Imagery and Utilization of an Area Model as a Way of Teaching Long Division: Meeting Diverse Student Needs

By: Kerri Richardson, Sarah Pratt and Stephanie Kurtts

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Article:

The teaching and learning of long division at the elementary level and beyond has presented a longstanding challenge for teachers and students alike. As mathematics teacher educators and as a specialized educator, we address the issue by analyzing some of the challenges involved in the teaching and learning of long division – particularly focusing on students who struggle in mathematics. Our inspiration comes from two shared experiences. First, a lesson taught by one of our graduate level, in-service special education teachers inspired us to consider how other teachers could consider teaching division by using an area model. The lesson that began our initial conversations will be shared later in this article to exhibit one teacher's use of area in teaching division in an interactive manner. Second, these conversations led to our collaborative work on a book chapter that centered on specialized mathematics education (Pratt, Richardson, & Kurtts, in press) In our chapter we focused on the significance of epistemological perspectives and how imagery relates to effective mathematics teaching and learning.

In this article we narrow our focus on the mathematical content of our work – area as a model for the teaching of long division as it relates to mathematical representation, specifically visualizing numbers, concepts, and images. We consider why some students come to "see" the mathematics picture, while others simply reproduce endless procedures, are unable to reason through problems, and ultimately lag behind their peers in the mathematics classroom. It is vital that struggling students whether identified as needing special education services or not, receive instructional approaches with imagery as a theme (National Mathematics Advisory Panel, 2008a, 2008b). We offer insights into some of the challenges students with learning disabilities face then we provide a description of an imagery-based division lesson to suggest alternative teaching strategies that could be utilized to help all learners.

Imagery

What do we mean by imagery and why do we choose to use this term? Wheatley (1998) describes imaging as an activity rather than as an isolated event that just "happens" in our mind. He notes that many perceive imaging as taking a picture in one's mind and then reproducing the image in the same way everyone else has "taken" the picture. Instead, he sees imaging as much more than that; he sees the process as constructing an image, re-presenting the image, and transforming the image. It is in constructing an image where many mathematics lessons stop, assuming everyone sees and interprets the same idea in the same way. Wheatley argues that it is more often n re-presenting and transforming an image where meaning making and reasoning occur. In re-presenting an image or developing one's own problem-solving map, a host of mental activity occurs between constructing and re-presenting. For example, several students draw and interpret a map in different ways yet still all arrive at the same place. Transformation of the image is where the manipulation of the image takes place, because the student then changes the image, experiments with it, and utilizes the image in a way that might lead to a solution for a particular problem.

In long division, however, images are often left out or utilized at a very basic level. Teachers are limited in their own images of the division algorithm, and often the incorporation of manipulatives is limited to telling students how to perform the algorithm with Base-10 Blocks before moving to the steps of the standard algorithm. We agree with Wheatley & Reynolds (1999) thatmanipulatives should only be used f they are effective n allowing participants to generate images that connect to mathematical concepts such as those related to long division (p. 26).

When a teacher mplements a lesson n which each participant is afforded opportunities to represent, explore, and describe mathematical concepts, meaning-making and imaging can occur. This leads to conceptual understanding, which is learning at a deeper level than merely memorizing. Manipulatives must be made available in a participatory manner, not through direct instruction, in order for this type of learning to occur. Imaging can happen when students are afforded these opportunities.

NCTM, Representations, and Long Division

The National Council of Teachers of Mathematics *Principles and Standards* (2000) for school mathematics also refer to imagery in a similar sense, using representation as one of their process standards. NCTM (2000) states:

Instructional programs from prekindergarten through grade 12 should enable all students to

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena. (p. 67)

NCTM's stance on representational thinking aligns with Wheatley's (1998) ideas on imaging as an activity. Both note the importance of an image or representation in mathematical problem solving and their definitions span all grade levels. The main difference is that Wheatley centralizes imagery while NCTM embeds it as one of their five processes. We believe, however, that these two perspectives are not necessarily in contradiction of one another, but instead offer varying perspectives that add layers to our thinking about mathematics.

