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Algebra is often described as the gateway to higher mathematics (Carpenter, Franke, & Levi, 2003; Kaput, 2008; Kaput & Blanton, 2001; Mason, 2008). Unfortunately, many students do not navigate this gateway successfully. Kaput (2008) and Mason (2008) suggested that this is due in part to the abrupt switch from arithmetic to algebra that occurs in late middle school to early high school. Many called for a change in the way mathematics is taught from kindergarten through high school (Blanton, 2008; Kaput, 2008; NCTM, 2000), which included a change to the introduction of early algebra. Carraher, Schliemann, and Schwartz (2008) caution that early algebra is different from algebra early. They suggested that this was a switch in the way the basic tenants of algebraic thinking is understood, including generalizing arithmetic and functional thinking (Brizuela & Lara-Roth, 2002; Warren, 2005b). Given this focus on functional reasoning in early elementary school, I conducted a study of students in the third through fifth grades. My purpose was to identify the characteristics of functional reasoning tasks that promoted the development of functional reasoning. I also wanted to discover the ways in which students used these characteristics as they worked on a task. Another important aspect of this study was to find any other influences on the types and frequencies of generalizations the students constructed. My study lasted for 10 weeks during an afterschool enrichment program called On Track Learn Math. The program occurred on two days during the week and students worked on a different task per week across two five-week sessions. This study analyzed 10 of those tasks. The data

demonstrated that the tasks contained specific characteristics that promoted the development of functional reasoning. The two most noteworthy considerations were that grade level and the complexity of the function rule affected the frequency with which students constructed explicit rules. The results of this study have implications on student learning, teacher practice, and curricular changes.

TASKS THAT PROMOTE FUNCTIONAL REASONING

IN EARLY ELEMENTARY SCHOOL STUDENTS

by

Nancy Tilley Payne

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Approved by

Sarah B. Berenson
Committee Co-Chair

Kerri Richardson
Committee Co-Chair

To Jeff and Megan

whose love and support made this possible

APPROVAL PAGE

This dissertation has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

Committee Co-Chair Sarah B. Berenson

Committee Co-Chair Kerri Richardson

Committee Members P. Holt Wilson

Heidi Carlone

March 13, 2012
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Date of Final Oral Examination

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CHAPTER I

INTRODUCTION

Problem Statement

"If reasoning ability is not developed in the student, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense" (Ross, 1998, p. 254). Kenneth Ross articulated well the importance of developing reasoning in mathematics. The National Council of Teachers of Mathematics (NCTM, 2000) asserted that analytical thinkers could identify structure, see and explore patterns, generalize and justify their thinking and make good arguments to the authenticity of their justifications. When students develop these types of skills, they come to see that mathematics makes sense and use these skills across content areas. Teachers can help students improve their reasoning skills by using relevant tasks and instructional practices that elicit students thinking (NCTM, 2000, Warren, 2005b).

Recognizing a need for mathematics reform, the NCTM (2000) content standards advocates the teaching of algebra to all grade levels with the goal to develop students who are capable of reasoning, problem solving, communicating their ideas, and using proof. Nevertheless, international studies show that there is still a gap between students in the United States and their peers, in countries with similar economics and development as the U.S., as students progress through school (NCES, 2004, 2007). In support of this are the recent Trends in International Mathematics and Science (TIMMS) study of fourth-grade students, which suggest that US students perform better when applying mathematical knowledge than when reasoning mathematically (NCES, 2007).

As a result, researchers advocate for introducing algebraic concepts in early elementary school in conjunction with arithmetic; not as a separate subject (Brizuela & Schliemann, 2004; Kaput, 2008; Warren, Cooper, & Lamb, 2006). The rationale for an early introduction to algebra is that when students begin a study of algebra in late middle or early high school, they only develop a superficial understanding of the subject (Kaput, 2008; Mason, 2008). Additionally, the introduction of algebra as a strand of thinking, along with problem solving, helps students develop a deeper understanding of the subject (Carpenter & Levi, 2000; Kaput, 2008).

While algebraic reasoning encompasses many concepts, functional reasoning is one aspect of algebraic reasoning that is critical to students' success in high-level mathematics (Brizuela & Lara-Roth, 2002; Warren, 2005b). Smith (2008) defined functions as "representational thinking that focuses on the relationship between two (or more) varying quantities" (p. 143). Many researchers report that students are capable of engaging in functional reasoning in early elementary school (e.g. Blanton & Kaput, 2004; Warren, 2005a). Also, some children develop an intuitive understanding of functions before any formal introduction of them occur (Eisenmann, 2009), but many others have a difficult time understanding algebraic functions even in higher levels of mathematics (Confrey & Smith, 1995; Warren et al., 2006). The research suggests that in order to develop functional reasoning, educators should begin the process early by engaging students in tasks that allow them to reason about the relationships between quantities and make generalizations (Blanton & Kaput, 2005; Confrey & Smith, 1994, Warren, 2005b).

The efforts of James Kaput, Maria Blanton, Elizabeth Warren, and Carolyn Maher and their colleagues have furthered the literature on functional reasoning. These researchers are on the forefront of the reform efforts in mathematics education. Much of the literature focuses on students' development of functional reasoning and student learning outcomes by using specific tasks. Certain types of tasks are particularly effective at developing students' functional reasoning. These include tasks that ask students to look for patterns in the data and to make generalizations about those patterns (Blanton & Kaput, 2004; Martinez & Brizuela, 2006; Smith, 2008; Warren et al., 2006).

On Track Learn Math is an afterschool enrichment program developed at the University of North Carolina at Greensboro by Dr. Sarah Berenson, principal investigator, and Dr. Kerri Richardson, co-principal investigator. The purpose of this program is to investigate early algebraic reasoning in grades three through five. The program's five sessions each ran for five weeks with students attending two days per week. Each afternoon begins with a mathematics inspired sports activity followed by students working on different reasoning activities.

Prior Research on this topic focused on justifying beginning a study of functions early, examining students' ability to reason about functions, and studying student solution strategies. While the research agreed on the importance of helping students develop their functional reasoning skills early, we still know very little about appropriate tasks that help develop these skills.

Purpose Statement and Research Questions

Algebra is historically considered the gateway to higher mathematics (Carpenter, Franke, & Levi, 2003; Kaput, 2008; Kaput & Blanton, 2001, Mason, 2008). Unfortunately, many students are not able to navigate successfully through it. Much of the literature on introducing algebra early falls into two major areas, generalization of arithmetic and developing functional reasoning. Generalization of arithmetic includes structuring activities to allow students to discover the properties of real numbers, work with unknown quantities, and use symbolic language to express their ideas (Blanton, 2008; Kaput, 2008). Another key focus is developing students' functional reasoning abilities. This type of reasoning is particularly difficult for students, because it requires them to develop a different set of skills that include noticing how quantities in a data set vary with respect to each other (Blanton, 2008; Martinez & Brizuela, 2006; Oehrtman, Carlson, & Thompson, 2008; Warren et al., 2006). These skills are important because they form the basis of higher-level mathematics.

According to Blanton (2008), developing students' functional reasoning does not require teaching additional topics, instead, teachers integrate it into the existing curriculum by giving students the opportunity to analyze and describe functional relationships. In order to do this, teachers need access to appropriate tasks. Therefore, this mixed methods study, analyzes the tasks used in the On Track Learn Math program and considers their impact on student learning. The following research questions drive this inquiry:

1. What are the functional reasoning characteristics of the On Track mathematical tasks?
2. What evidence suggests that the On Track tasks supported the development of students’ functional reasoning among students of varying age and ability levels?
3. Which tasks appear to promote students’ functional reasoning?
4. What effect does grade level have on students’ functional reasoning?
5. What types of function rules appear to promote students’ correspondence reasoning?

Significance of the Study

As stated previously an early introduction to algebra helps to support students’ success in mathematics. Existing literature maintains that young elementary aged children are capable of understanding functions and that teachers help students develop these skills, in part, by the tasks they select. This study is important for several reasons. First, this study adds to the scholarly research in algebraic reasoning by investigating the characteristics of tasks that help students develop functional reasoning. Second, it serves to support existing literature that young students are capable of understanding functional relationships. Third, it makes the tasks themselves the overall focus of the research and ties the tasks to student solution strategies. Lastly, it supports and informs teacher practice on introducing functions in elementary school.

Definition of Important Terms

Several important definitions will assist the reader to understand the concepts discussed in this dissertation. The following section will define those important terms.

Algebraic Reasoning

There exists multiple interpretations of algebraic reasoning; however, for the purpose of this study I will be using the definition of algebraic reasoning proposed by Kaput (2008). Kaput defined algebraic reasoning into two core aspects, the expression of generalizations and the use of symbols to act on these generalizations. Each of these core aspects appears across three strands, which include, but not limited to, algebra as the study of structures, symbols, functions, and relations. This study will focus on the generalization of rules that supports functional reasoning.

Functions

Throughout the history of mathematics, mathematicians have worked to define the concept of a function. Our modern day interpretation of a function is credited to both Dirichlet and Bourbaki and is now known as the Dirichlet-Bourbaki concept (Selden & Selden, 1991). Together, they added to the definition of a function which is now considered a correspondence between two sets such that each element in the first set corresponds to exactly one element in the second set (Vinner & Dreyfus, 1989). My research will use the modified definition given by Warren et al. (2006), "a function is denoted or expressed in terms of the relationship between a first variable quantity and a second variable quantity or in terms of the change from the second to the first" (pp. 208-209).

Generalization

According to Kaput (2008), one core aspect to algebraic reasoning is generalization. Generalization bridges the gap between arithmetic and algebra. When students generalize arithmetic, they make a general statement that covers many instances (Kaput, Blanton, & Moreno, 2008). For example, stating that an odd number plus an odd number is always an even number is a generalization about addition. Students' generalization takes on two forms, recursive and explicit (Lannin, Barker, & Townsend, 2006).

Recursive Generalizations

According to Lannin et al. (2006), when students use a recursive strategy, they build on previous terms to generate successive terms in a sequence. For example, given the Fibonacci Sequence (1, 1, 2, 3, 5, 8, . . .), a recursive generalization would state that you add the last two terms to get the next term or symbolically as $f_n = f_{n-1} + f_{n-2}$. When using input/output tables, students who use recursive strategies typically describe the change in successive outputs.

Explicit Functions

When students use an explicit strategy, they construct a rule that allows for an immediate calculation for any output value given a specific input value (Lannin et al., 2006). An example of an explicit rule in symbolic terms is $f(n) = 3n + 2$, however students tend to describe their explicit rules in words such as "multiply the input by three and add two to generate the output." This statement relates the output to the input value for any input value; it does not depend on the last output value as the recursive rule does.

Functional Reasoning

Functional reasoning is one strand of algebraic reasoning in which teachers can build generality into the content they teach (Blanton & Kaput, 2004, 2011). According to Smith (2008), when students focus representational thinking specifically on the relationship between two varying quantities, they begin to engage in functional thinking. The literature discusses both functional reasoning and functional thinking. In either instance, it refers to how students come to work with and understand functional relationships. Reasoning indicates more logical mental processing. Therefore, in my study I will be using the phrase functional reasoning.

Input/Output Table

An input/output table helps to organize student data. The table consists of two columns, with the input numbers (independent variable) in the first column and the output numbers (dependent variable) in the second column. The table shows horizontally which input numbers correspond to the output numbers. The following is an example of an input/output table.

Input	Output
1	6
2	12
3	18

Figure 1. Example of an Input/Output Table

Variational Reasoning

When students focus on a single data set and describe the change they see, they engage in single variational reasoning. When students look at two data sets and describe how they are changing in relation to each other, they engage in covariational reasoning. Students engage in variational reasoning when they look down the input/output table instead of across the table. Instead of distinguishing between single variational and covariational reasoning for simplicity, I will be using the term variational reasoning.

Correspondence Reasoning

According to Confrey and Smith (1994), correspondence reasoning occurs when students look for ways to relate the input value to its corresponding output value. Correspondence reasoning lead to rules that determines a unique output value for any given input value. For example, in a typical correspondence rule a student may give for the previous input/output table may be “multiply the input values by six to get the output values.” Students use correspondence reasoning when they look across the input/output table instead of down the table.

Study Organization

The remainder of this dissertation will present my study in detail and give implications for the results. In Chapter II, I review the relevant literature and establish a conceptual framework that informs this study. In Chapter III, I present my research design and methodology. This includes a discussion of the setting of the study, participants, and the methods of data collection and analysis. In Chapter IV, I present the results of my analysis. Finally, in Chapter V, I summarize the findings, discuss the implications of the study, and make recommendations for future research.

Summary

There is significant literature on students' difficulties in algebra (Kaput, 2008; Kaput & Blanton, 2001; Papic & Mulligan, 2007; Schifter, Monk, Russell, & Bastable, 2008). According to Kaput (2008), the curriculum in the United States historically introduced algebra at the completion of arithmetic. This "arithmetic-then-algebra" (p. 5) curriculum led to a late and abrupt approach to the learning of algebra. The consequences of this were high failure rates especially among disadvantaged students. The response was a push toward early algebra that included introducing opportunities into the elementary school curriculum for students to reason about generality. As Carraher et al. (2008) tell us, "early algebra is not the same as algebra early" (p. 235).

Early algebra takes two forms, generalizing arithmetic and the introductory study of functions (Brizuela & Lara-Roth, 2002; Warren, 2005b). Many studies show that young students are able to reason about functions when introduced through appropriate tasks that allows them to reason about data (Blanton, 2008; Blanton & Kaput, 2004; Carraher et al., 2008; Lannin, 2005; Martinez & Brizuela, 2006; Warren, 2005a, 2005b; Warren & Cooper, 2008a). However, the literature does not adequately detail the types of tasks that are appropriate to use nor suggests when they should be introduced. My study helps fill this void in the literature.

CHAPTER II

LITERATURE REVIEW

With an increasing academic focus on mathematical skills, there is a growing amount of concern about how to ensure student success in mathematics (Kaput, 2008). According to Kaput, one answer to this has been to introduce algebraic reasoning skills earlier in the curriculum. Even though many have called for an early exposure to algebra this does not mean schools will be teaching traditional algebra earlier. It does mean rethinking the elementary curriculum to introduce rich problems and promote thinking skills that leads to algebraic reasoning (Carraher et al., 2008). One aspect of algebraic reasoning, functional reasoning, is particularly difficult for students and can undermine their success in higher mathematics (NCTM, 2000; Martinez & Brizuela, 2006; Oehrtman et al., 2008; Warren et al., 2006). This study focuses on features of tasks that facilitate young students' development of functional reasoning. This review of the literature will situate the problem in context and provide evidence of the need for additional research.

Algebraic Reasoning

Many researchers have noted that a significant portion of students have difficulties in algebra (Kaput, 2008; Kaput & Blanton, 2001; Papic & Mulligan, 2007; Schifter et al., 2008) especially when it is introduced in late middle school or high school. Papic and Mulligan (2007) suggest that these difficulties are based partially on limited reasoning opportunities in elementary school. Also, according to Smith and Thompson (2008), too many students feel that mathematics has no useful relationship to them. It typically begins with numbers and procedures and ends with symbols and symbolic procedures. According to these researchers, there is a missing connection between the two. That connection is the inclusion of relevant and interesting problems and ideas that students can investigate.

Kaput (2008) describes the “algebra problem” as a historical separation of algebra from arithmetic. However, with the push toward developing a more technological nation, math and science has been brought to the forefront of educational issues. The expectation is that all students will graduate from high school having developed a good sense of mathematical literacy. This has brought about a push toward “mathematics for all” (Kaput, 2008, p. 6). Students develop a deeper understanding of mathematics by integrating algebra into the arithmetic curriculum as a strand of thinking (Carpenter & Levi, 2000; Kaput, 2008) and by the introduction of these concepts in early elementary school in conjunction with arithmetic and not as a separate subject (Brizuela & Schliemann, 2004; Kaput, 2008; NCTM, 2000; Warren et al., 2006). This had led to a significant body of research on algebraic reasoning in early elementary school.

Blanton and Kaput (2005) define algebraic reasoning as “a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentations, and express them in increasingly formal and age appropriate ways” (p. 413). While some researchers use a variation of Blanton and Kaput’s definition (Franke, Carpenter, & Battey, 2008, Smith &

Thompson, 2008), other researchers have included solving problems and using algebraic notations (Carpenter & Levi, 2000; Carraher & Schliemann, 2007; Schliemann, Carraher, & Brizuela, 2007); using representations and models (Schoenfeld, 2008; Smith, 2008); and relational and functional reasoning (Smith, 2008) to their definition.

Regardless of the definition used, the basic principles of algebraic reasoning include building generality into the curriculum. Russell, Schifter, and Bastable (2011) offer several ways to do this. First, students should understand the behavior of operations. This includes a conceptual approach to the development of the properties of arithmetic and multiplication. This includes an emphasis on noticing these properties while working on their computation skills. Second, teachers should prompt students to justify their ideas and to make their thinking clear. As students work on generalizations, there is a need to develop their language in order to describe these. Third, is extending the number system. Some discussions lead to ideas that are only true when working with the positive numbers. For example, when multiplying two whole numbers together the product is a larger number. This does not hold true for fractions. As students' number system expands it gives teachers the opportunity of discuss whether the students previous ideas hold. Lastly, students should use notations with meaning. However, teachers should be careful how they introduce notations. This should occur when students have developed their ideas in words and images. This allows them to maintain meaning when using symbols.

The NCTM (2000) content standards advocates in grades three through five algebraic reasoning should take the form of:

- identifying and building numerical and geometric patterns

- describe patterns verbally and represent them with tables or symbols
- look for and apply relationships between varying quantities to make predictions
- make and explain generalizations that seem to always work in particular situations
- use graphs to describe patterns and make predications
- explore number properties
- use invented notation, standard symbols, and variables to express a pattern, generalization, or situation (p. 159)

Blanton (2008) gives suggestions for uncomplicated ways to add algebraic reasoning into the elementary curriculum. These include attending to ways to add generality into instruction by purposefully designing computation problems that lead to generalizations. Teachers can also make known quantities unknown by taking out information. This encourages a discussion of how to solve for unknown quantities setting the stage to develop an understanding of variables later on. She also suggests that an essential way of integrating early algebra is by encouraging functional thinking.

The Historical Development of Functions

The following section will give a brief historical overview of the development of functions. It will also include a discussion of the treatment of functional reasoning in education. The purpose of including this section in the literature review is to give historical significance to this problem.

Rizzuti (1991) classified the concept of function into two perspectives, classical and modern. The classical definition of function is traced to the ancient philosophers who were interested in the study of motion, particularly that of astronomical motion (Atkinson, 2002). However, two paths of thought emerged, the study of moving things and the study of why things move. Aristotle (384-322 BCE) became interested in why things fall (change of place), however he did not try to explain this using velocity. The idea of velocity (change over time) first appeared during the middle ages, after clocks became commonplace. Nicole Oresme gained strides in the study of functions when he began using a horizontal line segment to represent the duration of a motion. For each point on the line, a vertical line segment represented the intensity of the motion. While not perfect, Oresme created what would later become a graphical representation of velocity. Later Galileo would expand on these ideas to develop the law of motion in the seventeenth century concept of functions (Boyer, 1991).

A more modern day notion of function grew out of the work to develop calculus. Initially, calculus was based on problem solving methods that were applicable to curves (Selden & Selden, 1992). Johann Bernoulli defined a function in 1718 that expressed it as a quantity that was dependent on other quantities (Selden & Selden, 1992) but Leonard Euler is credited with formalizing the definition of function in 1748 as an expression or formula representing a relation between variables. Additionally, he introduced the *f(x)* notation (Boyer, 1991). Fourier and Cauchy improved this definition around 1820 to include the property of uniqueness: “y is a function of x when to each value of x in a given interval, there corresponds a unique value of y” (Kline, 1972, p.950). In 1837, Dirichlet viewed functions as a special kind of correspondence between two sets, and Bourbaki created the ordered pair definition of functions in 1939. These two views make up the definition we currently use, called the Dirichlet-Bourbaki approach (Selden & Selden, 1992). In modern education, the concept of function serves as the foundation of most of the mathematical courses taught in secondary and post-secondary schools.

In a revision of its standards, NCTM (2000) added algebraic thinking to the elementary mathematics curriculum. The call to create a better understanding of algebra initiated this change. Additionally, there was a need to bridge the gap between the focus on arithmetic students studied in elementary school and the algebra introduced in middle school. However, no one was advocating that symbolic algebra, prevalent in high school mathematics, should be added to the elementary school curriculum. On the contrary, by recognizing, describing, and extending patterns of growth students can begin the process of learning to think algebraically. They also included the generalization of rules that govern pattern growth as an important precursor to functional thinking.

The Common Core State Standards (2010) initiative defines the knowledge and skills students need in order to succeed when they graduate from high school. These standards outline a mathematics curriculum from kindergarten through twelfth grade. Beginning in grade three, the standards, advocate the identification of arithmetic patterns, by grade four, there is an expectation that students can generate a number or shape pattern that follows a given rule. By grade five, students need to be able to use covariational and correspondence reasoning to identify relationships between corresponding terms, and form ordered pairs and graph in the coordinate plane.

The historical background on functions not only gives an overview of the development of these concepts, it gives a justification to the inclusion of these concepts in the modern day mathematics curriculum. The next section of this literature review will focus on tasks that elicit functional reasoning and other important concepts in the development of reasoning.

Functional Reasoning

Blanton (2008) describes functional thinking as the thinking process used when generalizing a set of data. Similarly, Smith (2008) defined functional thinking as students focusing representational thinking specifically on the relationship between two varying quantities. A function describes a relationship between two quantities, such that for any given input value there is a unique output value. For example, if your rate of pay is $10.00 per hour and you work 5 hours, then you will make $50.00. As long as the rule ($10.00 per hour) is consistent then the output $50.00 will always map to the input 5 hours. Encouraging students to reason about related data can integrate these skills into the elementary curriculum.

Functions can serve as one impediment to students' success in algebra and higher-level mathematics (Martinez & Brizuela, 2006; NCTM, 2000; Oehrtman et al., 2008; Warren et al., 2006). Oehrtman et al. (2008) wrote that students do not build a strong foundation in functional thinking when there is an over-emphasis on procedures without developing a deep understanding of concepts. In fact, students often relied on a memorized rule to support their thinking when asked about their misunderstandings involving functions.

Historically, the study of functions was limited to a more formal study of algebra that occurred in middle and high school. Now, many researchers are taking the stance that the study of functions should be more longitudinal beginning in early elementary school (Blanton & Kaput, 2011). Recent research shows that an early introduction to functions through tasks that are structured toward the development of these skills assist children as young as pre-k to reason about functions (Blanton & Kaput, 2004; Smith, 2003, 2008; Warren, 2005b; Warren et al., 2006). In fact, according to Eisenmann (2009), children's intuitive understanding of functions begins to develop before a formal introduction of functions. According to Blanton and Kaput (2004), teachers can help students develop their functional thinking by including specific tasks such as finding and generalizing patterns. In a study of urban elementary students, these researchers found that over time students were able to express mathematical relationships using tables, graphs, pictures, words, and symbols in increasingly sophisticated ways. By third grade, students could use symbols to represent varying quantities. These researchers noted that teachers were able to scaffold student thinking in such a way that the representational tools students used became a regular part of their mathematical vocabulary from an early age.

In order for students to develop functional reasoning, it is important that they first understand how quantities change in relation to each other (NCTM, 2000). In other words, they need to be able to recognize in what ways quantities vary. Students generally have little difficulty in determining how a single data set changes (single variation) but are challenged by how two data sets change in relation to each other (covariation) (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Warren, 2005a, 2005b; Warren & Cooper, 2008b). Since functions are dependent relationships between quantities and they exist in many domains, it is important for students to understand them and be able to use the tools of algebra to conceptualize and work with them (Kalchman & Koedinger, 2005). Smith (2008) proposed the following six activities that can form a foundation for the concept of function.

1. Engage in some type of physical or conceptual activity.
2. Identify two or more quantities that vary in the course of this activity and focus one's attention on the relationship between the two variables.
3. Make a record of the corresponding values of these quantities, typically tabular, graphical, or iconic.
4. Identify patterns in these records.
5. Coordinate the identified patterns with the actions involved in carrying out the activity.
6. Using this coordination to create a representation of the identified pattern in the relationship. (pp. 143-144)

The next section of this literature review will introduce the important components of functional reasoning that inform this study. These include ways in which students think about data sets including variational and correspondence reasoning.

