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Teachers' ability to elicit and use evidence of student thinking during instruction is critical in high-quality mathematics instruction. The practice of anticipating students' mathematics supports teachers in noticing and being prepared to respond to student thinking during in-the-moment instruction. To date, most research on anticipating has involved teachers in formal professional learning contexts such as university coursework or professional development programs and has focused on identifying and describing teachers' anticipations. However, few studies have investigated anticipating in the context of professional practice or how it is enacted by teachers with a variety of experiences.

This study examines three secondary mathematics teachers' practice of anticipating students' mathematics. Using a multi-case study design, it examines the ways in which these three teachers anticipate, the resources they draw upon, the purposes they have for engaging in the practice and highlights similarities and differences in their enactments. Findings indicate that when anticipating the ways students will engage with cognitively demanding mathematics tasks, teachers first consider their instructional contexts, including their curricular programs, academic standards, and the focal mathematical ideas for units of instruction. To varying degrees, they develop records of mathematical activity students might demonstrate and conjectures about ways students might think mathematically when engaging with these tasks. Beyond these similarities, teachers in this study drew upon different domains of knowledge and enacted the practice for different purposes. The results of this study have implications for mathematics teacher educators working to support mathematics teachers learning to anticipate as well as researchers investigating instructional practices in context.

DESCRIBING THE PRACTICE OF ANTICIPATING STUDENTS' MATHEMATICS OF SECONDARY MATHEMATICS TEACHERS: A MULTI-CASE STUDY

by

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DEDICATION

To my advisor, mentor, advocate, and champion. Your unwavering support and profound influence have been instrumental in shaping the trajectory of my career.

APPROVAL PAGE

This dissertation written by Emily B. Hare has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I: INTRODUCTION AND STATEMENT OF PROBLEM

For more than half a century, policy makers and education leaders in the United States have sought to improve the mathematical education of students. From the *National Defense Education Act* of 1958 in reaction to the USSR's launch of the first space satellite Sputnik to *A Nation at Risk* (National Commission on Excellence in Education [NCEE], 1983) and the *National Mathematics Panel* report (2008), there has been national recognition that the ways students learn mathematics must evolve and improve to ensure national security (NCEE, 1983) and economic prosperity (National Mathematics Advisory Panel [NMAP], 2008a, 2008b). Beyond these dominant narratives of global competition and security, critical scholars have illuminated how mathematics education in the United States perpetuates social inequities (Berry, Ellis, & Hughes, 2014; Gutiérrez, 2017) while also highlighting the role of an informed citizenry as foundational to the health of democracy (National Research Council [NRC], 1989). Regardless of perspective, there has been widespread agreement that an increasingly technological, information rich, and global society demands a degree of quantitative literacy that the US education system fails to support most students in developing (Darling-Hammond, 2010; Koedinger & Nathan, 2004; National Council of Teachers of Mathematics [NCTM], 2000; NRC, 1989; Ojose, 2011).

A fundamental goal of education reform is for every student to "learn with understanding" (Bransford et al., 2000, p. 16). Summarized in *Adding It Up* (NRC, 2001), a consensus document synthesizing a large volume of research on mathematics learning, learning mathematics with understanding leads to mathematical proficiency. In *Adding It Up*, the NRC's (2001) five strands of mathematical proficiency are five interconnected strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions, and all five are now expected outcomes for our learners in mathematics classrooms (NRC, 2001). For over 2 decades, learning scientists have generated knowledge about how students learn such proficiencies, the kinds of experiences that promote such learning, and the ways that their opportunities, experiences, and cultures interact with the understandings they create (Bransford et al., 2000; National Academies of Science, Engineering, and Medicine, 2018). In tandem, the mathematics education professional community has continued to engage in efforts to transform mathematics teaching into supportive environments where every student can develop such proficiencies.

In response to this progress in research on learning, mathematics education researchers have increased their attention to instructional practices that foster mathematical proficiency. Instructional practices that foster the five strands of mathematical proficiency are premised on the selection and implementation of cognitively demanding tasks (Stein et al., 1996; Stein & Lane, 1996). To implement such tasks in ways that maintain the cognitive demand for students (Stein & Lane, 1996), researchers have identified and described specific pedagogical practices such as discourse moves that assist teachers in encouraging and supporting students as they engage in mathematical discourse (Chapin & O'Connor; 2007; Hufferd-Ackles et al., 2004; Herbel-Eisenmann et al., 2013). Other practices such as launching complex tasks (Jackson et al., 2013), intentional questioning (Jacobs and Empson, 2016), and noticing student thinking (Jacobs et al., 2010) have emerged as high-leverage practices that support students in learning mathematics for understanding. In its seminal document *Principles to Action*, the National Council of Teachers of Mathematics (2014) synthesized research on instructional practices fostering mathematics learning with understanding with its Effective Mathematics Teaching Practices. These practices include establishing mathematics goals to focus reasoning;

implementing tasks that promote reasoning and problem solving; using and connecting mathematical representations; facilitating meaningful mathematical discourse; posing purposeful questions; building procedural fluency from conceptual understanding; supporting productive struggle in learning mathematics; and eliciting and using evidence of student thinking.

