PAYNE, CATHERINE, Ph.D. Approximation of Neutral Delay-Differential Equations. (2017)
Directed by Dr. Richard Fabiano. 110 pp.

We study questions related to stability and approximation of the system of linear neutral delay differential equations

$$\frac{d}{dt}\left[x(t) + \sum_{k=1}^{n} C_k x(t - r_k)\right] = Ax(t) + \sum_{k=1}^{n} B_k x(t - r_k)$$

with appropriate initial data. Here A, $B_1, B_2, \ldots, B_n$ and $C_1, C_2, \ldots, C_n$ are complex $m \times m$ matrices for a natural number m and $r_1, r_2, \ldots, r_n$ are positive numbers. We construct a new delay-independent sufficient condition for exponential stability of the solution semigroup associated with this equation. We obtain our condition by using the idea of renorming the state space to obtain a strong dissipative inequality on the generator of the solution semigroup. We also construct a new semidiscrete approximation scheme which yields convergence for both the solution semigroup and its adjoint. Finally, we discuss several examples to compare our results with existing results in the literature.

APPROXIMATION OF NEUTRAL DELAY-DIFFERENTIAL EQUATIONS

by

Catherine Payne

A Dissertation Submitted to
the Faculty of The Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Greensboro
2017

Approved by

____
Committee Chair

APPROVAL PAGE

This dissertation written by Catherine Payne has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

Committee Chair ______________________
Richard Fabiano

Committee Members ______________________
Maya Chhetri

____
Thomas Lewis

____
Ratnasingham Shivaji

____
Date of Acceptance by Committee

____
Date of Final Oral Examination

ACKNOWLEDGMENTS

I would like to thank my advisor, Richard Fabiano, for all of his help, support, and encouragement along my journey of writing this dissertation. Thanks also go to John Burns, who made time to read drafts and provide very helpful feedback. I greatly appreciate the other members of my committee Ratnasingham Shivaji, Maya Chhetri, and Tom Lewis, who provided suggestions on this document and a lot of support and help throughout my time at UNCG. Finally, a huge thank you to all of my family and friends who have become family–those who kept me going when I wasn't sure this would ever be finished.

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CHAPTER I

INTRODUCTION AND NOTATION

1.1 Introduction and Semigroup Formulation

A delay differential equation is a differential equation involving a function at a present time t as well as some past times. The terms involving past times are called delay terms. Such equations arise as models for applications in which the behavior of some physical mechanism depends natrually on lags or on past times. We refer to [Nic01], [Smi11], and [CL07] for examples of such applications. While these applications can lead to a variety of general delay equation models, including nonlinear models, partial delay differential equations, and equations with time-varying delays, in the present thesis we focus on the system of linear neutral delay differential equations of the form

$$\frac{d}{dt}\left[x(t)+\sum_{k=1}^{n} C_k x(t-r_k)\right] = Ax(t)+\sum_{k=1}^{n} B_k x(t-r_k), \tag{1.1}$$

with initial data

$$x(0)+\sum_{k=1}^{n} C_k x(-r_k) = \eta_0 \tag{1.2}$$

$$x(\theta) = f_0(\theta) \text{ on } [-r_n, 0)\,. \tag{1.3}$$

for given $\eta_0 \in \mathbb{C}^m$ and $f_0 \in L^2([-r_n, 0), \mathbb{C}^m)$. Here $B_k, C_k, A \in \mathbb{C}^{m\times m}$ for $k = 1, 2, \ldots, n$, and $0 = r_0 < r_1 < \cdots < r_n$. We refer to the points $-r_k$ as the delay points.

There are two main kinds of delay differential equations. Retarded delay equations contain delays only in the state (the terms $x(t - r_k)$), while neutral delay equations may also contain delays in the derivative (the terms $\frac{d}{dt}x(t - r_k)$). Since delay equations involve the past history of a function, an initial condition for a delay differential equation needs two parts: the function value at an initial point as well as function values on some past time interval. Generally speaking, the theory for neutral delay differential equations is a nontrivial extension of the theory for retarded delay differential equations (see [Hal77]). We are interested in two main problems:

(1) What conditions on the matrices in equation (1.1) guarantee that solutions will be exponentially stable?

(2) How can solutions of equation (1.1) be approximated?

Our approach to both problems is to reformulate the delay equation as a Cauchy problem $\dot{z}(t) = \mathcal{A}z(t)$ on an appropriate space and then use semigroup theory to address the questions related to stability and approximation. To proceed, we first recall the ideas in [BHS83] on how to reformulate the neutral equation as a Cauchy problem on the Hilbert space

$$M_2([-r_n, 0], \mathbb{C}^m) = \mathbb{C}^m \times L_2([-r_n, 0], \mathbb{C}^m),$$

although other choices are possible. This space has a nice structure for studying control problems related to neutral equations, and it has the advantage that much of the semigroup theory related to delay differential equations is well-established in M_2. We can reformulate the neutral differential equation (1.1) as

$$\begin{aligned} \dot{z}(t) &= \mathcal{A}z(t) \\ z(0) &= z_0 = (\eta_0, f_0(\theta)) \end{aligned} \tag{1.4}$$

on M_2, where $z(t) = (x(t) + \sum_{k=1}^{n} C_k x(t - r_k), x(t + \theta))$ and $z(0) = (\eta_0, f_0)$. Observe that

$$\begin{aligned} \mathcal{A}z(t) &= \dot{z}(t) \\ &= \left(\frac{d}{dt} \left(x(t) + \sum_{k=1}^{n} C_k x(t - r_k) \right), \frac{d}{dt} x(t + \theta) \right) \\ &= \left(Ax(t + 0) + \sum_{k=1}^{n} B_k x(t - r_k), \frac{d}{dt} x(t + \theta) \right), \end{aligned}$$

so we take $\mathcal{A}$ to be the operator

$$\mathcal{A}(\eta, f) = \left(Af(0) + \sum_{k=1}^{n} B_k f(-r_k), f' \right) \tag{1.5}$$

with domain

$$D(\mathcal{A}) = \left\{ (\eta, f) \in M_2 : f \in H^1([-r_n, 0], \mathbb{C}^m), \eta = f(0) + \sum_{k=1}^{n} C_k f(-r_k) \right\}.$$

It is known that $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ (see [BHS83] for details). We note that the adjoint operator for the retarded case is known (see

[Kat76], [NT96], [KS87], and [DM80]), but we did not find in the literature an explicit representation of the adjoint $\mathcal{A}^*$ for the general neutral case, and thus one of our first contributions will be to explicitly describe the adjoint operator.

1.2 The Stability Problem

With this semigroup formulation, our first problem can be restated as finding sufficient delay-independent conditions on the matrices in equation (1.1) to guarantee exponential stability of the solution semigroup. Although some results in the literature require the delays to be commensurable, our results will hold for incommensurable delays also. We will first address the history of problems related to stability of solutions of delay differential equations.

We recall that associated with (1.1) is the characteristic equation

$$\Delta(\lambda) \equiv \det[\lambda I - A - \sum_{i=1}^{n}(B_i e^{-\lambda r_i} + \lambda C_i e^{-\lambda r_i})] = 0. \qquad (1.6)$$

It is known (see [HL93, Corrollary 9.3.1] and [MN07, Proposition 1.20]) that (1.1) is asymptotically stable if

$$\sup\{\operatorname{Re}\lambda \,:\, \Delta(\lambda) = 0\} < 0. \qquad (1.7)$$

One standard approach to studying exponential stability of (1.1) is to instead study the roots of the characteristic equation (1.6) and verify (1.7). An early result using this approach is [BW67], although it is for the somewhat restrictive case of a single delay with the matrices A, B, and C all symmetric. In the research literature since then, this result has been steadily improved and generalized. This has led to progressively less restrictive sufficient conditions for stability of (1.1), for example in [Mor85], [Li88], [HH96], [HH97], and [HHC01]. Below we shall discuss several of these improved sufficient conditions in more detail.

A second standard approach to studying asymptotic stability of (1.1) is to apply Liapunov methods or Razumikhin type theorems. This approach typically leads to linear matrix inequality (LMI) sufficient conditions, such as those found in [Bli02], [GKC03], and [PPL05]. We will use a third approach in which sufficient conditions are derived to guarantee a strong dissipative inequality for the infinitesimal generator of the solution semigroup associated with (1.1), which in turn guarantees exponential stability of the semigroup and hence asymptotic stability of (1.1). The conditions we derive compare naturally with those derived using the first approach.

By exponential stability for the semigroup $T(t)$, we mean that there exist positive constants M and α such that

$$\|T(t)\| \leq Me^{-\alpha t} \tag{1.8}$$

for $t \geq 0$. If this condition holds, then the solution $z(t)$ of the Cauchy problem has the property that $z(t) \to 0$ as $t \to \infty$, and the convergence is exponentially fast (see Chapter 5 in [CZ95]). It is also known that the solution semigroup $T(t)$ satisfies the so-called spectrum determined growth condition, and that the spectrum of its infinitesimal generator $\mathcal{A}$ consists only of eigenvalues, which are precisely the roots of the characteristic equation (1.6). Thus $T(t)$ is exponentially stable if and only if (1.7) holds. Our approach, then, is to derive sufficient conditions which guarantee that $T(t)$ is exponentially stable, thus verifying (1.7) indirectly and without analysis of (1.6). The basic idea of our approach is to construct a new inner product norm $\|\cdot\|_w$ on M_2 which is equivalent to the usual (unweighted) norm and for which there exists $\omega < 0$ such that

$$\operatorname{Re}\langle \mathcal{A}(\eta, f), (\eta, f)\rangle_w \leq \omega \|(\eta, f)\|_w^2 \tag{1.9}$$

for all $(\eta, f) \in D(\mathcal{A})$. Condition (1.9) then implies there exists $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, which implies $T(t)$ is exponentially stable (note this also gives ω as an estimate of the decay rate). We point out an important motivation of our approach is that the norm which satisfies (1.9) can be used to construct semidiscrete Galerkin approximation schemes which preserve exponential stability uniformly in the discretization parameter. This stability, along with a connection to the adjoint operator, is sufficient to guarantee convergence of the feedback gain in LQR control problems.

There are a number of sufficient conditions for exponential stability of neutral equations already described in the literature. However, there is no clear best condition to date. All the conditions we study require that $\sum_{i=1}^{n} \|C_i\| < 1$ (or $\|C\| < 1$ where the paper studies only single delay problems). Many of them also involve the matrix measure $\mu(A)$, where

$$\mu(A) = \lim_{\theta \to 0} \frac{\|I + \theta A\| - 1}{\theta} \tag{1.10}$$

Properties of the matrix measure can be found in [DV75]. We will use only the matrix measure in the 2 norm, which is given by

$$\mu_2(A) = \max_i \left[\frac{\lambda_i \left(A + A^*\right)}{2} \right], \tag{1.11}$$

where $\lambda_i(A + A^*)$ is the ith eigenvalue of the Hermetian matrix $A + A^*$. We will compare our results against the following conditions:

- The result of Li in [Li88] (which is only for the single delay case): Equation (1.1) is asymptotically stable if $\|C\| < 1$ and

$$\mu(A) + \frac{\|B\| + \|A\| \, \|C\|}{1 - \|C\|} < 0. \tag{1.12}$$

- The result of Hu and Hu in [HH97]: Equation (1.1) is asymptotically stable if $\|C\| < 1$ and

$$\mu(A) + \sum_{i=1}^{n} \|B_i\| + \frac{\sum_{i=1}^{n} \|C_i A\| + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \|C_i B_j\| \right)}{1 - \sum_{i=1}^{n} \|C_i\|} < 0. \tag{1.13}$$

- A companion to the result of Hu and Hu. It is related to (1.13), and was derived in [FP]: Equation (1.1) is asymptotically stable if $\|C\| < 1$ and

$$\mu(A) + \sum_{i=1}^{n} \|B_i\| + \frac{\sum_{i=1}^{n} \|A C_i\| + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \|B_j C_i\| \right)}{1 - \sum_{i=1}^{n} \|C_i\|} < 0. \tag{1.14}$$

- The result of Fabiano and Turi in [FT07] (which is only for the single delay case): Let $H = \frac{1}{2}\left(A - \overline{A}^T\right)$ and $W = \frac{-1}{2}(A + \overline{A}^T)$. Equation (1.1) is asymptotically stable if there exists a number k such that $0 < k < |\mu(A)|$ and

$$W - C^T W C - \frac{1}{k} C^T H^T H C - \frac{1}{|\mu(A)| - k} B^T B > 0. \tag{1.15}$$

We note that condition (1.15) is easy to check because the left side of equation (1.15) is a self-adjoint matrix, so it suffices to check that all of its eigenvalues are positive. Also, condition (1.15) is valid only in the 2-norm because it was derived in a Hilbert space, whereas conditions (1.12), (1.13), and (1.14) hold in any norm. Practically speaking this means (1.12), (1.13), and (1.14) are also valid in the 1-norm and the ∞-norm, since these are the only norms besides the 2-norm for which the matrix measure is reasonable to compute.

Although there is no single clear best condition, there are some results for special cases. For example, it is known that in the single-delay case when $B = 0$ and A and C are both scalars, a necessary and sufficient condition for exponential stability of the solution semigroup is $A < 0$ and $|C| < 1$. In the single-delay case, both condition (1.12) and condition (1.13) require that $A < 0$ and $\|C\| < 1/2$. This is significantly more restrictive than $|C| < 1$. Our proposed condition attempts to partially correct this issue.

1.3 The Approximation Problem

We approximate solutions to the neutral delay-differential equation (1.1)–(1.2) by instead approximating solutions to the abstract formulation (1.4). By an approximation scheme for (1.1) we mean a sequence $\left\{X^N, \mathcal{A}^N\right\}_{N=1}^{\infty}$ consisting of finite-dimensional subspaces $X^N \subset X$ and linear operators $\mathcal{A}^N : X^N \to X^N$. Associated with such an approximation scheme are the orthogonal projections $P^N : X \to X^N$. Given an approximation scheme we then construct a sequence of finite-dimensional Cauchy problems

$$\dot{z}^N(t) = \mathcal{A}^N z^N(t) \tag{1.16}$$

$$z^N(0) = P^N z_0 \tag{1.17}$$

on X^N. The matrix representation for (1.16) is a large but finite system of linear ordinary differential equations which can be solved numerically. The approximation process is justified by proving a Trotter-Kato type of semigroup convergence. That is, we prove that $T^N(t)P^N \to T(t)$ strongly on X, uniformly in bounded t-intervals. Here $T^N(t)$ is the semigroup generated by $\mathcal{A}^N$ which, since X^N is finite-dimensional, is given by

$$T^N(t) = e^{\mathcal{A}^N t}. \tag{1.18}$$

The idea of using such semigroup-theoretic semidiscrete approximation schemes for delay equations goes back at least to the classic paper [BB78] of Banks and Burns. Theirs was the first paper to provide rigorous convergence results for this approach, and it paved the way for much subsequent research. If we recall that $X = M_2 = \mathbb{C}^m \times L^2((-r_n, 0), \mathbb{C}^m)$, it is clear that constructing subspaces X^N involves discretizing the function space $L^2(-r_n, 0)$. Thus X^N typically has the form $X^N = \mathbb{C}^m \times (Y^N)^m$, where Y^N is a finite-dimensional subspace of $L^2(-r_n, 0)$ and can therefore be expressed as the span of a finite set of basis functions. The index N is typically related to a mesh defined on $[-r_n, 0]$. In their classic paper, Banks and Burns used piecewise constant basis functions to define X^N and a finite difference approximation of the derivative in $\mathcal{A}$ to define $\mathcal{A}^N$. In their scheme the projection $P^N : X \to X^N$ essentially maps functions f in $L^2(-r_n, 0)$ to piecewise constant functions which are the average value of f over subintervals determined by the mesh. Consequently their method became known as the averaging approximation scheme (or AVE scheme) in the research literature. They proved convergence and used the scheme in several applications, including an optimal control problem, with good results.

At least partly because the AVE scheme is low order, in [BK79] Banks and Kappel developed higher order spline approximation schemes for delay equations. They used basis functions which are splines related to the mesh on $[-r_n, 0]$. However their particular spline scheme required the spaces X^N to be contained in the domain of $\mathcal{A}$, and they used the classical Galerkin idea to define the approximating operators by $\mathcal{A}^N = P^N \mathcal{A} P^N$. They proved semigroup convergence and demonstrated faster convergence rates than the AVE scheme, as expected. This faster convergence was observed by other researchers in applications involving simulation and parameter estimation for delay equations ([Ban81], [BBC81], [BD83]).

In a subsequent paper [BIR84] Banks, Ito, and Rosen applied both the spline and AVE scheme to the linear quadratic regulator (LQR) control problem for delay systems. They again observed faster convergence rates for the spline scheme, but for the approximation of the functional feedback gain the spline scheme yielded a (numerically observed) weaker convergence than the AVE scheme. (We observe related behavior for a similar problem in Chapter 5.)

Meanwhile other researchers were investigating convergence issues for semidiscrete approximation of general LQR control problems (not only those for delay equations). It was shown (see [Gib83]) that an approximation scheme $\left\{X^N, \mathcal{A}^N\right\}_{N=1}^{\infty}$ for such control problems should satisfy not only Trotter-Kato type semigroup convergence ($e^{\mathcal{A}^N t}P^N \to T(t)$ strongly), but also Trotter-Kato type convergence for the adjoint semigroup ($e^{\mathcal{A}^{N*} t}P^N \to T(t)^*$) as well as some form of exponential stability uniformly in the discretization parameter N. An example of the uniform stability property would be the existence of constants $M \geq 1$ and $\omega < 0$, independent of N, such that

$$\left\| e^{\mathcal{A}^N t} \right\| \leq M e^{\omega t} \tag{1.19}$$

for all $t \geq 0$. With these types of results in mind, it was conjectured that the Banks-Kappel spline scheme did not satisfy the adjoint semigroup convergence property. This conjecture seemed especially reasonable because it was known that the domains of $\mathcal{A}$ and $\mathcal{A}^*$ are not equal, and the Banks-Kappel subspaces X^N were constructed to be contained in the domain of $\mathcal{A}$. Subsequently Burns, Ito, and Propst proved in [BIP88] that the Banks-Kappel spline scheme did not have Trotter-Kato type convergence for the adjoint semigroup. We note that Gibson [Gib83] proved the AVE scheme does have the adjoint semigroup convergence property, and Salamon [Sal85] proved that the AVE scheme satisfies a uniform stability property.

Motivated by these developments, Kappel and Salamon [KS87] introduced a new spline-based approximation scheme for delay equations. They modified the Banks-Kappel basis functions so that the subspaces X^N contained sufficiently many elements of the domains of both $\mathcal{A}$ and $\mathcal{A}^*$. They proved their scheme satisfies Trotter-Kato convergence for both the semigroup and the adjoint semigroup. In a later paper [KS89], they proved their scheme has a type of uniform stability.

We are now in position to describe our contribution and its historical context. We note that all of the research cited in this section is for approximation of *retarded* delay equations. We point out that the state space formulation (1.4) on M_2, which is a required starting point for these approximation methods, was already known by Banks and Burns [BB78] for retarded delay equations, but it was only available for neutral delay equations after the work of Burns, Herdman, and Stech in [BHS83]. There has been some work extending the AVE scheme to neutral delay equations (see [BHZ13c],[BHZ13b],[BHZ13a]), as well as some spline approximations for neutral equations on spaces other than M_2 (see [KK81]). Our contribution will fill this gap by constructing an approximation scheme for neutral delay equations which has Trotter-Kato type convergence for both the semigroup and the adjoint semigroup. We will in fact be able to use the same basis functions as Kappel and Salamon, so our work can be viewed as the extension of their approximation scheme from retarded to neutral delay equations.

Some more recent work has been done to extend the scheme in [KS90] to neutral equations. In [Fab13], the scheme is extended to neutral equations, but the resulting scheme does not have adjoint convergence. Our work extends this scheme to neutral equations while maintaining the adjoint convergence. In order to do this, we must make several adjustments to the operators used to define the approximation scheme, and this is one of our main contributions. One of the main differences in our scheme and the one in [Fab13] is that we use splines which allow for the possibility of jumps at the delay points.

1.4 Renorming

The main idea behind our approach to the stability problem is to choose a new norm in which inequality (1.9) is satisfied. For the remainder of this discussion, any unsubscripted norm will refer to the usual Euclidean vector norm on $\mathbb{C}^m$ or its induced matrix norm on $m \times m$ matrices. To proceed, we recall that the usual norm on M_2 is given by

$$\|(\eta, f)\|_{M_2}^2 = \|\eta\|^2 + \int_{-r_n}^{0} \|f(\theta)\|^2 \, d\theta = \|\eta\|^2 + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \|f(\theta)\|^2 \, d\theta \tag{1.20}$$

with inner product

$$\langle (\eta, f), (\gamma, g) \rangle_{M_2} = \overline{\gamma}^T \eta + \int_{-r_n}^{0} \overline{g(\theta)}^T f(\theta) \, d\theta = \overline{\gamma}^T \eta + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T f(\theta) \, d\theta. \tag{1.21}$$

We use the following notation. Let

$$\chi_i(\theta) = \begin{cases} 1 & \text{if } \theta \in [-r_i, 0], \\ 0 & \text{otherwise} \end{cases} \tag{1.22}$$

denote the characteristic function on the interval $[-r_i, 0]$. We consider the class of equivalent norms on M_2 of the form

$$\begin{aligned} \|(\eta, f)\|_w^2 &= \|\eta\|^2 + \int_{-r_n}^{0} w(\theta)\, \|W^{1/2} f(\theta)\|^2\, d\theta \\ &= \|\eta\|^2 + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} w(\theta)\, \|W^{1/2} f(\theta)\|^2\, d\theta \end{aligned} \tag{1.23}$$

with compatible inner product

$$\begin{aligned} \langle (\eta, f), (\gamma, g) \rangle_w &= \overline{\gamma}^T \eta + \int_{-r_n}^{0} \overline{g(\theta)}^T W w(\theta) f(\theta)\, d\theta \\ &= \overline{\gamma}^T \eta + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T W w(\theta) f(\theta)\, d\theta. \end{aligned} \tag{1.24}$$

The weight matrix W is assumed to be a positive definite, self-adjoint matrix. The scalar weight function $w(\theta)$ is assumed to have the form $w(\theta) = \widetilde{w}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{w,i}$, where $k_{w,i} \geq 0$ for $i = 1, \ldots, n-1$, $\widetilde{w} \in C[-r_n, 0]$, and $\widetilde{w}$ is positive on $[-r_n, 0]$. Under these assumptions the norm $\|\cdot\|_w$ is equivalent to the norm (1.20). We observe that

$$w(\theta) = \begin{cases} \widetilde{w}(\theta) & \text{on } [-r_n, -r_{n-1}), \\ \widetilde{w}(\theta) + \sum_{j=i}^{n-1} k_{w,j} & \text{on } [-r_i, -r_{i-1}),\ i = 2, \ldots, n-1, \\ \widetilde{w}(\theta) + \sum_{j=1}^{n-1} k_{w,j} & \text{on } [-r_1, 0]. \end{cases} \tag{1.25}$$

Since $\widetilde{w}$ is continuous on $[-r_n, 0]$, the $k_{w,i}$ represents the jump at $-r_i$, and

$$w(-r_i^+) - w(-r_i^-) = k_{w,i} \tag{1.26}$$

for $i = 1, \dots, n-1$. For notational purposes we define

$$k_{w,n} \equiv w(-r_n) = \widetilde{w}(-r_n),$$

and notice that this implies

$$w(0) = \widetilde{w}(0) - \widetilde{w}(-r_n) + \sum_{i=1}^{n} k_{w,i} = \widetilde{w}(0) + \sum_{i=1}^{n-1} k_{w,i}. \tag{1.27}$$

Now define a space $V \subset M_2$ as follows:

$$V = \left\{ (\eta, f) | \eta \in \mathbb{C}^m, f(\theta) = \widetilde{f}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i}, \widetilde{f}(\theta) \in H^1(-r_n, 0) \right\}. \tag{1.28}$$

Note that functions in the second component of this space have a form similar to the weight functions described above but may be vector-valued. Similar to our notation for the weight function w, we define $k_{f,n} = f(-r_n)$. One main motivation for this space is that it will contain both the domain of the operator $\mathcal{A}$ and the domain of its adjoint. To create the approximation scheme, we will describe splines that come from the space V which can be used to approximate both operators. We will use the notation that for a function f, $f(c^-)$ represents the limit of $f(x)$ as x approaches c from the left, and $f(c^+)$ represents the limit of $f(x)$ as x approaches c from the right.

Finally, we need to define a sesquilinear form that will be related to both the operator $\mathcal{A}$ and its adjoint $\mathcal{A}^*$. For $u, v \in V$, where $u = (\eta, f(\theta))$ and $v = (\gamma, g(\theta))$, let $h(\theta) = Ww(\theta)f(\theta)$, and define $\sigma : V \times V \to \mathbb{C}$ by

$$\begin{aligned} &\sigma(u,v) \qquad (1.29) \\ = \;& \overline{\gamma}^T \left[A\eta - \sum_{i=1}^{n} \left(AC_i f(-r_i) - B_i f(-r_i) \right) \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T Ww(\theta) f'(\theta) d\theta \\ & + \overline{g(0)}^T Ww(0) \left[\eta - f(0) - \sum_{i=1}^{n} C_i f(-r_i) \right] + \sum_{i=1}^{n-1} \left(\overline{g(-r_i)}^T Ww(-r_i) - \overline{k_{h,i}}^T \right) k_{f,i}. \end{aligned}$$

It is important to note that this is not the only form related to the operator $\mathcal{A}$. Others can be found in [Fab13], for example. However, this one is unique in that it is related to both $\mathcal{A}$ and its adjoint $\mathcal{A}^*$.

In chapter 2, we will begin by considering questions related to the stability of solutions. In particular, we are interested in finding sufficient conditions on the matrices A, B_k, and C_k to guarantee stability of the solution semigroup. These are sometimes called delay-independent stability conditions. Delay-dependent stability conditions, which depend in addition on the delays, are not considered here, although they are actively studied by many researchers (for example [FS03], [HWSL04], [Par01], and [WHS04]).