Variational and Correspondence Reasoning

Variational reasoning occurs when students notice the change in one or more quantities. A review of variational reasoning begins with the work of Jere Confrey. Confrey and her colleagues looked at covariational reasoning as it relates to functional thinking and defined both the correspondence approach and covariational approach to reasoning about functions (Confrey & Smith, 1994). The correspondence approach relies on finding a rule that relates the independent and dependent variables ($y = f(x)$) (Confrey & Smith, 1994). However, this approach can be too abstract and emphasizes stating the rule explicitly (Confrey & Smith, 1995), which is difficult for younger students. The covariation approach entails generating two sequences from a pattern and looks at the juxtaposition of the two (Confrey & Smith, 1995), as in a table of values. When students notice the covariation between two quantities, they are able coordinate the variation of the two variables as they move up and down the table (Confrey & Smith, 1994). In other words, they can describe how one quantity changes in relation to the other. Other researchers have included the concept of single variation, especially when working with younger students. Warren (2005a), for example, studied young students' abilities to generalize rules for growing patterns. She found that while students have the ability to reason about functions, that single variation was cognitively easier for younger students.

Most of the current studies that exist on covariation cover topics in advanced mathematics and statistics. Many of these studies have focused on the benefits of covariational reasoning on students' development of the concept of functions and other related algebraic topics (Confrey & Smith, 1994, 1995; Saldanha & Thompson, 1998; Warren, 2005b). Others have looked at how covariational reasoning helps student develop ideas related to calculus topics (Carlson et al., 2002; Carlson, Larsen, & Jacobs, 2001; Oehrtman et al., 2008) and statistics (Zieffler & Garfield, 2009).

Few studies focus on how younger students come to understand covariation. Irwin (1996), in a teaching experiment, interviewed 107 children, age 4 to 7, about their understanding of covariance in part-whole relationships. In her experiment, she looked at students' notions of covariance and compensation. The first type of covariance tasks investigated students' understanding of part-whole relationships by looking at the effect on the whole when a quantity was added to one part, while another part was kept the same. The second type of covariation required the children to understand that effect on the whole, when a quantity was taken from one part while keeping another part the same. These researchers were interested to see whether students recognized the change in the whole part in relation to the parts. Compensation, on the other hand, required students to understand that if one part increased by a certain amount, while the other part decreased by the same amount, then the whole part remained unchanged. The tasks progressed from using concrete examples with manipulatives (e.g. blocks, buttons) and story problems (a trip to the zoo) to a more abstract example (equations). Some of the tasks involved uncounted quantities (piles of blocks, candies), to a counted quantities (buttons, zoo trip and equations). She found that the children's ability to explain the changes in amount increased across age and task difficulty level. It was more difficult to explain the changes in part /whole relationships in counted quantities than uncounted quantities. In addition, covariational tasks were not as difficult as compensation changes. Many of the younger children understood the effects on the whole as changes took place to the parts of uncounted quantities, yet less could explain the differences using the counted tasks. Most of the seven-year-old students demonstrated an understanding of both covariance and compensation. The preceding study is important because it looks at student initial notions of change, which is important in functional reasoning. Irwin found that students as young as four have a beginning understanding of covariation.

Moritz (2004) explored three areas dealing with covariation: translating a verbal statement into a graph, translating a scatter plot into a verbal statement, reading values, and interpolating. The participants in his study were boys and girls in third, fifth, seventh and ninth grade. Students worked several open ended questions and researchers coded the data based on four categories: 0—nonstatistical, 1—single aspect, 2—inadequate covariation and 3—appropriate covariation. The third- and fifth-grade students typically scored within the levels 2-3 range on translating a verbal statement into a graph; however, as the tasks became more complex these students scored in the lower levels. Students in the seventh and ninth grades scored in the higher levels. This finding suggests that students' ability to reason about covariation is age related and students' responses become more sophisticated as they progress through school.

Billings, Tiedt, and Slater (2008), in analyzing students' ability to reason about pictorial growth patterns, found that students typically engage in covariational reasoning then progress to correspondence reasoning. These researchers outlined five processes to analyze pictorial growth. The first three they defined as covariational processes: (a) analyzing change between consecutive figures, (b) use of the previous figure to generate the next and (c) use the pattern to identify what stays the same and what changes. The last two processes were correspondence processes: (d) Index the figure number with the changing aspect of the dependent variable; and (e) extend the figure to larger numbers of *n*. Their study agrees with Moritz (2004) that students' use of covariation and correspondence reasoning is developmental.

More studies focus on covariation as a cognitive stepping-stone in the development of functional reasoning. Blanton and Kaput (2004) found that kindergarteners could describe an additive relationship between input and output values and that by third grade students could symbolize a relationship as a functional correspondence. They also suggest that the current curriculum overemphasizes patterns finding with single variables, which may impede students' functional reasoning in later years. As a result, they suggest the K-5 curriculum should focus more on how quantities vary simultaneously. In a similar study, Warren (2005b) examined teachers' actions that enabled young children to visualize and describe growing patterns. She found that single variational thinking was much easier cognitively and noted that one reason for this is that it has become so entrenched in their earlier experiences that students naturally revert to this type of thinking.

We know from the literature how important an early introduction to algebra is for students' development of algebraic reasoning. It is also evident that in order for students to succeed in high school algebra they must first understand functions. Of critical importance to the notion of function is the concept of how the independent and dependent variables are related. When developing these notions, students use a variety of reasoning strategies, single variation, covariation, and correspondence. However, the literature is not arguing for students to experience symbolic algebra common in high school mathematics. Instead, it is suggesting that when introduced in meaningful ways, and by using appropriate tasks, young children are able to reason algebraically.

The literature also suggests that there are developmental levels in building students functional reasoning. Students often begin at an early age to look for patterns in sequences and extending growth patterns. From this, they move forward to describe changes in sequences or using variational reasoning. With the introduction of two related data sets, students begin to develop covariational reasoning. Finally, when students begin to look at the relationships in the ordered pairs they develop their correspondence reasoning.

The literature has suggested that students' reasoning about data is developmental. In addition, it is appropriate to introduce these concepts early in the elementary curriculum. However, the literature previously reviewed has not included ways to integrate these concepts into the existing curriculum. The next section will look at the aspects of tasks that develop students' reasoning in early elementary school.

Tasks

Academic Tasks

In their book *Designing and Using Mathematical Tasks*, John Mason and Sue Johnston-Wilder (2006) looked at the importance of mathematical tasks to student learning. According to these authors, the "heart of teaching lies in the interaction with the learner, with the aim that fruitful learning will take place" (p. 13). While these interactions can take many forms, mathematical learning begins with good task design. Designing good tasks includes choosing a topic, designing and implementing the task with a specific purpose in mind, and taking into account the abilities and motivations of the students.

Doyle (1983) acknowledged that tasks influence learning by focusing student attention on specific content and directing ways to process the information. He categorized academic tasks in terms of the cognitive operations that are involved in accomplishing the task. These categories included memory tasks (students simply reproduce information), procedural tasks (students use a given formula or algorithm), comprehension tasks (students apply prior knowledge to new situations) and opinion tasks (students state a preference for something).

Stein, Grover, and Henningsen (1996), using Doyle's categories, developed them in the area of mathematics. They defined mathematical tasks as activities whose purpose is to focus students' attention on a particular mathematical idea. They suggested instructional tasks pass through three phases: first as curricular materials, second as implemented by the teacher and third as implemented by the students. Each of the phases can have an effect on student learning outcomes. In addition, there are specific factors that influence how the task progresses to the next stage. For example, once the task is presented in the curricular materials, the goals of the teacher, teacher content knowledge, and teacher knowledge of the students all influence how the teacher sets up the materials. Once the materials have been set up, the classroom norms, task conditions, and teacher/student habits and dispositions affect how the student implements the task. All of these factors work together to influence students' learning.

Mathematical tasks fall into two broad categories, tasks that require procedural thinking and those that require conceptual thinking. The use of algorithms or formulas is indicative of tasks that require procedural thinking. There is little unpredictability in these tasks as the algorithms are very reliable in producing answers when there are no computational errors. Comprehension tasks, on the other hand, are accomplished by knowing why a procedure works, when to use it and how it relates to new situations. Procedural task requires students follow a set of steps while comprehension tasks require one to know why the procedure works (Doyle, 1983).

A complete understanding of mathematics requires students to do what mathematicians do, that is to solve problems, look for patterns, make conjectures, examine constraints, infer from data, explain, and justify (Stein et al., 1996). Therefore, task design, according to Francisco and Maher (2005), is crucial to developing students' mathematical reasoning. Francisco and Maher found that teachers often scaffold difficult mathematical problems into smaller, easier parts for students. For students to develop reasoning, they must have the ability to synthesize the parts into a cohesive whole. These researchers argue that this does not always occur and suggests that a better approach would be to allow the students to discover the complexity of the task and work through those difficulties for themselves.

This section gives the reader a beginning understanding of the importance of tasks in instruction. However, I do not want to leave the reader with the impression that functional reasoning depends solely on good tasks. As McClain and Cobb (2001) note, "it was the activities as they were realized in interaction in the classroom that supported students’ mathematical development” (p. 244). While there are multiple factors in the development of functional reasoning, I have chosen to highlight, in this dissertation, the effect of good tasks. In the next section, I will discuss the related literature on how tasks can promote students functional reasoning.

Tasks That Promotes Functional Reasoning

In this section, I take an in depth look at what the literature suggests to be appropriate tasks to use to help students develop their functional reasoning. In addition, I synthesize the literature to build a conceptual framework on the specific characteristics inherent in the tasks that supported and promoted students’ functional reasoning.

Tasks That Promotes Reasoning about Data

The literature suggests that the process of generating and reasoning about data promotes students’ functional reasoning. One way to integrate these activities into the existing curriculum is by varying arithmetic tasks (Blanton & Kaput, 2011). These researchers worked with teachers to take single numerical answer arithmetic problems and vary the parameters to allow for pattern finding and generalization. For example, they posed the following problem: “How many telephone calls could be made among 5 friends if each person spoke with each friend exactly once on the telephone?” (p. 17). In order to vary the parameter to change this task to a functional reasoning task, the teachers asked a series of questions changing the number of friends. For example, student might begin with five friends, but then consider four friends, then three friends. They could then look for patterns in the data and ask student to find a rule that would find the number of telephone calls for 100 friends or any number of friends.

Input/Output tables. Input/Output tables assist students in organizing data. These tables consist of two columns, one for the input values, and one for the output values. The table shows numbers in each row that have a correspondence with each other. The literature suggests that these types of tables are particularly effective in focusing student thinking about the relationships between corresponding values, generalizing rules, and promoting functional reasoning (Carraher & Earnest, 2003; Martinez & Brizuela, 2006; Smith, 2008; Warren et al., 2006).

Blanton (2008) uses input/output tables extensively in her research. According to her work, these types of representations help students organize their data in a way that preserves the relationship between the data and allow them to begin looking for patterns. However, she found that students typically use the data in the table to find a recursive rule by looking down the column of outputs. A remedy for this is teacher encouragement to look across the table for a rule of correspondence between the two quantities.

Most of the time input data is in numerical order. However, Warren et al. (2006), in a longitudinal study of fourth-grade students, found that by placing corresponding values in random order in the input/output table it forced students to focus on how the input relates to the output, rather than changes in the outputs. This encourages correspondence reasoning and often leads to students' generalization of explicit rules.

Generalization Tasks

When students generalize arithmetic, they make a general statement that covers many instances (Kaput et al., 2008). These generalizations can be express using words or symbols (Soares, Blanton, & Kaput, 2006). The studies that exist on generalization agree that young students have the cognitive ability to generalize their understandings in sophisticated ways (Bastable & Schifter, 2008; Kaput & Blanton, 2001; Lampert, 2001; Maher & Martino, 1996; Warren, 2005a) and come to view variables as dynamic quantities. It is through these types of generalizations students find strong connections between numbers and algebra that in turn leads to a deeper understanding of variables and symbols. Early work with children found that they are naturally curious about generalizing arithmetic facts and that instead of using symbols they may use diagrams or language to conceptualize the actions of the operations (Bastable & Schifter, 2008).

Lannin (2005) classified generalization into two categories: non-explicit and explicit. Non-explicit strategies involve counting or recursive strategies. When using a counting strategy, students draw a model or counts until they achieve the desired result. A recursive strategy consists of recognizing the change in the dependent variable (output value) and uses the value of one term to find the value at the next term (Lannin et al., 2006). When students use recursive strategies, they are not able to find any output value directly; they find subsequent values by first knowing the output value before it. Recursive strategies are not efficient strategies if the students are trying to find the output value for a very large input value. Explicit strategies on the other hand, are efficient. Lannin (2005) classified these types of strategies into three categories: whole-object, guess-and-check and contextual. When using the whole-object strategy student use multiplication to build larger units using a portion as a unit. The guess-and-check strategy involves guessing at the rule without regard as to why the rule works. The contextual strategy allows for direct calculations of any output given its input. Lannin et al. (2006)

modified these types of classification into two groups, recursive and explicit. This study will use these modified groups in looking at generalizations.

Bezuszka and Kenney (2008) also defined recursion as the process of recording step-by-step sequential change. They argue that recursive thinking is vital to the study of algebra. It begins informally when students study patterns and can help students learn to think inductively throughout mathematics. Despite this, generalizing sequences (rules) using recursion has its limitations. First, it can be difficult to find a rule when the sequence is out of order and second, finding output values for large numbers can be problematic. If the input number is significantly large, students cannot find a way to proceed, which impedes learning.

Student development of explicit rules is critical in the study of functions. Nevertheless, it can be difficult for students to develop explicit rules. Bezuszka and Kenney (2008) suggests that student can begin with a recursive rule as a starting point then use another strategy to find the explicit rule. Even though they outline specific strategies, the strategies they discuss are beyond the scope of the literature review because they are not appropriate to early algebra learners.

Pattern-finding Tasks

Current research points to pattern finding as a means to enhance students' abilities to reason algebraically (Bishop, Otto, & Lubinski, 2001; Kaput, 2008; NCTM, 2000; Richardson & McGalliard, 2010; Stacey, 1989; Store, Berenson, & Carter, 2010; Stump, 2011). In fact, these tasks are so important in developing skills for understanding mathematics that the NCTM advocates using them from pre-k through high school

(NCTM, 2000). Several studies supports this by showing young children successfully generalizing and justifying rules based on patterns (Bishop et al., 2001; Lannin, 2005; Stacey, 1989; Zazkis & Liljedahl, 2002) and working with unknown quantities (English & Warren, 1998).

The types of patterning tasks used and the integration of those tasks are important in the development of functional reasoning (Blanton & Kaput, 2004; Warren, 2005b). Ferrini-Mundy, Lappan, and Phillips (2002) looked at the growth of algebraic thinking using pattern tasks with students in kindergarten through sixth grade. These researchers used the same task with each age group but changed the nature of the questions. They posed the following task (see Figure 2). The following figure represents three versions of a square swimming pool. The black tiles represent the water and the white tiles represent the border around the pool.

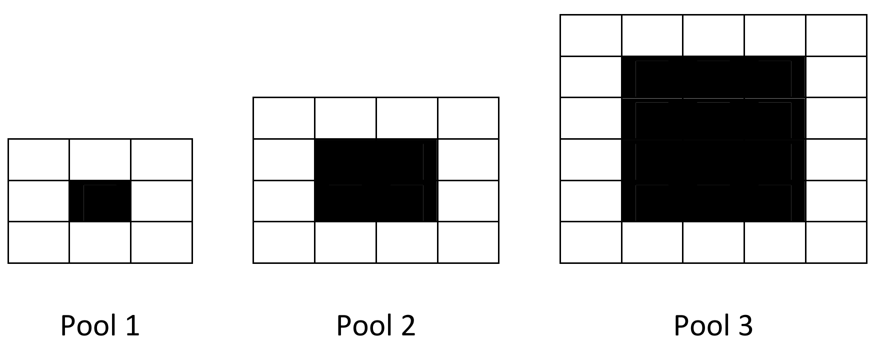


Figure 2. Pool Problem (Ferrini-Mundy et al., 2002)

For the K-2 students, the task consisted of sorting and counting the tiles, building the figures and describing any patterns they found. By grades 3-4, the questions encouraged students to use input/output tables to predict how many black or white tiles there would be in a larger figure. The tasks also asked the students if they could build the figure with *x* number of tiles. In grades 5-6, the researchers were emphasizing functions more in their questions. The tasks asked students to extend the figures and make a table of values for the first six pools. These students also explored the idea of variables, graphing and finding rules for larger inputs such as 25 or 100. Using the same task across grade levels, these researchers found that algebra emerges as a way to generalize and represent mathematical ideas and that these ideas can be developed from an early age. In addition, they proved that a single task, modified appropriately, could encourage reasoning among students of various age groups and grade levels.

Papic and Mulligan (2007) classified pattern tasks into three categories: repeating patterns, spatial structure patterns and growing patterns. With repeating patterns, students must identify the elements that repeat and continue the pattern. Spatial structure refers to mental organizations of objects or group of objects. These types of patterns often have a geometric component such as dots, blocks, grids, etc. Growing patterns increase or decrease systematically. These researchers conducted an intervention study on preschool students and found that young children can develop complex patterning skills prior to formal schooling. Their study encouraged children to see the structure in repeating patterns and to represent the pattern in different spatial forms. They also comment that pattern development promotes other mathematical processes such as transformational skills and multiplicative reasoning. Several other researchers have looked at how these specific types of patterns promote reasoning.

Stacey (1989) found that students could have difficulties finding rules for growing patterns. She asked students 8-13 years old to generalize a rule for a linear pattern and to find the output for a near input (e.g. 20) and for a far input (e.g. 100). She concluded that when the patterns were linear, students typically relied on finding the constant difference in the outputs. In other words, the students used recursive thinking to find the outputs. Moss and McNab (2011), in contrast, found that second grade students were very capable of finding two operation function rules without using recursive reasoning for both geometric and number patterns. Moss, Beatty, McNab, and Eisenband (2005), in a teaching experiment that focused on students' abilities to use patterns to generalize function rules, found students perceived relationships in geometric patterns but could move beyond recursive thinking in order to see the relationships between the input and output variables. In addition, these students often demonstrated the meaning of variables.

Similarly, Warren (2005b) argues that repeating patterns are cognitively easier for students than growing patterns; but she acknowledges that this may be due to overemphasis on these types of pattern tasks in the curriculum. Warren and Cooper (2008a) acknowledge that students spend a lot of time investigating repeating patterns but have little experience with geometric patterns. Papic and Mulligan (2007) disagreed with this finding and asserted that all patterns are valuable in student learning if teachers will attend to the underlying structure.

Smith (2003) claims that students do not see the relationship between patterns, functions, and algebra because they are often taught as separate subjects. He views patterns in two ways: stasis and change. Stasis involves seeing the structure of the pattern as it is while change involves viewing the pattern in terms of how it is repeated or extended. In addition, as we focus on patterns we often go back and forth between these two views. Smith formulated a framework for integrating patterns with functions and algebra. He suggests placing an emphasis on the relationship between the stasis and change by asking questions such as “How can you describe this pattern?” and “How can the pattern be repeated or extended?” (p. 143). Smith concludes that integrating patterns with early algebra and functional reasoning helps to lay the groundwork for higher levels of mathematics and that these ideas should be conceptually linked together from early elementary school through high school.

While most studies point to the effectiveness of pattern finding tasks, some researchers offer some cautions in the way they are used. According to Friel and Markworth (2009), geometric patterns can be a way of looking at patterns initially but to enhance functional reasoning students need exposure to patterns that use a variety of numbers and that are continuous. In addition, Billings et al. (2008) found that second graders had an easier time understanding pictorial growth patterns when objects were concrete, such as ice cream cones, rather than abstract objects, such as dots. Papic and Mulligan (2007) add that teachers should be careful when using repeating patterns, because they often ignore or misinterpret the underlying structure.

Lannin (2005) adds that patterning tasks should be of similar structure (isomorphic) to encourage students’ reflections on their justifications. In a recent study, Richardson, Berenson, and Staley (2009) asked pre-service teachers to find the perimeter of a pattern block train made up of one block, two blocks, three blocks, and write a rule for *n*-blocks. Their findings agreed with Lannin (2005), when the structure was the same throughout the problems, students could more easily generalize a pattern and justify their reasoning.

Warren and Cooper (2008a) addressed several barriers to students' success in pattern finding. First, students have imprecise language to describe their generalization rules. Students can often describe their patterns verbally but have a difficult time writing them down. Secondly, by overly focusing on single variational reasoning, students often fail to notice any missing steps within the pattern.

Function Machines

Introducing the concepts of functions via function machines is a strong cognitive representation, according to McGowen, DeMarois, and Tall (2000) and Tall, McGowen, and DeMarois (2000). They contend that since a vast portion of the brain is dedicated to vision it is natural for us to focus on objects and their properties. In this article, they define a cognitive root as the initial unit of core knowledge upon which students build more sophisticated understandings. Using a graphic to represent the idea that a machine acts on an input value to generate an output value serves as a strong cognitive root. These researchers also argue the introduction of the concept of domain and range is more natural since the students can visual number as physically going into the machine (inputs) and physically coming out of the machine (outputs). Students also seem to be able to develop the idea of a specific domain mapping to a specific range. For example, given the rule $y = 2x + 3$ for a function machine. Each time 2 goes into the machine a 7 comes out. Therefore, 2 maps to 7. Another benefit of using a function machine is that is appeals to a wide range of student ability levels. In a study of students in a developmental algebra course, Davis and McGowen (2002) found that students felt that a function machine assisted them in making sense of notation, helped to organize their thinking, and allowed them to produce equations.

Task Structures

The literature revealed that design of the task is important and can contribute to student gains on assessments. Stein and Lane (1996) found that student gains on assessments were greater when tasks included multiple solution strategies, multiple representations, and explanations. Student gains were less when tasks included single solution strategies, single representations and little to no mathematical communication.

The overall structure of a task makes a difference in student success. Task structures refer to the elements that make up the tasks. Prior research suggests that certain task structures can help students develop their reasoning skills. These include tasks that are open-ended and relevant, allow students to plan, organize, use conjecture, and make justifications. Further, the tasks should not rely on memorized algorithms (Femiano, 2003; Lampert, 2001; Lester & Kehle, 2003; Maher & Speiser, 1997; Schoenfeld, 1992). Additionally, students tend to be more successful when tasks are isomorphic (Greer & Harel, 1998) or connected (Richardson, Carter, & Berenson, 2010) and when the tasks are sequenced appropriately (Blanton, 2008).

Open-ended Tasks

There seems to be some disagreement on what constitutes an open-ended task. Zaslavsky’s (1995) view of an open-ended task is one that has multiple correct answers.

However, Cai, Jakabcsin, and Lane published a study in 1996 that described an open-ended task as those which asked students to justify their thinking, explain their answer, or to simply to show their work. For the purpose of this dissertation, I use the view espoused by Sullivan, Warren, and White (2000) that an open-ended task is one in which there are more than one possible pathway to a solution and there may be multiple responses, approaches, or lines of reasoning. In addition, when these open-ended tasks are content specific they engage the learner in a more meaningful way. Richardson et al. (2010) investigated the effect of open-ended, connected tasks with fifth-grade students. They concluded that the open-ended structure of their tasks promoted student reasoning by encouraging students to make predictions, conjecture, and justify their results.

Visual Images

Visual images take on many forms. They are pictures, diagrams, charts, graphs, symbols, and physical models that students typically use when solving problems. By creating these images, students learn to organize and communicate their ideas (Blanton, 2008; NCTM, 2000). Physical models, in particular, allow students to make sense of the problems they are solving by focusing their attention to the structure of the model (Blanton & Kaput, 2004; Clement, 2004; NCTM, 2000). In a teaching experiment with pre-service teachers, Richardson et al. (2009) found that while a tabular representation helped students generalize rules, it was the geometric model that assisted more in justifying their rules. Lannin (2005) in a study of fifth-grade students' abilities to generalize rules found that when students develop poor visual images, they often use guess and check as a solution strategy. This, however, often leads to incorrect generalizations.

Graphs. Learning to graph is one of the critical moments in learning mathematics, according to Leinhardt, Zaslavsky, and Stein (1990). In terms of functional reasoning, graphs allow students to explore relationships between quantities, visualize these relationships between quantities, and reason about the continuity of a function (Blanton, 2008; Carraher et al., 2008; Star & Rittle-Johnson, 2009). Since there is such an emphasis on function graphs in later mathematics, it is important that students begin to construct and use them in elementary schools (Blanton, 2008).

