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The focus of this thesis is to study positive solutions for classes of nonlinear reaction diffusion equations and systems. In particular, we consider three focuses. In Focus 1, we establish existence and uniqueness of positive solutions for a class of infinite semipositone problems with nonlinear boundary conditions. In Focus 2, we explore the consequences of fragmentation and trait-mediated dispersal on the coexistence of a system of two mutualists by employing a model built upon the reaction diffusion framework. We establish several coexistence and nonexistence results. Finally, in Focus 3, we develop and analyze a radial finite difference method that directly approximate solutions for classes of semipositone problems with Dirichlet boundary conditions.

Our existence results in Focus 1 and Focus 2 are achieved by methods of sub and supersolutions. In Focus 3, via computational methods, we obtain bifurcation diagrams describing the structure of positive solutions. Namely, we obtain these bifurcation diagrams via a modified finite difference method and MATLAB computations in the case when the domain is the unit disc in $\mathbb{R}^2$.

This dissertation aims to significantly enrich the mathematical and computational analysis literature on reaction diffusion equations and systems.

ANALYSIS OF POSITIVE SOLUTIONS FOR CLASSES OF NONLINEAR REACTION DIFFUSION EQUATIONS AND SYSTEMS

by

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Approved by

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Committee Chair

To my wife.

APPROVAL PAGE

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CHAPTER I

INTRODUCTION

Nonlinear steady state reaction diffusion equations play a crucial role in numerous applications such as chemical reactor theory, nonlinear heat generation, combustion theory and population dynamics (see [KC67], [CL70], [Par74], [BIS81], [Sem35], [Ske51], [Tur52], [Par61], [Ari69], [FK69], [Sat75], [Fif79], [KJD$^+$79], [Tam79], [ZBLM85], [OL01], [CC03] and [Mur03]). The time dependent models that arise are of the form:

$$\begin{cases} u_t = d\Delta u + f(u);\ x \in \Omega_0,\ t > 0, \\ u(x,0) = u_0(x);\ x \in \Omega_0, \\ Bu \equiv u = 0;\ x \in \partial\Omega_0,\ t > 0 \ \text{ or } \ Bu \equiv \dfrac{\partial u}{\partial \eta} + c(u)u = 0;\ x \in \partial\Omega_0,\ t > 0, \end{cases} \tag{1.1}$$

where $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u, $d > 0$ is the diffusion coefficient, $\Omega_0 \subset \mathbb{R}^N$ with $N > 1$, is a bounded domain with smooth boundary $\partial\Omega_0$ or $\Omega_0 = (0,1)$, the reaction term $f : (0,\infty) \to \mathbb{R}$ and $c : [0,\infty) \to (0,\infty)$ are continuous functions, and $\frac{\partial u}{\partial \eta}$ is the outward normal derivative of u. In the applications mentioned above, u describes a temperature distribution, a mass concentration or a population density, and in these cases, only non-negative solutions ($u \geq 0$ in $\overline{\Omega}_0$) are relevant. The steady states of (1.1) (if they exist) are of great importance in understanding the dynamics of the solutions of (1.1). For the case when $u = 0$; $x \in \partial\Omega_0$ (Dirichlet or hostile boundary condition), mathematicians have developed a rich literature, namely for

nonlinear elliptic boundary value problems of the form:

$$\begin{cases} -\Delta u = \lambda f(u); \ x \in \Omega_0, \\ \\ u = 0; \ x \in \partial\Omega_0, \end{cases} \tag{1.2}$$

where $\lambda = \frac{1}{d}$ is a positive parameter. When f is <u>*posi*</u>tive and mono<u>*tone*</u>, (1.2) is referred to in the literature as a "positone" problem. Classical examples arise in the theory of nonlinear heat generation (see [KC67] where the authors study the reaction term $f(u) = e^u$) and combustion theory (see [BIS81] where the authors study the reaction term $f(u) = e^{\frac{\alpha u}{\alpha+u}}; \alpha > 0$). For positone problems, we refer the reader to [KC67], [CL70], [Rab71], [Ama72], [CR73], [Par74], [CR75], [Ama76], [GNN79], [WL79], [BIS81], [Shi83], [CS84], [Ang85], [Dan86], [Rab86], [Shi87] and [Lio82]. In the case $f(0) = 0$, for each $\lambda > 0$, $u \equiv 0$ is a solution, and the study of a branch of positive solutions bifurcating from this branch of trivial solutions, and the global behavior of such a bifurcation branch, have been of great importance in applications such as population dynamics (see [OS98], [KS01], [DS06b], [GLS11], [OSS02], [CC06], [GMRS18], [CC07], [CFG$^+$19], [CC03], [Mur03] and [SS06]). In the case $f(0) < 0$ and eventually positive, (1.2) is referred to in the literature as a "semipositone" problem. Additionally, when $\lim_{s\to 0+} f(s) = -\infty$, (1.2) is referred to as an "inifinite semipositone" problem. Such boundary value problems arise in applications such as population dynamics with constant yield harvesting (see [OSS02]). The study of positive solutions of semipositone problems is considerably more challenging, since ranges of positive solutions must include regions where f is negative as well as where f is positive. For results related to positive solutions of semipositone and infinite semipositone problems, we refer the reader to [BS82], [CS88], [CS89a],

[BCS89], [CS89b], [CS93], [CHS95a], [AHS96], [CCS00], [LSY09b], [LSY09a] [ACS93], [AAB94], [CGS93], [CGS00], [CSS12], [CS98] [KRS15], [LSY10], [RSY07], [GILSS13], [HSS12], [CHS19] and [CHS20].

In this thesis we enrich these literatures. Namely, we consider the following focuses:

Focus 1: A uniqueness result for a class of infinite semipositone problems with nonlinear boundary conditions.

Focus 2: Modeling the effects of trait-mediated dispersal on coexistence of mutualists.

Focus 3: Radial finite difference methods for approximating solutions of sublinear semipositone problems in a ball.

1.1 Focus 1: A uniqueness result for a class of infinite semipositone problems with nonlinear boundary conditions

We study boundary value problems of the form:

$$\begin{cases} -u'' = \lambda h(t) f(u) \ ; \ (0,1) \\ \qquad u(0) = 0 \\ u'(1) + c(u(1))u(1) = 0, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter, $h \in C^1((0,1],(0,\infty))$ is a decreasing function, $f \in C^1((0,\infty),\mathbb{R})$ is an increasing concave function such that $\lim_{s\to\infty} f(s) = \infty$, $\lim_{s\to\infty} \frac{f(s)}{s} = 0$, $\lim_{s\to 0^+} f(s) = -\infty$ (infinite semipositone) and $c \in C([0,\infty),(0,\infty))$ is an increasing function. For classes of such h and f, we establish the existence and uniqueness of positive solutions for $\lambda \gg 1$.

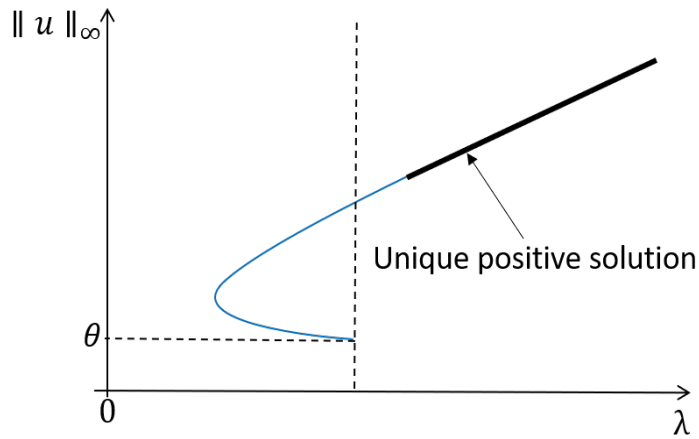


Figure 1. Bifurcation diagram

Boundary value problems of the form (1.3) with the assumption $\lim_{s\to 0^+} f(s) = -\infty$ are referred to in the literature as infinite semipositone problems (when $f(0)$ exists and is negative are referred to as semipositone problems). It is well documented (see [BCN96], [Lio82]) that the study of positive solutions to semipositone problems are considerably harder than in the case $\lim_{s\to 0^+} f(s) \geq 0$. In particular, uniqueness results are even more challenging. For uniqueness results in the semipositone non-singular case ($f(0) < 0$ and finite) see [CSS12], [KRS15]. Namely, in [CSS12] the Dirichlet boundary condition case $u(0) = 0 = u(1)$ was analysed, while in [KRS15] the nonlinear boundary condition as in (1.3) was discussed. However, for the more challenging infinite semipositone case, to date, there are only two earlier results in the literature. Namely, one for a two-point boundary value problem on $(0, 1)$ with Dirichlet boundary conditions $u(0) = 0 = u(1)$ (see [CHS19]), and the other for a study of radial solutions in the unit ball with Dirichlet boundary conditions which reduces to the study of a two-point boundary problem on $(0, 1)$ with boundary conditions $u'(0) = 0 = u(1)$ (see [CHS20]). For related existence and uniqueness results

with Dirichlet boundary conditions see [CHS95b], [CP89], [CRT77], [Dan86], [DS06a], [DRT77], [DH01], [GW94], [Hai11] and [HKS06].

<u>EXISTENCE FOR $\lambda \gg 1$</u>

We first establish the existence of a (maximal) positive solution for $\lambda \gg 1$ without assuming c is bounded (see Theorem 1.3). (For an existence result of positive solutions to (1.3) for $\lambda \gg 1$ when c is bounded, see [LSS16].)

We first introduce some hypotheses that we plan to use for establishing our results. For some $l > 0, \alpha > 0, \gamma > 0$ such that $\alpha + \gamma < 1$, f, h and c satisfy:

(H_1) $h : (0,1] \to (0,\infty)$ is C^1, decreasing and $\limsup_{s\to 0^+} s^\gamma h(s) < \infty$.

(H_2) $f : (0,\infty) \to \mathbb{R}$ in C^1, increasing, concave, $\lim_{s\to\infty} f(s) = \infty$, $\lim_{s\to\infty} \frac{f(s)}{s} = 0$, $\lim_{s\to 0^+} f(s) = -\infty$ (infinite semipositone) and $\lim_{s\to 0^+} s^{1+\alpha} f'(s) = l$.

(H_3) $c : [0,\infty) \to (0,\infty)$ is continuous and increasing.

We will establish Theorems 1.1-1.2 stated below, and use them to establish Theorem 1.3.

Consider the boundary value problem:

$$\begin{cases} -u'' = \bar{h} \ ; \ (0,1) \\ u(0) = 0 = u'(1) + c(u(1))u(1). \end{cases} \tag{1.4}$$

Theorem 1.1. *Let $\bar{h} \in L^1(0,1)$. Then (1.4) has a unique solution. Further, $T : L^1(0,1) \to C[0,1]$ such that $u = T(\bar{h})$ is the solution of (1.4), is a completely continuous operator.*

Next, we establish an existence result via a sub and supersolution method to the boundary value problem

$$\begin{cases} -u'' = g(t,u) \ ; \ (0,1) \\ u(0) = 0 = u'(1) + c(u(1))u(1) \end{cases} \tag{1.5}$$

when c satisfies (H_3), and when the following hypotheses hold:

(G_1) $g : (0,1) \times (0,\infty) \to \mathbb{R}$ is continuous and $\exists\ \Phi(\text{supersolution}), \Psi(\text{subsolution})$ $\in C^2(0,1) \cap C^1[0,1]$ with

$\Psi \leq \Phi;\ (0,1),\ \inf\limits_{(0,1)} \frac{\Psi}{p} > 0$ where $p(t) = \min(t, 1-t)$ such that

$$-\Psi'' \leq g(t,\Psi) \ ; \ (0,1)$$
$$\Psi(0) = 0, \Psi'(1) + c(\Psi(1))\Psi(1) \leq 0,$$

and

$$-\Phi'' \geq g(t,\Phi) \ ; \ (0,1)$$
$$\Phi(0) \geq 0, \Phi'(1) + c(\Phi(1))\Phi(1) \geq 0.$$

(G_2) $\exists\ \Gamma \in L^1(0,1)$ such that for all $\zeta \in [\Psi, \Phi]$, $|g(t,\zeta(t))| \leq \Gamma(t);\ (0,1)$.

Theorem 1.2. *Let* $(G_1) - (G_2)$ *hold. Then* (1.5) *has a positive solution* $u \in [\Psi, \Phi]$.

Now, we consider (1.3) when for some $l > 0, \alpha > 0, \gamma > 0$ such that $\alpha + \gamma < 1$, h and f satisfy:

$(\tilde{H}_1)$ $h : (0,1] \to (0,\infty)$ is $L^1(0,1)$ and $\limsup\limits_{s \to 0^+} s^\gamma h(s) < \infty$.

$(\tilde{H}_2)$ $f:(0,\infty)\to\mathbb{R}$ is C^1, $\lim\limits_{s\to\infty} f(s)=\infty$, $\lim\limits_{s\to\infty}\frac{f(s)}{s}=0$, $\lim\limits_{s\to 0^+} f(s)=-\infty$ and $\lim\limits_{s\to 0^+} s^{1+\alpha}f'(s)=l$.

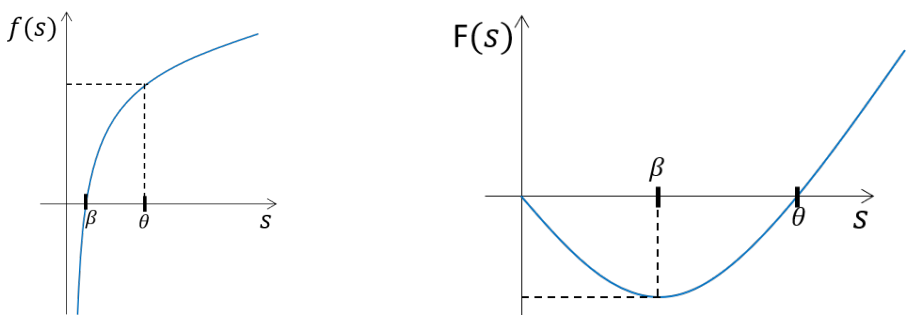
We establish:

Theorem 1.3. *Let* $(\tilde{H}_1)-(\tilde{H}_2)$ *hold. Then* (1.3) *has a maximal positive solution for* $\lambda\gg 1$.

<u>UNIQUENESS FOR $\lambda\gg 1$</u>

Next, we state Lemmas 1.4-1.12 which will be used to prove the uniqueness of positive solutions of (1.3) for $\lambda\gg 1$ (Theorem 1.13).

Let $F(s)=\int_0^s f(t)dt$. (H_2) implies that $\exists\ \beta,\theta$ such that $0<\beta<\theta$, $f(s)(s-\beta)>0$ for $s\neq\beta$ and $F(s)(s-\theta)>0$ for $s\neq\theta$.



(a) Graph of f (b) Graph of F

Figure 2. Graphs of f and F

Lemma 1.4. *Let* u *be a positive solution of* (1.3). *Then* $\exists\ t_0\in(0,1)$ *such that* u *is increasing on* $[0,t_0]$, *decreasing on* $[t_0,1]$, *and* $u(t_0)>\theta$.

For each $\sigma\in(0,\theta]$, let t_σ be the smallest positive number in $(0,1]$ such that $u(t_\sigma)=\sigma$. Let $\rho=\frac{\beta+\theta}{2}$.

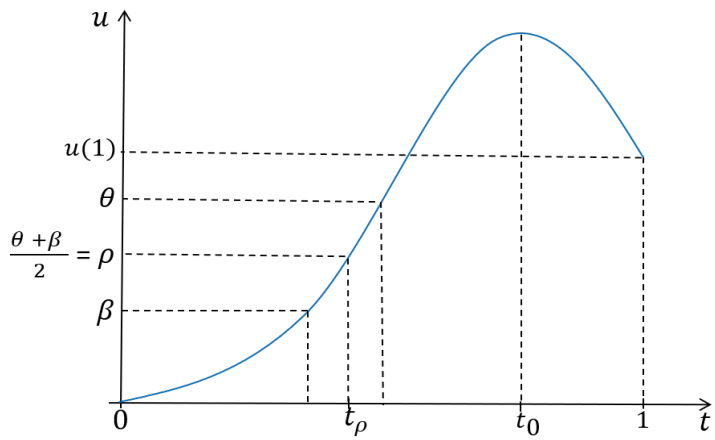


Figure 3. Graph of u

Lemma 1.5. *Let u be a positive solution of* (1.3)*. Then $\exists$ a constant $C > 0$ (independent of u) such that*

$$u(t) \geq C\lambda^{\frac{1}{1+\alpha}} \Big(\int_0^t \sqrt{h(s)} ds \Big)^{\frac{2}{1+\alpha}} \ ; \ [0, t_\rho].$$

In particular, $t_\rho \to 0$ as $\lambda \to \infty$.

Lemma 1.6. *Let u be a positive solution of* (1.3)*. Then $u(1) > \rho$ for $\lambda \gg 1$.*

Next, let $h_0(t) = \frac{h(t)}{\left(\int_0^t \sqrt{h(s)} ds\right)^{\frac{2\alpha}{1+\alpha}}}$ and $m = \inf_{[\rho,\infty)} f = f(\rho)$. Our hypotheses imply that $h_0 \in L^1$.

Lemma 1.7. *Let $K_0 > 0$ and $\bar{u}$ satisfy:*

$$-\bar{u}'' \geq \begin{cases} -\epsilon h_0(t) \ ; & (0, \frac{1}{4}] \\ mh(t) \ ; & (\frac{1}{4}, 1) \end{cases}$$

$$\bar{u}(0) \geq 0, \ \bar{u}'(1) + K_0 \bar{u}(1) \geq 0,$$

where $m_0 = \frac{m}{2}\int_{\frac{1}{4}}^{1} s(1-s)h(s)ds$ *and* $0 < \epsilon < m_0(\int_0^{\frac{1}{4}} h_0(s)ds)^{-1}$. *Then*

$$\bar{u}(t) \geq m_0 t \ ; \ [0, \tfrac{1}{4}].$$

Lemma 1.8. *Let* u *be a positive solution of* (1.3). *Then* $u(1) \to \infty$ *as* $\lambda \to \infty$.

Lemma 1.9. *Let* u *be a positive solution of* (1.3). *Then* $\inf_{[\frac{1}{4},1]} u \to \infty$ *as* $\lambda \to \infty$.

Now we state the Lemma 1.10 where we will obtain a sharp lower bounded for the solution u (see the Figure 4) when $\lambda \gg 1$. This plays a crucial role in the proof of the uniqueness result.

Lemma 1.10. *Let* $L > 0$. *Then* $\exists\ \lambda_0 > 0$ *such that for* $\lambda > \lambda_0$ *any positive solution* u *of* (1.3) *satisfies:*

$$u(t) \geq P_\lambda(t) = \begin{cases} \lambda L t \ ; \ [0, \frac{1}{4}] \\ \quad L \ ; \ (\frac{1}{4}, 1]. \end{cases}$$

Lemma 1.11. $\exists\ K_1 > 0, B > 0$ *such that* $f(s) - sf'(s) \geq K_1 - \dfrac{B}{s^\alpha}$ *for* $s > 0$.

