_0^1([0, 1])$  is compact. Hence  $S(\gamma, \cdot)$  is a compact perturbation of the identity. Clearly for  $\gamma > 0$ , if  $S(\gamma, \underline{w}) = \underline{0}$ , then  $\underline{w} = (w_1, w_2)$  is a solution of (2.2), and if  $S(0, \underline{w}) = \underline{0}$ , then  $\underline{w} = (w_1, w_2)$  is a solution of (2.3).

We first establish the following.

**Lemma 2.2** *There exists  $R > 0$  such that  $S(0, \underline{w}) \neq \underline{0}$  for all  $\underline{w} = (w_1, w_2) \in X$  with  $\|\underline{w}\|_X = R$  and  $\deg(S(0, \cdot), B_R(\underline{0}), \underline{0}) = 0$ .*

*Proof* Define  $S^l(0, \underline{w}) : X \rightarrow X$  by

$$S^l(0, \underline{w}) := \underline{w} - (K(b_1 h_1(s)|w_2 + l|^{q_1}), K(b_2 h_2(s)|w_1 + l|^{q_2}))$$

for  $l \geq 0$ . (Note  $S^0(0, \underline{w}) = S(0, \underline{w})$ .) By Lemma 2.1, if  $l \geq l_0$  then  $S^l(0, \underline{w}) \neq \underline{0}$  and if  $S^l(0, \underline{w}) = \underline{0}$  for  $l \in [0, l_0]$ , then  $\|\underline{w}\|_X \leq M$ . This implies that there exists  $R \gg 1$  such that  $S^l(0, \underline{w}) \neq \underline{0}$  for  $\underline{w} \in \partial B_R(\underline{0})$  for any  $l \geq 0$ . Also, since (2.1) has no solution for  $l \geq l_0$ ,  $\deg(S^{l_0}(0, \cdot), B_R(\underline{0}), \underline{0}) = 0$ . Hence, using the homotopy invariance of degree with the parameter  $l \in [0, l_0]$  we get

$$\deg(S(0, \cdot), B_R(\underline{0}), \underline{0}) = \deg(S^{l_0}(0, \cdot), B_R(\underline{0}), \underline{0}) = 0. \quad \square$$

Next we establish the following.

**Lemma 2.3** *There exists  $r \in (0, R)$  small enough such that  $S(0, \underline{w}) \neq \underline{0}$  for all  $\underline{w} = (w_1, w_2) \in X$  with  $\|\underline{w}\|_X = r$  and  $\deg(S(0, \cdot), B_r(\underline{0}), \underline{0}) = 1$ .*

*Proof* Define  $T^\tau(0, \underline{w}) : X \rightarrow X$  by

$$T^\tau(0, \underline{w}) := \underline{w} - (K(\tau b_1 h_1(s)|w_2|^{q_1}), K(\tau b_2 h_2(s)|w_1|^{q_2}))$$

for  $\tau \in [0, 1]$ . Clearly  $T^1(0, \underline{w}) = S(0, \underline{w})$ , and  $T^0(0, \underline{w}) = I$  is the identity operator. Note that  $T^\tau(0, \underline{w}) = \underline{0}$  if  $\underline{w} = (w_1, w_2)$  is a solution of

$$\left. \begin{aligned} -w_1''(s) &= \tau b_1 h_1(s)|w_2|^{q_1}, & 0 < s < 1, \\ -w_2''(s) &= \tau b_2 h_2(s)|w_1|^{q_2}, & 0 < s < 1, \\ w_1(0) &= w_1(1) = 0, & w_2(0) &= w_2(1) = 0, \end{aligned} \right\} \quad (2.4)$$

and for  $\tau = 1$ , (2.4) coincides with (2.3). Assume to the contrary that (2.4) has a solution  $\underline{w} = (w_1, w_2)$  with  $\|\underline{w}\|_X = \tilde{r} > 0$ . Without loss of generality assume  $\|w_1\|_\infty = \tilde{r}$ . Now,