Often referred to as *ambitious* (Lampert 2001, Lampert et al. 2010), *student*- or *learner-centered* (Cornelius-White & Harbaugh, 2009; Darling-Hammond, 1995), *core* (Forzani, 2014; Grossman et al., 2014; Jacobs & Spangler, 2017), or high quality (Munter, 2014), these teaching practices share a focus on eliciting and building upon student thinking, and researchers have discussed a number of benefits of dialogic classrooms resulting in students' deeper understanding of mathematics (Forzani, 2014; Franke, Kazemi, & Battey, 2007; Jacobs & Empson, 2016; Jacobs & Spangler, 2017; Lampert, 2001; NCTM, 2014). For example, the Cognitively Guided Instruction (CGI) research program demonstrates how teaching grounded in an understanding of students' intuitive mathematical thinking and problem-solving strategies promotes deep mathematical understanding and student engagement by valuing and building upon the natural ways students think about math (Carpenter & Fennema, 1992; Carpenter et al., 1996; Carpenter et al., 1999). Researchers have shown that teachers who participate in Cognitively Guided Instruction (CGI) professional development transform their beliefs and instructional methods, shifting their instructional focus from procedural knowledge to conceptual understanding and fluency (Fennema et al., 1996). An instructional orientation to students' thinking instruction is directly related to teachers' enhanced understanding of their students' mathematical thinking and higher student achievement in concepts and problem-solving (Fennema et al., 1996). CGI research suggests that focusing on students' mathematical thinking supports teachers in shifting to more dialogic instructional practices and has tangible impacts on students' mathematical proficiency.

For nearly 3 decades, mathematics education scholars have sought to identify and characterize instructional practices that make programs like CGI effective. The consensus from this line of research is that instructional practices that emphasize noticing, eliciting, and building upon students' ways of knowing and thinking are a key aspect of mathematics instruction that fosters learning with understanding (Jacobs & Spangler, 2017). The practice of anticipating students' mathematics prior to teaching has been identified as a way for teachers to prepare for noticing student thinking in the moment and facilitating mathematical discussion that advances student thinking towards a learning goal (Smith & Sherin, 2019). Anticipating students' mathematics decreases the cognitive load on teachers amid instruction and assists them in making sense of student thinking in the moment, adjusting instruction based on students' developing understandings, and teaching in ways that are responsive to students needs in the moment (Smith & Sherin, 2019). For mathematics teacher educators working to support teachers in organizing their instruction around student thinking, it is important to better understand the ways in which teachers make sense of and enact the practice of anticipating.

Supporting teachers in developing high leverage instructional practices like anticipating is not a trivial task. In their research involving 200 middle grade mathematics teachers, Munter and Correnti (2017) showed that incorporating new pedagogical approaches into teachers' existing practice is complex and entails more than merely expanding their repertoire of current teaching strategies. Instructional change involves both incorporating new pedagogical practices, repurposing others, and eliminating some that are well established and deeply embedded in their approaches to teaching. Such shifts require teachers to not only adopt innovative strategies for student engagement but also revise or discard practices they have long employed. Additionally, Heck et al.'s (2008) analysis of 48 Local Systemic Change projects funded by the National Science Foundation found that while positive effects of high-quality professional development were seen in teacher attitudes, preparedness, and teaching practice, these effects were modest. Taken together, research indicates that efforts focused on improving instruction face considerable obstacles in supporting teachers to learn and enact practices focused on student thinking (Heck et al., 2008; Munter & Correnti, 2017).

Stein et al. (2008) conceptualized a framework to support teachers in adopting an instructional model that honors, elicits, and responds to student thinking commonly referred to as the *5 Practices*. The framework describes five instructional practices teachers may use to engage students in cognitively demanding mathematics tasks. Cognitively demanding tasks are problems that promote mathematical reasoning, conceptual understanding, and stimulate a richer, more meaningful engagement among students with mathematics (Stein et al., 1996). Based on their work with the QUASAR project, Stein and Smith (1998) developed a framework for assessing and categorizing instructional tasks according to the cognitive demands they place on students. "Low demand" tasks involve memorization or rote application of mathematical procedures without connections to concepts or contexts and do not provide opportunities for students to engage in complex reasoning. Cognitively demanding tasks are those that require students to use mathematical procedures to connect concepts, offer no suggested solution paths, or call for exploration, conjecturing, and justification (Stein & Smith, 1998). Research from the QUASAR project showed that routine engagement with cognitively demanding mathematics tasks is associated with improved student outcomes (Stein & Lane, 1996).