Lemma 1.12. *Let* $L > 0$ *be such that*

$$\frac{B}{L^\alpha}\left\{ \int_0^{\frac{1}{4}} \frac{h(t)}{t^\alpha} dt + \int_{\frac{1}{4}}^{1} h(t)dt \right\} < \frac{b}{2}$$

where $b > 0$ *is such that* $z_1(t) \geq bp(t)$, z_1 *is the solution of*

$$-z_1'' = h(t)K_1 \ ; \ (0,1)$$
$$z_1(0) = 0 = z_1(1),$$

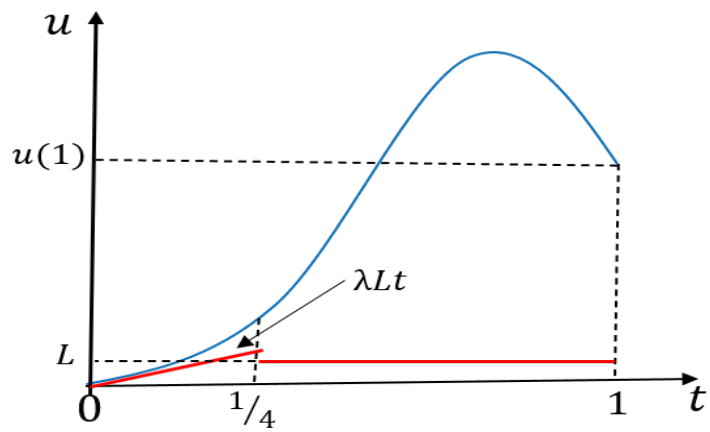


Figure 4. Graph of u with lower bound $P_\lambda(t)$

and $p(t) = \min\{t, 1-t\}$. *Let* $\lambda > \lambda_0$, *where* λ_0 *is defined as in Lemma* 1.10. *Let* z *be the solution of:*

$$-z'' = h(t)\Big[K_1 - \tfrac{B}{(P_\lambda(t))^\alpha}\Big] \ ; \ (0,1)$$
$$z(0) = 0 = z(1).$$

Then $z(t) \geq \frac{b}{2}p(t) \ ; \ [0,1]$.

Theorem 1.13. *Let* $(H_1) - (H_3)$ *hold. Then* (1.3) *has a unique positive solution for* $\lambda \gg 1$.

1.2 Focus 2: Modeling the effects of trait-mediated dispersal on coexistence of mutualists

Modelling aspect.

We study a system built upon the reaction diffusion framework which will model interactions of two mutualistic species that obey logistic growth laws and exhibit

trait-mediated dispersal. Reaction diffusion models have been used extensively in the literature, see [Lev74], [Lev81], [Fif79], [Oku81], [Mur03], [CC03], [HLV94] and references therein for a detailed history of the framework. In this model, $(u(t,x), v(t,x))$ represents the normalized density (i.e. carrying capacity is equal to one) of two different populations inhabiting the patch $\Omega_0 = \{lx \mid x \in \Omega\}$ with patch size $l > 0, \Omega \subset \mathbb{R}^n$ having unit measure (e.g. if $n = 2$ then the area of Ω is one) and smooth boundary, and $n = 1, 2, 3$. The patch is surrounded by a hostile matrix, denoted by $\Omega_M = \mathbb{R}^n \backslash \overline{\Omega}_0$. We also denote the boundary of Ω_0 by $\partial\Omega_0$. Here, the variable t represents time and x represents spatial location within the patch. The model is then:

$$\begin{cases} u_t = D_1 \Delta u + r_1 u(1-u); \ t > 0, x \in \Omega_0 \\ v_t = D_2 \Delta v + r_2 v(1-v); \ t > 0, x \in \Omega_0 \\ u(0,x) = u_0(x); \ x \in \Omega_0 \\ v(0,x) = v_0(x); \ \in \Omega_0 \\ D_1 \alpha_1(v) \frac{\partial u}{\partial \eta} + S_1^*[1-\alpha_1(v)]u = 0; \ t > 0, x \in \partial\Omega_0 \\ D_2 \alpha_2(u) \frac{\partial v}{\partial \eta} + S_2^*[1-\alpha_2(u)]v = 0; \ t > 0, x \in \partial\Omega_0 \end{cases} \tag{1.6}$$

where $D_i > 0$ represents patch diffusion rate, $r_i > 0$ the patch intrinsic growth rate, $u_0(x), v_0(x)$ initial population density distributions in the patch, and α_i the probability of an indivitual remaining in the patch upon reaching the boundary ($i = 1$ for u and $i = 2$ for v). The term $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative operator. Here, the parameter $S_i^* \geq 0$ is a measure of the hostility of the matrix towards the organism, has units of length by time, and can assume different forms depending upon the patch/matrix interface assumptions (see [CGS19]). If $\alpha_i \equiv 0$ then the boundary is absorbing, i.e. all individuals that reach the boundary will emigrate, whereas if

$\alpha_i \equiv 1$ then the boundary is reflecting, i.e. the emigration rate is zero. Also, a given relationship between density and emigration can be included in the model by selecting appropriate α_i's (see for example, [CC07], [CC06], [CGS19], [FGM+20] and [GMPS19]).

Now introducing the scaling

$$\tilde{x} = \frac{x}{l} \ \& \ \tilde{t} = r_1 t$$

dropping the tilde, (1.6) becomes:

$$\begin{cases} u_t = \frac{1}{\lambda}\Delta u + u(1-u); \ t > 0, x \in \Omega \\ v_t = \frac{D_0}{\lambda}\Delta v + r_0 v(1-v); \ t > 0, x \in \Omega \\ u(0,x) = u_0(x); \ x \in \Omega \\ v(0,x) = v_0(x); \ \in \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} g(v) u = 0; \ t > 0, x \in \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} h(u) v = 0; \ t > 0, x \in \partial\Omega \end{cases} \tag{1.7}$$

with corresponding steady state equation:

$$\begin{cases} -\Delta u = \lambda u(1-u); \ \Omega \\ -\Delta v = \lambda r v(1-v); \ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} g(v) u = 0; \ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} h(u) v = 0; \ \partial\Omega \end{cases} \tag{1.8}$$

where $\lambda = \frac{r_1 l^2}{D_1}$, $r_0 = \frac{r_2}{r_1}$, $D_0 = \frac{D_2}{D_1}$, $r = \frac{r_0}{D_0}$, $g(v) = \frac{S_1^*}{\sqrt{r_1 D_1}} \frac{1-\alpha_1(v)}{\alpha_1(v)}$, and $h(u) = \frac{S_2^*}{\sqrt{r_1 D_1 D_0}} \frac{1-\alpha_2(u)}{\alpha_2(u)}$ are all unitless. Also, recall that Ω has length, area, or volume of one.

Thus for fixed r_1, r_2, D_1, D_2, the composite parameter λ is proportional to patch size squared, $g(0)$ represents the effective matrix hostility towards u, and $h(0)$ the effective matrix hostility towards v. Here, we assume that α_1, α_2 are smooth increasing functions of v, u, respectively such that $g, h \in C^1([0,\infty),(0,\infty))$ are decreasing functions of v, u respectively. Also, r can be written as $r = \frac{\frac{r_2}{D_2}}{\frac{r_1}{D_1}}$ and interpreted as a means to compare the two species by their growth to diffusion ratio, defined as the ratio of patch intrinsic growth rate to patch diffusion rate. Thus, there are three cases: 1) if $r = 1$, then both growth to diffusion ratios are the same, 2) if $r > 1$ then v's growth to diffusion ratio is greater than u's, and 3) if $r < 1$ then u's ratio is greater than v's.

Now we discuss some eigenvalue problems for which our coexistence results are built upon, beginning with the eigenvalue problem:

$$\begin{cases} -\Delta z = Ez; \ \Omega \\ \frac{\partial z}{\partial \eta} + Mz = 0; \ \partial\Omega \end{cases} \tag{1.9}$$

for a given $M \in [0,\infty)$. In [RR19], the authors proved that for each $M \in [0,\infty)$, (1.9) has a positive principal eigenvalue $\bar{E}_1(M)$, $\bar{E}_1(0) = 0$, and the eigencurve $\bar{E}_1 : [0,\infty) \to [0,\infty]$ is Lipschitz continuous, strictly increasing and concave (see Figure 5). Furthermore, $\bar{E}_1(M) \to E_1^D$ as $M \to \infty$ where E_1^D is the principal eigenvalue to the Dirichlet eigenvalue problem:

$$\begin{cases} -\Delta z = Ez; \ \Omega \\ z = 0; \ \partial\Omega. \end{cases} \tag{1.10}$$

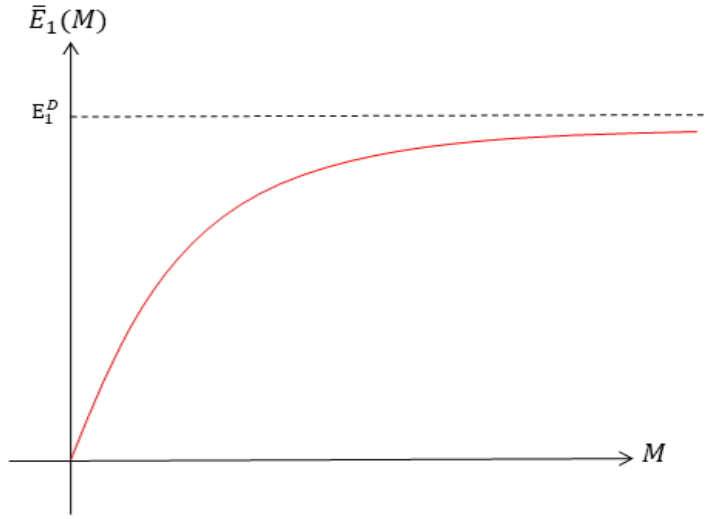


Figure 5. Principal eigencurve $\bar{E}_1(M)$.

Next, we recall the eigenvalue problem discussed in [GMRS18]:

$$\begin{cases} -\Delta z = rEz; \ \Omega \\ \frac{\partial z}{\partial \eta} + K\sqrt{E}z = 0; \ \partial\Omega \end{cases} \tag{1.11}$$

where K is a positive parameter. For fixed r and K, let $E_1(r, K)$ denote the principal eigenvalue of (1.11). See [GMRS18] for the existence of this principal eigenvalue. Further, $rE_1(r, K)$ is the y-coordinate of the intersection of the curves $\bar{E}_1(M)$ and $\frac{M^2}{K^2}$ (see Figure 6). Hence, for a fixed r, it is easy to see that $E_1(r, K)$ is an increasing function of K, $E_1(r, K) \to E_1^D$ as $K \to \infty$, and $E_1(r, K) \to 0$ as $K \to 0$. Also, since the intersection of $\bar{E}_1(M)$ and $\frac{M^2}{K^2}$ is independent of r, we must have

$$rE_1(r, K) = E_1(1, K); \text{ for all } r > 0. \tag{1.12}$$

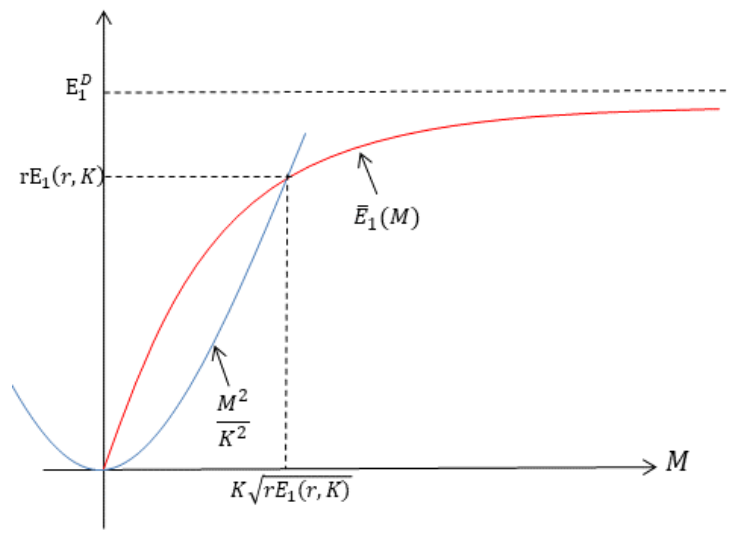


Figure 6. Existence of $E_1(r, K)$.

By (1.12), it is easy to see that for a fixed K, $E_1(r, K)$ is decreasing in r, $E_1(r, K) \to 0$ as $r \to \infty$, and $E_1(r, K) \to \infty$ as $r \to 0$. Furthermore, it is easy to see that the following lemma holds:

Lemma 1.14. *For any K_1, K_2 positive constants there exists $r^* = r^*(K_1, K_2) > 0$ such that $E_1(1, K_1) = E_1(r^*, K_2)$. Moreover, $r^* > 1$ when $K_1 < K_2$, $r^* < 1$ when $K_1 > K_2$, and $r^* = 1$ when $K_1 = K_2$.*

Next, we consider (1.8) in the cases when one population is present and the other is absent, namely:

$$\begin{cases} -\Delta w_1 = \lambda w_1(1 - w_1); \ \Omega \\ \frac{\partial w_1}{\partial \eta} + \sqrt{\lambda} g(0) w_1 = 0; \ \partial\Omega. \end{cases} \tag{1.13}$$

and

$$\begin{cases} -\Delta w_2 = \lambda r w_2(1-w_2);\ \Omega \\ \frac{\partial w_2}{\partial \eta} + \sqrt{\lambda} h(0) w_2 = 0;\ \partial\Omega. \end{cases} \tag{1.14}$$

In other words, (1.13) is the governing steady state equation for the species u in the absence of v and (1.14) is the governing steady state equation for the species v in the absence of u. The solution structure of (1.13) and (1.14) can be completely determined by consideration of the eigenvalue problems:

$$\begin{cases} -\Delta \phi_1 - \lambda \phi_1 = \sigma \phi_1;\ \Omega \\ \frac{\partial \phi_1}{\partial \eta} + \sqrt{\lambda} g(0) \phi_1 = 0;\ \partial\Omega. \end{cases} \tag{1.15}$$

and

$$\begin{cases} -\Delta \phi_2 - \lambda r \phi_2 = \sigma \phi_2;\ \Omega \\ \frac{\partial \phi_2}{\partial \eta} + \sqrt{\lambda} h(0) \phi_2 = 0;\ \partial\Omega. \end{cases} \tag{1.16}$$

Note that (1.15) and (1.16) are linearizations of (1.13) and (1.14) around $(0,0)$ respectively. Let $\sigma_1 = \sigma_1(\lambda, g(0))$ and $\sigma_2 = \sigma_2(\lambda, r, h(0))$ be the principal eigenvalues of (1.15) and (1.16) respectively with corresponding normalized eigenfunctions ϕ_1 and ϕ_2 chosen such that $\phi_1, \phi_2 > 0;\ \overline{\Omega}$.

In [GMRS18], the authors established the following lemma which guarantees the existence of the solutions of (1.13) and (1.14), which we will denote by $\tilde{w}_1$ and $\tilde{w}_2$, respectively.

Lemma 1.15. *We have the following:*

(1) If $\sigma_1 \geq 0$ then $w_1 \equiv 0$ is the only nonnegative solution of (1.13) *and is globally asymptotically stable.*

(2) If $\sigma_1 < 0$ then the trivial solution of (1.13) *is unstable. Furthermore, there exists a unique positive asymptotically stable solution $\tilde{w}_1$ of* (1.13).

(3) If $\sigma_2 \geq 0$ then $w_2 \equiv 0$ is the only nonnegative solution of (1.14) *and is globally asymptotically stable.*

(4) If $\sigma_2 < 0$ then the trivial solution of (1.14) *is unstable. Furthermore, there exists a unique positive asymptotically stable solution $\tilde{w}_2$ of* (1.14).

Moreover, $\tilde{w}_i \leq 1;\ \bar{\Omega}$ for $i = 1, 2$.

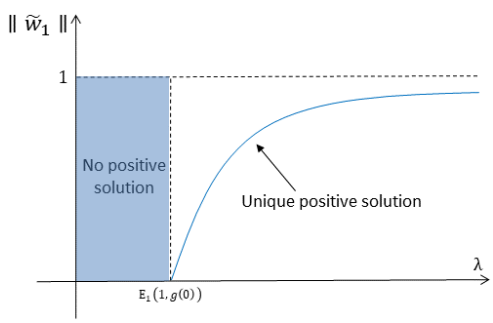


Figure 7. Bifurcation diagram of $\tilde{w}_1$.

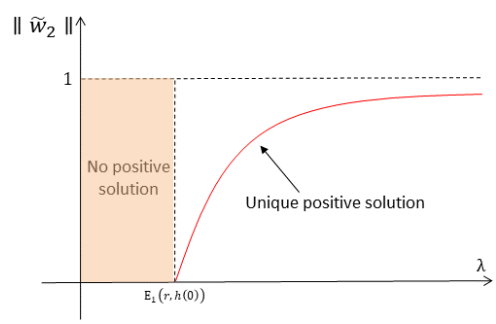


Figure 8. Bifurcation diagram of $\tilde{w}_2$.

Finally, we consider the eigenvalue problems:

$$\begin{cases} -\Delta\phi_3 - \lambda r\phi_3 = \sigma\phi_3;\ \Omega \\ \frac{\partial\phi_3}{\partial\eta} + \sqrt{\lambda}h(\tilde{w}_1)\phi_3 = 0;\ \partial\Omega. \end{cases} \tag{1.17}$$

and

$$\begin{cases} -\Delta\phi_4 - \lambda\phi_4 = \sigma\phi_4; \ \Omega \\ \frac{\partial\phi_4}{\partial\eta} + \sqrt{\lambda}g(\tilde{w}_2)\phi_4 = 0; \ \partial\Omega. \end{cases} \tag{1.18}$$

Note that (1.17) and (1.18) are linearizations of (1.8) about $(\tilde{w}_1, 0)$ and $(0, \tilde{w}_2)$ respectively. Let $\sigma_3 = \sigma_3(\lambda, r, h(\tilde{w}_1))$, $\sigma_4 = \sigma_4(\lambda, g(\tilde{w}_2))$ be the principal eigenvalues and ϕ_3, $\phi_4 > 0$; $\overline{\Omega}$ be the corresponding normalized eigenfunction of (1.17) and (1.18), respectively.

We establish Theorems 1.16-1.19 stated below.

Theorem 1.16. *Let $r > 0$ be given and define $\lambda^* = \max\{E_1(1, g(1)), E_1(r, h(1))\}$ and $\lambda^{**} = \max\{E_1(1, g(0)), E_1(r, h(0))\}$. Then the following hold:*

(a) If $\lambda < \lambda^*$ then (1.8) has no positive solutions, i.e coexistence is not possible.

(b) If $\lambda > \lambda^{**}$ then (1.8) has at least one positive solution. All positive solutions, (u^*, v^*), of (1.8) satisfy $(u^*, v^*) > (\tilde{w}_1, \tilde{w}_2); \Omega$. Furthermore, any solution of (1.7), (u, v), with small positive initial density will satisfy $(u(t, x), v(t, x)) > (\tilde{w}_1(x), \tilde{w}_2(x)); t > t_1, x \in \overline{\Omega}$, for some $t_1 > 0$ that depends on the initial density.