In terms of the arithmetic operation of division, NCTM continues to stand behind the importance of conceptual understanding (Gregg & Gregg, 2007; Martin, 2009). While examining both U.S. and Chinese elementary teachers' understandings of division, Ma (1999) notes that division is the most difficult of the four operations. Although her work focused on the division of fractions and not whole numbers, connections were made between the two. For example, she found that some Chinese teachers would refer to long division with whole numbers when explaining the algorithm used to divide two fractions. One teacher noted that the rule of "dividing by a number is equivalent to multiplying by its reciprocal" (p. 58) is not introduced until the division of fractions but applies to whole numbers as well (e.g. dividing by 2 is equivalent to multiplying by 1/2).

We now turn our attention to a different image of division, namely an area model. Below we share an example of a long division mathematics lesson taught by one of our graduate level, in-service special education teachers. She has taughtfor three years and while creating an imagery activity for her students, she made some interesting connections to the concept of long division. We conclude with why division as area has implications for students' development of number sense and algebraic understandings.

One Teacher's Example of Teaching Long Division

During her graduate methods course, Ms. Swett, a full-time special education teacher, began thinking critically about issues surrounding long division. One of her strongest critiques centered on the division algorithm and if it made sense – concluding that it did but only from a mathematical standpoint. (For her analysis, please see Pratt Richardson & Kurtts, in press.) As a result of her exposure to different approaches to teaching mathematics, she offered a well-considered, imagery-laden lesson to her special education students. The procedure of long division troubled her and her students. She reexamined her own way of teaching division and tried a conceptual approach, rather than procedural, and found this new method to be effective for her students.

Division as Fair Sharing

Ms. Swett began her lesson using the familiar "fair sharing" (measurement) approach to help students make connections to previous experiences with division. However, she took it several steps further by including a variety of discussion questions. For example, she first asked, "There are 73 pieces of fudge to be shared by 3 people. How many pieces of fudge does each person receive?" She asked the students to represent or build the number 73 and together determined that it looked similar to this: | | | | | | | ... (7 tens to represent 70 and 3 dots to represent 3). She then asked her students to share the fudge with three groups - each group receiving two sets of the tens each but with one group of 10 left out. This first discussion point encouraged the students to come to an understanding that the ten could not be shared equally unless it was exchanged for 10 ones. This exchange of one ten into ten ones became a powerful representation of the "borrowing" strategy often given to students with no meaning associated with it. By putting all exchanged ones with the other three ones, the students had 13 units to divide (fair share). In this step each of the three groups received four units, which left one of the units, which led to a rich discussion about the final piece. Together the students decided that this must become the remainder because the one could not be exchanged for anything smaller. Thus, the students each had constructed on their paper the image of fair sharing as: | |.... | | .... | |.... with a remainder of .. As a result, the students created a visual of 73 divided into 3 groups coming up with 24 and a remainder of 1.

Division as Area

Following this first imaging activity, Ms. Swett created an activity to help her students further their thinking about long division; this time she used an area model. Base-10 blocks were distributed, along with the question: "How many groups of 21 are in 483 units?" Still working in pairs, students created a rectangle (thus area) of 483 units in which one of the dimensions were 21 units long (or wide). Once this rectangle was constructed, the students counted that there were 23 units on the other dimension. (See Figure 1 for this image.)