We then turn to the creation of a new approximation scheme which will give good results both for simulation and control problems. In order to do this, we must first derive an explicit representation of the adjoint operator $\mathcal{A}^*$ in Chapter 3. In chapter 4, we prove that this scheme converges for both the operator $\mathcal{A}$ and its adjoint $\mathcal{A}^*$. Finally, in chapter 5 we close with the numerical implementation of our scheme and discuss its performance for some examples. The idea is to choose a finite-dimensional subspace of $M_2([-r_n, 0], \mathbb{C}^m)$ where we can approximate the action of the operator $\mathcal{A}$. Essentially we want to discretize the $L_2((-r_n, 0), \mathbb{C}^m)$ component of $M_2([-r_n, 0], \mathbb{C}^m)$. Particular properties of the subspace may help or hinder the approximation. Since splines form a finite dimensional subspace of $L_2([-r_n, 0], \mathbb{C}^m)$, we can use them to create a finite dimensional subspace of $M_2([-r_n, 0], \mathbb{C}^m)$ and this is the direction we choose to pursue. There are several different schemes for spline approximation for delay equations which will be discussed in more detail later. We project elements of $M_2([-r_n, 0], \mathbb{C}^m)$ onto the subspace, yielding a finite-dimensional Cauchy problem $\dot{z}^N(t) = \mathcal{A}^N z^N(t)$. We then use solutions of the finite dimensional problem to approximate solutions of the original equation.

CHAPTER II

THE STABILITY PROBLEM

2.1 Preliminaries

We now begin the work to create a sufficient condition to guarantee exponential stability of the solution semigroup. In order to do so, we will show that under certain conditions, we can create a weight function w so that equation (1.9) holds. However, we can actually prove a more general result about a dissipative inequality for the form σ. We then use the following lemma to relate the dissipative inequality for the form to $\mathrm{Re}\langle \mathcal{A}u, u\rangle$.

Lemma 2.1. *For any $u \in D(\mathcal{A})$ and $v \in V$, $\sigma(u, v) = \langle \mathcal{A}u, v\rangle$.*

Proof. Let $u = (\eta, f(\theta))$ and $v = (\gamma, g(\theta))$. Since $u \in D(\mathcal{A})$, we have that $\eta - f(0) - \sum_{i=1}^{n} C_i f(-r_i) = 0$ and that f is continuous. This means that $k_{f,i} = 0$. Also, $\eta = f(0) + \sum_{k=1}^{n} C_k f(-r_k)$, so $\eta - \sum_{k=1}^{n} C_k f(-r_k) = f(0)$. So we have

$$
\begin{aligned}
&\sigma(u, v) \\
&= \overline{\gamma}^T \left[A\eta - \sum_{i=1}^{n} (AC_i f(-r_i) - B_i f(-r_i)) \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T W w(\theta) f'(\theta) d\theta \\
&\quad + \overline{g(0)}^T W w(0) \left[\eta - f(0) - \sum_{i=1}^{n} C_i f(-r_i) \right] + \sum_{i=1}^{n-1} \left(\overline{g(-r_i)}^T W w(-r_i) - \overline{k_{h,i}}^T \right) k_{f,i} \\
&= \overline{\gamma}^T \left[A\eta - \sum_{i=1}^{n} (AC_i f(-r_i) - B_i f(-r_i)) \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T W w(\theta) f'(\theta) d\theta \\
&\quad + \overline{g(0)}^T W w(0) \left[0 \right] + \sum_{i=1}^{n-1} \left(\overline{g(-r_i)}^T W w(-r_i) - \overline{k_{h,i}}^T \right) 0
\end{aligned}
$$

$$= \overline{\gamma}^T \left[A\eta - \sum_{i=1}^{n} \left(AC_i f(-r_i) - B_i f(-r_i) \right) \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T Ww(\theta) f'(\theta) d\theta.$$

However, this is simply a representation of the norm on M_2, so we can write that

$$\begin{aligned} \sigma(u, v) &= \langle \left(A\eta - \sum_{i=1}^{n} \left(AC_i f(-r_i) - B_i f(-r_i) \right), f'(\theta) \right), (\gamma, g(\theta)) \rangle \\ &= \langle \left(Af(0) + \sum_{i=1}^{n} B_i f(-r_i), f'(\theta) \right), (\gamma, g(\theta)) \rangle \\ &= \langle \mathcal{A}u, v \rangle. \end{aligned}$$

Thus for any $u \in D(\mathcal{A})$ and $v \in V$, $\sigma(u, v) = \langle \mathcal{A}u, v \rangle$. □

We now prove several preliminary lemmas, to be used later in the proof of our main result. Toward this, let $u = (\eta, f)$ and $v = (\gamma, g)$ be elements of the space V. Let $h(\theta) = Ww(\theta)f(\theta)$. We approach the stability problem by searching for a bound on $\mathrm{Re}\sigma(u, u)$ of the form $\omega \|u\|^2$ with $\omega < 0$.

We first need to describe the relationship between the jumps in f, w, and h.

Lemma 2.2. *Suppose* $f(\theta) = \widetilde{f}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i}$, *where* $\widetilde{f}(\theta) \in C([-r_n, 0], \mathbb{C}^m)$ *and* $k_{f,i} \in \mathbb{C}^m$ *for* $i = 1, 2, \ldots, n-1$, *and* $w(\theta) = \widetilde{w}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{w,i}$, *where* $\widetilde{w}(\theta) \in C[-r_n, 0]$ *and* $k_{w,i} \in \mathbb{C}$ *for* $i = 1, 2, \ldots, n-1$. *Then* $h(\theta) = Ww(\theta)f(\theta)$ *can be written as* $h(\theta) = \widetilde{h}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{h,i}$, *where* $\widetilde{h}(\theta) \in C([-r_n, 0], \mathbb{C}^m)$ *and* $k_{h,i} = Ww(-r_i)k_{f,i} - Wk_{w,i}k_{f,i} + Wk_{w,i}f(-r_i)$.

Proof. First, note that since both w and f are continuous on all intervals $(-r_i, -r_{i-1})$, then so is h. Also, any function that is continuous on all such intervals $(-r_i, -r_{i-1})$ with finite jumps at each $-r_i$ can be written as $h(\theta) = \widetilde{h}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{h,i}$ for appropriate values $k_{h,i}$. The continuous function $\widetilde{h}(\theta)$ can be obtained by shifting each continuous piece on $(-r_i, -r_{i-1})$ so that all the pieces are connected, and the $k_{h,i}$ measure the size of the jump at $-r_i$ for $1 \leq i \leq n-1$. To find an expression for $k_{h,i}$, we consider $Ww(-r_i^+)f(-r_i^+) - Ww(-r_i^-)f(-r_i^-)$, where $w(-r_i^+)$ represents $\lim_{x\to -r_i^+} w(x)$ and $w(-r_i^-)$ represents $\lim_{x\to -r_i^-} w(x)$ (and similarly for f). By the way that we defined our characteristic functions, we have that $w(-r_i^+) = w(-r_i) = \widetilde{w}(-r_i) + \sum_{j=i}^{n-1} k_{w,j}$. Similarly, $w(-r_i^-) = w(-r_i) - k_{w,i} = \widetilde{w}(-r_i) + \sum_{j=i+1}^{n-1} k_{w,j}$. Similar results hold for f. Then we have

$$\begin{aligned}
k_{h,i} &= Ww(-r_i^+)f(-r_i^+) - Ww(-r_i^-)f(-r_i^-) \\
&= W\left[w(-r_i)f(-r_i) - (w(-r_i) - k_{w,i})(f(-r_i) - k_{f,i})\right] \\
&= W\left[w(-r_i)f(-r_i) - w(-r_i)f(-r_i) + f(-r_i)k_{w,i} + w(-r_i)k_{f,i} - k_{w,i}k_{f,i}\right] \\
&= W\left[f(-r_i)k_{w,i} + w(-r_i)k_{f,i} - k_{w,i}k_{f,i}\right] \\
&= Ww(-r_i)k_{f,i} - Wk_{w,i}k_{f,i} + Wk_{w,i}f(-r_i).
\end{aligned}$$

This completes the proof. □

Next, in the proof of our main result, we will perform integration by parts of some functions from the space V. In order to streamline the presentation of the main result, we present the details of the integration here.

Lemma 2.3. *If f and w are as in Lemma 2.2, then*

$$\begin{aligned}
&Re\left\{\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww(\theta)f'(\theta)\,d\theta\right\} \\
&= \frac{1}{2}\overline{f(0)}^T Ww(0)f(0) - \frac{1}{2}\overline{f(-r_n)}^T Ww(-r_n)f(-r_n)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{n-1}\Big[2Re\Big\{\overline{k_{f,i}}^T k_{w,i}f(-r_i)\Big\} - 2Re\Big\{\overline{k_{f,i}}^T Ww(-r_i)f(-r_i)\Big\}$$
$$-\overline{f(-r_i)}^T Wk_{w,i}f(-r_i) + \overline{k_{f,i}}^T Ww(-r_i)k_{f,i} - \overline{k_{f,i}}^T k_{w,i}k_{f,i}\Big]$$
$$-\sum_{i=1}^{n}\frac{1}{2}\int_{-r_i}^{-r_{i-1}} w'(\theta)\overline{f(\theta)}^T Wf(\theta)\, d\theta.$$

Proof. Let $h(\theta) = Ww(\theta)f(\theta)$. We will write $\overline{f(\theta)}^T Ww(\theta)f'(\theta)$ as $\overline{h(\theta)}^T f'(\theta)$ since $\overline{Ww(\theta)}^T = Ww(\theta)$ because $w(\theta)$ is a weight function, and thus real and scalar, and W is self-adjoint. The usual integration by parts formula tells us that since both f and h are continuous on $(-r_i, -r_{i-1})$,

$$\begin{aligned}
&\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww(\theta)f'(\theta)\, d\theta \\
&= \int_{-r_i}^{-r_{i-1}} \overline{h(\theta)}^T f'(\theta)\, d\theta \\
&= \overline{h(-r_{i-1}^-)}^T f(-r_{i-1}^-) - \overline{h(-r_i^+)}^T f(-r_i^+) - \int_{-r_i}^{-r_{i-1}} \overline{h'(\theta)}^T f(\theta)\, d\theta.
\end{aligned}$$

But since $h(\theta) = Ww(\theta)f(\theta)$, we know that

$$\overline{h'(\theta)}^T = \overline{f'(\theta)}^T Ww(\theta) + \overline{f(\theta)}^T Ww'(\theta).$$

We can now write

$$\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww(\theta)f'(\theta)\, d\theta \quad = \quad \int_{-r_i}^{-r_{i-1}} \overline{h(\theta)}^T f'(\theta)\, d\theta$$

$$= \overline{h(-r_{i-1}^-)}^T f(-r_{i-1}^-) - \overline{h(-r_i^+)}^T f(-r_i^+)$$
$$- \int_{-r_i}^{-r_{i-1}} \left(\overline{f'(\theta)}^T W w(\theta) + \overline{f(\theta)}^T W w'(\theta) \right) f(\theta)\, d\theta.$$

Now we see that

$$2\mathrm{Re}\left\{ \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f'(\theta)\, d\theta \right\}$$
$$= \overline{h(-r_{i-1}^-)}^T f(-r_{i-1}^-) - \overline{h(-r_i^+)}^T f(-r_i^+) - \int_{-r_i}^{-r_{i-1}} w'(\theta) \overline{f(\theta)}^T W f(\theta)\, d\theta.$$

Thus we can use the formula

$$\mathrm{Re}\left\{ \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f'(\theta)\, d\theta \right\}$$
$$= \frac{1}{2}\overline{f(-r_{i-1}^-)}^T W w(-r_{i-1}^-) f(-r_{i-1}^-) - \frac{1}{2}\overline{f(-r_i^+)}^T W w(-r_i^+) f(-r_i^+)$$
$$-\frac{1}{2}\int_{-r_i}^{-r_{i-1}} w'(\theta) \overline{f(\theta)}^T W f(\theta)\, d\theta.$$

We now consider $\mathrm{Re}\left\{ \sum_{i=1}^n \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f'(\theta)\, d\theta \right\}$. We will recall that $w(-r_i^+) = w(-r_i)$ and that $w(-r_{i-1}^-) = w(-r_{i-1}) - k_{w,i-1}$ for $2 \le i \le n$, and then expand terms.

$$\mathrm{Re}\left\{ \sum_{i=1}^n \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f'(\theta)\, d\theta \right\}$$
$$= \sum_{i=1}^n \mathrm{Re}\left\{ \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f'(\theta)\, d\theta \right\}$$
$$= \sum_{i=1}^n \left(\frac{1}{2}\overline{f(-r_{i-1}^-)}^T W w(-r_{i-1}^-) f(-r_{i-1}^-) - \frac{1}{2}\overline{f(-r_i^+)}^T W w(-r_i^+) f(-r_i^+) \right)$$
$$- \sum_{i=1}^n \frac{1}{2} \int_{-r_i}^{-r_{i-1}} w'(\theta) \overline{f(\theta)}^T W f(\theta)\, d\theta$$

Notice that by the way we have defined $\chi_i(\theta)$, we must have that

$$\overline{f(-r_0^-)}^T Ww(-r_0^-)f(-r_0^-) = \overline{f(0)}^T Ww(0)f(0).$$

Then separating the $i = 1$ terms gives us

$$\begin{aligned}
&\mathrm{Re}\left\{\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww(\theta)f'(\theta)\,d\theta\right\} \\
=\;& \frac{1}{2}\overline{f(0)}^T Ww(0)f(0) - \frac{1}{2}\overline{f(-r_1)}^T Ww(-r_1)f(-r_1) \\
& +\sum_{i=2}^{n}\left[\frac{1}{2}\overline{(f(-r_{i-1})-k_{f,i-1})}^T W(w(-r_{i-1})-k_{w,i-1})(f(-r_{i-1})-k_{f,i-1})\right. \\
& \left. -\frac{1}{2}\overline{f(-r_i)}^T Ww(-r_i)f(-r_i)\right] - \sum_{i=1}^{n}\frac{1}{2}\int_{-r_i}^{-r_{i-1}} w'(\theta)\overline{f(\theta)}^T Wf(\theta)\,d\theta \\
=\;& \frac{1}{2}\overline{f(0)}^T Ww(0)f(0) - \frac{1}{2}\overline{f(-r_1)}^T Ww(-r_1)f(-r_1) \\
& +\frac{1}{2}\mathrm{Re}\left\{\sum_{i=1}^{n-1}\left[\overline{(f(-r_i)}^T Ww(-r_i)f(-r_i) - \overline{f(-r_i)}^T Wk_{w,i}f(-r_i)\right.\right. \\
& -\overline{k_{f,i}}^T Ww(-r_i)f(-r_i) + \overline{k_{f,i}}^T k_{w,i}f(-r_i) \\
& -\overline{(f(-r_i)}^T Ww(-r_i)k_{f,i} + \overline{f(-r_i)}^T Wk_{w,i}k_{f,i} + \overline{k_{f,i}}^T Ww(-r_i)k_{f,i} - \overline{k_{f,i}}^T k_{w,i}k_{f,i} \\
& \left.\left. -\overline{f(-r_{i+1})}^T Ww(-r_{i+1})f(-r_{i+1})\right]\right\} - \sum_{i=1}^{n}\frac{1}{2}\int_{-r_i}^{-r_{i-1}} w'(\theta)\overline{f(\theta)}^T Wf(\theta)\,d\theta.
\end{aligned}$$

Then we use the fact that $\overline{x}^T x$ is real for any x and that $\mathrm{Re}\left\{a+\overline{a}^T\right\} = 2\mathrm{Re}\left\{a\right\}$ to write

$$\begin{aligned}
&\mathrm{Re}\left\{\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww(\theta)f'(\theta)\,d\theta\right\} \\
=\;& \frac{1}{2}\overline{f(0)}^T Ww(0)f(0) - \frac{1}{2}\overline{f(-r_1)}^T Ww(-r_1)f(-r_1)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=1}^{n-1}\left[\overline{(f(-r_i)}^T Ww(-r_i)f(-r_i) - \overline{f(-r_{i+1})}^T Ww(-r_{i+1})f(-r_{i+1})\right] \\
&+\frac{1}{2}\sum_{i=1}^{n-1}\Big[2\mathrm{Re}\left\{\overline{k_{f,i}}^T k_{w,i}f(-r_i)\right\} - 2\mathrm{Re}\left\{\overline{k_{f,i}}^T Ww(-r_i)f(-r_i)\right\} \\
&-\overline{f(-r_i)}^T Wk_{w,i}f(-r_i) + \overline{k_{f,i}}^T Ww(-r_i)k_{f,i} - \overline{k_{f,i}}^T k_{w,i}k_{f,i}\Big] \\
&-\sum_{i=1}^{n}\frac{1}{2}\int_{-r_i}^{-r_{i-1}} w'(\theta)\overline{f(\theta)}^T Wf(\theta)\,d\theta.
\end{aligned}
$$

We now use the fact that the first sum is a telescoping sum to obtain

$$
\begin{aligned}
&\mathrm{Re}\left\{\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww(\theta)f'(\theta)\,d\theta\right\} \\
&= \frac{1}{2}\overline{f(0)}^T Ww(0)f(0) - \frac{1}{2}\overline{f(-r_n)}^T Ww(-r_n)f(-r_n) \\
&\quad+\frac{1}{2}\sum_{i=1}^{n-1}\Big[2\mathrm{Re}\left\{\overline{k_{f,i}}^T k_{w,i}f(-r_i)\right\} - 2\mathrm{Re}\left\{\overline{k_{f,i}}^T Ww(-r_i)f(-r_i)\right\} \\
&\quad-\overline{f(-r_i)}^T Wk_{w,i}f(-r_i) + \overline{k_{f,i}}^T Ww(-r_i)k_{f,i} - \overline{k_{f,i}}^T k_{w,i}k_{f,i}\Big] \\
&\quad-\sum_{i=1}^{n}\frac{1}{2}\int_{-r_i}^{-r_{i-1}} w'(\theta)\overline{f(\theta)}^T Wf(\theta)\,d\theta.
\end{aligned}
$$

This completes the proof. □

We will make frequent use of the following lemma.

Lemma 2.4 (Cauchy-Schwartz)**.** *If x and y are vectors, then $Re\,\overline{x}^T y \leq \frac{\epsilon}{2}\|x\|^2 + \frac{1}{2\epsilon}\|y\|^2$ for any $\epsilon > 0$.*

Next we present several technical lemmas which will aid us in obtaining a bound on $\mathrm{Re}\langle \mathcal{A}u, u\rangle$ in order to obtain the desired dissipative inequality.

Lemma 2.5. *If* $c = \sum_{i=1}^{n} \|C_i\|$, *then*

$$\frac{1}{2} Re \sum_{i=1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W w(0) \left(\sum_{j=1, j \neq i}^{n} C_j f(-r_j) \right)$$
$$\leq \frac{1}{2} \left\| W^{1/2} \right\|^2 w(0) \sum_{i=1}^{n} \|C_i\| \left(c - \|C_i\|\right) \|f(-r_i)\|^2 .$$

Proof. We first use the fact that $\mathrm{Re}\left\{a + \overline{a}^T\right\} = 2\mathrm{Re}\left\{a\right\}$ to rewrite

$$\mathrm{Re} \sum_{i=1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W w(0) \left(\sum_{j=1, j \neq i}^{n} C_j f(-r_j) \right)$$

as $2\mathrm{Re} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W w(0) C_j f(-r_j)$. Thus we know

$$\frac{1}{2} \mathrm{Re} \sum_{i=1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W w(0) \left(\sum_{j=1, j \neq i}^{n} C_j f(-r_j) \right)$$
$$= \mathrm{Re} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W w(0) C_j f(-r_j).$$

We will next insert a convenient product and use Lemma 2.4. Taking $c = \sum_{i=1}^{n} \|C_i\|$ we have:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W w(0) C_j f(-r_j)$$
$$= w(0) \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W^{1/2} \frac{\|C_j\|^{1/2}}{\|C_i\|^{1/2}} \cdot \frac{\|C_i\|^{1/2}}{\|C_j\|^{1/2}} W^{1/2} C_j f(-r_j)$$
$$\leq w(0) \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{2} \left\| \overline{f(-r_i)}^T \overline{C_i}^T W^{1/2} \frac{\|C_j\|^{1/2}}{\|C_i\|^{1/2}} \right\|^2 + \frac{1}{2} \left\| \frac{\|C_i\|^{1/2}}{\|C_j\|^{1/2}} W^{1/2} C_j f(-r_j) \right\|^2$$

$$
\begin{aligned}
&= \frac{w(0)}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\|\frac{\|C_j\|^{1/2}}{\|C_i\|^{1/2}}W^{1/2}C_i f(-r_i)\right\|^2+\left\|\frac{\|C_i\|^{1/2}}{\|C_j\|^{1/2}}W^{1/2}C_j f(-r_j)\right\|^2\\
&\le \frac{w(0)}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\|C_j\|}{\|C_i\|}\left\|W^{1/2}\right\|^2\|C_i\|^2\|f(-r_i)\|^2+\frac{\|C_i\|}{\|C_j\|}\left\|W^{1/2}\right\|^2\|C_j\|^2\|f(-r_j)\|^2\\
&= \frac{w(0)}{2}\left\|W^{1/2}\right\|^2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\|C_j\|\|C_i\|\|f(-r_i)\|^2+\|C_i\|\|C_j\|\|f(-r_j)\|^2\\
&= \frac{w(0)}{2}\left\|W^{1/2}\right\|^2\sum_{i=1}^{n}\|C_i\|\|f(-r_i)\|^2(c-\|C_i\|).
\end{aligned}
$$

This completes the proof. □

Lemma 2.6. *Re* $\left\{\overline{\eta}^T A\eta\right\} = \frac{1}{2}Re\left\{\overline{\eta}^T\left(A+\overline{A}^T\right)\eta\right\}$.

Proof. To see this, we note that $\overline{\eta}^T A\eta+\overline{\eta}^T\overline{A}^T\eta = 2\mathrm{Re}\left\{\overline{\eta}^T A\eta\right\}$ and thus $\mathrm{Re}\left\{\overline{\eta}^T A\eta\right\} = \frac{1}{2}\mathrm{Re}\left\{\overline{\eta}^T\left(A+\overline{A}^T\right)\eta\right\}$. □

Lemma 2.7. $\frac{1}{2}\overline{C_i}^T Ww(0)C_i - \frac{1}{2}\left\|W^{1/2}\right\|^2 w(0)\|C_i\|^2 < 0.$

Proof. First, note that $\overline{C_i}^T Ww(0)C_i = w(0)\overline{C_iW^{1/2}}^T W^{1/2}C_i$. Then by Lemma 2.4, we can say that

$$
\overline{C_i}^T Ww(0)C_i \le w(0)\left(\frac{1}{2}\left\|\overline{W^{1/2}C_i}^T\right\|^2+\frac{1}{2}\left\|W^{1/2}C_i\right\|^2\right) = w(0)\left\|W^{1/2}C_i\right\|^2. \quad (2.1)
$$

We can now say that $\overline{C_i}^T Ww(0)C_i \le w(0)\left\|W^{1/2}C_i\right\|^2 \le w(0)\left\|W^{1/2}\right\|^2\|C_i\|^2$. This suffices to show that $\frac{1}{2}\overline{C_i}^T Ww(0)C_i - \frac{1}{2}\left\|W^{1/2}\right\|^2 w(0)\|C_i\|^2 < 0$. □

Lemma 2.8. *If λ_m, λ_M are the minimum and maximum eigenvalues of the weight matrix W, respectively, then $\lambda_m \le \left\|W^{1/2}\right\|^2 \le \lambda_M$. Thus for $1 \le i \le n$, we have $\overline{f(-r_i)}^T Wk_{w,i}f(-r_i) \le k_{w,i}\lambda_M\|f(-r_i)\|^2$.*

Proof. Since W is assumed to be positive definite, W is symmetric and bounded. Then by a theorem in [Kat76] (see problem 3.47 in chapter 5), we know that $\left\|W^{1/2}\right\| = \|W\|^{1/2}$. Also since W is symmetric, we know that $\|W\| = \lambda_M$. Thus we can say that $\lambda_m \leq \|W\| = \left\|W^{1/2}\right\|^2 = \lambda_M$. Finally, we can see that

$$\begin{aligned} \overline{f(-r_i)}^T W k_{w,i} f(-r_i) &= k_{w,i} \left\|W^{1/2} f(-r_i)\right\|^2 \\ &\leq k_{w,i} \left\|W^{1/2}\right\|^2 \|f(-r_i)\|^2 \\ &\leq k_{w,i} \lambda_M \|f(-r_i)\|^2 . \end{aligned}$$

Thus the result is proven. □

2.2 Main Result

We now turn to our main result. This is a new delay-independent condition that is sufficient to guarantee stability of the solution semigroup. In order to prove the the solution semigroup is exponentially stable, we will show that there is some $\omega < 0$ such that

$$\operatorname{Re} \sigma\left((\eta, f), (\eta, f)\right) \leq \omega \|(\eta, f)\|_w^2. \tag{2.2}$$

It will then follow from Lemma 2.1 that inequality (1.9) holds.

Theorem 2.9. *Let* $c = \sum_{i=1}^{n} \|C_i\|$, $b = \sum_{i=1}^{n} \|B_i\|$, $W = -(A + \overline{A}^T)/2$, *and* $H = (A - \overline{A}^T)/2$. *If* $c < 1$ *and*

$$\mu(A) + \frac{1}{2}\|W\|c^2 + \frac{1}{2}\sqrt{\|W\|^2 c^4 + 4(b + \|H\|c)^2} < 0, \tag{2.3}$$

then inequality (2.2) holds.