Remark. It is possible for λ^* to be greater than either $E_1(1, g(0))$ or $E_1(r, h(0))$, giving rise to a situation in which one organism could invade and survive in the patch regardless of the presence of the other species, but the patch size is still too small to foster coexistence. For simplicity of presentation, we only display the case that λ^* is less than both $E_1(1, g(0))$ and $E_1(r, h(0))$.

We now provide sufficient conditions, split into three cases depending on the comparison of $E_1(1, g(0))$ and $E_1(r, h(0))$, that will guarantee coexistence in the patch.

Following these results, we provide a table depicting the portions of the parameter space corresponding to each of the three cases.

Theorem 1.17. *Let $g(0), h(0)$ and r be fixed such that $E_1(1, g(0)) = E_1(r, h(0))$. Then*

(a) $\sigma_1, \sigma_2 \geq 0$ *for* $\lambda \leq E(1, g(0)$

(b) $\sigma_1, \sigma_2, \sigma_3, \sigma_4 < 0$ *for* $\lambda > E_1(1, g(0))$

(c) $\lambda^* < E_1(1, g(0))(= E_1(r, h(0)))$

A pictorial description of the case when $E_1(1, g(0)) = E_1(r, h(0))$ is given in Figure 10, and a bifurcation diagram showing one possibility for the complete solution structure of (1.8) is given in Figure 9.

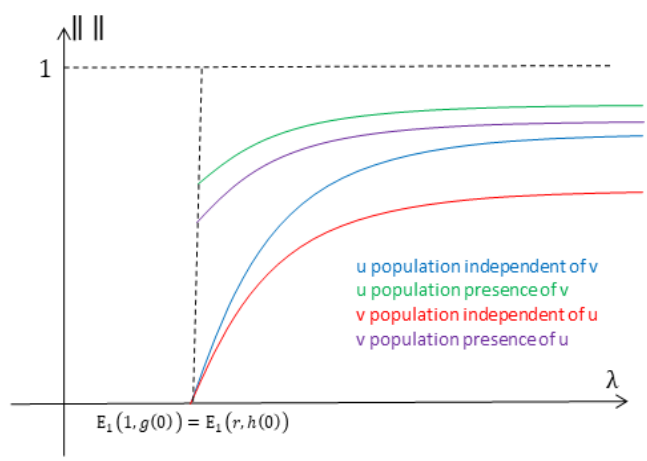


Figure 9. Example of a bifurcation diagram for the case $E_1(1, g(0)) = E_1(r, h(0))$

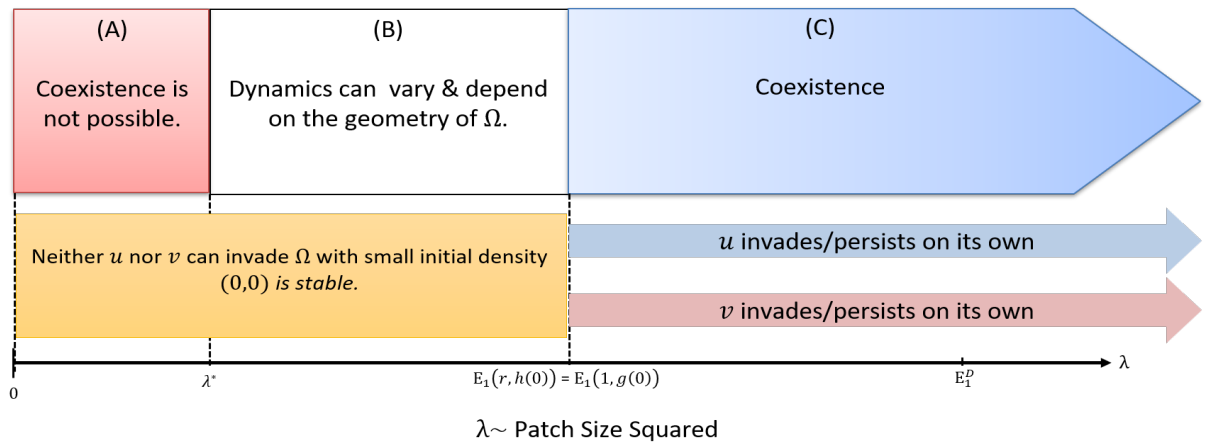


Figure 10. Existence vs λ when $E_1(1, g(0)) = E_1(r, h(0))$

Theorem 1.18. *Let $g(0), h(0)$ and r be fixed such that $E_1(r, h(0)) < E_1(1, g(0))$. Then there exist $\delta_1 = \delta_1(g(0), h(0), r)(> 0)$ and $\delta_2 = \delta_2(g(0), h(0), r)(> 0)$ such that*

(a) $\sigma_1 \geq 0$ for $\lambda \leq E_1(1, g(0))$ and $\sigma_1 < 0$ for $\lambda > E_1(1, g(0))$,

(b) $\sigma_2 \geq 0$ for $\lambda \leq E_1(r, h(0))$ and $\sigma_2 < 0$ for $\lambda > E_1(r, h(0)$,

(c) $\sigma_3 < 0$ for $\lambda > E_1(1, g(0))$,

(d) $\sigma_4 > 0$ for $\lambda \in (E_1(r, h(0)), E_1(r, h(0)) + \delta_1)$ and $\sigma_4 < 0$ for $\lambda > E_1(1, g(0)) - \delta_2$.

Moreover, for $\lambda \in (E_1(1, g(0)) - \delta_2, E_1(1, g(0)))$, (1.8) has a positive solution and there exists $\epsilon_1 > 0$ such that any solution of (1.7) with $(u(0, x)), v(0, x)) \geq (\approx)(\epsilon\phi_4(x), \tilde{w}_2(x)); \Omega$ and $\epsilon < \epsilon_1$ will converge to a positive coexistence steady state of (1.7) as $t \to \infty$ (see region (E) in Figure 11).

A pictorial description in the case when $E_1(1, g(0)) > E_1(r, h(0))$ is given in Figure 11, and a bifurcation diagram showing one possibility for the complete solution structure of (1.8) in Figure 12.

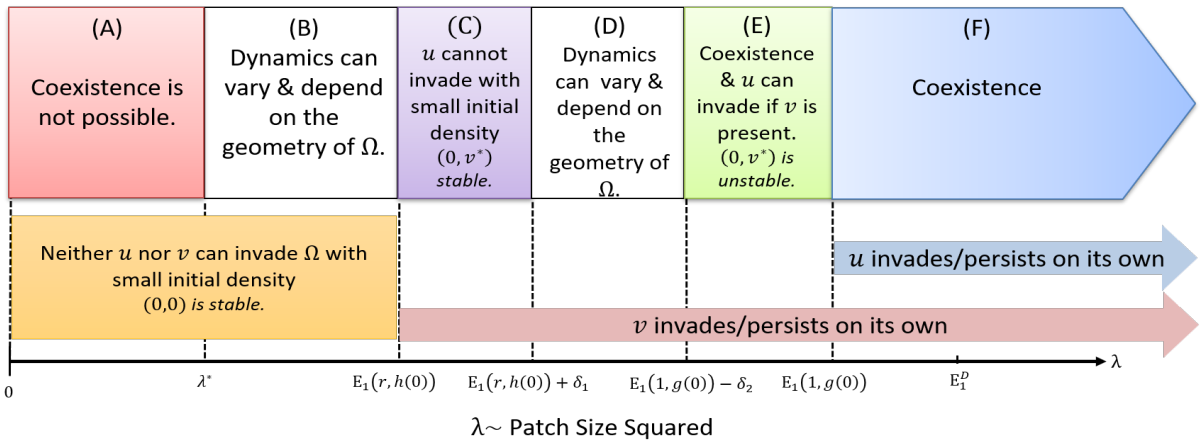


Figure 11. Existence vs λ when $E_1(1, g(0)) > E_1(r, h(0))$

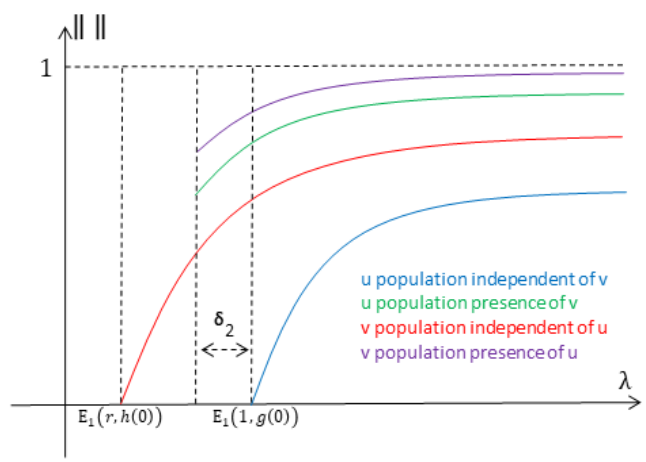


Figure 12. Example of a bifurcation diagram for the case $E_1(1, g(0)) > E_1(r, h(0))$

Theorem 1.19. *Let $g(0)$, $h(0)$ and r be fixed such that $E_1(1, g(0)) < E_1(r, h(0))$. Then there exist $\delta_1 = \delta_1(g(0), h(0), r)(> 0)$ and $\delta_2 = \delta_2(g(0), h(0), r)(> 0)$ such that*

(a) $\sigma_1 \geq 0$ for $\lambda \leq E_1(1, g(0))$ and $\sigma_1 < 0$ for $\lambda > E_1(1, g(0))$,

(b) $\sigma_2 \geq 0$ for $\lambda \leq E_1(r, h(0))$ and $\sigma_2 < 0$ for $\lambda > E_1(r, h(0)$,

(c) $\sigma_4 < 0$ *for* $\lambda > E_1(r, h(0))$,

(d) $\sigma_3 > 0$ *for* $\lambda \in (E_1(1, g(0)), E_1(1, g(0)) + \delta_1)$ *and* $\sigma_3 < 0$ *for* $\lambda > E_1(r, h(0)) - \delta_2$.

Moreover, for $\lambda > (E_1(r, h(0)) - \delta_2, E_1(r, h(0)))$, (1.8) *has a positive solution and there exists* $\epsilon_1 > 0$ *such that any solution of* (1.7) *with* $(u(0, x), v(0, x)) \geq (\approx)(\tilde{w}_1(x), \epsilon\phi_3(x)); \Omega$ *and* $\epsilon < \epsilon_1$ *will converge to a coexistence steady state of* (1.7) *as* $t \to \infty$ *(see region (E) in Figure* 13*)* .

A pictorial description in the case when $E_1(1, g(0)) < E_1(r, h(0))$ is given in Figure 13, and a bifurcation diagram showing one possiblity for the complete solution structure of (1.8) is given in Figure 14.

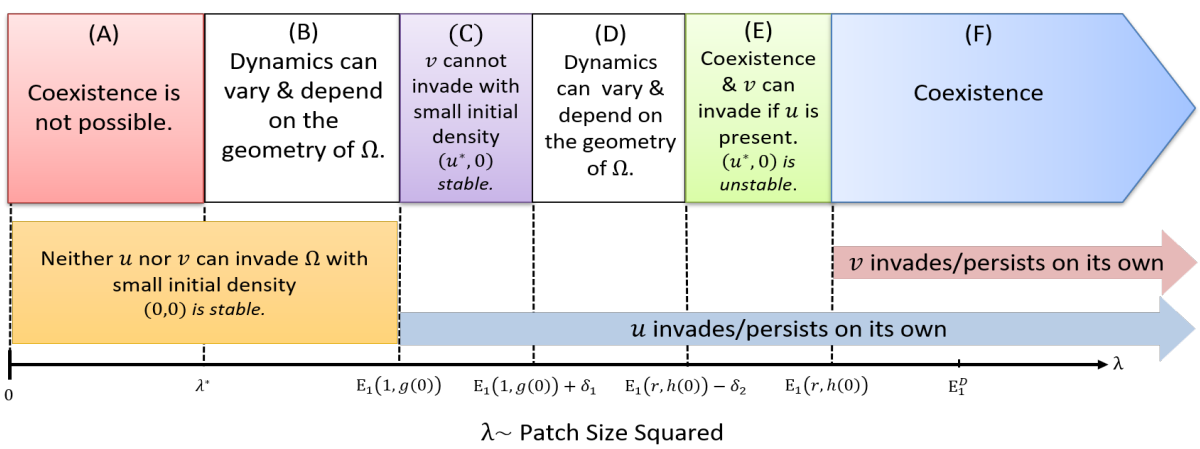


Figure 13. Existence vs λ when $E_1(1, g(0)) < E_1(r, h(0))$

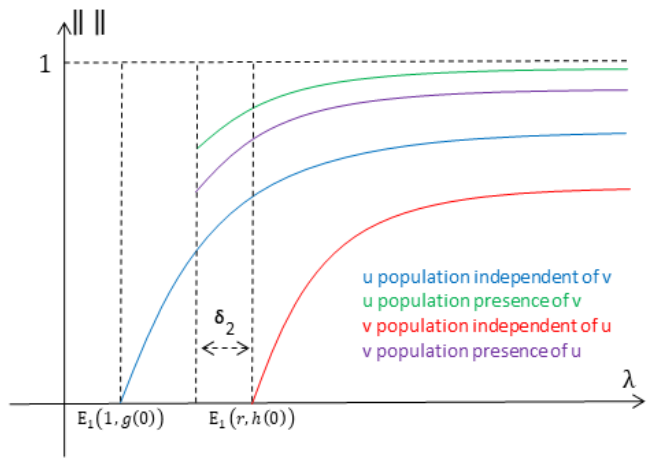


Figure 14. Example of a bifurcation diagram for the case $E_1(1, g(0)) < E_1(r, h(0))$

Finally, we present a table (see Figure 15) to help depict the portions of the parameter space corresponding to each of the three cases. It is easy to see that each of the three cases discussed previously are possible regardless of the difference in patch intrinsic growth rate and diffusion rate between the species. The main determining factor here appears to be the effective matrix hostility as represented in $g(0)$ for u and $h(0)$ for v.

BIOLOGICAL INTERPRETATION

In each of the three cases of comparison for $E_1(1, g(0))$ and $E_1(r, h(0))$, the model predicts coexistence for patch sizes large enough such that $\lambda > \lambda^{**}$ (see (C) in Figure 10, for example). This is due to the fact that sufficiently large patches have a core region large enough to ensure persistence, even in the presence of mortality from interacting with the hostile matrix at the patch boundary. Note that exactly the opposite effect is seen for patches so small that their λ-value falls below λ^* (see (A) in

<table>
<tr><th>$r = 1$</th><th>$r < 1$</th><th>$r > 1$</th></tr>
<tr><td>For $g(0) = h(0)$
$E_1(1, g(0)) = E_1(r, h(0))$</td><td>For $g(0) = h(0)$
$E_1(1, g(0)) < E_1(r, h(0))$</td><td>For $g(0) = h(0)$
$E_1(1, g(0)) > E_1(r, h(0))$</td></tr>
<tr><td rowspan="2">For $g(0) > h(0)$
$E_1(1, g(0)) > E_1(r, h(0))$</td><td>For $g(0) \neq h(0)$
There exists a $g^* > 0$, such that:</td><td>For $g(0) \neq h(0)$
There exists a $h^* > 0$, such that:</td></tr>
<tr><td>If $g(0) = g^*$, then
$E_1(1, g(0)) = E_1(r, h(0))$</td><td>If $h(0) = h^*$, then
$E_1(1, g(0)) = E_1(r, h(0))$</td></tr>
<tr><td rowspan="2">For $g(0) < h(0)$
$E_1(1, g(0)) < E_1(r, h(0))$</td><td>If $g(0) < g^*$, then
$E_1(1, g(0)) < E_1(r, h(0))$</td><td>If $h(0) < h^*$, then
$E_1(1, g(0)) > E_1(r, h(0))$</td></tr>
<tr><td>If $g(0) > g^*$, then
$E_1(1, g(0)) > E_1(r, h(0))$</td><td>If $h(0) > h^*$, then
$E_1(1, g(0)) < E_1(r, h(0))$</td></tr>
</table>

Figure 15. Depiction of the portions of the parameter space corresponding to each of the three cases for r.

Figure 10, for example), in that the model predicts extinction regardless of the initial population density.

In the case that $E_1(1, g(0)) = E_1(r, h(0))$, there is a stark change in model outcomes as λ drops below $E_1(1, g(0))$ (see (B) and (C) in Figure 10), with the model predicting that neither species is able to invade the patch with small positive initial density. Since $\lambda^* < E_1(1, g(0))$, a situation can arise where coexistence is still possible for patch sizes with corresponding λ-values in $(\lambda^*, E_1(1, g(0)))$. In that case, coexistence is due directly to the effect of trait-mediated dispersal since presence of the other organism causes a decrease in emigration out of the patch, i.e. we have a situation with dispersal-mediated coexistence.

Since the remaining two cases are symmetric, we will only give a biological interpretation for the case when $E_1(1, g(0)) > E_1(r, h(0))$. As previously noted, a mutualistic system can fall into this case regardless of which species is better able to invade and persist in the patch. In fact, all that is required is that v's effective matrix

hostility is sufficiently small compared with u's effective matrix hostility. For patch sizes corresponding to $\lambda \in (E_1(1, g(0)) - \delta_2, E_1(1, g(0)))$ (see (E) in Figure 11), the model predicts that, on its own, u is unable to successfully invade the patch, whereas v is able to invade the patch with small initial positive density independent of u. Most interestingly, the model predicts that u <u>is</u> able to invade with small initial population density and coexist in the patch as long as v is already established in the patch and near its steady state (independent of u), i.e. dispersal-mediated coexistence. The mechanism directly responsible for this outcome is trait-mediated dispersal. In other words, the presence of v in the patch decreases u's emigration rate low enough that u is able to persist in the patch. We also note that in this case, the role of coexistence facilitator can be reversed by increasing the facilitator's effective matrix hostility or decreasing the other species hostility. This reversal is in the same spirit as the one observed in a competitive system studied in [CCF98] and [CCL04], where changing the matrix hostility caused a competitive dominance reversal.

For patch sizes with $\lambda \in (\lambda^*, E_1(r, h(0))$ or $\lambda \in (E_1(r, h(0)) + \delta_1, E_1(1, g(0)) - \delta_2)$ (see (B) or (D) in Figure 11), our methods are not able to give general results regarding model predictions. In other words, the dynamics can vary greatly depending on the patch geometry and parameter values. Patch sizes giving a λ in $(E_1(r, h(0)), E_1(r, h(0)) + \delta_1)$ (see (C) in Figure 11) yield a model prediction that even with v being established in the patch is not enough to allow u to invade the patch and persist. However, our results do not exclude the possibility of dispersal-mediated coexistence for λ in $(E_1(r, h(0)), E_1(r, h(0)) + \delta_1)$ and certain ranges of r, $g(0)$, and $h(0)$. In the case of trait-mediated coexistence, the model would predict a type of Allee effect for λ in this range, as the semitrivial steady state $(0, v^*)$ and a positive co-

existence steady state would both be stable, giving rise to a dispersal-mediated Allee effect. Finally, for patch sizes corresponding to $\lambda \in (\lambda^*, E_1(r, h(0)))$ (see (B) in Figure 11), the model outcome is the same as in the case when $E_1(1, g(0)) = E_1(r, h(0))$.