$$w_1(s) = \tau \int_0^1 G(s, t) b_1 h_1(s) |w_2|^{q_1} ds.$$

Then  $\|w_1\|_\infty \leq \tilde{C} \|w_2\|_\infty^{q_1}$  for some constant  $\tilde{C} > 0$  independent of  $\tau \in [0, 1]$ . Similarly  $\|w_2\|_\infty \leq \hat{C} \|w_1\|_\infty^{q_2}$  for some constant  $\hat{C} > 0$ . This implies that

$$\tilde{r} = \|w_1\|_\infty \leq C \|w_1\|_\infty^{q_1 q_2} = C \tilde{r}^{q_1 q_2}$$

for some constant  $C > 0$ . But  $q_1 q_2 > 1$ , and hence this is a contradiction if  $\tilde{r} > 0$  is small. Thus there exists small  $r > 0$  such that (2.4) has no solution  $\underline{w}$  with  $\|\underline{w}\|_X = r$  for all  $\tau \in [0, 1]$ . Now using the homotopy invariance of degree with the parameter  $\tau \in [0, 1]$ , in particular using the values  $\tau = 1$  and  $\tau = 0$ , we obtain

$$\deg(S(0, \cdot), B_r(\underline{0}), \underline{0}) = \deg(T^1(0, \cdot), B_r(\underline{0}), \underline{0}) = \deg(T^0(0, \cdot), B_r(\underline{0}), \underline{0}) = 1. \quad \square$$

By Lemma 2.2 and Lemma 2.3, with  $0 < r < R$ , we conclude that

$$\deg(S(0, \cdot), B_R(\underline{0}) \setminus \overline{B_r(\underline{0})}, \underline{0}) = -1,$$

and hence (2.3) has a solution  $\underline{w} = (w_1, w_2)$  with  $w_1 > 0$ ,  $w_2 > 0$  in  $(0, 1)$ , and  $r < \|\underline{w}\|_X < R$ . Now we show that the solution obtained above (when  $\gamma = 0$ ) persists for small  $\gamma > 0$  and remains positive componentwise.

**Lemma 2.4** *Let  $R, r$  be as in Lemmas 2.2, 2.3, respectively. Then there exists  $\gamma_0 > 0$  such that:*

- (i)  $\deg(S(\gamma, \cdot), B_R(\underline{0}) \setminus \overline{B_r(\underline{0})}, \underline{0}) = -1$  for all  $\gamma \in [0, \gamma_0]$ .
- (ii) If  $S(\gamma, \underline{w}) = \underline{0}$  for  $\gamma \in [0, \gamma_0]$  with  $r < \|\underline{w}\|_X < R$ , then  $w_1 > 0$ ,  $w_2 > 0$  in  $(0, 1)$ .

*Proof of (i)* We first show that there exists  $\gamma_0 > 0$  such that  $S(\gamma, \underline{w}) \neq \underline{0}$  for all  $\underline{w} = (w_1, w_2) \in X$  with  $\|\underline{w}\|_X \in [R, r]$ , for all  $\gamma \in [0, \gamma_0]$ . Suppose to the contrary that there exists  $\{\gamma_n\}$  with  $\gamma_n \rightarrow 0$ ,  $S(\gamma_n, \underline{w}_n) = \underline{0}$  and  $\|\underline{w}_n\|_X \in [r, R]$ . Since  $\underline{K} = (K, K) : L^1(0, 1) \times L^1(0, 1) \rightarrow C_0^1([0, 1]) \times C_0^1([0, 1])$  is compact, and  $\{F(s, \gamma_n, w_{2n}), G(s, \gamma_n, w_{1n})\}$  are bounded in  $L^1(0, 1) \times L^1(0, 1)$ ,  $\underline{w}_n \rightarrow \underline{Z} = (Z_1, Z_2) \in C_0^1([0, 1]) \times C_0^1([0, 1])$  (up to a subsequence) with  $\|\underline{Z}\|_X = R$  or  $r$  and  $S(0, \underline{Z}) = \underline{0}$ . This is a contradiction to Lemma 2.2 or 2.3 and hence there exists a small  $\gamma_0 > 0$  satisfying the assertions. Now, by the homotopy invariance of degree with respect to  $\gamma \in [0, \gamma_0]$ ,

$$\deg(S(\gamma, \cdot), B_R(\underline{0}) \setminus \overline{B_r(\underline{0})}, \underline{0}) = \deg(S(0, \cdot), B_R(\underline{0}) \setminus \overline{B_r(\underline{0})}, \underline{0}) = -1$$

for all  $\gamma \in [0, \gamma_0]$ .