Proof. Let $u = (\eta, f(\theta))$ and for convenience, put $h(\theta) = Ww(\theta)f(\theta)$. It is sufficient to show that we can define a weight function w so that (2.2) holds for the weighted norm. Towards this end, let $u = (\eta, f) \in V$. We have:

$$\begin{aligned}
&\operatorname{Re}\{\sigma(u,u)\} \\
&= \operatorname{Re}\Bigg\{\overline{\eta}^T\left[A\eta - \sum_{i=1}^{n}\left(AC_if(-r_i) - B_if(-r_i)\right)\right] + \sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T Ww(\theta)f'(\theta)d\theta \\
&\quad + \overline{f(0)}^T Ww(0)\left[\eta - f(0) - \sum_{i=1}^{n} C_if(-r_i)\right] \\
&\quad + \sum_{i=1}^{n-1}\left(\overline{f(-r_i)}^T Ww(-r_i) - \overline{k_{h,i}}^T\right)k_{f,i}\Bigg\} \\
&= \operatorname{Re}\Bigg\{\overline{\eta}^T A\eta + \sum_{i=1}^{n}\overline{\eta}^T B_if(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T AC_if(-r_i) + \overline{f(0)}^T Ww(0)\eta \\
&\quad - \overline{f(0)}^T Ww(0)f(0) - \sum_{i=1}^{n}\overline{f(0)}^T Ww(0)C_if(-r_i) \\
&\quad + \sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T Ww(-r_i) - \overline{k_{h,i}}^T\right]k_{f,i}\Bigg\} + \sum_{i=1}^{n}\operatorname{Re}\left\{\int_{-r_i}^{-r_{i-1}}\overline{h(\theta)}^T f'(\theta)d\theta\right\}.
\end{aligned}$$

Then, using Lemma 2.3, we have

$$\begin{aligned}
&\operatorname{Re}\{\sigma(u,u)\} \\
&= \operatorname{Re}\Bigg\{\overline{\eta}^T A\eta + \sum_{i=1}^{n}\overline{\eta}^T B_if(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T AC_if(-r_i) + \overline{f(0)}^T Ww(0)\eta \\
&\quad - \overline{f(0)}^T Ww(0)f(0) - \sum_{i=1}^{n}\overline{f(0)}^T Ww(0)C_if(-r_i) \\
&\quad + \sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T Ww(-r_i) - \overline{k_{h,i}}^T\right]k_{f,i}\Bigg\} + \frac{1}{2}\overline{f(0)}^T Ww(0)f(0)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
&-\frac{1}{2}\sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T W w(-r_i) k_{f,i} + \overline{k_{h,i}}^T f(-r_i) - \overline{k_{h,i}}^T k_{f,i}\right] \\
= \quad & \mathrm{Re}\left\{\overline{\eta}^T A\eta + \sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T A C_i f(-r_i) + \overline{f(0)}^T W w(0)\eta \right.\\
&\left. - \sum_{i=1}^{n}\overline{f(0)}^T W w(0) C_i f(-r_i)\right\} - \frac{1}{2}\overline{f(0)}^T W w(0) f(0) \\
&-\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
&-\frac{1}{2}\sum_{i=1}^{n-1}\left[\overline{k_{h,i}}^T f(-r_i) - \overline{f(-r_i)}^T W w(-r_i) k_{f,i} + \overline{k_{h,i}}^T k_{f,i}\right].
\end{aligned}
$$

Then we use Lemma 2.2 about the jumps in $h(\theta)$ to obtain

$$
\begin{aligned}
&\mathrm{Re}\left\{\sigma(u,u)\right\} \\
= \quad & \mathrm{Re}\left\{\overline{\eta}^T A\eta + \sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T A C_i f(-r_i) + \overline{f(0)}^T W w(0)\eta \right.\\
&\left. - \sum_{i=1}^{n}\overline{f(0)}^T W w(0) C_i f(-r_i)\right\} - \frac{1}{2}\overline{f(0)}^T W w(0) f(0) \\
&-\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
&-\frac{1}{2}\sum_{i=1}^{n-1}\left[\left(\overline{k_{f,i}}^T W w(-r_i) - \overline{k_{f,i}}^T W k_{w,i} + \overline{f(-r_i)}^T W k_{w,i}\right) f(-r_i)\right. \\
&\left. - \overline{f(-r_i)}^T W w(-r_i) k_{f,i} + \left(\overline{k_{f,i}}^T W w(-r_i) - \overline{k_{f,i}}^T W k_{w,i} + \overline{f(-r_i)}^T W k_{w,i}\right) k_{f,i}\right] \\
= \quad & \mathrm{Re}\left\{\overline{\eta}^T A\eta + \sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T A C_i f(-r_i) + \overline{f(0)}^T W w(0)\eta \right.\\
&\left. - \sum_{i=1}^{n}\overline{f(0)}^T W w(0) C_i f(-r_i)\right\} - \frac{1}{2}\overline{f(0)}^T W w(0) f(0)
\end{aligned}
$$

$$-\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta.$$

$$-\frac{1}{2}\sum_{i=1}^{n-1}\Big[\mathrm{Re}\left\{\overline{k_{f,i}}^T W w(-r_i) f(-r_i) - \overline{f(-r_i)}^T W w(-r_i) k_{f,i}\right\}$$

$$+\mathrm{Re}\left\{\overline{f(-r_i)}^T W k_{w,i} k_{f,i} - \overline{k_{f,i}}^T W k_{w,i} f(-r_i)\right\}$$

$$+\overline{f(-r_i)}^T W k_{w,i} f(-r_i) + \overline{k_{f,i}}^T W w(-r_i) k_{f,i} - \overline{k_{f,i}}^T W k_{w,i} k_{f,i}\Big].$$

Next we use the fact that $\mathrm{Re}\{a - \overline{a}\} = 0$ to simplify and get

$$\begin{aligned}
&\mathrm{Re}\{\sigma(u,u)\} \\
&= \mathrm{Re}\left\{\overline{\eta}^T A\eta + \sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T A C_i f(-r_i) + \overline{f(0)}^T W w(0)\eta\right. \\
&\quad \left. - \sum_{i=1}^{n}\overline{f(0)}^T W w(0) C_i f(-r_i)\right\} - \frac{1}{2}\overline{f(0)}^T W w(0) f(0) \\
&\quad -\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
&\quad -\frac{1}{2}\sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T W k_{w,i} f(-r_i) + \overline{k_{f,i}}^T W w(-r_i) k_{f,i} - \overline{k_{f,i}}^T W k_{w,i} k_{f,i}\right] \\
&= \mathrm{Re}\left\{\overline{\eta}^T A\eta + \sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T A C_i f(-r_i)\right. \\
&\quad \left. +\overline{f(0)}^T W w(0)\left(\eta - \sum_{i=1}^{n} C_i f(-r_i)\right)\right\} - \frac{1}{2}\overline{f(0)}^T W w(0) f(0) \\
&\quad -\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
&\quad -\frac{1}{2}\sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T W k_{w,i} f(-r_i) + \overline{k_{f,i}}^T W\left(w(-r_i) - k_{w,i}\right) k_{f,i}\right].
\end{aligned}$$

Now we apply the Cauchy-Schwartz inequality to the term

$$\overline{f(0)}^T W w(0) \left(\eta - \sum_{i=1}^{n} C_i f(-r_i) \right)$$

and obtain

$$\begin{aligned}
\mathrm{Re}\{\sigma(u,u)\} \leq\ & \mathrm{Re}\left\{ \overline{\eta}^T A\eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T AC_i f(-r_i) \right\} \\
& + \frac{1}{2} \left\| \overline{f(0)}^T W^{1/2} w(0)^{1/2} \right\|^2 + \frac{1}{2} \left\| W^{1/2} w(0)^{1/2} \left(\eta - \sum_{i=1}^{n} C_i f(-r_i) \right) \right\|^2 \\
& - \frac{1}{2} \overline{f(0)}^T W w(0) f(0) - \frac{1}{2} \overline{f(-r_n)}^T W w(-r_n) f(-r_n) \\
& - \frac{1}{2} \sum_{i=1}^{n-1} \overline{f(-r_i)}^T W k_{w,i} f(-r_i) - \frac{1}{2} \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
=\ & \mathrm{Re}\left\{ \overline{\eta}^T A\eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T AC_i f(-r_i) \right\} \\
& + \frac{1}{2} \left(\overline{\eta}^T - \sum_{i=1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T \right) W w(0) \left(\eta - \sum_{i=1}^{n} C_i f(-r_i) \right) \\
& - \frac{1}{2} \overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2} \sum_{i=1}^{n-1} \overline{f(-r_i)}^T W k_{w,i} f(-r_i) \\
& - \frac{1}{2} \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta.
\end{aligned}$$

Next, expand the second line and use the fact that $\mathrm{Re}\{a + \overline{a}\} = 2\mathrm{Re}\{a\}$ to see that

$$\begin{aligned}
& \mathrm{Re}\{\sigma(u,u)\} \\
& \leq\ \mathrm{Re}\left\{ \overline{\eta}^T A\eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T AC_i f(-r_i) \right\} + \frac{1}{2} \overline{\eta}^T W w(0) \eta
\end{aligned}$$

$$
\begin{aligned}
& -\mathrm{Re}\left\{\frac{1}{2}\overline{\eta}^T W w(0)\sum_{i=1}^{n} C_i f(-r_i)\right\} - \mathrm{Re}\left\{\frac{1}{2}\left(\sum_{i=1}^{n}\overline{f(-r_i)}^T\overline{C_i}^T\right) W w(0)\eta\right\} \\
& +\frac{1}{2}\left(\sum_{i=1}^{n}\overline{f(-r_i)}^T\overline{C_i}^T\right) W w(0)\left(\sum_{i=1}^{n}\overline{f(-r_i)}^T\overline{C_i}^T\right) \\
& -\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n-1}\overline{f(-r_i)}^T W k_{w,i} f(-r_i) \\
& -\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
= \; & \mathrm{Re}\left\{\overline{\eta}^T\left[A + \frac{1}{2}W w(0)\right]\eta + \sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n}\overline{\eta}^T A C_i f(-r_i)\right\} \\
& -\mathrm{Re}\left\{\overline{\eta}^T W w(0)\sum_{i=1}^{n} C_i f(-r_i)\right\} \\
& +\frac{1}{2}\left(\sum_{i=1}^{n}\overline{f(-r_i)}^T\overline{C_i}^T\right) W w(0)\left(\sum_{i=1}^{n}\overline{f(-r_i)}^T\overline{C_i}^T\right) \\
& -\frac{1}{2}\overline{f(-r_n)}^T W w(-r_n) f(-r_n) - \frac{1}{2}\sum_{i=1}^{n-1}\overline{f(-r_i)}^T W k_{w,i} f(-r_i) \\
& -\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
= \; & \mathrm{Re}\left\{\overline{\eta}^T\left[A + \frac{1}{2}W w(0)\right]\eta + \overline{\eta}^T\sum_{i=1}^{n} B_i f(-r_i) - \overline{\eta}^T\left[A + W w(0)\right]\sum_{i=1}^{n} C_i f(-r_i)\right\} \\
& +\frac{1}{2}\mathrm{Re}\left\{\sum_{i=1}^{n-1}\overline{f(-r_i)}^T\left[\overline{C_i}^T W w(0)\sum_{j=1}^{n} C_j f(-r_j) - W k_{w,i} f(-r_i)\right]\right\} \\
& +\frac{1}{2}\mathrm{Re}\left\{\overline{f(-r_n)}^T\left[\overline{C_n}^T W w(0)\sum_{j=1}^{n} C_j f(-r_j) - W w(-r_n) f(-r_n)\right]\right\} \\
& -\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta.
\end{aligned}
$$

Now, let $c = \sum_{i=1}^{n} \|C_i\|$ and recall that we defined $k_{w,n} = w(-r_n)$. Separating the sums and using a Cauchy-Schwartz fact to eliminate the double sum gives

$$
\begin{aligned}
&\operatorname{Re}\{\sigma(u,u)\} \\
\le\ & \operatorname{Re}\left\{ \overline{\eta}^T \left[A + \frac{1}{2} W w(0)\right] \eta + \overline{\eta}^T \sum_{i=1}^{n} B_i f(-r_i) - \overline{\eta}^T \left[A + W w(0)\right] \sum_{i=1}^{n} C_i f(-r_i) \right\} \\
& + \frac{1}{2} \sum_{i=1}^{n} \overline{f(-r_i)}^T \left[\overline{C_i}^T W w(0) C_i - W k_{w,i} \right] f(-r_i) \\
& + \frac{1}{2} \sum_{i=1}^{n} \overline{f(-r_i)}^T \overline{C_i}^T W w(0) \left(\sum_{j=1, j \neq i}^{n} C_j f(-r_j) \right) \\
& - \frac{1}{2} \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
\le\ & \operatorname{Re}\left\{ \overline{\eta}^T \left[A + \frac{1}{2} W w(0)\right] \eta + \overline{\eta}^T \sum_{i=1}^{n} B_i f(-r_i) - \overline{\eta}^T \left[A + W w(0)\right] \sum_{i=1}^{n} C_i f(-r_i) \right\} \\
& + \frac{1}{2} \sum_{i=1}^{n} \overline{f(-r_i)}^T \left[\overline{C_i}^T W w(0) C_i - W k_{w,i} \right] f(-r_i) \\
& + \frac{1}{2} \left\| W^{1/2} \right\|^2 w(0) \sum_{i=1}^{n} \|C_i\| \left(c - \|C_i\|\right) \|f(-r_i)\|^2 \\
& - \frac{1}{2} \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta.
\end{aligned}
$$

The goal to finish the proof is to obtain a bound of the form

$$
\operatorname{Re}\{\sigma(u,u)\} \le \alpha_1 \|\eta\|^2 + \alpha_2 \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f(\theta)\, d\theta + \sum_{i=1}^{n} \beta_i \|f(-r_i)\|^2, \quad (2.4)
$$

where $\alpha_1 < 0$, $\alpha_2 < 0$, and $\beta_i \le 0$ for $i = 1, \ldots, n$. Rewriting some of the norms as products, we get

$$\begin{aligned}
&\operatorname{Re}\{\sigma(u,u)\} \\
\le\ & \frac{1}{2}\operatorname{Re}\left\{\overline{\eta}^T\left[A+\overline{A}^T+Ww(0)\right]\eta+\overline{\eta}^T\sum_{i=1}^{n}B_i f(-r_i)\right. \\
&\left.-\overline{\eta}^T\left[A+Ww(0)\right]\sum_{i=1}^{n}C_i f(-r_i)\right\} \\
&-\frac{1}{2}\sum_{i=1}^{n}\overline{f(-r_i)}^T\left[Wk_{w,i}-\overline{C_i}^T Ww(0)C_i-\left\|W^{1/2}\right\|^2\left\|C_i\right\|c\right. \\
&\left.+\left\|W^{1/2}\right\|^2\left\|C_i\right\|^2\right]f(-r_i)-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T Ww'(\theta)f(\theta)d\theta \\
\le\ & \frac{1}{2}\operatorname{Re}\left\{\overline{\eta}^T\left[A+\overline{A}^T+Ww(0)\right]\eta+\overline{\eta}^T\sum_{i=1}^{n}B_i f(-r_i)\right. \\
&\left.-\overline{\eta}^T\left[A+Ww(0)\right]\sum_{i=1}^{n}C_i f(-r_i)\right\} \\
&-\frac{1}{2}\sum_{i=1}^{n}\overline{f(-r_i)}^T\left[Wk_{w,i}-\left\|W^{1/2}\right\|^2\left\|C_i\right\|c\right]f(-r_i) \\
&-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T Ww'(\theta)f(\theta)d\theta.
\end{aligned}$$

Since our goal was to choose an appropriate weight function to obtain a dissipative inequality, recall that $W = \frac{-1}{2}\left(A + A^T\right)$. We will create a weight function $w(\theta)$ that satisfies

$$w(0) = 1 \quad \text{and} \quad w(\theta) = \widetilde{w}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta)\, k_{w,i}, \tag{2.5}$$

where $\widetilde{w}(\theta) = \beta e^{\gamma\theta}$. The positive numbers β, γ, and the non-negative numbers $k_{w,i},\ i = 1, \ldots, n-1$ will be specified later. Thus over each subinterval w satisfies

$$w'(\theta) = \gamma \widetilde{w}(\theta). \tag{2.6}$$

Recall from (1.27) the jumps must satisfy

$$1 = \beta + \sum_{i=1}^{n-1} k_{w,i}. \tag{2.7}$$

Making these replacements, we obtain

$$\begin{aligned}
\mathrm{Re}\left\{\sigma(u,u)\right\} \;\leq\; & \frac{1}{2}\overline{\eta}^T \left[A + \overline{A}^T - W\right]\eta + \mathrm{Re}\left\{\overline{\eta}^T \sum_{i=1}^{n} B_i f(-r_i)\right\} \\
& -\mathrm{Re}\left\{\overline{\eta}^T \left[A - W\right] \sum_{i=1}^{n} C_i f(-r_i)\right\} \\
& -\frac{1}{2}\sum_{i=1}^{n} \overline{f(-r_i)}^T \left[W k_{w,i} - \left\|W^{1/2}\right\|^2 \|C_i\|\, c\right] f(-r_i) \\
& -\frac{1}{2}\sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta.
\end{aligned}$$

We observe that $0 < -\mu(A) = \lambda_m \leq \lambda_M = \|W\|$. Since $\mathrm{Re}\,\overline{\eta}^T A\eta = \mathrm{Re}\,\overline{\eta}^T \overline{A}^T \eta$, we have

$$\begin{aligned}
\mathrm{Re}\,\overline{\eta}^T[A + \frac{1}{2}W] &= \mathrm{Re}\,\overline{\eta}^T[\frac{1}{2}(A + \overline{A}^T) - \frac{1}{4}(A + \overline{A}^T)]\eta \\
&= \mathrm{Re}\,\overline{\eta}^T \frac{1}{4}(A + \overline{A}^T)\eta \\
&= \frac{1}{2}\mathrm{Re}\,\overline{\eta}^T A\eta.
\end{aligned}$$

Also recall $H = \frac{1}{2}\left(A - \overline{A}^T\right)$. This gives us

$$\begin{aligned}
\text{Re}\left\{\sigma(u,u)\right\} &\leq \overline{\eta}^T\left[\frac{1}{2}\mu(A)\right]\eta + \text{Re}\left\{\overline{\eta}^T\sum_{i=1}^{n} B_i f(-r_i)\right\} \\
&-\text{Re}\left\{\overline{\eta}^T H \sum_{i=1}^{n} C_i f(-r_i)\right\} \\
&-\frac{1}{2}\sum_{i=1}^{n}\overline{f(-r_i)}^T\left[W k_{w,i} - \left\|W^{1/2}\right\|^2\|C_i\|\, c - \epsilon_i\right] f(-r_i) \\
&-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta. \qquad (2.8)
\end{aligned}$$

With (2.4) in mind, we shall next apply the Cauchy-Schwartz inequality to each $\eta B_i f(-r_i)$ term in (2.8). Fix $\epsilon > 0$. Notice if $\|B_i\| \neq 0$ then

$$\begin{aligned}
\text{Re}\,\overline{\eta}^T B_i f(-r_i) &\leq \frac{1}{2\epsilon\|B_i\|}\|\overline{\eta}^T B_i\|^2 + \frac{\epsilon\|B_i\|}{2}\|f(-r_i)\|^2 \\
&\leq \frac{1}{2\epsilon}\|B_i\|\,\|\eta\|^2 + \frac{\epsilon}{2}\|B_i\|\|f(-r_i)\|^2. \qquad (2.9)
\end{aligned}$$

But (2.9) also holds when $\|B_i\| = 0$, so we may apply it to each term in the first sum in (2.8) to get

$$\text{Re}\,\overline{\eta}^T\sum_{i=1}^{n} B_i f(-r_i) \leq \frac{b}{2\epsilon}\|\eta\|^2 + \frac{\epsilon}{2}\sum_{i=1}^{n}\|B_i\|\|f(-r_i)\|^2. \qquad (2.10)$$

By a similar argument, for any $\delta > 0$ we have

$$\mathrm{Re}\,\overline{\eta}^T H \sum_{i=1}^{n} C_i f(-r_i) \leq \frac{c}{2\delta}\|\eta\|^2 + \frac{\delta}{2}\|H\|^2 \sum_{i=1}^{n} \|C_i\|\|f(-r_i)\|^2. \tag{2.11}$$

Inequalities (2.10) and (2.11) are valid for any $\epsilon, \delta > 0$, and specific values will be chosen later. We may proceed from (2.8) to get

$$\begin{aligned} &\mathrm{Re}\,\{\sigma(u,u)\} \qquad (2.12)\\ &\leq \frac{1}{2}\big[\mu(A) + \frac{b}{\epsilon} + \frac{c}{\delta}\big]\|\eta\|^2 - \frac{\gamma}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W \widetilde{w}(\theta) f(\theta)\, d\theta \\ &\quad -\frac{|\mu(A)|}{2}\sum_{i=1}^{n}\big[k_{w,i} - \frac{\|W\|\|C_i\|c + \epsilon\|B_i\| + \delta\|H\|^2\|C_i\|}{|\mu(A)|}\big]\|f(-r_i)\|^2. \end{aligned}$$

Next we will show that ϵ and δ can be chosen so that both

$$\mu(A) < -\frac{b}{\epsilon} - \frac{c}{\delta} \tag{2.13}$$

and

$$\begin{aligned} \mu(A) &< -\sum_{i=1}^{n}\big[\|W\|\|C_i\|c + \epsilon\|B_i\| + \delta\|H\|^2\|C_i\|\big] \\ &= -\|W\|c^2 - \epsilon b - \delta\|H\|^2 c \qquad (2.14) \end{aligned}$$

hold. Observe that (2.14) implies

$$\sum_{i=1}^{n} \frac{\|W\|\|C_i\|c + \epsilon\|B_i\| + \delta\|H\|^2\|C_i\|}{|\mu(A)|} < 1. \tag{2.15}$$

Thus once ϵ and δ are chosen to make (2.13) and (2.14) true, we can define β, γ, and $k_{w,i}$, $i = 1, \ldots, n-1$ in order to make all the $\|f(-r_i)\|^2$ terms in (2.12) nonpositive. To do so, first define

$$k_{w,i} = \frac{\|W\|\|C_i\|c + \epsilon\|B_i\| + \delta\|H\|^2\|C_i\|}{|\mu(A)|}, \quad i = 1, \ldots, n-1. \tag{2.16}$$

This makes the terms $\|f(-r_i)\|^2$, $i = 1, \ldots, n-1$ equal to zero. For the $\|f(-r_n)\|^2$ term, we recall (2.5) and set $\beta = 1 - \sum_{i=1}^{n-1} k_{w,i}$, so $w(0) = 1$ is satisfied. Then (2.15) also implies

$$\beta > \frac{\|W\|\|C_n\|c + \epsilon\|B_n\| + \delta\|H\|^2\|C_n\|}{|\mu(A)|},$$

which means we can choose $\gamma > 0$ so that

$$k_{w,n} = \widetilde{w}(-r_n) = \beta e^{-\gamma r_n} > \frac{\|W\|\|C_n\|c + \epsilon\|B_n\| + \delta\|H\|^2\|C_n\|}{|\mu(A)|}.$$

This makes the $\|f(-r_n)\|^2$ term negative, so all the $\|f(-r_i)\|^2$ terms in (2.12) are less than or equal to zero. Since $\widetilde{w}(\theta) = \beta e^{\gamma\theta}$, both w and $\widetilde{w}$ are nondecreasing on $[-r_n, 0]$. Thus for all $\theta \in [-r_n, 0]$, we have

$$\frac{e^{\gamma r_n}}{\beta}\widetilde{w}(\theta) \geq \frac{e^{\gamma r_n}}{\beta}\widetilde{w}(-r_n) = 1 = w(0) \geq w(\theta),$$

so

$$-\frac{\gamma}{2}\widetilde{w}(\theta) \leq -\frac{\gamma\beta}{2}e^{-\gamma r_n}w(\theta). \tag{2.17}$$

Thus (2.12) becomes

$$\mathrm{Re}\,\{\sigma(u,u)\} \leq \frac{1}{2}\big[\mu(A) + \frac{b}{\epsilon} + \frac{c}{\delta}\big]\|\eta\|^2 - \frac{\gamma\beta}{2}e^{-\gamma r_n}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f(\theta)\,d\theta, \tag{2.18}$$

and it only remains to show we can choose ϵ and δ to make (2.13) and (2.14) true. We consider the cases $\|H\| \neq 0$ and $\|H\| = 0$ separately. When $\|H\| \neq 0$, define

$$\delta = \frac{-\|W\|c^2 + \sqrt{\|W\|^2c^4 + 4(b + \|H\|c)^2}}{2\|H\|(b + \|H\|c)} \tag{2.19}$$

and

$$\epsilon = \frac{-\|W\|c^2 + \sqrt{\|W\|^2c^4 + 4(b + \|H\|c)^2}}{2(b + \|H\|c)}. \tag{2.20}$$

It is straightforward to check that

$$\frac{b}{\epsilon} + \frac{c}{\delta} = \|W\|c^2 + \epsilon b + \delta\|H\|^2c = \frac{1}{2}\|W\|c^2 + \frac{1}{2}\sqrt{\|W\|^2c^4 + 4(b + \|H\|c)^2}. \tag{2.21}$$

Thus (2.13) and (2.14) each reduce to

$$\mu(A) < -\frac{1}{2}\|W\|c^2 - \frac{1}{2}\sqrt{\|W\|^2c^4 + 4(b + \|H\|c)^2} < 0, \tag{2.22}$$

and the result follows by the hypothesis (2.3). When $\|H\| = 0$ there are no δ terms because the second sum in (2.8) is zero so (2.11) is not needed. In this case (2.18) becomes

$$\operatorname{Re}\{\sigma(u,u)\} \leq \frac{1}{2}\big[\mu(A) + \frac{b}{\epsilon}\big]\|\eta\|^2 - \frac{\gamma\beta}{2}e^{-\gamma r_n}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f(\theta)\, d\theta. \quad (2.23)$$

Thus instead of (2.13) and (2.14), in this case we must show ϵ can be chosen so that both

$$\mu(A) < -\frac{b}{\epsilon} \quad (2.24)$$

and

$$\begin{aligned}\mu(A) &< -\sum_{i=1}^{n}\big[\|W\|\|C_i\|c + \epsilon\|B_i\|\big] \\ &= -\|W\|c^2 - \epsilon b \quad (2.25)\end{aligned}$$

hold. To do so, define

$$\epsilon = \frac{-\|W\|c^2 + \sqrt{\|W\|^2c^4 + 4b^2}}{2b}. \quad (2.26)$$

It can then be shown that

$$\frac{b}{\epsilon} = \|W\|c^2 + \epsilon b = \frac{1}{2}\|W\|c^2 + \frac{1}{2}\sqrt{\|W\|^2c^4 + 4b^2}. \quad (2.27)$$

Thus (2.24) and (2.25) each reduce to

$$\mu(A) < -\frac{1}{2}\|W\|c^2 - \frac{1}{2}\sqrt{\|W\|^2c^4 + 4b^2} < 0, \tag{2.28}$$

and the result again follows by the hypothesis (2.3). Thus in both cases we have shown how to define $w(\theta)$ (more specifically, how to define β, γ, and the jumps $k_{w,i}$) so that (2.18) or (2.23) is true, which implies

$$\begin{aligned} \mathrm{Re}\,\{\sigma(u,u)\} \;\leq\; & \frac{1}{2}\Big[\mu(A) + \frac{1}{2}\|W\|c^2 + \frac{1}{2}\sqrt{\|W\|^2c^4 + 4(b+\|H\|c)^2}\Big]\|\eta\|^2 \\ & -\frac{\gamma\beta}{2}e^{-\gamma r_n}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w(\theta) f(\theta)\,d\theta. \end{aligned}$$

Thus $\mathrm{Re}\,\{\sigma(u,u)\} \leq \omega\,\|u\|_w^2$, where

$$\omega = \max\Big\{\mu(A) + \frac{1}{2}\|W\|c^2 + \frac{1}{2}\sqrt{\|W\|^2c^4 + 4(b+\|H\|c)^2},\ -\frac{\gamma\beta}{2}e^{-\gamma r_n}\Big\} < 0.$$

Since $\sigma(u,u) = \langle \mathcal{A}u, u\rangle$ for $u \in D(\mathcal{A})$, the result follows. $\square$

2.3 Comparisons

There are a number of sufficient conditions for exponential stability of neutral equations already described in the literature. However, there is no clear best condition to date. The condition proposed here in Theorem 2.9 is independent of some previously studied delay-independent stability conditions for neutral equations, and improves another. Recall that all of the conditions we study require that $\sum_{i=1}^{n}\|C_i\| < 1$ (or $\|C\| < 1$ where the paper studies only single delay problems), so we will not list that part of the stability conditions. We will compare condition (2.3) used in Theorem 2.9 with the conditions found in equations (1.12), (1.13), (1.15), and (1.14).