Isomorphism

According to Greer and Harel (1998), isomorphic tasks share related structures and relationships. Isomorphic tasks enhance reasoning by allowing students to map corresponding relational structures from one task to another. Uptegrove and Maher (2005) found that young children have the ability to recognize related structures in tasks and use their knowledge in new situations. Richardson et al. (2009), in a study of pre-service teachers' abilities to generalize and justify rules, found that isomorphism was one aspect of task design that leads to the subjects' success in the tasks. Similarly, Richardson et al. (2010) found that students have greater success with tasks that are connected which means they have a shared relationship, context, property or operation.

Sequencing

Sequencing is an important aspect to curriculum design. Blanton (2008) suggested beginning with simple functional relationships. According to Merrienboer, Kirschner, and

Kester (2003), when using tasks that are highly complex, it creates a cognitive overload in students. Instead of achieving desired results, students simply shut down and fail to make any progress. Therefore, tasks should build in cognitive difficulty. Mason and Johnston-Wilder (2006) warn students may still see tasks as being disjointed even when they are sequenced appropriately. Therefore, teachers may need to make students aware of the connections within the tasks.

Function Rules

When generalizing function rules several studies point out that the structure of the function is an important consideration. English and Warren (1998) conducted a study looking at students' types of generalizations and found that function rules in the form of $x+c$ or ax were less difficult for students than those in the form of $ax+c$ or $ax-c$. Similarly, Tanish (2011) conducted a study of fifth-grade students as they worked with function machine tasks. He found that the students were more successful with function rules in the form of $y = 2x + a$ than with functions in the form $y = 2x - a$.

Lannin et al. (2006) also noted that the structure of the function rule was important in students' generalizations. In their study, they focused specifically on functions that were linear and non-linear and on functions that were increasing or decreasing. They found that students often generalized a rule when the function was linear. However, this study only focused on the work of two fifth-grade students. This may not have been enough data to determine what effect the type of function rule had on students' reasoning.

Similarly, Blanton (2008) suggests that tasks should begin with function rules that are simple enough for students to identify explicit rules by looking across the input/output table. Tasks should build in complexity by including function rules that combine operations in ways that are more complex. She argues that linear relationships are cognitively easier for students due to the constant rate of change; in particular the linear function rule in the form of $y = ax$. Quadratic rules are more challenging relationships for students to find due to the lack of a constant change. According to Blanton, when the function rules are too complex student may rely on guessing what the rule is. Appropriate scaffolding by the teacher can avoid this. Blanton also used tasks that generalized an exponential rule. Overall, she found that students were capable of writing explicit rules for each of these types of functions. Blanton does suggest that the tasks should originate from concrete situations and have simple function values.

Justifications

While generalization speaks to a students' ability to make conjectures about a wider context, justification attempts to answer the question "how do you know?" (Franke et al., 2008). Carpenter et al. (2003) posit that students cannot truly understand mathematics without engaging in some sort of justification. Students justify their thinking in a variety of ways including building cases, using contradictions, and direct arguments. In addition, these justifications are processed through conversations with peers so students can reflect on and revise their arguments if needed (Maher & Speiser, 1997; Mueller, Maher, & Yankelewitz, 2009). Schifter et al. (2008) add that justification in elementary-aged children should consist of more visual representations such as drawings, diagrams, and the use of manipulatives. While the literature does not suggest that all students can formalize a proof at this age, researchers do agree that young children can learn to give convincing arguments that their reasoning is sound (Carpenter et al., 2003; Lampert, 2001; Maher & Martino, 1996). However, teachers have to be diligent in getting students to justify all components of their rules in the context of the problem. Lannin (2005) found that students would often attempt to justify their results through examples and not by linking their rules to the context of the problem.

Implementing Tasks—Classroom Structures

Classroom Discourse

When students' ideas are emphasized over that of the teachers, students take ownership of their reasoning which leads to more successful problem solving. Collaboration and discourse helps students develop their reasoning skills (Francisco & Maher, 2005). The group members rely on each other to help generate and challenge each other's ideas. Using small group collaboration and whole group discussion, students share their ideas, clarify their own understanding and make their reasoning visible (Ball & Bass, 2003; NCTM, 2000). Together they build an understanding of the task.

Effective discourse helps to make students' mathematical reasoning visible and open for discussion. Teachers can promote discourse by engaging students in communicating about their mathematical ideas, making decisions about what directions these communications need to take to deepen understanding; questioning students to solidify or challenge these understanding and monitoring to make sure these communications are leading to mathematical reasoning (Walshaw & Anthony, 2008).

Teachers must also be able to recognize such opportunities in their curriculum and have the content knowledge to foster good in-depth classroom discussions and provide a "safe" environment so that students feel comfortable taking part in these discussions (Gutierrez, Mavrikis, & Pearce, 2008; Kaput & Blanton, 2001). Warren and Cooper (2008b) implied that there are specific teacher actions that support student thinking, such as the use of concrete materials, explicit questioning, using patterns where the relationship between the pattern and position are explicit, and generalizing from a small position number to a large position number.

The literature suggested that certain task characteristics promoted or supported functional reasoning. In synthesizing this information, I generated the following list of task characteristics:

- Open ended problems
- Allows students to generate data
- Encourages students to look for patterns within single or multiple sets of data or between corresponding values
- Uses and interpret graphs
- Uses input/output tables
- Makes generalizations (either recursive or explicit)
- Asks students to justify their reasoning
- Use of visual images
- Isomorphism in task designs

Summary

We know from the literature how important an early introduction to algebra is for students' development of algebraic reasoning. It is also evident that in order for students to succeed in algebra, they must first understand functions. Tasks are of critical importance in supporting students' functional reasoning. Teachers need to incorporate tasks that allow students to reason about data, look for patterns, generalize rules, and justify their reasoning. In addition, sequencing is important in task selection and classroom instruction must support students reasoning. Since functional reasoning is the backbone of advanced mathematics, researchers agree that in order to develop students' knowledge of functions, instruction should begin early and last over the course of several years.

The literature review outlined several issues regarding the development of functional reasoning. Failure to develop this understanding can limit student success in future mathematics. Several approaches can help student develop these skills but the tasks that teachers use is of critical importance. There seems to be a lack of research concerning the tasks and the effects of the tasks on developing students reasoning.

CHAPTER III

RESEARCH DESIGN AND METHODOLOGY

Introduction

The literature review gave insight in to how students develop their functional reasoning and the characteristics of appropriate tasks that can help to develop these skills. In Chapter II, I discussed in depth how each of these characteristics play a role in functional reasoning. These characteristics will serve as a conceptual framework with which I will frame my study. In this chapter, I will describe the study design and methodology I used to investigate the On Track tasks. First, I will discuss the overall design of the study as well as give a rationale for this type of design. I will then describe the schools and participants chosen for this study, methods of data collection and types of data collected. This chapter will also include an overall description of the tasks used. Lastly, I will discuss how I approached the analysis of the data and discuss possible validity threats, ethical concerns, and limitations of the study.

Study Design

This study is taken from a larger study on students' algebraic reasoning in an after school enrichment program entitled On Track Learn Math. I chose a mixed method, embedded design to investigate this phenomenon. A mixed method design incorporates both qualitative and quantitative approaches within a single study, in order to capitalize on the strengths of each (Johnson & Onwuegbuzie, 2004). In my study, I am using an embedded design, which according to Creswell and Plano-Clark (2007), is appropriate when researchers need either quantitative or qualitative data in order to answer research questions within a larger study using the other method. In my study, I have embedded quantitative data within a qualitative grounded theory study.

Sampling Procedures

There are two types of participants in this study: schools, and students within those schools. The school participants were chosen first using convenience sampling. According to Creswell (2005), in convenience sampling, participants are available and willing to participate in the study. Two local schools opted to participate along with four schools located in a nearby county. Within each of the schools, convenience sampling chose the student participants. For the first two schools, the participants were students who attended the afterschool childcare program. For the additional four schools, any interested students were given the option to participate. All schools limited participation to no more than fifteen students per school.

Participants

Six schools participated in the On Track program. Four of the schools are located in a rural area that has a population of less than 100,000 located in a southeastern state. Two of the schools are located in a large urban area whose population is approximately 475,000. The participants were students in grades three through five who self-selected to participate in the program. The participants from two of the schools attended the afternoon childcare program while their teachers invited the participants in the other four schools. Participation was limited to 15 students per school and student selection was on a first come/first serve basis. Total participation was less than 180 students ($n < 180$). The participants ranged in age from seven to ten years old and come from a variety of socio-economic backgrounds, ethnicities, and gender (NCDPI, 2011).

Table 1 shows basic demographics for each participating school. This is the latest release of school data and comes from the 2009-2010 school year. This table outlines the population of the school, type of school, whether is qualified as a Title 1 school, and the number of annual yearly progress (AYP) targets met in the latest reporting year. AYP is a measure of the yearly progress of different groups of students as outlined in the No Child Left Behind (NCLB) legislation. All public schools are required to collect and make this information public. The rational for including this information in the school demographic is to give a clearer picture of the participating schools. This shows that the schools in this study are concerned about continuing student improvement. While most of the schools met all of their AYP goals, they continue to be concerned about the success of their students. Also the data is included to show that the schools chosen to participate in the program are regular public elementary schools.

Table 1

On Track School Demographics

School	Location	Population	Type	AYP Targets Met
1	Urban	523	Pre K – 5 Title 1 School	21 of 21
2	Urban	713	Pre K – 5	21 of 21
3	Rural	508	Pre K – 5	13 of 13
4	Rural	220	Pre K – 5 Title 1 School	13 of 13
5	Rural	494	Pre K – 5	13 of 13
6	Rural	495	Pre K – 5 Title 1 School	20 of 21

On Track Learn Math

On Track Learn Math was an afterschool enrichment program for students in grades three to five. The purpose of the program was to help students develop mathematical reasoning skills. Students attended On Track two days per week for a total of five weeks. There were five, 5-week sessions. Data for this study came from the first two sessions. The curriculum and materials used were the same for each of the six schools. Each daily session began with thirty minutes of an outdoor sports activity, to help students expend some pent up energy from the day. Afterwards, the participants spent forty-five minutes working on either a functional reasoning or combinatorial reasoning task.

Classroom teachers volunteered to participate in the On Track program. The research team recruited one teacher from each participating school to teach the afterschool enrichment program. Prior to beginning On Track, they attended a series of four professional development sessions to work each of the tasks and to discuss ways to encourage and support students' functional reasoning.

Tasks

Each daily session of On Track consisted of students working on a function machine tasks, followed by either a functional reasoning or combinatorial task. My study will focus solely on the functional reasoning tasks. Table 2 outlines each task by session, day, and type of reasoning used. Appendix A contains each of the functional reasoning tasks analyzed.

Table 2

On Track Tasks

Number	Task	Session, Day	Type	Rule
		On Track Session One		
1	How many different towers can you build that are 3 blocks tall	1.1	Combinatorial Reasoning	N/A
2	How many different towers can you build that are 4 blocks tall	1.2	Combinatorial Reasoning	N/A
3	Compare the number of towers that are 2, 3, 4 blocks tall; find a rule for 100 blocks tall	1.3	Functional Reasoning	$f(n) = 2^n$
4	How many triangles will fit around 100 connected squares	1.4	Function Reasoning	$f(n) = n + 2$
5	Square Train Tables	1.5	Function Reasoning	$f(n) = 2n + 2$
6	Generalize a perimeter	1.6	Functional Reasoning	N/A
7	Square series and Perimeter patterns	1.7	Functional Reasoning	$f(n) = 4n$
8	One Dice Game	1.8	Combinatorial Reasoning	N/A
9	Two Dice Game	1.9	Combinatorial Reasoning	N/A
10	Skeleton Tower Task	1.10	Functional Reasoning	$f(n) = n(2n - 1)$
		On Track Session Two		
11	Square Numbers	2.1	Functional Reasoning	$f(n) = n^2$
12	Rectangular Numbers	2.2	Functional Reasoning	$f(n) = n(n + 1)$
13	Triangle Train Tables	2.3	Functional Reasoning	$f(n) = n + 2$
14	Hexagon Train Tables	2.4	Functional Reasoning	$f(n) = 4n + 2$
15	Pentagon Train Tables	2.5	Functional Reasoning	$f(n) = 3n + 2$
16	Triangle Numbers	2.6	Functional Reasoning	$f(n) = \frac{1}{2}n(n + 1)$
17	Handshake Problem	2.7	Functional Reasoning	$f(n) = \frac{1}{2}n(n - 1)$
18	Cuisenaire Rods Task	2.8	Functional Reasoning	N/A
19	Addend Pairs Task	2.9	Functional Reasoning	$f(n) = n + 1$
20	Pizza Task	2.10	Combinatorial Reasoning	N/A

The On Track tasks asked students to generate a set of data and to look for patterns in their data. The students were then asked to find a rule that would generate the sequence of output values and eventually find the output value for a relatively large input value (ex. $n = 100$). The functional reasoning tasks rules consisted of linear, quadratic, and exponential functions. In addition, the rules were one and two operations. In addition, each task contained some type of visual image to assist students reasoning. Students could use manipulatives supplied by the teacher to build a model that extended the pattern for several stages or many chose to draw the next stages.

Data Collection

Each daily session of On Track asked students to work on various tasks. The research team, including myself, videotaped students as they worked in both small group and large group settings while they worked on these tasks. In addition, researchers interviewed individual students to gauge their thinking as they worked. While there were multiple types of data collected, the primary sources of date were the tasks before students had a chance to work the problems and the student work samples afterwards. The videotape data served as a secondary data and I used this data to clarify any misunderstanding from the work samples.

I collected data for this study during the first two sessions of On Track for 20 afterschool enrichment sessions while students worked on their tasks. In addition, the research team collected work samples from each individual student during each day of attendance.

Confidentiality

Student confidentiality and anonymity is of the upmost importance. We kept the data in a secure locked location. In addition, I maintained students’ anonymity by assigning each student a unique identification number during the data analysis and used pseudo names when writing about them individually. In addition, I did not identify any of the teacher, schools, or locations of those skills when writing this manuscript.

Analysis Part 1

The analysis of the data occurred in two phases based on the research questions. The purpose of phase 1 was to answer the first two research questions.

Research Question 1: What are the functional reasoning characteristics of the On Track mathematical tasks?

To answer question 1, I began analyzing each of the tasks based on the conceptual framework. The purpose was to identify the functional reasoning characteristcs inherant in the task. I organized this information in a table by noting with an X which characteristics were present. I established coding reliability by asking peers to code randomly selected tasks. The reliability was 100%. In the following section, I discuss the characteristics I found.

Open-ended Task

Open-ended tasks include those who have no clear set way of solving them. In looking at the On Track tasks, it was important to notice whether the tasks asked students questions that led to multiple solution pathways or solution strategies. Overall, the tasks needed to allow students the opportunity to explore the problem and choose their own methods to solve the problems.

Generates Data

In order to develop functional reasoning students should focus on the relationship in data (Blanton, 2008). Therefore, data is an important characteristic of functional reasoning tasks. Moreover, when students generate their own sets of data they are able to focus on the structure of the pattern and observe the relationship among the changing quantities. For example, in the Square Tables Task, students began with one square table and counted the number of students who could sit around it. Then, the students added another table and counted the number of people. By focusing on this structure (joining the tables together), they are able to see that when one adds a table instead of adding four people, one actually take two away from where the tables are joined.

Pattern Finding

In the literature review, Smith (2003) stated that students should begin looking for patterns in data in early elementary school. In addition, finding patterns helps to lay the groundwork for higher levels of mathematics. Patterns finding tasks can be either numeric data or geometric structures.

Use Input/Output Tables

Input/Output tables help students organize their data. The use of these tables served several purposes, not only did they help to organize data they also offered a chance for students to explore the ideas of independent and dependent variables, naming those variables, and using symbolic notation in place of the name (Blanton, 2008).

Generalizations

According to Blanton (2008), functional reasoning is the process about which one generalizes rule that describe patterns in data. Students develop their functional reasoning by looking for and understanding how these rules work. Lannin et al. (2006) suggests these generalizations take two forms, explicit and recursive.

Use and Interpret Graphs

While the literature review identified the use of graphs as promoting functional reasoning, the On Track tasks did not include this characteristic.

Justifications

Students justify their thinking by explaining their results both in writing and in the classroom discussions. The On Track teachers were encouraged to ask students, “How can you convince me . . . ?”

Visual Images

Giving students visual images to work with is important in building their functional reasoning. Visual images can take different forms. They can be any type of representations that aid the student in visualizing the structure of the pattern. They can also be images that the students create.

Isomorphism

Isomorphic task share the same underlying structure.

The purpose of Research Question 1 was to identify the characteristics of each of the On Track tasks based on the conceptual framework. However, just identifying the tasks did not tell me how the students used these components when working on the tasks. The second research question looked specifically at students’ work samples to investigate this further.

Research Question 2: What evidence suggests that the On Track tasks supported the development of students’ functional reasoning among students of varying age and ability levels?

In order to answer this question, I chose three target students and examined their work samples across the session(s) they attended. The intent of this was to determine if the characteristics aided in their development of functional reasoning and if so in what ways. I felt it was important to choose these students based on grade level, sessions attended, attendance, and legibility of their work. The first student, Brandon, was a third-grade student sho attended the first session of On Track. The second student, Jackie, was a fourth-grade student who participarted in the second session of On Track. The last student, Zack, was a fifth-grade student who participated in both sessions.

During this phase qualitative methods were used to analyze in what ways the students made use of the specific characteristics. Also of interest was whether there were any overall changes in the students work as they particpated in the session. I was interested in knowing if the students increased in the amount of work shown, language used, and if students' generalizations became more sophisicated across the session.

The purpose of the first phase of the analysis was to focus on the functional reasoning characteristics that were components of the On Track tasks. Also important in this phase was to see how the students used these characteristics and whether they aided the students in their development of functional reasoning. In the second phase, I wanted to determine if the tasks encouraged students to write generalizations and if so what types of generalizations they wrote.

Analysis Part 2

I began this phase of the analysis by developing a coding scheme for students' types of generalizations. During the literature review, I identified two specific types of generalizations that are important in functional reasoning, recursive, and explicit (Lannin et al., 2006). A recursive rule describes how the output values (dependent variable) are changing. An explicit rule describes the relationship between the input values (independent variable) and output values (dependent variable).

Initial Coding

Lannin et al.'s (2006) definitions of recursive and explicit generalizations formed the basis of the data-coding scheme. When developing the codes I considered Lannin's definitions, classroom observations and an initial analysis of student works. This led to the development of 7 codes: no response, recursive incorrect, recursive transitional, recursive correct, explicit incorrect, explicit transitional, and explicit correct. I used the no response code to identify which students were present on the day of the task but left the generalization question blank or wrote an unintelligible answer. A transitional rule occurred when the student used correct reasoning but left out a specific part of the response such as the name of the input or output. Essentially the generalization is correct but could be more specific. Correct, incorrect, and no response rules, on the other hand, are exactly what their name implies.

After completing the initial coding, I arranged the data in a spreadsheet. I then calculated the percentages of each type of response by taking the number of student responses in each category and dividing by the total number of responses (*n*) for that task. Since each task had a different number of responses, the *n* value of each task was different.

In order to make the coding scheme clearer, I will use the following input/output table (see Figure 3) and give examples of each code.

Input	**Output**
1	7
2	14
3	21

Figure 3. Input/Output Table Example

Table 3 represents the range of possible responses that I saw while analyzing the student generalizations.

Table 3

Example Responses and Codes

Response	**Code**
The output is doubling	Recursive Incorrect
Add 7	Recursive Transitional
The output is increasing by 7	Recursive Correct
Add 7 to each input	Explicit Incorrect
Multiply by 7	Explicit Transitional
Each input is multiplied by 7	Explicit Correct

I established inter-rater reliability for this coding scheme by choosing 10 students' work samples at random and asked two peers to rate each sample. We achieved 90% reliability on the coding scheme.

Research Question 3: Which tasks appeared to promote students functional reasoning?

After looking at the percentages of students' generalization across each task, no patterns emerged, so I decided to collapse the categories of codes. The new codes were no response, incorrect, recursive, and explicit. The no response category remained the same. I grouped the categories recursive incorrect and explicit incorrect into one category, incorrect. I grouped the categories recursive transitional and recursive correct into one category named recursive. Lastly, I combined the categories explicit transitional and explicit correct into one group named explicit.

After collapsing the codes, some patterns did emerge. To answer research question three, I used bar graphs to compare the percentages of students' responses across each session of On Track. The purpose was to observe any overall trends in the student percentages of writing explicit rules. I was particularly interested in identifying the specific tasks that students wrote either a high or a low percentage of explicit rules. I was also interested in the overall trends in writing recursive rules.

Research Question 4: What effect does grade level have on students' functional reasoning?

In my classroom observations, I observed that the third-grade students had trouble with certain tasks; therefore, I felt it was important to see if grade level affected students' chances of generalizing explicit rules. To disaggregate the data based on grade level, I sorted my main spreadsheet by grade level, and then looked at each grade level across individual sessions.

This analysis was important to determine how each grade level compared to one another on each specific task. The purpose was to compare and contrast results based on grade level and to determine if students wrote higher percentages of explicit rules per task. After comparing student results by grade level, it was difficult to tell whether grade level significantly contributed to students' ability to write explicit rules. Therefore, I identified a need for a more robust analysis.

Using the collapsed data, I converted the categorical data to numerical data by letting 0 = no response, 1 = incorrect, 2 = recursive, and 3 = explicit. The literature supports the argument that recursive generalizations are less sophisticated than explicit generalizations, therefore the numerical codes does order them on a continuum. Next, I grouped individual responses by grade level and performed a single factor ANOVA with post hoc tests. The purpose of this test was to compare the means of each group. The null hypothesis for this test was that there was no difference in the means for each group ($H_0 = \mu_1 = \mu_2 = \mu_3$).

After determining the effect grade level had on students' percentage of explicit generalizations, I wanted to know if the type of function rule had an effect on student reasoning. The literature review revealed that the type of function rule determined students' success on generalizing rules. Linear function rules are easier for young children to generalize (Blanton, 2008; English & Warren, 1998, Lannin et al., 2006; Tanish, 2011). In addition, Blanton found that quadratic and exponential functions were more complex than the linear ones.

At this point in the analysis, it became clear that students' ability to generalize function rules is promoted by type of task and age level of the students. However, it was not clear whether these were the only factors. With Research Question 5, I wanted to discover whether the number of operations of the function had an effect on students' generalizations as well.

Research Question 5: What type of function rules appears to promote students' correspondence reasoning?

In order to answer this question, I began by looking at the percentages of explicit rules for each task and ordered the tasks from highest to lowest. While this suggested that one term rules were easier for students, it was not enough to confirm Lannin et al. (2006) and Blanton's (2008) conclusions. Therefore, I used the original data and calculated the percentages of students responses based on type of function rule. I then compared the results based on the following designations: one-operation linear rules, two-operation linear rules, one-operation quadratic rules, two-operation quadratic rules, and one-operation exponential rules.

Validity

Validity refers to how well a study measures what it is supposed to measure (Creswell & Plano Clark, 2007). Johnson and Turner (2003) add that validity refers to trustworthiness of the research. Minimizing validity threats determines whether the research is of high quality (Onwuegbuzie & Johnson, 2006). I identified the following validity threats in this research.

Researcher Bias

According to Johnson (1999), this type of bias comes from selective observations and data collections. In addition, the personal views of the researcher can affect how data is interpreted. I avoided this validity threat by using all of the student work samples for the tasks I included in the study. Furthermore I noted previously the only tasks I eliminated were those that had such a low response rate, that calculating percentages of student responses would be misleading.

Interpretive Validity

To ensure that I interpreted the results of my study correctly; I began by having a clear conceptual framework, based on peer reviewed research, and thorough research questions. Then I presented my findings to the group of teachers who participated in On Track. Without telling them my interpretations, I asked for their opinions of the findings. They all agreed with my interpretation of the data. In addition, I solicited feedback from the other members of the On Track research team throughout the study.

