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OPERATIONAL LEVELS OF COGNITIVE STAGE ACHIEVEMENT AND REPRESENTATIONS OF COGNITIVE STRUCTURES USED IN MATHEMATICAL PROBLEM SOLVING BY YOUNG ADULT PROSPECTIVE TEACHERS

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OPERATIONAL LEVELS OF COGNITIVE STAGE ACHIEVEMENT AND REPRESENTATIONS OF COGNITIVE STRUCTURES USED IN MATHEMATICAL PROBLEM SOLVING BY YOUNG ADULT PROSPECTIVE TEACHERS

by

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APPROVAL PAGE

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ABSTRACT

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The differential achievement among concrete, transitional, and formal operational levels of cognitive stage in young adults was investigated. Also, the relations were tested among (a) cognitive stage achievement, (b) mathematical problem solution success, and (c) problem-solving strategies used in mathematics (spontaneous figure labeling, chosen solution strategy, and its match with actual strategy).

The cognitive interviews replicated those of Piaget and Inhelder (1975) on notions of chance and probability with the use of Green's (1978) quasi-standardized procedural and methodological suggestions. The relations among cognitive achievement, success, and strategies used with math problem solving were tested with an inventory for representations of mathematical cognitive structure (Clark & Reeves, in press). Forty subjects represented prospective teachers. The mean age was 25.5 years.

The results of the cognitive interview analysis revealed that 27% of the young adults achieved consolidated formal operations, 40% demonstrated a partial or transitional achievement towards formal

thought, and 33% did not perform beyond concrete operations. Kendall's Tau correlation revealed that cognitive stage achievement and successful mathematical problem solving were not independently related. The math problem solving strategies were tested to be independently related both to cognitive stage achievement and to math problem solution success.

The conclusions were that (a) young adults demonstrated differentiated cognitive stage achievement among concrete, transitional, and formal operational levels; (b) cognitive stage achievement and problem solving success in mathematics are not independent constructs; and (c) problem solving strategies of spontaneous figure labeling, identifying solution strategy, and matching it with employed strategy are not related to the constructs in (a) and (b). Research was recommended for cognitive stage achievement of young adults, its relation with discipline-specific cognitive structures, and experimental curricula in higher education using cognitive developmental goals and problem-solving methodologies.

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CHAPTER I

INTRODUCTION

According to Piagetian theory, adults are capable of abstract reasoning. For example, they should be able to solve simple word problems from algebra and geometry and to explain the strategies they use. This expectation would be true especially for prospective mathematics teachers. The purpose of this research was to investigate cognitive stage achievement in young adults and its relation to mathematical problem solving.

Jean Piaget has been the most referenced spokesman on cognitive development. He observed and interviewed children to assess the quality of their reasoning. From this research he formulated a theory of cognitive development. Early studies were concerned with language and thought (1926b), judgment and reason (1926a), conceptions of the world and its causes (1929); and the scope broadened to include children's moral judgment (1932) and the origins of intelligence (1952). His research on various qualities of intellectual development has been reported in approximately 200 publications. He and Barbel Inhelder updated interpretations of the research in refining the cognitive stage theory (1969b).

Piaget established a qualitative distinction between the thinking of children and adults. Inhelder and Piaget initiated the investigation of the cognitive functioning of adolescents and adults (1958). They have refined the analysis through studying how adolescents reason in different areas of knowledge. One example was the investigation of formal operations, the highest stage of cognitive development, in adolescents relating to notions of chance and probability (first reported in 1961, published in English in 1975). Two of these interview tasks were replicated in this research using young adult subjects.

The distinction of formal operations achievement has not been clear between adolescents and adults, much less among various ways to group adults. Historically, scientific theories of human development orginated with a child's conception and stopped at the conclusion of puberty. The assumption was that the development of intelligence and other human qualities followed the pattern of the more obvious physical development. Legal adult status has been defined at physical maturity, for example, when citizens can vote, serve in the military, work without legal restrictions, and own property. Strong suspicion emerged recently that further refinement among adolescent and adult levels of thinking was needed: from researchers (Baltes & Labouvie, 1973; Erikson, 1968; Gould, 1978, Horn, 1970; Horn & Catell, 1967; Kohlberg, 1973; Levinson, 1978; Vaillant, 1977), popular literature (Sheehy, 1976), and life-span texts in developmental psychology (Craig, 1980; Kastenbaum, 1979; Schell & Hall, 1979; Vander Zanden, 1978). Arlin (1975) speculated that a stage beyond formal operations existed to account for what appeared to be the creative intelligence of some scientists and artists. In an effort to contribute to the understanding of cognitive operations in young adults, this research replicated studies of operational behavior in which adolescent subjects were used.

Nature of the Problem

This research investigated formal operational reasoning in young adults. What was known about adolescent cognition originated with Inhelder and Piaget (1958) and has been modified in replication research. Piagetian descriptions of adolescent reasoning have dominated the field. For example, his theory and research were the singular source of knowledge on adolescent cognition in the current proliferation of texts in child psychology, human development and developmental psychology (Alexander, Roodin, & Gorman, 1980; Ausubel & Sullivan, 1970; Craig, 1980; Fein, 1978; Gallagher & Mansfield, 1980; Gardner, 1978; Learner & Spanier, 1980; Papalia & Olds, 1979; Rice, 1978; Vander Zanden, 1978).

Piaget was fascinated with how children reason. He observed their behavior, interviewed them for their understandings, and set up simple, manipulative experiments for them to describe their experiences. These behaviors have been interpreted as representing certain mental abilities or unseen mental structures. He organized his research into a theory of cognitive development, an explanation of how humans refine their mental behavior as a function of physical maturation and environmental experiences. The research on the thinking of children (1926a, b, 1952) established qualitative distinctions with adult thought. The descriptions have been accepted among both theoretical (Ausubel & Sullivan, 1970; Elkind & Flavell, 1969; Flavel, 1977) and educational psychologists (Bruner, 1966, 1973; Furth, 1970; Furth & Wachs, 1975).

Piaget's pioneering research on the cognitive structures of adolescence (with Inhelder, 1958) created more questions than answers. Adolescents reasoned differently from children. They considered multiple options simultaneously, reasoned abstractly with the aid of symbols, and deliberated hypothetical paradoxes. Yet this period was perplexing: choices became overwhelming, emotions appeared to conflict with reason, and an emerging introspection reflected an awkward self- and social-consciousness (Elkind, 1974).

Piaget used the research on adolescents to define the formal operational stage of cognitive development. Piaget's formal operational characteristics of adolescents have been validated through replication (Jackson, 1965; Kuhn & Anageler, 1975; Lovell, 1961b; Neimark, 1975b; Webb, 1974). His interpretations, however, of adolescent thinking have not been accepted as universally as his conclusions on young children.

For example, Berry and Dasen (1974) in a cross-cultural replication did not verify the formal operational characteristics of youngsters in nonliterate cultures. Dulit (1975) discovered that only highly gifted males achieved formal operations in a study that included females. Neimark (1975a) noted wide variation in thinking among adolescents, especially higher cognitive functioning for those with former experiences with the research tasks; he concluded that the stage was inadequately defined. Piaget (1972b; & Inhelder, 1973) admitted that the investigation of adolescent thought was incomplete. Undetected prior experiences may have been necessary to formal operations achievement. Other researchers (Kohlberg & Meyer, 1972; Wason & Johnson-Laird, 1972) doubted that all adults achieve formal operations.

The study was conducted to identify the developmental relation between cognitive stage and mathematical problem solving in young adults. The extent of young adults' formal operations achievement was investigated. And the relations between operational levels of cognitive stage and mathematical problem solving were identified.

Background for the Study

Adolescent and Young Adult Reasoning

Based on his research findings Piaget theorized that normal human beings could develop the thinking behaviors of formal operations by approximately age 15 (Flavell, 1977; Ginsburg & Opper, 1978; Inhelder &

Piaget, 1958; Piaget, 1967, 1972b). Assuming his analysis to be accurate, many educators organized the educational experiences of adolescents according to the descriptions of formal operations (Bruner, 1960, 1966, 1973; Combs, 1974; Elkind, 1976; Furth, 1970, 1975; Gwynn & Chase, 1969; Holt, 1967; Kohlberg & Mayer, 1972; Macdonald, Wolfson, & Zaret, 1973; Rogers, 1969; Schwab, 1962; Silberman, 1970). For example, abstract and logical reasoning was expected of a high school geometry student. Hypothetical interpretations of intangible symbols were required in a college literature course. A high school history teacher expected students to analyze the multiple details of events, documents and human motives, to resolve contradictory interpretations, and to defend a contemporary legal position. Although Piaget did not specify how to apply the research and theory of formal operations into educational settings, many high school and college educators used Piaget to justify their instructional decisions for adolescents and young adults.

There appeared to be an experimental disparity over the achievement of formal operations in young adults between Inhelder and Piaget (1958) and a host of studies in the 1970's (Blasi & Hoffel, 1974; Danner & Day, 1977; Papalia & Bielby, 1974; Schwebel, 1975; Tomlinson-Keasey, 1972). For example, Kuhn, Langer, Kohlberg, and Haan (1977) studied the cognitive stage achievement of 265 adolescents and adults, aged 12 to 50 years. They reported that only 30-50% of the sample demonstrated any formal operational reasoning on each of several problems. Nearly one-fourth of the sample performed at the concrete operational level on all tasks, while the remaining portion of the sample performed transitionally between the concrete operational and formal operational levels. Such findings indicated that adolescents and young adults varied in the level of their cognitive attainment.

Adolescent and Young Adult Problem-Solving Experiences

Problematic inquiry has been a characteristic associated with formal operational thought (Ausubel & Sullivan, 1970; Scandura, 1977). In schools, teachers of adolescents have chosen learning goals that require abstract reasoning. English teachers asked students to imagine multiple, possible explanations about the symbolic whiteness of Moby Dick or of snow in a Robert Frost poem and to document their interpretation logically. Language teachers asked for intuitive, cultural explanations of idioms. A college math student could have mistaken an assignment of a tautological proof as originating in a philosophy class.

The cognitive structures of problem solving were interdisciplinary. The rigor of the scientific method was based on it (Kerlinger, 1973). Its precision was valued as the essence of mathematical learning (Scandura, 1977). Some educational philosophers have adopted inquiry as the essence of an education: "a problem well put is half solved" (Dewey, 1938, p. 108).

Both the process of problem solving (Bruner, 1960; Parker & Rubin, 1966) and the structure of problem solving (Ford & Pugno, 1964) have been hailed as the interdisciplinary purpose of all subjects. Dewey's pedagogical creed of 1897 anticipated the problematic, structural essence of education: "I believe, finally, that education must be conceived as the continuing reconstruction of experience; that the process and the goal of education are one and the same thing" (Dewey, 1915, p. xix). Dewey's progressive education movement of the 1930's in America was rekindled by the open school movement (1965-1975) inspired by Piaget. For both men, problem solving was both a description and goal of human achievement.

The problematic thought of formal operations was the least researched of Piaget's cognitive stages. Teachers of adolescents and young adults were assumed to be qualified if they were proficient only in their respective subject matter. Knowing how formal operational thinkers reasoned appeared to be irrelevant. Secondary (grades 7-12) teaching certifications generally required only one course in educational psychology, which was usually preoccupied with testing practices. No expectation existed for teachers in higher education to understand how young adults reasoned. Problem solving, as synthetic learning (Bloom, 1956), appeared unamenable to the analytic (piecemeal) tradition of college syllabi and texts. Major blame has been traced to the training of prospective teachers

(Piaget, 1970c). Problem-solving curricula were more work for both teachers and students, and the learning was difficult to assess (Bruner, 1960).

Problem Solvin and Mathematics Education

Ausubel and Sullivan (1970) described problem solving as "any activity in which both the cognitive representations of prior experience and the components of a current problem situation are reorganized in order to achieve a designated objective" (p. 630). They further identified three levels of problem solving: (a) Trial-and-error was characterized by a series of random or systematic choices until the successful "right answer" appeared to work. (b) An underlying principle was assumed to exist. One created a rule that would explain the problem in a way that could be tested. (c) Alternatively, one sought to discover a law or system of interrelationships indemic to the solution of the problem. Such insight involved a model transferred from a previous experience or a fundamental cognitive restructuring. Insights emerged suddenly or discontinuously. Thinking was a special case of problem solving, "When the activity is limited to the manipulations of images, symbols, and symbolically formulated propositions, and does not involve overt manipulation of objects" (p. 630). Thinking appeared to describe the abstract process of formal operations, a process that neither depended upon nor excluded tangible manipulations to solve problems.

Ausubel and Sullivan (1970) reinforced Piagetian notions of cognitive development that were reflected in problem solving approaches. Reorganization of both cognitive representations and problem conceptualizations occurred as a function of cognitive achievement. For example, as egocentricity and subjectivity of children's thought declined with age, qualitative increases in thinking and problem solving occurred. Agreement was uniform in the research on this point, perhaps diminishing the contradictory evidence of stage achievement.

Problem solving was a multidimensional process not to be equated with any one of the factors that influenced its development. Intelligence often correlated positively to trial-and-error (Nelson, 1936) and insightful problem solving (Stevenson et al., 1968); yet it was only a part of a problem-solving ability (Gallagher, 1964). Other factors that influenced problem solving included: grade in school, cumulative experience, cognitive traits like flexibility and curiosity, cognitive style, motivational traits, and success experiences.

Piaget (1967) described an essential distinction between formal (abstract) and concrete operations as the ability to solve problems in different ways. Concrete problem solving strategies involved the manipulation of objects. Formal thinking, or what Piaget preferred to label "hypothetico-deductive" (p. 62) thinking, involved the mental executions and reflections selected among several possible operations: "Formal operations engender a 'logic of propositions' in contrast to the logic of relations, classes, and numbers engendered by concrete operations" (p. 63). Only formal operations, begun around age 11 or 12, detached and liberated its thinking from concrete reality in order to reflect and theorize.

Achieving an optimum match between cognitive structure and content structure became a cardinal principle of developmental psychology and educational pedagogy. Developmentalists (Flavell, 1977; Piaget, 1952a, 1972b) argued the principle from the learner's perspective: mental structures determined the environmental structures one perceived. Bruner, in defining structure as "how things are related" (1960, p. 7) described that what was perceived in learning a discipline was its structures. Bloom's (1956) taxonomy of cognitive development influenced educators in demonstrating the cognitive levels of learners and in offering a guide for describing content goals at higher levels of cognitive reasoning than accumulation of information. The cognitive match between the learner and the content has been argued developmentally. And the curricular match among educational goals, instruction and evaluation has been argued pedagogically. Mathematics educators defined structure as the interrelationships of mathematical concepts which were achieved through an optimum match with the cognitive structure of the learner (Lovell, 1976; Moyer, 1978; Scandura, 1967; Suppes, 1967). Mathematics learning was conceptualized as "understanding the structure of inquiry" (Schulman, 1970, p. 70).

Teaching and Learning with a Piagetian Model of Inquiry

Elkind (1976) described how Piagetian ideas have had great impact on the mathematics emphasis ("new math") of Piagetian problem solving tasks and conceptual hypothetical relationships. Elkind agreed with Piaget's description of the problem with teaching children about numbers without their understanding the logical relations of the problems which the numbers represented:

> Experiments that we have been able to carry out on the development of mathematical and physical ideas have demonstrated that one of the basic causes of passivity in children in such fields, instead of the free development of intellectual activity they should provide, is due to the insufficient dissociation that is maintained between questions of logic and numberical and metric questions. In a problem of velocities, for example, the student must simultaneously manage reasoning concerning the distances covered and the lengths utilized, and carry out a computation with the numbers that express these quantities. While the logical structures of the problem is not solidly assured, the numerical considerations remain without meaning, and on the contrary, they obscure the system of relationship between each element. Since the problem rests precisely on these numbers, the child often tries all sorts of computations by gropingly applying the procedures that he knows, which has the effect of blocking his reasoning powers (Piaget, 1976, pp. 99-100).

Research concerning teaching and learning with a Piagetian model of inquiry (Renner, Stafford, Lawson, McKinnon, Friot & Kellogg, 1976) reported on the transition from concrete to formal operations of high school and college students. Renner, Stafford and Friot discovered that consolidation of concrete operations occurred more frequently in problem solving and in active learning experiences than in so called traditional, passive learning curricula. The research illustrated great gains in operational reasoning by an inquiry curricular approach. The assumption was that children were capable of the respective concrete and formal operational reasoning by an inquiry curricular approach. The assumption was that children were capable of the respective concrete and formal operational behavior at the age ranges specified by Piaget and that they needed problem solving opportunities to consolidate their cognitive potential. McKinnon discovered that less than 50% of the college students had achieved formal operations. Other educational psychologists (Furth, 1970; Furth & Wachs, 1975) have created curricula organized around Piaget's descriptions of how children develop mentally through inquiry or problem solving experiences.