1.3 Focus 3: Radial finite difference methods for approximating solutions of sublinear semipositone problems in a ball.

Here, we study problems of the form:

$$\begin{cases} -\Delta u = \lambda h(x,y) f(u) \ ; & \Omega, \\ u(0) = 0 \ ; & \partial\Omega, \end{cases} \tag{1.19}$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ (pictured in Figure 16),

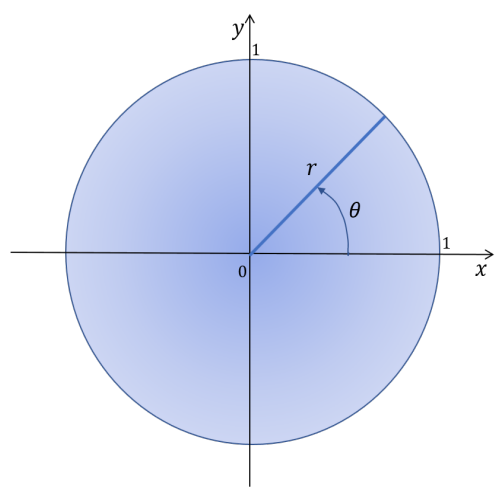


Figure 16. Domain Ω

$\lambda > 0$ (parameter), h is a positive bounded continuous weight function, and $f : [0, \infty) \to \mathbb{R}$ is a C^2 nondecreasing function such that $f(0) < 0$ (semipositone), $f''(0) <$

0, and $\lim_{s\to\infty} \frac{f(s)}{s} = 0$ (sublinear at ∞). Known analytical results when $h \equiv 1$ (see [ACS93] and [AAB94]) establish the existence of a unique positive radial solution for $\lambda \gg 1$ suggesting the following $(\lambda, \|u\|_\infty)$ bifurcation curve for positive radial solutions pictured in Figure 17.

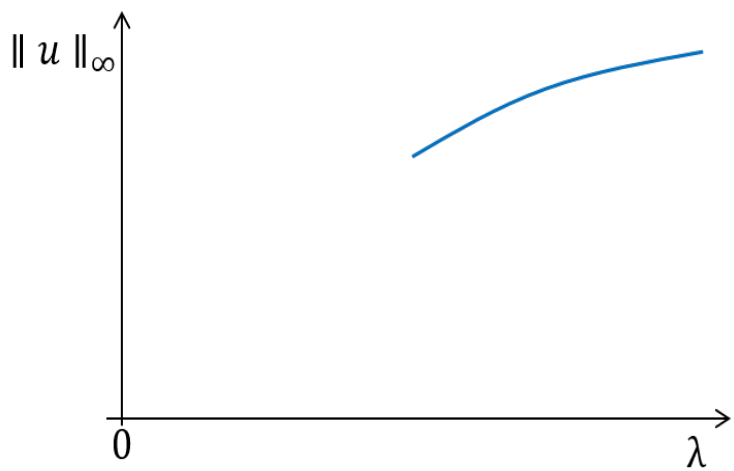


Figure 17. Bifurcation diagram corresponding to the results in [ACS93] and [AAB94]

Here, via computational methods, for various examples, we obtain accurate approximate bifurcation diagrams for positive solutions. In particular, we will analyze whether the bifurcation diagrams are as described in Figure 18, featuring a region with two positive solutions.

We now formulate a finite difference (FD) method for approximating (1.19). We develop and analyze FD methods that directly approximate solutions to (1.19) without any additional assumptions such as radial symmetry. We also work directly in polar coordinates on the disc since many of the theoritical results assume a smooth boundary. To this end, we discretize the polar form of the Laplacian operator given

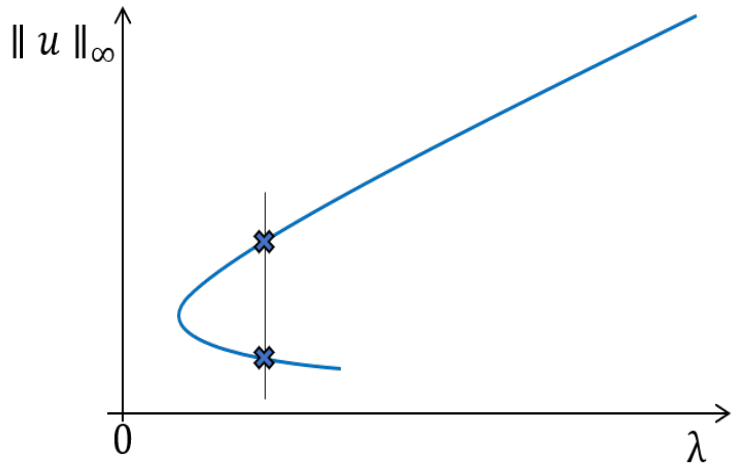


Figure 18. Approximate bifurcation diagram for (1.19)

by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

where $0 < r \leq 1$ and $0 \leq \theta \leq 2\pi$. Let $\Omega = \{(r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 \mid 0 \leq r < 1, 0 \leq \theta \leq 2\pi\}$. We define FD approximations over grids that are uniformly spaced with respect to the coordinates r and θ. Let N_r and N_θ be positive integers, and let $h_r = \frac{1}{N_r}$ and $h_\theta = \frac{2\pi}{N_\theta}$. Define $N = 1 + N_r N_\theta$, and let the set of index pairs be defined by $\mathbb{N}_N = \{(i,j) \mid 0 \leq i \leq N_r, 0 \leq j \leq N_\theta\}$. Then we define a polar grid $\mathcal{T}_N$ over Ω by $\mathcal{T}_N = \{(r_i, \theta_j) = (ih_r, jh_\theta) \mid (i,j) \in \mathbb{N}_N\}$. By conversion, we only consider grids where N_θ is divisible by 4. This ensures the grid aligns with the xy-axes as seen in Figure 19.

Define the (second order) central difference operators for approximating first order and second order partial derivatives for a function $v : \mathbb{R}^2 \to \mathbb{R}$ with respect to

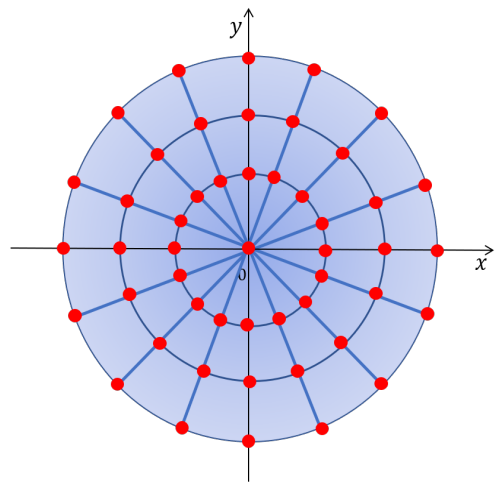


Figure 19. Mesh $\mathcal{T}_{49}$ with $N_r = 3$ and $N_\theta = 16$

r at a point (r_i, θ_j) by

$$\frac{\delta v}{\delta r}(r_i, \theta_j) \equiv \frac{v(r_i + h_r, \theta_j) - v(r_i - h_r, \theta_j)}{2h_r} \tag{1.20}$$

and

$$\frac{\delta^2 v}{\delta r^2}(r_i, \theta_j) \equiv \frac{v(r_i + h_r, \theta_j) - 2v(r_i, \theta_j) + v(r_i - h_r, \theta_j)}{{h_r}^2}, \tag{1.21}$$

respectively. Define the (second order) central difference operator for approximating second order partial derivatives for a function $v : \mathbb{R}^2 \to \mathbb{R}$ with respect to θ at a point (r_i, θ_j) by

$$\frac{\delta^2 v}{\delta \theta^2}(r_i, \theta_j) \equiv \frac{v(r_i, \theta_j + h_\theta) - 2v(r_i, \theta_j) + v(r_i, \theta_j - h_\theta)}{{h_\theta}^2}. \tag{1.22}$$

Define the (second order) central difference operator for approximating second order partial derivatives for a function $v : \mathbb{R}^2 \to \mathbb{R}$ with respect to x and y at the origin by

$$\frac{\delta^2 v}{\delta x^2}(0,0) \equiv \frac{v(h_r,0) - 2v(0,0) + v(h_r,\pi)}{{h_r}^2} \tag{1.23}$$

and

$$\frac{\delta^2 v}{\delta y^2}(0,0) \equiv \frac{v(h_r,\frac{\pi}{2}) - 2v(0,0) + v(h_r,\frac{3\pi}{2})}{{h_r}^2}, \tag{1.24}$$

respectively, where we have assumed v is defined in polar coordinates.

Using (1.20), (1.21), (1.22), (1.23) and (1.24), we define the (second order) central discrete Laplacian operator Δ_N at the point $(r_i,\theta_i) \in \mathcal{T}_N \cap \Omega$ by

$$\Delta_N v(r_i,\theta_j) = \begin{cases} \dfrac{\delta^2 v}{\delta r^2}(r_i,\theta_j) + \dfrac{1}{r_i}\dfrac{\delta v}{\delta r}(r_i,\theta_j) + \dfrac{1}{{r_i}^2}\dfrac{\delta^2 v}{\delta \theta^2}(r_i,\theta_j) & \text{if } (r_i,\theta_j) \neq (0,0), \\ \dfrac{\delta^2 v}{\delta x^2}(0,0) + \dfrac{\delta^2 v}{\delta y^2}(0,0) & \text{if } (r_i,\theta_j) = (0,0). \end{cases} \tag{1.25}$$

The FD method for approximating (1.19) is defined as finding a grid function $U : \mathcal{T}_N \to \mathbb{R}$ such that

$$-\Delta_N U_{ij} = \lambda h(r_i,\theta_j) f(U_{ij}) \ ; \ (r_i,\theta_j) \in \mathcal{T}_N \cap \Omega, \tag{1.26}$$

$$U_{ij} = 0 \ ; \qquad (r_i,\theta_j) \in \mathcal{T}_N \cap \partial\Omega, \tag{1.27}$$

where U_{ij} is the approximation for $u(r_i,\theta_j)$ and h is defined in polar coordinates. We seek FD approximations such that $U_{ij} > 0$ for all $(r_i,\theta_j) \in \mathcal{T}_N \cap \Omega$.

We establish Theorems 1.20-1.24 in which we prove the existence of a FD approximation:

Theorem 1.20. *Let $V : \mathcal{T}_N \to \mathbb{R}$ and suppose $-\Delta_N V_{ij} \leq 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \Omega$ and $V_{ij} \leq 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \partial\Omega$. Then $V_{ij} \leq 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \overline{\Omega}$.*

Corollary 1.21. *Let $V : \mathcal{T}_N \to \mathbb{R}$ and suppose $-\Delta_N V_{ij} \geq 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \Omega$ and $V_{ij} \geq 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \partial\Omega$. Then $V_{ij} \geq 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \overline{\Omega}$.*

Corollary 1.22. *There exists a unique solution $V : \mathcal{T}_N \to \mathbb{R}$ to the problem*

$$\begin{cases} -\Delta_N V_{ij} = s_{ij} \ ; & \mathcal{T}_N \cap \Omega, \\ V_{ij} = g_{ij} \ ; & \mathcal{T}_N \cap \partial\Omega, \end{cases} \tag{1.28}$$

where $s_{ij}, g_{ij} \in \mathbb{R}$ for all $(i, j) \in \mathbb{N}_N$.

Theorem 1.23. *Let $A \in \mathbb{R}^{N \times N}$ be a matrix representation of $-\Delta_N$ with Dirichlet boundary conditions. Then A is a monotone matrix.*

Theorem 1.24. *There exists at least one grid function U such that U solves (3.9). Furthermore, there exists a constant C independent of h_r and h_θ such that $\|U\|_{l^\infty(\mathcal{T}_N)} < C$.*

Solving the algebraic System (1.26)**-**(1.27) **and forming bifurcation curves**

In [LMZ], the authors have provided a guaranteed solver that can be used to test for the existence of positive solutions to the problem (1.26)-(1.27) and to generate bifurcation curves. For $\lambda > 0$ fixed, this solver will converge monotonically provided f in (1.19) is Lipschitz continuous and the initial guess is given by a discrete subsolution or a discrete supersolution.

Consider the fixed point iteration

$$U^{(n+1)} = \mathcal{M}_K U^{(n)} \tag{1.29}$$

for all $n \geq 0$, where K is a Lipschitz constant for f in (1.19) and $\mathcal{M}_K$ is defined such that

$$-\Delta_N U_{ij}^{(n+1)} + \lambda K U_{ij}^{(n+1)} = \lambda f(U_{ij}^{(n)}) + \lambda K U_{ij}^{(n)} \ ; \ (r_i, \theta_j) \in \mathcal{T}_N \cap \Omega, \tag{1.30}$$

$$U_{ij}^{(n+1)} = 0 \ ; \ (r_i, \theta_j) \in \mathcal{T}_N \cap \partial\Omega, \tag{1.31}$$

where $U^{(0)}$ is an initial guess. Now we recall some results in [LMZ]:

Lemma 1.25. *Let U be a solution to* (1.26)-(1.27)*. If $U^{(0)} \leq U$ is a subsolution of* (1.26)-(1.27)*, then $U^{(1)} = \mathcal{M}_K U^{(0)}$ is a subsolution of* (1.26)-(1.27) *with $U^{(0)} \leq U^{(1)} \leq U$. If $U^{(0)} \geq U$ is a supersolution of* (1.26)-(1.27)*, then $U^{(1)} = \mathcal{M}_K U^{(0)}$ is a supersolution of* (1.26)-(1.27) *with $U \leq U^{(1)} \leq U^{(0)}$.*

Theorem 1.26. *Let U be a solution to* (1.26)-(1.27)*. If $U^{(0)} \leq U$ is a subsolution of* (1.26)-(1.27) *with $U_{ij}^{(0)} > 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \Omega$, then the sequence $U^{(n)}$ defined by* (1.29) *converges to a solution of* (1.26)-(1.27)*. If $U^{(0)} \geq U$ is a supersolution of* (1.26)-(1.27)*, then the sequence $U^{(n)}$ defined by* (1.29) *converges to a solution of* (1.26)-(1.27)*.*

The iteration (1.29) can be used to determine if the nonlinear system (1.26)-(1.27) has a positive solution for a given value of λ. Indeed, letting $U^{(0)}$ to be a discrete positive supersolution (see Chapter VI for existence of this supersolution), we have

$U^{(0)} > \vec{0}$, and the iteration monotonically converges to some $\tilde{U}$. If $\tilde{U} > 0$, then it represents the maximal positive solution of (1.26)-(1.27). If not we conclude that (1.26)-(1.27) has no positive solution for the given value of λ.

Given a positive subsolution of (1.26)-(1.27), the mapping $\mathcal{M}_K$ can naturally be used to find a minimal solution and a maximal solution to (1.26)-(1.27). As such, the methods can be applied to assist in generating $(\lambda, \|.\|_{\max})$-bifurcation curves since they can be used to find the minimal branch and the maximal branch. If the two solutions agree, then we can conclude uniqueness for the given λ value.

The convergence proof used by the authors in [LMZ] can also be extended to (1.26)-(1.27) ensuring that the FD approximations converge to PDE solutions as $N_r, N_\theta \to \infty$.

Finally we test the FD methods by approximating solutions to the examples:

(1) $h(x, y) = 1,\ f(s) = \sqrt{s+1} - 2,$

(2) $h(x, y) = 2 + y,\ f(s) = \sqrt{s+1} - 2.$

Our computational results show that the bifurcation diagrams are of the form in Figure 18 (see Chapter VII).

We now describe the plan for the rest of this dissertation. In Chapter II, we state some preliminaries that we use in the proofs of our results. In Chapters III and IV, we provide the proofs of results stated in Focus 1. Namely, we provide proofs of Theorems 1.1 - 1.3 in Chapter III, and proofs of Lemmas 1.4 - 1.12 and Theorem 1.13 in Chapter IV. Chapter V is devoted to the proofs of results stated in Focus 2. Namely, we provide proofs of Theorems 1.16 - 1.19. In Chapter VI, we provide proofs of Theorems 1.20, 1.23 - 1.24 and Corollary 1.21 - 1.22 stated in Focus 3. Chapter

VII is dedicated to computational results for examples in Focus 3. Finally, in Chapter VIII, we provide conclusions and future directions.

CHAPTER II

PRELIMINARIES

2.1 Schauder Fixed Point Theorem

Let X be a Banach Space and K be a closed, bounded and convex subset of X. If $T : K \to K$ is a completely continuous operator then T has a fixed point.

2.2 Arzela-Ascoli Theorem

Let F be a subset of $C[\alpha, \beta]$ where $\alpha, \beta \in \mathbb{R}$. Then F is compact if and only if F is closed, bounded and equicontinuous.

2.3 Method of Sub and Supersolutions

Consider the system:

$$\begin{cases} -\Delta u = \lambda u(1-u); \ \Omega \\ -\Delta v = \lambda r v(1-v); \ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} g(v) u = 0; \ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} h(u) v = 0; \ \partial\Omega \end{cases} \tag{2.1}$$

where λ is a positive parameter and $g, h \in C^1([0,\infty),(0,\infty))$ are decreasing functions of v, u respectively.

By a subsolution of (2.1), we mean a pair of functions $(\psi_1, \psi_2) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies:

$$\begin{cases} -\Delta\psi_1 \leq \lambda\psi_1(1-\psi_1);\ \Omega \\ -\Delta\psi_2 \leq \lambda r\psi_2(1-\psi_2);\ \Omega \\ \frac{\partial\psi_1}{\partial\eta} + \sqrt{\lambda}g(\psi_2)\psi_1 \leq 0;\ \partial\Omega \\ \frac{\partial\psi_2}{\partial\eta} + \sqrt{\lambda}h(\psi_1)\psi_2 \leq 0;\ \partial\Omega. \end{cases}$$

By a supersolution of (2.1), we mean a pair of functions $(z_1, z_2) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies:

$$\begin{cases} -\Delta z_1 \geq \lambda z_1(1-z_1);\ \Omega \\ -\Delta z_2 \geq \lambda r z_2(1-z_2);\ \Omega \\ \frac{\partial z_1}{\partial\eta} + \sqrt{\lambda}g(z_2)z_1 \geq 0;\ \partial\Omega \\ \frac{\partial z_2}{\partial\eta} + \sqrt{\lambda}h(z_1)z_2 \geq 0;\ \partial\Omega. \end{cases}$$

Then the following result holds (see [Pao92]):

Lemma 2.1. *Let (ψ_1, ψ_2) and (z_1, z_2) be a subsolution and a supersolution of (2.1), respectively, such that $(\psi_1, \psi_2) \leq (z_1, z_2)$. Then (2.1) has a solution $(u, v) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$.*

2.4 *Z*-matrix

A matrix $A = [a]_{ij}$ is called a Z-matrix if $a_{ij} \leq 0$ for all $i \neq j$.

2.5 *M*-matrix

- An M-matrix is a Z-matrix with eigenvalues whose real parts are nonnegative.