Although there is no single clear best condition, there are some results for special cases. For example, it is known that in the single-delay case when $B = 0$ and A and C are both scalars, a necessary and sufficient condition for exponential stability of the solution semigroup is $A < 0$ and $|C| < 1$. In the single-delay case, both condition (1.12) and condition (1.13) require that $A < 0$ and $|C| < 1/2$. This is significantly more restrictive than $|C| < 1$. In this case, condition (2.3) reduces to $A < \frac{-1}{2} A |C|^2 + \frac{1}{4} A |C|^2 + \frac{1}{4} A |C|^2$ and $|C| < 1$, which means $A < 0$ and does not further restrict C. In this sense, the proposed condition is better than some existing conditions.

We will now describe the relationships between all the listed stability conditions. It can be shown that condition (1.14) is independent of the condition (1.13) and (1.15), but we do not include the proofs here since they are similar to existing proofs. It is also easy to check that if equation (1.12) is satisfied, so are equations (1.13) and (1.14). Thus condition (1.12) implies both condition (1.14) and (1.13). Also, Fabiano and Turi showed that condition (1.12) implies equation (1.15). Thus all three conditions are stronger than condition (1.12). In addition Fabiano and Turi showed that (1.15) is independent of condition (1.13). One can then show that (2.3) is independent of both (1.13) and (1.14). It improves (1.12). We will show this with a collection of single-delay examples with 2×2 matrices.

We now turn to comparisons with (2.3), and we will first show independence. We can use a single strategic example for both condition (1.13) and (1.14). Take

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

B the zero matrix, and C any 2×2 matrix with $\frac{1}{2} < \|C\| < 1$. Then we have $\mu(A) = -1$, $\|A\| = 1$, and $\|CA\| = \|AC\| = \|C\|$. Then the left side of (1.13) reduces to $\mu(A)\,(1 - \|C\|) + \|CA\| = 0$, and the left side of (1.14) reduces to $\mu(A)\,(1 - \|C\|) + \|AC\| = 0$. Meanwhile, the left side of (2.3) reduces to $\mu(A) + \|A\|\,\|C\|^2 = -1 + \|C\|^2 < 0$, so conditions (1.13) and (1.14) are not satisfied, while (2.3) is satisfied. Note that in this example, $\|C\| > \dfrac{1}{2}$, which shows that condition (1.13) is more restrictive than necessary in some cases. For the other direction, take

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -9 \end{bmatrix}, C = \begin{bmatrix} \frac{5}{12} & 0 \\ 0 & 0 \end{bmatrix},$$

and B again the zero matrix. We have $\mu(A) = -1$, $\|A\| = 9$, $\|C\| = \frac{5}{12}$, and $\|CA\| = \|AC\| = \frac{5}{12}$. Then the left side of (1.13) and (1.14) reduces to $\frac{-2}{12}$, and the left side of (2.3) reduces to $\frac{9}{16}$, so the new condition (2.3) is not satisfied while conditions (1.13) and (1.14) are both satisfied.

(2.3) is an improvement on condition (1.12). To see that they are not equivalent, we recall that since condition (1.12) implies condition (1.13), the above example which satisfies condition (2.3) but fails condition (1.13) will also fail condition (1.12).

To see that the new condition is an improvement, we have the following theorem. Note that (1.12) is valid only for single-delay equations.

Theorem 2.10. *If the coefficients of equation (*1.1*) satisfy condition (*1.12*), then they also satisfy (*2.3*).*

Proof. To see this, we will show that the left side of (2.3) is bounded above by (1.12). We first show that

$$\|W\|^2 c^4 + 4b^2 + 8b\,\|H\|\,c + 4\,\|H\|^2 c^2 < \left[-\|W\|\,c^2 + 2\,|\mu(A)|\,c + 2\,\|A\|\,c + 2b\right]^2. \tag{2.29}$$

This follows in part from $\|W\|^2 c^4 = (-\|W\|\,c^2)^2$ and $4b^2 = (2b)^2$. Also, since $H = \frac{1}{2}(A - \overline{A}^T)$, we have $\|H\| \leq \|A\|$, and thus $4\,\|H\|^2 c^2 \leq (2\,\|A\|\,c)^2$. Finally, using the same idea, we have $8b\,\|H\|\,c \leq 2(2\,\|A\|\,c)(2b)$. Since

$$\left[\|W\|\,c^2 + 2\,|\mu(A)|\,c + 2\,\|A\|\,c + 2b\right]^2$$

consists of these right hand terms plus other positive terms, we have the desired inequality. Taking square roots on both sides of (2.29) and multiplying by $\frac{1}{2}$ yields

$$\frac{1}{2}\sqrt{\|W\|^2 c^4 + 4(b + \|H\|\,c)^2} \leq \frac{-1}{2}\,\|W\|\,c^2 + |\mu(A)|\,c + \|A\|\,c + b. \tag{2.30}$$

To finish the proof, we simply move the $\|W\|\,c^2$ term to the other side and add $\mu(A)$ on both sides, which gives us

$$\mu(A) + \frac{1}{2}\,\|W\|\,c^2 + \frac{1}{2}\sqrt{\|W\|^2 c^4 + 4(b + \|H\|\,c)^2}$$

$$\begin{aligned} &\leq \mu(A) + |\mu(A)|\, c + \|A\|\, c + b \\ &= \mu(A)(1-c) + \|A\|\, c + b. \end{aligned}$$

Since the left hand side is (2.3), and the right hand side is equivalent to (1.12), we are done. □

We now address the relationship between condition (2.3) and (1.15). The two conditions are not equivalent, as demonstrated by the following example. Take

$$A = \begin{bmatrix} -0.8211 & -0.4917 \\ -0.6011 & -0.6262 \end{bmatrix}, C = \begin{bmatrix} 0.1811 & 0.2720 \\ 0.2917 & 0.2505 \end{bmatrix},$$

and B the zero matrix. Then the smallest eigenvalue of (1.15) is .15846, and the left side of (2.3) reduces to 0.15335, so the condition (1.15) holds and condition (2.3) does not hold. In fact, condition (2.3) implies condition (1.15), formalized below.

Theorem 2.11. *If the coefficients of equation (*1.1*) satisfy the condition (*2.3*), then they also satisfy (*1.15*).*

Proof. Suppose that condition (2.3) holds. This is equivalent to

$$-\,|\mu(A)|^2 + |\mu(A)|\, \|W\|\, c^2 + (b + \|H\|\, c)^2 < 0. \tag{2.31}$$

To see this, we note that (2.3) is equivalent to

$$\sqrt{\|W\|^2 c^4 + 4(b + \|H\|\, c)^2} < -2\mu(A) - \|W\|\, c^2.$$

Then squaring both sides and rearranging terms leads to $-4\mu(A)^2 - 4\mu(A)\|W\| c^2 + 4(b + \|H\| c)^2 < 0$, and then dividing by 4 yields $-|\mu(A)|^2 + |\mu(A)| \|W\| c^2 + (b + \|H\| c)^2 < 0$. We know that since W is positive definite and self-adjoint and $W = \frac{-1}{2}(A + \overline{A}^T)$, we have

$$|\mu(A)| \|x\|^2 \leq \overline{x}^T W x \leq \|W\| \|x\|^2$$

for any $x \in \mathbb{C}^n$. Using properties of norms we also have

$$\overline{x}^T \overline{C}^T W C x = \overline{Cx}^T W C x \leq \|W\| \|C\|^2 \|x\|^2 \tag{2.32}$$

$$\overline{x}^T \overline{B}^T B x = \overline{Bx}^T B x \leq \|B\|^2 \|x\|^2 \tag{2.33}$$

$$\overline{x}^T \overline{C}^T \overline{H}^T H C x = \overline{HCx}^T H C x \leq \|H\|^2 \|C\|^2 \|x\|^2 \tag{2.34}$$

for all $x \in \mathbb{C}^n$. Therefore, in order to verify that condition (1.15) holds, it suffices to show that

$$|\mu(A)| > \|C\|^2 \|W\| + \frac{1}{k} \|C\|^2 \|H\|^2 + \frac{1}{|\mu(A)| - k} \|B\|^2 \tag{2.35}$$

for some k with $0 < k < |\mu(A)|$. For now, suppose $B \neq 0$ and $H \neq 0$. Then we take

$$k = \frac{|\mu(A)| \|C\| \|H\|}{\|B\| + \|C\| \|H\|}.$$

$B \neq 0$ and $H \neq 0$ will guarantee that $0 < k < |\mu(A)|$. With this choice of k, equation (2.35) becomes

$$|\mu(A)| > \|C\|^2 \|W\| + \frac{1}{|\mu(A)|} \left(\|B\| + \|H\| \|C\|\right)^2. \tag{2.36}$$

However, multiplying both sides by $|\mu(A)|$ will show that this is equivalent to equation (2.31), and since (2.31) holds by assumption, so does (2.36).

When $H = 0$, (2.31) becomes $-|\mu(A)|^2 + |\mu(A)| \|W\| c^2 + b^2 < 0$ and it suffices to show that

$$|\mu(A)| > \|C\|^2 \|W\| + \frac{1}{|\mu(A)| - k} \|B\|^2, \tag{2.37}$$

or equivalently, $|\mu(A)|^2 > |\mu(A)| \|C\|^2 \|W\| + \frac{|\mu(A)|}{|\mu(A)| - k} \|B\|^2$, for an appropriate k. Since we know $|\mu(A)| \|W\| c^2 + b^2 < |\mu(A)|^2$, we must simply choose k close enough to $|\mu(A)|$ so that

$$|\mu(A)| \|W\| c^2 + b^2 - r < |\mu(A)| \|C\|^2 \|W\| + \frac{|\mu(A)|}{|\mu(A)| - k} \|B\|^2 < |\mu(A)|^2.$$

Similarly, when $B = 0$, (2.31) becomes $-|\mu(A)|^2 + |\mu(A)| \|W\| c^2 + \|H\|^2 c^2 < 0$, which implies that $|\mu(A)) > \|W\| c^2 + \frac{1}{|\mu(A)|} \|H\|^2 c^2$. It suffices to show that

$$|\mu(A)| > \|C\|^2 \|W\| + \frac{1}{k} \|C\|^2 \|H\|^2 \tag{2.38}$$

for an appropriate choice of k. Here again, we choose k close enough to $|\mu(A)|$ so that

$$\|W\| c^2 + \frac{1}{|\mu(A)|} \|H\|^2 c^2 < \|W\| c^2 + \frac{1}{k} \|H\|^2 c^2 < |\mu(A)| .$$

This completes the proof. □

Although this shows that condition (1.15) is better in some cases, we note that it holds only for equations with one delay. Thus, our condition is still a significant improvement in the sense of allowing for multiple delays. It is possible that a similar (but so far unknown) condition could be constructed which would be equivalent to the condition (1.15) in the single delay case.

2.4 Open Questions

Although the condition proposed here does improve upon existing conditions in some ways, it is unclear whether there is a stability condition for equations with multiple delays that will reduce to the (known) good condition for the scalar case. Also, numerical experiments suggest that it may be possible to bound the real part of eigenvalues of the operator $\mathcal{A}$ away from 0 for a wider class of delay equations than just the ones satisfying equation (2.3). This suggests that we may be able to obtain stability results for a wider class of equations. Finally, we have not yet addressed the question of what delay equations may be stablizable, but not stable.

CHAPTER III

ADJOINT OPERATOR FOR NEUTRAL SYSTEMS

We will now calculate the adjoint of the operator $\mathcal{A}$ given in equation (1.5). This operator is essential to the construction of our approximation scheme. We first consider the computation of $\mathcal{A}^*$ in the usual norm, when W is the identity matrix and $w(\theta) = 1$. Then it will be easy to describe the adjoint in the weighted inner product in terms of $\mathcal{A}^*$. First, recall from the definition of an adjoint operator that the adjoint operator $\mathcal{A}^*$ has domain

$$\begin{aligned} D(\mathcal{A}^*) &= \{(\gamma, g) \in M_2 : \text{ there exists } (\beta, z) \in M_2 \text{ such that} \\ &\qquad \langle \mathcal{A}(\eta, f), (\gamma, g)\rangle = \langle (\eta, f), (\beta, z)\rangle \text{ for all } (\eta, f) \in D(\mathcal{A})\}, \end{aligned} \tag{3.1}$$

and in this case $\mathcal{A}^*(\gamma, g) = (\beta, z)$.

Recall the space V defined in equation (1.28). We will also define the space

$$S = \left\{(\gamma, g) \in V : \left(\overline{B_i}^T - \overline{C_i}^T \overline{A}^T\right)\gamma - k_{g,i} = \overline{C_i}^T g(0) \text{ for } i = 1, 2, \cdots, n\right\}. \tag{3.2}$$

Note that S is a subset of V, and we will show that S is in fact $D(\mathcal{A}^*)$.

Theorem 3.1. *On M_2 endowed with the inner product (1.21), the adjoint of the operator $\mathcal{A}$ is given by*

$$\mathcal{A}^*(\gamma, g) = \left(\overline{A}^T \gamma + g(0), -\widetilde{g}'(\theta)\right) \tag{3.3}$$

with domain $D(\mathcal{A}^*) = S$.

Proof. We will prove that $S = D(\mathcal{A}^*)$ in the usual way, by showing each set is a subset of the other, and while doing this we will verify that (3.3) holds. We first prove $S \subset D(\mathcal{A}^*)$, so suppose $(\gamma, g) \in S$. Then using the definition of the operator $\mathcal{A}$ we have

$$\langle \mathcal{A}(\eta, f), (\gamma, g) \rangle = \left\langle Af(0) + \sum_{i=1}^{n} B_i f(-r_i), \gamma \right\rangle + \int_{-r_n}^{0} \overline{g(\theta)}^T f'(\theta) d\theta \tag{3.4}$$

for all $(\eta, f) \in D(\mathcal{A})$.

Note that for $\theta \in [-r_i, -r_{i-1}]$ we have that $g(\theta) = \widetilde{g}(\theta) + \sum_{j=i}^{n-1} k_{g,j}$, because $\chi_j(\theta) = 0$ for any $j < i$ and $\chi_j(\theta) = 1$ for any $j \geq i$. Using this fact to rewrite the integral gives us

$$\begin{aligned}
&\langle \mathcal{A}(\eta, f), (\gamma, g) \rangle \\
&= \overline{\gamma}^T \left(Af(0) + \sum_{i=1}^{n} B_i f(-r_i) \right) + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T f'(\theta) d\theta \\
&= \overline{\gamma}^T \left(Af(0) + \sum_{i=1}^{n} B_i f(-r_i) \right) + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} [\overline{\widetilde{g}(\theta)}^T + \sum_{j=i}^{n-1} \overline{k_{g,j}}^T] f'(\theta) d\theta \\
&= \overline{\gamma}^T \left(Af(0) + \sum_{i=1}^{n} B_i f(-r_i) \right) + \int_{-r_n}^{0} \overline{\widetilde{g}(\theta)}^T f'(\theta) d\theta \\
&\quad + \sum_{i=1}^{n-1} \int_{-r_i}^{-r_{i-1}} \left(\sum_{j=i}^{n-1} \overline{k_{g,j}}^T \right) f'(\theta) d\theta \\
&= \overline{\gamma}^T \left(Af(0) + \sum_{i=1}^{n} B_i f(-r_i) \right) + \overline{\widetilde{g}(0)}^T f(0) - \overline{\widetilde{g}(-r_n)}^T f(-r_n) \\
&\quad - \int_{-r_n}^{0} \overline{\widetilde{g}'(\theta)}^T f(\theta) d\theta + \sum_{i=1}^{n-1} \left(\sum_{j=i}^{n-1} \overline{k_{g,j}}^T \right) [f(-r_{i-1}) - f(-r_i)]
\end{aligned}$$

where we integrated by parts over each interval and used the fact that $f \in H^1([-r_n, 0], \mathbb{C}^m)$. We continue to get

$$
\begin{aligned}
&\langle \mathcal{A}(\eta, f), (\gamma, g)\rangle \\
&= \left(\overline{\gamma}^T A + \overline{\widetilde{g}(0)}^T\right) f(0) + \sum_{i=1}^{n-1} \overline{\gamma}^T B_i f(-r_i) + \left(\overline{\gamma}^T B_n - \overline{\widetilde{g}(-r_n)}^T\right) f(-r_n) \\
&\quad + \left(\sum_{i=1}^{n-1} \overline{k_{g,i}}^T\right) f(0) - \sum_{i=1}^{n-1} \overline{k_{g,i}}^T f(-r_i) + \int_{-r_n}^{0} \overline{[-\widetilde{g}'(\theta)]}^T f(\theta) d\theta \\
&= \left(\overline{\gamma}^T A + \overline{\widetilde{g}(0)}^T + \sum_{i=1}^{n-1} \overline{k_{g,i}}^T\right) f(0) + \sum_{i=1}^{n} \left(\overline{\gamma}^T B_i - \overline{k_{g,i}}^T\right) f(-r_i) \\
&\quad + \int_{-r_n}^{0} \overline{[-\widetilde{g}'(\theta)]}^T f(\theta) d\theta.
\end{aligned}
$$

We now use $g(0) = \widetilde{g}(0) + \sum_{i=1}^{n-1} k_{g,i}$ and $f(0) = \eta - \sum_{i=1}^{n} C_i f(-r_i)$ and collect terms to see that

$$
\begin{aligned}
&\langle \mathcal{A}(\eta, f), (\gamma, g)\rangle \\
&= \left(\overline{\gamma}^T A + \overline{g(0)}^T\right) \left(\eta - \sum_{i=1}^{n} C_i f(-r_i)\right) + \sum_{i=1}^{n} \left(\overline{\gamma}^T B_i - \overline{k_{g,i}}^T\right) f(-r_i) \\
&\quad + \int_{-r_n}^{0} \overline{[-\widetilde{g}'(\theta)]}^T f(\theta) d\theta \\
&= \left(\overline{\gamma}^T A + \overline{g(0)}^T\right) \eta + \sum_{i=1}^{n} \left(\overline{\gamma}^T B_i - \overline{k_{g,i}}^T - \overline{\gamma}^T A C_i - \overline{g(0)}^T C_i\right) f(-r_i) \\
&\quad + \int_{-r_n}^{0} \overline{[-\widetilde{g}'(\theta)]}^T f(\theta) d\theta.
\end{aligned}
$$

Then since $(\gamma, g) \in S$, $\overline{\gamma}^T B_i - \overline{k_{g,i}}^T - \overline{\gamma}^T A C_i - \overline{g(0)}^T C_i = 0$ for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}\langle \mathcal{A}(\eta, f), (\gamma, g) \rangle &= \left(\overline{\gamma}^T A + \overline{g(0)}^T \right) \eta + \int_{-r_n}^{0} \overline{[-\widetilde{g}'(\theta)]}^T f(\theta) d\theta \\ &= \langle (\eta, f(\theta)), \left(\overline{A}^T \gamma + g(0), -\widetilde{g}'(\theta) \right) \rangle.\end{aligned}$$

It follows that $\langle \mathcal{A}(\eta, f), (\gamma, g) \rangle = \langle (\eta, f), (\alpha, h) \rangle$ for all $(\eta, f) \in D(\mathcal{A})$, where $(\alpha, h) = (\overline{A}^T \gamma + g(0), -\widetilde{g}'(\theta))$. Thus $(\gamma, g) \in D(\mathcal{A}^*)$ and $\mathcal{A}^*(\gamma, g) = \left(\overline{A}^T \gamma + g(0), -\widetilde{g}'(\theta) \right)$. This shows that $S \subset D(\mathcal{A}^*)$.

To show $D(\mathcal{A}^*) \subset S$, first fix λ sufficiently large so that λ is in the resolvent set of $\mathcal{A}$ and hence the range of $\mathcal{A} - \lambda I$ is the entire space M_2. Such a λ exists by the Hille-Yosida theorem since $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup. To simplify notation, let $E_i = \overline{C_i}^T \overline{A}^T - \overline{B_i}^T$. Also choose λ large enough that the determinant of

$$A - \lambda I - e^{-\lambda r_n} \left(E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T \right) - \sum_{i=1}^{n-1} e^{-\lambda r_1} \left(E_i - \overline{C_i}^T A + \lambda \overline{C_i}^T \right) \tag{3.5}$$

is nonzero. Now suppose $(\gamma, g) \in D(\mathcal{A}^*) = D(\mathcal{A}^* - \lambda I)$. Then there exists some (α, h) such that $\langle (\mathcal{A} - \lambda I)(\eta, f), (\gamma, g) \rangle = \langle (\eta, f), (\alpha, h) \rangle$ for all $(\eta, f) \in D(\mathcal{A})$. Our argument proceeds as follows: we construct (β, z) such that $(\beta, z) \in S$ and $\langle (\mathcal{A}^* - \lambda I)(\eta, f), (\beta, z) \rangle = \langle (\eta, f), (\alpha, h) \rangle$ for all $(\eta, f) \in D(\mathcal{A})$. This implies that $\langle (\mathcal{A}^* - \lambda I)(\eta, f), (\beta, z) - (\gamma, g) \rangle = 0$, which then implies $(\beta, z) = (\gamma, g)$ since λ is in the resolvent set of $\mathcal{A}$, and hence $(\gamma, g) \in S$. Our recipe for constructing (β, z) comes from the set of equations which arises when one formally writes $(\mathcal{A}^* - \lambda I)(\beta, z) = (\alpha, h)$. To proceed, in order to construct β and z, we will need to specify values for β and $k_{z,i}$

for $i = 1, 2, \ldots, n-1$ and a function $\widetilde{z}(\theta)$. In order to have $(\mathcal{A}^* - \lambda I)(\beta, z) = (\alpha, h)$, we must have $-\widetilde{z}'(\theta) - \lambda z(\theta) = h(\theta)$. We consider this equation on the interval $[-r_i, -r_{i-1})$ and multiply by $e^{\lambda\theta}$ to get

$$\left(\widetilde{z}(\theta)e^{\lambda\theta}\right)' = -h(\theta)e^{\lambda\theta} - \lambda e^{\lambda\theta}\sum_{j=1}^{n-1} k_{z,j}. \tag{3.6}$$

Integrating both sides tells us that on $[-r_i, -r_{i-1})$, $\widetilde{z}(\theta)$ must have the form

$$\widetilde{z}(\theta) = d_i e^{-\lambda\theta} - e^{-\lambda\theta}\int_{-r_i}^{\theta} e^{\lambda s}h(x)\,ds - \sum_{j=i}^{n-1} k_{z,j} + e^{-\lambda(\theta+r_i)}\sum_{j=i}^{n-1} k_{z,j}. \tag{3.7}$$

Thus we will first describe the function $z(\theta)$ on $[-r_1, 0]$ by choosing d_1 and $k_{z,i}$, and then proceed to describe it on the other intervals.

