Boundedly Simple Groups Have Trivial Bounded Cohomology

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The goal of this short note is to observe that the singular part of the second bounded cohomology group of boundedly simple groups constructed in [3] is trivial. Recall that a group G is called m-boundedlv simple if every element of G can be represented as a product of at most m conjugates of g or g^{-1} for any $g \in G$.

We recall that bounded cohomology $H_b^*(G)$ of a group G (we will be considering only cohomology with coefficients in the additive group of reals $\mathbb{R}$ with trivial action, so in our notations for cohomology the coefficient module will be omitted) is defined using the complex

$$\cdots \longleftarrow C_b^{n+1}(G) \xleftarrow{\delta_b^n} C_b^n(G) \longleftarrow \cdots \longleftarrow C_b^2(G) \xleftarrow{\delta_b^1} C_b^1(G) \xleftarrow{\delta_b^0=0} \mathbb{R} \xleftarrow{\delta_b^{-1}=0} 0$$

of bounded cochains $f\colon G \times \cdots \times G \to \mathbb{R}$, and $\delta_b^n = \delta^n|_{C_b^n(G)}$ is the bounded differential operator. Since $H_b^0(G) = \mathbb{R}$ and $H_b^1(G) = 0$ for any group G, investigation of bounded cohomology starts in dimension 2. One observes that $H_b^2(G)$ contains a subspace $H_{b,2}^2(G)$ (called the *singular part* of the second bounded cohomology group), which has a simple algebraic description in terms of quasicharacters and pseudocharacters, and the quotient space $H_b^2(G)/H_{b,2}^2(G)$ is canonically isomorphic to the bounded part of the ordinary cohomology group $H^2(G)$. See [2] for background and available results on bounded cohomology of groups.

A function $F\colon G \to \mathbb{R}$ is called a *quasicharacter* if there exists a constant $C_F \geqslant 0$ such that

$$|F(xy) - F(x) - F(y)| \leqslant C_F \quad \text{for all } x, y \in G.$$

A function $f\colon G \to \mathbb{R}$ is called a *pseudocharacter* if f is a quasicharacter and in addition

$$f(g^n) = nf(g) \quad \text{for all } g \in G \text{ and } n \in \mathbb{Z}.$$

We use the following notation: $X(G)$ = the space of additive characters $G \to \mathbb{R}$; $QX(G)$ = the space of quasicharacters; $PX(G)$ = the space of pseudocharacters; $B(G)$ = the space of bounded functions. Then

$$H_{b,2}^2(G) \cong QX(G)/(X(G) \oplus B(G)) \cong PX(G)/X(G) \tag{1}$$

as vector spaces (cf. [2, Proposition 3.2 and Theorem 3.5]). Special interest in $H_{b,2}^2$ is motivated in part by its connections with other structural properties of groups such as commutator length [1] and bounded generation [2].

Theorem 1 *If G is a boundedly simple group, then $H_{b,2}^2(G) = 0$.*

Proof. In view of (1) it suffices to show that the group G does not have any nontrivial pseudocharacters. First, we observe that every pseudocharacter is constant on conjugacy classes. Indeed, suppose that $f \in PX(G)$ and $|f(gxg^{-1}) - f(x)| = a > 0$ for some x, $g \in G$. Then on the one hand

$$|f(gx^n g^{-1}) - f(x^n)| = |f(gx^n g^{-1}) - f(x^n) - f(g) - f(g^{-1})| \leqslant 2C_f$$

is bounded independent of n, on the other hand

$$|f(gx^n g^{-1}) - f(x^n)| = n|f(gxg^{-1}) - f(x)| = na \to \infty \quad \text{as } n \to \infty,$$

whence a contradiction.

Suppose that G is m-boundedly simple. Then every element x of G can be written in the form

$$x = g_1 \cdots g_k$$

where $k \leqslant m$ and every g_i is a conjugate of either g or g^{-1} for some fixed $g \in G$, whence $|f(g_i)| = |f(g)|$ for all $i = 1, \ldots, k$. Then

$$\begin{aligned} |f(x)| &= |f(g_1 \cdots g_k) - f(g_1) - \cdots - f(g_k) + f(g_1) + \cdots + f(g_k)| \\ &\leqslant |f(g_1 \cdots g_k) - f(g_1) - \cdots - f(g_k)| + |f(g_1)| + \cdots + |f(g_k)| \\ &\leqslant (m-1)C_f + m|f(g)| \end{aligned}$$

which implies that f is bounded on G, hence must be trivial. □

References

[1] Ch. Bavard, Longueur stable des commutateurs, *Enseign. Math.* **37** (1991), no. 1–2, 109–150.

[2] R.I. Grigorchuk, Some results on bounded cohomology, *London Math. Soc. Lecture Note Ser.* **204** (1995), 111–163.

[3] A. Muranov, Diagrams with selection and method for constructing boundedly generated and boundedly simple groups, Preprint, 2004.

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