*Proof of (ii)* Assume to the contrary that there exists  $\gamma_n \rightarrow 0$  and a corresponding solution  $\underline{w}_n = (w_{1n}, w_{2n})$  such that  $r < \|\underline{w}_n\|_X < R$  and

$$\Omega_n := \{x \in (0, 1) \mid w_{1n}(x) \leq 0 \text{ or } w_{2n}(x) \leq 0\} \neq \emptyset.$$

Arguing as before,  $\underline{w}_n \rightarrow \underline{Z} \in C_0^1([0, 1]) \times C_0^1([0, 1])$  with  $S(0, \underline{Z}) = \underline{0}$  (up to a subsequence). Note that  $\underline{Z} \neq \underline{0}$  since  $\|\underline{Z}\|_X \geq r > 0$ . By the strong maximum principle  $Z_1 > 0$ ,  $Z_2 > 0$ ,  $Z_1'(0) > 0$ ,  $Z_2'(0) > 0$ ,  $Z_1'(1) < 0$  and  $Z_2'(1) < 0$ . Now suppose there exists  $\{x_n\} \in (0, 1)$  with  $\{x_n\} \in \Omega_n$  and  $w_{1n}(x_n) \leq 0$ . Then  $\{x_n\}$  must have a subsequence (renamed as  $\{x_n\}$  itself) such that  $x_n \rightarrow \tilde{x} \in [0, 1]$ . But  $Z_1 > 0$  in  $(0, 1)$  implies that  $\tilde{x} \in [0, 1]$ . Suppose  $\tilde{x} = 0$ . Since  $w_{1n}(x_n) \leq 0$  and  $w_{1n}(0) = 0$ , there exists  $y_n \in (0, x_n)$  such that  $w_{1n}'(y_n) \leq 0$ , and hence taking the limit as  $n \rightarrow \infty$  we will have  $Z_1'(0) \leq 0$ , which is a contradiction since  $Z_1'(0) > 0$ . A similar contradiction follows if  $\tilde{x} = 1$ , using the fact that  $Z_1'(1) < 0$ . Further, contradictions can



be achieved if there exists  $\{x_n\} \in \Omega$  with  $\{x_n\} \in \Omega_n$  and  $w_{2n}(x_n) \leq 0$  using the facts that  $Z'_2(0) > 0$  and  $Z'_2(1) < 0$ . This completes the proof of the lemma.  $\square$

We now easily conclude the proof of Theorem 1.1. From Lemma 2.4, since  $\underline{w} = (w_1, w_2)$  is a positive solution of (2.2) for  $\gamma$  small,  $(u, v) = (\gamma^{-1}w_1, \gamma^{-\theta}w_2)$  with  $\theta = \frac{q_2+1}{q_1+1}$  is a positive solution of (1.3) for  $\lambda = \gamma^\delta$  where  $\delta = \frac{q_1q_2-1}{q_1+1}$ . Further, since  $w_1 > 0$  and  $w_2 > 0$  in  $(0, 1)$  for  $\gamma \in [0, \gamma_0]$ ,  $\|u\|_\infty \rightarrow \infty$  and  $\|v\|_\infty \rightarrow \infty$  as  $\lambda (= \gamma^\delta) \rightarrow 0$ . This completes the proof of Theorem 1.1.  $\square$