Assumptions

Cognitive structures were hypothetical constructs of cognitive developmental stage theory (Piaget, 1970b). Theory could not be tested directly. In this research cognitive stage achievement was measured by observing the reasoning behaviors of subjects who were assumed to represent cognitive activity. The behavioral tasks selected (Inhelder & Piaget, 1975) were assumed to be appropriate representations of cognitive structures in young adults. Likewise, representations of mathematical cognitive structures were measured by judging the success and procedures used in solving mathematical problems (Geeslin & Shavelson, 1975). The selection of prospective mathematics teachers who were college juniors was assumed to be an appropriate sample for studying the relation between cognitive stage achievement and mathematical problem solving structures in young adults.

Research Questions

Cognitive stage achievement and its relationship to mathematical cognitive structures in young adults were examined by answering the four questions below.

1. Did young adults demonstrate differential achievement among concrete, transitional, and formal operational levels of cognitive stage?

2. Was there a significant ($p < .05$) relationship between the operational levels of cognitive stage achievement and successful mathematical problem solving in young adults?

3. Was there a significant ($p < .05$) relationship between the operational levels of cognitive stage achievement and problem solving characteristics in young adults?

4. Was there a significant ($p < .05$) relationship between successful mathematical problem solving and problem solving characteristics in young adults?

Definitions

Operational Levels of Cognitive Achievement

Cognitive development was a gradual and continuous process of reorganization of mental structures (Ginsburg & Opper, 1979). Piaget (1970b) accounted for qualitatively different levels of reasoning, knowing, or demonstrated intelligence in adolescents through the identification of the concrete and formal operational stages. Concrete operations utilized direct physical experience to organize mental responses. Formal operations was abstract reasoning. Piaget categorized one's cognitive achievement based on the highest level of one's performance. Kuhn et al. (1977) found this scoring criterion too broad, and they identified a transitional group of young adults achieving between consolidated concrete and consistent formal operations. Cognitive achievement in this research was defined operationally by the combined scoring on two replicated Piagetian tasks. Cognitive achievement was categorized as concrete, transitional, or formal operational level.

Cognitive Structures Used in Mathematical Problem Solving

Piaget conceptualized a mental structure as "a form of organization of experience . . . tools of one's behavior . . . forms equilibrium toward which the intellectual coordinations tend" (in Batto, 1973). Cognitive structures were mental organizations of behavior. The relationship between cognitive achievement levels and selected cognitive structures of problem solving were examined in this research. Ausubel and Sullivan (1970) defined problem solving as the insightful discovery of a system of relationships underlying the solution of a problem. Problem-solving structures were defined operationally in this research by successful solution, labeling solving procedure, and identifying solution strategy. Mathematical cognitive structures were defined as mathematical cocepts (Geeslin & Shavelson, 1975) that were measured in six selected mathematical problems.

Limitations and Delimitations

This research was limited to (a) replicating Piagetian interviews to measure levels of cognitive achievement in young adults, and (b) describing the relationship between cognitive stage achievement and selected cognitive structures used in mathematical problem solving. The examination and categorization of young adults' cognitive achievement were determined by the combined performance on two cognitive developmental tasks. The cognitive structures used in problem solving were determined by an analysis of six mathematical problems. The research sample was student volunteers from the three sections of a college mathematics education course.

CHAPTER II

REVIEW OF THE LITERATURE

The key constructs of this research were cognitive stage achievement and selected cognitive structures used in mathematical problem solving. The research literature that refined these notions is presented in this chapter. Emphasis is given to the interrelation of the constructs.

Cognitive Achievement and Cognitive Structure

Cognitive development is a reorganization of mental structures that is gradual and continuous (Ginsburg & Opper, 1979). A study of cognitive development seeks to describe the changes of reasoning and knowing that occur within an individual over time and the variations of such changes among people (Baltes, Reese, & Nesselroade, 1977). A cognitive perspective on the human experience "is conceptualized as a complex system of interacting processes which generate, code, transform, and otherwise manipulate information of diverse sorts" (Flavell, 1977).

Piaget (1970b) cited four interacting factors that explain how thinking develops: (a) the biological maturation of the nervous system, (b) active, physical experiences with objects and events in the environment, (c) the social transmission influence of education, language, and culture, and (d) the equilibration or self regulation of the assimilation-accommodation that accounts for the individual's interaction with the environment and one's subsequent adaptation or developmental change. According to Piaget, cognitive development resulted in the formation of general structures of knowledge that are common to all members of the species; the general structures permitted learning, which was always specific, to take place.

Cognitive development is the changing process of acquiring and using knowledge. Within this context, the terms intelligence and knowledge are synonymous with cognition. Piaget's (1976) research resulted in the following conclusions about cognitive development. The interrelationships of these concepts formed a theoretical explanation of cognitive behavior in humans.

1. Knowledge is constantly linked with actions (physical, emotional, and social) or mental operations, that is, with transformations. Knowing is an active process of interaction (subject with object), rather than a passive interpretation or perception of knowledge as external to the self.

2. The natural consequence of interactions is construction. Construction implies two types of independent activity: (a) the coordination of actions themselves and (b) the interrelations between objects.

What is constructed as a person acts are structures (conceptualizations for organizing, classifying, coordinating, etc.). Structures are hypothetical constructs of human activity used to explain how a person knows and, therefore, how he thinks.

3. Actions that become internalized (so that they may be used in more ways than initially constructed, that are reversible, and are seen as relational to other structures) are transformed into operations. Operational structurations (subdivided into concrete operational and formal operational stages) are dependent upon the transformation of a person's activity.

4. A general theory of the development of intelligence focuses on the fundamental relations among the biological theory of adaptation by self-regulation, developmental psychology and genetic epistomology. Especially clear is this interdependence in the development of logico-mathematical structures. The relations of inclusion, order and correspondence appear in biologic origin in genetic (DNA) programming of embryologic development and in physiologic organization. They become fundamental structures of behavior in early development, and they become refined with logic and mathematics in more abstract thought.

5. The process of assimilation, both in a biological and intellectual sense, explains the integration of external elements into a person's evolving or complete structures. Assimilation accounts for

the understanding of new physical and mental acitivity within existing schemes.

6. Accommodation is the cognitive complement of assimilation. Accommodation modifies an assimilatory scheme or structure by the elements it assimilates, thereby creating a new scheme. Accommodation is cognitive adaptation.

7. Assimilation and accommodation are interdependent, are mutually subordinate and are present in all activity. Intelligent behavior is defined as the achievement of a balance--a state of equilibrium--between assimilation and accommodation. When assimilation outweighs accommodation, one behaves egocentrically, even autistically. Conversely, one imitates literally without meaning or schematic transformation. Achieving equilibrium between assimilation and accommodation in intellectual behavior is analagous to achieving a balance between content (substance, meaning) and form in the fine arts.

8. Equilibrium is more progressively achieved as the individual develops. The emergence and gradual achievement of reversible operations and decentration allow an individual to refine the egocentric assimilation and incomplete accommodation of early childhood into an increasing harmony between the two.

9. Stages of cognitive development exist under these two conditions:

> (a) that they must be defined to guarantee a constant order of succession, and (b) that the definition allow for progressive construction without entailing total preformation. These two conditions are necessary because knowledge obviously involves learning by experience, which means an external contribution in addition to that involving internal structures, and the structures seen to evolve in a way that is not entirely predeterminal (Piaget, 1976, p. 22).

The Optimal Match in Mathematics Education: Mathematical Structure (Concepts) and the Cognitive Structure of the Learner

In a recent nation-wide review of current practices in mathematics education, Gibney and Karnes (1979) reported that instructional methods embodied mostly traditional content derived primarily from a narrow range of relatively standard mathematics texts. Likewise, in their comprehensive review of educational instruction, Stake and Easley (1978) discovered that mathematics teaching at all levels consisted predominantly of going over the problems assigned with either the teacher or the student working at the chalkboard while others observed. This trend was surprising in light of alternative, active learning curriculum projects of the last decade. It was concluded that the instructional pendulum appears to be swinging back to one teaching source--the textbook.

In contrast to the above practice of limiting consideration of effective mathematical instruction to traditional ways of responding to the textbook, many educators looked at how students and teachers organize mathematical concepts as a source of knowledge that had important implications for mathematics instruction. Suppes (1967), for example, clearly proclaimed that the ultimate objective of basic research in mathematics education was to understand how students learn mathematics and to use this understanding to outline more effective ways of organizing the curriculum. There appeared to be a need to discover a more specific understanding of the relationships among modes of mathematical representatives, cognitive-developmental levels of organizing knowledge, and educational methodology.

This position was outlined by Moyer (1978), who believed that considerable learning difficulties arose if the instructor did not take into account the cognitive structure of the learner. Schulman (1970) amplified this pcint, noting that "If the two structures (mathematical and cognitive) are consonant the new principle or concept can be taught; if they are dissonant, they cannot" (p. 42). This point was made lucidly by Scandura's (1967) initial observation:

> Any reasonable complete understanding of mathematical learning and performance will depend on (1) the identification of those 'ideal' competencies underlying various kinds of mathematical behavior . . . and (2) an understanding of how inherent psychological capacities and subject matter competencies already developed by a learner interact with external stimulation to produce mathematical learning and performance (p. 121).

Further, Gagne (1967) made almost an identical observation in identifying two categories of variables:

> (1) knowledge, that is, the capabilities the individual possesses at any given stage in the learning; (2) and instruction, the content of the communications presented within the frames a learning program (p. 7).

The reasoning was clear: examination of the relationship between learners' cognitive structures and mathematical structures to be taught was essential in determining appropriate mathematical curricula. Two related issues emerged from this position: (1) what was the nature of the mathematical representation or structure which was to be learned, and (2) what was the nature of the spontaneously occurring cognitive organizers which furnished representations and meaning for the mathematical content?

The notion of an optional match between mathematics instruction and learner knowledge or learner cognitive structure had important implications for mathematics education since the research in mathematics has focused almost exclusively on the former (mathematical concepts) (Suppes, 1967). There seems to be an implicit suggestion that because research has tended to focus on mathematical concepts per se and students' solutions to problems, there has not developed a sufficient data base from which textbook manufacturers could alter their organization of textbook materials. Concerning the second issue raised above, one may inquire about the nature of cognitive organizers and spontaneous cognitive processes involved in learners' understanding of mathematical concepts.

Over a decade ago Scandura (1967) reported that such processes as mental imagery, cognitive organization, and mental representations involved in thinking and understanding any domain of mathematics had scarcely begun to be studied from a scientific standpoint. Since then Wittrock (1974) identified two general features which appear to be involved in the natural organization of mathematical understanding: (a) organizational structures for storing and retrieving information and (b) processes for relating new information to the stored information. Three other theorists--Piaget (1970b), Bruner (1966), and Rosch (1975)--developed and reported comprehensive investigations of various cognitive processes which both encompassed and elaborated Wittrock's distinction.

For Piaget, thinking was an action that transformed one reality state into another, thereby leading to knowledge of the state. To understand a state (or a mathematics problem) was to understand the transformations from which the state resulted. These implicit mental actions or the covert transformations were of special interest to mathematics education.

For Piaget, the notion of a cognitive structure or stage entailed specific properties. Cognitive stages were organizations of transformation characterized by properties of interdependency, and hence, reversibility. These stages took the form of self-regulating systems, and because any element of the system was logically derived from other elements of the system, they were said to represent structures. Furthermore, stages formed a hierarchical sequence such that higher stages necessarily incorporated lower stages through the construction of more comprehensive systems of organization.

Piaget further distinguished between two kinds of knowledge: operative knowledge (derived from transformations and coordinations of relationship) and figurative knowledge (derived from mental representations of objects, events, and reality states). Pinard and Laurendeau (1969) made an explicit and detailed analysis of the stage construct used by Piaget, which elucidated aspects of operative knowledge. Furth (1969) also provided a sketch of figurative aspects of knowledge.

Both aspects of knowledge were important for mathematical learning. When a mathematical idea to be learned depended on a level of thought beyond that which the learner possessed or which required a cognitive organizer not present in the learner's mental repertoire, the idea was either partially learned or learned with much difficulty. This notion was made quite explicit by Piaget and Inhelder (1969) who stated, "The subject requires an instrument of assimilation which takes in the essential aspects of the concept, failing which he cannot assimilate it" (p. 200). Lovell (1976) reiterated this point from an empirical viewpoint noting that "all experience at Leeds suggests that, in the case of able pupils, once it is available in one type of task, the logical instrument soon becomes available in related tasks" (p. 167).

Defined as instruments of assimilation, cognitive structures took on four forms; each represented a stage in the ontogenesis of intellectual development. This research was concerned only with the higher structures of intelligence, the concrete operational and formal operational stages. According to Piaget, concrete operations was characterized by a system of logico-mathematical groupings which operated on concrete materials, that is, materials able to be experienced or manipulated. Formal operations, on the other hand, involved the formation of hypothetico-deductive thought which transcended the concrete world of objects and events by subsuming such experiences under the realm of possible transformations.

Formal operational thought emerged from about 12 years of age in the brightest pupils and from 14-15 years of age in more ordinary students. One pervasive characteristic of this thought which had import for learning mathematics was the ability to construct and manipulate second-degree operations (regulations of transformations between relations). Students at this stage could structure relations between relations, for example, as in metric proportion involving the recognition of the equivalence of two ratios (Lovell, 1976).

A variety of studies by Lovell (1961), Lunzer (1965), and Lovell and Butterworth (1966) with British students, and of Steffe and Parr (1968) and Gray (1970) with American students, confirmed that apart from the ablest 12-year-olds, it was from the beginning of junior high school onwards that the facility was acquired to understand metric proportions. To the regret of high school mathematics teachers, many students could not do this until 14-15 years of age, and some were never able to do it.

Evidence from recent research indicated that as many as half of the adult population failed to attain the formal operational level (Blasi & Hoeffel, 1974; Danner & Day, 1977; Papalia & Bielby, 1974: Schwebel, 1975; Tomlinson-Keasey, 1972). One study of 265 adolescents and adults, aged 12 to 50 years (Kuhn, Langer, Kohlberg, & Haan, 1977), reported that only 30-50% of the sample demonstrated any formal operational reasoning on each of several problems. Nearly one-fourth of the same sample performed at the concrete operational level on all tasks, with the remaining portion of the sample performing transitionally between the concrete operational and formal operational levels. Such findings indicated that adolescents and young adults varied in the level of their cognitive attainment.

The issue of mature cognitive attainment was in need of further elaboration. If all adults did not routinely operate at the formal level, one might have asked what factors were influential. Piaget (1972) discussed this problem from the point of view of developmental psychology. According to his reasoning, different types of schooling and social experiences may have influenced the rate of development, and disadvantageous environmental conditions may have precluded advanced forms of thought from appearing altogether. Factors such as career motivations and aptitudes also played a role in determining the manner in which formal thought became manifest, by the very fact that certain individuals naturally followed interests that inherently led to problems which transcended the field of immediate experience.

Lovell (1976) discussed implications of failing to achieve formal thought:

> this is a matter of great consequence; it has repercussions in the teaching of physics and chemistry. This inability to handle metric proportion until these ages clearly shows the dependence of the growth of mathematical understanding on the growth of the general ways of knowing (p. 171).

Such a body of research had clear implications: If learning metric proportion was constrained by cognitive stage, other related mathematical concepts were probably influenced. Evidence indicated that it was the general ways of knowing (cognitive stages) which determine the manner in which material was understood (Green, 1979).

Equally important, however, was the limitation placed on learning by a student's figurative knowledge. Figurative knowledge took a variety of forms--for example, mental images, perceptions, imitations, and language. Figurative knowledge, or representational modes were characterized by Bruner (1966) as enactive (concrete-manipulable), iconic (pictorial-figural), and symbolic (algebraic-semantic). It was one or more of these forms that instructional representations of concepts and problems took. The relationship between instructional mode and cognitive factors was the specific focus of recent studies by Hancock (1972, 1975) and Geeslin and Shavelson (1975). Hancock (1972, 1975) compared the cognitive preferences of ninth-grade students and college students for verbally oriented material. Differences in achievement were found between a verbal instructional program and a figural program among college students. No differences were found among the ninth graders. Hancock (1975) suggested that perhaps these ninth graders had not yet developed a cognitive preference for (or at least an adaptability to) material that was verbally oriented.

In a related study, Geeslin and Shavelson (1975) analyzed the representation of a mathematical structure in students' cognitive structures. The results of their study indicated that, among eighth-grade students, the learning of mathematical structure might not be related to solving problems involving that mathematical structure. The finding suggested the possibility that the cognitive organizations of teachers and students could correspond and still have students unsuccessful at solving problems relevant to the mathematical structure.