The *5 Practices* framework was developed to assist teachers in preparing to implement and facilitate productive mathematics discussions of cognitively demanding mathematics tasks in instruction (Stein et al., 2008). It outlines a set of sequential and contingent instructional practices teachers can use to remove some of the uncertainty of organizing instruction around students' mathematical ideas while using those ideas to meet their learning goals. Through *anticipating student responses*, teachers consider the ways students might mathematically engage with a cognitively demanding task prior to instruction. As students engage with the task individually or in small groups, *monitoring student responses* involves making sense of their mathematical thinking and work to determine the "learning potential of particular strategies or representations" (p. 326) for the class. Prior to a culminating class discussion, *selecting and sequencing student responses* allows teachers to choose and order students' mathematical approaches, solutions, or representations for the class to consider. During the discussion, the practice of *connecting student responses* ensures key mathematical ideas and connections are highlighted across students' approaches and solutions that assist teachers in meeting their learning goals for the lesson. In describing these practices, Stein et al. (2008) note the embedded relationships among them and argue that the successful enactment of each is predicated on the previous. For example, identifying important mathematical connections across student responses requires that ideas shared in the discussion were purposefully selected and intentionally sequenced for presentation. Similarly, the variety of mathematical approaches a teacher might notice and understand when monitoring may be limited by the number and quality of approaches they anticipated prior to the lesson. For this reason, the practice of anticipating can be viewed as foundational to other instructional practices that use students' thinking as resources for their teaching.

In recent years, the practice of anticipating has had a growing presence in mathematics education curricula, practice, and policy. New and popular curricular programs, such as *Illustrative Mathematics* (2019), *Open Up Resources* (2023), and the *Mathematics Vision Project* use the *5 Practices* framework as an underlying instructional model. Multiple professional development programs and materials have also emphasized the *5 Practices* to support teachers in learning about and using student thinking in instruction, including NCTM's (2014) *Principles to Action Toolkit*. Teacher preparation programs have also come to include the *5 Practices* framework and leveraged anticipating as a tool for supporting teacher candidates in learning to elicit and use student thinking (Didiş & Erbaş, 2021; Hunt et al., 2022). The practice of anticipating has also emerged as a standard expectation for mathematics teachers. Professional examinations such as the *Educative Teacher Performance Assessment* (edTPA) (Stanford Center for Assessment, Learning, and Equity, 2015) and the *National Board for Professional Teaching Standards* (National Board for Professional Teaching Standards, 2010) require teachers to provide evidence of how they are anticipating students' mathematics. Additionally, the Association of Mathematics Teacher Educators (2019) recently adopted *Standards for Preparing Teachers of Mathematics*, which includes anticipating as a standard and argues that the practice is integral to understanding students as learners of mathematics.

While the explicit identification of anticipating as an instructional practice is relatively new in mathematics education, some researchers have noted the importance of anticipating for over 30 years. For example, Leinhardt (1988) described how expert teachers used their knowledge of how previous students engaged with mathematics when planning for their lesson. Schoenfeld (1998) discussed the idea of teachers' ability to envision obstacles to learning and ways to respond using the term "lesson image" (p. 136). More recently, Stein et al.'s (2008) definition of anticipating relates most closely to Lampert's (2001) description of an aspect of her lesson planning stating:

> I was anticipating where my students might get stuck or distracted as well as what might provoke productive work. I needed to think of all the things they would or could do when presented with the problem. This kind of preparation showed me what words might be useful in talking about their solutions, as well as what drawings they or I might use to support their studies. To respond to their work in a thoughtful way, I needed to be able to anticipate what they might be able to do independently and where they would need information from me to process productively. (p. 103)

While the practice of anticipation may not be new, the popularity of the *5 Practices* framework has led to more attention and emphasis on anticipating. In their framework, Stein et al. (2008) define the practice of anticipating student responses as follows:

> Anticipating students' responses involves developing considered expectations about how students might mathematically interpret a problem, the array of strategies—both correct and incorrect—they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn (Lampert, 2001; Schoenfeld, 1998; Yoshida, 1999, cited in Stigler & Hiebert, 1999). (pp. 322–323)

In a series of practitioner books used to support teachers in learning and using the *5 Practices* framework, authors have offered several elaborations to assist teachers in anticipating, including questions such as:

> How do you solve the task? How might students approach the task? What challenges might students face as they solve the task? What assessing questions will you ask to draw

> out student thinking? What advancing questions will help you move student thinking forward? What strategies do you want to be on the lookout for as students work on the task? (Smith & Sherin, 2019, p. 38)

While the addition of key questions is a helpful guide when learning how to anticipate, this conceptualization is still insufficient to capture the complexity of how teachers enact the practice of anticipating students' mathematics in their daily practice at varying levels of experience. Additionally, this conceptualization of anticipating relies heavily on investigations into the personal experiences of teachers selected for their success in mathematics teaching (Lampert, 2001; Schoenfeld, 1998; Stein et al., 2008, Smith & Stein, 2011, 2018; Smith & Sherin, 2019). These explorations of expert teachers' anticipations have produced a model of expert practice. However, models of expert practice alone are insufficient to support novices in learning (Dreyfus & Dreyfus, 1986). Mathematics teacher educators working to support teachers in centering student thinking in their instruction would benefit from a characterization of anticipating that includes images of the practice at varying levels of proficiency.