- A real square matrix A is monotone if for all real vectors v, $Av \geq 0$ implies $v \geq 0$.
- A real square matrix A is inverse-positive if A^{-1} exists and $[A^{-1}]_{ij} \geq 0$.
- Note that a nonsingular Z-matrix is an M-matrix if and only if it is monotone if and only if it is inverse-positive.

CHAPTER III

PROOFS OF THEOREMS 1.1 - 1.3 STATED IN FOCUS 1

3.1 Proof of Theorem 1.1

By integrating (1.4) we obtain

$$u'(t) = A - \int_0^t \bar{h}(s)ds \ ; \ (0,1) \tag{3.1}$$

$$u(t) = At - \int_0^t \Big(\int_0^s \bar{h}(v)dv \Big) ds \ ; \ (0,1) \tag{3.2}$$

where $A \in \mathbb{R}$ is such that $u'(1) + c(u(1))u(1) = 0$, i.e., $G(A) = 0$ where

$$G(z) = z - \int_0^1 \bar{h}(s)ds + c\Big(z - \int_0^1 \Big(\int_0^s \bar{h}(v)dv \Big) ds \Big) \Big(z - \int_0^1 \Big(\int_0^s \bar{h}(v)dv \Big) ds \Big).$$

Since $G : \mathbb{R} \to \mathbb{R}$ is increasing and $\lim_{z\to\infty} G(z) = \infty$, $\lim_{z\to-\infty} G(z) = -\infty$, $\exists_1 \ A \in \mathbb{R}$ such that $G(A) = 0$. Thus (1.4) has a unique solution $u \equiv T(\bar{h})$. Note that $|A| \leq \int_0^1 |\bar{h}|ds$. (If $z > \int_0^1 |\bar{h}|ds$ then $G(z) > 0$ while if $z < -\int_0^1 |\bar{h}|ds$ then $G(z) < 0$.) Hence (3.1) and (3.2) give:

$$|u|_{C^1} = \max(\|u\|_\infty, \|u'\|_\infty) \leq 2\|\bar{h}\|_{L^1}. \tag{3.3}$$

Let $\{\bar{h}_n\}_{n\in\mathbb{N}}$ be such that $\|\bar{h}_n\|_L^1 < M \ ; \ M > 0 \ \forall n$. We will show that $\{u_n = T(\bar{h}_n)\}_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous. Then by using the Arzela-

Ascoli Theorem there exists a subsequence of u_n that converge uniformly in $C[0,1]$. So T maps bounded sets in $L^1(0,1)$ into relatively compact subsets of $C[0,1]$. It remains to show that T is continuous. Let $\bar{h}_n \subset L^1(0,1)$ be such that $\bar{h}_n \to \bar{h}$ in $L^1(0,1)$ and let $u_n = T(\bar{h}_n)$, $u = T(\bar{h})$. $-(u_n'' - u'') = \bar{h}_n - \bar{h}$. So (3.3) gives $|u_n - u|_{C^1} \leq 2\|\bar{h}_n - \bar{h}\|_{L^1} \to 0$. Thus Theorem 1.1 is proven.

3.2 Proof of Theorem 1.2

For $v \in C[0,1]$ define $u = S(v)$ to be the unique solution of:

$$\begin{cases} -u'' = g(t, \tilde{v}) \ ; \ (0,1) \\ u(0) = 0 = u'(1) + c(u(1))u(1), \end{cases} \tag{3.4}$$

where $\tilde{v}(s) = \min\big(\max(v(s), \Psi(s)), \Phi(s)\big)$. Note that by (G_2), $\exists\ \Gamma \in L^1(0,1)$ such that $|g(t,\tilde{v})| \leq \Gamma(t)$ and so (3.4) has a unique solution by Theorem 1.1. Since $\tilde{T} : C[0,1] \to L^1(0,1)$ defined $\tilde{T}(v) = \tilde{v}$ is continuous and $S = T \circ \tilde{T}$ (here T is as defined in Theorem 1.1), it follows that $S : C[0,1] \to C[0,1]$ is completely continuous. Since $S(C[0,1])$ is bounded, S has a fixed point u by the Schauder Fixed Point Theorem. Finally use $(G_1) - (G_2)$ to show that $u \in [\Psi, \Phi]$; $[0,1]$, and so $\tilde{u} = u$, i.e., u is a solution of (1.5).

3.3 Proof of Theorem 1.3

By Theorem 1.3 in [LSS11], the Dirichlet boundary value problem:

$$-w'' = \lambda h(t) f(w) \ ; \ (0,1)$$
$$w(0) = 0 = w(1)$$

has a positive solution w for $\lambda \gg 1$. Clearly this w satisfies the role of Ψ in (G_1) since $w'(1) \leq 0$. Let e be the unique positive solution of

$$-e'' = h \ ; \ (0,1)$$
$$e(0) = 0 = e'(1) + c(0)e(1),$$

and choose $z(t) = M(\lambda)e(t) \ ; \ [0,1]$ where $M(\lambda) \gg 1$ so that

$$M(\lambda) \geq \lambda \tilde{f}(M(\lambda)\|e\|_\infty)$$

where $\tilde{f}(s) = \max\limits_{(0,s]} f(t)$ (which is possible since $\lim\limits_{s\to\infty} \frac{f(s)}{s} = 0$). Then

$$-z'' = M(\lambda)h(t) \geq \lambda h(t)\tilde{f}(M(\lambda)\|e\|_\infty) \geq \lambda h(t)\tilde{f}(M(\lambda)e(t)) \geq \lambda h(t)f(z); \ (0,1).$$

Also $z(0) = 0$ and

$$z'(1) + c(z(1))z(1) \geq M(\lambda)[e'(1) + c(0)e(1)] = 0$$

since c is increasing. Now choose $M(\lambda) \gg 1$ so that $z \geq \Psi(= w)$. Then z satisfies the role of Φ in (G_1) and hence (G_1) hold. Also, for $\zeta \in [\Psi, \Phi]$, $\exists\ C > 0$ such that

$$\begin{aligned} |g(t,\zeta)| = \lambda h(t)|f(\zeta)| &\leq \lambda \frac{Ch(t)}{\zeta^\alpha} \\ &\leq \lambda \frac{Ch(t)}{\Psi^\alpha} \equiv \Gamma. \end{aligned}$$

Note that $\Gamma \in L^1(0,1)$ since $\inf\limits_{(0,1)} \frac{\Psi}{p} > 0$ and $\limsup\limits_{s\to 0^+} s^\gamma h(s) < \infty$. So (G_2) holds and by Theorem 1.2, (1.3) has a positive solution for $\lambda \gg 1$. To show the existence of a

maximal solution, let u be a positive solution of (1.3). Then

$$-u'' = \lambda h(t) f(u) \leq \lambda h(t) \tilde{f}(u) \leq \lambda h(t) \tilde{f}(\|u\|_\infty)$$
$$u(0) = 0, u'(1) + c(0)u(1) \leq 0.$$

This implies

$$u(t) \leq \lambda \tilde{f}(\|u\|_\infty) e(t).$$

Since $\lim_{s\to\infty} \frac{f(s)}{s} = 0 = \lim_{s\to\infty} \frac{\tilde{f}(s)}{s}$, $\exists\ c_\lambda > 0$ such that

$$\|u\|_\infty \leq c_\lambda,$$

and hence

$$u(t) \leq \lambda \tilde{f}(c_\lambda) e(t).$$

This implies u is bounded by a supersolution of the form $M(\lambda)e(t)$ for $M(\lambda) \gg 1$, and the existence of the maximal positive solution follows.

CHAPTER IV

PROOFS LEMMAS 1.4 - 1.12 AND THEOREM 1.13 STATED IN FOCUS 1

Let $F(s) = \int_0^s f(t)dt$. (H_2) implies that $\exists\ \beta, \theta$ such that $0 < \beta < \theta$, $f(s)(s - \beta) > 0$ for $s \neq \beta$ and $F(s)(s - \theta) > 0$ for $s \neq \theta$. We shall verify these properties of f and F. Indeed, since $\lim\limits_{s \to 0^+} s^{1+\alpha} f'(s) = l$, L'Hôpital's rule gives

$$\lim_{s \to 0^+} s^{\alpha} f(s) = \lim_{s \to 0^+} \frac{f(s)}{s^{-\alpha}} = \lim_{s \to 0^+} \frac{f'(s)}{-\alpha s^{-\alpha-1}} = -\frac{l}{\alpha},$$

which implies the existence of F. Since f is increasing with $\lim\limits_{s \to 0^+} f(s) = -\infty$ and $\lim\limits_{s \to \infty} f(s) = \infty$, there exists a unique $\beta > 0$ such that $f(\beta) = 0$. Hence $f < 0$ on $(0, \beta)$ and $f > 0$ on (β, ∞) i.e. $f(s)(s - \beta) > 0$ for $s \neq \beta$. Since $F < 0$ on $(0, \beta]$ and F is increasing on $[\beta, \infty)$ with $\lim\limits_{s \to \infty} F(s) = \infty$, there exists a unique $\theta > \beta$ such that $F(\theta) = 0$. Clearly $F < 0$ on $(0, \theta)$ and $F > 0$ on (θ, ∞) i.e. $F(s)(s - \theta) > 0$ for $s \neq \theta$.

4.1 Proof of Lemma 1.4

Since $u(0) = 0$ and $f < 0$ for $u \approx 0$, $u'' > 0$ for $t \approx 0$. But $u'(0) \geq 0$. Hence $u' > 0$ for $t \approx 0$. Note that $u'(1) \leq 0$ since $u'(1) + c(u(1))u(1) = 0$. Let $t_0 \in (0, 1]$ be the first point such that $u'(t_0) = 0$. Now multiplying the differential equation in (1.3) by u' and integrating we obtain

$$0 \leq \frac{[u'(0)]^2}{2} = \lambda h(t_0) F(u(t_0)) - \lambda \int_0^{t_0} h'(s) F(u) ds$$

since by our hypotheses $\lim_{s\to 0^+} h(s)F(u(s)) = 0$. Hence if $u(t_0) \leq \theta$ then we have a contradiction. This implies $u(t_0) > \theta$. Note that $t_0 \neq 1$ since $u'(1) + c(u(1))u(1) = 0$. Thus u is decreasing near t_0 (to the right) since $u''(t_0) = -\lambda h(t_0)f(u(t_0)) < 0$. Now suppose $\exists\ t_1, t_2 \in (t_0, 1)$ such that $t_1 < t_2$ and $u(t_2) \geq u(t_1)$. Let $t^* \in (t_0, t_2)$ be such that $u'(t^*) = 0$ and $u''(t^*) \geq 0$. Then $u(t^*) \in (0, \beta]$ and hence $F(u(t^*)) < 0$. Let $\hat{t}$ be such that $F(u) < 0$ on $(\hat{t}, t^*)$ with $F(u(\hat{t})) = 0$ (note that $F(\theta) = 0$). Integrating (1.3) we get

$$0 \leq \frac{[u'(\hat{t})]^2}{2} = \lambda h(t^*)F(u(t^*)) - \int_{\hat{t}}^{t^*} h'(s)F(u)ds < 0$$

which is a contradiction. Hence Lemma 1.4 is proven.

4.2 Proof of Lemma 1.5

Multiplying the differential equation in (1.3) by u' and integrating on $[0, s]$; $s \leq t_\rho$, we get

$$\frac{[u'(s)]^2}{2} = \frac{1}{2}[u'(0)]^2 - \lambda h(s)F(u(s)) + \lambda \int_0^s h'(t)F(u)dt,$$

again using the fact that $\lim_{s\to 0^+} h(s)F(u(s)) = 0$, which follows from our hypotheses. This implies

$$u'(s) \geq \sqrt{2\lambda}\sqrt{-F(u(s))}\sqrt{h(s)},$$

and integrating on $[0, t]$; $t \leq t_\rho$, we get

$$\int_0^{u(t)} \frac{dz}{\sqrt{-F(z)}} \geq \sqrt{2\lambda} \int_0^t \sqrt{h(s)}ds. \tag{4.1}$$

Let $K > 0$ be such that

$$-F(z) \geq Kz^{1-\alpha} \ ; \ [0, \rho]. \tag{4.2}$$

(Note that $\lim_{z \to 0^+} [\frac{-F(z)}{z^{1-\alpha}}]$ exists and is positive, and since $\rho < \theta$, $[\frac{-F(z)}{z^{1-\alpha}}]$ must have a positive lower bound on $[0, \rho]$). Hence

$$\int_0^{u(t)} \frac{dz}{\sqrt{-F(z)}} \leq \frac{1}{\sqrt{K}} \int_0^{u(t)} z^{-\frac{(1-\alpha)}{2}} dz = \frac{2}{\sqrt{K}(1+\alpha)} [u(t)]^{\frac{1+\alpha}{2}}.$$

Combining this with (4.1) we obtain

$$u(t) \geq C\lambda^{\frac{1}{1+\alpha}} \Big(\int_0^t \sqrt{h(s)} ds \Big)^{\frac{2}{1+\alpha}} \ ; \ [0, t_\rho]$$

where $C = \Big(\sqrt{\frac{K}{2}} (1+\alpha) \Big)^{\frac{2}{1+\alpha}}$. In particular, when $t = t_\rho$ we obtain

$$\rho \geq C\lambda^{\frac{1}{1+\alpha}} \Big(\int_0^{t_\rho} \sqrt{h(s)} ds \Big)^{\frac{2}{1+\alpha}}$$

and hence $t_\rho \to 0$ as $\lambda \to \infty$. Hence Lemma 1.5 is proven.

4.3 Proof of Lemma 1.6

Assume $u(1) \leq \rho$. Let $\tilde{t}_\theta$ be the largest point such that $u(\tilde{t}_\theta) = \theta$. Multiplying the differential equation in (1.3) by u' and integrating on $[\tilde{t}_\theta, 1]$ we obtain

$$\frac{[u'(1)]^2}{2} = \frac{[u'(\tilde{t}_\theta)]^2}{2} + \lambda \int_{\tilde{t}_\theta}^1 h'(s) F(u) ds - \lambda h(1) F(u(1)).$$

This implies $-u'(1) \geq \sqrt{2\lambda h(1)}\sqrt{-F(u(1))}$ and hence by the boundary condition at $t = 1$ we obtain $c(u(1))u(1) \geq C_0\sqrt{\lambda K}(u(1))^{\frac{1-\alpha}{2}}$ where $C_0 = \sqrt{2h(1)}$ and K is as before (see (4.2)). This implies $c(u(1))u(1)^{\frac{1+\alpha}{2}} \geq C_0\sqrt{\lambda K} \to \infty$ as $\lambda \to \infty$. Hence, if $u(1) \leq \rho$ we obtain a contradiction. Thus $u(1) > \rho$ for $\lambda \gg 1$ and Lemma 1.6 is proven.

4.4 Proof of Lemma 1.7

Since $\bar{u}'(t) = \bar{u}'(1) - \int_t^1 \bar{u}''(s)ds$ we have $\bar{u}'(t) \geq -K_0\bar{u}(1) - \int_t^1 \bar{u}''(s)ds$, and integrating on $[0, t]$ we obtain

$$\bar{u}(t) \geq -K_0\bar{u}(1)t - \int_0^t \Big(\int_s^1 \bar{u}''(z)dz\Big)ds.$$

In particular, setting $t = 1$ and changing the order of integration we get

$$\begin{aligned}(1 + K_0)\bar{u}(1) &\geq -\int_0^1 z\bar{u}''(z)dz = -\int_0^{\frac{1}{4}} z\bar{u}''(z)dz - \int_{\frac{1}{4}}^1 z\bar{u}''(z)dz \\ &\geq -\epsilon\int_0^{\frac{1}{4}} h_0(z)dz + m\int_{\frac{1}{4}}^1 zh(z)dz > 0.\end{aligned} \tag{4.3}$$

Hence by the comparison principle, $\bar{u} \geq v$, where v is the solution of:

$$-v'' = \begin{cases} -\epsilon h_0(t) \ ; \ (0, \frac{1}{4}] \\ mh(t) \ ; \ (\frac{1}{4}, 1) \end{cases} = \tilde{h}(t) \text{ (say)}$$
$$v(0) = 0 = v(1).$$

Note that $v(t) = \int_0^1 G(t,s)\tilde{h}(s)ds$ where $G(t,s) = \begin{cases} s(1-t)\ ;\ s \le t \\ t(1-s)\ ;\ s > t. \end{cases} \le t\ \forall\ s,t.$

Hence for $t \le \frac{1}{4}$,

$$\begin{aligned} v(t) &\ge -\epsilon \int_0^{\frac{1}{4}} t h_0(s)ds + m \int_{\frac{1}{4}}^1 t(1-s)h(s)ds \\ &\ge \big[m \int_{\frac{1}{4}}^1 s(1-s)h(s)ds - \epsilon \int_0^{\frac{1}{4}} h_0(s)ds\big]t \ge m_0 t. \end{aligned}$$

Thus $u(t) \ge m_0 t\ ;\ [0, \frac{1}{4}]$ and Lemma 1.7 is proven.

Note that Lemma 1.7 holds also when the condition $\bar{u}'(1) + K_0\bar{u}(1) \ge 0$ is replaced by $\bar{u}(1) \ge 0$.

4.5 Proof of Lemma 1.8

Let $M > 0$ be such that $u(1) \le M$. Then $c(u(1)) \le c(M)$. Since $\lim_{z\to 0^+} z^\alpha f(z) < \infty$ (which is explained at the beginning of this section), $\exists\ A_0 > 0$ such that

$$f(z) \ge -\frac{A_0}{z^\alpha} \text{ for all } z > 0, \tag{4.4}$$

and hence by Lemma 1.7 for $\lambda \gg 1$ we obtain

$$h(t)f(u(t)) \ge -\frac{A_0 h(t)}{C^\alpha \lambda^{\frac{\alpha}{1+\alpha}} \big(\int_0^t \sqrt{h(s)}ds\big)^{\frac{2\alpha}{1+\alpha}}\big)} \ge -Ah_0(t)\ ;\ (0, t_\rho] \tag{4.5}$$

where $A = \frac{A_0}{C^\alpha}$. Thus for $\lambda \gg 1$ we have

$$-u'' \geq \begin{cases} -\lambda A h_0(t) \ ; \ (0, t_\rho] \\ \lambda m h(t) \ ; \ (t_\rho, 1] \end{cases}$$
$$u(0) = 0, \ u'(1) + c(M)u(1) \geq 0.$$

(Recall here that for $\lambda \gg 1$, by Lemma 1.6, we have $u(1) > \rho$, and $m = \inf_{[\rho,\infty)} f$.) Let $\bar{u} = \lambda^{-1} u$. Then $\bar{u}$ satisfies:

$$-\bar{u}'' \geq \begin{cases} -A h_0(t) \ ; \ (0, t_\rho] \\ m h(t) \ ; \ (t_\rho, 1] \end{cases}$$
$$\bar{u}(0) = 0, \ \bar{u}'(1) + c(M)\bar{u}(1) \geq 0.$$

Now as in the proof of Lemma 1.7 (see (4.3)), we get

$$\begin{aligned} \bar{u}(1) &\geq \frac{1}{1 + c(M)} \big[- A \int_0^{t_\rho} h_0(z) dz + m \int_{t_\rho}^1 z h(z) dz \big] \\ &\geq \frac{1}{1 + c(M)} \Big(\frac{m}{2} \Big) \int_{\frac{1}{4}}^1 z h(z) dz = M_0 \text{ (say)}, \end{aligned}$$

if $\lambda \gg 1$ so that $t_\rho < \frac{1}{4}$ and $m \int_{t_\rho}^1 z h(z) dz > 2A \int_0^{t_\rho} h_0(z) dz$. This is possible since $t_\rho \to 0$ as $\lambda \to \infty$ (see Lemma 1.5). This implies $u(1) = \lambda \bar{u}(1) \geq \lambda M_0 > M$ for $\lambda > \frac{M}{M_0}$ which is a contradiction. Hence $u(1) \to \infty$ as $\lambda \to \infty$, and Lemma 1.8 is proven.