Research which addressed the relationship between cognitive representations and meaning emerged in the recent work of Rosch (1973, 1975; Heider, 1971, 1972; Heider & Olivier, 1972). Rosch's central thesis embodied the proposition that psychological categories had an internal structure which regulated meaning as a function of degree rather than truth value (the traditional logical/mathematical orientation). According to Rosch, psychological categories were organized around central, prototypical members or exemplars which were "good" examples of that category. Peripheral members of the same category differed from the prototypical member not in terms of identify/nonidentity, but in terms of degree of relatedness. Degree of relatedness, in turn, corresponded to notions such as proximity, centrality, and representativeness.

In summary, three emergent themes have been highlighted from the research related to mathematics education: (a) cognitive stage was clearly implicated in constraining the manner in which students at different levels learned mathematics; (b) the relationship between cognitive organizers and instructional representations of mathematical knowledge remained much in need of preliminary investigation; and (c) natural categories or naturally occurring concepts had properties which were conceptualized in terms of degree of relatedness, proximity, preference, correspondence, and centrality.

Cognitive Achievement and Cognitive Structure in Piagetian Research on Chance and Probability in Adolescents

Piaget and Inhelder's (1975) treatment of children's concepts of chance and probability provided a taxonomy of response patterns associated with developmental stages published earlier (Inhelder & Piaget, 1958; Piaget, 1965, 1970b, 1972b). Piaget and Inhelder characterized the preoperational stage as the absence of reversibility in thought. Reasoning at this level was typically intuitive, prelogical, transductive, syncretic, and generally insensitive to logical contradiction. The child could not distinguish between logical or physical certainty and uncertainty. Neither was he able to distinguish between what was possible and what was necessary. With time and experience, preoperational intuitions developed with more differentiated modes of thinking.

The preoperational child could not suspect the true nature of random mixture. In Inhelder and Piaget's (1975) interview tasks, preoperational children tried to find within the disorder some hidden order based on common properties of elements (e.g., spatial, geometrical, or temporal similarity) or subjective features of one's own thought (e.g., desires, interests). These children lacked a system of reversible operations. Their intuition was a logic dependent upon past experiences.

<u>Concrete Operations</u>

Concrete operations was characterized by the presence of reversible thought. Logical classification, relation and quantification became possible. The youngster discovered some combinations and permutations empirically and incompletely. The possibilities could be quantified only if they were small in number and static in nature. The mental operations were limited to concretely experienced or observed reality.

During this stage the appearance of logico-mathematical reasoning enabled the first real discovery of the idea of chance. Random mixture was conceptualized as an empirical process, no longer as an apparent disorder or accidental veiling of hidden order. The formation of deductive operations enabled the youngster to distinguish between the necessary and the possible. Given a class of objects B, where B = A + A' (two subclasses):

> If x is a member of B, it is then either in A or in A'. This disjunction in its concrete form implies both the necessary (if x is in B, it is necessarily in A or in A') and the possible (if x is in B, it can be in A, but it is also possible that it is in A') (Piaget and Inhelder, 1975, p. 226).

The discovery of deductive necessity allowed the child to conceive of the nondeducible character of isolated and uncertain events and to distinguish between the necessary and the possible.

Physical chance occurred in the form of interactions between independent causal sequences. The random mixing of black and white marbles within a closed container (Task 1) illustrated the interactions between elements, each of which taken alone explained a deductive prediction of a final position. However, when all the elements were considered simultaneously, no single element's position could be deduced, because the whole was rendered indeterminate due to the chance interactions among individual elements. The elements interacted when the marbles collided. The youngster who became capable of deductive reasoning took into account this logical indeterminacy (logical uncertainty), and this discovery was the source of the idea of chance.

Only quasi-probabilistic judgments, i.e., a synthesis of chance and quantitative operations, could be produced during concrete operations. These judgments were based on simple, static considerations of either (a) the comparison of certain parts within the whole, or (b) the comparison of certain parts with the whole, but not both (a) and (b) simultaneously. Consequently, an evaluation of proportions could not yet be produced, because proportions represented a comparative evaluation between part-whole combinations or relations between two relations (not tied to concrete reality). In the random drawing of chips task (Task 2), youngsters realized that some colors contained more discs than other colors, but they were unable both to compute the ratio between each color and the total and to compare these ratios to each other (proportion = ratio between ratios).

<u>Formal Operations</u>

During the formal operational period, judgments of probability became organized on an abstract level. The use of hypothetical-deductive thought (reasoning from hypotheses by deducing logical implications from them) characterized this stage. Reasoning at this level was freed from the constraints of concrete reality by the ability to reason propositionally. Propositions were subject-predicate structures which combined classes and relations. The description "operations on operations" described formal operational thought. Interpropositional operations (transformations of negation, inversion) were performed on statements whose intrapropositional content consisted of class and relational operations (Dulit, 1975; Flavell, 1963; Ginsburg & Opper, 1979). Possible propositions could be formed and transformed. Reasoning transcended concrete reality, or according to Piaget, reality became subsumed under possibility. Proportions could be quantified and translated into probabilities.

Piaget and Inhelder described formal operational thinking as inductive reasoning which notions of chance and probability represented.

Induction was the abstraction of a general principle from specific events, reasoning from particular to general. Deduction reasoned from a general principle to specific events, and the thinking was limited to the concrete reality of that general rule. Inductive processes had no necessary constraint. Formal thought was characterized by a search for principles to explain observed operations (not events).

The final arrangement of black and white marbles (Task 1) was unpredictable and indeterminate because of the innumerable possible collisions to set into motion by the tipping of the box. Nevetheless, once the adolescent learned the operations of permutations, he could calculate the possible combinations. By combining proportions with permutations, he reasoned about the most likely distribution of marbles to predict outcomes and to demonstrate a formal operational understanding of probability. And it was the indeterminability of the outcome that signified the formal operational understanding of chance.

Piaget and Inhelder summarized this process:

> But since chance is not an operative system, we understand that the invisible permutations remain fortuitous, that is, (a) instead of being effected according to a systematic order, they move pell-mell in all directions, and especially (b) instead of being complete, they are able to achieve only certain of the possibilities simultaneously, chance realizes only certain possibilities, but creates nothing new and remains necessarily within the framework of deduced possibilities, its only originality is being disordered and incomplete. In the same way, the drawing of pairs of discs from a sack containing several colored sets . . .

can be done according to all the combinations of pairs. Operations predict all the possibilities, some of them coming about by chance, which are thus observed ones, but they also must be among the original possible cases. In short, instead of letting chance keep its unpredictable character, incomprehensible as such, the mind translates it into the form of a system of operations, which are incomplete and effected with no order. Chance subsists, therefore, but has been moved to the plane of operations where it gains intelligibility . . . . The operations lead to the determination of all possible cases, even though each of them remains indeterminate for its particular realization. Probabilitiy then consists in judging isolated cases a fractional coefficient of realization. Probability, therefore, conforms with its actual definition, a fraction of determination (Piaget and Inhelder, 1975, pp. 232-3).

CHAPTER III

METHODS OF PROCEDURE

The research was a developmental study that described the cognitive stage achievement and cognitive structures used in mathematical problem solving of specified prospective teachers. A quasi-standardized interview determined the protocol for collecting and analyzing the cognitive achievement data. A content analysis of solving mathematical problems indicated the usage of selected cognitive structures. Correlations (Kendall's Tau) measured the strength of association among cognitive achievement, mathematical problem-solving success, and three cognitive structures of problem solving. The research design was a variation of the one-shot case study in which two observations were made on a specified population (Campbell & Stanley, 1963). It was pre-experimental in that it did not use an experimental treatment, randomized sample selection, and group comparison.

Sample

Forty undergraduate students volunteered for the study. They were enrolled in a mathematics education course at a state university in North Carolina. The subjects represented prospective teachers. There were four males and 36 females in the sample. The mean age was 25.5 years, and the age range was from 19 to 45 years. The mathematics problems were given to all students during a reserved class period. The cognitive tasks were conducted in personal interviews outside of class at times convenient to the students.

Research Instruments

Description of the Cognitive Tasks

Task One: Random Mixture of Marbles. This task replicated Inhelder and Piaget's (1975, ch. 1) interview concerning notions of chance. Subjects were seated at a table upon which rested a rectangular box (11" x 6 1/2" x 3/4") with a fulcrum under it so that it could be tipped back and forth in a seesaw motion. Five black marbles were initially placed on the right side of one end, and five white marbles, on the left side.

A subject was asked to predict the position of the ten marbles after being tipped forward and back one time. The prediction was recorded, and the experimenter tipped the box. The subject was asked to observe any difference between the prediction and the outcome. This procedure was repeated twice.

A tape recorder was turned on for the cognitive interview. Questions asked were taken from Inhelder and Piaget (1975) and Green (1977). The questions focused on the subject's notions about chance as it related to predicting the final positions of the tipped marbles. Exemplar questions included the following:

> Will the marbles get more or less mixed up if ten more were added, 100 more? What would happen if it were tipped ten more times, 100 more? Will the marbles ever return to their original position, could they, would they ever have to? What does mixing mean any way? How does it work? What controls it? Could a very smart person predict accurately and consistently the final positions of the tipped marbles? Why? Are some arrangements more likely than others?

The experimenter probed for full explanations of the questions. Task one interview directions and prediction sheet were attached as Appendix A, and transcriptions of three exemplar interviews, as Appendix B.

Task Two: Random Drawing of Chips. The order of cognitive task presentation was reversed for each successive interview. The poker chip task replicated Inhelder and Piaget's (1975, ch. 5) interview concerning notions of probability. Subjects were seated at a table upon which were arranged four horizontal rows of colored discs (1 1/2" poker chips), each row being stacked two discs high. Rows contained the following sets of discs: row one (furthest from the subject), nine stacks of yellow discs; row two, six stacks of red discs; row three, two stacks of white discs; and row four, one stack of blue discs. The experimenter asked the subject to pick the top chip from each stack, to deposit it into a paper bag, and to shake the bag to mix the chips up. The remaining discs were left on the table as a model. The experimenter did not mention them again, especially anything about the one-to-one correspondence between the chips on the table and those in the bag.

A tape recorder was turned on for the cognitive interview. The subject was asked to predict the colors of the two chips that he would be asked to pull out of the bag. The predictions were recorded. The subject was instructed to draw out two chips together without looking. The procedure was repeated eight more times until all chips were selected. The questions after each selection focused on the subject's notions about probability as it related to predicting the colors of selected chips (Inhelder & Piaget, 1975; Green, 1977). Exemplar questions included the following:

> Why did you choose that color(s)? What makes it more or less likely than another pair? What are other possibilities? Which is the best choice, why? What makes predictions work? How many chips are left in the bag? How do you know? Could a smart person predict accurately each time, why? What would he need to know? Does knowing what you'll get in the long run affect what you may draw on a single turn?

The experimenter probed for the explanations of the questions. The interview directions and the prediction sheet for task two were attached as Appendix C, and transcriptions of three exemplar interviews, as Appendix D.

Cognitive Tasks Scoring

Green's (1978) cognitive scoring instrument was adapted for use with young adult subjects in this research. Green replicated Inhelder and Piaget's (1975) interview tasks with children and adolescent subjects.

He developed a scoring procedure to overcome these criticisms of the Genevan research: (a) the reliability of the results have been questioned due to lack of methodological and quantitative rigor; (b) Piaget's clinical method capitalized on individual differences in children, making developmental comparisons or generalizations difficult; and (c) a lack of specific scoring criteria made replication difficult.

Green (1978) minimized these scoring problems by (a) standarizing procedures used in all interviews, (b) identifying specific test items in the form of interview questions that represented the stage-related features reported by Piaget and Inhelder, and (c) creating specific scoring criteria and decision rules for interpreting the cognitive interview items. Green's manual comprised twenty items, fifteen verbal and five behavioral items. Five items tested for preoperational thought, seven for concrete operations, and six for formal reasoning.

The six concrete-verbal items and the five formal-verbal items were selected for use in this research with Green's permission. The items and scoring procedures were modified to test for concrete, transitional, and formal operations of young adults. The cognitive interview items and the scoring instructions developed for this research were attached as Appendix E.

Reliability and Validity. Green (1978) reported an interrater reliability for the 20 cognitive items calculated by Cohen's (1960) computation of k. The results ranged from .72 to 1.0 on the concrete items and from .66 to 1.0 on the formal items (all significant at $p < .01$). Green selected procedures, described above, to increase the reliability of the results, namely, (a) a quasi-standardized procedure-question sequence used during the cognitive interviews, (b) specific test items and scoring criteria for judging cognitive performance, and (c) testing the same children on multiple tasks. Piagetian concepts of cognitive stage achievement determined the development of these procedures as a measure of construct validity. These procedures were followed in this research.

Mathematical Problem Solving Inventory

Clark and Reeves (in press) developed problems to measure mathematical cognitive structures defined as mathematical concepts (Geeslin & Shavelson, 1977). Six mathematical concepts were selected from the inventory for use in this research: from algebra, (1) one-to-one correspondence, (2) proportion-mixture, (3) spatial logic; and from geometry, (4) volume measure, (5) area measure, (6) partitioning by intersecting lines. The mathematical problems selected to test these concepts were used with the authors' permission and described in Appendix F. Clark and Reeves identified these problems as the most reliable and valid measures of the respective mathematical concepts used with high school students.

Subjects were given the inventory during a reserved class period prior to the scheduling of the cognitive interviews. Subjects were allowed five minutes to solve a given problem. They were told to write whatever they wanted on the problem page as part of the solution. Secondly, subjects were asked to select a strategy from four listed choices that best represented their approach to the mathematical problem; a minute was allowed for the choice. Therefore, six minutes was allowed for each problem, a total of 36 minutes for completion of the inventory.

Four measures were judged in an analysis of the subjects' written solutions: (a) successful solution, (b) spontaneous labeling of problem-solving procedure, (c) subject's choice of mathematical strategy employed, and (d) comparison of strategy choice with strategy demonstrated. The scoring decision for (a) successful solution was either no, partial or total success. Scoring decisions for (b) labeling and (d) matching strategies were judged as yes or no. Five scoring categories indicated (c) subject's identified solution strategy: no choice; concrete, pictorial figure; abstract, geometric figure; algebraic, numerical representation; and verbal, logical description.

Data Analysis Methodology

The results of the cognitive interviews were used to answer the first research question, the differentiation of cognitive stage achievement in young adults among concrete, transitional, and formal operational levels. Three measures of reliability were estimated. An interjudge scoring reliability (Cohen's _k_) was established. A correlation (Kendall's Tau) was calculated for the two cognitive tasks. And a homogeneity of performance (Hoyt's _R_) was derived. Mean and modal scores were reported for each of the three operational levels.

Success in mathematical problem solving was tested with cognitive stage achievement to answer the second research question. The correlation (Kendall's Tau _b_) measured the strength of relationship between them.

The three mathematical problem-solving characteristics of labeling, strategy selection, and matching strategies were tested with cognitive stage achievement to answer the third research question. The correlation (Kendall's Tau _c_) measured the strength of relationship between them. And the same characteristics were tested with successful solutions also by Kendall's Tau _c_ in answering research question four.

CHAPTER IV

RESULTS

Cognitive Stage Achievement Levels

Thirteen (33%) of the 40 young adult subjects demonstrated a concrete operational level of cognitive achievement on the combined two cognitive interview tasks. Sixteen (40%) of the 40 subjects performed at a transitional level. And eleven (27%) of the 40 young adults achieved formal operational reasoning. The sample age mean was 25.5 years.

Concrete operational subjects had a mean score of 4.3 on six concrete items. Transitional thinkers averaged 5.1, and formal achievers 5.7 on the concrete items. The sample mean was 5.0 for the concrete items. The mean scores on the five formal items revealed a larger distinction: concrete subjects averaged .5, transitional subjects 2.5, and formal subjects 4.4. The mean formal item score for the 40 subjects was 2.4. The results were summarized in Table 1. Raw subject data and raw scores on the cognitive tasks were listed in Appendices G, H, and I.

Cognitive Items Interjudge Reliabilities

Interjudge scoring reliabilities were computed using Cohen's Kappa (1960). It was used for nominal data of the item criteria (pass, fail); it measured the agreement between judges after

TABLE 1

OPERATIONAL LEVELS OF COGNITIVE ACHIEVEMENT: COMBINED SCORES FROM COGNITIVE TASKS ONE AND TWO

Level	N	Age	6 Concrete Items		5 Formal Items	
		Mean	Mean	Mode	Mean	Mode
Concrete	13	24.0	4.3	3, 4 5, 6	.5	1
Transitional	16	26.9	5.1	5	2.5	2, 3
Formal	11	25.7	5.7	6	4.4	4
Total	40					
Mean		25.5	5.0		2.4	

controlling for the proportion of agreement expected by chance. The scores were listed in Table 2. The items were judged reliably at .005 level of significance or lower.