Construct Validity

Dellinger and Leech (2007) defined construct validation as the process of continually collecting evidence to support the meaning of data. They contend that it is a "continuous process of negotiation of meaning. This is accomplished through argument as dialogue, criticism, and objection" (p. 320). Several issues fall within this category: design quality, legitimation, and interpretive rigor. In order to minimize these types of threats I worked closely with a committee, which gave me feedback on the design of this study and made suggestions for improvement.

Ethics

No harm could possibly come to students or teachers from participating in my research. This research does not seek to look for deficits in students learning or teacher practice. It seeks to establish a link between mathematical tasks and student learning. This research will maintain the anonymity of all participants throughout this study. Additionally, all participants used in this study will be informed of how the data will be used and will be asked to signed consent form.

CHAPTER IV

ANALYSIS AND FINDINGS

Overview

Functions form the basis of higher-level mathematics and model many real world phenomena. Students engage in functional reasoning when they build, describe, and think about the relationships between sets of data. Functional reasoning is a subset of algebraic reasoning because students form generalizations about data (Blanton, 2008). When students form these generalizations, they are showing an understanding of the mathematical formula that produces the data. According to Blanton, the power in these generalizations is that it allows students to find the value of an output for any given input.

The purpose of my study is to investigate the features of the On Track tasks that helped students develop their functional reasoning skills. Previous research (Blanton, 2008; Blanton & Kaput, 2004; Carraher et al., 2008; Lannin, 2005; Martinez & Brizuela, 2006; Warren, 2005a, 2005b; Warren et al., 2006) shows that early elementary students can develop functional reasoning by working on appropriate tasks that support this development. While observing the afterschool On Track sessions, I noticed that some tasks appeared to be more difficult for the students than others. In session one, the students experienced difficulty with the first task. However, for most of the students this was their first experience with tasks of this nature. As the students became more skilled with these types of problems, they improved in their abilities to generalize function rules.

In addition, they become very adept at generating data and using input/output tables to help them find these generalizations. In session two, I expected similar results that overall students would improve in their abilities as the sessions continued. However, this was not the case, as the session progressed there seemed to be some tasks that were particularly difficult. These observations suggest that there were several factors involved in the development of functional reasoning.

I began by analyzing each of the On Track functional reasoning tasks to identify the characteristics the literature suggested helped students develop functional reasoning. I then chose three target students and followed their work across the sessions they attended to show how these characteristics aided in the development of functional reasoning. The purpose of this analysis was to look specifically at the students' growth of understanding as they participated in this program, and to see if the characteristics helped or hindered them in generalizing rules. In part two, I compared all of the On Track student results across sessions, grade levels and tasks to find other characteristics that may have an effect on the student outcome that were not identified in phase 1.

Analysis Part 1

Each daily session of On Track focused on a different reasoning task. The emphasis of these tasks was either functional reasoning or combinatorial reasoning. My study analyzed the functional reasoning tasks. Table 4 gives important information about the On Track tasks. The first column refers to the daily number of the task. For example, 1.3 refers to the task given on the third day of the first session. Task 2.6 refers to the task given on the sixth day of the second session. The second column in the table gives the name of the task, which I will be referring to in the analysis instead of the number of the task. This makes the analysis clearer for the reader. This column also gives a brief description of the pattern the students were looking for. The last column gives the expected explicit generalization of the pattern.

Table 4

On Track Tasks, Patterns and Generalizations

No	Task Name / Pattern	Student Generalizations
	On Track Session One	
1.3	Towers - Compare the number of towers that are 2, 3, 4 blocks tall; find a rule for 100 blocks tall	$b = 2^h$ where b is the number of blocks and h is the height of the tower.
1.4	Triangles and Square - How many Triangles will fit around 100 connected squares?	t = 2s + 2 where t is the number of Triangles and s is the number of squares.
1.5	Square Tables – How many people can sit around a train of 100 tables?	p = 2t + 2 where p is the number of people and t is the number of tables.
1.7	Perimeter Task – What is the Perimeter of the “L” shaped figure that adds one square to the bottom and one to the top?	p = 4n where p is the Perimeter and n is the numbers of square.
	On Track Session Two	
2.1	Square Numbers – What rule will generalize the Square Numbers?	$n = s^2$ where n is the number of dots (Square Number) and s is the stage number.
2.2	Rectangular Numbers – What rule will generalize the rectangular numbers?	n = s (s+1) where n is the number of dots (rectangular number) and s is the stage number.
2.3	Triangle Train Tables – How many people can sit around a train of 100 Triangle tables?	p = t + 2 where p is the number of people and t is the number of tables.
2.4	Hexagon Train Tables- How many people can sit around a train of 100 Hexagon Tables?	p =4t + 2 where p is the number of people and t is the number of tables.
2.5	Pentagon Train Tables -How many people can sit around a train of 100 Triangle tables?	p = 3t + 2 where p is the number of people and t is the number of tables.
2.6	Triangular Numbers – What rule generalizes the Triangular Numbers?	n = ½ s(s + 1) where n is the number of dots (Triangular Number) and s is the stage number.
2.7	Handshake Problem – How many handshakes are there between 100 people?	h = ½ p (p – 1) where h is the number of handshakes and p is the number of people.

Research Question 1

What are the functional reasoning characteristics of the On Track mathematical tasks?

The literature review outlined the nature of functional reasoning tasks and provided a list of nine characteristics. In order to answer Research Question 1, I compared each of the On Track tasks to these known characteristics. In Table 5, I list each of the characteristics and use an X to shows whether the characteristics was present in the task.

Table 5

Functional Reasoning Characteristics Exhibited by the On Track Tasks

	First Session					Second Session							
Characteristic	1.3	1.4	1.5	1.7	1.10	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.9
1. Open Ended	X	X	X	X	X	X	X	X	X	X	X	X	X
2. Generates Data	X	X	X	X	X	X	X	X	X	X	X	X	X
3. Pattern Finding	X	X	X	X	X	X	X	X	X	X	X	X	X
4. Graphs													
5. Input/Output Tables	X	X	X	X	X	X	X	X	X	X	X	X	X
6. Generalizations	X	X	X	X	X	X	X	X	X	X	X	X	X
7. Justification	X	X	X	X		X	X						
8. Visual images	X	X	X	X	X	X	X	X	X	X	X	X	X
9. Isomorphism													

The analysis showed that of the nine characteristics, six were present in all of the On Track tasks. These tasks were open ended, and they asked students to generate data. Students looked for pattern and used input/output tables to organize the data, then they were encouraged to make generalizations to describe these patterns. They also used visual images such as pattern blocks or unifix cubes, which were provided. In addition, some of the tasks shared isomorphic qualities. However, understanding which characteristics were present in the tasks was not sufficient information to understand why the students struggled with some tasks over others.

After identifying the characteristics of the On Track tasks, I scrutizined the student work samples and noted ways in which the students developed their functional reasoning over time. I was particularly interested in how the characteristics identified in the first research question aided in the students' responses. Additionally, I wanted to examine the development of students' reasoning as they gained experience during the sessions.

Research Question 2

What evidence suggests that the On Track tasks supported the development of students' functional reasoning among students of varying age and ability levels?

The students who participated in On Track entered the program with a variety of ability levels both across grade levels and within grade levels. Regardless, the student work samples showed that the students were able to make sense of the tasks and worked to answer the problems to the best of their abilites. In addition, while all students were not able to generalize a rule for all tasks, the students still made strides toward developing their functional reasoning. To illustrate this, I have chosen three different students, one in each grade level, and tracked their progress on the On Track tasks across the session(s) in which they participated. I was also interested to see how the functional reasoning characteristics played a role in student learning.

The students were chosen purposefully to illustrate their growth. I felt it was important to choose one student from each grade level. Furthermore, the students attendance in the program needed to be consistent during the session(s) they attended. Also, I wanted one student who only attended the first session, one who only attended the second session and one who attended both sessions in order to compare their results. Lastly, their work samples needed to be legible and show a good effort in solving the problem. The first student, Brandon, a third-grade student, attended the first session of On Track. The second student, Jackie, was a fourth-grade student who participarted in the second session of On Track. The last student, Zack, was a fifth-grade student who participated in both sessions.

Brandon, Third Grade

Brandon attended the first session of On Track that was held early in the school year. As I observed him during this session, I noticed that he was enthusiatic about participating in this program and was tenatious in trying to find an answer for the given questions. However, he tended to become so involved with trying to find an answer using one method, he would not look for any alternate patterns in the data. While working on the function machine tasks, prior to the functional reasoning tasks, he liked to use brute force to try to find the output for a large input. He would often ask for extra paper because he ran out of room on his worksheet for all his computations.

During this session of On Track, Brandon worked on four of the functional reasoning tasks. All of these tasks were open-ended, required students to generate data, organize data in input/outut tables, look for patterns, generalize a rule to find the output for a large input (e.g. 100) and to justify their generalizations. In addition, these tasks used visual images as aids in problem solving. On the Towers Task (1.3), Brandon continued to exhibit the type of thinking that I observed as he worked on the function table tasks (see Figure 4). He was still trying to use brute force to answer the question. His work sample showed the ineffeciency of his methods. He was unable to generalize an appropriate rule, either recursive or explict due to his over emphasis on using variational resoning and brute force to try to find the number of towers 100 blocks tall.

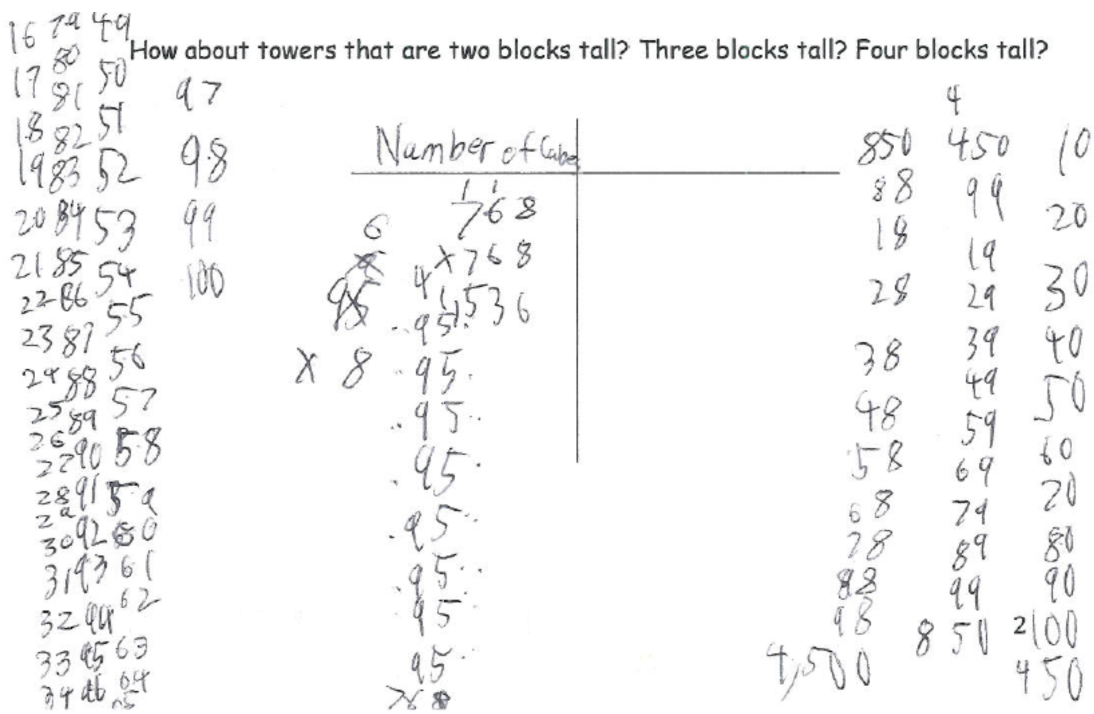


Figure 4. Brandon's Work Sample for the Towers Task

As Brandon participated in the enrichment program, he learned alternate ways of thinking about the tasks. While working on the Squares and Triangles Task (1.4), Brandon made use of his pattern block model to help him find the number of triangles that could fit around 100 connected squares. As the following transcript of the video will show, Brandon began by finding the output for 100 incorrectly. Initially, he finds the answer by finding the number of triangles for 10 squares then multiplying that by 10. After the researcher brings his focus back to the pattern block model, he was able to find the correct answer.

> Researcher: Alright, what if I had 100? I'm going to get a number so big that you can't figure it out by adding to every time.
>
> Researcher: I want you to try something that's in a different way. Look at your model.
>
> Researcher: Let's see you've got one, two tables, three tables, four tables, ten tables.
>
> Brandon: Twenty-two times 10, 220.
>
> Researcher: Where is twenty-two coming from?
>
> Brandon: From ten.
>
> Researcher: How? Show me.
>
> Brandon: ‘Cause, ‘cause we were doing ten times twelve.
>
> Researcher: I want you to show me on this model.
>
> Researcher: Where would twenty-two triangles be?
>
> Brandon: One right here, one right here.
>
> Researcher: Mhm. You mean around here?
>
> Brandon: It would be (points at bottom side of squares.)
>
> Researcher: How many go on this side? (points to the bottom of the triangle train)
>
> Brandon: Ten?

Researcher: How many on this side? (points to the top of the triangle train)

Brandon: Ten? Ohhh. You mean like 202 right?

Researcher: For 100 tables?

Brandon: Yeah.

Researcher: What do you think happened?

Brandon: It’s a number, times two plus two.

Researcher: What’s times two?

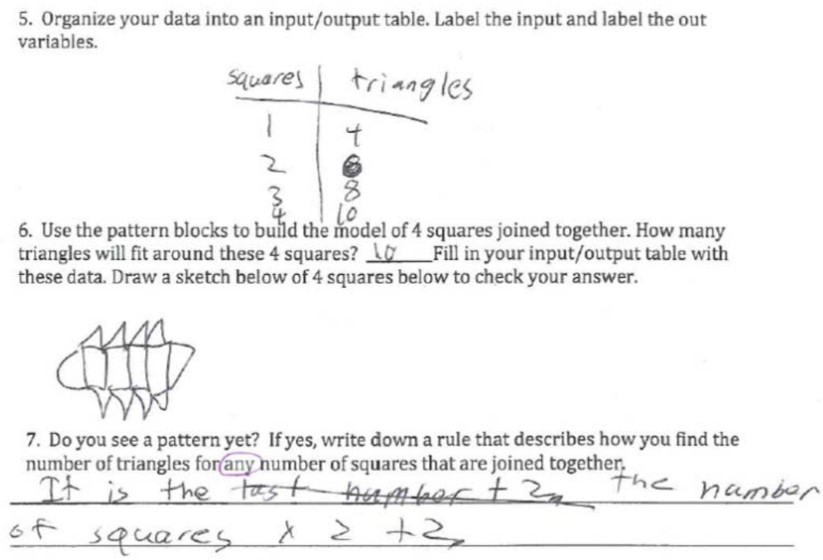
Brandon: It is the number times two and then plus two.

Researcher: What do you mean by the number? The number of what?

Brandon: The number of squares.

Brandon’s work sample for this task also showed that he had learned to organize his data in an input/output table. In contrast to his work on the Towers Task, his work sample on the Squares and Triangles Task (1.4) showed almost no reliance on brute force in order to answer the question. Brandon showed a legible input/output table and a drawing of the model he used to find a rule (see Figure 5).

On the Perimeter Task (1.7), which was the last of the session 1 functional reasoning tasks, Brandon work sample showed that not only could he write a correct explicit rule, he was able to write the rule using proper language and notation (see Figure 6).


5. Organize your data into an input/output table. Label the input and label the out variables.

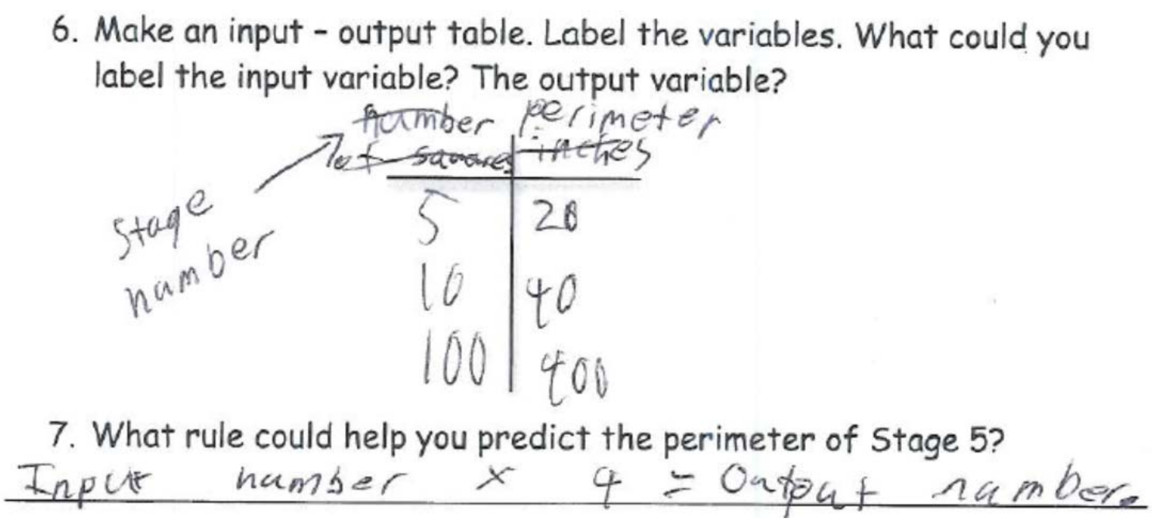
squares	triangles
1	4
2	6
3	8
4	10

6. Use the pattern blocks to build the model of 4 squares joined together. How many triangles will fit around these 4 squares? 10 Fill in your input/output table with these data. Draw a sketch below of 4 squares below to check your answer.

7. Do you see a pattern yet? If yes, write down a rule that describes how you find the number of triangles for any number of squares that are joined together.

It is the ~~last number~~ + 2, the number of squares x 2 +2

Figure 5. Brandon's Work Sample for the Triangles and Squares Task


6. Make an input - output table. Label the variables. What could you label the input variable? The output variable?

~~number~~ ~~of squares~~	perimeter ~~inches~~
5	20
10	40
100	400

stage number

7. What rule could help you predict the perimeter of Stage 5?

Input number x 4 = Output number

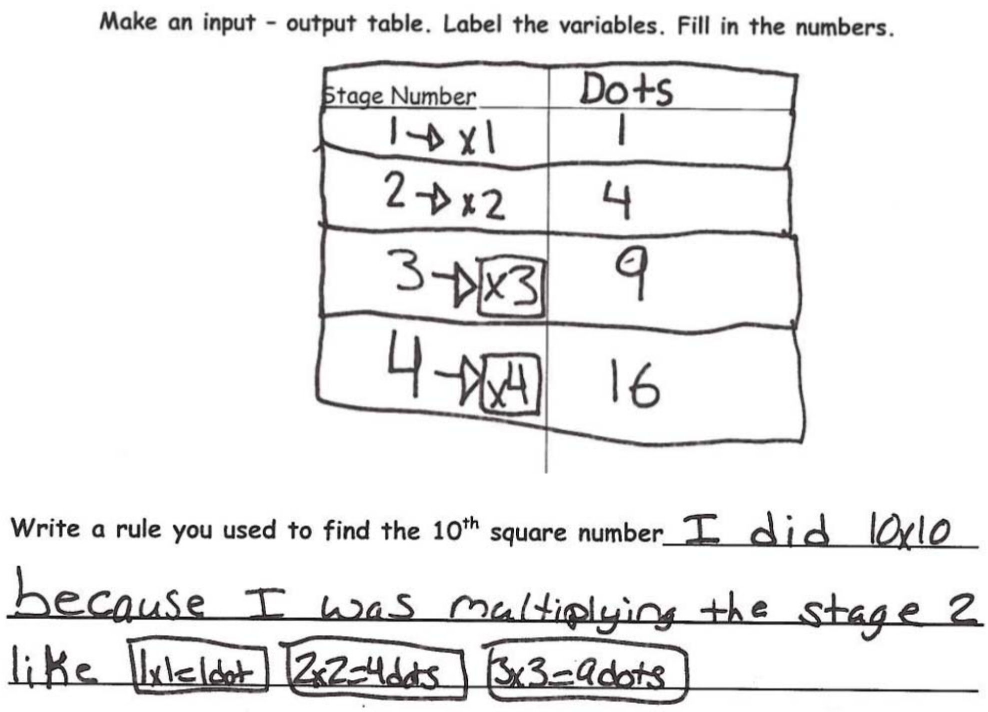
Figure 6. Brandon's Work Sample for the Perimeter Task

Initially, Brandon relied solely on using brute force computations to find the output for large inputs. The open-ended characteristics of the tasks allowed Brandon to explore various ways to solve the problem. As the session progressed, Brandon learned several new techniques to add to his toolkit for problem solving. These included how to generate data, how to use the input/output table to organize the data, and how to look for patterns in the data. In addition, by the end of the first session, Brandon was finding explicit generalizations to describe his patterns. The visual images also helped Brandon work toward a generalization. In the video tape data, he uses the pattern block to discover the structure of the pattern that led to the development of an explicit rule. Both the work sample data and the video tape data suggests that the characteristics of the task aided in Brandon's development of functional reasoning.

Jackie, Fourth Grade

Jackie was a fourth-grade student, who only attended the second session of On Track. However, during this session she exhibited substantial progress in using her functional reasoning skills. In the first task of the session, the Square Numbers Task (2.1), she had difficulty writing a rule to find the 10th Square Numbers. Jackie easily recognized the relationship between the input and output values, but was unable to translate this into a written rule. Instead, she showed how she found the 10th Square Numbers using computations (see Figure 7). This response indicated that she was developing her ideas of how to generalize a rule.

For the second task, the Rectangular Numbers Task (2.2), Jackie showed that she was learning to use the input/output table in various ways to look for patterns. On this task, she attempted to find a recursive rule by looking at the differences in the output values. Since the function rule for this task was quadratic, the differences in the output values were not consistent. She also attempted to find a correspondence between the input and output values by looking for a number that could be multiplied to the input to generate an output. This led her to noticing that each input times the next input number generated the corresponding output. She used this rule find the 99th rectangle number but instead of writing out the rule in words, she used multiplication to show the answer (see Figure 8). Jackie has a sense of the relationship between the input and outputs values but does not yet understand how to put her rule into words.



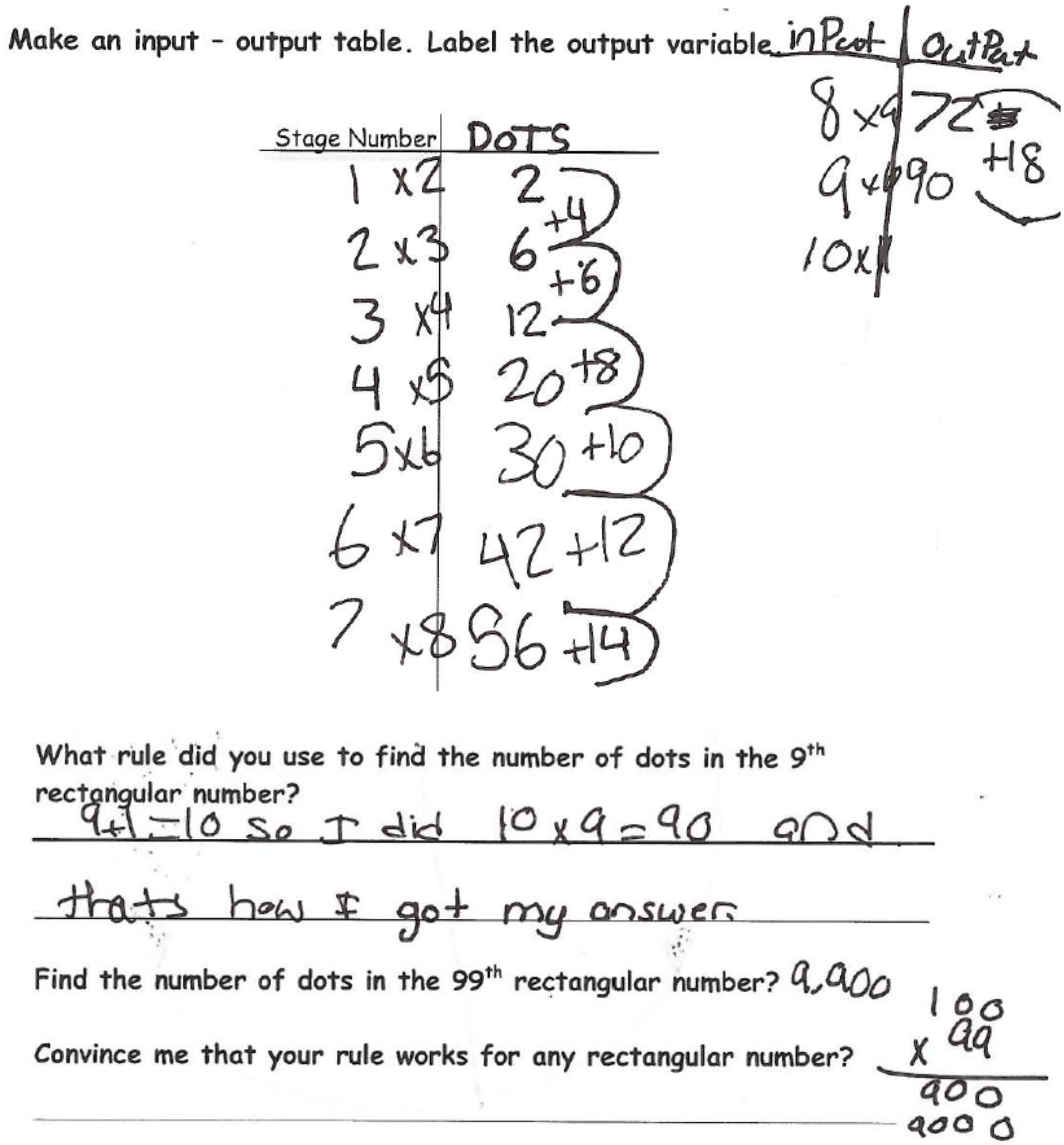
Make an input - output table. Label the variables. Fill in the numbers.