Despite its increased presence in mathematics education practice and policy, research on the practice of anticipating is only beginning to emerge. Some existing studies have demonstrated that anticipating supports other practices of high-quality and equitable mathematics teaching, such as monitoring student progress in small groups and selecting and sequencing student responses for discussion (Janike, 2019; Vale et al., 2019). Others have focused on characterizing what teachers anticipate when enacting the practice, including student strategies, representations, and errors (Didiş Kabar & Erbaş, 2021; Hughes, 2007; Kartal et al., 2020; Krause et al., 2016; Morrissey et al., 2019; Rupe, 2019; Şen; Şen Zeytun et al., 2010; Vale et al., 2019; Wilson et al., 2024). Researchers have also shown that teachers' anticipations may also attend to students' ways of thinking and reasoning such as the mathematical conceptions students may use, misconceptions they may have, and difficulties they may experience when solving a task (Didiş Kabar & Erbaş, 2021; Morrissey et al., 2019; Nickerson & Masarik, 2010; Şen Zeytun et al., 2010).

While studies documenting what teachers anticipate and how anticipations support other instructional practices that are responsive to student thinking have begun to provide empirical support for Stein et al.'s (2008) conceptual development of the practice of anticipating, there are several limitations of this emerging knowledge base. First, the majority of investigations have taken place in formal professional learning contexts such as university coursework or professional development settings where the practice of anticipating was either an explicit goal for teacher learning (e.g., Hallman-Thrasher, 2017) or as a means of understanding how teachers learned and used knowledge on student thinking in their teaching (e.g., Wilson et al., 2015). Second, existing research has predominantly focused on elementary grades teachers' anticipations for tasks, often within a specific domain of mathematics such as fractions (Krause et al., 2016) or whole number operations (Morrisey et al., 2019). Lastly, existing research has largely reported on the outcomes of anticipating exclusively without attention to the day-to-day enactments of mathematics teachers including what actions they take, what resources they use, and their reasons for engaging in anticipating students' mathematics. Additional empirical evidence about secondary mathematics teachers' practices of anticipating in their day-to-day teaching is needed to better inform mathematics teacher educators' designs for supporting teachers in learning and incorporating the practice in their instruction.

Statement of Research Purpose and Potential Significance

The purpose of this descriptive multi-case study is to provide a deeper and enhanced understanding of secondary mathematics teachers' practices of anticipating students' mathematics. More specifically, this study characterizes the practice of anticipating through the actions three teachers take when anticipating, the resources they draw upon when doing so, and the purposes they have for enacting the practice. Using this characterization, this study examines the similarities and differences in how teachers prepare to facilitate cognitively demanding mathematics tasks with their students. Findings of this study provide a more comprehensive understanding what the practice entails in daily professional contexts and describes variations in enactment in relation to their individual experiences, access to curricular resources, and access to professional learning opportunities. Results have the potential to contribute to the field mathematics teacher education by adding perspectives on the ways teachers who have varied experiences and understandings of the *5 Practices* framework enact the practice of anticipating as part of their daily instruction with students.

Outline of Dissertation

This dissertation is organized into five chapters. In Chapter I, I provide a background of mathematics education reform, its focus on instructional practices that center on eliciting and using student thinking, and the practice of anticipating students' mathematics. Chapter II summarizes the research literature on anticipating and provides the definitions around instructional practices used to frame the study. In Chapter III, I justify my use of case study methods and describe the context, participants, data sources, and methods of analysis. In Chapter IV, I present a portrait of each case teacher's enactment of the practice of anticipating students' mathematics and detail findings from an analysis of the similarities and differences among the teachers' practices. I conclude by answering my research questions, situating my findings within the existing literature, and discussing future research and implications for mathematics teacher educators and researchers in Chapter V.

CHAPTER II: REVIEW OF THE LITERATURE AND CONCEPTUAL FRAMEWORK

This chapter provides a review of the research literature related to the practice of anticipating students' mathematics. I begin by presenting a conceptual frame for defining and studying a teaching practice. Next, I review the literature on planning and anticipating students' mathematics by describing the field's current understandings of the practice, how other researchers have studied this practice, and a summary of the current limitations of the research base. I end with framing the practice of anticipating students' mathematics as the shared meaning from the broader mathematics education community that is helpful when considering the distinction of individual teachers' understandings and enactments of the practice in later chapters.

Conceptual Framework

To frame my investigation of a particular teaching practice, I reviewed the literature and compiled a set of definitions that other scholars have used to define *a practice*. I examined these definitions and identified similarities in how others have conceptualized *a practice* and distilled these similarities into components shared among the definitions. I then used these components to develop a definition for a mathematics teaching practice that I used to conceptualize the practice of anticipating students' mathematics for this study.