TABLE 2

COGNITIVE ITEM INTERJUDGE RELIABILITES USING COHEN'S KAPPA

Cognitive Item	k/k_{max}	σ_k	Z	p
Concrete Items: Task 1 and 2				
C1	.48	.16	3.0	.001
C2	.90	.35	2.6	.005
C3	.80	.15	5.3	$<$ .001
C4	.68	.02	34.0	$<$ .001
C5	1.0	.0	∞	$<$ .001
C6	1.0	.0	∞	$<$ .001
Formal Items: Tasks 1 and 2				
F1	.45	.04	11.3	$<$ .001
F2	.71	.16	4.4	$<$ .001
F3	.78	.16	4.9	$<$ .001
F4	.95	.19	5.0	$<$ .001
F5	1.0	.0	∞	$<$ .001

Cognitive Tasks Correlations

Correlations between the two cognitive tasks tested task independence. Kendall's rank-order correlation coefficient was used since the cognitive achievement variable was tabulated as ordinal level of measurement (formal achievers scoring more formal items correctly).

The Tau coefficient for the concrete items of the two tasks was .53, p = .001. The formal items correlation was .40, p = .004. The moderate correlations indicated that the tasks were not measuring identical cognitive structures of cognitive achievement, and therefore, that a combined usage provided a more complete assessment of cognitive performance. The results were shown in Table 3.

TABLE 3

CORRELATIONS FOR COGNITIVE INTERVIEW TASKS

Cognitive Stage Level	Marble-Colored Chips Tasks	
	Kendall Tau	p
Concrete Stage Items	.53	.001
Formal Stage Items	.40	.004

Homogeneity of Performance

The operational level homogeneity of performance on the concrete and formal task items was tested by Hoyt's R (1941). Hoyt adapted the Kudor-Richardson formula #20 to test the consistency of performance for each subject within a given category. Hoyt's reliability formula was designed for binary (pass, fail) data used in the cognitive task item scoring. The results were displayed in Table 4. The concrete subjects' performance on the concrete items showed a large within-cell variation (1.73) which indicated a large range of cognitive

TABLE 4

COGNITIVE OPERATIONAL LEVEL HOMOGENEITY OF PERFORMANCE ON COGNITIVE ITEMS (TASKS ONE AND TWO COMBINED) MEASURED BY HOYT'S R

Operational Level	Concrete Items		Formal Items	
	W/in Var.	R	W/in Var.	R
Concrete	1.73	.73	.27	.99
Transitional	.47	.93	.26	.99
Formal	.42	.93	.25	.99
Between Var.:	6.41		43.69	
Unweighted Mean W/in Var.:	.87		.26	
Unweighted Mean R:		.86		.99

achievement. Transitional and formal subjects showed only moderate variation (.47, .42). The mean within-cell variation on the concrete items was moderate (6.41) which indicated moderate variation among the three groups of subjects on the concrete items. The low within-cell variation of all groups (.27, .26, .25) on the formal items indicated high homogeneity of performance within each group, and the between variance (43.69) indicated large differences between the performance of the three groups on the formal items (partially explained by the scoring instrument that identified the groupings). The R values were high and indicated homogeneous performance of the three groups on both concrete and formal items.

Cognitive Achievement and Mathematical Problem Solving Success

The relationship between cognitive achievement and successful mathematical problem solving was measured using Kendall's Tau b since the variables were scored as ordinal level measures. A cross-tabulation of achievers and problem solvers was computed for each of the six mathematical problems. Table 5 displayed the cross-tabulation of problem #4 as an example (using SPSS format, Nie, Hull, Jenkins, Steinbrenner, & Bent, 1975). In the example, the test statistic resulted in a significant moderately high correlation (.36, $p < .01$) which indicated that the two variables (levels of cognitive achievement and degree of math success) were not independently related

TABLE 5

A CROSS-TABULATION EXAMPLE OF COGNITIVE STAGE AND SUCCESSFUL MATHEMATICAL PROBLEM SOLVING

Mathematical Problem #4: Partitioning by Intersecting Lines

N = 40	Mathematical Problem Solving Success			
Cognitive Stage Levels	None	Partial	Complete	Total
Concrete	2	11	0	13
Transitional	1	14	1	16
Formal	0	8	3	11
Total	3	33	4	40

Kendall's Tau B = .36 Significance = .007

in solving this math problem. Table 6 listed the correlations and levels of significance for all six problems; raw scores were listed in Appendix J. Since four of the six problems showed a significantly ($p < .05$) strong relation, cognitive achievement levels and mathematical problem solving success were not independent of each other.

TABLE 6

INDEPENDENCE OF COGNITIVE STAGE AND SUCCESS IN MATHEMATICAL PROBLEM SOLVING

Mathematical Problems	Cognitive Stage and Success in Mathematical Problem Solving	
	Kendall's Tau b	p
Algebraic		
Prob. 3	.24	.04*
Prob. 5	.25	.04*
Prob. 6	-.13	.19
Geometric		
Prob. 1	.19	.09
Prob. 2	.36	.01*
Prob. 4	.36	.01*

*Significant, $p < .05$

TABLE 7

INDEPENDENCE OF COGNITIVE STAGE AND USAGE OF LABELING IN MATHEMATICAL PROBLEM SOLVING

Mathematical Problems	Cognitive Stage and Usage of Labeling	
	Kendall's Tau C	p
Algebraic		
Prob. 3	-.04	.34
Prob. 5	-.06	.20
Prob. 6	.05	.37
Geometric		
Prob. 1	.28	.05*
Prob. 2	-.11	.24
Prob. 4	-.01	.49

*Significant, $p < .05$

Cognitive Achievement and Mathematical Problem-Solving Characteristics

The relation was tested between cognitive operational levels and three cognitive characteristics of mathematical problem solving: (a) subject's spontaneous labeling of a figure while solving a problem,

TABLE 8

INDEPENDENCE OF COGNITIVE STAGE AND MATHEMATICAL PROBLEM SOLVING STRATEGIES

Mathematical Problems	Cognitive Stage and Solution Strategies	
	Kendall's Tau C	p
Algebraic		
Prob. 3	-.08	.28
Prob. 5	.08	.27
Prob. 6	.21	.05*
Geometric		
Prob. 1	-.19	.09
Prob. 2	.04	.37
Prob. 4	.04	.37

*Significant, $p < .05$

(b) subject's choice of a solution strategy, and (c) the match between the solution strategy and one's chosen strategy. Kendall's Tau C was the test statistic used to measure the strength of the relationships. Tables 7, 8, and 9 displayed the results of the correlational tests for cognitive achievement and the three characteristics, respectively. The raw

TABLE 9

INDEPENDENCE OF COGNITIVE STAGE AND MATCHING OF MATHEMATICAL PROBLEM-SOLVING STRATEGIES USED AND CHOSEN

Mathematical Problems	Cognitive Stage and Matching Strategies	
	Kendall's Tau C	p
Algebraic		
Prob. 3	.01	.49
Prob. 5	.03	.42
Prob. 6	-.24	.05*
Geometric		
Prob. 1	.01	.47
Prob. 2	-.04	.41
Prob. 4	.17	.15

*Significant, $p < .05$

scores for these relationships were attached in Appendices K, L, and M.

The results indicated that cognitive achievement was independent and not related to the characteristics of mathematical problem solving (labeled figure, chosen solution strategy, and its match with used strategy).

TABLE 10

INDEPENDENCE OF SUCCESS AND USAGE OF LABELING IN MATHEMATICAL PROBLEM SOLVING

Mathematical Problems	Success and Labeling in Mathematical Problem Solving	
	Kendall's Tau C	p
Algebraic		
Prob. 3	-.04	.37
Prob. 5	-.05	.21
Prob. 6	.12	.22
Geometric		
Prob. 1	.32	.03*
Prob. 2	.02	.45
Prob. 4	.04	.36

*Significance, $p < .05$

Only one of six problems in each of the three tests showed a strong relationship, $p < .05$.

Success and Characteristics in Mathematical Problem Solving

The relation was tested between mathematical problem-solving success and the problem-solving characteristics described above.

TABLE 11

INDEPENDENCE OF SUCCESS AND STRATEGIES USED IN MATHEMATICAL PROBLEM SOLVING

Mathematical Problems	Success and Strategies Used in Mathematical Problem Solving	
	Kendall's Tau C	p
Algebraic		
Prob. 3	.19	.09
Prob. 5	.15	.09
Prob. 6	.18	.08
Geometric		
Prob. 1	-.26	.03*
Prob. 2	.09	.29
Prob. 4	-.03	.39

*Significance, $p < .05$

Kendall's Tau C was used. The results were summarized in Tables 10, 11, and 12. The results indicated an independent relationship between success and the selected strategies of mathematical problem solving.

TABLE 12

INDEPENDENCE OF SUCCESS AND MATCHING STRATEGIES (USED AND CHOSEN) IN MATHEMATICAL PROBLEM SOLVING

Mathematical Problems	Success and Matching Strategies in Mathematical Problem Solving	
	Kendall's Tau C	p
Algebraic		
Prob. 3	.31	.03*
Prob. 5	.07	.26
Prob. 6	.39	.01*
Geometric		
Prob. 1	.05	.38
Prob. 2	-.16	.13
Prob. 4	.21	.03*

*Significant, $p < .05$

The tests with labeled figures and selected solution strategies revealed only one of six problems significant at the .05 level; the test for matched strategy showed three of six problems significant at .05.

CHAPTER V

DISCUSSION OF THE RESULTS

Three operational levels of cognitive stage achievement were described in this research using cognitive developmental interviews. The relation among cognitive achievement, successful mathematical solution, and three characteristics of mathematical problem solving were tested using math problems representing six math concepts. Young adults, specified as prospective teachers, comprised the sample.

Cognitive Stage Achievement Levels

The combined scoring of two cognitive interview tasks on chance and probability replicated Green's (1978) and Piaget and Inhelder's (1975) investigations of child and adolescent reasoning. The results of this research with 40 young adult subjects revealed that only 27% achieved a consolidated level of formal operations. The findings contradicted the initial observations of adolescents by Inhelder and Piaget (1958) who theorized that all normally developed humans achieved formal operations capability by age 15. Jackson (1965), Lovell (1961b), and Webb (1974) have replicated Piagetian constructs of formal operations successfully. Yet Berry and Dasen (1974) and Dulit (1975) reported a lack of formal behavior in cross-sectional studies. Neimark (1975b) concluded that the formal operations stage was described incompletely. Piaget (1972) refined his initial observation in acknowledging the lack of abstract reasoning among many adolescents. He attributed this fact to a lack of appropriate environmental experiences rather than an impossiblity of cognitive development. The results of this research with late adolescents and young adults were compatible with earlier studies that reported incomplete formal operational achievement by the completion of adolescence.

Forty percent of the college subjects demonstrated partial formal operations and represented a transitional operational level. This finding related to the results of Kuhn et al. (1975), who reported a distinct group of adolescents whose thinking was transitional between concrete and formal thought. Thirty-three percent of the subjects did not achieve beyond concrete operations. This finding was similar to a series of cognitive achievement studies by Renner et al. (1976), who reported that approximately one-half of the college freshmen interviewed reasoned below formal operations.

The cognitive interview instrument was adapted from Green (1978) for use with young adult subjects. It provided a quasi-standardized approach to the interview protocol, cognitive test items and scoring, thereby increasing procedural reliability. Additionally, three measures of reliability were estimated. An interjudge scoring reliability ($p < .01$) indicated a consistent understanding of cognitive item constructs and its application used in scoring. Correlations between the two cognitive tasks indicated a moderate degree of relationship (.53 for concrete items, .40 for formal items, $p < .01$); the tasks measured different but associated traits of operational thought. The combined usage provided a more complete assessment of cognitive performance. Homogeneity of performance tests indicated each of the three operational level groups behaved consistently and differentiated its thinking behavior from the other two levels.

Success in Mathematical Problem Solving

The relation between the operational levels of cognitive achievement and successful mathematical problem solving was examined in this research. The results showed that the variables were not independently related as measured by an inventory of six mathematics problems representing different math concepts. Four of the six problems resulted in moderate correlations (range .24 to .36, $p < .05$), two of three algebra concepts and two of three geometry concepts. Cognitive stage achievement and successful math problem solving were associated for young adults.

Bruner (1960) implied that anything could be learned if it were explained in a structure which matched the cognitive level of the student. Geeslin (1972) and Shavelson (1972) defined mathematical cognitive structure as the structure inherent in mathematical content or concepts. The content structure of the mathematical problems used in the research instrument (Clark & Reeves, in press) consisted of abstract word problems which approximated the formal operational cognitive structures. The formal achievers were the most successful problem solvers, and the concrete thinkers were least successful.

The results indicated that the matching of content structure with the more general cognitive structures appropriate at the subjects' cognitive stage level resulted in successful mathematical solutions. The match of content structure and cognitive structure was demonstrated in other studies (Lovell, 1976; Moyer, 1978; Scandura, 1967; Schulman, 1970; Suppes, 1967). This match has been argued pedagogically by educators (Bloom, 1956; Gagne, 1967; Macdonald et al., 1973; Schwab, 1962).

Three Characteristics of Problem Solving

Both cognitive stage achievement and successful math problem solving were tested for the strength of association with three characteristics of problem solving: labeled figure, chosen strategy, and its match with used strategy. They were measured by six mathematical problems representing different math concepts. The results showed that the problem solving characteristics were independent of both cognitive stage achievement and mathematical problem solving success.

The three characteristics were selected from Clark & Reeves's (in press) list of strategies that help children learn mathematics.

Specifically, the operational definitions of the strategies were found to be independent of cognitive stage and successful math solutions: (a) the spontaneous labeling of a math problem or figure while deriving the solution, or not doing so; (b) choosing a strategy to represent how one solved a problem from five possibilities: no choice, pictorial figure, geometric figure, numeric formula, verbal description; (c) the matching of one's chosen strategy with the judge's identification of the strategy demonstrated.

The nonrelatedness of these operationally defined mathematical strategies with the constructs cognitive achievement and mathematical cognitive structure could have accounted for the results. Geeslin and Shavelson (1975) analyzed the representation of a mathematical structure in students' cognitive structures and did not find that learning mathematical structure (concepts) guaranteed successful solution. Specific strategies did not always lead to singular right answers.

As another explanation, Ausubel and Sullivan (1970) discussed problem solving as a multidimensional process that could not be equated with any one factor that influenced its development or limited to any one of its three levels--trial-and-error, hypothesis testing, or insight. Gallagher (1964) and Stevenson et al. (1968) found intelligence to be only a part of problem solving ability. The test for problem-solving characteristics using only the three characteristics was not a multidimensional operational definition of problem solving. Piaget (1967) described an essential distinction between formal and concrete operations as the ability to solve problems in different ways. The result from a paper-and-pencil test used in the research did not indicate that there was a difference among operational levels of cognitive achievement in the use of spontaneous labeling, pattern of chosen solution strategy, or congruence in chosen and used strategies. Distinctions might have appeared in a cognitive interview in which the subject was asked to display the reasoning instead of relying on an analysis of a written test to reveal it.

CHAPTER VI

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Three operational levels of cognitive stage achievement have been identified by Piagetian cognitive interviews for 40 young adult (age mean = 25.5 years) prospective teachers: 27% achieved formal operations, 40% demonstrated a partial or transitional achievement towards formal thought, and 33% did not perform beyond concrete operations. The relation among cognitive stage achievement, successful mathematical problem solutions, and three characteristics of problem solving used in mathematics have been tested using Kendall's Tau. Cognitive stage achievement and successful mathematical problem solution were not independently related. Three characteristics of problem solving used in mathematics (spontaneous figure labeling, solution strategy selection and its use) were tested as independently related both to cognitive stage achievement levels and to degree of success in mathematical problem solving.

Conclusions

1. Young adults demonstrated differential achievement among concrete, transitional, and formal levels of cognitive stage.

2. Cognitive stage achievement and successful mathematical problem solving were associated constructs.

3. The three characteristics of problem solving used in mathematics--spontaneous figure labeling, chosen solution strategy, and its match with demonstrated strategy--are not related to cognitive stage achievement or successful mathematical problem solving.

Recommendations

1. A cross-sectional investigation of cognitive stage achievement in young adults.

2. A longitudinal investigation of cognitive stage achievement in young adults through middle adulthood.

3. An investigation of cognitive stage achievement in young adults using multiple cognitive tasks.

4. A study of the relation among (a) cognitive stage achievement of teachers, (b) their mastery of cognitive structures (concepts) of their disciplines, and (c) their applied understanding of the cognitive structures used by their pupils.