We will choose β, d_1, and the $k_{z,i}$ to be column vectors which are a solution of the system

$$\begin{cases} \left(e^{-\lambda r_n} I + \overline{C_n}^T\right) d_1 + E_n\beta + \overline{C_n}^T \sum_{i=1}^{n-1} e^{-\lambda r_1} k_{z,i} &= \overline{C_n}^T \int_{-r_1}^{0} e^{\lambda s}h(s)ds \\ &\quad - e^{-\lambda r_n}\int_{-r_n}^{-r_1} e^{\lambda s}h(s)ds \\ d_1 + (A - \lambda I)\,\beta + \sum_{i=1}^{n-1} e^{-\lambda r_1} k_{z,i} &= \alpha + \int_{-r_1}^{0} e^{\lambda s}h(s)ds \\ \overline{C_i}^T d_1 + E_i\beta + k_{z,i} + \sum_{j=1}^{n-1} e^{-\lambda r_1}\overline{C_i}^T k_{z,j} &= \left(-\int_{-r_1}^{0} e^{\lambda s}h(s)ds\right)\overline{C_i}^T \end{cases}$$

Notice that the last equation is really a collection of n equations, one for each $i = 1, 2, \cdots, n$. In this system, the first and last equations will guarantee that $\left(\overline{B_i}^T - \overline{C_i}^T\overline{A}^T\right)\beta - k_{z,i} = \overline{C_i}^T z(0)$ so that $(\beta, z) \in S$. The construction of z will guarantee that $(\beta, z) \in V$ and that $-\widetilde{z}'(\theta) - \lambda z(\theta) = h(\theta)$. In order to have

$(\mathcal{A}^* - \lambda I)(\beta, z) = (\alpha, h)$, we must also have $(\overline{A}^T - \lambda I)\beta + z(0) = \alpha$, and this will come from the equation

$$d_1 + (A - \lambda I)\,\beta + \sum_{i=1}^{n-1} e^{-\lambda r_1} k_{z,i} = \alpha + \int_{-r_1}^{0} e^{\lambda s} h(s) ds.$$

We must first determine if a solution exists, so we consider the following matrix given in block form:

$$M = \begin{bmatrix} e^{\lambda r_n} I + \overline{C_n}^T & E_n & e^{-\lambda r_1}\overline{C_n}^T & e^{-\lambda r_1}\overline{C_n}^T & \cdots & e^{-\lambda r_1}\overline{C_n}^T \\ I & A - \lambda I & e^{-\lambda r_1} I & e^{-\lambda r_1} I & \cdots & e^{-\lambda r_1} I \\ \overline{C_1}^T & E_1 & I + e^{-\lambda r_1}\overline{C_1}^T & e^{-\lambda r_1}\overline{C_1}^T & \cdots & e^{-\lambda r_1}\overline{C_1}^T \\ \overline{C_2}^T & E_2 & e^{-\lambda r_1}\overline{C_2}^T & I + e^{-\lambda r_1}\overline{C_2}^T & \cdots & e^{-\lambda r_1}\overline{C_2}^T \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{C_{n-1}}^T & E_{n-1} & e^{-\lambda r_1}\overline{C_{n-1}}^T & e^{-\lambda r_1}\overline{C_{n-1}}^T & \cdots & I + e^{-\lambda r_1}\overline{C_{n-1}}^T \end{bmatrix}.$$

We will multiply this matrix by several others in order to simplify it. First, we multiply by a block matrix with each block being a $m \times m$ matrix with I's on the diagonal, and $-\overline{C_n}^T$ in the $1, 2$ position. This gives:

$$\begin{bmatrix} I & -\overline{C_n}^T & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & 0 \cdots & & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \end{bmatrix} M =$$

$$\begin{bmatrix} e^{\lambda r_n} I & E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T & 0 & 0 & \cdots & 0 \\ I & A - \lambda I & e^{-\lambda r_1} I & e^{-\lambda r_1} I & \cdots & e^{-\lambda r_1} I \\ \overline{C_1}^T & E_1 & I + e^{-\lambda r_1} \overline{C_1}^T & e^{-\lambda r_1} \overline{C_1}^T & \cdots & e^{-\lambda r_1} \overline{C_1}^T \\ \overline{C_2}^T & E_2 & e^{-\lambda r_1} \overline{C_2}^T & I + e^{-\lambda r_1} \overline{C_2}^T & \cdots & e^{-\lambda r_1} \overline{C_2}^T \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{C_{n-1}}^T & E_{n-1} & e^{-\lambda r_1} \overline{C_{n-1}}^T & e^{-\lambda r_1} \overline{C_{n-1}}^T & \cdots & I + e^{-\lambda r_1} \overline{C_{n-1}}^T \end{bmatrix}$$

In a similar manner, we will multiply by another block matrix with block I's on the diagonal, and $-\overline{C_i}^T$ in the $i+2, 2$ position for $i = 1, 2, \cdots, n-1$. We notice that both of these matrices have determinant 1 since they are triangular and have identity matrices on the diagonal, so multiplying by these matrices does not change the determinant of M. After these multiplications, we have

$$M' = \begin{bmatrix} e^{\lambda r_n} I & E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T & 0 & 0 & \cdots & 0 \\ I & A - \lambda I & e^{-\lambda r_1} I & e^{-\lambda r_1} I & \cdots & e^{-\lambda r_1} I \\ 0 & E_1 - \overline{C_1}^T A + \lambda \overline{C_1}^T & I & 0 & \cdots & 0 \\ 0 & E_2 - \overline{C_2}^T A + \lambda \overline{C_2}^T & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & E_{n-1} - \overline{C_{n-1}}^T A + \lambda \overline{C_{n-1}}^T & 0 & 0 & \cdots & I \end{bmatrix}.$$

We now multiply by one final matrix to simplify the second row:

$$\begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ -e^{\lambda r_n} I & I & -e^{-\lambda r_1} & -e^{-\lambda r_1} & \cdots & -e^{-\lambda r_1} \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \end{bmatrix} M'$$

$$= \begin{bmatrix} e^{\lambda r_n} I & E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T & 0 & 0 & \cdots & 0 \\ 0 & D & 0 & 0 & \cdots & 0 \\ 0 & E_1 - \overline{C_1}^T A + \lambda \overline{C_1}^T & I & 0 & \cdots & 0 \\ 0 & E_2 - \overline{C_2}^T A + \lambda \overline{C_2}^T & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & E_{n-1} - \overline{C_{n-1}}^T A + \lambda \overline{C_{n-1}}^T & 0 & 0 & \cdots & I \end{bmatrix} = M'',$$

where $D = A - \lambda I - \sum_{i=1}^{n-1} e^{-\lambda r_1} \left(E_i - \overline{C_i}^T A + \lambda \overline{C_i}^T \right)$. Notice that again the determinant of the multiplication matrix is 1 because

$$\begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ -e^{\lambda r_n} I & I & -e^{-\lambda r_1} & -e^{-\lambda r_1} & \cdots & -e^{-\lambda r_1} \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ -e^{\lambda r_n} I & I & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & -e^{-\lambda r_1} & -e^{-\lambda r_1} & \cdots & -e^{-\lambda r_1} \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \end{bmatrix}$$

and both of these matrices are triangular with ones on their diagonals, so their determinants are both 1.

We use the following two lemmas about the determinant of a block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

to calculate the determinant of M'.

Lemma 3.2. *[LT85, Exercise 2.4.2]*

$$\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det A \det D$$

Lemma 3.3. *If A is invertible, then*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det \left(D - CA^{-1}B \right).$$

Lemma 3.3 follows from Lemma 3.2 by writing

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}. \tag{3.8}$$

Notice that we can view M'' as a 2×2 block matrix, where the upper right block is 0, and the lower right block is an identity, the upper left block is

$$\begin{bmatrix} e^{\lambda r_n} I & E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T \\ I & D \end{bmatrix}$$

and the lower left block is

$$\begin{bmatrix} 0 & E_1 - \overline{C_1}^T A + \lambda \overline{C_1}^T \\ 0 & E_2 - \overline{C_2}^T A + \lambda \overline{C_2}^T \\ \vdots & \vdots \\ 0 & E_{n-1} - \overline{C_{n-1}}^T A + \lambda \overline{C_{n-1}}^T \end{bmatrix}.$$

Thus

$$\begin{aligned} \det M' &= \det \begin{bmatrix} e^{\lambda r_n} I & E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T \\ I & D \end{bmatrix} \det I \\ &= \det \begin{bmatrix} e^{\lambda r_n} I & E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T \\ I & D \end{bmatrix} \\ &= \det \left(e^{\lambda r_n} I\right) \det \left(D - I \left(e^{\lambda r_n} I\right)^{-1} \left(E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T\right)\right) \end{aligned}$$

$$\begin{aligned} &= e^{\lambda r_n} \det \Bigg(A - \lambda I - \sum_{i=1}^{n-1} e^{-\lambda r_1} \left(E_i - \overline{C_i}^T A + \lambda \overline{C_i}^T \right) \\ &\quad - e^{-\lambda r_n} \left(E_n - \overline{C_n}^T A + \lambda \overline{C_n}^T \right) \Bigg) \\ &\neq 0 \end{aligned}$$

where the last step holds by our choice of λ.

Now we can define

$$\widetilde{z}(\theta) = d_1 e^{-\lambda\theta} - e^{-\lambda\theta} \int_{-r_1}^{\theta} e^{\lambda s} h(s) ds - \sum_{j=1}^{n-1} k_{z,j} + e^{-\lambda(\theta + r_1)} \sum_{j=1}^{n-1} k_{z,j}$$

on $[-r_1, 0]$. Similarly, we let $d_2 = d_1 + \int_{-r_2}^{-r_1} e^{\lambda s} h(s) ds$, and define

$$\widetilde{z}(\theta) = d_2 e^{-\lambda\theta} - e^{-\lambda\theta} \int_{-r_2}^{\theta} e^{\lambda s} h(s) ds - \sum_{j=2}^{n-1} k_{z,j} + e^{-\lambda(\theta + r_2)} \sum_{j=2}^{n-1} k_{z,j}$$

on $[-r_2, -r_1]$. Proceeding similarly, we can let $d_i = d_1 + \int_{-r_i}^{-r_1} e^{\lambda s} h(s) ds$ and define $\widetilde{z}(\theta)$ on $[-r_i, -r_{i-1}]$ for $1 \leq i \leq n-1$ by

$$\widetilde{z}(\theta) = d_i e^{-\lambda\theta} - e^{-\lambda\theta} \int_{-r_i}^{\theta} e^{\lambda s} h(s) ds - \sum_{j=i}^{n-1} k_{z,j} + e^{-\lambda(\theta + r_i)} \sum_{j=i}^{n-1} k_{z,j}.$$

Finally, we set

$$\widetilde{z}(\theta) = d_n e^{-\lambda\theta} - e^{-\lambda\theta} \int_{-r_n}^{\theta} e^{\lambda s} h(s) ds$$

on $[-r_n, -r_{n-1}]$ with the same choice of d_n. Then $z(\theta) = \widetilde{z}(\theta) + \sum_{i=1}^{n-1} \chi_{[-r_i,0]}(\theta) k_{z,i}$. To show $(\beta, z) \in S$, we show that

i) $z(\theta) = \widetilde{z}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{z,i}$,

ii) $\widetilde{z}(\theta) \in H^1(-r_n, 0)$,

iii) $\left(\overline{B_i}^T - \overline{C_i}^T \overline{A}^T\right) \beta - k_{z,i} = \overline{C_i}^T z(0)$ for $i = 1, 2, \cdots, n$.

Criteria i) and ii) are satisfied by the way we have constructed $\widetilde{z}(\theta)$ and z. Criteria iii) is satisfied because β, d_1, and the $k_{z,i}$ are solutions to the above system of equations. We can see that $z(-r_n) = d_1 e^{\lambda r_n} + e^{\lambda r_n} \int_{-r_n}^{-r_1} e^{\lambda s} h(s) ds$ and that $z(0) = d_1 - \int_{-r_1}^{0} e^{\lambda s} h(s) ds + e^{-\lambda r_1} \sum_{j=1}^{n-1} k_{z,i}$ from our definition of z. Then the first equation says that

$$\left(e^{-\lambda r_n} I + \overline{C_n}^T\right) d_1 + E_n \beta + \overline{C_n}^T \sum_{i=1}^{n-1} e^{-\lambda r_1} k_{z,i}$$
$$= \overline{C_n}^T \int_{-r_1}^{0} e^{\lambda s} h(s) ds - e^{-\lambda r_n} \int_{-r_n}^{-r_1} e^{\lambda s} h(s) ds,$$

so therefore

$$E_n \beta - \left(d_1 e^{\lambda r_n} \int_{-r_n}^{-r_1} e^{\lambda s} h(s) ds\right) = \overline{C_n}^T \left(d_1 - \int_{-r_1}^{0} e^{\lambda s} h(s) ds + e^{-\lambda r_1} \sum_{j=1}^{n-1} k_{z,j}\right),$$

and therefore $E_n \beta - z(-r_n) = \overline{C_n}^T z(0)$ and the last criteria is satisfied. The third criteria is really a collection of $n-1$ criteria, and these are satisfied because of the last $n-1$ equations in the system, $\overline{C_i}^T d_1 + E_i \beta + k_{z,i} + \sum_{j=1}^{n-1} e^{-\lambda r_1} \overline{C_i}^T k_{z,j} = \left(-\int_{-r_1}^{0} e^{\lambda s} h(s) ds\right) \overline{C_i}^T$. Therefore $(\beta, z) \in S$.

Next, I claim that $(\mathcal{A}^* - \lambda I)(\beta, z) = (\alpha, h)$. By our previous work, since $(\beta, z) \in S$ we know that $(\mathcal{A}^* - \lambda I)(\beta, z) = (\overline{A}^T \beta + z(0) - \lambda\beta, -\widetilde{z}'(\theta) - \lambda z)$. By the second equation in the system, we have that $d_1 + (A - \lambda I)\,\beta + \sum_{i=1}^{n-1} e^{-\lambda r_1} k_{z,i} = \alpha + \int_{-r_1}^{0} e^{\lambda s} h(s)ds$, so therefore $\overline{A}^T \beta + z(0) - \lambda\beta = \alpha$. For the second component, we have that $-\widetilde{z}'(\theta) - \lambda z = h$ by our choice of the function z. We can see that on $[-r_i, -r_{i-1}]$ we have that

$$\widetilde{z}'(\theta) = -\lambda d_i e^{-\lambda\theta} + \lambda e^{-\lambda\theta} \int_{-r_i}^{\theta} e^{\lambda s} h(s)ds - h(\theta) - \lambda(\sum_{j=i}^{n-1} k_{g,i}) e^{-\lambda(\theta + r_i)}.$$

Then

$$\begin{aligned} -\widetilde{z}'(\theta) - \lambda z &= \lambda d_i e^{-\lambda\theta} - \lambda e^{-\lambda\theta} \int_{-r_i}^{\theta} e^{\lambda s} h(s)ds + h(\theta) + \lambda(\sum_{j=i}^{n-1} k_{z,i}) e^{-\lambda(\theta + r_i)} - \lambda d_i e^{-\lambda\theta} \\ &= h(\theta). \end{aligned}$$

Thus we have that $(\mathcal{A}^* - \lambda I)(\beta, z) = (\alpha, h)$.

Finally, I want to show that we must have $(\beta, z) = (\gamma, g)$. But since $(\mathcal{A}^* - \lambda I)(\beta, z) = (\alpha, h)$ and $\langle(\mathcal{A} - \lambda I)(\eta, f), (\gamma, g)\rangle = \langle(\eta, f), (\alpha, h)\rangle$, we have that

$$\begin{aligned} \langle(\mathcal{A} - \lambda I)(\eta, f), (\gamma, g)\rangle &= \langle(\eta, f), (\mathcal{A}^* - \lambda I)(\beta, z)\rangle \\ &= \langle(\mathcal{A} - \lambda I)(\eta, f), (\beta, z)\rangle. \end{aligned}$$

Since the range of $\mathcal{A} - \lambda I$ is the entire space for λ large enough, this shows that $(\beta, z) = (\gamma, g)$, so $(\gamma, g) \in S$ as desired. This suffices to show that $D(\mathcal{A}^*) = S$. $\square$

We can now consider the adjoint in a weighted norm. Let $\langle u,v\rangle_w$ denote the inner product on M_2 with weight function $w(\theta)$, so that $\langle u,v\rangle_1$ denotes the inner product used above. Since a weighted norm will be equivalent to the norm with $w(\theta)=1$, there must be some operator Q such that $\langle u,v\rangle_w = \langle Qu,Qv\rangle_1$. The operator given by

$$Q(\eta,f(\theta)) = \left(\eta, \sqrt{w(\theta)}W^{1/2}f(\theta)\right)$$

satisfies this condition because

$$\begin{aligned}
\langle Q\left(\eta,f(\theta)\right), Q\left(\gamma,g(\theta)\right)\rangle_1 &= \langle\left(\eta,\sqrt{w(\theta)}W^{1/2}f(\theta)\right), \left(\gamma,\sqrt{w(\theta)}W^{1/2}g(\theta)\right)\rangle_1 \\
&= \overline{\gamma}^T\eta + \int_{-r_n}^{0} \overline{\sqrt{w(\theta)}W^{1/2}g(\theta)}^T \sqrt{w(\theta)}W^{1/2}f(\theta)d\theta \\
&= \overline{\gamma}^T\eta + \int_{-r_n}^{0} \overline{g(\theta)}^T w(\theta)Wf(\theta)d\theta \\
&= \langle\left(\eta,f(\theta)\right),\left(\gamma,g(\theta)\right)\rangle_w .
\end{aligned}$$

The next-to-last equality holds because W is positive definite and self-adjoint, and thus its square root has the same properties, so $\overline{W^{1/2}}^T = W^{1/2}$ (see [Kat76, Theorem V.3.35].

We let $\mathcal{A}^*$ represent the adjoint when $w(\theta)=1$ and $\mathcal{A}^{*w}$ represent the adjoint in a general weighted norm. We can then describe $\mathcal{A}^{*w}$ in terms of $\mathcal{A}^*$ and Q.

Theorem 3.4. *For a general weighted norm, we have*

$$\mathcal{A}^{*w} = Q^{-2}\mathcal{A}^*Q^2$$

where $D(\mathcal{A}^{*w}) = \left\{ (\gamma, g(\theta)) : g(\theta) = \dfrac{h(\theta)}{w(\theta)} \text{ for some } (\gamma, h(\theta)) \in D(\mathcal{A}^*) \right\}$.

Proof: For any u and v in M_2, we have:

$$\begin{aligned}
\langle u, \mathcal{A}^* Q^* Q v \rangle_1 &= \langle u, (Q\mathcal{A})^* Q v \rangle_1 \\
&= \langle Q\mathcal{A} u, Q v \rangle_1 \\
&= \langle \mathcal{A} u, v \rangle_w \\
&= \langle u, \mathcal{A}^{*w} v \rangle_w \\
&= \langle Q u, Q\mathcal{A}^{*w} v \rangle_w \\
&= \langle u, Q^{*w} Q \mathcal{A}^{*w} v \rangle_w .
\end{aligned}$$

Since this holds for all u and v, we have that $\mathcal{A}^* Q^* Q = Q^{*w} Q \mathcal{A}^{*w}$. Since Q is self-adjoint, this means that $\mathcal{A}^* Q^2 = Q^2 \mathcal{A}^{*w}$. Since Q is invertible, we can now say that $\mathcal{A}^{*w} = Q^{-2} \mathcal{A}^* Q^2$. Since the domain of Q is all of the space M_2, this means that the domain of $\mathcal{A}^{*w}$ is all $(\gamma, g(\theta))$ such that $Q^2 (\gamma, g(\theta))$ is in the domain of $\mathcal{A}^*$. But this is precisely the set of pairs $(\gamma, g(\theta))$ such that $g(\theta) = \dfrac{h(\theta)}{w(\theta)}$ for some $(\gamma, h(\theta)) \in D(\mathcal{A}^*)$.

CHAPTER IV

THE APPROXIMATION PROBLEM

We now turn to defining a new approximation scheme to approximate solutions of Equation (1.1). Our goal is to use a Trotter-Kato style theorem to prove the convergence of our scheme for both the operator $\mathcal{A}$ and its adjoint $\mathcal{A}^*$. We use the following version, found in [Fab13].

Theorem 4.1. *Suppose V and X are Hilbert spaces, with V densely and continuously embedded in X, and let $c_V > 0$ satisfy*

$$\|x\|_X \leq c_V \|x\|_V \ \forall x \in V.$$

Assume $\mathcal{A} : dom\mathcal{A} \subset V \subset X \to X$ is the infinitesimal generator of a C_0 semigroup $T(t)$ on X, and there is a sesquilinear form $\sigma : V \times V \to \mathbb{C}$ and a fixed $\omega \in \mathbb{R}$ satisfying

$$\sigma(u, v) = \langle \mathcal{A}u, v \rangle_X \ \forall x \in dom\mathcal{A}, v \in V, \tag{4.1}$$

and

$$\text{Re}\sigma(u, u) \leq \omega \|u\|_X^2 \ \forall u \in V. \tag{4.2}$$

Let $\left\{X^N\right\}_{N=1}^{\infty}$ *be a sequence of finite dimensional subspaces of* V*, and let* P^N *denote the orthogonal projection of* X *onto* X^N*. For each* N *define the operator* $\mathcal{A}^N : X^N \to X^N$ *by*

$$\left\langle \mathcal{A}^N u, v \right\rangle_X = \sigma(u,v) \; \forall u, v \in X^N.$$

If there are constants $s \geq 1$ *and* $L > 0$ *such that for all* $v \in dom \mathcal{A}^s$ *and all* $N = 1, 2, \ldots$ *there exists* $v^N \in X^N$ *satisfying*

$$\left|\sigma(u, v - v^N)\right| \leq L \left\|u\right\|_X \left\|v - v^N\right\|_V \;\; \forall u \in V,$$

and

$$\lim_{N\to\infty} \left\|v - v^N\right\|_V = 0,$$

then $T^N(t)P^n \to T(t)$ *strongly on* X*. Here* $T^N(t) = e^{t\mathcal{A}^N}$ *is the semigroup on* X^N *generated by* $\mathcal{A}^N$*.*

4.1 Preliminaries

Our main goal is to use Theorem 4.1 to extend the scheme from [KS90] to neutral equations while preserving adjoint convergence. In order to use this theorem, we must first define the required sesquilinear forms and spaces. Recall the form σ in equation (1.29). We define a related form by

$$\tau(u,v) = \overline{\sigma(v,u)} \tag{4.3}$$

We will prove some facts about σ and τ, to be used later in our convergence proof. We have already shown that σ is related to $\mathcal{A}$. We will also show that τ is related to $\mathcal{A}^*$, followed by several preliminary lemmas. Next, we prove a dissipative inequality for σ for any $u \in V$, and then sharpen it in the case that A is a symmetric matrix.

Lemma 4.2. *For any $u \in D(\mathcal{A}^*)$ and $v \in V$, $\tau(u,v) = \langle \mathcal{A}^* u, v\rangle$.*

Proof. Let $u = (\eta, f(\theta))$ and $v = (\gamma, g(\theta))$. We have already shown that $\sigma(u,v) = \langle \mathcal{A}u, v\rangle$ for any $u \in D(\mathcal{A})$ and $v \in V$. So, let $u \in D(\mathcal{A}^*)$ and $v \in V$. We note that by properties of the inner product and the relationship between τ and σ, we already know that $\tau(u,v) = \langle \mathcal{A}^* u, v\rangle$ for $u \in D(\mathcal{A}^*)$ and $v \in D(\mathcal{A})$. However, we need the relationship between τ and $\mathcal{A}^*$ to hold for all $v \in V$, so there is something to show here. We have $h(\theta) = Ww(\theta)f(\theta)$ and note that an explicit representation of $\tau(u,v)$ is given by

$$
\begin{aligned}
&\tau(u,v)\\
&= \overline{\gamma}^T \overline{A}^T \eta + \overline{\gamma}^T W w(0) f(0) - \overline{g(0)}^T W w(0) f(0)\\
&\quad - \sum_{i=1}^{n} \left[\overline{g(-r_i)}^T \left(\overline{C_i}^T \overline{A}^T - \overline{B_i}^T \right) \eta - \overline{g(-r_i)}^T \overline{C_i}^T W w(0) f(0) \right]\\
&\quad + \sum_{i=1}^{n-1} \left[\overline{k_{g,i}}^T W w(-r_i) f(-r_i) - \overline{k_{g,i}}^T k_{h,i} \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g'(\theta)}^T W w(\theta) f(\theta) d\theta.
\end{aligned}
$$

Next, using the definition of $\mathcal{A}^{*w}$, we have

$$
\langle \mathcal{A}^{*w} u, v\rangle = \overline{\gamma}^T \left[\overline{A}^T \eta + W w(0) f(0) \right] - \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g(\theta)}^T h'(\theta) d\theta.
$$

Here we are using the fact that $h'(\theta) = \widetilde{h}'(\theta)$ on each interval $(-r_i, -r_{i-1})$. Using integration by parts on each delay interval, we can rewrite this as

$$
\begin{aligned}
&\langle \mathcal{A}^{*w} u, v\rangle \\
&= \overline{\gamma}^T \overline{A}^T \eta + \overline{\gamma}^T W w(0) f(0) - \overline{g(0)}^T h(0) + \overline{g(-r_1)}^T h(-r_1) \\
&\quad - \sum_{i=2}^{n} \left[\overline{g(-r_{i-1}^-)}^T h(-r_{i-1}^-) - \overline{g(-r_i)}^T h(-r_i) \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g'(\theta)}^T h(\theta) d\theta \\
&= \overline{\gamma}^T \overline{A}^T \eta + \overline{\gamma}^T W w(0) f(0) - \overline{g(0)}^T h(0) + \overline{g(-r_1)}^T h(-r_1) \\
&\quad - \sum_{i=2}^{n} \left[\left(\overline{g(-r_{i-1})}^T - \overline{k_{g,i-1}}^T \right) (h(-r_{i-1}) - k_{h,i}) - \overline{g(-r_i)}^T h(-r_i) \right] \\
&\quad + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g'(\theta)}^T h(\theta) d\theta \\
&= \overline{\gamma}^T \overline{A}^T \eta + \overline{\gamma}^T W w(0) f(0) - \overline{g(0)}^T h(0) + \overline{g(-r_n)}^T h(-r_n) \\
&\quad - \sum_{i=1}^{n-1} \left[-\overline{k_{g,i}}^T h(-r_i) - \overline{g(-r_i)}^T k_{h,i} + \overline{k_{g,i}}^T k_{h,i} \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{g'(\theta)}^T h(\theta) d\theta.
\end{aligned}
$$

To show that $\tau(u, v) = \langle \mathcal{A}^{*w} u, v\rangle$, it now suffices to show that

$$
\begin{aligned}
&\overline{g(-r_n)}^T h(-r_n) - \sum_{i=1}^{n-1} \left[-\overline{k_{g,i}}^T h(-r_i) - \overline{g(-r_i)}^T k_{h,i} + \overline{k_{g,i}}^T k_{h,i} \right] \\
&= \sum_{i=1}^{n} \overline{g(-r_i)}^T \left[\left(\overline{B_i}^T - \overline{C_i}^T \overline{A}^T \right) \eta - \overline{C_i}^T h(0) \right] + \sum_{i=1}^{n-1} \left[\overline{k_{g,i}}^T h(-r_i) - \overline{k_{g,i}}^T k_{h,i} \right].
\end{aligned}
$$

But this holds because $u \in D(\mathcal{A}^{*w})$, so therefore $k_{h,i} = \left(\overline{B_i}^T - \overline{C_i}^T \overline{A}^T \right) \eta - \overline{C_i}^T h(0)$ for $i = 1, \ldots, n-1$ and $h(-r_n) = \left(\overline{B_n}^T - \overline{C_n}^T \overline{A}^T \right) \eta - \overline{C_n}^T h(0)$. This completes the proof that $\tau(u, v) = \langle \mathcal{A}^{*w} u, v\rangle$.