Stage Number	Dots
1 → x1	1
2 → x2	4
3 → $\boxed{x3}$	9
4 → $\boxed{x4}$	16

Write a rule you used to find the 10th square number I did 10x10 because I was multiplying the stage 2 like $\boxed{1x1=1dot}$ $\boxed{2x2=4dots}$ $\boxed{3x3=9dots}$

Figure 7. Jackie's Rule for the Square Numbers Task

As the function rules became more challenging, Jackie developed in several ways. She became more engaged in the tasks as evidenced by the amount of work she produced, the quality of work she did and the language that she used (see Figure 9). In this example, she discovered both a recursive rule (+ 4 for the output) and an explicit rule (x4+2). In her explicit rule, Jackie does not state the independent variable explicitly but it is implied. She also used her input/output table to show how her rule works.



Make an input – output table. Label the output variable inPeot | OutPat

Stage Number	DOTS
1 x2	2 +4
2 x3	6 +6
3 x4	12
4 x5	20 +8
5x6	30 +10
6 x7	42 +12
7 x8	56 +14

8 x9	72
9 x10	90 +18
10x11	

What rule did you use to find the number of dots in the 9th rectangular number?

9+1=10 so I did 10 x 9 = 90 and thats how I got my answer

Find the number of dots in the 99th rectangular number? 9,900

Convince me that your rule works for any rectangular number?

100
x 99
900
9000

Figure 8. Jackie's Work Sample for the Rectangular Numbers Task

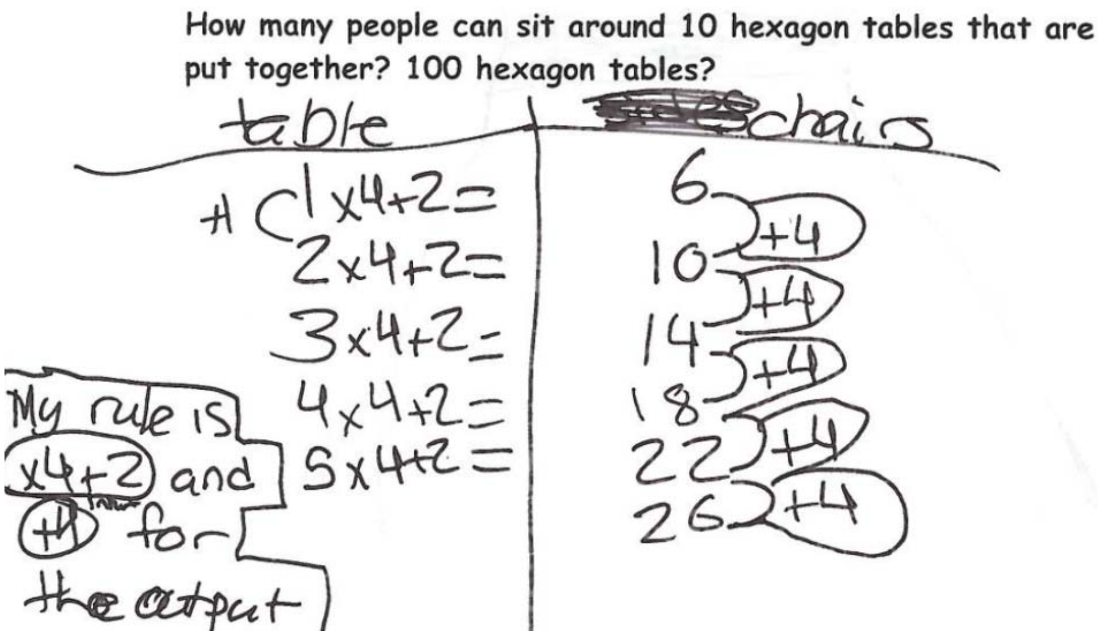


Figure 9. Jackie's Work Sample for the Hexagon Tables Task

While Jackie was not able to write a number sentence for the explicit rule for all of the session two tasks, she understood the computations needed to find the answers and increased her understanding of functions.

On the Triangular Numbers Task (2.6), which was one of the most difficult functional reasoning tasks of the session, Jackie attempted to find a relationship between the input and output numbers (see Figure 10).

She looked for a rule in several ways, using variational and correspondence reasoning. After failing to find a pattern in the output values, Jackie tried looking for an explicit rule by finding a combination of operations on the input that generated the output. In her work sample, she showed in the first input cell that she was trying to multiply by 2 then subtract 1. While this rule actually worked for the first two inputs, it failed for the third. She then showed another attempt at an explicit rule (x0 + 1) that also did not work. Finally, she discovered that if she takes the output and adds that value to the next input, it would generate the next output. This rule is evidence that Jackie can use both variational and correspondence reasoning when working with the input/output table. She described this as crisscrossing the input and output and adding the numbers.

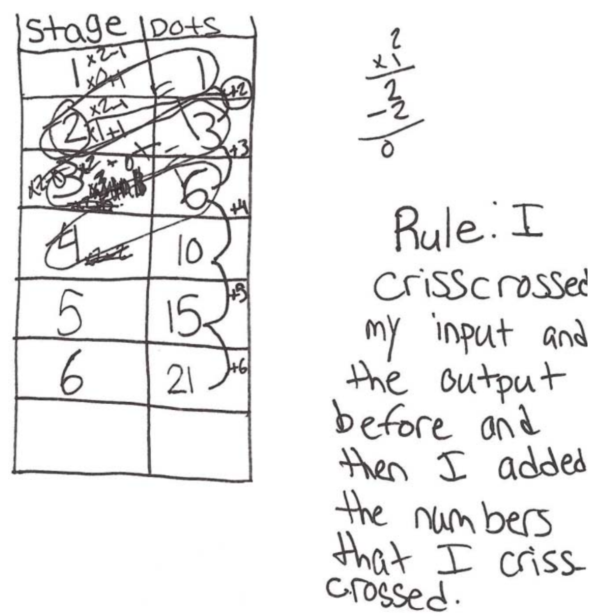


Figure 10. Jackie's Work Sample for the Triangular Numbers Task

I chose to highlight Jackie's work during session two because as a fourth-grade student, she had little experience using input/output tables or looking for patterns in growing data. In addition, she consistently attended this session and showed a more organized way of solving the problems from the first task to the last. Jackie's work samples indicated the characteristics of the tasks were an important component in developing her functional reasoning. For Jackie, the input/output tables were especially useful. As the session progressed, Jackie learned how to utilize the input/output tables to look for many different types of rules. This type of organization helped her to visualize the relationships between the inputs and outputs. The use of an input/output rule led Jackie to develop a recursive rule for some tasks and an explicit rule for others. Nevertheless, she continued to make progress.

Zack, Fifth Grade

Zack, a fifth-grade student, was one of the few students who attended both sessions of On Track. He experienced difficulty finding any rule for the Towers Task (1.3). In his attempt, he began by finding a number that when added to each input generated the corresponding output (see Figure 11). However, after trying several input values, he failed to find an patterns and subsequently did not generate a recursive or an explicit rule using this line of thinking.

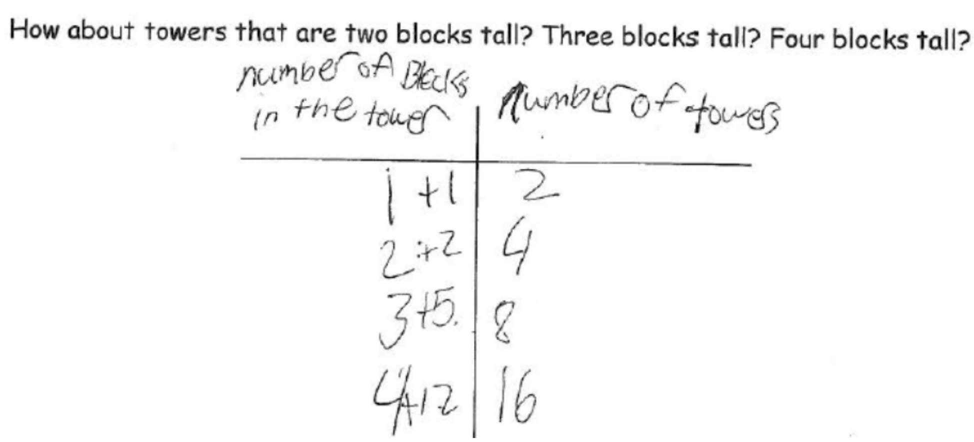


Figure 11. Zack’s Input/Output Table for the Towers Task

Zack experienced similar results on the second task (Triangles and Squares). On this task, he generated the input/output table but was unable to find a pattern in the data. By the third functional reasoning task (Square Tables), he tried to write an explicit rule (see Figure 12); however, his rule was incorrect (due to the order of operations). Also at this point in the sessions, he was still unable to articulate his reasoning using the appropriate mathematical language.

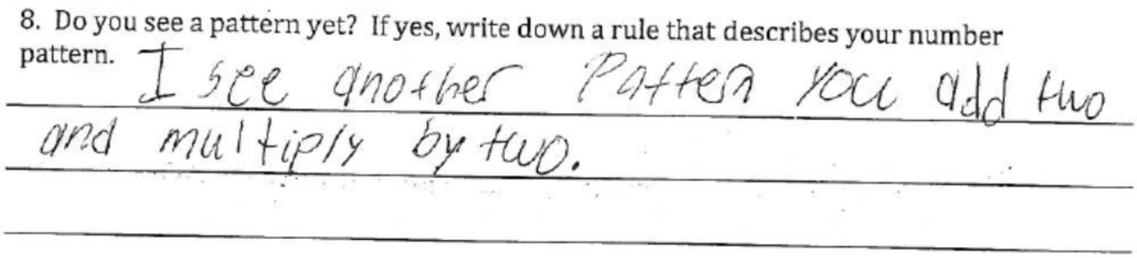

8. Do you see a pattern yet? If yes, write down a rule that describes your number pattern. I see another Patten you add two and multiply by two.

Figure 12. Zack's Generalization for the Square Tables Task

In session two, while working on the Square Numbers Task (2.1), Zack linked this task to previous knowledge and wrote an explicit rule (see Figure 13). Not only did he describe his rule as stage number times itself, he goes on to describe that the model was like an array (used in multiplication). On this task, he was wrote an explicit rule and showed that he is beginning to understand how to use appropriate language.

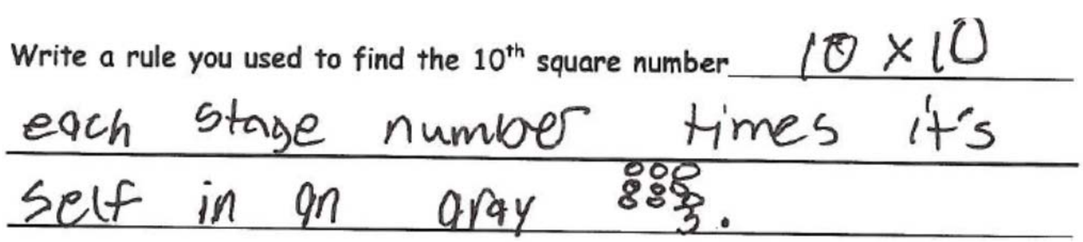

Write a rule you used to find the 10th square number 10 x 10 each stage number times it's self in an aray.

Figure 13. Zack's Generalization for the Square Numbers Task

On the Rectangular Numbers Task (2.2), Zack wrote an explicit rule using correspondence reasoning (see Figure 14). He wrote the following function rule "I did the stage number times another number (another number goes up by one each time) and got the output." This rule showed an understanding of how he used the input numbers to generate the output numbers. This example showed that Zack used his correspondence reasoning and that he was beginning to develop appropriate language to describe the rule.

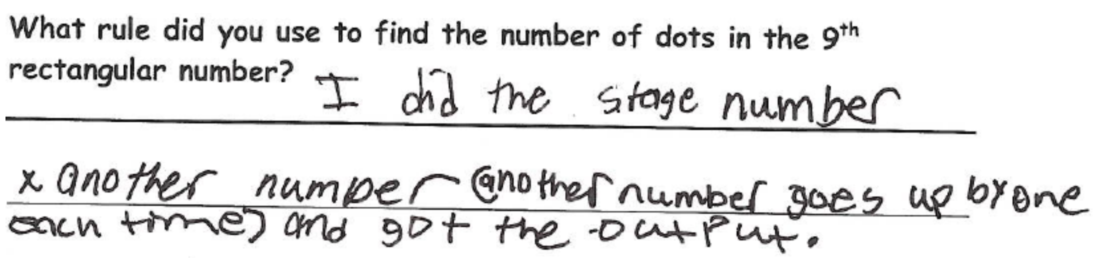

What rule did you use to find the number of dots in the 9th rectangular number? I did the stage number x another number (another number goes up by one each time) and got the output.

Figure 14. Zack's Rule for the Rectangular Numbers Task

On the Triangle Tables Task (2.3), Zack exhibited growth of functional reasoning in several areas. He generated an appropriate input/output table and labeled the input as "triangle" and the output as "people." He then began looking for rules in the pattern. By looking down the column labeled "people," he noticed that each output values increased by one. This led him to write the recursive rule "people +1 gets next person" (see rule #1, Figure 15). He further analyzed his input/output table to find a correspondence pattern. In rule #2 (Figure 15), he wrote the explicit rule "Triangles +2 gets next person."

Across the two sessions of On Track, Zack grew in his understanding of how to work with input/output tables, generalize patterns and to write generalizations in increasingly sophisticated ways. Early in session one, while using input/output tables Zack showed only a minimal number of inputs, and labeled his input and output using multi-word descriptions. By the end of the second session, he increased the amount of data he worked with and labeled his input and outputs more succinctly. As his experience increased, he began to show more and more work indicating an increased focus on the task. In the beginning, he barely wrote anything on his paper, but in session two, he often used his entire paper along with the margins to work on the problem.

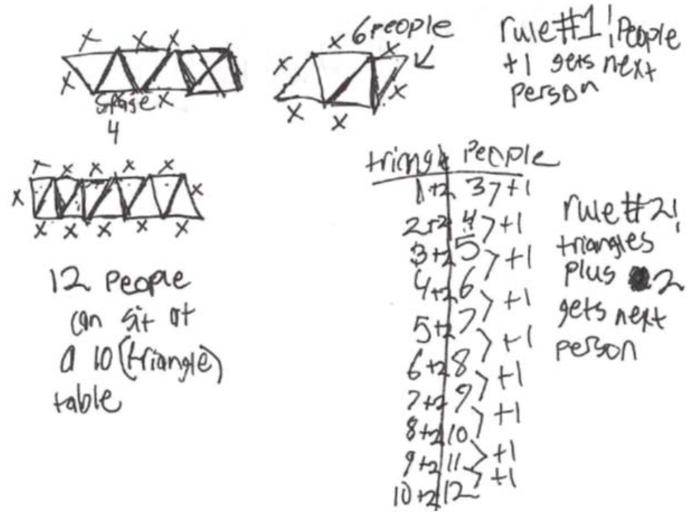


Figure 15. Zack's Work Sample for the Triangular Tables Task

Lastly, Zack showed an increase in the mathematical terms he used to write his generalizations. Early in the session, he stated that he added two and multiplied by two when describing his rule for the Square Tables Task. This did not make explicit what he was adding his number to or the value that he was generating. By the end of the sessions, he was writing multiple rules and using language that made explicit his understanding. While his rules may not have been as sophisticated as others, there was a significant difference in the way he wrote his rule early in the session compared to his later rules.

The purpose of the first part of the analysis was to identify the characteristics of the On Track tasks and to see if the students' reasoning developed as they participated in the sessions. I was also interested in what ways participation in On Track enhanced the students' problem solving skills. At first, the students approached these tasks in random, trial and error strategies. However, as they became more organized and systematic in their work, they were able to develop functional reasoning. The analysis of the student work samples also found that the characteristics of the On Track tasks were significant in the development of functional reasoning. While each of the work samples do not overtly show the students using each of these characteristics, the following section will discuss the role each of the identified characteristics played in the students' work.

Open-ended. This characteristic allowed the students to approach the On Track tasks in multiple ways. Since the problem did not specify a method to solve the problem, the students explored the problem in any way they wanted. This characteristic also gave the students freedom to explore the problem without threat of being wrong. Brandon, while working on the Towers Task, showed reliance on brute force in order to find a solution but on later tasks uses his input/output table to analyze the data. Jackie also shows a reliance of using computations to try to solve the Square Numbers and Rectangular Numbers Tasks but learned other ways to approach the problems as they session progressed.

Generates data. The development of functional reasoning depends on writing generalizations for patterns in data (Blanton, 2008). All of the On Track tasks expected students to generate a set of data that represented a growing pattern. It was important that the students themselves generate the data in order to investigate the structure of the problem rather than having the data given to them. Manipulatives such as pattern blocks and Unifix cubes assisted the students in this process or the students could choose to draw the figures.

Input/output tables. Input/output tables allow the students to organize their data. According to Blanton (2008), these tables help to make the dependency visible to the students because the student records the independent value (input) in the left column and the dependent value (output) in the right column. This visual display helps students organize the data into meaningful parts. In each of the student work samples, the student showed increasing use of their input/output tables as they worked toward a generalization. In many cases, such as Jackie's (see Figure 7), the students used the input/output tables to identify the patterns they saw. According to Blanton, one of the students' first responses is to look down the right hand column to find a pattern in the sequence of output values. This variational reasoning leads to recursive rules as the student notice the differences between the output values. Both Jackie and Zack show this type of reasoning in their work samples. The students quickly find this type of reasoning inefficient when looking for the output value for a large input value. With encouragement students begin looking across the table in order to look for a correspondence between the input and output values.

Pattern finding. Smith (2003) suggested that pattern finding helps to lay the groundwork for higher levels of mathematics and that tasks should ask students to describe patterns beginning in early elementary school. The On Track tasks asked students to find and describe patterns they found in the data. The students were not limited in the patterns they found. This led some students to use variational reasoning to find patterns in the output data or correspondence reasoning to find patterns in the relationships between the input and output values. In some cases, the students used both variational and correspondence reasoning to find a relationship between the input and output values, which led to a recursive rule such as the one Jackie found in Figure 10 (crisscross pattern).

Generalizations. Finding a generalization lies at the heart of functional reasoning. According to Blanton (2008), functional reasoning is the process about which one generalizes about data. Therefore, to develop students' functional reasoning, tasks must contain a set of data and ask students to find a rule to describe patterns in the data. The student work samples showed that the students could find generalizations that were either recursive or explicit. Both types of rules are important in developing functional reasoning. Students came to understand that recursive rules are not efficient and it is this understanding that leads them to look for other efficient rules. Therefore, recursive rules are stepping-stones to the development of explicit rules as Blanton (2008) and others suggests. These students also showed that even though they may not be able to write their rule in words they were still able to show, using computations, that they understood a rule (see Jackie's work sample in Figures 7 and 8).

Visual Images. Visual images were an important component of the On Track tasks. The visual images helped the students generate their data and served as a geometric model. The data showed that students used their visual images to help them investigate and reason about the tasks. In the videotape data Brandon, showed how the pattern blocks helped him to see the explicit generalization for the Squares and Triangles Task.

The first phase of the analysis concentrated on the overall structure of the tasks used in On Track. The purpose was to determine if these structures supported student reasoning and to determine in what ways the students came to understand the tasks as the sessions progressed. However, the categories were not sufficient to differentiate between the tasks. The next analysis highlighted student work samples to show in what ways students developed their reasoning across the sessions. The first phase found that over time the structure of the tasks promoted student learning in a variety of ways. For example, the student work samples showed an increase in students' abilities to use their input/output tables to find patterns, find generalizations, and use mathematical terms when describing their rules. This phase gives important information regarding the tasks but it does not pinpoint which specific tasks promoted students' abilities to write explicit rules. The second phase of the analysis took a more in-depth look at the types and frequencies of generalizations that the students wrote.

Analysis Part 2

This phase of the analysis began with the development of a coding scheme. The literature review found that variational and correspondence reasoning are two ways in which students looked for patterns in data. When students focus on how the output values are changing, they are using variational reasoning, which can lead to the development of a recursive rule. When the students focus on the relationship between the input and output values they are using correspondence reasoning which can lead to the development of an explicit rule. Both types of reasoning are important in understanding functions. In addition, we know from the literature that there are levels in the development of functional reasoning. Students tend to use recursive strategies first then move toward explicit strategies when prompted (Billings et al., 2008; Kaput & Blanton, 2004; Tanish, 2011).

Developing the Codes

Lannin et al.'s (2006) definitions of recursive and explicit generalizations formed the basis of the data-coding scheme. In order to develop the codes, I considered Lannin's definitions, classroom observations and an initial analysis of student works. This lead to the development of seven codes, no response, recursive incorrect, recursive transitional, recursive correct, explicit incorrect, explicit transitional and explicit correct. I used the no response code to identify which students were present on the day of the task but left the generalization question blank or wrote an unintelligible answer. A transitional code referred to a generalization that was on the right track but lacked a critical word or phrase such as the variable name. I established a 90% reliability rate by randomly selecting work samples and comparing my results to several of my peers.

After coding the students work samples, I arranged the data in a spreadsheet and found the percentages of each type of response by taking the number of student responses in each category and dividing by the total number of responses (*n*) for that task. Since each task had a different number of responses, the *n* value of each task was different. The *n* value changed depending on the task since not all students attended each daily session. However, *n* ranged in value from 12 to 86. Table 6 shows the percentages of generalizations students wrote based on the initial coding.

Table 6

On Track Percentage of Generalizations by Task

	Tasks												
	1.3	1.4	1.5	1.7	1.1	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.9
No Response	53.5	21.5	6.5	19.3	25.0	2.1	7.8	10.4	20.0	2.7	32.1	48.9	26.3
Recursive Incorrect	4.7	6.3	5.2	4.8	0.0	0.0	3.9	0.0	3.6	5.4	5.4	2.1	0.0
Recursive Transitional	5.8	17.7	23.4	6.0	0.0	0.0	0.0	0.0	9.1	5.4	16.1	0.0	15.8
Recursive Correct	11.6	15.2	18.2	9.6	8.3	0.0	0.0	4.2	25.5	13.5	25.0	29.8	31.6
Explicit Incorrect	7.0	1.3	7.8	1.2	25.0	0.0	11.8	2.1	7.3	8.1	7.1	8.5	0.0
Explicit Transitional	2.3	21.5	23.4	33.7	0.0	19.1	15.7	20.8	18.2	24.3	0.0	2.1	0.0
Explicit correct	15.1	16.5	15.6	25.3	41.7	78.7	60.8	62.5	16.4	40.5	14.3	8.5	26.3
N	86	79	78	83	12	47	51	48	55	37	56	48	15

Research Question 3

Which tasks appeared to promote students functional reasoning?