Figure 1. Using an Area Model

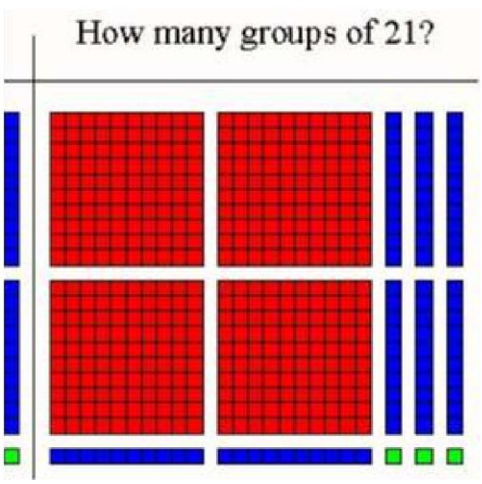

Figure 2. The Division Problem

21) 483	
− 420	20 groups of 21
63	
− 63	+ 3 groups of 21
0	23 groups of 21

In keeping with Wheatley and Abshire's (2005) argument that manipulatives must be meaningful and connect with the formal mathematics that they are displaying, Ms.Swett concluded her lesson with a different interpretation of the division algorithm, one that connects directly with the blocks. As demonstrated in *Figure 2*, she led the class in the division algorithm to calculate 483 divided by 21, first by counting the number of groups of tens, then the number of ones, and completing this with no remainder. By concluding with this analysis of the division algorithm, Ms. Swett connected the formal algorithm with the area model of the blocks and the fair share model to provide her students opportunities for imaging.

Why This Impacts Conceptual Understanding

After her lesson, Ms. Swett shared that when she tried a new approach as a result of her epistemological belief changes, her students learned with ease and confidence. She believes this new form of mathematical engagement worked in her classroom because:

> the child has visually seen what it looks like to divide. Many of experiences with this concrete method of dividing will eventually lead them to understand the actual algorithm and then be able to use it, which is exactly how they learned to add and subtract— concretely first.

Ms. Swett was transformed as a teacher in this process. She now sees the remarkable benefit of allowing students to explore mathematical concepts in imaging activities so they could display and communicate their own understandings of the concept. As a teacher, she can listen and adapt her instructional strategies as she deems appropriate for her students. She claims that this exploration into the division algorithm excitedboth her and her students that they can all do math. Her work is a specific example that affirms Martin's (2007) and Wheatley's (1998) arguments, in that the use of imaging to understand the standard division algorithm was an effective strategy for her students.

The Importance of the Area Mode in Teaching Long Division to Students with Learning Disabilities in Mathematics

Why is t important to include the area model n teaching long division to students who struggle with mathematical concepts? Students with learning disabilities in mathematics and students who have difficulty in grasping higher level mathematical skills and concepts need to have clear and explicit instruction that helps them "see" the mathematics picture we are teaching. The National Mathematics Advisory Panel (2008b) has identified (a) systematic and explicit instruction, in which teachers guide students with a clearly defined instructional sequence; (b) self-instruction, in which students implement such tools as self-monitoring checklists to prompt them to remember problem-solving steps (see also Maccini & Gagnon, 2006); (c) peer tutoring involving pairs of students working together on mathematical tasks; and, (d) visual representation and imagery as four approaches to instruction that have a research base to support their implementation for students with learning disabilities. Students with learning disabilities in mathematics struggle with cognitive processing, all of which have an effect on visual processing, visual memory, and visual-spatial relationships. These difficulties can impact mathematics proficiency, specifically conceptual understandings and procedural fluency (Kilpatrick et al., 2001).

These evidence-based instructional practices support effective instruction delivered to students with learning disabilities (Butler, Beckingham, & Lauscher 2005) The four instructional approaches as listed above that should be used in teaching students who struggle with mathematics will aid students in developing computational skills, problem-solving, and thinking about meaningful applications of mathematics to their daily lives. Here we have highlighted the use of visual representation and imagery to enhance the concept of long division. Clearly, the concrete framework assists students' understandings of the foundations of process, as well as the steps necessary to solve problems.

Connecting Number Sense to Algebraic Understandings and Beyond

While we focused our example on the standard long-division algorithm, we believe that the use of an area model for division is the first step in connecting to understanding algebraic representations of division. Smith & Thompson (2008) argue that "what is often missing is any linkage between numbers and symbols and situations" (p. 96). Specifically, the concept of factoring in algebra is often disassociated from the concept of numerical division. If, instead, factoring is seen as an extension of division students can make connections between numerical sense and algebraic understandings.