Defining a Teaching Practice

Social practice theory is an approach to understanding how practices – the routine behaviors and activities of individuals – form and evolve over time (Lave, 1988; Holland & Lave, 2001). Rather than focus on interpretations of individual behavior, the theory considers practice in relation to social structures and cultural norms. From this perspective, practices consist of not just physical activities but also of tools, knowledge and skills, cultural meanings, and norms.

To conceptualize a teaching practice, I reviewed the literature in an effort to understand how other scholars have defined and operationalized a teaching practice for empirical study. Table 1 summarizes the definitions of *practice* I collected from this search and contains direct quotations from each. In examining these definitions, I identified five salient components shared across definitions. *Patterned actions,* referred to as actions for short, alludes to repeated behaviors and activities associated with a practice. *Meanings* refer to the shared interpretations of a practice. *Purposes* describe the reasons and the goal or goals one has for enacting a practice. *Resources* refers to the personal, social, or material tools one draws upon when enacting a practice. Finally, *values* are the collective norms and shared understandings that guide and emerge from the routine actions within a community.

Table 1. Definitions of a Practice

Reference	Definition of *a practice*	Component
Barnes (2001)	"socially recognized forms of activity, done on the basis of what members learn from others, and capable of being done well or badly, correctly or incorrectly" (p. 19)	actions meanings purposes
Scribner and Cole (1981)	"… assign a social dimension to practice and associate it with tool use: Whether defined in broad or narrow terms, practice always refers to socially developed and patterned ways of using technology and knowledge to accomplish tasks. Conversely, tasks that individuals engage in constitute a social practice when they are directed to socially recognized goals and make use of a shared technology and knowledge system." (p. 236)	meanings purposes resources

Reference	Definition of *a practice*	Component
Grossman et al. (2009)	"Practice in complex domains involves the orchestration of understanding, skill, relationship, and identity to accomplish particular activities with others in specific environments. Practice can be understood in terms of its goals, its activities, and its historical tradition (Chaiklin & Lave, 1996)." (p. 6)	actions meanings purposes resources values
Cook and Seely-Brown (1999)	"… we intend the term 'practice' to refer to the coordinated activities of individuals and groups in doing their 'real work' as it is informed by a particular organizational or group context." (p. 386)	actions purposes
Orlikowski (2000)	"A practice lens to examine how people, as they interact with a technology in their ongoing practices, enact structures which shape their emergent and situated use of that technology." (p. 404)	actions meanings resources
Kelly (2005)	"… practice is constituted by a patterned set of actions, typically performed by members of a group based on common purposes and expectations, with shared cultural values, tools, and meanings." (p. 2)	actions meanings purposes resources values

Based on this review, I adapt Kelly's (2005) definition of practice in the context of teaching and conceptualize *a teaching practice* as *a patterned set of actions, performed by teachers with shared meanings, values, and resources, for a common purpose*. Carlone et al. (2011) also used a similar definition to operationalize the *practice* of "being a science person" which guided the identification and analysis of normative scientific practices within two different science classroom settings. This definition assisted investigators in highlighting the regularities of shared practice essential for being considered competent in each classroom (Carlone et al., 2011).

Yet for my examination of how secondary mathematics teachers enact a teaching practice in their professional contexts, it is imperative to distinguish between the shared understanding and meanings of a practice, in this case derived from the broader mathematics education community, and the ways in which a teacher takes up and enacts the practice. Teachers with varied backgrounds and experiences enacting such practices may or may not identify with the broader community of mathematics educators and thus not have shared community values and meanings. However, teachers that may not have this shared meaning still have patterned actions that play out in how they enact a practice, a purpose for engaging in the practice, and resources that they draw upon. Thus, to investigate an individual's teaching practice in their professional context, I also define a *teacher's distinct practice* as a teacher's patterned actions performed while enacting their unique understanding of a teaching practice, the resources they draw upon, and their purpose(s) for enacting the teaching practice.

The Teaching Practice of Anticipating Students' Mathematics

To define the teaching practice of anticipating students' mathematics, I begin reviewing the literature on anticipating students' mathematics by first differentiating between lesson planning and anticipating. Next, I describe the field's current understandings of the practice, how other researchers have studied this practice, and a summary of the current limitations of the research base. Together, results of these studies show that mathematics teachers attend to a variety of aspects of ways students might engage with cognitively demanding mathematics tasks when anticipating students' mathematics. They also suggest that the depth and quality of teachers' anticipations vary in relation to their context, content knowledge, and prior experiences. To conclude this chapter, I use the literature to frame the community's practice of anticipating students' mathematics and draw distinctions between it and the distinct practice of teachers who are anticipating students' mathematics that will be explored in this study.