After looking at Table 6, no obvious patterns emerged, so I decided to collapse the categories. I kept the following categories: no response, incorrect, recursive and explicit. The no response category remained the same. I grouped the categories recursive incorrect and explicit incorrect into one category, incorrect. Next, I collapsed the categories recursive transitional and recursive correct into one category, recursive. Lastly, I combined the categories explicit transitional and explicit correct into one group, explicit. Table 7 organizes the data by the new collapsed categories.

Table 7

On Track Percentages of Generalizations—Collapsed Codes by Task

	Tasks												
	1.3	**1.4**	**1.5**	**1.7**	**1.10**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.9**
No Response	53.5	21.5	6.5	19.3	25.0	2.1	7.8	10.4	20.0	2.7	32.1	48.9	26.3
Incorrect	11.7	7.6	13	6	25	0.0	15.7	2.1	10.9	13.5	12.5	10.6	0.0
Recursive	17.4	32.9	41.6	15.6	8.3	0.0	0.0	4.2	34.6	18.9	41.1	29.8	47.4
Explicit	17.4	38	39	59	41.7	97.8	76.5	83.3	34.6	64.8	14.3	10.6	26.3
N	86	79	78	83	12	47	51	48	55	37	56	48	15

After collapsing the categories, several patterns became evident. In session one, the percent of student writing an explicit rule increased across the tasks from 17.4 % in task 1.3 to 59% in task 1.5. The exception was task 1.10. Since there were so few responses for this task (*n=12*), I eliminated it from further analysis. In session two, the percentage of correct generalizations was very high for the first task (97.8%) then declined slightly across tasks 2.2 - 2.4, then increased for task 2.5 then declined again to a low of 10.6% in task 2.7. In addition, for session two, I eliminated task 2.9 from further analysis because of the low response rate (*n=15*).

I expected that the percentage of students writing explicit rules would increase across the second session of On Track. This happened during session one; however, this was not the case in session two. The next section will investigate the differences between the two sessions.

Session One Tasks

The first session of On Track included four functional reasoning tasks that I included in my analysis, the Towers Task (1.3), the Squares and Triangles Task (1.4), the Square Numbers Task (1.5), the Perimeter Task (1.7). Figure 16 shows a comparison of students' generalizations across these tasks. The bar graph gives a better picture of the general trend of student responses across this session and indicates that the students learned to write explicit rules as the session progressed.

This session began with students working on the Towers Task. This task asked them to use Unifx cubes and build towers three blocks tall using just two colors. The next day's task asked the students to build towers four blocks tall using two colors. The third task in this sequence, and the first functional reasoning task, asked the students to generate data for the number of towers they could build using two colors if the towers increased in height (1 block, 2 blocks, 3 blocks, etc.). The task then prompted students to look for patterns in the data and to find a rule to generate the number of towers built using two colors if the towers were 100 blocks tall.

The Towers Task was the students' first experience using this type of reasoning. The data showed a low percentage of students who could write either an explicit or a recursive rule. In addition, the generalization rule for this task was an exponential function. Over 50% of the students did not write any type of generalization and over 10% tried to find a rule but the rule was incorrect. In the following section, I will highlight some of the student responses to this task.

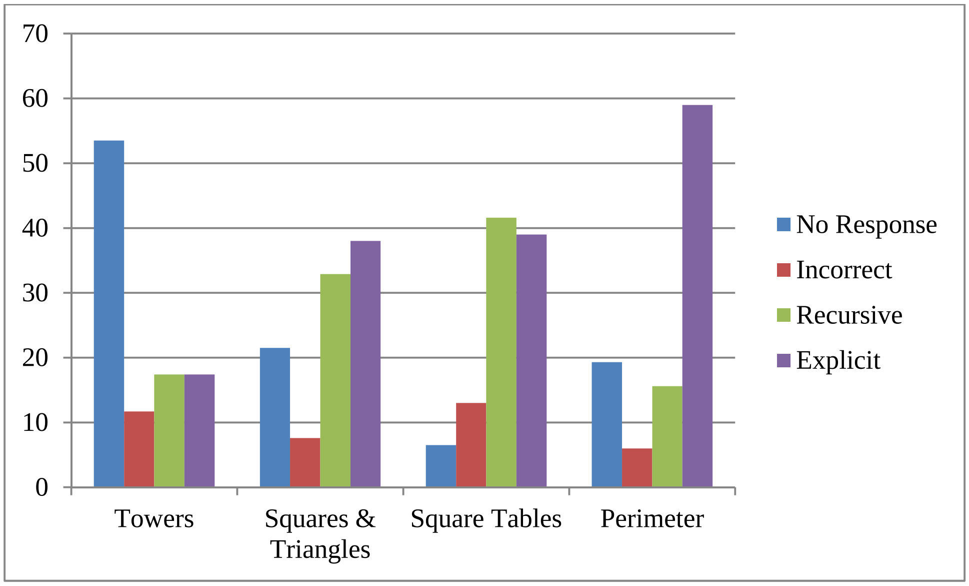


Figure 16. Students' Percentage of Generalizations in Session One

Overall, the student work samples revealed that many students were able to fill in their input/output tables appropriately but often left the question regarding the rule to find 100 towers blank. While others misinterpreted the question, Misty, for example, when asked to write a rule for 100 towers wrote, "I used my head and hands to think." Seventeen percent of students were able to write a recursive rule that showed the number of towers doubling for each input. At this point, the students also lacked the language to describe the rule they found. Kayte, a third-grade student, for example, was able to fill in her input/output table appropriately and wrote her rule on the side of the table (see Figure 17). She found the recursive rule "add the number with itself" which she means to find the next output by doubling the previous output. Many other students wrote similar rules.

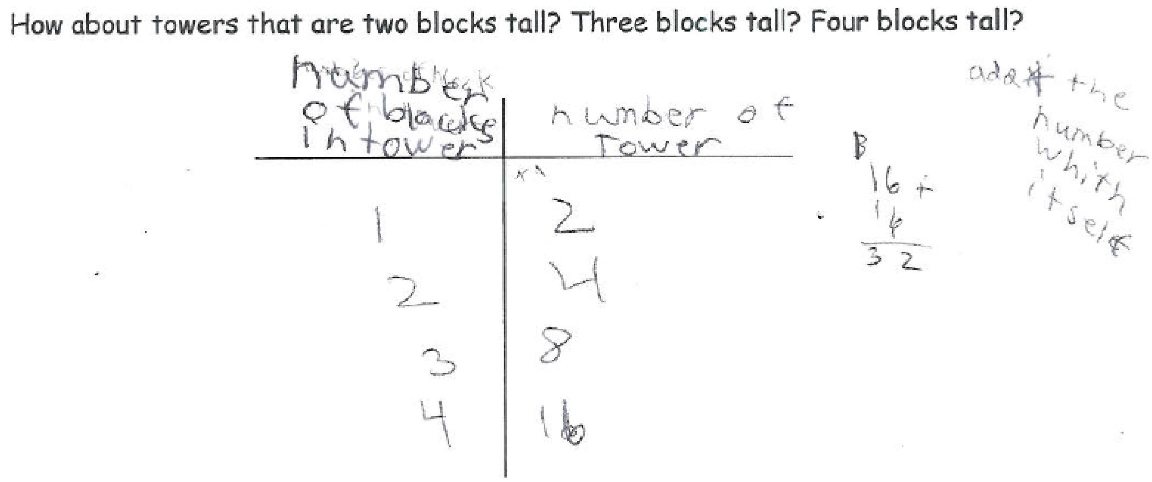


Figure 17. Kayte’s Work Sample for the Towers Task

Jacob, a fifth-grade student, was one of the few students who were able to generalize an explicit function rule. Jacob’s work sample (see Figure 19) indicated an attempt to find a recursive rule (x2 beside each output number). He then goes on to show his work toward finding an explicit function rule. He noticed that for three towers the output of 8 could be found by (2^3=2x2x2), the output for 4 towers could be found using (2^4= 2x2x2x2). This analysis led him to his explicit rule “each # of different towers is 2 to the exponent of the height.”

The remainder of the tasks in session one were linear function rules. The Squares and Triangles Task and the Square Tables Task both generalized the same rule, which explains the students’ similar results on these two tasks. The last task, the Perimeter Task, also generalized a linear function rule in the form p = 4n. In session one the students generalized the highest percent of explicit rules on the Perimeter Task (59%).

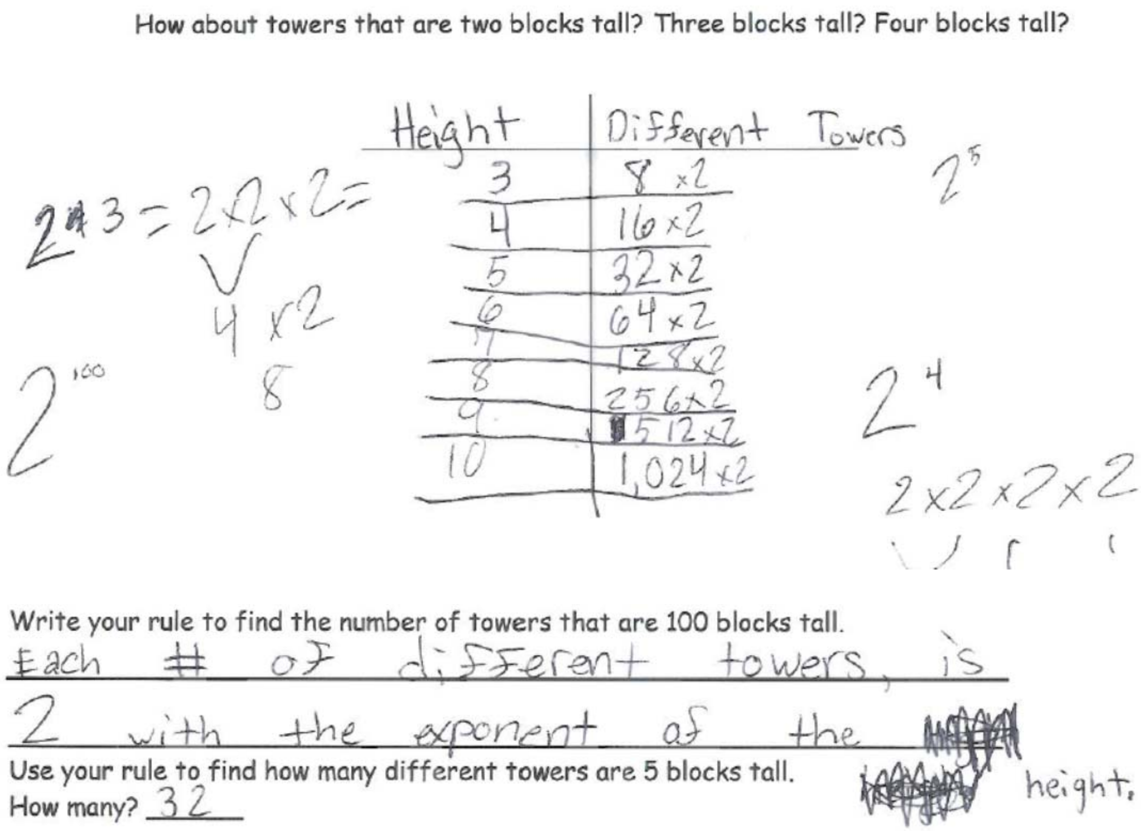


Figure 18. Jacob's Work Sample for the Towers Task

I expected that students would have a higher percentage of explicit rules as they progressed through the first session of On Track and they did. As they worked on their tasks students were encouraged to find multiple patterns in the data. Not only did they learn to look down the output values they also began to look across the input/output table to find explicit rules. Once the students realized an explicit rule would give them the output for a relative large input easier than a recursive rule they were more interested in finding them. This encouraged the students to switch from variational reasoning to correspondence reasoning.

Session Two Tasks

Session 2 consisted of six functional reasoning tasks, the Square Numbers Task (2.1), the Rectangular Numbers Task (2.2), the Triangle Tables Task (2.3), the Hexagon Tables Task (2.4), the Pentagon Tables Task (2.5), the Triangular Numbers Task (2.6), and the Handshake Problem (2.7). The first two tasks in session two contained quadratic function rules. The representation used for both of these tasks was geometric much like the multiplication arrays that students use in class. The next sequence of functional reasoning tasks were similar to the Square Tables Task from session one. The task asked the students to find the number of people who could sit around a train of 100 tables. The difference between the tasks was the polygon shape of the table changed from triangles, to hexagons to pentagons. The last two tasks in this session (Triangular Numbers Task and the Handshake Problem) were both quadratic rules that included fractional coefficients.

During session 2, the data showed that the students' percentages of explicit rules did not consistently increase across the session as expected (see Figure 19). Instead, the students' percentage of explicit rules for the first three tasks were relatively high (>75%), but after these tasks, the percentages were not as consistent. The last two tasks (Triangular Numbers Task and the Handshake Problem) were much lower than expected, particularly since these were the last of the functional reasoning tasks. The Triangular Numbers Task formula was $n = \frac{1}{2} s(s + 1)$ where n is the number of dots (Triangular Numbers Task) and s is the stage number, while the formula for the Handshake Problem was $h = \frac{1}{2} p (p - 1)$ where h is the number of handshakes and p is the number of people.

Both of these tasks required students to use the fraction ½ or divide their rule by 2. The fraction in this rule may have been problematic, especially for the younger students. In addition, these tasks are isomorphic to each other, which could explain why the students showed an increase in writing explicit rules between them.

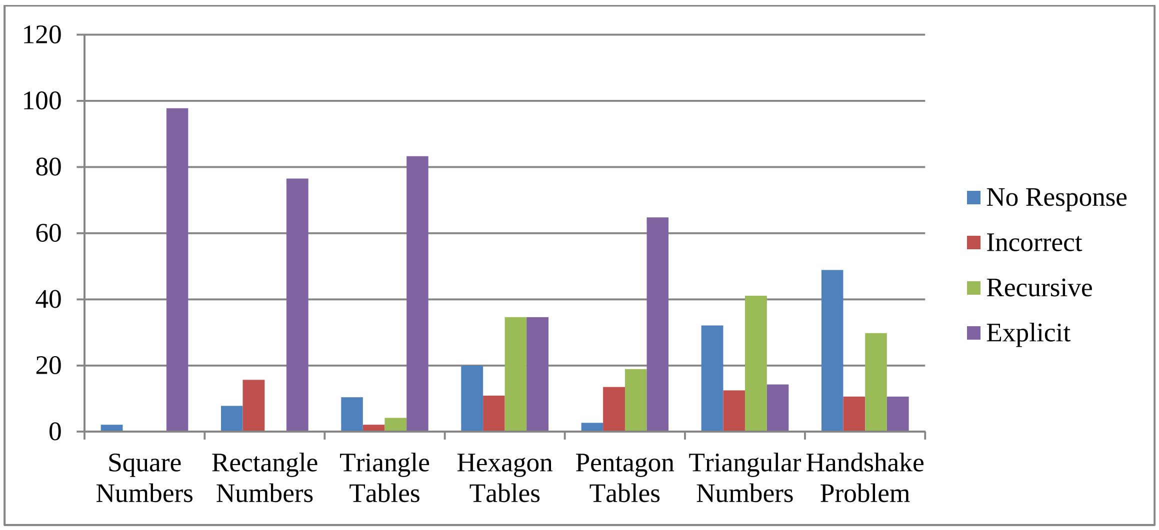


Figure 19. On Track Tasks Session 2

The data across each sessions showed that some tasks promoted students' functional reasoning over others. In session one, those tasks were the Squares and Triangles Task, the Square Tables Task and the Perimeter Task. In session 2, those tasks were the Square Numbers Task, Rectangular Numbers Task, Triangle Tables Task, and the Pentagon Tables Task. The tasks did share several common characteristics. First, the Squares and Triangles Task, Square Tables Task, Perimeter Task, Triangle Tables Task and the Pentagon Tables Task all had linear function rules. Second, the Perimeter Task, the Triangle Tables Task, and the Square Numbers Task had function rules that contained only one operation. Third, the students were encouraged to use pattern blocks for all the table tasks. Lastly, the presentation of the Square Numbers Task and Rectangular Numbers Task used dots that were similar to the arrays students currently use when learning multiplication. While this question identified several important factors related to these tasks, it is also important to determine whether grade level was a contributing factor.

In comparing the data across both sessions, the data does indicate a significant increase in the percentage of explicit rules, especially in the beginning of the session two. This may be due in part to the continuing participation by many of the session one students.

Research Question 4

What Effect Does Grade Level Have on Students' Functional Reasoning?

In my classroom observations, I observed and talked with students from different grade levels. Early on, it appeared that the third-grade students seemed to have more difficulty writing explicit rules on some tasks than the older students. In this section of the analysis, I disaggregated the data based on grade level. First, I sorted my main spreadsheet by grade level, and then looked at each grade level across individual sessions.

Third Grade

Table 8 summarizes the data for the third-grade students in session 1 and session 2. In general, the third-grade students' percentage of explicit generalizations increased during session 1, the only exception was the Square Tables Task (1.5). On this task, the third-grade students actually wrote recursive rules more often (42.4%), which was surprising since this task was so similar to the Squares and Triangles Task. A more specific look at the students work showed that 34 third-grade students responded to the Squares and Triangles Task while 33 students responded to the Square Tables task. Of those, 14 students wrote an explicit rule for the Squares and Triangles Task and 12 wrote explicit rules for the Square Tables Task (1.5). Thirteen students had incorrect rules or no response on the Squares and Triangles Task (1.4), yet there were only nine incorrect or no response on the Square Tables Task (1.5).

Table 8

Third Grade Percentages of Generalizations

	Session 1 Tasks				**Session 2 Tasks**						
	1.3	**1.4**	**1.5**	**1.7**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**
No Response	65.7	29.4	12.1	25.6	0	9.5	19.1	36	5.3	34.6	57.1
Incorrect	8.6	8.8	12.1	10.3	0	19.1	4.8	8	10.5	11.5	14.3
Recursive	11.4	23.5	42.4	12.8	0	0	9.5	36	15.8	50	23.8
Explicit	14.3	38.2	33.3	51.3	100	71.4	66.7	20	68.4	3.9	4.8
n	35	34	33	39	21	21	21	25	19	26	21

Most of these students wrote recursive rules, which added to the percentage on the Square Tables Task (1.5). This shows that many of the students who fell into the no response and incorrect categories on the Squares and Triangles Task (1.4) were able to write recursive and explicit rules for the Square Tables Task (1.5). However, many more students fell into the recursive category, which pushed up the percentage of recursive generalizations on the Square Tables Task.

In session two, the third-grade students showed strong reasoning with 100% of the students able to write an explicit rule for the Square Numbers Task even though this rule was quadratic. However, after the first task their explicit generalizations dropped off for the next several tasks. The percent of generalizations jumped for the Pentagon Tables Task but then fell significantly for the last two tasks.

One surprising result in looking at this data was the response rate on the table tasks, especially the sharp decline from the Triangle Tables Task to the Hexagon Tables Task. The Triangle Tables, Hexagon Tables, and Pentagon Tables Tasks were a series of isomorphic tasks that were similar in design. Each of the function rules were linear with two operations, each ending in +2, which represented the number of people who were sitting on the end of the train of tables.

In comparing student work samples from the Triangle Tables Task and the Hexagon Tables Task, one possible explanation stood out. When using the input/output table, generating an explicit rule for the Triangle Tables Task was much easier than the Hexagon Tables Task. For the Triangle Tables Task the students could easily find the additive rule (input +2) which became more difficult with the Hexagon Tables Task due to the fact the rule became more complex (input x4 + 2). After working on the Hexagon Tables Task, the students' percentage of explicit rules significantly increased for the next task, the Pentagon Tables Task (68.4%). When looking at the student work samples and the structure of these tasks, their results begin to make sense. The third-grade students found the additive rule easily to find, then experienced difficulty on the next task but learned from this and did quite a bit better on the next task.

When working on the last two tasks, the percentage of recursive rules were higher than the percentage of explicit rules. The explicit rule [input (input -1) / 2] proved very difficult for the students to find. Instead, the third-grade students relied on recursive rules to describe their patterns. The presentation of the Triangular Numbers Task made finding a recursive rule easier than finding an explicit rule.

The third-grade students were the largest population in the study and showed an overwhelming increase in the percentages of explicit rules for many of the tasks. However, as the tasks grew more difficult to find, they relied more on recursive strategies to find their rules. In general, the third-grade students wrote explicit generalizations more often when the function rule was linear or contained only one operation.

Fourth Grade

Like the third-grade students, the fourth-grade students' percentages of explicit rules increased across the session one (see Table 9). The starkest contrast was between the Squares and Triangles Task and the Square Tables Task (1.5). The data showed that the percent of student writing explicit rules increased between these two tasks, while the percent of students writing recursive rules decreased. These results were not surprising given the fact these tasks were similar.

In session two, the fourth-grade students did particularly well with writing explicit rules for the first five tasks and had especially high percentages on the first three. In addition, the percentage of no response and incorrect responses were very low on these tasks. Again, these results are not surprising, as I would expect the fourth-grade students to do better than the third-grade students did. However, unlike the third-grade students the fourth-grade students' gain in percentages for writing explicit rules did not increase from the Hexagon Tables Task to the Pentagon Tables Task, although the percentage of recursive rules did increase. While working on the last two tasks of this session, the fourth-grade student results were very similar across both tasks. There was approximately a 1% increase in the percentage of explicit generalizations and a decrease in the percentage of recursive generations.

Table 9

Percentages of Fourth-grade Generalizations

	Session 1 Tasks				Session 2 Tasks						
	1.3	**1.4**	**1.5**	**1.7**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**
No Response	44.83	13.04	0.0	8.70	7.69	0.00	6.25	5.88	0.00	27.78	29.41
Incorrect	17.24	4.35	17.99	0.00	0.00	22.23	0.00	5.88	25.00	11.12	11.76
Recursive	17.24	56.52	36.37	21.74	0.00	0.00	0.00	29.41	16.67	44.44	41.18
Explicit	20.69	26.09	50.0	69.57	92.31	77.78	93.75	58.82	58.33	16.67	17.65
n	29	23	22	23	13	18	16	17	12	18	17

For most of the tasks in session one, the fourth-grade students wrote explicit rules more often than the third-grade students. The exception to this was the Squares and Triangles Task, Square Numbers Task and the Pentagon Tables Task. However, the difference in these percentages was not very large. A possible explanation for this may be due to a lower participation rate for fourth-grade students and lower rates of no response, which showed that a higher percentage of fourth-grade students attempted these problems.

Fifth Grade

Table 10 summarizes the data for the fifth-grade students in session 1 and session 2. The fifth-grade students began the first session similar to the other two groups with the highest percentage falling into the no response category on the Towers Task (1.5). In addition, the percentage of recursive rules was still greater than the percentage of explicit rules on this task. But, across the rest of the tasks the fifth-grade students did not perform as expected, their percentages of explicit rules did not continue to increase for each task, instead their percentage dropped off during the Square Tables Task (1.5). It is interesting that the students' results changed between the Squares and Triangles Task (1.4) to the Square Tables Task (1.5). I expected that the results between the two tasks would be similar or that the percentage of explicit rules would increase. However, this did not happen. I analyzed the data further to determine whether this was a significant event.

When compared to the other two groups, I expected that the fifth-grade students would have smaller percentages in the no response and incorrect category and higher percentages in the recursive and explicit categories. This was not necessary true. In fact, for each of the tasks in session one the fourth-grade students had a lower percentage of no response than either the third- or fifth-grade students did. Neither did the fifth-grade students have the lowest percentages in the incorrect category. When writing explicit rules, the fifth-grade students also did not consistently have higher percentage rates. It is important to note that the fifth grade was the smallest population in the study so the sample size may have been a contributing factor to these results.

Table 10

Percentages of Fifth-grade Generalizations

	Session 1 Tasks				Session 2 Tasks						
	1.3	**1.4**	**1.5**	**1.7**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**
No Response	44.45	18.18	4.35	19.05	0.00	16.67	0.00	7.69	0.00	33.33	60.00
Incorrect	9.09	9.09	17.39	4.76	0.00	0.00	0.00	23.07	0.00	16.16	0.00
Recursive	27.27	22.73	43.44	14.28	0.00	0.00	0.00	38.46	33.33	16.16	20.00
Explicit	18.19	50.00	34.78	61.90	100.00	83.33	100.00	30.77	66.67	33.33	20.00
n	22	22	23	21	13	12	11	13	6	12	10

Across session two, many fifth-grade students wrote explicit rules for most of the tasks. However, they had relatively low percentages of explicit rules for the Hexagon Tables, Triangular Numbers, and the Handshake Problems. Much like the other groups, the Handshake Problem was the most difficult with only 20% of the fifth graders finding an explicit rule. In fact, on this problem 60% of the fifth-grade students had no response at all. It is also interesting that on the Rectangular Numbers Task the fifth-grade students either wrote an explicit rule or had no response. No students fell into any other categories.