For example, dividing 132 by 11 can be represented with base-10 blocks (see Figure 3). 132 tiles are placed as a rectangle, with one dimension of the rectangle as 11. The other dimension then works out to be 12. Therefore, the image generates the solution of $132 \div 11 = 12$.

Now replace the problem with the algebraic question of what is $(x^2 + 3x + 2)$ divided by $(x + 1)$. The use of

Algeblocks (or algebra tiles) replaces the base-10 blocks. The rectangle of $x^2 + 3x + 2$ is created, with one dimension of $(x + 1)$. The other dimension becomes. (See Figure 4.) The solution then, is that $(x^2 + 3x + 2) \div (x + 1) = (x + 2)$.

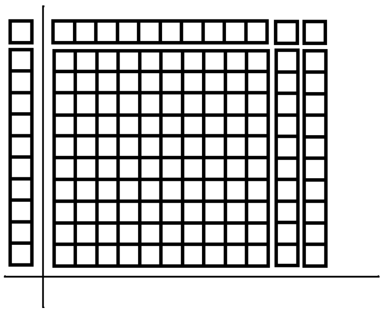

Figure 3. Area Model of Division of 132 ÷ 11.

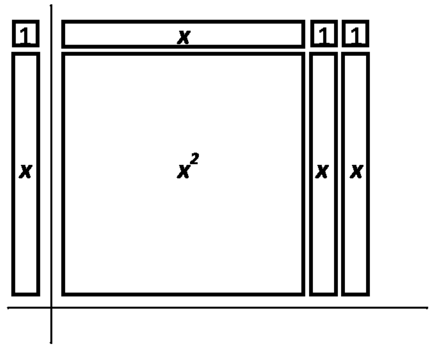

Figure 4. Area Model of Division of $(x^2 + 3x + 2) \div (x + 1)$.

These two examples are chosen intentionally to demonstrate similarities between the concepts. The images of 132 and $x^2 + 3x + 2$ are the same in structure, as are 11 and $x + 1$ and 12 and $x + 2$, respectively. The algebraic problem allows the base value of 10 in the numeric problem to be replaced with an unknown variable quantity of x. The image created is a powerful tool to provide opportunities for students and teachers alike to make connections between what is known about division of whole numbers and division of polynomials. Furthermore, geometric understandings of dimensions (length and area) can also be generated and enriched. Together, the image and the symbolic representations supply numerous ways of conceptually understanding mathematical ideas. Further mathematical ideas, such as integer multiplication and division, volume, rate of change, and accumulation can extend what is developed in the activity of imaging with division as area.

In this article we echo the sentiment of Smith & Thompson who "advocate an early emphasis on developing children's ability to conceive of, reason about, and manipulate complex ideas and relationships, as an equal complement to numerical reasoning and computation" (p. 98). We have worked with and noticed remarkable changes in the dynamics and understandings of mathematical ideas, specifically division, when we first provide imaging activities that will engage participants in developing numerical sense.

These types of activities need to start early and build through all grades. Elementary teachers should follow MacGregor & Stacey (1999), who suggest that there are five aspects related to numerical understandings that will assist students in the transition to algebraic reasoning. The aspects are: "understanding equality; recognizing the operations; using a wide range of numbers; understanding important properties; and, describing patterns and functions" (p. 78). Middle school teachers should build on these components when working on conceptual understandings of rational numbers (e.g., Ma, 1999; Azim, 2002) High school teachers should continue by utilizing imaging activities as they engage in mathematical concepts in the secondary curriculum (e.g., Vinogradova, 2007; Quinn, 2003). Students who engage in imaging activities throughout their educational experiences will succeed in achieving what NCTM (2000) outlines in the content and process standards as important components of mathematical learning.

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