4.6 Proof of Lemma 1.9

Let $\lambda \gg 1$ so that $t_\rho < \frac{1}{8}$. Since $u'' \leq 0$ on $(\frac{1}{8}, 1)$, u is concave on $[\frac{1}{8}, 1]$. Hence

$$u(t) \geq \frac{1-t}{(\frac{7}{8})} u(\tfrac{1}{8}) + \frac{(t-\frac{1}{8})}{(\frac{7}{8})} u(1) \geq \tfrac{1}{7} u(1) \ ; \ [\tfrac{1}{4}, 1].$$

Thus $\inf_{[\frac{1}{4},1]} u \geq \frac{1}{7} u(1) \to \infty$ as $\lambda \to \infty$ by Lemma 1.8. Hence Lemma 1.9 is proven.

4.7 Proof of Lemma 1.10

Let A be defined as in (4.5) and let $L_1 > 0$ be large enough so that $A < L_0 \big(\int_0^{\frac{1}{4}} h_0(s) ds \big)^{-1}$ where $L_0 = \frac{L_1}{2} \int_{\frac{1}{4}}^1 s(1-s) h(s) ds$ and $L_0 > L$. Assume $\lambda \gg 1$ so that $t_\rho < \frac{1}{8}$. Let u be a positive solution of (1.3). Then by (4.5) we obtain $h(t) f(u(t)) \geq -A h_0(t)$; $(0, \frac{1}{4})$. (Note that if $t_\rho < t \leq \frac{1}{4}$ then $u(t) > \rho$ which implies $f(u(t)) > 0$.) Since $\lim_{z \to \infty} f(z) = \infty$, it follows from Lemma 1.9 that for $\lambda \gg 1$, we have $h(t) f(u(t)) \geq L_1 h(t)$; $(\frac{1}{4}, 1]$. Let $\bar{u} = \lambda^{-1} u$. Then $\bar{u}$ satisfies:

$$-\bar{u}'' \geq \begin{cases} -A h_0(t) \ ; \ (0, \frac{1}{4}] \\ L_1 h(t) \ ; \ (\frac{1}{4}, 1) \end{cases}$$
$$\bar{u}(0) = 0, \ \bar{u}(1) \geq 0.$$

Taking $\epsilon = A, m = L_1$, Lemma 1.7 implies $\bar{u}(t) \geq L_0 t \geq L t$; $[0, \frac{1}{4}]$, i.e. $u(t) \geq \lambda L t$; $[0, \frac{1}{4}]$. Also, Lemma 1.9 implies $\inf_{[\frac{1}{4},1]} u \to \infty$ as $\lambda \to \infty$. Hence $u(t) \geq L$; $(\frac{1}{4}, 1]$ and Lemma 1.10 is proven.

4.8 Proof of Lemma 1.11

Let $H(s) = f(s) - s f'(s)$. Clearly for $s_1 > s_2$, $H(s_1) - H(s_2) = [f(s_1) - f(s_2)] - s_1 f'(s_1) + s_2 f'(s_2) \geq f'(s_1)(s_1 - s_2) - s_1 f'(s_1) + s_2 f'(s_2) = s_2 (f'(s_2) - f'(s_1)) \geq 0$

since f is concave. Suppose $c \in (-\infty, \infty]$ is such that $\lim_{s\to\infty} H(s) = c$. If $c \leq 0$ then $sf'(s) - f(s) \geq 0$ for $s > 0$. Hence $\left(\frac{f(s)}{s}\right)' \geq 0$ which contradicts $\lim_{s\to\infty} \frac{f(s)}{s} = 0$ and $\lim_{s\to\infty} f(s) > 0$. Hence $c > 0$ and $\exists\ K_1 > 0, \delta > 0$ such that

$$H(s) \geq K_1 \ ; \ s > \delta. \tag{4.6}$$

Next, $\lim_{s\to 0^+} s^\alpha H(s) = \lim_{s\to 0^+} s^\alpha f(s) - \lim_{s\to 0^+} s^{1+\alpha} f'(s) = -\dfrac{l}{\alpha} - l = -\dfrac{l}{\alpha}(1+\alpha)$. Hence $\exists$ $B_1 > 0$ such that

$$H(s) \geq -\frac{B_1}{s^\alpha} \ ; \ s \in (0, \delta]. \tag{4.7}$$

Let $B = B_1 + K_1\delta^\alpha$. Then from (4.6) and (4.7) we obtain $H(s) \geq K_1 - \dfrac{B}{s^\alpha}$; $s > 0$ and Lemma 1.11 is proven.

4.9 Proof of Lemma 1.12

Let z_2 be the solution of:

$$\begin{aligned} -z_2'' &= \tfrac{Bh(t)}{(P_\lambda(t))^\alpha} \ ; \ (0,1) \\ z_2(0) &= 0 = z_2(1). \end{aligned}$$

Then $z_2(t) \leq B\|\frac{h}{P_\lambda^\alpha}\|_{L_1} p(t)$ for all $t \in [0,1]$ where $p(t) = \min\{t, 1-t\}$. This is true since

$$\begin{aligned} z_2(t) = \int_0^1 G(t,s)\frac{Bh(s)}{(P_\lambda(s))^\alpha}ds &\leq \int_0^t s(1-t)\frac{Bh(s)}{(P_\lambda(s))^\alpha}ds + \int_t^1 t(1-s)\frac{Bh(s)}{(P_\lambda(s))^\alpha}ds \\ &\leq B\Big\|\frac{h}{P_\lambda^\alpha}\Big\|_{L_1} t(1-t) \leq B\Big\|\frac{h}{P_\lambda^\alpha}\Big\|_{L_1} p(t) \end{aligned}$$

where

$$G(t,s) = \begin{cases} t(1-s)\ ; & t < s \\ s(1-t)\ ; & t \geq s. \end{cases}$$

Next note that

$$B\|\frac{h}{P_\lambda^\alpha}\|_{L_1} = \frac{B}{\lambda^\alpha L^\alpha}\int_0^{\frac{1}{4}} \frac{h(t)}{t^\alpha}dt + \frac{B}{L^\alpha}\int_{\frac{1}{4}}^1 h(t)dt < \frac{b}{2} \text{ for } \lambda > 1.$$

This implies $z_2(t) \leq \frac{b}{2}p(t)\ ;\ [0,1]$. Now since $z = z_1 - z_2$, it follows that $z(t) \geq \frac{b}{2}p(t)\ ;\ [0,1]$. Hence Lemma 1.12 is proven.

4.10 Proof of Theorem 1.13

Let u be the maximal positive solution for $\lambda \gg 1$ (see Theorem 1.3 for the existence of this maximal solution), and v be any other positive solution of (1.3). Then for $w = u - v$ we have (by the mean value theorem and using f is concave)

$$\begin{cases} -w'' = \lambda h(t)[f(u) - f(v)] \\ \qquad \leq \lambda h(t) f'(v) w\ ;\ (0,1) \\ w(0) = 0, w(1) \geq 0. \end{cases} \tag{4.8}$$

Multiplying the inequality in (4.8) by v and integrating we obtain

$$-\int_0^1 w''v ds \leq \lambda \int_0^1 h(s)wvf'(v)ds$$

i.e.

$$-w'v\Big|_0^1 + \int_0^1 w'v'ds \leq \lambda \int_0^1 h(s)vwf'(v)ds.$$

Now since $v(0) = 0$,

$$\begin{aligned} -w'v\Big|_0^1 &= -\,w'(1)v(1) = -[u'(1) - v'(1)]v(1) \\ &= [c(u(1))u(1) - c(v(1))v(1)]v(1). \end{aligned}$$

Thus we obtain

$$c(u(1))u(1)v(1) - c(v(1))[v(1)]^2 + \int_0^1 w'v'ds \leq \lambda \int_0^1 h(s)vwf'(v)ds. \tag{4.9}$$

Next, multiplying $-v'' = \lambda h(t)f(v)$ by w and integrating we obtain

$$-\int_0^1 v''wds = \lambda \int_0^1 h(s)f(v)wds$$

i.e.

$$-v'w\Big|_0^1 + \int_0^1 v'w'ds = \lambda \int_0^1 h(s)f(v)wds.$$

But since $w(0) = 0$,

$$-v'w\Big|_0^1 = -v'(1)w(1) = c(v(1))v(1)[u(1) - v(1)].$$

Thus we obtain

$$c(v(1))v(1)u(1) - c(v(1))[v(1)]^2 + \int_0^1 v'w'ds = \lambda \int_0^1 h(s)f(v)wds. \tag{4.10}$$

Now since $c(v(1)) \leq c(u(1))$ by (H_3), subtracting (4.9) from (4.10) we obtain

$$\lambda \int_0^1 h(s)[f(v) - vf'(v)]wds \leq 0,$$

and using Lemma 1.11 we get

$$\int_0^1 h(s)\left[K_1 - \frac{B}{v^\alpha}\right]wds \leq 0.$$

Next by Lemma 1.10 we have $v(t) \geq P_\lambda(t)$ for $\lambda \gg 1$ and thus

$$\int_0^1 h(s)\left[K_1 - \frac{B}{(P_\lambda(s))^\alpha}\right]wds \leq 0.$$

This implies $-\int_0^1 z''wds \leq 0$ where z is defined in Lemma 1.12, and hence $-z'(1)w(1)+\int_0^1 z'w'ds \leq 0$, i.e.

$$-z'(1)w(1) - \int_0^1 zw''ds \leq 0 \quad (\text{since } z(0) = 0 = z(1)). \tag{4.11}$$

Note that $z(t) \geq \frac{b}{2}p(t)$; $[0,1]$ by Lemma 1.12. Hence $z'(1) < 0$, and since $w(1) \geq 0$, it follows from (4.11) that

$$-\int_0^1 zw''ds = \lambda \int_0^1 h(s)[f(u) - f(v)]zds \leq 0,$$

from which it easily follows that $u \equiv v$ (since f is increasing, $h(s) > 0$; $(0, 1]$ and $z(t) > 0$; $(0, 1)$). Thus Theorem 1.13 is proven.

CHAPTER V

PROOFS OF THEOREMS 1.16 - 1.19 STATED IN FOCUS 2

First we consider the eigenvalue problem:

$$\begin{cases} -\Delta\phi = \mu\phi; \ \Omega \\ \frac{\partial\phi}{\partial\eta} + \beta(x)\phi = 0; \ \partial\Omega \end{cases} \tag{5.1}$$

where β is a nonnegative continuous function on $\partial\Omega$. Let $\mu_1 = \mu_1(\beta)$ denote the principal eigencurve of (5.1). Note that μ_1 increases in β (see [CC03]). For a fixed $K > 0$, when $\beta = \sqrt{\lambda}K$, by Lemmas 2 & 3 in [GMRS18], the principal eigencurve $\mu_1(\sqrt{\lambda}K) = \bar{E}_1(\sqrt{\lambda}K)$ is strictly increasing and concave in λ. Moreover, $\mu_1(0) = 0$ and $\mu_1(\sqrt{\lambda}K) \to E_1^D$ as $\lambda \to \infty$ where E_1^D is the principal eigenvalue of the eigenvalue problem (1.10). Thus, comparing (1.15), (1.16), (1.17) and (1.18) with (5.1) we obtain:

$$\begin{aligned} \sigma_1 &= \mu_1(\sqrt{\lambda}g(0)) - \lambda, \\ \sigma_2 &= \mu_1(\sqrt{\lambda}h(0)) - \lambda r, \\ \sigma_3 &= \mu_1(\sqrt{\lambda}h(\tilde{w}_1)) - \lambda r, \\ \sigma_4 &= \mu_1(\sqrt{\lambda}g(\tilde{w}_2)) - \lambda, \end{aligned} \tag{5.2}$$

respectively.

Note that the proof of Theorem 1.19 is not provided since it is very similar to the proof of Theorem 1.18.

5.1 Proof of Theorem 1.16

(a) We first introduce the eigenvalue problems:

$$\begin{cases} -\Delta\psi_1 - \lambda\psi_1 = \sigma\psi_1; \ \Omega \\ \frac{\partial\psi_1}{\partial\eta} + \sqrt{\lambda}g(1)\psi_1 = 0; \ \partial\Omega \end{cases} \tag{5.3}$$

and

$$\begin{cases} -\Delta\psi_2 - \lambda r\psi_2 = \sigma\psi_2; \ \Omega \\ \frac{\partial\psi_2}{\partial\eta} + \sqrt{\lambda}h(1)\psi_2 = 0; \ \partial\Omega. \end{cases} \tag{5.4}$$

Let $\tilde{\sigma}_1 = \tilde{\sigma}_1(\Omega, \lambda, g(1))$, $\tilde{\sigma}_2 = \tilde{\sigma}_2(\Omega, \lambda, r, h(1))$ be the principal eigenvalues and ψ_1, $\psi_2 > 0$; $\overline{\Omega}$ the corresponding normalized eigenfunction of (5.3) and (5.4), respectively. Comparing (5.3) and (5.4) with (5.1), we obtain

$$\tilde{\sigma}_1 = \mu_1(\sqrt{\lambda}g(1)) - \lambda \tag{5.5}$$

$$\tilde{\sigma}_2 = \mu_1(\sqrt{\lambda}h(1)) - \lambda r. \tag{5.6}$$

Note that at $\lambda = E_1(1, g(1))$, $\mu_1(\sqrt{E_1(1, g(1))}g(1)) = E_1(1, g(1))$ and at $\lambda = E_1(r, h(1$ $))$, $\mu_1(\sqrt{E_1(r, h(1))}h(1)) = rE_1(r, h(1))$. Since $\mu_1(\sqrt{\lambda}K)$ is concave, we must have that $\mu_1(\sqrt{\lambda}g(1)) > \lambda$ for $\lambda < E_1(1, g(1))$ and $\mu_1(\sqrt{\lambda}h(1)) > \lambda r$ for $\lambda < E_1(r, h(1))$. Then (5.5) and (5.6) imply that $\tilde{\sigma}_1 > 0$ for $\lambda < E_1(1, g(1))$ and $\tilde{\sigma}_2 > 0$ for $\lambda < E_1(r, h(1))$, respectively. Assume to the contrary that (u, v) is a positive solution of (1.8) for $\lambda < \lambda^*$.

Case I: $E_1(r, h(1)) \leq E_1(1, g(1))$

By Green's second identity, we obtain

$$\begin{aligned}\int_\Omega [(\Delta u)\psi_1 - (\Delta\psi_1)u]dx &= \int_{\partial\Omega} [(-\sqrt{\lambda}g(v)u)\psi_1 + (\sqrt{\lambda}g(1)\psi_1)u]ds \\ &= \int_{\partial\Omega} \sqrt{\lambda}(g(1) - g(v))u\psi_1 ds < 0\end{aligned}$$

while

$$\begin{aligned}\int_\Omega [(\Delta u)\psi_1 - (\Delta\psi_1)u]dx &= \int_\Omega [(-\lambda u(1-u))\psi_1 + ((\lambda + \tilde{\sigma}_1)\psi_1)u]dx \\ &= \int_\Omega (\lambda u + \tilde{\sigma}_1)u\psi_1 dx > 0\end{aligned}$$

giving rise to a contradiction.

Case II: $E_1(r, h(1)) > E_1(1, g(1))$

Now again by the Green's second identity, we obtain

$$\begin{aligned}\int_\Omega [(\Delta v)\psi_2 - (\Delta\psi_2)v]dx &= \int_{\partial\Omega} [(-\sqrt{\lambda}h(u)v)\psi_2 + (\sqrt{\lambda}h(1)\psi_2)v]ds \\ &= \int_{\partial\Omega} \sqrt{\lambda}(h(1) - h(v))v\psi_2 ds < 0\end{aligned}$$

while

$$\begin{aligned}\int_\Omega [(\Delta v)\psi_2 - (\Delta\psi_2)v]dx &= \int_\Omega [(-\lambda rv(1-v))\psi_2 + ((\lambda r + \tilde{\sigma}_2)\psi_2)v]dx \\ &= \int_\Omega (\lambda rv + \tilde{\sigma}_2)v\psi_2 dx > 0\end{aligned}$$

providing a contradiction.

(b) It is clear that when $\lambda > \lambda^{**}$, $\sigma_1, \sigma_2 < 0$. By Lemma 1.15, the unique positive solution $\tilde{w}_1$ of (1.13) exists, and the unique positive solution $\tilde{w}_2$ of (1.14) exists. Now

since g and h are decreasing functions, $\tilde{w}_1$ and $\tilde{w}_2$ satisfy:

$$\begin{aligned} -\Delta\tilde{w}_1 &= \lambda\tilde{w}_1(1-\tilde{w}_1);\ \Omega, \\ \frac{\partial\tilde{w}_1}{\partial\eta} &= -\sqrt{\lambda}g(0)\tilde{w}_1 < -\sqrt{\lambda}g(\tilde{w}_2)\tilde{w}_1;\ \partial\Omega \end{aligned}$$

and

$$\begin{aligned} -\Delta\tilde{w}_2 &= \lambda r\tilde{w}_2(1-\tilde{w}_2);\ \Omega, \\ \frac{\partial\tilde{w}_2}{\partial\eta} &= -\sqrt{\lambda}h(0)\tilde{w}_2 < -\sqrt{\lambda}h(\tilde{w}_1)\tilde{w}_2;\ \partial\Omega. \end{aligned}$$

Thus, $(\tilde{w}_1, \tilde{w}_2)$ is a strict subsolution of (1.8). Since g and h are positive functions it is easy to see that $(1,1)$ is a supersolution of (1.8). Hence, by Lemma 2.1 there exists a positive solution (u^*, v^*) of (1.8) such that $(\tilde{w}_1, \tilde{w}_2) \leq (u^*, v^*) \leq (1,1);\ \overline{\Omega}$. Since $(\tilde{w}_1, \tilde{w}_2)$ is a strict subsolution of (1.8), all positive solutions of (1.8), (u^*, v^*), must satisfy $(u^*, v^*) > (\tilde{w}_1, \tilde{w}_2);\ \overline{\Omega}$. Finally, the furthermore statement holds since, by an analogous version of Theorem 5.1 in Chapter 10 of [Pao92], any solution of (1.7), (u, v), with small positive initial density will converge to a minimal positive steady state of (1.7) in $\langle(\tilde{w}_1, \tilde{w}_2), (1,1)\rangle$ as $t \to \infty$. This completes the proof.