Planning

The practice of planning is a multi-dimensional and deliberate process fundamental to high-quality teaching, particularly within mathematics education. At its core, planning is characterized through a cycle of designing, implementing, reflecting, and refining instructional experiences (Cevikbas et al., 2023; König et al., 2021; Yinger, 1980). This iterative process is adaptive, responding to the dynamic nature of classroom settings and the diverse needs of learners, thereby underscoring the critical role of both cognitive and affective components in shaping instructional methodologies (König et al., 2021).

Planning encompasses both individual and collaborative dimensions. Individually, teachers engage in reflective practices, drawing upon their professional knowledge, beliefs, and experiences to tailor lessons in alignment with their instructional vision and the unique attributes of their students (Yinger, 1980). Collaboratively, planning harnesses the collective understandings, skills, and innovative potential of professional learning communities (Cevikbas et al., 2023; Munthe & Conway, 2017). This collaborative aspect enriches the planning process, fostering a shared commitment to student learning and achievement (Cevikbas et al., 2023; Munthe & Conway, 2017; Yinger, 1980).

Anticipating as a Practice of Planning

Anticipating students' mathematics occurs before facilitating instruction during teacher planning (Smith & Sherin, 2019) and is identified by Cevikbas et al. (2023) as a means of producing high-quality lesson plans. Additionally, in Akyuz et al.'s (2013) particularistic case study focusing on the planning practices of an expert seventh-grade mathematics teacher before and during instruction on integers, investigators described how anticipating was central to the teacher's planning and supported them in implementing high-quality mathematics lessons that facilitated student learning. They describe anticipating as an integral practice within the broader context of planning for teaching where planning is outlined as a comprehensive process of preparation, reflection, assessment, and revision (Akyuz et al., 2013). While the practice of anticipating involves predicting student reactions to materials, foreseeing potential difficulties, and envisioning multiple approaches students might take, planning encompasses specific planned responses to one's anticipations (Akyuz et al., 2013; Cevikbas et al., 2023).

This conceptualization of anticipating as a part of a teacher's planning practices follows the conceptual development (Stein et al., 2008) and subsequent presentations of the *5 Practices* framework (Smith & Stein, 2011, 2018; Smith & Sherin, 2019; Smith et al., 2020), in which Stein et al. (2008) describe the practice of anticipating student responses as:

> … developing considered expectations about how students might mathematically interpret a problem, the array of strategies—both correct and incorrect—they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn. (pp. 322–323)

For this study, I expand this description of anticipating to also encompass ways of mathematical thinking and other considerations teachers have when preparing to use cognitively demanding mathematics in their teaching and refer to this practice as anticipating students' mathematics.

In the last two decades, several studies have investigated a variety of aspects of mathematics teachers' anticipations. In what follows, I review these studies, highlight their contributions to the emerging body of literature on anticipating, and note several limitations of the knowledge base that warrant this study.

Anticipating and Its Relation to High Quality and Equitable Mathematics Instruction

Researchers investigating mathematics teaching have highlighted the central role of teachers' anticipations for enacting high-quality and equitable mathematics instruction (Akyuz et al., 2013; Hallman-Thrasher, 2015; Janike, 2019; Vale et al., 2019). In Vale and colleagues' (2019) study of nine teachers and coaches involved in a lesson study, researchers collected lesson plans, observed instruction, and collected student work to analyze the critical factors that attributed to effective lessons. Their investigation showed that teachers' documentation of anticipated solutions supported them in more effectively monitoring student engagement and progress during the lesson and selecting student responses to be shared for class discussion. In a study of prospective elementary grades teachers as a part of their professional preparation program, Hallman-Thrasher (2017) found that teachers had difficulties in responding to unanticipated student thinking during in-the-moment instruction. Together, these studies provide evidence that the practice of anticipating supports teachers in preparing mathematics lessons and enacting other practices that elicit and build upon student thinking.

Some research focused on teacher learning in professional development contexts have examined teachers' anticipations as a means of understanding how teachers learn and make use of frameworks of students' mathematical thinking in their teaching (Edgington, 2012; Krause, Empson, Pynes, & Jacobs, 2016; Wilson et al., 2015). Wilson and colleagues (2015) conducted a year-long professional development program on mathematics learning trajectories (LTs) with 19 elementary school teachers. The program included the *5 Practices* framework as a collection of pedagogical practices that would assist them in eliciting student thinking that they could use the LTs to interpret. When examining teachers' instruction, they examined teachers' enactments of the *5 Practices*, including anticipating student responses, and concluded that LTs assisted teachers in anticipating specific details about what students might do, know, or not know. Similarly, Krause et al. (2016) assessed the ability of 18 upper elementary grades teachers to anticipate in the context of a professional development program sharing frameworks for children's strategies for solving equal sharing fraction problems. Teachers' anticipations were analyzed for their variety and validity as an indication of the flexibility of their knowledge. The findings revealed that teachers with more flexible knowledge could anticipate strategies that align with how children typically approach fraction problems, as opposed to merely relying on taught procedures and conventions.