When looking at each grade level's data, it was difficult to compare the percentage of explicit generalizations across both sessions. I created a bar chart for just explicit rules (see Figure 20). This chart shows that the students wrote explicit rules more often for the Square Numbers, Rectangular Numbers, and the Triangle Tables Task. The also had the most difficultly on the Towers Task, the Triangular Numbers and the Handshake Problem. In looking at Figure 20, it is difficult to determine if grade level is a significant factor in students’ success in writing explicit rules. In some tasks, the third-grade students underperform the other grades but in other tasks, this group performs better than the other two groups. In addition, the fourth-grade students seem to outperform both groups in many of the tasks. Lastly, the fifth-grade students seem to perform significantly better on the tasks with the complex quadratic rules.

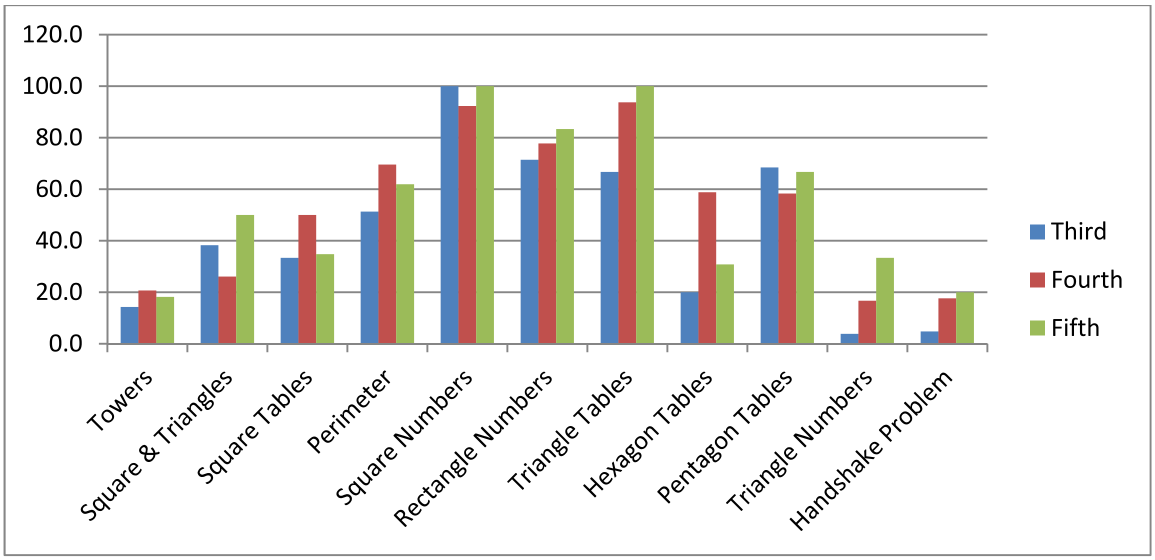


Figure 20. Student Explicit Generalizations by Grade Level

Summary

Overall, all groups showed a growth in the percentage of explicit rules they wrote during the first session of On Track. The data showed that all three groups shared the same overall trend across session one. In session two, all groups performed similarly on the Square Numbers Task and the Rectangular Numbers Task when compared to the other tasks in the session. However, there was variation in tasks when comparing each grade level. I expected that the fifth-grade students would write explicit rules more often than either the third- or fourth-grade students. This was not true, in fact, the third grade students wrote explicit rules more often than the other two group on the Pentagon Tables Tasks and matched the fifth grade students on the Square Numbers Task. The fourth grade students out performed both groups on the Towers Task, Square Tables Task, Perimter Task and the Hexagon Tables Task. The fifth grade students wrote explicit generalizations more often with the Squares and Triangles Task, the Rectangle Numbers Task, the Triangle Tables Task, the Triangle Numbers Task and the Handshake Problem. While this analysis gave important insights into how each grade level performed across the sessions and how they compared to the other grade level groups, it did not answer the question adequately whether grade level was significant factor.

Examining the percentages in order to compare and contrast grade level with each individual task was helpful in determining if there were trends either in grade level performance or between individual tasks. While this analysis began to develop a picture of students' abilities by grade level, I needed a more robust test to determine if grade level significantly affected student achievement. Using the original collapsed data I converted the categorical data to numerical data letting 0 = no response, 1 = incorrect, 2 = recursive and 3 = explicit. I grouped responses by grade level and performed a single factor ANOVA with post hoc tests to compare the means of each group. The null hypothesis for this test was that there was no difference in the means for each group ($H_0 = \mu_1 = \mu_2 = \mu_3$). Table 11 gives the results of the ANOVA.

Table 11

Analysis of Variance for Grade Level Performance

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
Column 1	296	512	1.72973	1.580944
Column 2	208	436	2.096154	1.188777
Column 3	156	327	2.096154	1.390695

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	21.91882	2	10.95941	7.758872	0.000467	3.009434
Within Groups	928.013	657	1.412501			
Total	949.9318	659				

$p < .05$

This test shows that there is a significant difference between the means of each group $F(2, 657) = 7.75$, $p = .00046$. Therefore, I rejected the null hypothesis and determined that the means are significantly different and that grade level does affect students’ abilities to generalize function rules. However, this test does not tell us which grade level is significantly different or how they are different. To determine this, a post hoc Tukey test was performed and the results determined that the third-grade students’ means were significantly lower than the other groups ($M = 1.7$, $SD = 1.26$) but that the fourth-grade ($M = 2.096$, $SD = 1.09$) and fifth-grade ($M = 2.096$, $SD = 1.179$) groups were not significantly different from each other.

Research Question 5

What types of function rules appear to promote students’ correspondence reasoning?

Confrey and Smith (1994) explained that correspondence reasoning occurs when students look for the relationships between the input value and the corresponding output value. One indication that students are developing their correspondence reasoning is their ability to write explicit rules. Therefore, in order to answer Research Question 5, I focused on students' percentages of explicit rules.

Table 12 orders the tasks by percentage of explicit rules from highest to lowest. I conjectured that the type of function rule could be a factor in students' percentage of explicit rules. However, there was not enough evidence to determine what types of function rules were more effective. Therefore, I used the original data and calculated the percentages of students' responses based on type of function rule. I used the following designations: one-operation linear rules, two-operation linear rules, one-operation quadratic rules, two-operation quadratic-rules, and one-operation exponential rules.

To clarify, when I discuss a one-operation function rule, I am not talking about number of terms. For example, the generalization for the Triangle Tables Task is a one-operation function rule because 2 is being added to the input (x value). In contrast, the Square Tables Task is a two-operation linear function rule due to the fact the input (x) is multiplied by 2 then 2 is added to this product. In addition, the Handshake Problem is considered a two-operation quadratic rule because the input value is multiplied by the previous input value then the product is divided by two.

The data showed that the type of function rule determined what type of generalizations the students wrote (see Figure 21). The students overwhelming wrote explicit rules for the one-operation quadratic rule tasks followed by the one-operation linear rule tasks. The two-operation linear and the quadratic rule tasks provided similar results, however, the linear rule tasks showed a higher percentage of recursive rules. The exponential one-rule tasks had the smallest percentages of explicit rules.

Table 12

Task Ordered by Highest Percent of Explicit Generalizations

Task	Type	Rule	Percentage
2.1	Square Numbers	$f(n) = n^2$	97.8
2.3	Triangle Tables	$f(n) = n + 2$	83.3
2.2	Rectangular Numbers	$f(n) = n(n + 1)$	76.5
2.5	Pentagon Tables	$f(n) = 3n + 2$	64.8
1.7	Perimeter	$f(n) = 4n$	59.0
1.5	Square Tables	$f(n) = 2n + 2$	39.0
1.4	Squares and Triangles	$f(n) = 2n + 2$	38.0
2.4	Hexagon Tables	$f(n) = 4n + 2$	34.6
1.3	Towers	$f(n) = 2^n$	17.4
2.6	Triangular Numbers	$f(n) = \frac{1}{2}n(n + 1)$	14.3
2.7	Handshake	$f(n) = \frac{1}{2}n(n - 1)$	10.6

One surprise in this analysis was that the students wrote explicit rules more often for the one-operation quadratic rule task. However, one explanation may be that the students easily recognized the explicit rules due to their familiarity with the square numbers. In addition, the explicit rule could be described in two ways, either as the output = $(\text{input})^2$ or output = input * input. Other than this, the results were in line with my expectations. Another explanation is that many students from session one attended session two and had already sharpened their reasoning abilities.

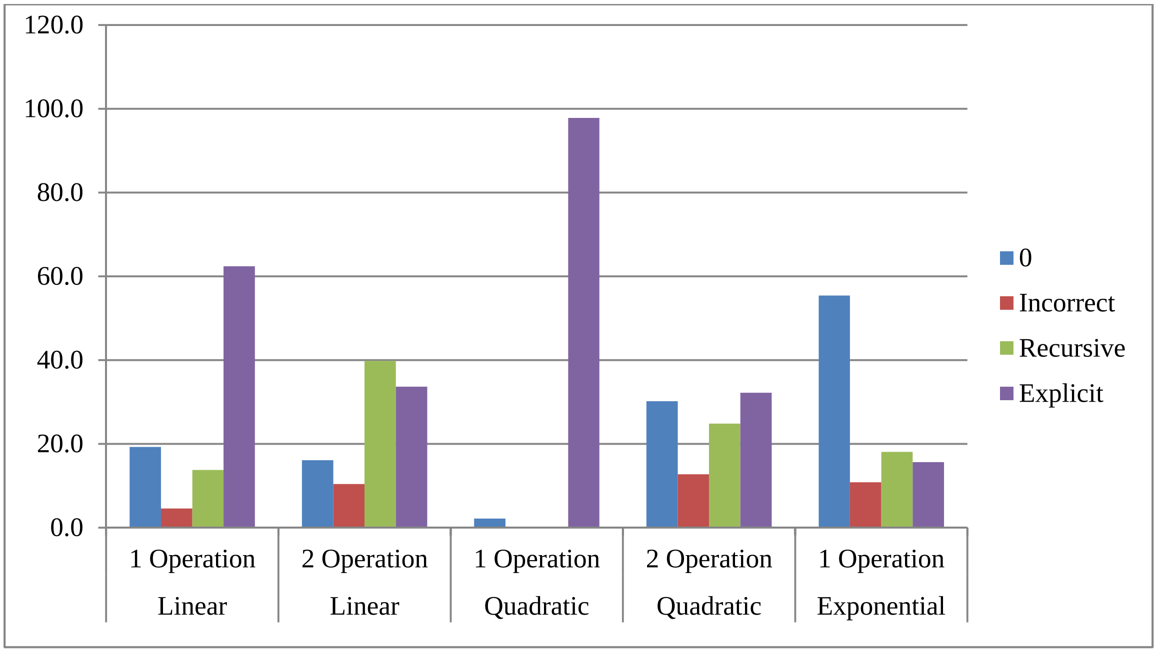
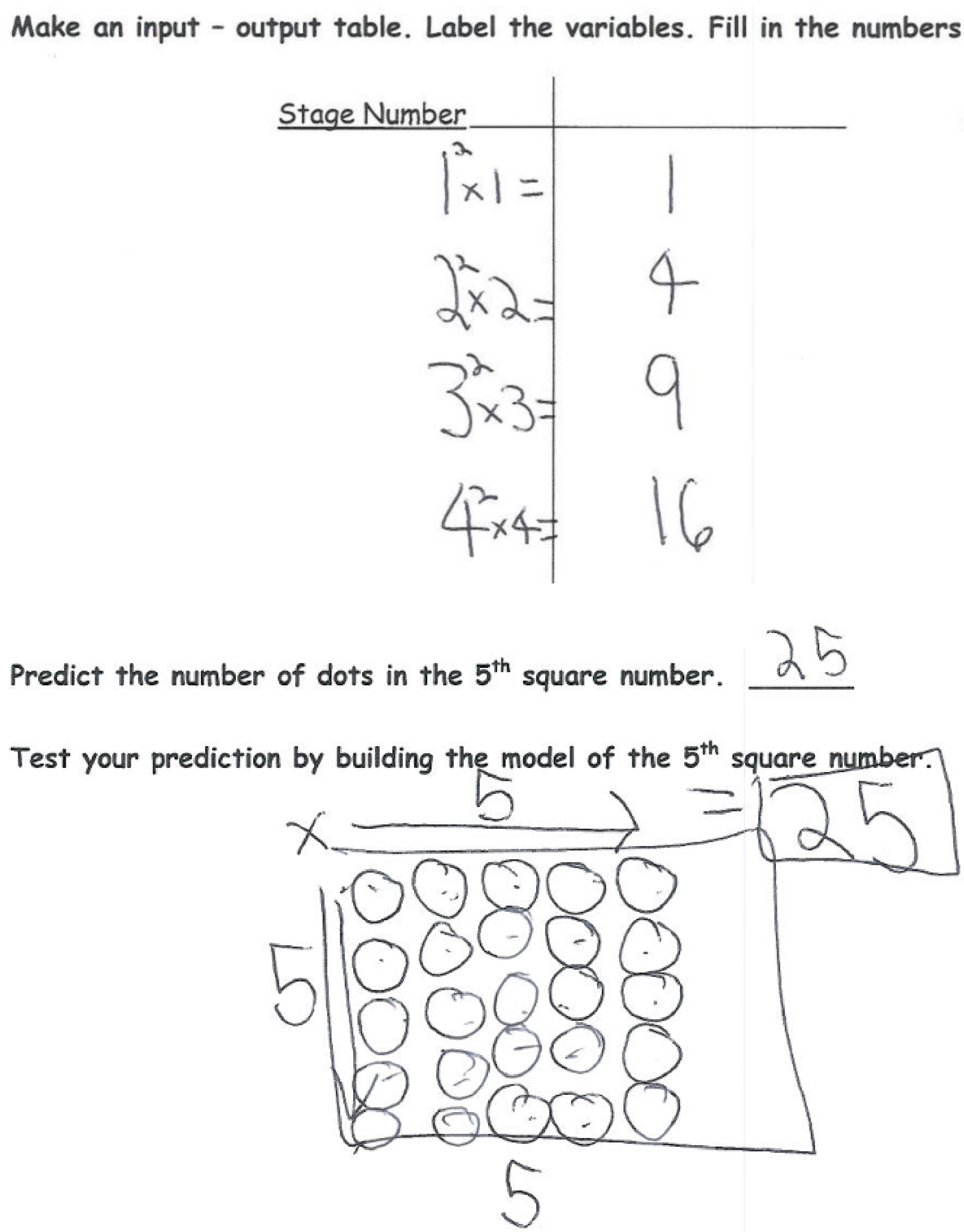


Figure 21. Students' Percentages of Explicit Generalizations by Function Type and Operation

Below is an example of one of the fourth-grade students' work on this task (see Figure 22). In this example, Alex noticed that each output was the product of the input times itself. When asked to draw the model for the fifth square number, Alex's work showed that he is using the array or area model commonly learned in elementary multiplication. He draws a diagram showing the model is five across and five down.



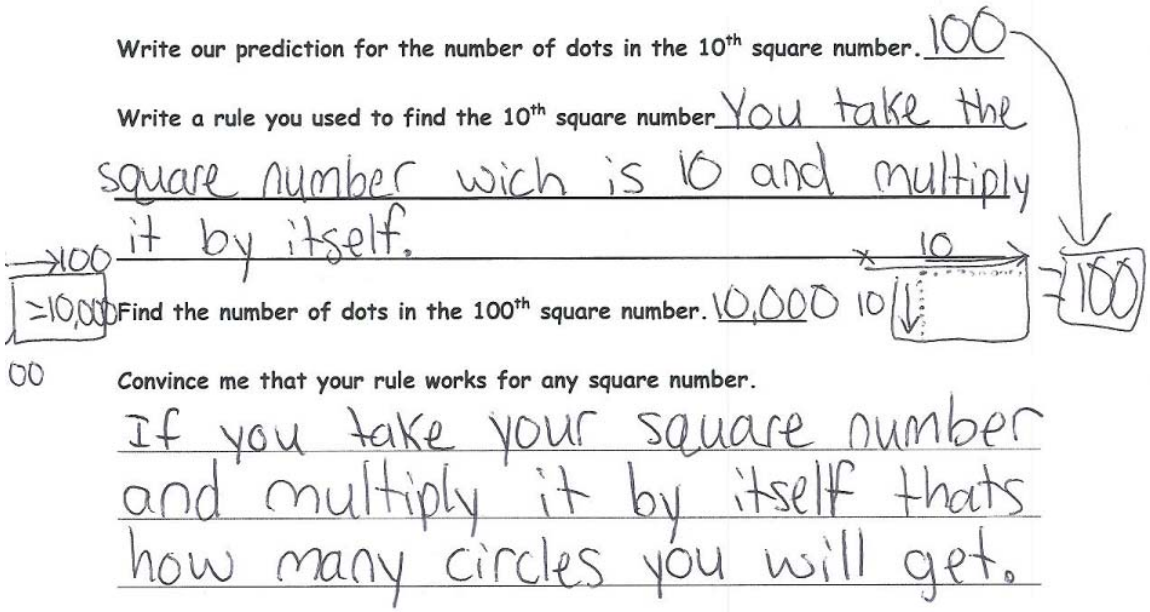
Make an input – output table. Label the variables. Fill in the numbers.

Stage Number	
$1^2 \times 1 =$	1
$2^2 \times 2 =$	4
$3^2 \times 3 =$	9
$4^2 \times 4 =$	16

Predict the number of dots in the 5th square number. 25

Test your prediction by building the model of the 5th square number.

Figure 22. Alex's Work Sample on the Square Numbers Task



Write our prediction for the number of dots in the 10th square number. 100

Write a rule you used to find the 10th square number You take the square number wich is 10 and multiply it by itself.

→100 ×10 10 ↓ = 100

=10,000 Find the number of dots in the 100th square number. 10,000

Convince me that your rule works for any square number.

If you take your square number and multiply it by itself thats how many circles you will get.

Figure 22 (cont.)

Summary

Students have an inate ability to think and reason about functions. However, they need to have access to tasks that help them to use and develop their reasoning by working on open ended problems and generating data for two quanties that are related. In order to understand functions, students need to first understand how these data vary with respect to each other. By looking for patterns in the data, they begin to make this conection. Even if they they find a recurisive rule, this is still considered a stepping stone to the development of functional reasoning. With encouragement the students will attempt to find explicit rules.

After completing the analysis, the following major themes emerged.

1. *The characteristics of the tasks assisted in the development of functional reasoning.*

2. *The function rules of the tasks should increase in complexity in terms of type and number of operations.*
3. *The complexity of the functional reasoning tasks may affect whether students use correspondence or variational reasoning.*
4. *Students in grades three through five are capable of reasoning about functions, but grade level is a consideration when introducing functional reasoning tasks.*

CHAPTER V

DISCUSSION

Introduction

I began this journey to develop ideas about young children's ability to reason about functions and to study the conditions that promote their growth of understanding. The On Track Learn Math afterschool enrichment program gave me an opportunity to observe students as they worked on open-ended functional reasoning tasks. During this program, students had both the time and opportunity to work on problems with which they had no prior experience. Admittedly, it is difficult to get students to sign up for an extra hour of math afterschool but the students who attended did so enthusiastically. Not only did they enjoy working on these problems, they were eager to share their results and ideas with the other students. Some students were so engrossed in finding a solution to their problems at the end of the daily session they asked if they could take their work home with them.

Summary

The development of reasoning is a critical component of students' success in mathematics (NCTM, 2000). In particular, students' lack of functional reasoning skills hinders their advancement in mathematics. Many researchers advocate introducing reasoning about functions in early elementary school (Blanton & Kaput, 2004, Carraher et al., 2008; Lannin, 2005; Martinez & Brizuela, 2006; Warren, 2005a, 2005b; Warren &

Cooper, 2008a, 2008b). However, these researchers primary focus is on students' solution strategies when reasoning about functions. The conclusions drawn from these studies agree that students are capable of reasoning about functions in early elementary school. However, little research looks at the nature of specific functional reasoning tasks.

The purpose of my study was to take an in depth look at materials that supports students' development of functional reasoning. This research was unique because I analyzed 11 different functional reasoning tasks use in an afterschool enrichment program and developed a theory about their role in students' understanding of functions. In this case, the students' solution strategies served to validate the effectiveness of the task.

During the course of this investigation, I collected and analyzed data concerning the On Track tasks. I sought to answer five research questions using two phases of analysis. They were:

1. What are the functional reasoning characteristics of the On Track mathematical tasks?
2. What evidence suggests that the On Track tasks supported the development of students' functional reasoning among students of varying age and ability levels?
3. Which tasks appear to promote students' functional reasoning?
4. Does grade level have an effect on students' functional reasoning?
5. What types of function rules appear to promote students' correspondence reasoning?

Findings and Interpretations

The authors of the conceptual framework for this study found that students begin to develop their functional reasoning skills in early elementary schools when tasks include specific characteristics. The results of my analysis determined that the On Track tasks contained many of these characteristics. In addition, I identified several additional factors of the tasks that supported students' development of functional reasoning, the most important of which is the mathematical structure of the function. This section outlines the major findings of this study, previous research that supports or negates the findings, and implications of the findings to practice.

1. *The characteristics of the tasks assisted in the development of functional reasoning.*

The literature review determined the characteristics of effective tasks used to study functional reasoning. This allowed for the planning and development of the tasks used in the On Track program. We modeled the tasks from previous literature and edited them to meet the needs of the program. Overall, these 10 functional reasoning tasks were a mixture of different geometric representations, function rules, and mathematical structures. My research found that specific characteristics assisted the students to reason about functions and allowed them to generalize function rules successfully.

Overall, this finding is in agreement with the work of Richardson et al. (2010) who observed that connected tasks that have open-ended questions promotes reasoning. Others had similar findings regarding the characteristics related to the demands of the tasks. The use of input/output tables helped students organize their data and look for patterns in the data (Blanton & Kaput, 2004; Warren & Cooper, 2005a, Carraher, Schliemann, & Brizuela, 2000). In fact, Blanton and Kaput (2004) found that third-grade students used input/output charts fluently. My study found similar results, that the use of input/output tables assisted the students in reasoning about data.

Generating data, looking for patterns in data and generalizing rules for these patterns are characteristics of effective functional reasoning tasks. Several researchers agree that these are necessary components in the development of functional reasoning (e.g. Blanton, 2008; Blanton & Kaput, 2004; Lannin, 2005; Lannin et al., 2006; Smith, 2008; Warren, 2005a, 2005b). My findings support the previous studies.

The NCTM Content Standards (2000) included working with patterns as a way of increasing students' algebraic reasoning. They advocated that students in grades three through five should, identify and build numerical and geometric patterns and describe patterns verbally and represent them with tables or symbols. Several studies reported that pattern finding supports students algebraic reasoning and in particular their functional reasoning (Bishop et al., 2001; Blanton & Kaput, 2004; Ferrini-Mundy et al., 2002; Kaput, 2008; NCTM, 2000; Richardson & McGalliard, 2010; Stacey, 1989; Store et al., 2010; Stump, 2011, Warren, 2005b).

Warren (2005b) and Stacey (1989) agree that growth patterns are difficult for students. Stacy found that students had a particularly difficult time with linear patterns in the form $y = mx + b; b \neq 0,$ and students typically found only the constant difference in the outputs. In other words, the students used variational reasoning to generate a recursive rule. Warren commented that repeating patterns were cognitively easier for students but that there was an overemphasis on these types of patterns in the curriculum.

In contrast to this, my findings agree with Moss and McNab (2011) who found that second-grade students found two-operation function rules without using recursive strategies. In fact, for many of the On Track tasks, the students saw the explicit rule without first finding a recursive rule.