5.2 Proof of Theorem 1.17

Parts (a) and (c) are clear. To prove (b), we consider three cases, namely, $r = 1$, $r < 1$ and $r > 1$. Note that we only provide proofs of the cases $r = 1$ and $r < 1$, as the case when $r > 1$ is almost identical to the $r < 1$ case.

Case I: $r = 1$ (which implies that $g(0) = h(0)$)

When $\beta = \sqrt{\lambda}g(0) = \sqrt{\lambda}h(0)$, we have $\mu_1(\sqrt{\lambda}g(0)) = \mu_1(\sqrt{\lambda}h(0))$. Also, we have that

$\mu_1(\sqrt{E_1(1,g(0))}g(0)) = E_1(1,g(0))$ at $\lambda = E_1(1,g(0))$ and $\mu_1(\sqrt{E_1(1,h(0))}h(0)) = E_1(1,h(0))$ at $\lambda = E_1(1,h(0))$. Now, since $\mu_1(\sqrt{\lambda}K)$ is concave in λ and by (5.2), we obtain $\sigma_1 = \mu_1(\sqrt{\lambda}g(0)) - \lambda \geq 0$, $\sigma_2 = \mu_1(\sqrt{\lambda}h(0)) - \lambda \geq 0$ for $\lambda \leq E_1(1,g(0))$ and $\sigma_1 < 0$, $\sigma_2 < 0$ for $\lambda > E_1(1,g(0))$. Since g and h are decreasing, we have $\mu_1(\sqrt{\lambda}g(\tilde{w}_2)) < \mu_1(\sqrt{\lambda}g(0))$ and $\mu_1(\sqrt{\lambda}h(\tilde{w}_1)) < \mu_1(\sqrt{\lambda}h(0))$. This implies $\sigma_3 = \mu_1(\sqrt{\lambda}g(\tilde{w}_2)) - \lambda < 0$ and $\sigma_4 = \mu_1(\sqrt{\lambda}h(\tilde{w}_1)) - \lambda < 0$ for $\lambda > E_1(1,g(0))$. Figure (20) illustrates the main argument of this case.

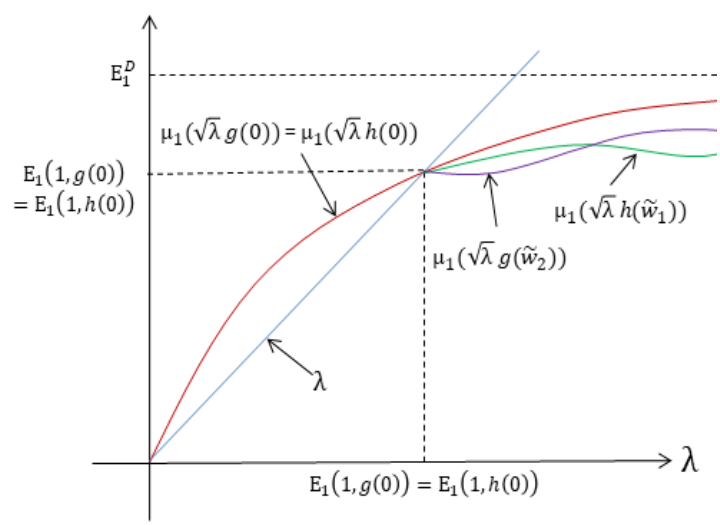


Figure 20. Principal eigencurves $\mu_1(\sqrt{\lambda}g(0))$ and $\mu_1(\sqrt{\lambda}h(0))$ with $r = 1$.

Case II: $r < 1$ (which implies that $g(0) > h(0)$)

When $\beta = \sqrt{\lambda}g(0) = \sqrt{\lambda}h(0)$, we have $\mu_1(\sqrt{\lambda}g(0)) > \mu_1(\sqrt{\lambda}h(0))$. We also have that $\mu_1(\sqrt{E_1(1,g(0))}g(0)) = E_1(1,g(0))$ at $\lambda = E_1(1,g(0))$ and $\mu_1(\sqrt{E_1(r,h(0))}h(0)) = rE_1(r,h(0))$ at $\lambda = E_1(r,h(0))$. Now, since $\mu_1(\sqrt{\lambda}K)$ is concave in λ and by (5.2), we obtain that $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ for $\lambda \leq E_1(1,g(0))$ and $\sigma_1 < 0$, $\sigma_2 < 0$ for $\lambda >$

$E_1(1, g(0))$. Since g and h are decreasing, we have that $\mu_1(\sqrt{\lambda}g(\tilde{w}_2)) < \mu_1(\sqrt{\lambda}g(0))$ and $\mu_1(\sqrt{\lambda}h(\tilde{w}_1)) < \mu_1(\sqrt{\lambda}h(0))$. This implies $\sigma_3 = \mu_1(\sqrt{\lambda}g(\tilde{w}_2)) - \lambda < 0$ and $\sigma_4 = \mu_1(\sqrt{\lambda}h(\tilde{w}_1)) - \lambda r < 0$ for $\lambda > E_1(1, g(0))$. Figure (21) provides an illustration of the main argument of this case.

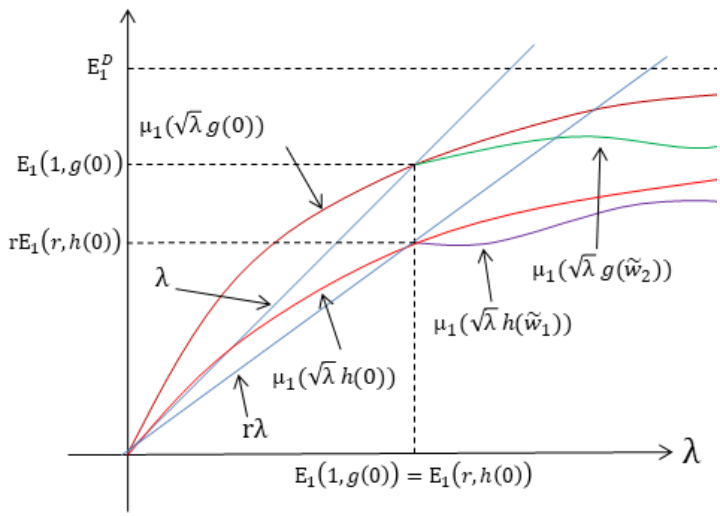


Figure 21. Principal eigencurves $\mu_1(\sqrt{\lambda}g(0))$ and $\mu_1(\sqrt{\lambda}h(0))$ with $r < 1$.

This completes the proof.

5.3 Proof of Theorem 1.18

Proof of (a) and (b) are clear. Next, we prove part (c). By part (a), $\sigma_1 < 0$ for $\lambda > E_1(1, g(0))$. Now, by Lemma 1.15, $\tilde{w}_1$ exists for $\lambda > E_1(1, g(0))$. Here, we note that at $\lambda = E_1(r, h(0))$,

$$\mu_1(\sqrt{E_1(r, h(0))}h(0)) = rE_1(r, h(0)).$$

Then, since $\mu_1(\sqrt{\lambda}h(0))$ is concave in λ,

$$\mu_1(\sqrt{\lambda}h(0)) < r\lambda \text{ for } \lambda > E_1(r, h(0)). \tag{5.7}$$

Since h is decreasing, the inequality

$$\mu_1(\sqrt{\lambda}h(\tilde{w}_1)) < \mu_1(\sqrt{\lambda}h(0)) \tag{5.8}$$

holds for $\lambda > E_1(1, g(0))$. Now (5.7) and (5.8) imply that $\mu_1(\sqrt{\lambda}h(\tilde{w}_1)) < r\lambda$ for $\lambda > E_1(1, g(0))$. Thus, $\sigma_3 = \mu_1(\sqrt{\lambda}h(\tilde{w}_1)) - r\lambda < 0$ for $\lambda > E_1(1, g(0))$. This completes the proof of (c).

In order to prove (d), we must consider three cases: $g(0) < h(0)$, $g(0) = h(0)$, and $g(0) > h(0)$. However, we will only provide a proof for the first case, as the remaining cases are almost identical to the first.

Case I: $(g(0) < h(0))$

First, we note that

$$E_1(1, g(0)) = \mu_1(\sqrt{E_1(1, g(0))}g(0)).$$

Since $\mu_1(\sqrt{\lambda}g(0))$ is concave in λ, the inequality

$$\lambda < \mu_1(\sqrt{\lambda}g(0)) \tag{5.9}$$

holds for $\lambda < E_1(1, g(0))$. This implies that

$$E_1(r, h(0)) < \mu_1(\sqrt{E_1(r, h(0))}g(0)). \tag{5.10}$$

Note that $\mu_1(\sqrt{\lambda}g(\tilde{w}_2)) < \mu_1(\sqrt{\lambda}g(0))$ for $\lambda > E_1(r, h(0))$ and $\mu_1(\sqrt{\lambda}g(\tilde{w}_2)) \to \mu_1(\sqrt{E_1(r, h(0))}g(0))$ as $\lambda \to E_1(r, h(0))^+$. This with (5.10) and by continuity imply that there exists $\delta_1(r, g(0), h(0))$ such that $\lambda < \mu_1(\sqrt{\lambda}g(\tilde{w}_2))$ for $\lambda \in (E_1(r, h(0)), E_1(r, h(0)) + \delta_1)$. This implies $\sigma_4 = \mu_1(\sqrt{\lambda}g(\tilde{w}_2)) - \lambda > 0$ for $\lambda \in (E_1(r, h(0)), E_1(r, h(0)) + \delta_1)$.

Since $E_1(1, g(0)) = \mu_1(\sqrt{E_1(1, g(0))}g(0))$ and $\mu_1(\sqrt{E_1(1, g(0))}g(\tilde{w}_2)) < \mu_1(\sqrt{E_1(1, g(0))}g(0))$, by continuity, there exists $\delta_2(g(0), h(0), r) > 0$ such that $\mu_1(\sqrt{\lambda}g(\tilde{w}_2)) < \lambda$ for $\lambda > E_1(1, g(0)) - \delta_2$. This implies $\sigma_4 = \mu_1(\sqrt{\lambda}g(\tilde{w}_2)) - \lambda < 0$ for $\lambda > E_1(1, g(0)) - \delta_2$. Also see Figure (22). In this case, $E_1(1, g(0)) < E_1(1, h(0))$ since $g(0) < h(0)$. Using (1.12) we obtain $E_1(1, g(0)) < rE_1(r, h(0))$. Now this with $E_1(1, g(0)) > E_1(r, h(0))$ implies $r > 1$.

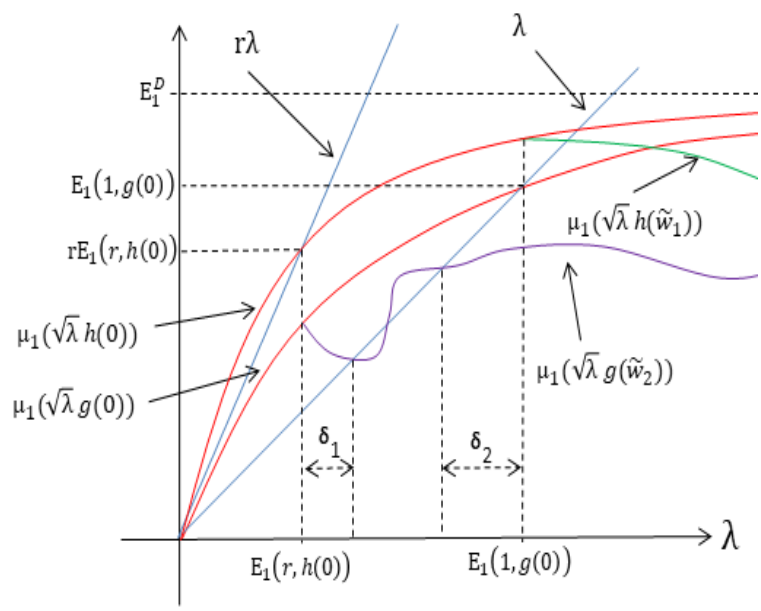


Figure 22. Principal eigencurves $\mu_1(\sqrt{\lambda}g(0))$ and $\mu_1(\sqrt{\lambda}h(0))$ when $E_1(r, h(0)) < E_1(1, g(0))$ and $g(0) < h(0)$.

Next, we provide a proof of coexistence. Since $\lambda > E_1(r, h(0))$ we have that $\sigma_2 < 0$. Thus, by Lemma 1.15, the unique solution $\tilde{w}_2$ of (1.14) exists. Also, part (d) implies that $\sigma_4 < 0$ for $\lambda > E_1(1, g(0)) - \delta_2$. Let $y_1 = \epsilon\phi_4$ and $y_2 = \tilde{w}_2$. Now choose $\epsilon > 0$ and $\epsilon \approx 0$ such that $\sigma_4 \leq -\lambda\epsilon\phi_4$ and $\epsilon\phi_4 \leq 1$. Now by (1.18), y_1 satisfies:

$$\begin{aligned}
-\Delta y_1 &= \epsilon(\lambda + \sigma_4)\phi_4 \leq \epsilon(\lambda - \lambda\epsilon\phi_4)\phi_4 = \lambda y_1(1 - y_1);\ \Omega, \\
\frac{\partial y_1}{\partial \eta} &= -\sqrt{\lambda} g(\tilde{w}_2)\epsilon\phi_4 \\
&= -\sqrt{\lambda} g(y_2) y_1;\ \partial\Omega.
\end{aligned}$$

Further, since h is decreasing, y_2 satisfies:

$$\begin{aligned}
-\Delta y_2 &= (-\Delta\tilde{w}_2) = \lambda\tilde{w}_2(1 - \tilde{w}_2) = \lambda y_2(1 - y_2);\ \Omega \\
\frac{\partial y_2}{\partial \eta} &= -\sqrt{\lambda} h(0)\tilde{w}_2 \\
&\leq -\sqrt{\lambda} h(\epsilon\phi_4) y_2 \\
&= -\sqrt{\lambda} h(y_1) y_2;\ \partial\Omega.
\end{aligned}$$

Thus $(\epsilon\phi_4, \tilde{w}_2)$ is a subsolution of (1.8). Since g and h are positive functions it is easy to see that $(1, 1)$ is a supersolution of (1.8). Hence, by Lemma 2.1 there exists a positive solution (u, v) of (1.8) such that $(\epsilon\phi_4, \tilde{w}_2) \leq (u, v) \leq (1, 1);\ \overline{\Omega}$. Finally, by an analogous version of Theorem 5.1 in Chapter 10 of [Pao92], any solution of (1.7), (u, v), with $(u(0, x), v(0, x)) \geq (\approx)\ (\epsilon\phi_4(x), \tilde{w}_2(x));\ \Omega$ and $\epsilon < \epsilon_1$ will converge to a minimal positive steady state of (1.7) in $\langle(\epsilon\phi_4(x), \tilde{w}_2), (1, 1)\rangle$ as $t \to \infty$. This completes the proof.

CHAPTER VI

PROOFS OF THEOREMS 1.20, 1.23 - 1.24 AND COROLLARIES 1.21 - 1.22 STATED IN FOCUS 3

6.1 Proof of Theorem 1.20

Let (i,j) be the index for the maximum of V and suppose $V_{ij} > 0$. Let $N_{ij} \subset \mathbb{N}_N$ represent the indices for neighboring nodes of $x_{ij} \in \mathcal{T}_N$, and define $V_{max} = \max_{(k,l)\in N_{ij}} V_{kl}$. Note that $r_i \geq h_r$ for all $(i,j) \in \mathbb{N}_N$. Now we consider two cases:
Case 1: $(i,j) \neq (0,0)$. Then $-\Delta_N V_{ij} \leq 0$ implies

$$\begin{aligned}(\frac{2}{h_r^2} + \frac{2}{h_\theta^2 r_i^2})V_{ij} &\leq (\frac{1}{h_r^2} - \frac{1}{2h_r r_i})V_{i-1j} + (\frac{1}{2h_r r_i} + \frac{1}{h_r^2})V_{i+1j} + (\frac{1}{h_\theta^2 r_i^2})V_{ij-1} + (\frac{1}{h_\theta^2 r_i^2})V_{ij+1} \\ &\leq (\frac{2}{h_r^2} + \frac{2}{h_\theta^2 r_i^2})V_{max}.\end{aligned} \tag{6.1}$$

Case 2: $(i,j) = (0,0)$. Then $-\Delta_N V_{ij} \leq 0$ implies

$$\begin{aligned}\frac{4}{h_r^2}V_{ij} &\leq \frac{1}{h_r^2}V_{i-1j} + \frac{1}{h_r^2}V_{i+1j} + \frac{1}{h_r^2}V_{ij-1} + \frac{1}{h_r^2}V_{ij+1} \\ &\leq \frac{4}{h_r^2}V_{max}.\end{aligned} \tag{6.2}$$

Now (6.1) and (6.2) imply $V_{max} = V_{ij}$. After removing V_{max} from the neighbors of V_{ij} and cancelling the corresponding coefficients for V_{ij}, repeating the same argument for the remaining neighbors implies $V_{kl} = V_{ij}$ for all $(k,l) \in N_{ij}$. Extending this argument, we get $V_{kl} = V_{ij} > 0$ for all $(r_k, \theta_l) \in \mathcal{T}_N \cap \overline{\Omega}$, a contradiction. Hence $V_{ij} \leq 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \overline{\Omega}$. Note that Corollary 1.21 can be proven similarly.

6.2 Proof of Corollary 1.22

Let $A\vec{V} = \vec{b}$ be a corresponding linear system of equations representing (1.28). Suppose $s_{ij} = 0$ and $g_{ij} = 0$. Then $A\vec{V} = 0$, and by Theorem 1.20 and Corollary 1.21, there holds $\vec{V} \leq \vec{0}$ and $\vec{V} \geq \vec{0}$. Thus $\vec{V} = \vec{0}$, and it follows that A is nonsingular. Thus Corollary 1.22 is proven.

6.3 Proof of Theorem 1.23

First we prove that $A \in \mathbb{R}^{N\times N}$ is a Z-matrix. For given $k \in \{1, 2, .., N\}$, let (i_k, j_k) be the corresponding index pair in $\mathbb{N}_N$. It is easy to see from (1.25) that $a_{kk} = \dfrac{2}{h_r^2} + \dfrac{2}{h_r^2 h_\theta^2} > 0$ if $(i_k, j_k) \neq (0, 0)$ and $a_{kk} = \dfrac{4}{h_r^2} > 0$ if $(i_k, j_k) = (0, 0)$. Moreover, off-diagonal elements of A consists only of $\dfrac{-1}{r_{i_k}^2 h_\theta^2} < 0$, $-\dfrac{1}{h_r^2} - \dfrac{1}{2r_{i_k} h_r^2} < 0$ and $\dfrac{-1}{h_r^2} + \dfrac{1}{2r_{i_k} h_r} < 0$ for $(i_k, j_k) \neq (0, 0)$. The last term is negative since $r_{i_k} \geq h_r$. For $(i_k, j_k) = (0, 0)$, the only off-diagonal term is $\dfrac{-1}{h_r^2} < 0$. Thus, all off-diagonal elements of A are negative, and hence A is a Z-matrix.