□

4.2 Approximation Scheme

We now turn to defining the necessary spaces for use with Theorem 4.1 and our approximation scheme. We let $X = M_2$ with the inner product defined earlier, and V be as in equation (1.28). The main issue is that of discretizing the continuous part of the function component f of elements (η, f) of V. Note that the portion of the function $\sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i}$ is already finite-dimensional since it is piecewise constant on finitely many delay intervals. We choose to use splines to discretize the necessary function. We will implement this scheme later using piecewise linear splines, but the theory proceeds for higher-order splines. For a discretization parameter N, suppose J^N is a finite-dimensional subspace of $H^1(-r_n, 0)$ with the property that for each $f \in H^2(-r_n, 0)$ there is a function $I_f^N(\theta) \in J^N$ satisfying

$$I_f^N(-r_i) = f(-r_i), i = 1, 2, \ldots, n, \tag{4.4}$$

$$\left\| f(\theta) - I_f^N(\theta) \right\|_{L_2} \leq \frac{C_1}{N^2} \left\| f''(\theta) \right\|_{L_2}, \tag{4.5}$$

and

$$\left\| f'(\theta) - I_f^{N\prime}(\theta) \right\|_{L_2} \leq \frac{C_2}{N} \left\| f''(\theta) \right\|_{L_2}, \tag{4.6}$$

where C_1 and C_2 are constants independent of N. If $f(\theta) = \big(f_1(\theta), f_2(\theta), \ldots, f_m(\theta)\big)$ is a vector-valued function, then we use $I_f^N(\theta) = \big(I_{f,1}^N(\theta), I_{f,2}^N(\theta), \ldots, I_{f,m}^N(\theta)\big)$. We will explicitly describe the meshpoints and the linear splines we use later.

To obtain a finite-dimensional subspace of V, let $X^N = \mathbb{C}^m \times (J^N)^m \times (\text{span}\,\{\chi_i\})^m$. We will define approximating operators $\mathcal{A}^N$ and $\mathcal{A}^{*N}$ by

$$\langle \mathcal{A}^N u, v\rangle = \sigma(u, v) \tag{4.7}$$

for $u \in X^N, v \in V$ and

$$\langle \mathcal{A}^{*N} u, v\rangle = \tau(u, v). \tag{4.8}$$

With this notation, we let $T(t)$ represent the semigroup generated by the operator $\mathcal{A}$, and $T^N(t)$ be the semigroup generated by $\mathcal{A}^N$. Similarly, $T^*(t)$ represents the semigroup generated by the operator $\mathcal{A}^*$, and $T^{*N}(t)$ represents the semigroup generated by $\mathcal{A}^{*N}$. In fact, $\mathcal{A}^{*N}$ is the adjoint of $\mathcal{A}^N$ by construction, and $T^{*N}(t)$ is the adjoint of $T^N(t)$. We can approximate the semigroup $T(t)$ by $T^N(t)$ and $T^*(t)$ by $T^{*N}(t)$. We now turn to showing that the approximating semigroups converge to $T(t)$ and $T^*(t)$, respectively.

4.3 Semigroup Convergence Results

Now we are ready to prove our main result for the approximation problem:

Theorem 4.3. *Let $T(t)$ represent the semigroup generated by the operator $\mathcal{A}$ and approximating operators $\mathcal{A}^N$ and $\mathcal{A}^{*N}$ be as defined in equations (4.7) and (4.8). Then each $\mathcal{A}^N$ is the infinitesimal generator of a semigroup $T^N(t)$, and $T^N(t)P^N \to T(t)$ strongly, uniformly on bounded t-intervals. Also, each $\mathcal{A}^{*N}$ is the infinitesimal generator of the adjoint semigroup $T^{*N}(t)$, and $T^{*N}(t)P^N \to T^*(t)$ strongly, uniformly on bounded t-intervals.*

4.3.1 A Dissipative Inequality

In order to use the Trotter-Kato theorem to obtain convergence, we need to have a dissipative inequality of the form $\text{Re}\sigma(u,u) \leq \omega \|u\|_X^2 \ \forall u \in V$. Although we already have such an inequality under certain conditions in Theorem 2.9, we can obtain a cruder inequality under less restrictive conditions here, although we do not get a negative ω this time.

Theorem 4.4. *If the scalar weight function $w(\theta)$ has the form $\widetilde{w}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{w,i}$ and is strictly positive with $\widetilde{w}(\theta)$ continuously differentiable on each interval $[-r_{i-1}, -r_i)$ and it satisfies*

$$\frac{1}{2} k_{w,i} \lambda_m - w(0) \|W\| \, n \, \|C_i\|^2 - \frac{1}{2} \geq 0 \tag{4.9}$$

for $i = 1, \ldots, n$, then there exists $\omega \in \mathbb{R}$ such that $\text{Re}\sigma(u,u) \leq \omega \|u\|^2$ for all $u \in V$.

Proof. Let $u = (\eta, f(\theta))$ and for convenience, put $h(\theta) = Ww(\theta)f(\theta)$. We have:

$$\begin{aligned}
&\text{Re}\{\sigma(u,u)\} \\
&= \text{Re}\Bigg\{ \overline{\eta}^T \left[A\eta - \sum_{i=1}^{n} (AC_i f(-r_i) - B_i f(-r_i)) \right] + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww(\theta) f'(\theta) d\theta \\
&\quad + \overline{f(0)}^T Ww(0) \left[\eta - f(0) - \sum_{i=1}^{n} C_i f(-r_i) \right] \\
&\quad + \sum_{i=1}^{n-1} \left(\overline{f(-r_i)}^T Ww(-r_i) - \overline{k_{h,i}}^T \right) k_{f,i} \Bigg\} \\
&= \text{Re}\Bigg\{ \overline{\eta}^T A\eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T AC_i f(-r_i) + \overline{f(0)}^T Ww(0)\eta \\
&\quad - \overline{f(0)}^T Ww(0) f(0) - \sum_{i=1}^{n} \overline{f(0)}^T Ww(0) C_i f(-r_i)
\end{aligned}$$

$$+\sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T Ww(-r_i)-\overline{k_{h,i}}^T\right]k_{f,i}\Bigg\}+\sum_{i=1}^{n}\mathrm{Re}\left\{\int_{-r_i}^{-r_{i-1}}\overline{h(\theta)}^T f'(\theta)d\theta\right\}.$$

Then, using Lemma 2.3, we have

$$\begin{aligned}
&\mathrm{Re}\{\sigma(u,u)\}\\
=\;&\mathrm{Re}\Bigg\{\overline{\eta}^T A\eta+\sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i)-\sum_{i=1}^{n}\overline{\eta}^T AC_i f(-r_i)+\overline{f(0)}^T Ww(0)\eta\\
&-\overline{f(0)}^T Ww(0)f(0)-\sum_{i=1}^{n}\overline{f(0)}^T Ww(0)C_i f(-r_i)\\
&+\sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T Ww(-r_i)-\overline{k_{h,i}}^T\right]k_{f,i}\Bigg\}+\frac{1}{2}\overline{f(0)}^T Ww(0)f(0)\\
&-\frac{1}{2}\overline{f(-r_n)}^T Ww(-r_n)f(-r_n)-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T Ww'(\theta)f(\theta)d\theta\\
&-\frac{1}{2}\sum_{i=1}^{n-1}\left[\overline{f(-r_i)}^T Ww(-r_i)k_{f,i}+\overline{k_{h,i}}^T f(-r_i)-\overline{k_{h,i}}^T k_{f,i}\right]\\
=\;&\mathrm{Re}\Bigg\{\overline{\eta}^T A\eta+\sum_{i=1}^{n}\overline{\eta}^T B_i f(-r_i)-\sum_{i=1}^{n}\overline{\eta}^T AC_i f(-r_i)+\overline{f(0)}^T Ww(0)\eta\\
&-\sum_{i=1}^{n}\overline{f(0)}^T Ww(0)C_i f(-r_i)\Bigg\}-\frac{1}{2}\overline{f(0)}^T Ww(0)f(0)\\
&-\frac{1}{2}\overline{f(-r_n)}^T Ww(-r_n)f(-r_n)-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^T Ww'(\theta)f(\theta)d\theta\\
&-\frac{1}{2}\sum_{i=1}^{n-1}\left[\overline{k_{h,i}}^T f(-r_i)-\overline{f(-r_i)}^T Ww(-r_i)k_{f,i}+\overline{k_{h,i}}^T k_{f,i}\right].
\end{aligned}$$

Then we use Lemma 2.2 about the jumps in $h(\theta)$ to get

$$\mathrm{Re}\{\sigma(u,u)\}$$

$$
\begin{aligned}
= \quad & \operatorname{Re}\left\{ \overline{\eta}^T A\eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T AC_i f(-r_i) + \overline{f(0)}^T Ww(0)\eta \right. \\
& \left. - \sum_{i=1}^{n} \overline{f(0)}^T Ww(0)C_i f(-r_i) \right\} - \frac{1}{2}\overline{f(0)}^T Ww(0)f(0) \\
& -\frac{1}{2}\overline{f(-r_n)}^T Ww(-r_n)f(-r_n) \\
& -\frac{1}{2}\sum_{i=1}^{n-1} \left[\left(\overline{k_{f,i}}^T Ww(-r_i) - \overline{k_{f,i}}^T Wk_{w,i} + \overline{f(-r_i)}^T Wk_{w,i} \right) f(-r_i) \right. \\
& \left. -\overline{f(-r_i)}^T Ww(-r_i)k_{f,i} + \left(\overline{k_{f,i}}^T Ww(-r_i) - \overline{k_{f,i}}^T Wk_{w,i} + \overline{f(-r_i)}^T Wk_{w,i} \right) k_{f,i} \right] \\
& -\frac{1}{2}\sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww'(\theta)f(\theta)d\theta \\
= \quad & \operatorname{Re}\left\{ \overline{\eta}^T A\eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T AC_i f(-r_i) + \overline{f(0)}^T Ww(0)\eta \right. \\
& \left. - \sum_{i=1}^{n} \overline{f(0)}^T Ww(0)C_i f(-r_i) \right\} - \frac{1}{2}\overline{f(0)}^T Ww(0)f(0) \\
& -\frac{1}{2}\overline{f(-r_n)}^T Ww(-r_n)f(-r_n) \\
& -\frac{1}{2}\sum_{i=1}^{n-1} \left[\operatorname{Re}\left\{ \overline{k_{f,i}}^T Ww(-r_i)f(-r_i) - \overline{f(-r_i)}^T Ww(-r_i)k_{f,i} \right\} \right. \\
& +\operatorname{Re}\left\{ \overline{f(-r_i)}^T Wk_{w,i}k_{f,i} - \overline{k_{f,i}}^T Wk_{w,i}f(-r_i) \right\} \\
& \left. +\overline{f(-r_i)}^T Wk_{w,i}f(-r_i) + \overline{k_{f,i}}^T Ww(-r_i)k_{f,i} - \overline{k_{f,i}}^T Wk_{w,i}k_{f,i} \right] \\
& -\frac{1}{2}\sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T Ww'(\theta)f(\theta)d\theta.
\end{aligned}
$$

Next we use the fact that $\operatorname{Re}\{a - \overline{a}\} = 0$ to simplify

$$
\begin{aligned}
& \operatorname{Re}\{\sigma(u,u)\} \\
& \quad = \quad \operatorname{Re}\left\{ \overline{\eta}^T A\eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T AC_i f(-r_i) + \overline{f(0)}^T Ww(0)\eta \right.
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{i=1}^{n} \overline{f(0)}^T W w(0) C_i f(-r_i) \Bigg\} - \frac{1}{2} \overline{f(0)}^T W w(0) f(0) \\
& -\frac{1}{2} \overline{f(-r_n)}^T W w(-r_n) f(-r_n) \\
& -\frac{1}{2} \sum_{i=1}^{n-1} \left[\overline{f(-r_i)}^T W k_{w,i} f(-r_i) + \overline{k_{f,i}}^T W w(-r_i) k_{f,i} - \overline{k_{f,i}}^T W k_{w,i} k_{f,i} \right] \\
& -\frac{1}{2} \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta \\
= \ & \mathrm{Re} \left\{ \overline{\eta}^T A \eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T A C_i f(-r_i) \right. \\
& \left. + \overline{f(0)}^T W w(0) \left(\eta - \sum_{i=1}^{n} C_i f(-r_i) \right) \right\} - \frac{1}{2} \overline{f(0)}^T W w(0) f(0) \\
& -\frac{1}{2} \overline{f(-r_n)}^T W w(-r_n) f(-r_n) \\
& -\frac{1}{2} \sum_{i=1}^{n-1} \left[\overline{f(-r_i)}^T W k_{w,i} f(-r_i) + \overline{k_{f,i}}^T W \left(w(-r_i) - k_{w,i} \right) k_{f,i} \right] \\
& -\frac{1}{2} \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{f(\theta)}^T W w'(\theta) f(\theta) d\theta .
\end{aligned}
$$

We will now use a version of a Cauchy-Schwartz inequality (see Lemma 2.4) to bound several parts of the above equation. First, we have

$$
\begin{aligned}
& \mathrm{Re} \left\{ \overline{\eta}^T A \eta + \sum_{i=1}^{n} \overline{\eta}^T B_i f(-r_i) - \sum_{i=1}^{n} \overline{\eta}^T A C_i f(-r_i) \right\} \\
\leq \ & \|\eta\|^2 \left(\|A\| + \frac{n\beta}{2} \right) + \frac{1}{2} \sum_{i=1}^{n} \|f(-r_i)\|^2 ,
\end{aligned}
$$

where $\beta = \max \|B_i - AC_i\|$. We can also see that

$$
\mathrm{Re} \left\{ \overline{f(0)}^T W w(0) \left(\eta - \sum_{i=1}^{n} C_i f(-r_i) \right) \right\}
$$

$$
\begin{aligned}
&\leq \;\; w(0)\left\|W\right\|\left\|\eta\right\|^2 + \frac{1}{4}w(0)\left\|W^{1/2}f(0)\right\|^2 + \frac{1}{4}w(0)\left\|W^{1/2}f(0)\right\|^2 \\
&\quad\; + w(0)\left\|\sum_{i=1}^{n} W^{1/2}C_i f(-r_i)\right\|^2 \\
&\leq \;\; w(0)\left\|W\right\|\left\|\eta\right\|^2 + \frac{1}{2}w(0)\left\|W^{1/2}f(0)\right\|^2 + w(0)n\left\|W\right\|\sum_{i=1}^{n}\left\|C_i\right\|^2\left\|f(-r_i)\right\|^2 .
\end{aligned}
$$

We also note that

$$
-\frac{1}{2}\overline{f(0)}^T W w(0) f(0) = -\frac{1}{2}w(0)\left\|W^{1/2}f(0)\right\|^2 .
$$

We can bound the remaining terms as follows:

$$
-\overline{f(-r_i)}^T W k_{w,i} f(-r_i) \leq k_{w,i}\lambda_m \left\|f(-r_i)\right\|^2 ,
$$

where λ_m is the minimum eigenvalue of W. To simplify notation, we have made the convention that $k_{w,n} = w(-r_n)$. Finally, we note that $w(-r_i) - k_{w,i} = w(-r_i^-)$, and therefore this must be a positive number. This means that the term

$$
-\frac{1}{2}\sum_{i=1}^{n-1}\overline{k_{f,i}}^T W \left(w(-r_i) - k_{w,i}\right) k_{f,i}
$$

is negative. Thus we can write

$$
\begin{aligned}
&\mathrm{Re}\left\{\sigma(u,u)\right\} \\
&\quad \leq \;\; \left\|\eta\right\|^2\left(\left\|A\right\| + \frac{n\beta}{2}\right) + \frac{1}{2}\sum_{i=1}^{n}\left\|f(-r_i)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+w(0)\left\|W\right\|\left\|\eta\right\|^2+\frac{1}{2}w(0)\left\|W^{1/2}f(0)\right\|^2+w(0)n\left\|W\right\|\sum_{i=1}^{n}\left\|C_i\right\|^2\left\|f(-r_i)\right\|^2\\
&-\frac{1}{2}w(0)\left\|W^{1/2}f(0)\right\|^2-\frac{1}{2}\sum_{i=1}^{n}k_{w,i}\lambda_m\left\|f(-r_i)\right\|^2\\
&-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^TWw'(\theta)f(\theta)d\theta\\
=\ &\left\|\eta\right\|^2\left(\left\|A\right\|+\frac{n\beta}{2}+w(0)\left\|W\right\|\right)-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^TWw'(\theta)f(\theta)d\theta\\
&-\sum_{i=1}^{n}\left\|f(-r_i)\right\|^2\left(\frac{1}{2}k_{w,i}\lambda_m-w(0)\left\|W\right\|n\left\|C_i\right\|^2-\frac{1}{2}\right).
\end{aligned}
$$

But since $\frac{1}{2}k_{w,i}\lambda_m-w(0)\left\|W\right\|n\left\|C_i\right\|^2-\frac{1}{2}\geq 0$ by assumption, this means we have

$$
\begin{aligned}
\operatorname{Re}\left\{\sigma(u,u)\right\} \ \leq\ &\left\|\eta\right\|^2\left(\left\|A\right\|+\frac{n\beta}{2}+w(0)\left\|W\right\|\right)\\
&-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^TWw'(\theta)f(\theta)d\theta.
\end{aligned}
$$

Finally, since $w(\theta)$ is positive and continuously differentiable on $[-r_{i-1},-r_i)$ for $i=1,\ldots,n$, both $w(\theta)$ and $w'(\theta)$ are bounded. Therefore we can say that

$$
-\frac{1}{2}\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^TWw'(\theta)f(\theta)d\theta\leq T\sum_{i=1}^{n}\int_{-r_i}^{-r_{i-1}}\overline{f(\theta)}^TWw(\theta)f(\theta)d\theta, \tag{4.10}
$$

where

$$
T=\max_{\theta\in[-r_n,0]}\frac{|w'(\theta)|}{2w(\theta)}.
$$

We then take

$$\omega = \max\left\{T, \|A\| + \frac{n\beta}{2} + w(0)\|W\|\right\},$$

and we have $\text{Re}\sigma(u,u) \leq \omega \|u\|^2$ as desired. □

4.3.2 Convergence for $\mathcal{A}$

We will first show that the approximation scheme converges for $\mathcal{A}$, using the Trotter-Kato theorem. We let $X = M_2$ with the inner product defined earlier, and V be as in equation (1.28). X is a Hilbert space because both $\mathbb{C}^m$ and $L_2(-r_n, 0)$ are Hilbert spaces. Note that we can also write $V = \mathbb{C}^m \times H_1(-r_n, 0) \times \text{span }\{\chi_i\}_{i=1}^{n-1}$, and that span $\{\chi_i : i = 1, 2, \ldots, n-1\}$ is a Hilbert space because it is a finite-dimensional subspace of $L_2(-r_n, 0)$. Therefore V is also a Hilbert space since it is a product of Hilbert spaces. The inner product on X is the weighted inner product defined earlier, and the inner product on V is

$$\begin{aligned}\langle(\eta, f), (\gamma, g)\rangle_V &= \overline{\gamma}^T\eta + \int_{-r_n}^0 \overline{\widetilde{g}(\theta)}^T W w(\theta)\widetilde{f}(\theta) + \overline{\widetilde{g}(\theta)'}^T W w(\theta)\widetilde{f}(\theta)' d\theta \\ &\quad + \int_{-r_n}^0 \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i} k_{f,i} d\theta. \end{aligned} \tag{4.11}$$

Equivalently, we can write

$$\langle(\eta, f), (\gamma, g)\rangle_V = \langle\eta, \gamma\rangle_{\mathbb{C}^m} + \langle\widetilde{f}(\theta), \widetilde{g}(\theta)\rangle_{H_1} + \langle\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}, \sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\rangle_{L_2},$$

where the H_1 and L_2 inner products are taken to be weighted inner products.

To check that the corresponding norm on V is indeed a norm, first note that for any $(\eta, f) \in V$, $\|(\eta, f)\| \geq 0$ because it is the square root of the sum of three other norms. Also if $\|(\eta, f)\|^2 = 0$, then $\|\eta\|_{\mathbb{C}^m}^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \left\|\sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i}\right\|_{L_2}^2 = 0$, so we must have $\|\eta\|_{\mathbb{C}^m} = 0$, $\left\|\widetilde{f}(\theta)\right\|_{H_1}^2 = 0$, and $\left\|\sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i}\right\|_{L_2}^2 = 0$, and therefore $\eta = 0$, $\widetilde{f}(\theta) = 0$, and $\sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} = 0$. This means that $(\eta, f) = (0, 0)$. It is just as straightforward to check that $\|\alpha(\eta, f)\| = |\alpha| \, \|(\eta, f)\|$. The triangle inequality takes a little more work.

Recall that any unsubscripted norm refers to the usual Euclidean vector norm and any unsubscripted inner product refers to the usual Euclidean inner product. First, note that

$$
\begin{aligned}
&[\|(\eta, f)\|_V + \|(\gamma, g)\|_V]^2 \\
&= \left[\left(\|\eta\|^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2}^2 \right)^{1/2} \right. \\
&\quad \left. + \left(\|\gamma\|^2 + \|\widetilde{g}(\theta)\|_{H_1}^2 + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i} \right\|_{L_2}^2 \right)^{1/2} \right]^2 \\
&= \|\eta\|^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2}^2 + \|\gamma\|^2 + \|\widetilde{g}(\theta)\|_{H_1}^2 + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i} \right\|_{L_2}^2 \\
&\quad +2 \left[\left(\|\eta\|^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2}^2 \right) \left(\|\gamma\|^2 + \|\widetilde{g}(\theta)\|_{H_1}^2 \right. \right. \\
&\quad \left. \left. + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i} \right\|_{L_2}^2 \right) \right]^{1/2} .
\end{aligned}
$$

We will need to show the following fact:

$$
\begin{aligned}
&2\left(\|\eta\|\,\|\gamma\| + \left\|\widetilde{f}(\theta)\right\|_{H_1} \|\widetilde{g}(\theta)\|_{H_1} + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i},\right\|_{L_2}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}\right)\\
\leq\;& 2\left[\left(\|\eta\|^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\right)\left(\|\gamma\|^2 + \|\widetilde{g}(\theta)\|_{H_1}^2\right.\right.\\
&\left.\left.+\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2\right)\right]^{1/2}.
\end{aligned}
$$

Equivalently, we will show that

$$
\begin{aligned}
&\left(\|\eta\|\,\|\gamma\| + \left\|\widetilde{f}(\theta)\right\|_{H_1} \|\widetilde{g}(\theta)\|_{H_1} + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i},\right\|_{L_2}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}\right)^2\\
\leq\;& \left(\|\eta\|^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\right)\left(\|\gamma\|^2 + \|\widetilde{g}(\theta)\|_{H_1}^2\right.\\
&\left.+\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2\right).
\end{aligned}
$$

Expanding the left side of the above inequality gives us

$$
\begin{aligned}
&\left(\|\eta\|\,\|\gamma\| + \left\|\widetilde{f}(\theta)\right\|_{H_1} \|\widetilde{g}(\theta)\|_{H_1} + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i},\right\|_{L_2}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}\right)^2\\
=\;& \|\eta\|^2\|\gamma\|^2 + 2\|\eta\|\,\|\gamma\|\left\|\widetilde{f}(\theta)\right\|_{H_1}\|\widetilde{g}(\theta)\|_{H_1}\\
&+2\|\eta\|\,\|\gamma\|\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}\\
&+\left\|\widetilde{f}(\theta)\right\|_{H_1}^2\|\widetilde{g}(\theta)\|_{H_1}^2 + 2\left\|\widetilde{f}(\theta)\right\|_{H_1}\|\widetilde{g}(\theta)\|_{H_1}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}
\end{aligned}
$$

$$+\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2.$$

Then, we use the Cauchy-Schwartz inequality on the terms with coefficient 2 and factor to obtain

$$\begin{aligned}
&\left(\|\eta\|\,\|\gamma\|+\left\|\widetilde{f}(\theta)\right\|_{H_1}\|\widetilde{g}(\theta)\|_{H_1}+\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i},\right\|_{L_2}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}\right)^2\\
\leq\;& \|\eta\|^2\|\gamma\|^2+\left\|\widetilde{f}(\theta)\right\|_{H_1}^2\|\widetilde{g}(\theta)\|_{H_1}^2+\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2\\
&+\|\eta\|^2\left\|\widetilde{f}(\theta)\right\|_{H_1}^2+\|\gamma\|^2\|\widetilde{g}(\theta)\|_{H_1}^2+\|\eta\|^2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\\
&+\|\gamma\|^2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2+\left\|\widetilde{f}(\theta)\right\|_{H_1}^2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\\
&+\|\widetilde{g}(\theta)\|_{H_1}^2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2\\
=\;&\left(\|\eta\|^2+\left\|\widetilde{f}(\theta)\right\|_{H_1}^2+\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\right)\Bigg(\|\gamma\|^2+\|\widetilde{g}(\theta)\|_{H_1}^2\\
&+\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2\Bigg),
\end{aligned}$$

and this proves the fact.