### 3 Non-existence result

We first recall from [13] that, when  $(H_5)$  is satisfied, one can prove via an energy analysis that a nonnegative solution  $u$  of (1.4) must be positive in  $(0, 1)$  and have a unique interior maximum with maximum value greater than  $\theta$ , where  $\theta$  is the unique positive zero of  $\tilde{F}(s) = \int_0^s \tilde{f}(y) dy$ . Further, for  $\lambda \gg 1$  and  $s_1, \hat{s}_1 \in (0, 1)$  such that  $\hat{s}_1 > s_1$ ,  $u(s_1) = u(\hat{s}_1) = \beta$  (see Figure 1), where  $\beta > 0$  is the unique zero of  $\tilde{f}$ , there exists a constant  $C$  such that  $s_1 \leq C\lambda^{-\frac{1}{2}}$  and  $(1 - \hat{s}_1) \leq C\lambda^{-\frac{1}{2}}$ . Hence we can assume  $(\hat{s}_1 - s_1) > \frac{1}{2}$  for  $\lambda \gg 1$ . Now we provide the proof of Theorem 1.2.

*Proof of Theorem 1.2* Let  $v := u - \beta$ . Then  $v > 0$  in  $(s_1, \hat{s}_1)$  and satisfies

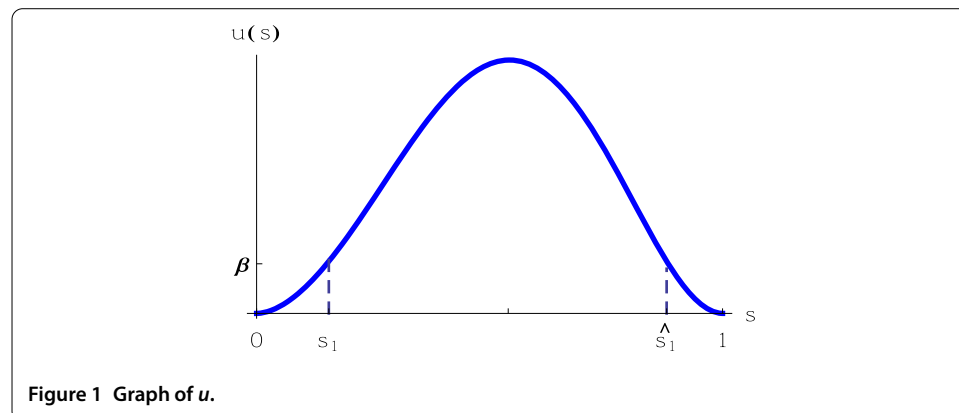
$$\left. \begin{aligned} -v'' &= \lambda h_1(s) \frac{\tilde{f}(u)}{u-\beta} v, \quad s_1 < s < \hat{s}_1, \\ v(s_1) &= v(\hat{s}_1) = 0. \end{aligned} \right\}$$

Note that  $\phi(s) = -(\sin(\frac{\pi(s-s_1)}{(\hat{s}_1-s_1)})) > 0$  in  $(s_1, \hat{s}_1)$ ,  $\phi(s_1) = \phi(\hat{s}_1) = 0$ , and it satisfies  $-\phi'' = \frac{\pi^2}{(\hat{s}_1-s_1)^2} \phi$  in  $(s_1, \hat{s}_1)$ . Hence using the fact that  $\int_{s_1}^{\hat{s}_1} (-\phi v'' + v \phi'') ds = 0$ , we obtain

$$\int_{s_1}^{\hat{s}_1} \left( \lambda \frac{\tilde{f}(u)}{u-\beta} h_1(s) - \frac{\pi^2}{(\hat{s}_1-s_1)^2} \right) v \phi ds = 0.$$

In particular,

$$\lambda \frac{\tilde{f}(u(s_\lambda))}{u(s_\lambda) - \beta} h_1(s_\lambda) = \frac{\pi^2}{(\hat{s}_1 - s_1)^2}, \quad \text{for some } s_\lambda \in (s_1, \hat{s}_1). \quad (3.1)$$