Teachers' Anticipations

Though studies of mathematics teaching and teacher learning in professional development settings suggest anticipating students' mathematics is an essential practice of instruction that is high-quality and equitable, the body of research focusing on anticipating is only beginning to emerge. The majority of these studies focus on describing teachers' anticipations in formal professional learning contexts such as university coursework or professional development programs (Didiş et al., 2021; Kartal, Morrisey, & Popovic, 2020; Krause et al., 2016; Lin & Chiu, 2010; Lin, 2016; Morrissey et al., 2019; Nickerson & Masarik, 2010; Rupe, 2019; Vale et al., 2019; Webb, 2006) with elementary and middle grades teachers, with fewer investigations of high school teachers (Hughes, 2007; Janike, 2019; Kabar & Erbaş, 2021; Şen Zeytun et al. 2010; Wilson et al., 2024). Predominantly, these studies have utilized document analysis and interviews as their primary methods of inquiry, with relatively few employing observations (Vale et al., 2019; Webb, 2006) to examine the considerations teachers have when preparing for instruction. In what follows, I review these studies to highlight what teachers consider when anticipating students' mathematics.

Student Misconceptions

When anticipating students' mathematics, studies indicate that teachers often consider student misconceptions or errors (Akyuz et al., 2013; Didaş et al., 2021; Hughes, 2007; Janike, 2019; Kabar & Erbaş, 2021; Kartal, Morrisey, & Popovic, 2020; Lin & Chiu, 2010; Lin, 2016; Nickerson & Masarik, 2010; Şen Zeytun, Çetinkaya, Erbaş, 2010; Webb, 2006; Wilson et al., 2024). These studies highlight that teachers anticipate difficulties students may encounter when engaging with specific mathematical tasks (Morrissey et al., 2019; Nickerson & Masarik, 2010). For example, Şen Zeytun et al. (2010) examined five secondary mathematics teachers' anticipations enrolled in a graduate university mathematics course focused on mathematical modeling. They documented how the teachers believed students would struggle with modeling tasks, particularly with creating representations, identifying variables, and making conjectures about the problem's context. Akyuz et al. (2013) highlighted how anticipating students' incorrect solutions and other difficulties was a critical component of expert teachers' preparations to respond to students' thinking during instruction.

Students' Mathematical Activity and Thinking

In addition to focusing on student misconceptions, studies also suggest that teachers attend to the strategies, representations, solutions might use to engage with a task and what these approaches suggest about their thinking when anticipating (Didaş et al., 2021; Hughes, 2007; Kartal et al., 2020; Krause et al., 2016; Morrissey et al., 2019; Rupe, 2019; Şen Zeytun et al., 2010; Vale et al., 2019; Wilson et al., 2024). A substantial portion of the empirical studies in this review is devoted to examining how teachers anticipate specific elements of student work, including solutions, strategies, and other discernible aspects of their mathematical engagement with certain tasks or within a given mathematical domain. For example, Kartal et al. (2020)

examined 88 prospective elementary grades teachers' anticipations for students engaging with problem solving tasks. They reported that teachers considered strategies that were both conceptual and procedural in nature. Krause et al. (2016) investigated described above that identified distinct flexible strategies a child could use to solve a fraction problem that leads to a correct answer.

Wilson, McCulloch, Wonsavage, Hare, and Baucom (2024) described secondary mathematics teachers' anticipations by distinguishing *students' mathematical actions* – defined as "observable mathematical actions students might do, use or say" (p.7) – from *students' mathematical* thinking, which they defined as "responses focused on inferences of student thinking based on assumptions about the cognitive resources students will bring to the task" (p.7). In their study of 127 secondary mathematics teachers' anticipations to 17 cognitively demanding mathematics tasks, they report that anticipations focused on students' mathematical actions include how students might represent a mathematical situation, strategies for solving a problem, and working with physical or social resources (i.e., manipulatives and peers). Anticipations focused on students' mathematical thinking included assumptions about prior knowledge students may be able to utilize or mathematical connections they may make during the problem-solving process. Of the 283 distinct anticipations examined in their study, 80% focused on either the mathematical actions a student might make or aspects of their thinking.

Student Affective Responses

Two studies of mathematics teachers' anticipations report a focus on affective responses that students might exhibit in reaction to a mathematics task (Webb, 2006; Wilson et al., 2024). Webb (2006) utilized a case study methodology to investigate four preservice secondary mathematics teachers' anticipations when participating in a lesson study. They found that the teachers included a focus on students' potential emotional reactions to the lesson tasks, such as motivation, engagement, enjoyment, and the value students place on their own work. In Wilson et al.'s (2024) study, 6% (n = 17) of the anticipations they examined attended to students' affect, predominantly related to student "struggle."