Now we consider $\|(\eta,f)+(\gamma,g)\|_V^2$. We first use the inner product definition and the fact that $\|x+y\|^2=\langle x+y,x+y\rangle=\|x\|^2+\|y\|^2+2\mathrm{Re}\langle x,y\rangle$ to simplify.

$$
\begin{aligned}
&\|(\eta, f)+(\gamma, g)\|_V^2 \\
&= \|\eta+\gamma\|^2 + \left\|\widetilde{f}(\theta)+\widetilde{g}(\theta)\right\|_{H_1}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i} + \sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2 \\
&= \|\eta\|^2 + \|\gamma\|^2 + 2\mathrm{Re}\langle\eta,\gamma\rangle + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \|\widetilde{g}(\theta)\|_{H_1}^2 + 2\mathrm{Re}\langle\widetilde{f}(\theta),\widetilde{g}(\theta)\rangle_{H_1} \\
&\quad + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2 + 2\mathrm{Re}\langle\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}, \sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\rangle_{L_2}.
\end{aligned}
$$

Next, we use the fact that $\mathrm{Re}\langle x,y\rangle \leq \|x\|\,\|y\|$ and the fact above to see that

$$
\begin{aligned}
&\|(\eta, f)+(\gamma, g)\|_V^2 \\
&\leq \|\eta\|^2 + \|\gamma\|^2 + 2\,\|\eta\|\,\|\gamma\|_+ \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \|\widetilde{g}(\theta)\|_{H_1}^2 + 2\left\|\widetilde{f}(\theta)\right\|_{H_1}\|\widetilde{g}(\theta)\|_{H_1} \\
&\quad + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2 + 2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i},\right\|_{L_2}\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2} \\
&\leq \|\eta\|^2 + \|\gamma\|^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \|\widetilde{g}(\theta)\|_{H_1}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2 \\
&\quad +2\left[\left(\|\eta\|^2 + \left\|\widetilde{f}(\theta)\right\|_{H_1}^2 + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2\right)\left(\|\gamma\|^2 + \|\widetilde{g}(\theta)\|_{H_1}^2\right.\right. \\
&\quad \left.\left. + \left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{g,i}\right\|_{L_2}^2\right)\right]^{1/2} \\
&= \left[\|(\eta,f)\|_V + \|(\gamma,g)\|_V\right]^2.
\end{aligned}
$$

Therefore we have $\|(\eta, f)+(\gamma, g)\|_V \leq \|(\eta,f)\|_V + \|(\gamma,g)\|_V$, and this norm is a proper norm.

Next, we need to show that V is densely and continuously embedded in X. V is dense in X because H_1 is dense in L_2, so therefore

$$H_1(-r_n, 0) \times \text{ span } \{\chi_i : i = 1, 2, \ldots, n-1\}$$

must also be dense in $L_2(-r_n, 0)$. Since the first components of both spaces are $\mathbb{C}^m$, and the second component of V is dense in the second component of X, V is dense in X. The continuous embedding is the identity map. To see continuity, given any $\epsilon > 0$, take $\delta = \epsilon$. I will show that if $u, v \in V$ and $\|u - v\|_V < \delta$, then $\|u - v\|_X < \epsilon$. For any $u = (\eta, f) \in V$, I claim that $\|u\|_X \leq \|u\|_V$. To see this, we simply write the norm as its component parts and again use $\|x + y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\text{Re}\langle x, y\rangle$.

$$\begin{aligned}
&\|(\eta, f)\|_X^2 \\
&\quad = \|\eta\|^2 + \|f\|_{L_2}^2 \\
&\quad = \|\eta\|^2 + \left\| \widetilde{f}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2}^2 \\
&\quad \leq \|\eta\|^2 + \left\| \widetilde{f}(\theta) \right\|_{L_2}^2 + 2 \left\| \widetilde{f}(\theta) \right\|_{L_2} \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2} + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2}^2 .
\end{aligned}$$

We now use a Cauchy-Schwartz inequality on the product and continue.

$$\begin{aligned}
&\|(\eta, f)\|_X^2 \\
&\quad \leq \|\eta\|^2 + \left\| \widetilde{f}(\theta) \right\|_{L_2}^2 + \left\| \widetilde{f}(\theta) \right\|_{L_2}^2 + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2}^2 + \left\| \sum_{i=1}^{n-1} \chi_i(\theta) k_{f,i} \right\|_{L_2}^2
\end{aligned}$$

$$
\begin{aligned}
&= \|\eta\|^2 + 2\left\|\widetilde{f}(\theta)\right\|_{L_2}^2 + 2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2 \\
&\leq 2\|\eta\|^2 + 2\left\|\widetilde{f}(\theta)\right\|_{H^1}^2 + 2\left\|\sum_{i=1}^{n-1}\chi_i(\theta)k_{f,i}\right\|_{L_2}^2 \\
&= 2\|(\eta, f)\|_V^2 .
\end{aligned}
$$

For the form defined in equation (1.29), we have already shown that $\sigma(u, v) = \langle \mathcal{A}u, v\rangle_X$ for all $x \in \mathrm{dom}\mathcal{A}$ and $v \in V$. We have also shown that if we choose our weight function $w(\theta)$ appropriately, we have Re $\sigma(u, u) \leq \omega \|u\|_X^2$ for all $u \in V$, where ω is a real number. To prove convergence of the approximation scheme, it remains to show that there are constants $s \geq 1$ and $L > 0$ such that for all $v \in D(\mathcal{A}^s)$ and all $N = 1, 2, \ldots$, there exists $v^N \in X^N$ satisfying $\left|\sigma(u, v - v^N)\right| \leq L \|u\|_x \left\|v - v^N\right\|_v \ \forall u \in V$, and $\lim_{N\to\infty} \left\|v - v^N\right\|_V = 0$.

Let $s = 3$. Then $v = (\eta, f) \in D(\mathcal{A}^3)$ means that $v \in D(\mathcal{A})$ also, so $v = (f(0) + \sum_{i=1}^n C_i f(-r_i), f(\theta))$. We must also have that $\mathcal{A}^2(v) \in D(\mathcal{A})$, so $\mathcal{A}^2(v) = (Af'(0) + \sum_{i=1}^n B_i f'(-r_i), f''(\theta)) \in D(\mathcal{A})$. Therefore we must have that $f''(\theta) \in H^1(-r_n, 0)$, which implies that $f(\theta) \in H^3(-r_n, 0)$. This means that f' and f'' are continuous on $(-r_n, 0)$ and that f has finite L_2 norm.

We recall the functions $I_f^N(\theta) \in J^N$, and define

$$
v^N = \left(I_f^N(0) + \sum_{i=1}^{n} C_i I_f^N(-r_i), I_f^N(\theta)\right).
$$

Note that $v^n \in X^N$. Also, recall that since $I_f^N(\theta)$ satisfies (4.4), this allows us to show that

$$\begin{aligned}
& v - v^N \\
&= \left(f(0) + \sum_{i=1}^{n} C_i f(-r_i), f(\theta)\right) - \left(I_f^N(0) + \sum_{i=1}^{n} C_i I_f^N(-r_i), I_f^N(\theta)\right) \\
&= \left(f(0) - I_f^N(0) + \sum_{i=1}^{n} C_i \left(f(-r_i) - I_f^N(-r_i)\right), f(\theta) - I_f^N(\theta)\right) \\
&= (0, f(\theta) - I_f^N(\theta)).
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
\left\|v - v^N\right\|_V^2 &= \left\|f(\theta) - I_f^N(\theta)\right\|_{H^1}^2 \\
&= \sum_{j=1}^{m} \left\|f_j(\theta) - I_{f,j}^N(\theta)\right\|_{H^1}^2 \\
&= \sum_{j=1}^{m} \left\|f_j(\theta) - I_{f,j}^N(\theta)\right\|_{L_2}^2 + \left\|f_j'(\theta) - I_{f',j}^N(\theta)\right\|_{L_2}^2 .
\end{aligned}$$

But each term of this sum goes to zero as $N \to \infty$ by equations (4.5) and (4.6), so thus $\left\|v - v^N\right\|_V \to 0$ as $N \to \infty$.

We now show that there exists $L > 0$, independent of v and v^N, such that $\left|\sigma(u, v - v^N)\right| \leq L \left\|u\right\|_X \left\|v - v^N\right\|_V$ for all $u \in V$. Let $u = (\gamma, g(\theta))$. Recall that $w(\theta)$ is piecewise exponential function, so that $w'(\theta) = \alpha w(\theta)$ for some number α. Let

$L = 2\,|\alpha|$. Then using the definition of σ and the fact that the first component of v is zero, we have

$$\begin{aligned}
&\left|\sigma(u, v - v^N)\right| \\
&= \left|\sigma\big((\gamma, g(\theta)), (0, f(\theta) - I_f^N(\theta))\big)\right| \\
&= \Bigg|0 \left[A\gamma - \sum_{i=1}^{n} (AC_i g(-r_i) - B_i g(-r_i))\right] \\
&\quad + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{\big(f(\theta) - I_f^N(\theta)\big)}^T W w(\theta) g'(\theta) d\theta \\
&\quad + \overline{\big(f(0) - I_f^N(0)\big)}^T W w(0) \left[\gamma - g(0) - \sum_{i=1}^{n} C_i g(-r_i)\right] \\
&\quad + \sum_{i=1}^{n-1} \left(\overline{\big(f(-r_i) - I_f^N(-r_i)\big)}^T W w(-r_i) - \overline{k_{h,i}}^T\right) k_{g,i}\Bigg|.
\end{aligned}$$

Next, we can use the fact that $f(-r_i) = I_f^N(-r_i)$ and integration by parts to get

$$\begin{aligned}
&\left|\sigma(u, v - v^N)\right| \\
&= \Bigg|0 + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{\big(f(\theta) - I_f^N(\theta)\big)}^T W w(\theta) g'(\theta) d\theta \\
&\quad + (0) W w(0) \left[\gamma - g(0) - \sum_{i=1}^{n} C_i g(-r_i)\right] + \sum_{i=1}^{n-1} \left(0 - \overline{k_{h,i}}^T\right) k_{g,i}\Bigg| \\
&= \Bigg|\sum_{i=1}^{n} \overline{\big(f(\theta) - I_f^N(\theta)\big)}^T W w(\theta) g(\theta)\Big]_{-r_i}^{-r_{i-1}} \\
&\quad - \int_{-r_i}^{-r_{i-1}} \frac{d}{d\theta}\left[\overline{\big(f(\theta) - I_f^N(\theta)\big)}^T W w(\theta)\right] g(\theta) d\theta\Bigg| \\
&= \Bigg|\sum_{i=1}^{n} (0) W w(-r_{i-1}) g(-r_{i-1}) - (0) W w(-r_i) g(-r_i)
\end{aligned}$$

$$
\begin{aligned}
& - \int_{-r_i}^{-r_{i-1}} \frac{d}{d\theta} \left[\overline{\left(f(\theta) - I_f^N(\theta)\right)}^T W w(\theta) \right] g(\theta) d\theta \Bigg| \\
= & \left| \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \frac{d}{d\theta} \left[\overline{\left(f(\theta) - I_f^N(\theta)\right)}^T W w(\theta) \right] g(\theta) + \overline{\left(f(\theta) - I_f^N(\theta)\right)}^T W w'(\theta) g(\theta) d\theta \right| .
\end{aligned}
$$

Next, we take derivatives and use that $w'(\theta) = \alpha w(\theta)$.

$$
\begin{aligned}
& \left|\sigma(u, v - v^N)\right| \\
& \quad = \left| \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \frac{d}{d\theta} \left[\overline{\left(f(\theta) - I_f^N(\theta)\right)}^T \right] W w(\theta) g(\theta) \right. \\
& \qquad \left. + \alpha \left[\overline{\left(f(\theta) - I_f^N(\theta)\right)}^T \right] W w(\theta) g(\theta) d\theta \right| \\
& \quad \leq |\alpha| \left| \langle g(\theta), (f(\theta) - I_f^N(\theta)) + \frac{d}{d\theta} \left(f(\theta) - I_f^N(\theta)\right) \rangle_{L_2} \right| \\
& \quad \leq |\alpha| \left\| g(\theta) \right\|_{L_2} \left\| f(\theta) - I_f^N(\theta) + \frac{d}{d\theta} \left(f(\theta) - I_f^N(\theta)\right) \right\|_{L_2} \\
& \quad \leq |\alpha| \left\| g(\theta) \right\|_{L_2} \left(\left\| f(\theta) - I_f^N(\theta) \right\|_{L_2} + \left\| \frac{d}{d\theta} \left(f(\theta) - I_f^N(\theta)\right) \right\|_{L_2} \right) .
\end{aligned}
$$

Finally, we use the relationship between L_2 norms and H_1 norms and $L = 2\,|\alpha|$ to see that

$$
\begin{aligned}
& \left|\sigma(u, v - v^N)\right| \\
& \quad \leq |\alpha| \left\| g(\theta) \right\|_{L_2} \left(\left\| f(\theta) - I_f^N(\theta) \right\|_{H^1} + \left\| \frac{d}{d\theta} \left(f(\theta) - I_f^N(\theta)\right) \right\|_{H^1} \right) \\
& \quad = L \left\| g(\theta) \right\|_{L_2} \left\| f(\theta) - I_f^N(\theta) \right\|_{H^1} \\
& \quad \leq L \left\| u \right\|_X \left\| v - v^N \right\|_V .
\end{aligned}
$$

By the Trotter-Kato result (Theorem 4.1), we now conclude that $T^N(t)P^N \to T(t)$ strongly on X.

4.3.3 Convergence for $\mathcal{A}^*$

We will now use the same method to prove that the approximation scheme for $\mathcal{A}^*$ converges. We will use the same spaces X, X^N and V that we used for $\mathcal{A}$. We know that $\mathcal{A}^*$ is the infinitesimal generator of a C_0 semigroup. We will make use of the sesquilinear form τ defined in equation (4.3), and we have already shown that $\tau(u,v) = \langle \mathcal{A}^* u, v\rangle_X \ \ \forall x \in \text{dom}\mathcal{A}^*, v \in V$. Also, since we know that Re $\sigma(u,u) \le \omega \|u\|_X^2 \ \ \forall u \in V$ and Re $\tau(u,u) =$ Re $\overline{\sigma(u,u)} =$ Re $\sigma(u,u)$, we also have that Re $\tau(u,u) \le \omega \|u\|_X^2 \ \ \forall u \in V$. It remains to show that there are constants $s \ge 1$ and $L > 0$ such that for all $v \in D(\mathcal{A}^{*s})$ and all $N = 1, 2, \ldots$, there exists $v^N \in X^N$ satisfying $\left|\tau(u, v - v^N)\right| \le L \|u\|_x \left\|v - v^N\right\|_v \ \ \forall u \in V$, and $\lim_{N\to\infty} \left\|v - v^N\right\|_V = 0$.

Let $s = 3$. Then $v = (\gamma, g) \in D(\mathcal{A}^{*3})$ means that $v \in D(\mathcal{A}^*)$. We must also have that $\mathcal{A}^{*2}(v) \in D(\mathcal{A}^*)$, so

$$\mathcal{A}^{*2}(v) = \mathcal{A}^*\left(\overline{A}^T\gamma + g(0), -\widetilde{g}'(\theta)\right) = \left((\overline{A}^T)^2\gamma + \overline{A}^T g(0) - \widetilde{g}'(0), \widetilde{\widetilde{g}}'(\theta)\right) \in D(\mathcal{A}^*),$$

where $\widetilde{\widetilde{g}}(\theta)$ represents the continuous part of $\widetilde{g}'(\theta)$. Therefore we must have that $\widetilde{\widetilde{g}}'(\theta)$ has the form $\widetilde{\widetilde{\widetilde{g}}}(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{\widetilde{\widetilde{g}},i}$, where $\widetilde{\widetilde{\widetilde{g}}}(\theta)$ is an $H^1(-r_n, 0)$ function. This implies that $\widetilde{\widetilde{g}}(\theta)$ has the form $h(\theta)+p(\theta)$, where $h(\theta) \in H^2(-r_n, 0)$ and $p(\theta)$ is piecewise linear continuous function, since the derivative of a piecewise linear continuous function will be $\sum_{i=1}^{n-1} \chi_i(\theta) k_{\widetilde{\widetilde{g}},i}$. Since $\widetilde{\widetilde{g}}(\theta)$ represents the continuous part of $\widetilde{g}'(\theta)$, we have that $\widetilde{g}'(\theta)$ has the form $h(\theta)+p(\theta)+\sum_{i=1}^{n-1} \chi_i(\theta) k_{\widetilde{g},i}$. Using the same argument we have $\widetilde{g}(\theta)$ has the form $h(\theta)+q(\theta)+p(\theta)$, where $h(\theta) \in H^3(-r_n, 0)$, $q(\theta)$ is a piecewise continuous quadratic function, and $p(\theta)$ is as before. To define v^N, we will approximate $h(\theta)+q(\theta)$ by functions in J^N. We have that $h(\theta)+q(\theta) \in H^3(-r_n,0)$, so $h'(\theta)+q'(\theta)$ and $h''(\theta)+q''(\theta)$ are continuous on $(-r_n,0)$ and that $h(\theta)+q(\theta)$ has finite L_2 norm.

Now we define

$$v^N = \left(\gamma, I_g^N(\theta) + I_p^N(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i}\right).$$

Note that $v^n \in X^N$. Also, recall that since $I_g^N(\theta)$ satisfies (4.4), we can show that

$$\begin{aligned}
&\left\|v - v^N\right\|_V^2 \\
&= \left\|(\gamma, g(\theta)) - \left(\gamma, I_g^N(\theta) + I_p^N(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i}\right)\right\|_V^2 \\
&= \left\|\left(0, h(\theta) + q(\theta) + p(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i} - (I_g^N(\theta) + I_p^N(\theta) + \sum_{i=1}^{n-1} \chi_i(\theta) k_{g,i})\right)\right\|_V^2 \\
&= \left\|h(\theta) + q(\theta) - I_g^N(\theta) + p(\theta) - I_p^N(\theta)\right\|_{H^1}^2 \\
&\le \sum_{j=1}^{m} \left\|g_j(\theta) - I_{g,j}^N(\theta)\right\|_{H^1}^2 + \left\|p_j(\theta) - I_{p,j}^N(\theta)\right\|_{H^1}^2 \\
&= \sum_{j=1}^{m} \left\|g_j(\theta) - I_{g,j}^N(\theta)\right\|_{L_2}^2 + \left\|g_j'(\theta) - I_{g',j}^N(\theta)\right\|_{L_2}^2 + \left\|p_j(\theta) - I_{p,j}^N(\theta)\right\|_{L_2}^2 \\
&\quad + \left\|p_j'(\theta) - I_{p',j}^N(\theta)\right\|_{L_2}^2 .
\end{aligned}$$

But each term of this sum goes to zero as $N \to \infty$, so thus $\left\|v - v^N\right\| \to 0$ as $N \to \infty$.

We now show that for $L > 0$, we have that $\left|\tau(u, v - v^N)\right| \le L \left\|u\right\|_X \left\|v - v^N\right\|_v$ for all $u \in V$. Let $u = (\eta, f(\theta))$. Recall that $w(\theta)$ is piecewise exponential function, so that $w'(\theta) = \alpha w(\theta)$ for some number α. Let $L = 2|\alpha|$. Then:

$$
\begin{aligned}
&\left|\tau(u, v - v^N)\right| \\
&= \left|\tau\big((\eta, f(\theta)), (0, h(\theta) + q(\theta) - I_g^N(\theta))\big)\right| \\
&= \left|\sigma\big((\eta, f(\theta)), (0, h(\theta) + q(\theta) - I_g^N(\theta))\big)\right| \\
&= \left| 0 + \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{\left[h(\theta) + q(\theta) - I_g^N()\right]}^T W w(\theta) f'(\theta) d\theta \right. \\
&\quad + \overline{\left[h(0) + q(0) - I_g^N(0)\right]}^T W w(\theta) \left[\eta - f(\theta) - \sum_{i=1}^{n} C_i f(-r_i)\right] \\
&\quad \left. + \sum_{i=1}^{n-1} \left[\overline{\left(h(-r_i) + q(-r_i) - I_g^N(-r_i)\right)}^T W w(-r_i) - \overline{k_{wg,i}}^T\right] k_{f,i} \right| \\
&= \left| \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{\left[h(\theta) + q(\theta) - I_g^N()\right]}^T W w(\theta) f'(\theta) d\theta \right. \\
&\quad \left. + [0]\, W w(\theta) \left[\eta - f(\theta) - \sum_{i=1}^{n} C_i f(-r_i)\right] + \sum_{i=1}^{n-1} \left[0 W w(-r_i) - \overline{k_{wg,i}}^T\right] k_{f,i} \right| \\
&= \left| \sum_{i=1}^{n} \int_{-r_i}^{-r_{i-1}} \overline{\left[h(\theta) + q(\theta) - I_g^N()\right]}^T W w(\theta) f'(\theta) d\theta - \sum_{i=1}^{n-1} \overline{k_{wg,i}}^T k_{f,i} \right|.
\end{aligned}
$$

From here we proceed exactly as in the proof for $\mathcal{A}$, and get that

$$
\left|\tau(u, v - v^N)\right| \le L \|u\|_X \left\|v - v^N\right\|_V
$$

as desired. Therefore by the Trotter-Kato result (Theorem 4.1), we now conclude that $T^{*N}(t)P^N \to T^*(t)$ strongly on X.

4.4 A Linear Spline Scheme

Our implementation of this scheme uses linear splines for simplicity, with the same number of splines in each delay interval. However, the theory is the same for higher-order splines. For a discretization parameter $N = 1, 2, \ldots$, choose the meshpoints

$$\theta_{k,j}^N = -r_{k-1} - j\frac{R_k}{N},$$

for $k = 1, 2, \ldots, n$, and $j = 0, 1, \ldots, N$, where $R_k = r_k - r_{k-1}$. These points are equally spaced within each delay interval, although they may not be equally spaced throughout $[-r_n, 0]$. Also, each delay point is a meshpoint, and we have a total of $Nn + 1$ distinct meshpoints. Next, define a set of first order splines across the entire interval $[-r_n, 0]$ by combining splines defined on each subinterval. Note that these are exactly the same first order splines used for retarded systems in [KS90, section 5.3]. The specific splines are described below. We take

$$b_{1,0}^N = \begin{cases} \dfrac{N}{R_1}(\theta - \theta_{1,1}^N) & \text{if } \theta_{1,1}^N \le \theta \le 0 \\ 0 & \text{otherwise,} \end{cases}$$

and for $k = 2, \ldots, n$ define

$$b_{k,0}^N = \begin{cases} \dfrac{N}{R_k}(\theta - \theta_{k,1}^N) & \text{if } \theta_{k,1}^N \le \theta < \theta_{k,0}^N \\ 0 & \text{otherwise.} \end{cases}$$

Also for $k = 1, 2, \ldots, n$ define

$$b_{k,N}^N(\theta) = \begin{cases} -\dfrac{N}{R_k}(\theta - \theta_{k,N-1}^N) & -r_k \le \theta \le \theta_{k,N-1}^N \\ 0 & \text{otherwise.} \end{cases}$$

This completes the definition of the first and last spline on each delay interval. These splines can be thought of as 'half-hat' functions. To construct the 'hat' functions on the interior of each delay interval, for $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, N-1$, define

$$b_{k,j}^N = \begin{cases} -\dfrac{N}{R_k}(\theta - \theta_{k,j-1}^N) & \text{if } \theta_{k,j}^N \le \theta \le \theta_{k,j-1}^N \\ \dfrac{N}{R_k}(\theta - \theta_{k,j+1}^N) & \text{if } \theta_{k,j+1}^N \le \theta < \theta_{k,j}^N \\ 0 & \text{otherwise.} \end{cases}$$

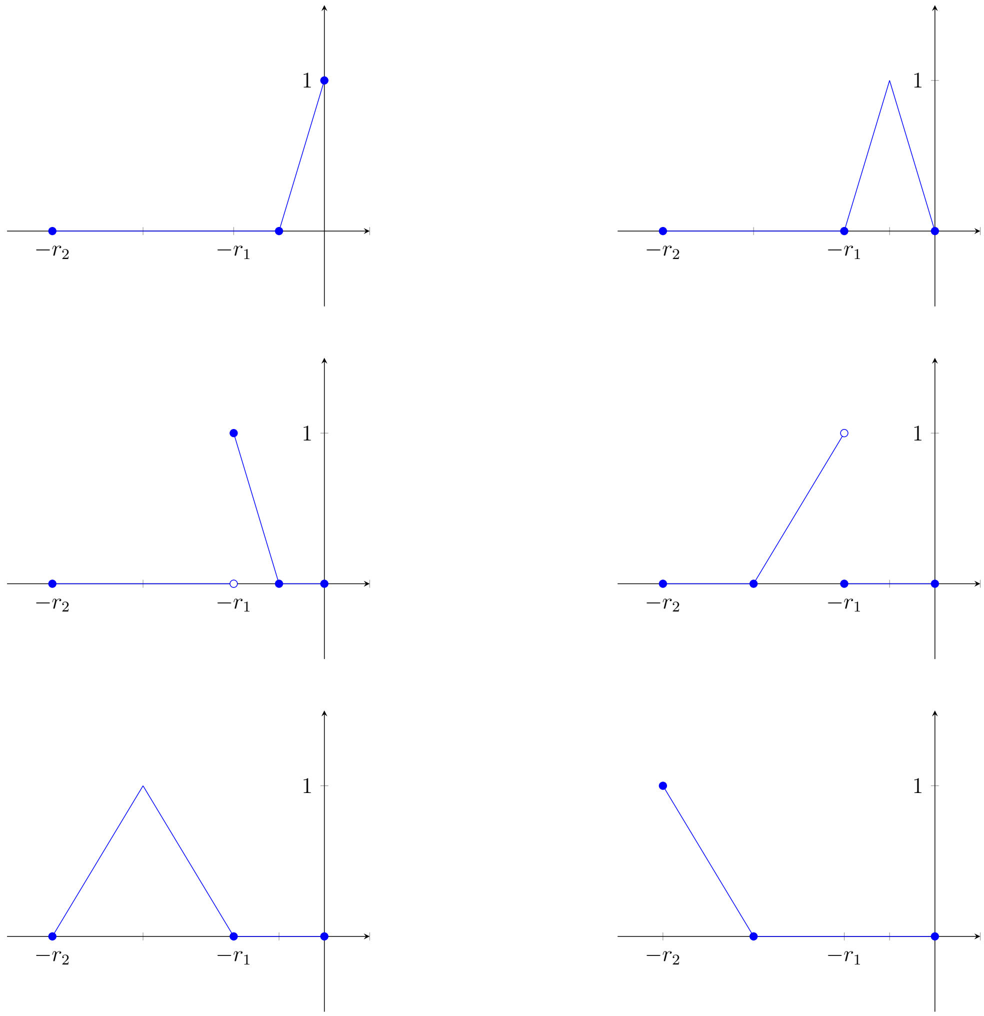
There are $n(N+1)$ of these first order splines, and they are used to discretize the function component of the state space. To complete the construction, let $e_i \in \mathbb{C}^m$ denote the standard Euclidean basis vector (all zeros except for the value 1 in the ith position). Define

$$\mathcal{E}_l = \begin{cases} (e_l, 0) & \text{if } l = 1, 2, \ldots, m, \\ (0, b_{k,j}^N(\theta)e_i) & \text{if } l = m[(k-1)(N+1)+1] + jm + i, \end{cases}$$

for $k = 1, 2, \ldots, n$, $j = 0, 1, \ldots, N$, and $i = 1, 2, \ldots, m$.