5. Development, implementation, and evaluation of experimental curriculum in higher education with explicit developmental goals for cognitive achievement through problem-solving methodologies.

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APPENDIX A

DIRECTIONS AND LIST OF INTERVIEW QUESTIONS FOR COGNITIVE TASK ONE: NOTIONS OF CHANCE

Cognitive Task One replicates Inhelder and Piaget's (1975, ch. 1) experiment. Marbles of two colors (five black and five white) are tilted back and forth inside a rectangular box. Subjects are to discuss the possible marble arrangements and their understanding of related causes.

Steps for Marble Task

1. Calibrate tipping apparatus.

2. Tell subject there are no wrong/right answers; we're just trying to find out how people differ in their perceptions of what happens.

3. Make sure tipping apparatus is ready to go (with marbles arranged), and blank apparatus and tape recorder have been tested.

4. Tell subject, "I've got something here I want to talk to you about. I want you to watch what happens, and later on I'll talk with you about what happens."

5. "Before we start, I'd like you to show me how you think the marbles would look if we tipped it back and forth just one time. You can use these marbles (give to S) and place them in this other table (point to blank apparatus). RECORD PREDICTION.

6. "Now watch what happens when I tip it one time." TIP APPARATUS BACK AND FORTH.

7. "Does your prediction look like how these marbles came back?" OK, now watch while I tip it once more." TIP APPARATUS A SECOND TIME.

8. "Before I tip it again, use those other marbles to show me how you think these will look after I tip it another time." RECORD PREDICTION, THEN TIP APPARATUS AGAIN.

9. "Do those look like these?" "OK, now watch while I tip it again." TIP APPARATUS.

10. "OK, now you can use those other marbles again to show me how they'll look after I tip it one more time." RECORD PREDICTION. THEN TIP APPARATUS.

11. "OK, now I'm going to tip it one last time." TIP APPARATUS AGAIN.

12. "Now I'm going to talk with you a little about what you saw happening." TURN ON TAPE RECORDER.

<u>Tape Recorded Interview</u>

Discuss the following items relative to the marble Task.

1. Do you think the marbles will get more or less mixed up if we added ten more marbles? What if we added a hundred more? PROBE.

2. What would happen if we continued tipping it for ten more times? A hundred more times? PROBE.

3. Do you think the marbles would ever come back to where they started, with the blacks on their side and the whites on their side? PROBE.

4. Will they ever cross over to the other side, like they were swapping sides? If we tipped it long enough, would they <u>have</u> to? PROBE.

5. What does mixing up mean anyway? How does mixing work? What controls it? What happens to the marbles that makes them get mixed up?

6. Could a very smart person ever guess exactly where the marbles would end up on each tip, you know, like a mathematics professor? How would they do that? PROBE.

7. Are some arrangements more likely than others? What makes them (more likely/all) just the same? PROBE. HAVE SUBJECT SHOW MOST LIKELY AND LEAST LIKELY ARRANGEMENTS, THEN PROBE FOR REASONS.

8. Does knowing how they'll turn out in the long run have any relation to how they turn out after any single tipping?

9. Note: "What's your notion?" is a good question to get subject to talk. Also: "I'm not sure I understand what you mean."

COGNITIVE TASK ONE: PREDICTION SHEET

Marble Task

Subject's id. no:

Name:

Age:

Date of Interview:

Directions: Draw circles when you think the five white marbles will end up, and draw circles with an X inside when you think the five black marbles will end up.

First Prediction: ______________________________

Second Prediction: ______________________________

Third Prediction: ______________________________

Draw the most likely arrangement: ______________________________

Draw the least likely arrangement: ______________________________

Comments:

APPENDIX B

THREE TRANSCRIPTIONS OF EXEMPLAR INTERVIEWS:

COGNITIVE TASK ONE,

MARBLE TASK ON NOTIONS OF CHANCE

Subject #15

DO YOU THINK THAT IF WE ADDED 10 MORE MARBLES TO THE BOX, THE ARRANGEMENT WOULD GET MORE OR LESS MIXED UP?

I think less mixed up. Because there'd be less chance of movement in there, 'cause there'd be more in the space.

WHAT DO YOU MEAN BY CHANCE?

Well, the fewer objects that you have in a space, there's more room to move around and switch. Whereas if you've got a whole lot, they wouldn't switch around as much, 'cause there wouldn't be as much room to roll.

OK, HOW ABOUT IF WE ADDED A HUNDRED MORE MARBLES? DO YOU THINK THEY'D GET MORE OR LESS MIXED UP THEN?

Less, 'cause it'd be kind of hard for them to move around.

WHAT ABOUT IF WE ENLARGED THE SIZE OF THE BOX?

If you enlarged the size of the box, then it's be about the same, the amount of mixture. I don't think it would be really more or less.

OK, WHAT DO YOU THINK WOULD HAPPEN IF WE KEPT TIPPING THIS BOX, SAY TEN MORE TIMES? DO YOU THINK THAT THE MARBLES WOULD GET MORE OR LESS MIXED UP?

The same number that's in there?

YES, USING THE TEN MARBLES WE STARTED WITH.

I think they would become, um, less mixed up. I think they would be more segregated as to which color were on each side.

HOW COME?

I guess its just the way I um, just a guess. Its just the law of averages would have it so it would even back out.

WHAT DOES GUESSING HAVE TO DO WITH THIS?

I guess it just has to do with the way I think about it. Because I don't know what its going to do.

ARE YOU SAYING THE BEST WAY TO DO THIS WOULD BE TO JUST GUESS?

For me it would, 'cause I don't think mathematically or in "n" dimensions or stuff.

YOU MENTIONED THE LAW OF AVERAGES. WHAT DO YOU MEAN BY THAT?

Just that um, the way things seem to do. The way they come out. I don't know how to put it into words.

OK, WHAT IF WE TIPPED THIS BOX A HUNDRED MORE TIMES, DO YOU THINK IT WOULD BE MORE OR LESS MIXED UP?

I think it would be about the same. Like it would switch around, but by the end of a hundred times, it'd be about the same as far as mix.

COULD THE MARBLES EVER COME BACK JUST LIKE YOU PREDICTED OVER THERE? LIKE TO THEIR ORIGINAL SPOT?

I think they could, but I don't know how many turns it would take.

HOW ABOUT, DO YOU THINK THEY COULD EVER COMPLETELY CROSS OVER TO THE OTHER SIDE, WHERE THE WHITES WERE OVER HERE AND THE BLACKS OVER HERE?

Yeah. I don't know why, but I do.

DO YOU THINK BOTH OF THOSE ARE POSSIBLE ARRANGEMENTS THEN?

Umhum (yes).

COULD WE JUST TIP IT RIGHT NOW AND HAVE THAT HAPPEN?

I have no idea. Its a chance, but I have no idea.

OK, WHAT DOES MIXING UP MEAN TO YOU? HOW WOULD YOU DESCRIBE MIXING UP?

Mixing up would be to put different colors together, different numbers of the marbles together, not necessarily the same number on each side, but different colors.

OK, HOW DOES THAT WORK? HOW DOES THAT OCCUR?

By tipping the box.

IS THAT THE ONLY THING THAT AFFECTS IT?

I imagine the amount of pressure that's put on it when you tip it.

OK, WHAT CONTROLS WHERE THEY LAND OR HOW THEY MIX?

I would think whether or not they hit up here and just rolled part way, with what force and what angle.

COULD YOU EXPLAIN THAT A LITTLE FURTHER?

I don't know, let's see. Um, I think it would just be when you tilted it if they hit each other or when they hit the sides, and the angle at which they hit, and the force would determine where they would land.

WELL, DO YOU THINK THAT SOMEBODY REALLY SMART, LIKE A MATHEMATICS PROFESSOR COULD JUST LOOK AT THIS AND MAKE A PREDICTION THAT WOULD BE RIGHT EACH TIME IT WAS TIPPED?

Yeah, I think so. I think they could look at um, just because they've learned to think that way or they tend to think that way, like the dimensions and the number of marbles and how hard they hit, and figure up an average of how it would turn back or whatever.

SO THOSE ARE THINGS HE WOULD HAVE TO KNOW TO BE ABLE TO PREDICT?

I would think so, or either be a good guesser (laughter).

DO YOU THINK THERE'S OTHER WAYS TO LOOK AT IT. WHILE YOU GUESS, DO YOU BELIEVE THE MATHEMATICS PROFESSOR COULD WORK OUT A FORMULA FOR IT?

Yeah, I think there's ways to work out things like that, but I think the average person on the street guesses.

OK, UM DO YOU THINK SOME ARRANGEMENTS ARE MORE LIKELY THAN OTHERS?

Hmm, I hadn't thought about it. If you're looking at it mathematically, there probably are, but if you look at it the way I look at it well . . . there aren't.

SO YOU THINK ANY ARRANGEMENT IS LIKELY?

Right.

(Shows likely arrangement)

(Show less likely arrangement)

OK, WHY DID YOU CHOOSE THAT?

I just think that unless you purposely manipulated it it would be hard for them all to be on one side.

OK, THAT'S INTERESTING. TELL ME, DO YOU THINK THAT KNOWING HOW THINGS WOULD TURN OUT IN THE LONG RUN WOULD GIVE YOU ANY INDICATION OF WHAT MIGHT TURN UP ON A SINGLE TIP?

You mean like if we knew after one hundred times that it would end up this way?

RIGHT.

Yeah, I think you'd be more accurate with it.

SO AFTER WATCHING IT ONE HUNDRED TIMES, WE'D KNOW WHAT THE 101st TIP WOULD LOOK LIKE?

Well, like two out of three times. 'Cause on one tip it would just be chance.

THERE'S THAT WORD "CHANCE" AGAIN. CAN YOU TELL ME A LITTLE MORE ABOUT THAT?

Well it would just be, like you um can do one certain thing a hundred times, and that may be the result. But if you do it a hundred times again, you may get a different result.

SO WHAT WOULD CAUSE THE DIFFERENT RESULTS?

Well, the line of the marbles, how they were lined up. Like if you started from a totally different starting point. Do it would change it the next hundred times, or it might not change it.

Subject #16

WILL THEY GET MORE MIXED IF WE ADDED TEN?

About the same.

100?

More mixed up; they'd be more to run into. That doesn't agree with what I said earlier. With more marbles there'd be more action.

IF WE TIPPED IT MORE TIMES?

Yes, the more you tipped it the more it'd get confused. It'd have more chances, but whether it'd do that.

TIPPED IT 100 MORE TIMES?

?

WHAT CAUSES THEM TO GET MIXED UP?

When you tilt the board, and they hit, if they are not level, if . . . .

I HAD CHECKED IT OUT BEFORE WE STARTED. WHAT HAPPENS TO THE MARBLES?

They roll down and back. They probably wouldn't come back to the same place. Abnormalities of the experiment would cause them to go sideways.

THE ONLY THING THAT CAUSES THEM TO GET MIXED UP IS ONCE THEY DEVIATE FROM THE STRAIGHT AND NARROW PATH, THEN OTHER THINGS START TO HAPPEN.

It causes a chain reaction.

IF WE CONTINUED, WOULD THEY EVER COME BACK TO THE ORIGINAL POSITION?

Yes.

HAVE TO?

No.

WHAT WOULD MAKE IT HAPPEN?

Patience. Eventually, the probability of the odds aren't so far off that you couldn't do it.

WHAT DO PROBABILITY AND ODDS HAVE TO DO WITH THIS?

It's probability not level, like there's a speck of dust. Probability is like the chips you can't tell what's going to happen. It might hit something you don't see.

WHAT ABOUT SWITCHING SIDES?

Yes, it'd happen.

MORE OR LESS LIKELY TO HAPPEN THAN RETURNING TO THE ORIGINAL POSITION?

Less likely to switch sides. More likely to come back to the original side if the board were flat. I don't know. They'd probably come out equal.

IF WE HAD THE MATH PROF, DO YOU THINK HE COULD BE ABLE TO PREDICT IT ACCURATELY EACH TIME?

No. He knows lots about math, but not about the physical properties of the roll. They'd be left up to chance. He would be guessing if he got it right.

SHOW ME . . . (TAPE CUT OFF)

Subject #20

CINDY, DO YOU THINK THE MARBLES WOULD GET MORE OR LESS MIXED UP IF WE ADDED TEN MORE MARBLES TO THE BOX?

Less.

HOW COME?

'Cause there'd be more in there and they'd be all scrunched up together.

SO DO YOU THINK THE MOVEMENT OF THE MARBLES HAS SOMETHING TO DO WITH HOW THEY END UP WHEN WE TIP IT?

Yes.

OK, COULD YOU EXPLAIN TO ME HOW THAT WORKS?

If there was more marbles, then there wouldn't be as much room for them to move, 'cause the marbles would take up more space, and they'd be more likely to stay where they was at. Maybe a few of them would.

OK WHAT DO YOU THINK WE'D GET IF WE ADDED A HUNDRED MORE MARBLES?

Less mixed up, 'cause then they would't hardly have anyplace to move. Just kinda stay where they was at. Just move a little bit.

OK, HOW ABOUT IF WE STARTED BACK WITH OUR ORIGINAL BOX WITH TEN MARBLES AND WENT ON TO TIP IT TEN MORE TIMES, THEN DO YOU THINK THEY'D GET MORE OR LESS MIXED UP?

More.

HOW COME?

'Cause they got more chances to get mixed up.

I SEE. WHAT DOES CHANCE HAVE TO DO WITH THIS?

Well if you only do it two times, then a few of them will move. But if you do it ten times, then I think more of 'em would move around.

OK, WELL WHAT DO YOU THINK THIS NOTION OF CHANCE HAS TO DO WITH HOW THE MARBLES WILL LOOK IN THE END?

Well, you really don't know how they're gonna go. It just happens.

I SEE. IS THERE ANYTHINGS THAT CONTROLS IT?

I don't think so. I think they just go.

WELL, WHAT IF WE TIPPED IT A HUNDRED MORE TIMES? WOULD IT BE MORE OR LESS MIXED UP THEN?

Less.

HOW COME?

Because they can only mix up so many times.

WHAT DO YOU MEAN?

Have so many whites over here and so many blacks over here, and um, so many on top of the other ones.

WHAT DOES MIXING MEAN, ANYWAY?

They're not like in a straight line, and they're not like the same color all together. Just mixed up.

DO YOU THINK IT WOULD BE POSSIBLE FOR THEM TO CHANGE SIDES FROM HOW THEY STARTED OUT? COULD THE BLACK MARBLES ALL END UP OVER HERE AND THE WHITE MARBLES END UP OVER HERE?

Yeah, its possible, but not likely.

NOT LIKELY. WHAT MAKES SOMETHING LIKELY?

I guess according to what's happened to it before.

OK, UM, DO YOU MEAN THAT YOU THINK THERE IS A PATTERN TO THE WAY THESE MARBLES FALL?

No, it just, um, there sorta was a pattern, but there sorta wasn't. Like there wasn't a whole lot of whites over here and a whole lot of whites over here.

OK, UM, DO YOU THINK THE MARBLES WOULD EVER COME BACK LIKE THEY STARTED OUT, WITH WHITES OVER HERE AND BLACKS HERE?

Eventually, after a while. You know, after you just kept on doing it.

WOULD IT BE LIKELY?

No.

DO YOU THINK A REALLY SMART PERSON, LIKE A MATHEMATICS PROFESSOR, WOULD HE BE ABLE TO PREDICT HOW EACH TIP WOULD COME OUT?

Yes.

OK, HOW WOULD HE DO IT? WHAT KINDS OF THINGS WOULD HE HAVE TO KNOW?

He'd have to include probability.

OK, WHAT DO YOU MEAN BY PROBABILITY?

Like how many times, u. First I think he would just experiment and look at how they went. And then decide by how many times he did it what the chances were that it would come out one way or the other.

HOW WOULD HE FIGURE OUT THESE CHANCES AND PROBABILITY?

Well, you have five of each color. And if you, say, do it fifteen times, you figure out what the, how many times the blacks would be like have a chance to mix up with the white, you know, for the first one. And like that.

DO YOU THINK IF HE DID THIS FOR A LONG TIME, AND FOUND OUT WHAT WOULD HAPPEN IN THE LONG RUN, THAT THAT WOULD HAVE ANY RELATION TO THE VERY NEXT TIP?

Say that one more time.

OK, IF HE DID THIS FOR A LONG PERIOD OF TIME, WOULD THAT HAVE ANY RELATION TO HOW THESE MARBLES WOULD TURN OUT AFTER ONE TIP?

If he did it a whole lot and kept watching it.

THEN YOU THINK KNOWING THIS IN THE LONG RUN WOULD TELL YOU WHAT THE NEXT TIP WAS GOING TO BE?