Smith (2003) suggested that focusing on the underlying structure of the pattern enhanced students' reasoning. Lannin (2005) also added that multiple patterning tasks should be of similar structure (isomorphic). In a recent study of preservice teachers, Richardson et al. (2009), agreed with Lannin (2005), and found that when the structure of the pattern was the same throughout a series of tasks the students could more easily generalize a rule. Structure played an important role in students' functional reasoning. The students who participated in On Track were encouraged to focus on the structure of the pattern by physically building models with pattern blocks or drawing the figures on their paper. This anchored the students thinking on the physical model and assisted them in finding patterns. There were several isomorphic tasks used in On Track, however not all of the tasks were used in the same session. Therefore, the data is inconclusive as to whether isomorphism was a factor in students' ability to find patterns.

Visual images, according to Lannin et al. (2006), are important in generalizing explicit rules. In their study of fifth-grade students' generalization strategies, these researchers found students' poor visual images led to guess and check strategies, which were often incorrect. However, when the students connected their visual images to the context of the problem they usually wrote an explicit rule. My findings agree that visual images were important in students' generalizations. All of the On Track tasks allowed the students to connect their rules to a visual image. For example, in the polygon table tasks the students used pattern blocks to investigate the nature of the problem. By building the structures, students identified that adding a new table increased the number of people at a constant rate. This led to many students writing recursive rules. Others noticed that the numbers of people who could sit on each of the long sides of the tables were multiples of the number of tables plus two people on each end.

Lastly, my results showed that tasks that encouraged the generalization of rules supported the development of functional reasoning (Kaput, 2008; Lannin, 2005; Lannin et al., 2006; Warren et al., 2006). Generalizations are the heart of functional reasoning. According to Kaput (2008), functional reasoning involves building, describing, and reasoning with and about functions by making generalizations about how data are related.

2. The structure of the function rules of the tasks should increase in complexity.

First, my study found that students' development of functional reasoning depended on the structure of the function rules. The rules were either one or two operation linear, quadratic, or exponential functions. For example, the Triangle Tables Task was a one-operation, linear function rule ($p = t + 2$), the Rectangular Numbers Task was a two-operation quadratic rule ($n = s^2 + s \; or \; n = s(s+1)$), and the Towers Task was a one operation exponential rule ($b = 2^h$). The results revealed that the students found a higher percentage of explicit rules for the one-operation quadratic rules, followed by the one-operation linear rules. Next, the students' percentages of explicit rules were highest on the two-operation linear function rules followed by the two-operation quadratic rules. The most difficult group of tasks, in terms of students writing explicit rules, was the two-operation quadratic rules with fractional coefficients (Triangular Numbers Task and the Handshake Problem) and the one-operation exponential rule (Towers Task). However, even though a lower percentage of students wrote explicit rules for these tasks, they are still valuable in supporting students' reasoning by supporting a range in the development of functional reasoning found among students in grades 3-5.

While there is little research on students' reasoning on complex functional relationships, my findings agree with Blanton (2008) who suggested that functional reasoning tasks should increase in complexity. Combining operations in the function rule increases the complexity of the function, for example, a linear function rule$f(x) = 4x + 2$ is more complex than $f(x) = 4x$. In addition, she suggested beginning with linear relationships, which are the least complex because they grow at a constant rate of change. Quadratic relationships on the other hand, are more complex than linear relationships due to the lack of a constant rate of change. Blanton states that students often rely on guess and check methods to find the rules but that appropriate scaffolding by the teacher can help students see the underlying structure.

In contrast to Blanton (2008), the students in my study actually wrote a higher percentage of explicit rules for the Square Numbers Task, a one-operation quadratic rule. My findings suggested that functional reasoning tasks should begin with one-operation function rules, either linear or quadratic. Since, one-operation multiplicative or additive rules are less complex they help students focus on the relationship between the input and output variable. I also argue that the Square Numbers Task is a good starting point. Since this task had a one-operation quadratic rule the lack of a constant rate of change supports students emerging correspondence reasoning by encouraging students to look across the input/output table.

Carraher, Schliemann, and Brizuela (2000), in a teaching experiment with third-grade students, found that addition and subtraction functions were easier for this population of students. I am also in agreement with this finding. The only task with an additive rule On Track was Triangle Tables Task. The third-grade students' percentage of explicit rules was 100%. In order for students to continue to develop their functional reasoning skills, they must work on tasks that are appropriately challenging. By increasing the complexity of the function rules, it increases the challenge of the task. In addition, students are encouraged to look at the data in multiple ways using both variational and correspondence reasoning. Both recursive and explicit rules are important in developing reasoning. When the students failed to find an explicit rule, they relied on recursive rules instead.

3. *The complexity of the functional reasoning tasks dictates whether students use correspondence or variational reasoning.*

Several studies focused on the use of correspondence and variational reasoning and the role they play in developing functional reasoning. First, Warren (2005a) conducted a teaching experiment with 45 fourth-grade students and found that the students had a tendency to use variational reasoning when searching for patterns in input/output tables. In a later study with third-grade students, Warren and Cooper (2008a) found that single variational reasoning was cognitively easier for this group. However, the Warren/Cooper study only asked students to generalize rules for geometric patterns and students were not encouraged to use input/output tables to organize their data. Smith (2003), while working with fifth-grade students on a growing pattern that generalized the square numbers, concluded that younger students more commonly used a variational approach when generalizing this rule. All of these studies suggested that students' use of variational reasoning over correspondence reasoning depends more on age and developmental factors.

Lannin et al. (2006) studied students' use of both recursive (variational reasoning) and explicit (correspondence reasoning) strategies in generalizing function rules. They found that students' tended to use recursive strategies when the input values were close together and when students used recursion on previous tasks. Explicit rules were used when input values were larger and more spread out, task structure included linear increasing or decreasing relationships and when the students needed a more efficient strategy.

My findings indicated that age and type of function rule are factors in students' generalizations. While the third-grade students' probability of writing explicit rules was lower than the other two grade levels, there is no support for the idea that recursive rules are cognitively easier for younger students. In fact, the third-grade students showed an overwhelming capacity for writing explicit rules especially for the linear function tasks. In addition, over 50% of all students wrote explicit rules for the Square Numbers Task, Rectangular Numbers Task, Triangle Table Task, Perimeter Task, and the Pentagon Table Task. This suggests that for these tasks the majority of students used correspondence reasoning. The students' percentages of recursive rules were higher on the more complex function rules. These tasks included the Triangular Numbers Task, and the Handshake Problem. This suggested that as the students failed to find explicit rules they relied on recursive ones instead. In light of this, I concluded that students' use of recursive and correspondence reasoning depends on the structure of the task.

The previous section linked students' use of recursive or explicit rules to the types of functional reasoning tasks. This section examines the use of variation as supporting correspondence reasoning. According to Smith (2003), a covariational approach to functional reasoning is a tool that leads to correspondence reasoning. Lannin et al. (2006) suggested that students should be encouraged to use multiple reasoning strategies in order to reflect on the advantages and limitations of each. These researchers also stated little research exists on what influences students to use particular types of generalizations. Bezuszka and Kenney (2008) asserted that recursive thinking is valuable to the study of algebra because it helps students think inductively. In addition, when students cannot find a way to proceed with their task, it impedes learning. These views hold that variation reasoning is an important cognitive support for correspondence reasoning.

My research somewhat supported these findings. I noticed that when a task asked students to find an output value for a large input value (e.g. 100) they quickly discovered that their recursive rules were inefficient and began to look for explicit ones. In addition, when the function rules became too complex for students to find explicit rules (such as the Triangular Numbers Task), the students' looked for recursive rules instead. More research is needed in order to say definitively in what ways recursive strategies support correspondence reasoning.

4. *Students in grades three through five are capable of reasoning about functions, but grade level is a consideration when introducing functional reasoning tasks.*

My results agreed with other studies that found that young students are capable of reasoning about functions and the study of functions should begin in early elementary school (Bastable & Schifter, 2008; Blanton & Kaput, 2004; Kaput & Blanton, 2001; Lampert, 2001; Maher & Martino, 1996; Smith, 2003, 2008; Warren, 2005a, 2005b; Warren et al., 2006). In addition, the results did find a significant difference between the third-grade students' ability to generalize rules versus the fourth- and fifth-grade students. This does not mean that third-grade students should not be involved in this type of tasks, but it does mean that teachers should take care in choosing tasks that are more appropriate for the third-grade students. For example, the third-grade students had significantly more problems generalizing explicit rules on the Handshake Problem and the Triangular Numbers Task. Not only were these tasks quadratic function rules, they also included fractions as coefficients. Therefore, it may be more appropriate to use tasks such as these with older students.

The literature also suggested that grade level affects students' use of variational versus correspondence reasoning. Warren (2005a) asserted that single variation was cognitively easier for younger students. In addition, her study supported others that found young children tend to focus on additive strategies, in other words, to look down the column of output values. However, Warren began her study looking at repeating patterns then moved to growing patterns. This focus on repeating patterns first may have affected students' strategies with the growing patterns. Therefore, while Warren's assertion may be true, one should not assume that this occurs in all cases. My results showed that for some tasks, the students moved quickly from recursive rules to explicit rules. This was due in part to asking them to find the output for the 100^{th} input. In fact, on many of the tasks, the students in my study wrote a higher percentage of explicit rules than recursive rules.

Limitations

Attendance

The first limitation I identified was student attendance. Not all of the students attended both sessions. Sixty-three students attended the first session. Thirty-one students attended the second session. Thirty-eight students attended both sessions. Since the tasks were not repeated between session one and session two, I cannot say that student attendance across both sessions was a factor in their generalization success. In addition, not all students attended every day so that accounts for some missing data. Since attendance was beyond the scope of this study, I did not compare the results of the 38 students who attended both sessions to those who attended only a single session.

Lack of Prior Research

The second limitation I identified was the lack of prior research on this subject. In my literature review, I pointed out the fact that little research has focused on types of functional reasoning tasks. In fact, I can only identify two sources where tasks are a major focus in the study (Blanton, 2008; Lannin et al., 2006). Since there was a lack of information on appropriate tasks, I had no other sources with which to compare my results.

Sample Size

The third limitation was an imbalance in sample size. The largest population in the study was the third-grade students. Sixty third-grade students, 47 fourth-grade students, and 32 fifth-grade students participated in the first two sessions of On Track. With unequal sample size, small changes in responses creates larger changes in percentages, which can make comparisons more difficult.

Implications

Student Learning

There are implications from the results of my study that can improve student learning in the area of algebraic reasoning. The NCTM Standards (2000) and the Common Core State Standards (2010) advocate the study of algebra beginning in kindergarten. Since functional reasoning is a subset of algebraic reasoning, it is appropriate to use tasks that help students develop these skills.

My study concludes that the On Track tasks are both effective and appropriate at helping students develop functional reasoning. In addition, the characteristics of the tasks support students' understanding and give them the opportunity to generate data and analyze patterns in the data. In particular, the use of an input/output table allows the students to look at the data in multiple ways. The results also suggest that grade level is a factor in students' generalization of rules. In particular, the third-grade students experienced more difficulty on the complex quadratic rules. Therefore, the introduction of each task should occur based on the mathematical structure of the task and grade level.

Teacher Practice

The results of this study can inform teacher practice in several ways. According to Lannin (2005), the introduction of algebraic reasoning in the curriculum creates a challenge to classroom teachers. Hiebert (2003) claims that despite the research into the value of problem solving strategies, classroom teachers tend to rely on procedural knowledge as the basis for instruction. The On Track program supplies ready-made tasks that can integrate functional reasoning into the existing curriculum of the classroom teachers. In addition, the tasks serve as models for other functional reasoning tasks that would make it easier for teachers to make up their own tasks. In addition, the results appear to acknowledge that the complexity of the function rules with respect to the mathematical structure has implications for teachers in ordering tasks for instruction appropriately.

Curriculum Changes

The findings of this study also have implications for curricular changes in early elementary school mathematics. Functional reasoning is the basis of higher-level mathematics (NCTM, 2000) and it takes years for students' functional reasoning to develop (Warren et al., 2006). By beginning the process early, students have a greater chance of success in mathematics later in school. The characteristics of these tasks allow students to explore them in multiple ways while developing their ideas about data.

Suggestions for Future Research

Sequencing

Future research can determine whether there is an optimum way to order the tasks to maximize student learning and the development of function reasoning. According to Blanton (2008), sequencing of tasks should move from simple to more complex function rules. The focus of this study was not to determine what the optimum order should be, but future research could look at which tasks are more appropriate for each age group.

The Role of Justification

The second suggestion is for future research to examine the ways in which students justify their reasoning. Several studies indicate that justification is an important factor in developing students' algebraic reasoning (Martino & Maher, 1999; Mueller et al., 2009; Store & Berenson, 2010). Even though many of the tasks asked students to justify their rules, few students answered these questions appropriately or not at all. Justification as a characteristic of these tasks is another area for future investigation.

Teacher Practice

How teachers implemented the tasks is beyond the scope of this study, therefore the effects of teacher practice on students' functional reasoning is another area of future inquiry. It is important to know whether the teachers implemented the tasks as planned and in what ways they encouraged students' thinking.

Conclusion

According to Maria Blanton (2008), "building classrooms that develop children's algebraic thinking is about access to ideas . . . It is about all children having the opportunity to learn to think and reason mathematically" (p. 158). When I began this journey, I had no idea how the students would respond to these tasks. In fact, I was skeptical about whether the students would be able to find generalizations for the complex function rules task. However, it did not take long for me to see that not only did the students have a strong capacity to reason about functions but that they had creative ways to reason about these rules. These tasks give concrete ways for teachers to change what they teach and how they teach. The On Track tasks also offer support for students' development of functional reasoning.

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APPENDIX A

ON TRACK TASKS

On Track Task 1.3 (Towers Task)

ON TRACK–LEARN MATH

NAME: ________________________ School: ________________________

Towers Rule Task.

MATERIALS: 24 Unifix Cubes, paper and pencil.

Today we want to find a rule that tells how many different towers you can build with two colors if the towers are 100 blocks tall.

First let's guess how many different towers you can build with two colors that are 100 blocks tall? ____________

The output variable is the number of different towers. What name will you put on the table for this variable? __________

What variable changes the number of towers? ______________

Make an input – output table. Let's label the variables.

Let's start with an easy input, towers that are one block tall. How many different towers can you build with two colors that are one block tall? Fill in your table.

How about towers that are two blocks tall? Three blocks tall? Four blocks tall?

Write your rule to find the number of towers that are 100 blocks tall.

____

____

Use your rule to find how many different towers are 5 blocks tall.
How many? ________

One student in another school used this pattern to find all the 4-block towers. Explain how this student organized the colors.

____

____

B	R	R	B	B	B	R	B	B	R	B	R	B	R	R	R
B	R	B	R	B	B	R	R	B	B	R	B	R	B	R	R
B	R	B	B	R	B	B	R	R	B	B	R	R	R	B	R
B	R	B	B	B	R	B	B	R	R	R	B	R	R	R	B

==

Use paper and pencil to convince me that your rule works for 5 block towers. Label the two colors R and B.

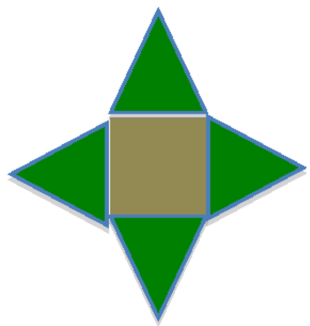
Can you use your rule to find the number of different towers that are 100 blocks tall? ________ How many? ______________

On Track Task 1.4 (Squares and Triangles Task)

ON TRACK-LEARN MATH

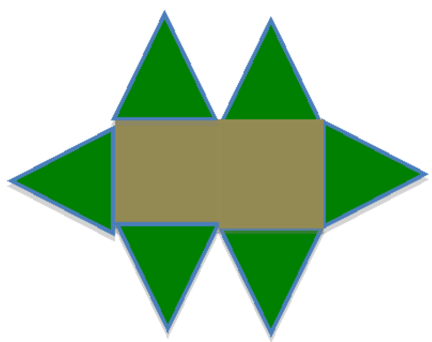
NAME: ____________________________ School: ____________________________

1. If you have one square, how many triangles will fit around the square?



How many triangles? ___________

2. If you put two squares together, how many triangles will fit around the squares?



How many triangles? ___________

3. If you put three squares together, how many triangles will fit around these squares? Draw the model here.

How many triangles? ___________

4. Do you see a pattern yet? If yes, write down a description of your number pattern.

5. Organize your data into an input/output table. Label the input and label the out variables.

6. Use the pattern blocks to build the model of 4 squares joined together. How many triangles will fit around these 4 squares? __________ Fill in your input/output table with these data. Draw a sketch below of 4 squares below to check your answer.

7. Do you see a pattern yet? If yes, write down a rule that describes how you find the number of triangles for any number of squares that are joined together.

8. Will your rule help you find how many triangles you could fit around a 10 squares that are joined together? _______ What answer did you get when you used your rule? _________ How could you convince me that your rule is correct?

9. Use your rule to find how many triangles you need to fit around a 100 squares that are joined together. Number of triangles? ___________

10. How could you convince me that your rule works?

On Track Task 1.5 (Square Tables Task)

ON TRACK-LEARN MATH

NAME: ____________________________ School: ____________________________

1. If you have one square table, how many chairs will fit around the table if you have one chair on each side of the square? __________

How many chairs? ______

2. If you put two square tables together, how many chairs will fit around the new rectangular table? ____________

How many chairs? ______

3. If you put three square tables together, how many chairs will fit around this new table? ________________

How many chairs? ______

4. Do you see a pattern yet? If yes, write down a description of your number pattern.

5. Organize your data into an input/output table. Label the input variable and the output variable.

6. Sketch a model of a 4-square table below. How many chairs will fit around a 4-square table? Fill in your input/output table with these data.

7. How many chairs do you think will fit around a 5-square table? ________ Sketch a model of a 5-square table below. Was your answer correct? _________ Fill in your table with these data.

8. Do you see a pattern yet? If yes, write down a rule that describes your number pattern.

9. Will your rule help you find how many chairs you could fit around a 10-block table? _____What answer did you get when you used your rule? _____ How could you convince me that your rule is correct?

10. Use your rule to find how many chairs you could fit around a 100-block table. _______

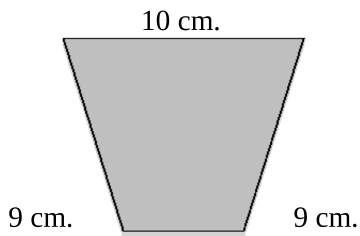
On Track Task 1.7 (The Perimeter Task)

ON TRACK-LEARN MATH

NAME: ____________________________ School: ____________________________

TASK: PERIMETER REVIEW

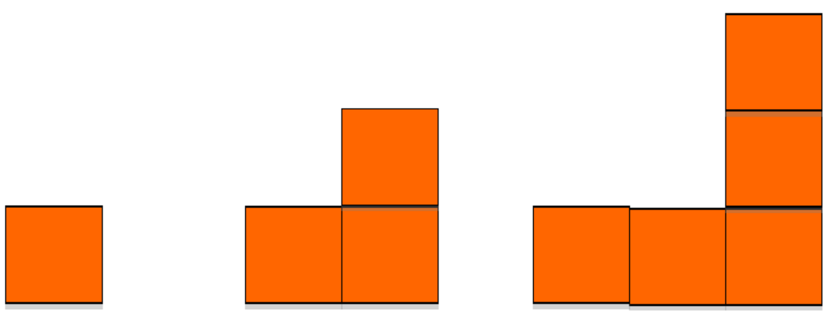
1. Find the perimeter of this figure?



2. Write the rule you used to find the perimeter.

TASK: SQUARE SERIES AND PERIMETER

This is the pattern of the first three stages.



Stage 1 Stage 2 Stage 3

3. Build Stage 4 of this pattern next to stage 3.

4. How would you describe this pattern to a friend? ________
____

5. If one side of the square is 1 inch, what is the perimeter of stage 1 figure? ______Stage 2 figure?_____ Stage 3 figure? _____ Stage 4 figure______?

6. Make an input – output table. Label the variables. What could you label the input variable? The output variable?

7. What rule could help you predict the perimeter of Stage 5?

____

8. Use your rule to find the perimeter of Stage 5. __P = ?
9. Test your rule by drawing Stage 5.

10. Find the perimeter of stage 10. ___________
11. Find the perimeter of stage 100. ___________
12. How could you convince me that your rule works?

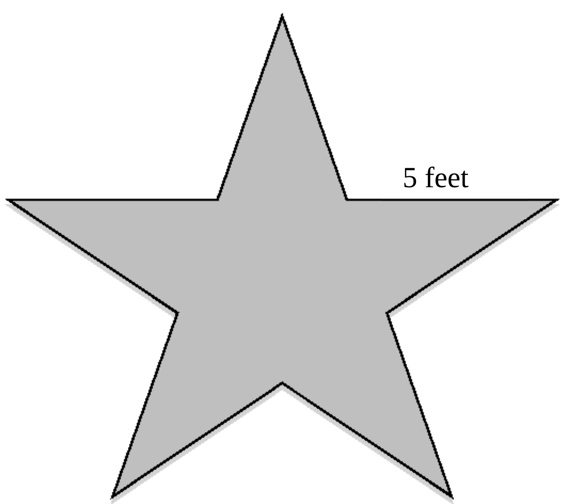
Use perimeter in a sentence. ___________________________________

GENERALIZING A PERIMETER RULE

Write a rule to find the perimeter of a shape.

How can you convince us that your rule works?

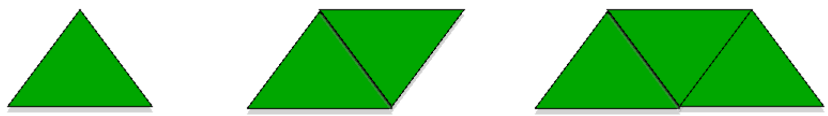
Each side of the star measures 5 feet. What is the perimeter of the star? Explain your thinking.



On Track Task 2.3 (The Triangle Tables Task)

NAME____________________SCHOOL________________________GRADE____________

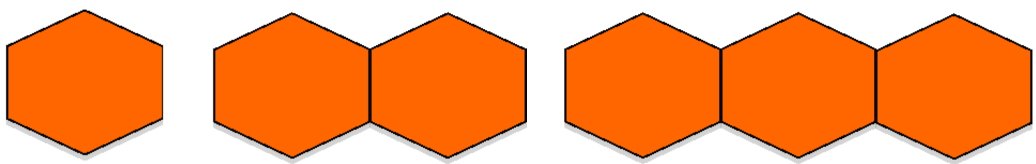
This is pattern block triangle train of 1, 2, and 3 triangle train tables. One person can sit on each side of the table. How many people can sit at a 10 triangle train table? How many people can sit at a 100 triangle train table?



On Track Task 2.4 (Hexagon Tables Task)

NAME____________________SCHOOL________________________GRADE____________

Suppose you make a train of hexagon tables. If one person can sit on one side of the hexagon, how many people can sit at 1 hexagon table? A train of two hexagon tables? A train of three hexagon tables?



How many people can sit around 10 hexagon tables that are put together? 100 hexagon tables?

On Track Task 2.5 (Pentagon Tables Task)

NAME____________________SCHOOL________________________GRADE____________

Suppose you make a train of pentagon tables. If one person can sit on one side of the pentagon, how many people can sit at 1 pentagon table? A train of two pentagon tables? A train of three pentagon tables?

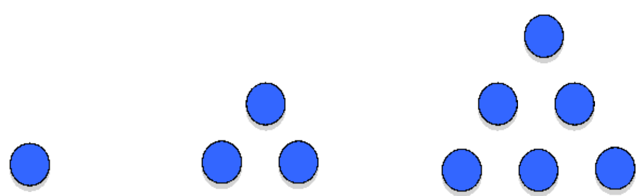
How many people can sit around a train of 10 pentagon tables that are put together? How many people can sit around a train of 100 pentagon tables?

On Track Task 2.6 (Triangular Numbers Task)

NAME____________________SCHOOL________________________GRADE____________

This pattern is called the triangular numbers. The first, second and third triangular numbers are shown below. Draw the 4th triangular number on your paper.

Write down your prediction for the number of dots in the 10th triangular number. The 100th triangular number?



1 2 3 4

On Track Task 2.7 (Handshake Problem)

NAME____________________SCHOOL________________________GRADE____________

Suppose there is a party. Everyone at the party shakes hands once with all the other people at the party. How many handshakes are there when 1 person is at the party? When 2 people are at the party? When 3 people are at the party? When 4 people are at the party?

Find a rule that will tell you how many handshakes when 10 people are at the party. When 100 people are at the party.