We then take $X^N = \text{span}\,\{\mathcal{E}_l\}$. Observe that X^N is exactly the same subspace used by Kappel and Salamon in [KS90].

Figure 1. Kappel-Salamon Basis Functions

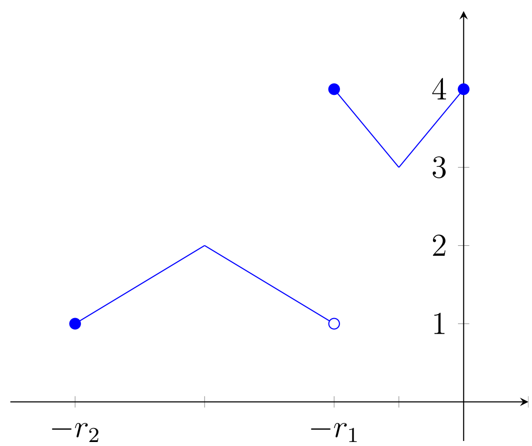


Although we did not use the characteristic functions $\{\chi_i\}_{i=1}^{n-1}$ explicitly to define X^N, it turns out that X^N can also be written as

$$X^N = \mathbb{C}^m \times (J^N)^m \times (\text{span}\,\{\chi_i\})^m \tag{4.12}$$

and hence the convergence results from the previous chapter apply. Here we are using linear splines, and the linear spline interpolant satisfies (4.5) and (4.6) by Theorem 2.5 in [Sch73].

Figure 2. Sample Function Component of X^N

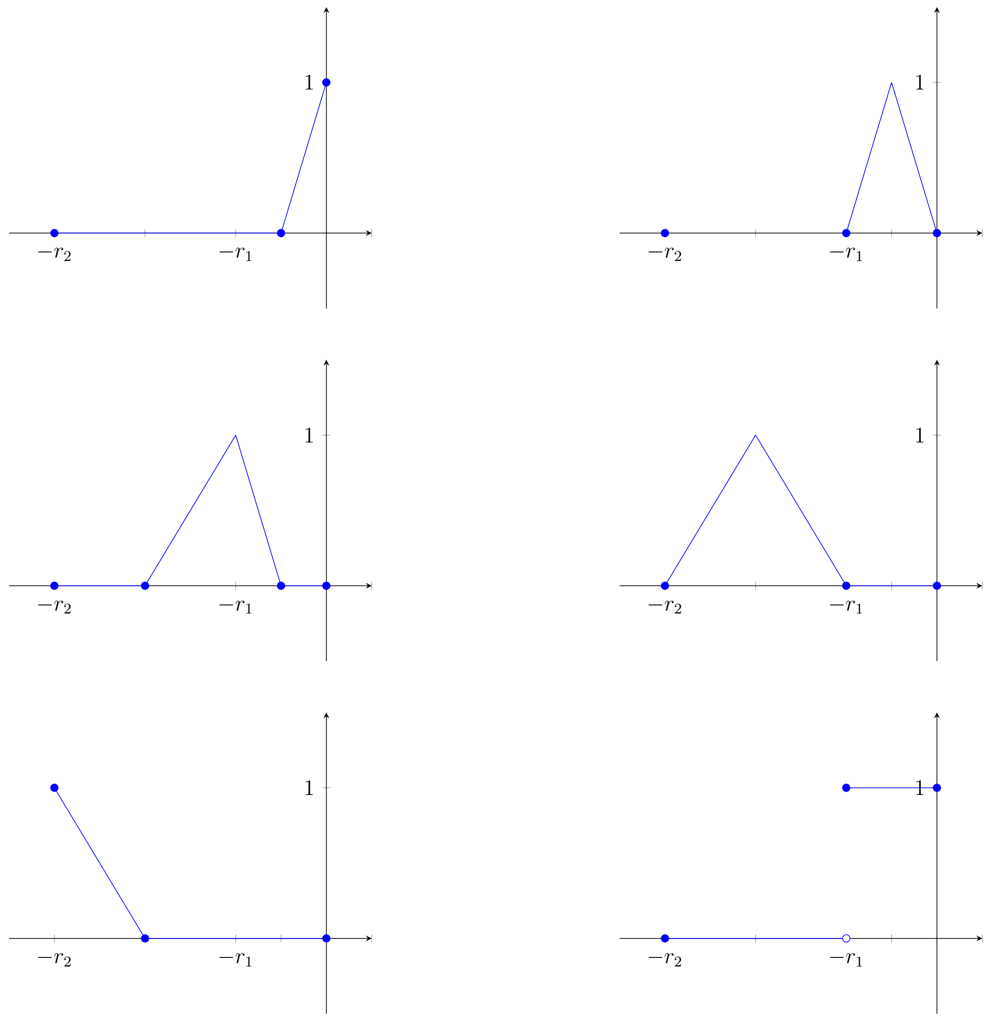


In other words, we can represent X^N in two different ways, because the function component in X^N can be written as the span of two different sets of basis functions. We illustrate this point graphically by considering the scalar two delay case with $N = 2$. Then

$$X^N = \text{span}\{\mathcal{E}_l\} = \mathbb{C} \times J^2 \times \text{span}\{\chi_i\}, \tag{4.13}$$

and a typical function in $J^2 \times \text{span}\{\chi_i\}$ is shown in Figure 2. However, $J^2 \times \text{span}\{\chi_i\}$ is the span of the basis function in Figure 1 (like the Kappel-Salamon basis functions) and is also the span of the basis function in Figure 3 (which includes χ_1). Thus we use the representation (4.12) for X^N in order to apply our convergence result, and we use the representation $X^N = \text{span}\{\mathcal{E}_l\}$ in order to construct matrix representations.

Figure 3. Basis Functions Using Characteristic Functions



We have already defined operators $\mathcal{A}^N$ and $\mathcal{A}^{*N}$ (see equations (4.7) and (4.8)). In order to implement an approximation scheme, we need a matrix representation of the form σ and the inner products of the basis numbers. The mass matrix M is given by $M_{ij} = \langle \mathcal{E}_i, \mathcal{E}_j \rangle$, where the inner product is the weighted inner product on M_2. The stiffness matrix K is given by $K_{ij} = \sigma(\mathcal{E}_i, \mathcal{E}_j)$. Then, in order to compute an approximate solution to $\dot{z}^N(t) = A^N z^N(t)$, we will let $q^N(t)$ denote the vector representation of the coefficients of $z^N(t)$ in the basis. Our work is then to solve the differential equation

$$M\dot{q}^N(t) = Kq^N(t).$$

However this can be easily solved in MATLAB by rewriting it as

$$\dot{q}^N(t) = (KM^{-1})^T q^N(t).$$

Upon finding an approximation solution $q^N(t)$ to this equation, we recall that the second component of $z(t)$ is the function $x(t+\theta)$. So, to approximate the solution of Equation (1.1), we will use $q^N(t)$ to find the value of the second component of $z^N(0)$.

4.5 Open Problems

Although this new approximation scheme does much to improve existing schemes, there are several open questions remaining. For example, we currently we have uniform stability in the discretization parameter only for neutral equations which satisfy equation (2.3). Can we get stability results for a wider class of equations? Also, we have not yet explored how changing the weight function used may affect the convergence of the scheme. Are some weight functions better than others? How much of this theory and scheme can be extended to related delay equations, such as equations with distributed delays? Finally, there is a scheme by Ito and Kappel in [IK91], [IK87], and [Kap91] which performs very well for retarded delay equations and has a uniform differentiability property. It would be interesting to see if this scheme could be extended to neutral equations also.

CHAPTER V

NUMERICAL IMPLEMENTATION AND EXAMPLES

We now turn to some numerical examples to demonstrate how this scheme improves upon existing schemes. We consider some examples related to the scalar delay equation

$$\dot{x}(t) + \frac{1}{4}\dot{x}(t - \frac{1}{2}) = -x(t) + \frac{1}{4}x(t - 1) \tag{5.1}$$

for $t \geq 0$, with initial data given by $x(t) = -t$ for $-1 \leq t \leq 0$. Note that this problem does satisfy the criteria for asymptotic stability. The true solution is given by

$$x(t) = h(t) = \frac{1}{4}(3 - t - 3e^{-t}) \text{ for } t \in [0, 1/2] \tag{5.2}$$

$$x(t) = h(t) - \frac{3}{16} + \frac{1}{32}(9 - 6t)e^{-(t-\frac{1}{2})} \text{ for } t \in [1/2, 1]. \tag{5.3}$$

We will use this problem to study the behavior of different approximation schemes as well as different weight functions. In particular, we will use the approximation scheme defined in Chapter 4 based upon the Kappel-Salamon splines, and compare with a modified spline scheme for neutral equations found in [Fab13]. For the two-delay example under consideration, the main difference between the two schemes is that the modified scheme combines the two Kappel-Salamon splines surrounding the interior delay into a single continuous spline. Thus the basis functions for the modified scheme are continuous throughout the interval.

The modified scheme uses a different space V_{mod} and form σ_{mod}, and in particular dom $\mathcal{A}^*$ is not contained in V_{mod} (recall the adjoint domain contains elements whose functional component has jump discontinuity at the interior delay, and the modified splines are continuous at the interior delay). Thus while the semigroup convergence $T^N(t)P^N \to T(t)$ holds for the modified scheme (as proved in [Fab13]), we do not expect adjoint semigroup convergence for the modified scheme. These two schemes allow us to compare the effects of adjoint semigroup convergence, especially for the LQR control problem. We will also study the effect of the choice of weight function on the approximation schemes. In particular, we consider a piecewise linear weight function which is sufficient to guarantee semigroup convergence, and a piecewise exponential weight function which gives a dissipative inequality that guarantees exponential stability uniformly in the discretization parameter. The piecewise linear weight function for this problem is

$$w_{\text{lin}}(\theta) = \begin{cases} -4\theta + 1 & \text{for } -1/2 \leq \theta \leq 0 \\ -\theta + 1/2 & \text{for } -1 \leq \theta < -1/2, \end{cases}$$

and the piecewise exponential weight function is

$$w_{\text{exp}}(\theta) = \begin{cases} \dfrac{1}{4} + \dfrac{3}{4} e^{(\ln \frac{3}{2})\theta} & \text{for } -1/2 \leq \theta \leq 0, \\ \dfrac{3}{4} e^{(\ln \frac{3}{2})\theta} & \text{for } -1 \leq \theta < -1/2. \end{cases}$$

These computational examples are very consistent with the theory: in Example 1 both schemes with both weights do well for the simulation problem, where neither adjoint convergence nor uniform stability is required by theory. In Example 2 we study a related LQR problem in which the modified scheme (without adjoint convergence) does poorly. We note in Example 2 that neither scheme obtained convergence with the piecewise linear weight, indicating the importance of uniform stability for control problems. We compare both schemes with the AVE scheme found in [BB78], which is known to have good convergence for both simulation and optimal control problems.

Example 1 We use the two schemes to approximate the solution of (5.1). To do this we reformulate (5.1) in state space form as

$$\frac{d}{dt}z(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \tag{5.4}$$

and then use the schemes to construct finite dimensional problems

$$\frac{d}{dt}z^N(t) = \mathcal{A}^N z^N(t) \tag{5.5}$$

on X^N. We take matrix representations and solve the resulting differential equations in Matlab. In Figures 4 and 5 we plot the approximate and exact solution on $0 \leq t \leq 1$ for both schemes and both weight functions. In both figures, the red line indicates the exact solution, and the blue line represents the approximation solution. These illustrate the semigroup convergence for each scheme.

The weight function did not affect convergence, but in Figures 6 and 7 we plot the eigenvalues of $\mathcal{A}^N$ to illustrate that with the piecewise exponential weight function the scheme is exponentially stable uniformly in the discretization parameter (it can be shown that the eigenvalues of $\mathcal{A}^N$ stay bounded away from the imaginary axis uniformly in N), but the same is not true for the piecewise linear weight function.

The plots are for the Kappel-Salamon splines, but the same behavior is observed for the modified splines.

Figure 4. Simulation Problem, Kappel-Salamon Splines (With Jumps)

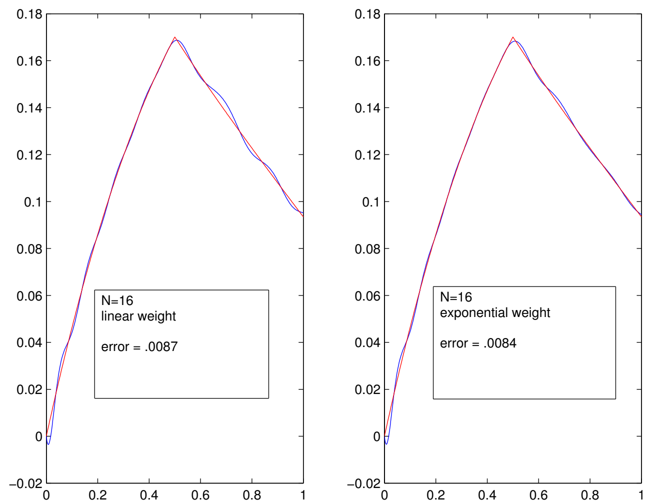


Figure 5. Simulation Problem, Modified Scheme (No Jumps)

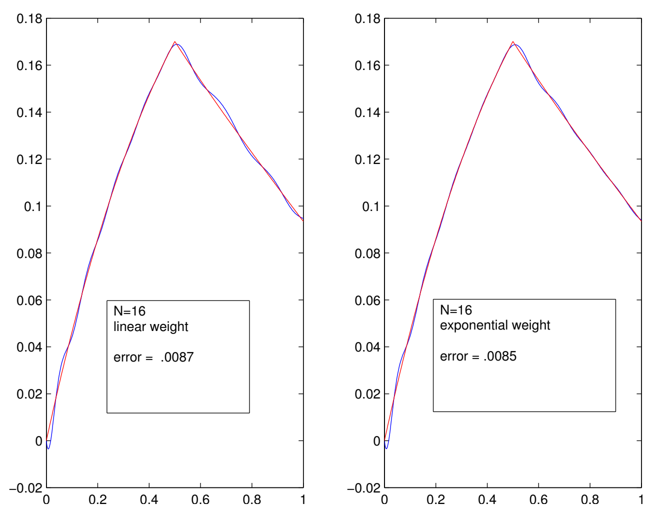


Figure 6. Eigenvalues for Piecewise Linear Weight Function

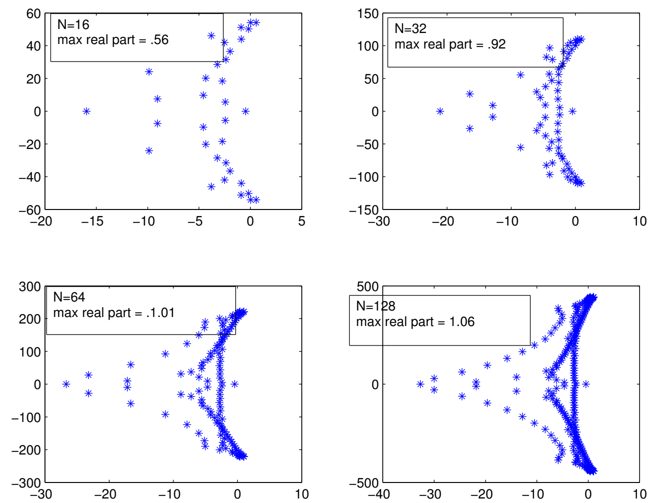
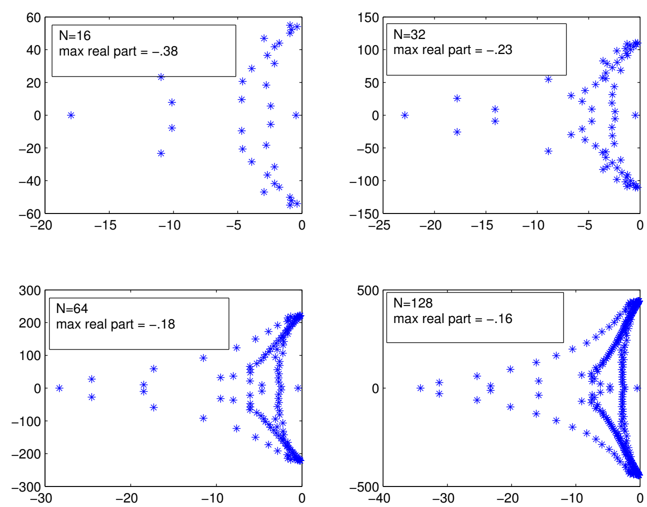


Figure 7. Eigenvalues for Piecewise Exponential Weight Function



Example 2 We now turn to an example of an LQR control problem, which will demonstrate the advantages of the present scheme. We are interested in the problem of

minimizing the cost functional

$$J(u) = \int_0^\infty (\|Qx(t)\|^2 + R|u(t)|^2)\, dt, \tag{5.6}$$

subject to dynamics governed by

$$\frac{d}{dt}[x(t) + \sum_{k=1}^{n} C_k x(t - r_k)] = Ax(t) + \sum_{k=1}^{n} D_k x(t - r_k) + b\, u(t). \tag{5.7}$$

Here $b \in \mathbb{C}^m$ and we take the controller u to be one dimensional for simplicity of exposition, but the ideas easily extend to higher dimensions. Using the same method that we used for the delay equation, this problem can be reformulated as

$$\frac{d}{dt} z(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \tag{5.8}$$

where $\mathcal{B} : \mathbb{C} \to X$ is defined by $\mathcal{B}u = (bu, 0)$.

Specifically, we consider the LQR problem of minimizing the cost functional

$$\int_0^\infty (|x(t)|^2 + |u(t)|^2)\, dt$$

subject to dynamics governed by

$$\dot{x}(t) + \frac{1}{4}\dot{x}(t - \frac{1}{2}) = -x(t) + \frac{1}{4}x(t-1) + u(t).$$

When we reformulate this in Cauchy form, the cost functional becomes

$$\int_0^\infty (\|\mathcal{Q}z(t)\|_X^2 + |u(t)|^2)\,dt.$$

Here we have $R = 1$ and $\mathcal{Q}$ is given by

$$\mathcal{Q}(\eta, f) = Q\left(\eta - \sum_{i=1}^{n} C_k f(-r_k)\right). \tag{5.9}$$

We can then implement the approximation scheme to yield an LQR problem with finite dimensional dynamics. The solution is given in feedback form $u(t) = Kz(t)$, and the gain operator is a bounded linear functional on Z. Thus by the Riesz theorem the $L^2(-r_n, 0; \mathbb{C})$ component of the gain can be represented by a function $k(\theta)$, called the feedback functional gain. We compute the approximate gains $k^N(\theta)$. We compute the approximate gains $k^N(\theta)$ for the two spline schemes as well as the averaging (AVE) scheme. We do not have the exact solution $k(\theta)$, but as we increase the discretization parameter we can observe convergence behavior for all three schemes. This is illustrated in Figures 8, 9 and 10. Figure 11 shows the feedback gain for all three schemes on one axis.

The modified scheme, which does not have adjoint semigroup convergence, does not perform as well as the Kappel-Salamon scheme or the averaging scheme. In this example the piecewise exponential weight was used, so both spline schemes do have preservation of stability uniformly in the discretization parameter. Although we do not have an exact solution for the control problems, we observe that the current scheme demonstrates convergence as N increases.

Figure 8. Control Problem, Kappel-Salamon Splines

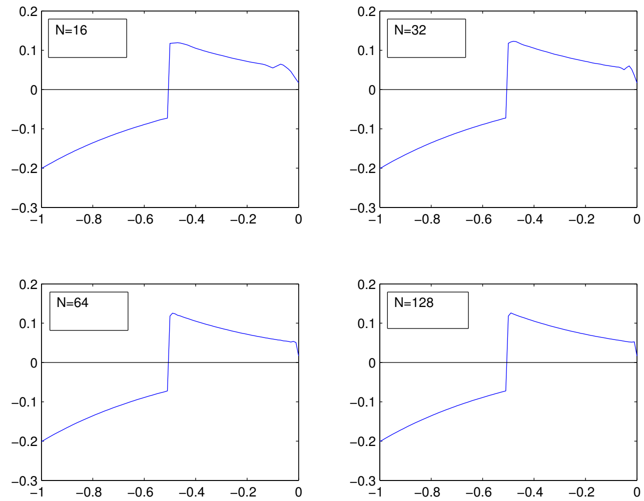


Figure 9. Control Problem, Modified Scheme

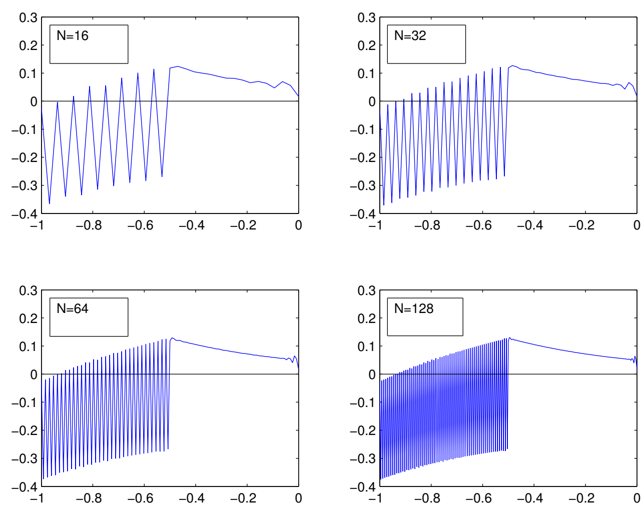


Figure 10. Control Problem, AVE Scheme

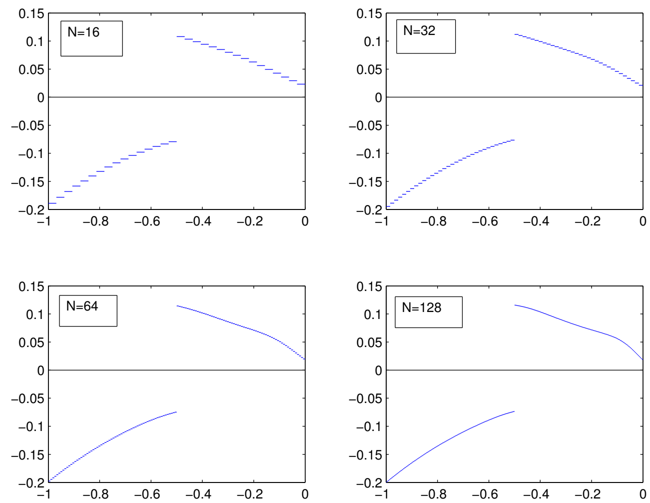
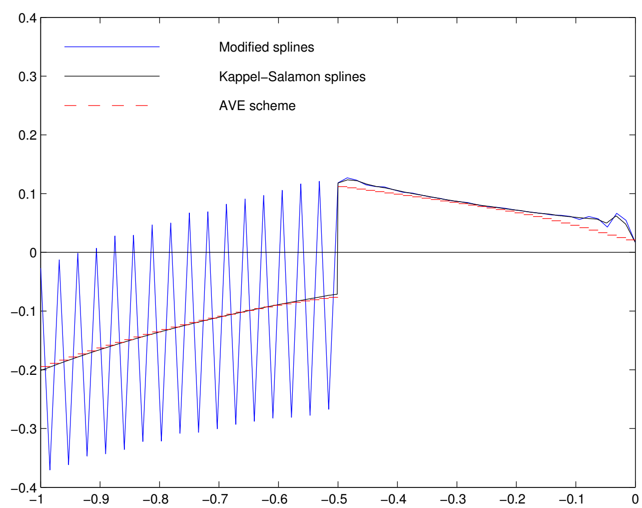


Figure 11. Control Problem, All Schemes



However, the plots for the modified scheme show oscillations, indicating the importance of having adjoint convergence in the control problem. Both schemes were implemented with a piecewise exponential weight function in order to preserve stability uniformly in the discretization parameter. We can still obtain convergence behavior for other weight functions with appropriately chosen jumps at the delay points.

CHAPTER VI

CONCLUSION

Delay differential equations arise in a variety of applications. We have proposed a new delay-independent condition to guarantee exponential stability for the solution semigroup of equation (1.1), as well as a new approximation scheme for solutions of (1.1). In both cases, we reformulate the delay equation as an abstract Cauchy problem (1.4) and proceed using semigroup theory. The new stability condition is given in terms of the matrices in equation (1.1). This condition is obtained by renorming the space M_2, by adding a weight function of a particular form. We then proved a dissipative inequality which guarantees stability by choosing an appropriate weight function to satisfy equation (1.9). The new stability condition in (2.3) is independent of several existing conditions in the literature, and improves another condition.

Our new approximation scheme extends the scheme by Kappel and Salamon in [KS90] from retarded to neutral delay differential equations. Convergence was shown using a Trotter-Kato style theorem. In order to do this, we defined new forms σ and τ which are used to define approximating operators $\mathcal{A}^N$ and $\mathcal{A}^{N*}$. A key feature of our scheme is that it converges for both the operator $\mathcal{A}$ and $\mathcal{A}^*$. We then implemented this scheme with linear splines and considered some example computations to show that the new scheme performs well for both simulation and optimal control LQR problems. In particular, the new scheme shows convergence of the feedback gain for the LQR problem, which was not present in many existing spline schemes.

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