To prove A is diagonally dominant, we consider two cases:

Case 1: $(i_k, j_k) \neq (0, 0)$. For a fixed i_k, we obtain

$$\begin{aligned}\sum_{l=1, l\neq k}^{N} |a_{kl}| &\leq \frac{2}{r_{i_k}^2 h_\theta^2} + \frac{1}{h_r^2} + \frac{1}{2h_r r_{i_k}} + \frac{1}{h_r}\left(\frac{1}{hr} - \frac{1}{2r_{i_k}}\right) \\ &= \frac{2}{r_{i_k}^2 h_\theta^2} + \frac{2}{h_r^2} \leq a_{kk}.\end{aligned}$$

Case 2: $(i_k, j_k) = (0, 0)$.

$$\sum_{l=1, l\neq k}^{N} |a_{kl}| = \frac{1}{h_r^2} + \frac{1}{h_r^2} + \frac{1}{h_r^2} + \frac{1}{h_r^2} = a_{kk}.$$

Thus A is diagonally dominant, and by the Gershgorin circle theorem, it follows that the real parts of all of the eigenvalues of the matrix A are nonnegative. Furthermore, A is nonsingular by Corollary 1.22. Thus A is an M-matrix, and we have $[A^{-1}]_{ij} \geq 0$ for all components. This implies A is monotone. The proof is complete.

Now we create discrete sub and supersolutions for (3.9). First, we assume that $f(s) = f(0)$ for $s < 0$. Define the grid function E as the unique solution to

$$\begin{aligned} -\Delta_N E_{ij} &= 1 \quad \text{if } (r_i, \theta_j) \in \mathcal{T}_N \cap \Omega, \\ E_{ij} &= 0 \quad \text{if } (r_i, \theta_j) \in \mathcal{T}_N \cap \partial\Omega. \end{aligned}$$

Define $\phi(r, \theta) = \frac{1}{4}(1-r^2)$. Then for $(i, j) \neq (0, 0)$, we have $-\Delta_N \phi(r_i, \theta_j) = \frac{1}{4}(2+\frac{2r}{r}) = 1$, and for $(i, j) = (0, 0)$, we also have $-\Delta_N \phi(0, 0) = \frac{1}{4}(2+2) = 1$. Since $\phi(1, \theta) = 0$, it follows that $E_{ij} = \phi(r_i, \theta_j)$ for all $(r_i, \theta_j) \in \mathcal{T} \cap \overline{\Omega}$. Then $E_{ij} > 0$ for all $(r_i, \theta_j) \in \mathcal{T}_N \cap \Omega$.

We define $\underline{U}_{ij} = -cE_{ij}$ for $c > 0$ and $\overline{U}_{ij} = ME_{ij}$ for $M > 0$. Now,

$$-\Delta_N \underline{U}_{ij} - \lambda h(r_i, \theta_j) f(\underline{U}_{ij}) = -c - \lambda h(r_i, \theta_j) f(0) < 0$$

for $c \gg 1$ independent of N since h is bounded and f is not infinite semipositone. Thus $\underline{U}$ is a discrete subsolution of (3.9) for $c \gg 1$. Also, since f is sublinear at infinity, we get

$$-\Delta_N \overline{U}_{ij} - \lambda h(r_i, \theta_j) f(\overline{U}_{ij}) = M - \lambda h(r_i, \theta_j) f(M) > 0$$

for $M \gg 1$ independent of N. Thus $\overline{U}$ is a discrete supersolution of (3.9) for $M \gg 1$.

6.4 Proof of Theorem 1.24

Let $S(\mathcal{T}_N)$ denote the space of all grid functions defined over $\mathcal{T}_N$. Choose $\rho > 0$, and define the mapping $\mathcal{M}_\rho : S(\mathcal{T}_N) \to S(\mathcal{T}_N)$ by

$$\hat{U} = \mathcal{M}_\rho U, \tag{6.3}$$

where

$$\hat{U}_{ij} = U_{ij} - \rho[-\Delta_N U_{ij} - \lambda h(r_i, \theta_j) f(U_{ij})] \quad \text{if } (r_i, \theta_j) \in \mathcal{T}_N \cap \Omega, \tag{6.4-a}$$

$$\hat{U}_{ij} = 0 \qquad \text{if } (r_i, \theta_j) \in \mathcal{T}_N \cap \partial\Omega. \tag{6.4-b}$$

Clearly a fixed point of (6.3) is also a solution to (3.9).

By the definition of Δ_N and the monotonicity of f, we have (6.4-a) is non-decreasing with respect to U_{kl} for all $(k, l) \neq (i, j)$. Furthermore, for ρ sufficently small, (6.4-a) is increasing with respect to U_{ij}. Thus the mapping $\mathcal{M}_\rho$ is monotone for all $\rho > 0$ sufficiently small. Let $\leq$ denote the partial ordering for vectors, and let $W \in S(\mathcal{T}_N)$ such that $\underline{U} \leq W \leq \overline{U}$. Define $\vec{F}_0$ and $\vec{F}^0$ by

$$[\vec{F}_0]_{ij} \equiv h(r_i, \theta_j) f(\underline{U}_{ij}) \equiv h(r_i, \theta_j) f(0), \qquad [\vec{F}^0]_{ij} \equiv h(r_i, \theta_j) f(\overline{U}_{ij}).$$

There holds

$$\mathcal{M}_\rho W \geq \mathcal{M}_\rho \underline{U} = \underline{U} - \rho\big[-\Delta_N \underline{U} - \lambda[\vec{F}_0]\big] \geq \underline{U},$$
$$\mathcal{M}_\rho W \leq \mathcal{M}_\rho \overline{U} = \overline{U} - \rho\big[-\Delta_N \overline{U} - \lambda[\vec{F}^0]\big] \leq \overline{U},$$

by the monotonocity of $\mathcal{M}_\rho$ and the fact that $\underline{U}$ and $\overline{U}$ are discrete sub and supersolutions, respectively. Thus $\underline{U} \leq \hat{W} \leq \overline{U}$, and it follows by the Brouwer Fixed Point Theorem that $\mathcal{M}_\rho$ has a fixed point U such that $\underline{U} \leq U \leq \overline{U}$. Lastly, we have

$$\|U\|_{l^\infty(\mathcal{T}_N)} \leq \max\{\|\overline{U}\|_{l^\infty(\mathcal{T}_N)}, \|\underline{U}\|_{l^\infty(\mathcal{T}_N)}\} < C$$

for some constant C independent of N.

CHAPTER VII

COMPUTATIONALLY GENERATED BIFURCATION CURVES AND SOLUTIONS IN DIMENSION N=2 FOR EXAMPLES IN FOCUS 3

Example 1: Let $f(u) = \sqrt{u+1} - 2$ and $h(x, y) = 1$.

We obtained the following bifurcation diagram (Figure 23) and the outward normal derivative plot (Figure 24) for (1.19) which provide evidence that the lower solution starts changing its sign after reaching a certain value of λ.

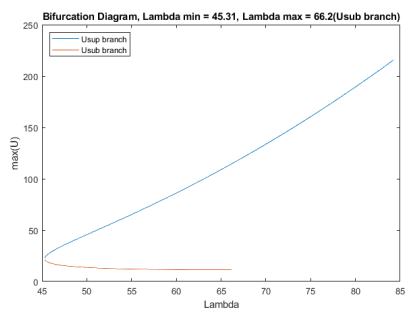


Figure 23. Bifurcation diagram

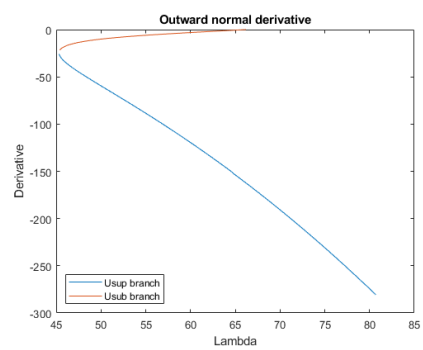


Figure 24. Outward normal derivative

The approximate lower and upper solutions at $\lambda = 65$ are graphed in Figures 25 and 26, respectively,

and corresponding plots of 2D slices can be found in Figures 27 and 28, respectively.

Example 2: Let $f(u) = \sqrt{u+1} - 2$ and $h(x, y) = 2 + y$.

We obtained the following bifurcation diagram (Figure 29) and outward normal derivative plot (Figure 30) for (1.19).

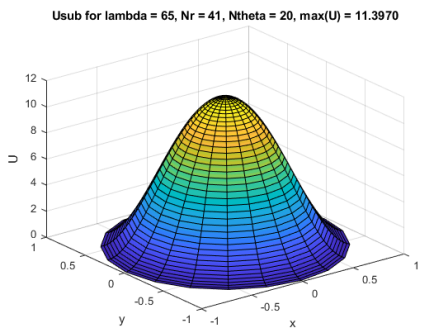


Figure 25. Lower solution at $\lambda = 65$

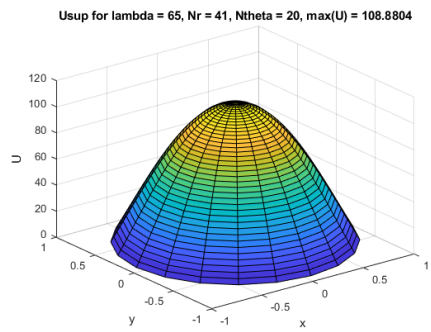


Figure 26. Upper solution at $\lambda = 65$

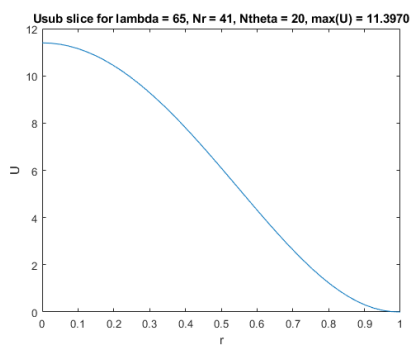


Figure 27. Slice along $\theta = 0$ of the lower solution at $\lambda = 65$

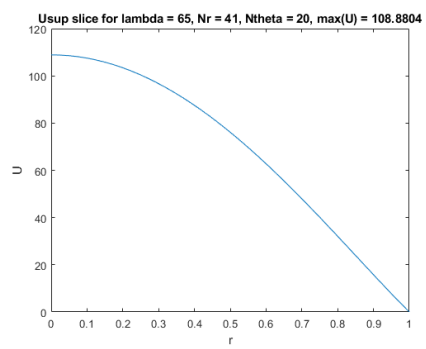


Figure 28. Slice along $\theta = 0$ of the upper solution at $\lambda = 65$

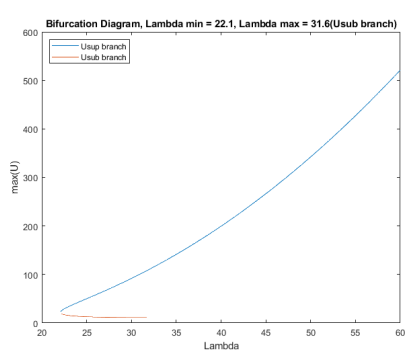


Figure 29. Bifurcation diagram

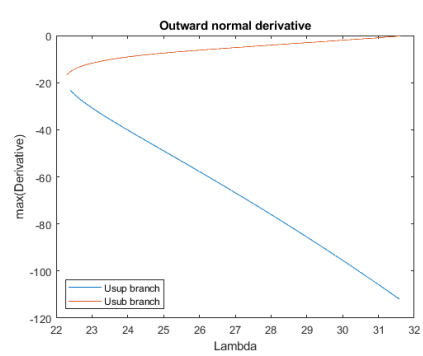


Figure 30. Outward normal derivative

The approximate lower and upper solutions at $\lambda = 26$ are graphed in Figures 31 and 32, respectively,

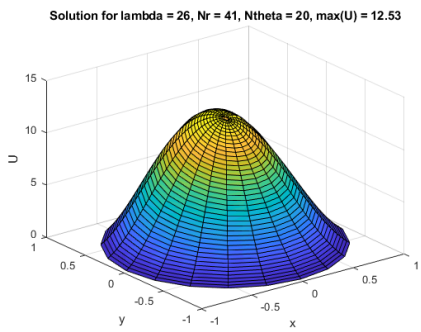


Figure 31. Lower solution at $\lambda = 26$

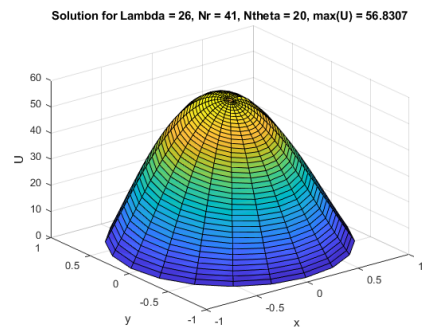


Figure 32. Upper solution at $\lambda = 26$

and corresponding plots of 2D slices can be found in Figures 33 and 34, respectively.

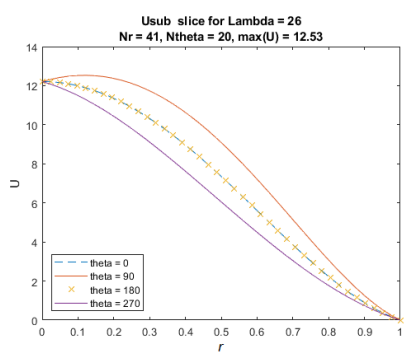


Figure 33. Slices of the lower solution at $\lambda = 26$

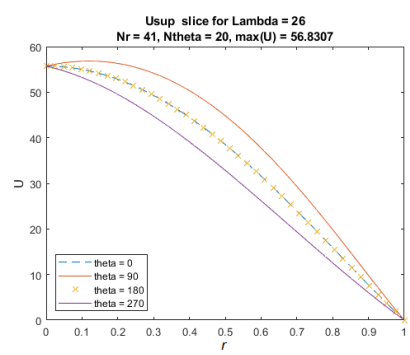


Figure 34. Slices of the upper solution at $\lambda = 26$

CHAPTER VIII

CONCLUSIONS AND FUTURE DIRECTIONS

8.1 Conclusions

In this dissertation, we analyzed positive solutions for classes of nonlinear reaction diffusion equations and systems. In Focus 1, we established new existence and uniqueness results for positive solutions for a class of infinite semipositone problems with nonlinear boundary conditions. In Focus 2, we explored the consequences of fragmentation and trait-mediated dispersal on the coexistence of a system of two mutualists by employing a model built upon the reaction diffusion framework. Our mathematical analysis demonstrates that trait-mediated dispersal can have important impacts on coexistence of mutualists. Specifically, one species can invade and persist because it has either a high growth rate, low diffusion rate, or a lower effective matrix hostility. Either way, the population for that species can build up in the patch which can favor the coexistence of the other species by reducing that species likelihood of leaving the patch. We also found that the role of coexistence facilitator can be reversed by increasing the facilitator's effective matrix hostility or decreasing the other species hostility. Finally, in Focus 3, we developed and analyzed a radial finite difference method that directly approximates solutions for classes of finite semipositone problems with Dirichlet boundary conditions. Using this method, computationally, for various examples, we obtained approximate bifurcation diagrams for positive solutions. In particular, our computational results show the uniqueness of positive solutions when a parameter is large, existence of two positive solutions for

a certain range of the parameter, and non-existence of positive solutions when the parameter is small.

8.2 Future Directions

8.2.1 Existence and uniqueness results for p-Laplacian infinite semipositone problems with nonlinear boundary conditions

We plan to extend the results in Focus 1 for the corresponding p-Laplacian problem:

$$\begin{cases} -(\phi(u'))' = \lambda h(t) f(u) \ ; \ (0,1) \\ u(0) = 0 \\ u'(1) + c(u(1))u(1) = 0, \end{cases} \tag{8.1}$$

where $\phi(s) = |s|^{p-2}$, $p > 1$.

Here, we again plan to analyse the case $\lim_{s \to 0^+} f(s) = -\infty$ (infinite semipositone) and aim to prove existence and uniqueness of a positive solution for $\lambda \gg 1$.

8.2.2 Trait-mediated dispersal on coexistence of competitors

We plan to extend the results in Focus 2 for systems of the form:

$$\begin{cases} -\Delta u = \lambda u(1-u); \ \Omega \\ -\Delta v = \lambda r v(1-v); \ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} g(v) u = 0; \ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} h(u) v = 0; \ \partial\Omega, \end{cases} \tag{8.2}$$

where $\lambda > 0$, $r > 0$ are parameters and $g, h \in C^1([0,\infty),(0,\infty))$ are **increasing** functions of v, u respectively (u and v are competitive on the boundary). For (8.2),

we plan to obtain nonexistence and coexistence of positive solutions for certain ranges of λ and uniqueness of a positive solution for $\lambda \gg 1$.

8.2.3 Ecological model of two species with competition in the interior of the patch

We plan to study the model:

$$\begin{cases} -\Delta u = \lambda u(1-u-b_1 v);\ \Omega \\ -\Delta v = \lambda r v(1-v-b_2 v);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0;\ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_2 v = 0;\ \partial\Omega, \end{cases} \tag{8.3}$$

where $\lambda > 0$, $r > 0$, γ_1 and γ_2 are positive parameters, and $b_1 > 0$, $b_2 > 0$ are two constants. For this ecological model, we plan to obtain nonexistence and coexistence of positive solutions for certain ranges of λ and uniqueness of a positive solution for $\lambda \gg 1$.

8.2.4 Exact bifurcation diagrams for semipositone problems with nonlinear boundary conditions

We plan to extend the results in Focus 3 for semipositone problems with nonlinear boundary conditions of the form:

$$\begin{cases} -\Delta u = \lambda h(x,y) f(u)\ ; & \Omega, \\ u = 0\ ; & r = a, \\ \frac{\partial u}{\partial \eta} + g(u)u = 0\ ; & r = b, \end{cases} \tag{8.4}$$

where $a, b \in \mathbb{R}^+$, $\Omega = \{(x, y) \in \mathbb{R}^2 \mid a < r = \sqrt{x^2 + y^2} < b\}$, $\lambda > 0$ (parameter), h is a positive bounded continuous weight function, and $f : [0, \infty) \to \mathbb{R}$ is a C^2 nondecreasing function such that $f(0) < 0$ (finite semipositone) and $g \in C([0, \infty), (0, \infty))$.

8.2.5 Exact bifurcation diagrams for infinite semipositone problems

We plan to extend the results in Focus 3 for infinite semipositone problems of the form:

$$\begin{cases} -\Delta u = \lambda h(x, y) f(u) \ ; & \Omega, \\ \quad u = 0 \ ; & r = a, \\ \beta \frac{\partial u}{\partial \eta} + g(u) u = 0 \ ; & r = b, \end{cases} \tag{8.5}$$

where $a, b \in \mathbb{R}^+$, $\Omega = \{(x, y) \in \mathbb{R}^2 \mid a < r = \sqrt{x^2 + y^2} < b\}$, $\lambda > 0$ (parameter), $\beta \in \{0, 1\}$, h is a positive bounded continuous weight function, and $f : (0, \infty) \to \mathbb{R}$ is a C^2 nondecreasing function such that $\lim_{s \to 0^+} f(s) = -\infty$ (infinite semipositone) and $g \in C([0, \infty), (0, \infty))$.

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