Close to it, maybe.

I SEE. WELL, DO YOU THINK THAT SOME ARRANGEMENTS ARE MORE LIKELY THAN OTHER ARRANGEMENTS TO OCCUR?

Yes.

(Shows likely and unlikely arrangements)

OK, WHAT MAKES THAT UNLIKELY?

Well, when you start out, with the blacks and whites on their own side, and when you tip it, its just not likely that the black ones are gonna git right between the white ones like that.

APPENDIX C

DIRECTIONS AND LIST OF INTERVIEW QUESTIONS FOR COGNITIVE TASK TWO: NOTIONS OF PROBABILITY

Cognitive Task Two replicates Inhelder and Piaget's (1975, ch. 5) experiment. Poker chips of four colors (9 yellow, 6 red, 2 white, 1 blue) all arranged as shown below. Subjects are to discuss the possible drawings of pairs of chips and their understanding of how it changes in subsequent drawings.

Poker Chip Task Directions

Poker Chip Task

Set up chips like this:

YYYYYYYYY	(9)
RRRRRR	(6)
WW	(2)
B	(1)

MAKE SURE RECORDER IS RECORDING PROPERLY.

Tell subjects: "This is kind of a game. I'll take the top chip off of every stack here and put it into this paper bag (do this). Now would you mix those up really well for me (have S shake bag). Now I want you to draw out two chips at a time without looking, but before you do, tell me which color you think each chip is most likely to be."

Tape Recorded Interview

RECORD FIRST PREDICTION: Ask: Why did you choose that color/those colors? What makes it/them the more likely? Are there any other possible pairs that would be more likely? Are there any other possible pairs that are unlikely? Which ones? How come? What makes something the most/least likely anyway? PROBE. Allow subject to draw two chips and record drawing.

RECORD SECOND AND THIRD PREDICTIONS AND DRAWINGS.

On Trial #4 - repeat above questions after subject's prediction. Also ask, What makes predictions work?

RECORD FIFTH AND SIXTH PREDICTIONS AND DRAWINGS.

On Trial #7 - ask all of the above questions. In addition, ask: How many chips of each color are left in the bag? How could you tell? Would a very smart person be able to predict which colors he'd get each time? How come? What would that person need to know, how . . . What determines which color(s) get drawn each time? How does that work? What's the best way to figure out which colors are most likely to get drawn? Does knowing what you'll get in the long run have any affect on what you draw for a single turn?

RECORD EIGHT AND NINTH PREDICTIONS AND DRAWINGS.

QUESTION AT END: (1) When we first begin are we more likely to get YY or YR? How would you figure out whether one pair was more likely than another pair of colors? (2) Does it make any difference drawing chips out one at a time rather than two at a time?

COGNITIVE TASK TWO: PREDICTION SHEET

Random Drawing of Chips Task

Subject's id. no.:

Name:

Age:

Date of Interview:

Directions: Record predictions of pair of chips (like YY, RY, . . .) in first column and the actual draw in the second column. After interview, fill in BD (best draw) prediction available from actual chips remaining, for scoring purposes.

Trials	Predicts	Draws	BD
1*			
2			
3			
4*			
5			
6			
7*			
8			
9			

Comments

*Tape record discussion points.

APPENDIX D

THREE TRANSCRIPTIONS OF EXEMPLAR INTERVIEWS: COGNITIVE TASK TWO, POKER CHIPS TASK ON NOTICNS OF PROBABILITY

Subject #15

WHAT COLORS OR WHAT COMBINATION OF COLORS IS MOST LIKELY FOR YOU TO DRAW OUT?

Ok, um, yellow and red.

OK, WHY DID YOU CHOOSE THAT?

Because they're my two favorite colors.

OK (laughs), DOES THAT MAKE THEM THE MOST LIKELY?

No.

OK, WHAT DOES MAKE SOMETHING MOST LIKELY?

Well, there are more of the yellow ones and there are more of the red ones, and so that would make a difference.

WELL, THERE ARE ACTUALLY MORE YELLOW ONES THAN THERE ARE RED. HOW COME YOU DIDN'T CHOOSE TWO YELLOW ONES?

I guess 'cause I don't usually tend to pick two of the same thing. I usually pick something different.

Red and Blue.

HOW COME?

Well the odds are going down on the yellow, 'cause I picked three of those. The odds are going down on the red and the white.

WHAT DO YOU MEAN BY ODDS?

There's three of these eliminated now; there's two of these eliminated now. So that only makes this many more, and that many more (manipulating chips) and one of these. And this one (blue) is starting to get down to where I could possibly pick up a blue one.

ARE THERE ANY OTHER PAIRS THAT WOULD BE JUST AS LIKELY?

Oh yeah, I could get two reds, two yellows, the white and the blue, the red and the blue, the yellow and the blue.

OK, SO YOU THINK THE RED AND BLUE IS THE MOST LIKELY PAIR?

Yeah, I don't know why, just a hunch.

HOW DO HUNCHES PLAY INTO THIS?

Hunches play in with anything I look at. (laughter)

WELL THIS IS KIND OF LIKE PLAYING CARDS OR SLOT MACHINES.

I usually lose at cards; my hunches aren't really all that great.

OK, YOU MENTIONED IN YOUR SECOND DRAW THAT THE PROBABILITY OF SOMETHING. WHAT DID YOU MEAN BY THAT?

I don't remember my statement.

IT WASN'T RECORDED, BUT YOU MENTIONED IT WHEN YOU MADE YOUR SECOND DRAW.

Oh, I think it works with the number of objects you're working with of each individual color. You're going to have more in there of one color than another color to pick out. It might not work out though, cause you could get those on the first time. Or it could be that there are more of those colors, so you pick those.

Red and yellow.

OK, WHY DID YOU CHOOSE THAT?

I have more of those left than I do the white one.

WHAT MAKES THEM THE MOST LIKELY?

There's more of them in there.

OK, ARE ANY OTHER COMBINATIONS JUST AS LIKELY?

Well, it could be red and white, it could be red and red, it could be red and yellow, it could be yellow and yellow, it could be yellow and white.

OK, HOW MANY CHIPS ARE LEFT IN THE BAG OF EACH COLOR?

Uh, two yellows, three reds, and one white.

HOW COULD YOU TELL?

I already counted them.

OK, WOULD A REALLY SMART PERSON, LIKE OUR MATHEMATICS PROFESSOR WE WERE TALKING ABOUT A WHILE AGO, BE ABLE TO PREDICT WHICH COLORS HE'D GET ON EACH DRAW?

I don't think so.

HOW COME?

'Cause he wouldn't know how they fell into the bag when you put 'em in. Each person, um shakes the bag differently, and there'd be just so many factors in there that I don't think they could determine it.

I SEE. WELL WHAT DETERMINES WHICH COLORS GET DRAWN ANYWAY?

The person putting their hand in and picking it out, or the availability of the colors. Like they're not going to pick a black one if there aren't any black ones in there.

THAT'S INTERESTING. LET'S SEE. IF YOU WERE TRYING TO EXPLAIN TO SOMEBODY ELSE, HOW WOULD YOU EXPLAIN TO THEM THE BEST WAY TO FIGURE OUT WHAT IS MOST LIKELY TO GET DRAWN?

I wouldn't be able to tell, because I really don't know.

OK, WELL, WHAT DID YOU DO TO FIGURE THIS OUT?

Well part of the time I look at how many are left of each color, and I usually go with at least one of the colors that has more in there. And the other one is just that I like variety. You know, its like, its just a personal thing, its not anything really scientific.

OK, DOES KNOWING WHAT YOU'LL GET IN THE LONG RUN HAVE ANY AFFECT ON WHAT YOU'LL GET ON A SINGLE DRAWING?

If you use the same exact colors in the same numbers, and you had a control group type study, you could come up with a pretty sure answer if they all came up with the same last two colors left. But on the draws before the last one there'd be no way to control it for sure which order they got drawn out in.

Subject #16

First prediction would be red and yellow since there are more than blue and white.

OTHER PAIRS JUST AS LIKELY?

Oh, yes, any combination of the four.

JUST AS LIKELY?

Yes. You could just as probably get one as another.

HOW DOES PROBABILITY WORK?

Like a ratio: 9 to 6 to 2 to 1. I'd get a yellow 9 times more than I'd get a blue.

IF YOU HAVE MORE ODDS OF GETTING A YELLOW ONE THAN A RED ONE, HOW DOES THAT RELATE TO YOUR PREDICTION OF GETTING A RED AND A YELLOW.

There's more yellow and red than blue and white.

FOURTH PREDICTION?

Three yellows, red, white and a blue out of the bag now. Nine minus three equals six yellows left. Five reds left and one white and a blue. Chances are almost even for a yellow and a red.

OTHER COMBINATIONS JUST AS LIKELY?

Yes. White and red or yellow and white, but not just as likely.

WHAT CAUSES A DIFFERENT LIKELIHOOD?

Because of the number of chips left in the bag makes me think that I'll pick one of those first.

Two yellows, three reds and one white left, therefore, more chances for a red. One chance in six to get a white. I'll still have to go with red and yellow.

IF A PERSON WANTS TO GO BY NUMBER, THEN THIS IS A GOOD WAY TO MAKE A PREDICTION? IS THAT THE BEST WAY TO FIGURE OUT THE MOST LIKELY?

Yes, basing it on the number of chips left in the bag.

IS THERE A RELATIONSHIP BETWEEN WHAT'S THE MOST LIKELY TO GET PICKED AND WHAT YOU MIGHT ACTUALLY GET?

Chance determines it. You never know what it is going to be.

CHANCE?

Any of the chips could come out of the bag. You couldn't tell what would come out. There is an equal chance for each chip to come out, any color at any time.

IS THERE AN EQUAL CHANCE FOR EACH COLOR?

Yes.

EARLIER YOU TOLE ME THAT RED AND YELLOW HAD A BETTER CHANCE: NOW YOU ARE TELLING ME THAT ALL HAVE AN EQUAL CHANCE?

When we started there were more red and yellow, there are more chances for them. Odds would be the way that things stack up. Since there are more yellow chips in there, I think the odds are better. I don't know why I'd pick yellow over blue if both are in there, but I would because there's more yellows in there.

WOULD A SMART MATH PROF WHO KNEW ALL ABOUT PROBABILITY, WOULD HE BE ABLE TO GET EVERY PREDICTION RIGHT?

Probably not.

LOT OF LUCK INVOLVED?

Right.

WHY COULDN'T HE GET THEM RIGHT WITH ALL OF HIS MATH KNOWLEDGE?

Fate, chance, luck, destiny--I don't know. No one could know exactly what you are going to pull out.

DOES IT HELP ON A SINGLE DRAW IF YOU KNOW HOW THINGS WILL TURN OUT IN THE LONG RUN?

Yeah, depending on what's left in the bag.

WHAT DETERMINES WHICH COLORS ACTUALLY GET DRAWN?

I just get the first two I touch.

THE COLORS?

I'm not sure those are under control.

WHEN WE FIRST BEGAN, WHERE WE MORE LIKELY TO GET TWO YELLOWS OR A RED AND YELLOW ON THE FIRST DRAW?

Yellow and a red because you have more three more yellows than red.

WHY WOULDN'T IT BE TWO YELLOWS.

I put them in a one to one correspondence, and there are three left over of the yellows. There are no more reds to correspond.

SO YOU HAVE A BETTER CHANCE OF GETTING A SECOND CHIP OF RED THAN YELLOW?

No. It's how you think about it. If you line them up equally, and you draw--I'd take yellow-yellow now.

HOW WOULD YOU FIGURE OUT IF ONE PAIR OF COLORS WAS MORE LIKELY THAN ANOTHER?

By the number left in the bag.

DOES IT MAKE ANY DIFFERENCE IN DRAWING CHIPS TWO OR ONE AT A TIME?

Yes, you'd have more combinations in two at a time. Part of me says you'd have an equal chance in drawing out one at a time, and the other part of me says no.

DIFFERENCE?

Only single colors in the one draw, one of four chances. It's more complex with two.

Subject #20

Two yellow ones.

OK, HOW COME?

Because there's more yellow ones, and the chances of picking it out are better than the other ones.

AND WHAT DO YOU MEAN BY CHANCE HERE?

Well, if you've got nine yellow ones, and six red ones, two white ones, and one blue one, and you stick your hand in the bag, there's more yellow ones. And even thought they're mixed up, you'd probably draw one.

WOULD ANY OTHERS BE JUST AS LIKELY?

The red ones might, but you've still got more yellows.

OK, TELL ME WHICH COLOR YOU THINK EACH CHIP IS MOST LIKELY TO BE THIS TIME?

Red and yellow.

HOW COME?

They're getting down pretty even.

YES.

I still have more red and yellows than I do white, and still one more yellow than red, but they're pretty close.

WHAT MAKES THEM THE MOST LIKELY?

There's more of them.

Red and white.

OK, HOW COME?

Tired of the yellow and the red.

IS THAT WHAT MAKES IT MOST LIKELY?

No. There's only one white left, and two yellows, three reds. The chances would be better probably with the red and yellow than red and white.

WHY DO YOU SAY THE CHANCES ARE BETTER?

'Cause there's two yellows and only one white.

OK, THEN WHICH IS MORE LIKELY, YOUR GUESS OF RED AND WHITE OR A GUESS OF RED AND YELLOW?

Well, red and yellow are more likely, but I still want red and white.

BUT WHAT WE'RE AFTER HERE IS THE COLORS THAT ARE MOST LIKELY.

Ok, I'll change it then. Red and yellow.

OK. WELL, WOULD A REALLY SMART PERSON BE ABLE TO PREDICT WHICH COLORS HE'D GET EACH TIME?

Maybe not each time, but sort of have an idea.

HOW WOULD HE GET THAT IDEA?

According to how many colors of each chips he has.

HOW DOES CHANCE WORK IN A CASE LIKE THIS?

Well, unless there's just a few of them. Well, like with blue and white, there'd be three. There'd be fifteen with the yellow and the reds. So those three are gonna get, like lost with the yellow and reds, 'cause there's so many of them.

WHAT DETERMINES WHAT COLOR GETS DRAWN EACH TIME.

The number that's in the bag of each color.

OK, COULD YOU EXPLAIN THAT A LITTLE FURTHER?

Well, since there's nine yellows and six whites (sic), and you stick your hand in the bag. You don't have to draw what's on top, you can just grab around in there and pick anything. And since there's more yellow ones than the others, like there'd be a yellow, maybe not two yellows on one draw, but there'd be like one yellow in there on the first time. You can just use probability.

I SEE. WELL WHAT DO YOU MEAN BY PROBABILITY?

Like you see how many yellows, and how many reds, and how many blue, and the whites. And then you can look and see which is most likely to get.

OK, WELL WHAT ABOUT THIS? HOW WOULD YOU EXPLAIN IT IF YOU DIDN'T DRAW A YELLOW THE FIRST TIME, EVEN THOUGH THERE'S MORE YELLOWS IN THERE?

The yellow ones were hid over the other ones (laughter).

APPENDIX E

COGNITIVE INTERVIEW ITEMS AND SCORING INSTRUCTIONS

Operational Scoring Definitions

1. Concrete Operational Stage Achievement: Subjects passing none or one of the formal items.

2. Transitional Operational Stage Achievement: Subjects passing two or three of the five formal items.

3. Formal Operational Stage Achievement: Subjects passing four or five of the formal items.

Concrete Operational Items

C1. Task 1: Does the subject logically explain either: (a) that more tippings will increase the mixture, or (b) that after some initial tippings all resulting arrangements are equally mixed?

C2. Task 1: Does the subject affirm the possibility of the marbles returning to their initial distribution or the possibility of crisscrossing sides without invoking tautological arguments or the influence of external agents which act differentially on the various marbles?

C3. Task 1: Do explanations of smart person's inability to correctly predict marble arrangements make reference to indeterminacy of the outcomes?

C4. Task 2: Do explanations of a smart person's inability to correctly predict pairwise drawings make reference to indeterminacy of the outcomes?

C5. Task 2: Are frequency or quantitative criteria used to logically justify more than one prediction?

C6. Task 2: Are three or more predictions the most probable?

Formal Operational Items

F1. Task 1: In explaining the relationship between mixture and more marbles/tippings, does the subject either: (a) discuss more permutations or more actual collisions, or (b) discuss relative equality of "mixed up" among permutations which deviate from the initial order?

F2. Task 1: Does the subject explain a smart person's inability to correctly predict marble arrangements by appealing to the origin of indeterminacy?

F3. Task 2: Does the subject explain a smart person's inability to correctly predict pairwise drawings of chips by appealing to the origin of indeterminacy or the probability/odds of undrawn colors?