But  $\hat{h} = \inf_{(0,1)} h_1(s) > 0$ , and  $(\hat{s}_1 - s_1) > \frac{1}{2}$  for  $\lambda \gg 1$ . Thus clearly (3.1) can hold when  $\lambda \rightarrow \infty$ , only if  $Z = u(s_\lambda) \rightarrow \infty$  with  $\frac{\tilde{f}(u(s_\lambda))}{u(s_\lambda) - \beta} \rightarrow 0$ . But by  $(H_4)$ , this is not possible since  $\lim_{Z \rightarrow \infty} \frac{\tilde{f}(Z)}{Z} \geq m_0 > 0$ . Hence the nonnegative solution cannot exist for  $\lambda \gg 1$ .  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

#### Author details

<sup>1</sup>Department of Mathematics and Statistics, The University of North Carolina at Greensboro, Greensboro, USA.

<sup>2</sup>Department of Mathematics & NTIS, University of West Bohemia, Univerzitní 22, Plzeň, 30614, Czech Republic.

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#### References

1. Ambrosetti, A, Arcoya, D, Buffoni, B: Positive solutions for some semi-positone problems via bifurcation theory. *Differ. Integral Equ.* **7**(3-4), 655-663 (1994)
2. Maya, C, Girg, P: Existence and nonexistence of positive solutions for a class of superlinear semipositone systems. *Nonlinear Anal.* **71**(10), 4984-4996 (2009)
3. Berestycki, H, Caffarelli, LA, Nirenberg, L: Inequalities for second-order elliptic equations with applications to unbounded domains. I. *Duke Math. J.* **81**(2), 467-494 (1996). A celebration of John F. Nash, Jr
4. Lions, P-L: On the existence of positive solutions of semilinear elliptic equations. *SIAM Rev.* **24**(4), 441-467 (1982)
5. Brown, KJ, Shivaji, R: Simple proofs of some results in perturbed bifurcation theory. *Proc. R. Soc. Edinb., Sect. A* **93**(1-2), 71-82 (1982/1983)
6. Castro, A, Shivaji, R: Nonnegative solutions for a class of nonpositone problems. *Proc. R. Soc. Edinb., Sect. A* **108**(3-4), 291-302 (1988)
7. Castro, A, Shivaji, R: Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric. *Commun. Partial Differ. Equ.* **14**(8-9), 1091-1100 (1989)
8. Hai, DD, Sankar, L, Shivaji, R: Infinite semipositone problems with asymptotically linear growth forcing terms. *Differ. Integral Equ.* **25**(11-12), 1175-1188 (2012)
9. Lee, EK, Sankar, L, Shivaji, R: Positive solutions for infinite semipositone problems on exterior domains. *Differ. Integral Equ.* **24**(9-10), 861-875 (2011)
10. Lee, EK, Shivaji, R, Ye, J: Classes of infinite semipositone systems. *Proc. R. Soc. Edinb., Sect. A* **139**(4), 853-865 (2009)
11. Sankar, L, Sasi, S, Shivaji, R: Semipositone problems with falling zeros on exterior domains. *J. Math. Anal. Appl.* **401**(1), 146-153 (2013)
12. Maya, C, Girg, P: Existence of positive solutions for a class of superlinear semipositone systems. *J. Math. Anal. Appl.* **408**(2), 781-788 (2013)
13. Castro, A, Sankar, L, Shivaji, R: Uniqueness of nonnegative solutions for semipositone problems on exterior domains. *J. Math. Anal. Appl.* **394**(1), 432-437 (2012)
14. Brown, KJ, Castro, A, Shivaji, R: Nonexistence of radially symmetric nonnegative solutions for a class of semi-positone problems. *Differ. Integral Equ.* **2**(4), 541-545 (1989)
15. Shivaji, R, Ye, J: Nonexistence results for classes of  $3 \times 3$  elliptic systems. *Nonlinear Anal.* **74**(4), 1485-1494 (2011)
16. Ko, E, Lee, EK, Shivaji, R: Multiplicity results for classes of singular problems on an exterior domain. *Discrete Contin. Dyn. Syst.* **33**(11-12), 5153-5166 (2013)

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