F4. Task 2: Does the subject's explanation of the most likely combination of chips refer to a fraction or ratio of the total chips undrawn?

F5. Task 2: Are seven or more of the subject's predictions the most probable?

General Scoring Instructions

1. There are 11 items in the interview battery. The first set of six items distinguishes concrete operations; the second set of five items distinguishes formal operations. There are no scorable items which could discriminate levels of ability below concrete operations; this is not meant to preclude the possibility.

2. Each item is constituted by subject response to an initial probe question and answers to subsequent related probes. Decisions about the relatedness of probes, and hence about the inclusion or exclusion of any single response under an item must be left to the discretion of each judge in relation to these scoring procedures. Follow-up probe questions may vary slightly in content and wording.

3. In scoring each of the items, the judge must decide if subject responses display the logical character of at least the respective level of cognitive achievement. Items are scored "1" (pass) or "0" (fail).

4. Judges are advised to consider each subject on the basis of the optimum conceptual performance on a given task. In a case where data appears to be in conflict or marginally ambiguous, the higher level response will generally be scored.

5. Every effort should be made to score items either pass or fail. Occasionally data may be ambiguous or missing. In such cases,

if the judge cannot score the item as either pass or fail, then it is to be scored "X" (indeterminate).

6. After items have been scored independently, judges should meet to resolve their scoring disagreements through discussion. In the event that two judges cannot agree after discussing the responses to an item, the higher score is to be assigned. Ranking scores from highest to lowest goes from pass to fail to indeterminate. For final indeterminate scores, alternately substitute a pass and fail so the subject can be categorized. The number of scoring agreements between judges prior to making any resolutions about their disagreements provides the basis for computation of reliabilities.

APPENDIX F

MATHEMATICAL PROBLEM SOLVING INVENTORY

The following six mathematical problems were selected from those in the _Mathematical Representations of Cognitive Structures Inventory_ (Clark & Reeves, in press). The solutions to the problems represent cognitive structures (mathematical concepts) used in problem solving. Each of the six problems tests for a specific mathematical concept and, synonymously, for representations of a specific mathematical cognitive structure. The concepts or structures can be categorized into the two broad mathematical areas of algebra and geometry; three problems were selected from each area, as described in the table below.

TESTS FOR MATHEMATICAL CONCEPTS

(REPRESENTATIONS OF MATHEMATICAL STRUCTURES)

Problem Number

	Three	Five	Six
Algebra	1.1 Correspondence	Proportion, Mixture	Spatial Logic
Geometry	One	Two	Four
	Volume Measure	Area Measure	Partitioning by intersecting Lines

Directions

Completion of the Mathematical Representations of Cognitive Structures Inventory requires 36 minutes. Each problem is allotted five minutes for solving. One minute is allotted for completing the second page of each problem. Therefore, each problem requires six minutes. A total of 36 minutes is required for all six problems.

All work is to be done on the respective test page. Subjects are to work only on the page as directed without turning pages.

Problem 1

Try to solve the following problem. Below the problem there is space for you to make any necessary drawings and to show your work. Do not go to the next page until you are instructed to do so.

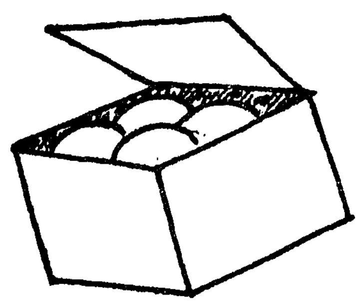
****

A dozen balls are tightly packed in a box of 3 rows, 4 balls in each row. If each ball has a radius of 2 inches, what are the inside dimensions of the box?

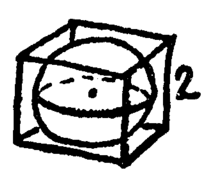
Problem 1--Page 2

Instructions: The four items shown below (A, B, C, D) are related to the problem you have just tried to solve. One or more of these items may remind you of your understanding of the problem. Put a "1" beside the letter for the item which you think best fits the way you thought about the problem in order to solve it. Put a "2" beside the item which would be Next Best. Put "3" and "4" beside the items of your next two choices. "0" for any not fitting.

__________A



__________B



Problem 1--Page 3

__________C H = 3 x 2

L = 2 + 2

W = 2 + 2

__________D The height of the box is the same as the sum of three diameters. So the height of the box is six inches. The base of the box is square and its sides have length

Problem 2

Try to solve the following problem. Below the problem there is space for you to make any necessary drawings and to show your work. Do not go to the next page until you are instructed to do so.

****

A rectangular floor is to be constructed of square tiles, all the same size. The dimension of the floor is 15 1/2 feet long and 8 1/3 feet wide. If the floor is to be made only of whole pieces of tile, find a size of tile which will work. How many of your tiles would be needed to cover the floor?

Problem 2--Page 2

Instructions: The four items shown below (A, B, C, D) are related to the problem you have just tried to solve. One or more of these items may remind you of your understanding of the problem. Put a "1" beside the letter for the item which you think best fits the way you thought about the problem in order to solve it. Put a "2" beside the items of your next two choices. "0" for any not fitting.

__________ A Each tile is to be a square so I need to find the area of the floor and try to subdivide it into a whole number of little squares. There is more than one solution to the problem and one of these solutions may be found by first finding a common denominator

__________ B Area = 15 1/2 x 8 1/3

= 31/2 x 25/3

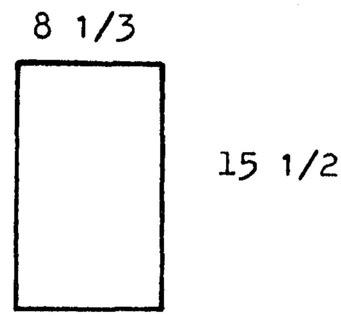
= 93/6 x 50/6

= 93 x 50 x $(1/6)^2$

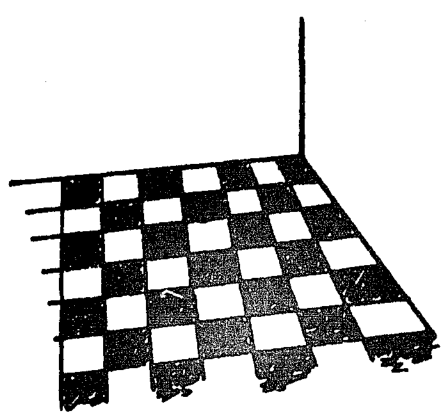
= 4650 x $(1/6)^2$; So,

Problem 2--Page 3

__________C



__________D



Problem 3

Try to solve the following problem. Below the problem there is space for you to make any necessary drawings and to show your work. Do not go to the next page until you are instructed to do so.

****

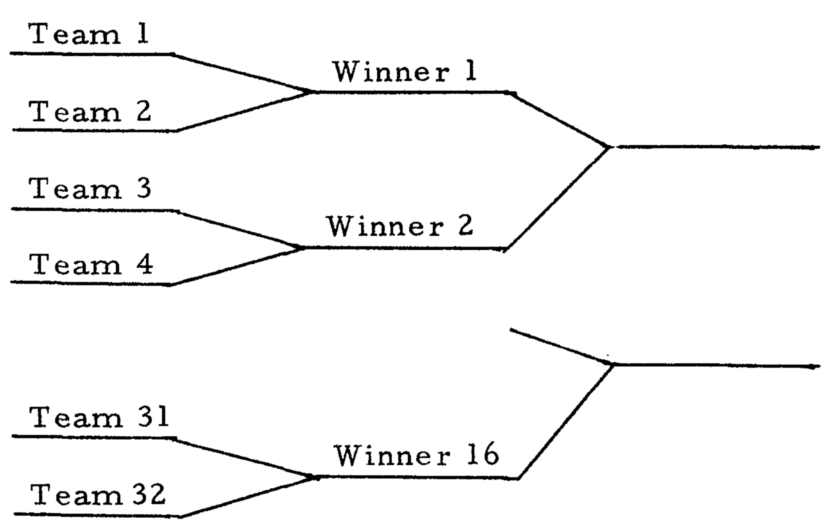
Thirty-two teams are to play in a single-elimination basketball tournament--single-elimination means if a team loses any game, they are no longer in the tournament. How many games must be played in order for a winner to be declared?

Problem 3--Page 2

Instructions: The four items shown below (A, B, C, D) are related to the problem you have just tried to solve. One or more of these items may remind you of your understanding of the problem. Put a "1" beside the letter for the item which you think best fits the way you thought about the problem in order to solve it. Put a "2" beside the item which would be Next Best. Put "3" and "4" beside the items of your next two choices. "0" for any not fitting.

	Number of teams	Number of games	
______ A	2	1	
	3	2	
	4	3	
	32	G	
	50		G = ?

______ B



Problem 3--Page 3

__________ C

1st Round	2nd Round	3rd Round
16 games	8 games	4 games

__________ D

Every time a game is played a team is eliminated from the tournament. Since there are 32 teams in the tournament, 31 teams must lose, so the number of games needed is

Problem 4

Try to solve the following problem. Below the problem there is space for you to make any necessary drawings and to show your work. Do not go to the next page until you are instructed to do so.

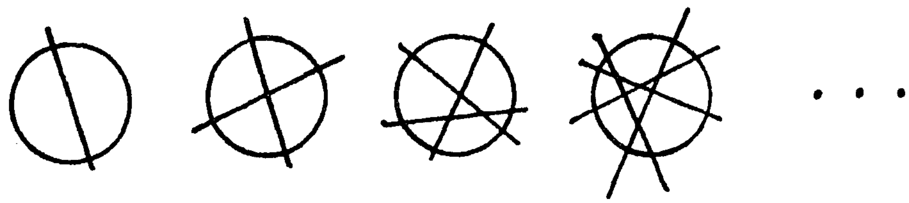
****

Suppose you are going to slice a pie with 6 straight cuts of a knife. What is the largest number of pieces you can get by doing this-- and don't worry if some of the pieces are smaller than others!

Problem 4--Page 2

Instructions: The four items shown below (A, B, C, D) are related to the problem you have just tried to solve. One or more of these items may remind you of your understanding of the problem. Put a "1" beside the letter for the item which you think best fits the way you thought about the problem in order to solve it. Put a "3" and a "4" beside the items of your next two choices. "0" for any not fitting.

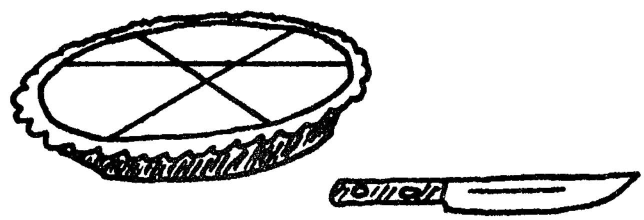
__________A



__________B In order to get the greatest possible number of pieces on each cut it is necessary to cross all previous cuts. Furthermore, the cuts must not cross at the intersection of two previous cuts. Each time a cut crosses a previous cut there is an increase of one more piece

Problem 4--Page 3

__________C



Number of cuts	Number of pieces
1	2
2	4
3	7
4	11
5	?
6	?

__________D

Number of pieces P = ____?____

Problem 5

Try to solve the following problem. Below the problem there is space for you to make any necessary drawings and to show your work.

****

Suppose you have 3 cups and 10 coins. Arrange them in such a way as to have an odd number of coins in each cup. Use all 10 coins in the arrangement. Your task is to explain or show how this can be done.

Problem 5--Page 2

Instructions: The four items shown below (A, B, C, D) are related to the problem you have just tried to solve. One or more of these items may remind you of your understanding of the problem. Put "1" beside the letter for the item which you think best fits the way you thought about the problem in order to solve it. Put "2" beside the item which you think fits NEXT BEST. Put "3" and "4" beside the item of your next two choices. "0" if none seems to fit.

__________ A

__________ B

1 + 3 + 6 = 10

3 + 5 + 2 = 10

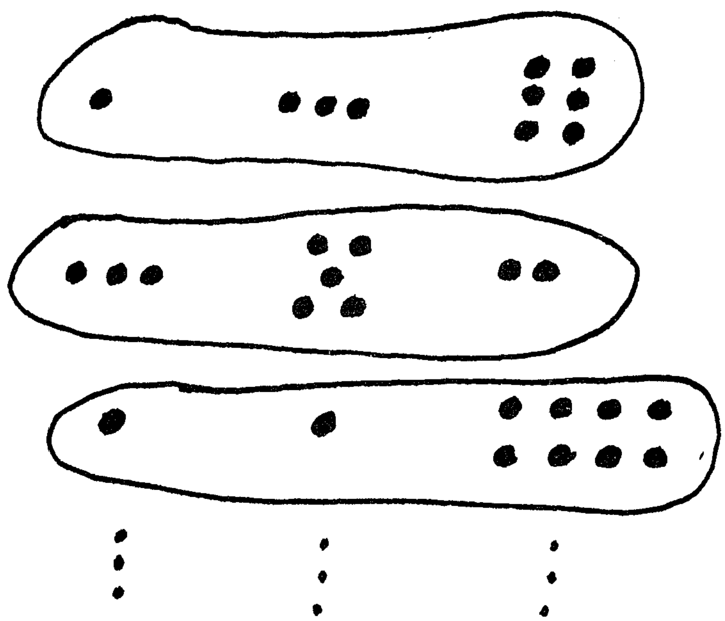
1 + 1 + 8 = 10

1 + 7 + 2 = 10

Problem 5--Page 3

__________C There are several things that might be "arranged" in this problem: The coins and the cups. Since the sum of any three odd numbers is always an odd number (i.e., not 10), I need to arrange the cups as well as the coins. So, by placing

__________D



Problem 6

Try to solve the following problem. Below the problem there is space for you to make any necessary drawings and to show your work. Do not go to the next page until you are instructed to do so.

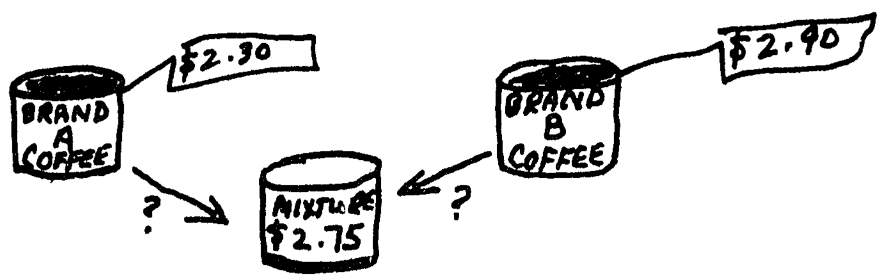
****

Brand A coffee retails for $2.30 per pound and Brand B coffee retails for $2.90 per pound. In a mixture of just these two brands, what percentage of the mixture should be Brand A if the retail price of the mixture is to be $2.75 per pound. Assume that the retail price of the mixture is entirely dependent upon the retail prices of the two brands.

Problem 6--Page 2

Instructions: The four items shown below (A, B, C, D) are related to the problem you have just tried to solve. One or more of these items may remind you of your understanding of the problem. Put a "1" beside the letter for the item which you think best fits the way you thought about the problem in order to solve it. Put "2" beside the item which would be NEXT BEST. Put "3" and "4" beside the items of your next two choices. Put "0" beside any which does not fit at all.

__________A



__________B

Coffee	Amount (lb)	Price/lb
A	X	2.30
B	?	2.90
Mix	1	2.75

Problem 6--Page 3

__________C The fractional portion of Brand A in one pound of the required mixture is a certain ratio. That ratio is the difference between the price of one pound of Brand B and one pound of the mixture compared to the difference between the price of one pound of Brand B and one pound of Brand A.

__________D

$$2.30\,(X) + (1-X)\,2.90 = 2.75$$

$$\text{or } X = \frac{2.90 - 2.75}{2.90 - 2.30}$$

APPENDIX G

RAW SAMPLE DATA: AGE, SEX, RANDOM ORDER OF COGNITIVE TASK PRESENTATION

Subject	Age	Sex	Random Order of 2 Cognitive Tasks
Concrete Operational			
04	21	F	1, 2
06	19	F	1, 2
12	19	F	2, 1
13	21	F	1, 2
15	25	F	1, 2
17	20	F	2, 1
19	21	F	1, 2
23	23	F	1, 2
24	20	F	1, 2
25	35	F	2, 1
29	42	F	2, 1
31	23	F	2, 1
38	23	F	1, 2

n=13 mean= 24.0

Subject	Age	Sex	Random Order of 2 Cognitive Tasks
Transitional Operational			
02	30	F	1, 2
03	45	F	1, 2
08	40	F	1, 2
09	21	F	2, 1
20	20	F	1, 2
21	22	F	2, 1
22	19	F	1, 2
27	36	F	1, 2
28	19	F	1, 2
30	22	F	1, 2
32	26	F	1, 2
34	20	F	1, 2
35	36	M	1, 2
36	21	F	1, 2
37	19	F	1, 2
40	35	M	2, 1
n=16	mean= 26.9		
Formal Operational			
01	25	F	2, 1
05	19	F	2, 1
07	32	F	2, 1
10	21	F	1, 2
11	28	F	1, 2
14	34	F	1, 2
16	28	M	2, 1
18	20	F	2, 1
26	32	M	1, 2
33	20	F	2, 1
39	20	F	2, 1
n±11	mean= 25.7		

APPENDIX H

RAW SCORES* ON COGNITIVE INTERVIEW TASK ONE: MARBLE TASK ON NOTIONS OF CHANCE

Subjects	Concrete Operational Items			Formal Operational Items	
	C1	C2	C3	F1	F2
Concrete Operational					
04	X	1	1	0	0
06	1	1	0	0	0
12	1	1	0	0	0
13	1	1	1	0	0
15	1	1	0	0	0
17	X	1	0	0	0
19	X	1	0	0	0
23	1	1	0	0	0
24	X	1	0	0	0
25	1	1	1	0	0
29	1	1	0	1	0
31	1	1	0	0	0
38	1	1	1	1	0
n = 13 mean =	1.0	1.0	.3	.2	.0

* 1 = Pass
0 = Fail
X = Indeterminable

Subjects	Concrete Operational Items			Formal Operational Items	
	C1	C2	C3	F1	F2
Transitional Operational					
02	X	1	1	0	1
03	1	1	0	0	0
08	0	1	1	0	1
09	X	1	1	0	1
20	1	1	0	1	0
21	1	1	1	1	0
22	1	0	1	1	0
27	1	1	1	1	0
28	0	1	1	0	0
30	1	1	1	1	1
32	1	1	1	1	1
34	0	1	1	0	1
35	1	1	1	1	0
36	1	1	X	1	X
37	1	1	1	1	1
40	X	1	1	0	1
n = 16	mean= .7	.9	.9	.6	.5
Formal Operational					
01	1	1	1	1	1
05	1	1	1	1	1
07	1	1	1	1	1
10	X	1	1	0	1
11	1	1	1	1	1
14	1	1	1	1	1
16	1	1	1	1	1
18	X	0	1	1	1
26	1	1	1	1	1
33	1	1	1	1	0
39	1	1	1	1	1
n = 11	mean = 1.0	.9	1.0	.9	.9

APPENDIX I

RAW SCORES* ON COGNITIVE INTERVIEW TASK TWO: POKER CHIP TASK ON NOTIONS OF PROBABILITY

Subjects	Concrete Operational Items			Formal Operational Items		
	C4	C5	C6	F3	F4	F5
Concrete Operational						
04	1	1	1	0	0	0
06	1	1	0	0	0	0
12	1	1	1	0	1	0
13	1	1	1	0	0	1
15	1	1	0	0	0	0
17	0	1	0	0	0	0
19	0	1	1	0	0	1
23	0	1	1	0	0	0
24	0	1	1	0	0	0
25	1	1	1	0	0	1
29	1	1	1	0	0	0
31	1	1	0	1	0	0
38	1	1	1	0	0	0
n = 13	.6	1.0	.7	.1	.1	.2

*1 = Pass
0 = Fail
X = Indeterminable

Subjects	Concrete Operational Items			Formal Operational Items		
	C4	C5	C6	F3	F4	F5
Transitional Operational						
02	0	1	1	0	1	0
03	1	1	1	1	1	1
08	1	1	1	0	0	1
09	1	1	1	1	0	0
20	1	1	1	0	1	0
21	1	1	1	1	1	0
22	1	1	0	0	1	0
27	1	1	1	0	1	1
28	1	1	1	1	1	0
30	1	1	1	0	0	0
32	1	1	1	0	0	0
34	1	1	1	1	1	0
35	1	1	1	1	1	0
36	1	1	1	1	1	0
37	1	1	1	1	0	0
40	1	1	1	0	1	1
n = 16	.9	1.0	.9	.5	.7	.3
Formal Operational						
01	1	1	1	1	1	0
05	1	1	1	1	1	0
07	1	1	1	1	1	0
10	1	1	1	1	1	1
11	1	1	1	1	1	1
14	1	1	1	1	1	0
16	1	1	1	1	1	1
18	1	1	1	1	1	0
26	1	1	1	1	1	1
33	1	1	1	1	1	1
39	1	1	1	1	1	1
n = 11	1.0	1.0	1.0	1.0	1.0	.5

APPENDIX J

RAW SCORES FOR COGNITIVE STAGE LEVELS AND PROBLEM-SOLVING SUCCESS

Scoring

Degree of problem solving success is measured on the six problems below and is served according to the following code:

0 = no success in mathematical problem solving
1 = partial success in mathematical problem solving
2 = total success in mathematical problem solving

Subject	Algebraic Problems Prob. 3	5	6	Geometric Problems Prob. 1	2	4	Mean Score
Concrete Operational							
04	1	1	1	2	0	1	1.0
06	1	0	1	0	0	0	.3
12	2	0	1	0	0	1	.7
13	0	0	0	1	0	1	.3
15	2	0	1	1	0	1	.8
17	0	0	0	0	0	1	.2
19	1	0	1	1	1	1	.8
23	2	0	1	1	0	1	.8
24	0	0	0	1	0	1	.3
25	1	0	0	1	0	1	.5
29	0	0	0	2	1	1	.7
31	0	0	0	0	0	0	.0
38	2	1	1	1	1	2	1.3
n = 13 mean=	.9	.2	.5	.8	.2	.9	0.6

Subject	Algebraic Problems Prob.3	5	6	Geometric Problems Prob.1	2	4	Mean Score
Transitional Operational							
02	2	1	2	1	1	2	1.5
03	0	0	0	1	0	1	.3
08	0	0	1	0	1	0	.3
09	1	0	1	1	0	1	.7
20	0	0	0	0	1	1	.3
21	2	1	1	1	0	1	1.0
22	1	0	1	2	1	1	1.0
27	1	0	1	1	0	1	.7
28	2	2	1	1	0	1	1.2
30	2	0	0	0	0	1	.5
32	0	0	0	0	0	1	.2
34	1	1	1	2	1	1	1.2
35	2	0	0	0	1	1	.7
36	0	0	1	0	0	1	.3
37	0	0	0	0	0	1	.2
40	2	1	1	0	1	1	1.0
n= 16	mean=1.0	.4	.7	.6	.4	1.0	0.7
Formal Operational							
01	2	0	0	1	1	1	.8
05	2	0	0	1	0	1	.7
07	1	1	0	2	1	1	1.0
10	0	0	0	2	1	1	.7
11	2	0	0	1	0	1	.7
14	2	2	1	2	1	2	1.7
16	2	1	2	1	1	2	1.5
18	0	0	0	1	1	1	.5
26	2	0	0	1	0	1	.7
33	2	0	0	1	0	1	.7
39	2	1	1	1	1	2	1.3
n= 11	mean=1.5	.5	.4	1.3	.6	1.3	0.9

Combined Stage Level Totals

Cognitive Level	Raw Scores			Percentages			Totals	
	0	1	2	0	1	2	Raw Score	%
Concrete	38	33	7	48.7	42.3	9.0	78	32.5
Transitional	41	44	11	42.7	45.8	11.5	96	40.0
Formal	21	29	16	31.8	43.9	24.3	66	27.5
Totals	100	106	34				240	
Percentage	41.7	44.2	14.1					100%

APPENDIX K

RAW SCORES FOR COGNITIVE STAGE LEVELS AND LABELED FIGURES

Scoring

Labeled figure is measured on the six problems below and is scored according to the following code:

0 = Subject did not label figure
1 = Subject labeled figure spontaneously as part of the problem solving process

Subject	Algebraic Problems Prob. 3	5	6	Geometric Problems Prob. 1	2	4	Total Labels (1)/SS
Concrete Operational							
04	0	0	1	0	0	0	1
06	0	0	0	0	1	0	1
12	0	0	0	0	1	1	2
13	1	0	0	1	1	1	4
15	0	0	0	0	1	1	2
17	0	1	0	0	1	1	3
19	0	0	0	1	1	1	3
23	0	0	0	1	1	0	2
24	0	0	0	0	0	0	0
25	0	0	1	1	1	1	4
29	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0
38	0	0	0	1	1	1	3
							mean = 1.9
choice 0=	12	12	11	8	4	6	
choice 1=	1	1	2	5	9	7	
n=13 total choices=	13	13	13	13	13	13	

Subject	Algebraic Problems			Geometric Problems			
Prob.	3	5	6	Prob. 1	2	4	Total Labels (1)/SS
Transitional Operational							
02	0	0	1	0	0	0	1
03	0	0	1	1	1	0	3
08	1	0	0	1	1	0	3
09	0	0	1	1	0	0	2
20	0	1	1	1	1	1	5
21	1	0	0	0	1	0	2
22	0	0	1	1	1	0	3
27	0	0	1	0	0	1	2
28	0	0	1	1	1	1	4
30	0	0	0	0	1	0	1
32	0	0	0	0	1	0	1
34	1	0	0	1	1	0	3
35	1	0	0	0	1	1	3
36	0	0	1	0	0	1	2
37	0	0	0	0	1	1	2
40	0	0	0	1	1	1	3
							mean=2.5
choice 0=	12	15	8	8	4	9	
choice 1=	4	1	8	8	12	7	
total n=16 choices=	16	16	16	16	16	16	
Formal Operational							
01	0	0	1	1	1	1	4
05	0	0	0	1	1	0	2
07	0	0	0	1	1	0	2
10	0	0	0	1	0	1	2
11	0	0	0	0	0	1	1
14	0	0	0	1	0	0	1
16	0	0	0	1	1	1	3
18	0	0	1	0	0	0	1
26	0	0	0	1	0	1	2
33	0	0	0	1	1	1	3
39	0	0	0	0	1	0	1
							mean=2.0
choice 0=	11	11	9	3	5	5	
choice 1=	0	0	2	8	6	6	
total n=11 choices=	11	11	11	11	11	11	

APPENDIX L

RAW SCORES FOR COGNITIVE STAGE LEVELS AND CHOSEN PROBLEM-SOLVING STRATEGY

Scoring

Preference of mathematical conceptual representation of one's problem solving strategy is measured on the six problems below and is scored according to the following code:

0 = No choice of a problem solving strategy was selected as representing how the subject solved the problem

1 = Concrete, pictorial figure

2 = Abstract, geometric figure

3 = Algebraic, numerical representation

4 = Verbal, logical description

Subject	Algebraic Problems			Geometric Problems		
Prob.	3	5	6	1	2	4
Concrete Operational						
04	2	4	3	1	4	1
06	1	3	1	3	2	4
12	1	1	1	1	2	2
13	3	1	1	1	2	1
15	4	3	3	2	4	2
17	1	3	1	1	2	4
19	3	1	3	2	2	1
23	4	4	1	2	1	2
24	1	4	2	3	4	1
25	2	1	1	2	2	1
29	4	4	3	1	4	3
31	3	1	1	4	4	1
38	4	1	1	1	2	2
Choice 0=	0	0	0	0	0	0
Choice 1=	4	6	8	6	1	6
Choice 2=	2	0	1	4	7	4
Choice 3=	3	3	4	2	0	1
Choice 4=	4	4	0	1	5	2
n=13 Totals	13	13	13	13	13	13

Subject	Algebraic Problems			Geometric Problems		
Prob.	3	5	6	1	2	4
Transitional Operational						
02	3	4	4	4	0	1
03	0	1	1	2	2	2
08	3	1	1	1	4	1
09	1	1	1	1	2	1
20	0	1	1	1	2	1
21	1	1	4	1	3	1
22	1	1	1	2	2	2
27	2	4	1	4	2	4
28	2	1	1	1	2	2
30	1	1	1	4	2	1
32	1	0	4	1	1	1
34	2	4	3	1	2	2
35	2	3	1	2	2	2
36	2	2	4	1	3	2
37	3	3	1	2	2	2
40	2	1	1	3	4	2
Choice 0 =	2	1	0	0	1	0
Choice 1 =	5	9	11	8	1	7
Choice 2 =	6	1	0	4	10	8
Choice 3 =	3	2	1	1	2	0
Choice 4 =	0	3	4	3	2	1
n= 16 Total	16	16	16	16	16	16

Subject	Algebraic Problems Prob. 3	5	6	Geometric Problems Prob. 1	2	4
Formal Operational						
01	4	1	3	2	2	2
05	3	4	1	3	2	3
07	2	1	3	1	4	2
10	0	0	4	0	4	1
11	2	4	1	3	3	1
14	1	4	3	0	4	2
16	4	4	4	3	2	2
18	3	4	1	1	2	2
26	2	4	3	0	4	2
33	1	1	2	1	2	2
39	3	4	3	0	2	1
Choice 0 =	1	1	0	4	0	0
Choice 1 =	2	3	3	3	0	3
Choice 2 =	3	0	1	1	6	7
Choice 3 =	3	0	5	3	1	1
Choice 4 =	2	7	2	0	4	0
n=11 Totals	11	11	11	11	11	11

Combined Stage Level Totals

Cognitive Level	Raw Score: Choice					Percentage of Choice					Totals	
	0	1	2	3	4	0	1	2	3	4	Raw Score	%
Concrete	0	31	18	13	16	0.0	39.7	23.1	16.7	20.5	78	32.5
Transitional	4	41	29	9	13	4.2	42.7	30.2	9.4	13.5	96	40.0
Formal	6	14	18	13	15	9.1	21.2	27.3	19.7	22.7	66	27.5
Totals	10	86	65	35	44						240	
Percentages	4.2	35.8	27.1	14.6	18.3							100%

APPENDIX M

RAW SCORES FOR COGNITIVE STAGE LEVELS AND MATCHING CHOSEN AND USED PROBLEM-SOLVING STRATEGIES

Scoring

Matching (a) problem solving strategies actually employed and (b) one's stated preference of a mathematical strategy that best describes one's reasoning in understanding and solving the problem is measured on the six problems below and is scored according to the following code:

0 = (a) and (b) do not match

1 = (a) and (b) match: the subject's preferred the same problem solving strategy that he/she actually used

Subject	Algebraic Problems			Geometric Problems			Total
Prob.	3	5	6	Prob.1	2	4	Matches (1)/SS
Concrete Operational							
04	1	0	1	1	0	0	3
06	1	0	1	0	1	0	3
12	1	0	1	1	1	1	5
13	1	0	1	1	1	1	5
15	0	0	1	1	0	1	3
17	0	0	1	1	1	0	3
19	0	0	1	0	1	0	2
23	1	0	1	1	1	1	5
24	0	0	0	0	0	1	1
15	1	0	1	0	1	1	4
29	1	1	0	1	1	0	4
31	0	0	0	0	0	0	0
38	0	0	1	1	1	1	4
							mean=3.2
Totals:							
No match (0)	=6	12	3	5	4	6	
Matched (1)	= 7	1	10	8	9	7	
n = 13	13	13	13	13	13	13	

Subject	Algebraic Problems Prob. 3	5	6	Geometric Problems Prob. 1	2	4	Total Matches (1)/SS
Transitional Operational							
02	1	1	1	0	0	1	4
03	0	0	0	0	1	1	2
08	0	0	1	1	0	0	2
09	1	0	1	1	1	1	5
20	0	1	1	1	1	0	4
21	1	0	1	0	1	0	3
22	0	0	1	0	1	1	3
27	1	1	1	1	0	1	5
28	1	0	1	1	1	1	5
30	0	0	1	1	1	0	3
32	0	0	0	0	1	1	2
34	1	0	1	1	1	1	5
35	0	0	1	0	1	1	3
36	1	1	1	1	1	1	6
37	0	0	1	0	1	1	3
40	1	0	1	1	1	1	5
							mean=3.8
Totals:							
No Match (0)=8		12	2	7	3	4	
Matched (1) = 8		4	14	9	13	12	
n=16	16	16	16	16	16	16	

Subject	Algebraic Problems			Geometric Problems			Total
Prob.	3	5	6	Prob. 1	2	4	Matches (1)/SS
Formal Operational							
01	0	0	0	1	1	1	3
05	1	0	0	1	1	0	3
07	1	0	0	1	0	1	3
10	0	0	0	0	0	1	1
11	1	0	1	1	1	0	4
14	1	1	1	0	0	1	4
16	0	0	0	1	1	1	3
18	0	0	1	0	0	1	2
26	1	0	1	0	1	1	4
33	0	0	0	1	1	1	3
39	1	0	1	1	1	0	4
							mean = 3.1
Totals							
No Match (0)=5		10	6	4	4	3	
Matched (1) = 6		1	5	7	7	8	
n=11	11	11	11	11	11	11	