Variations in the Complexity of Anticipations

These and other studies indicate that not only do mathematics teachers' anticipations differ in focus, but that their anticipations also vary in amount and quality (Hughes, 2006; Kartal et al., 2020; Krause et al., 2016; Morrissey et al., 2019; Nickerson & Masarik, 2010; Şen Zeyton et al., 2010). For example, results from Kartal et al.'s (2020) investigation of 88 elementary grades teacher candidates showed that while most teachers anticipated solutions that integrated both conceptual and procedural understanding, only a fraction offered distinctly mathematical anticipations. Similarly, Didaş Kabar & Erbaş (2021) noted that although most of the 25 secondary teacher candidates they studied anticipated student strategies and possible challenges in tackling four modeling problems, the depth and specificity of their predictions about students' mathematical reasoning varied across different tasks. Kastan's (2009) case study showed that teacher candidates' anticipations made in a university setting varied significantly with those they made in their field experiences, suggesting that contextual factors like available resources and granted authority substantially influenced their enactment of the practice. Lastly, Şen Zeyton et al. (2010) discussed how teachers' anticipations may be related to their mathematical understanding. Analyses of five secondary mathematics teachers' anticipations for mathematical modelling tasks involving covariational reasoning indicated that while the teachers were able to articulate some possible errors and strategies, their own understanding of covariation limited the depth and accuracy of their anticipations.

Together, results of these studies show that mathematics teachers attend to a variety of aspects of ways students might engage with cognitively demanding mathematics tasks when anticipating students' mathematics. They also suggest that the depth and quality of teachers' anticipations vary in relation to their context, content knowledge, and prior experiences. Next, I highlight the limitations of the existing research on mathematics teachers' anticipations and present a conceptualization of a teaching practice that together served as foundations for this study.

Limitations of Current Knowledge of Anticipating

While emerging research has begun to outline the contours of the practice of anticipating students' mathematics, three limitations of the knowledge base make space for the current investigation. First, there are very few studies of secondary mathematics teachers, and most of those are focused on prospective teachers. More research is needed, particularly at the secondary level, to enhance our understanding of what constitutes the practice of anticipating students' mathematics, how this practice is learned, and how it can be leveraged for instructional improvements focused on student thinking.

Second, most research on the practice of anticipating has taken place within structured environments, like university courses or targeted professional development programs, which focus on the practice and include support to teachers for learning to enact the practice. Further investigation into how teachers anticipate in their daily professional contexts and practice—without specific encouragement, resources, or feedback to guide this practice—would offer valuable perspectives on teachers' personal interpretations of the practice and its role in preparing for high-quality and equitable mathematics instruction. Such perspectives would be beneficial for mathematics teacher educators in creating learning opportunities that build upon teachers' natural inclinations toward anticipation, allowing for the development of more effective professional learning tools and experiences.

Last, while Wilson et al.'s research (2024) was one of the first investigations to take a broader look at teachers' anticipations outside of a highly structured professional learning environment, their study, like most others, focused solely on the outcome of teachers engaged in the practice of anticipating. That is, studies of anticipations predominantly focus on teachers' anticipations articulated while preparing for facilitation of a task or lesson. However, just as selecting and sequencing entails more than just the order in which teachers choose to present students' work (Dunning, 2023), the practice of anticipating students' mathematics encompasses more than a list of expected strategies or solutions that teachers consider prior to instruction. Additional research that describes the practice of anticipating—beyond what it yields—and how it is enacted by secondary mathematics teachers in different contexts and with different levels of experiences would inform efforts to design learning experiences to teachers in developing and enacting the practice of anticipating students' mathematics, but ultimately, high-quality and equitable mathematics instruction.

Framing the Practice of Anticipating Students' Mathematics

In this section, I use my framework, the literature, and the most recent work from Smith and Sherin (2019) to summarize the mathematics education community's practice of anticipating by describing the patterned actions, purposes, and resources that encompass the practice of anticipating students' mathematics. This practice has meaning and value to the community because it is essential for eliciting and responding to students' thinking during instruction and is an essential practice of high-quality and equitable math instruction (NCTM, 2014).

Patterned Actions of Anticipating Students' Mathematics

The mathematics *teaching practice* of *anticipating students' mathematics* used in this study begins with Stein et al.'s (2008) definition of anticipating student responses. They state:

> Anticipating students' responses involves developing considered expectations about how students might mathematically interpret a problem, the array of strategies—both correct and incorrect—they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn. (pp. 322–323)

This conception of anticipating focuses on the behaviors and activities of teachers when enacting the practice; that is, it addresses the patterned actions associated with the practice of anticipating students' mathematics and includes what teachers anticipate as summarized in the previous section. Additionally, Smith and Sherin (2019) have refined their conceptualization of anticipating student responses by including "getting inside the problem" as the first step of enacting the anticipating practice. They identify three key questions to ask when anticipating that encourage teachers to reflect on how they would solve a task, how students might do so, and what challenges students might face as they solve. Smith and Sherin (2019) include an example "monitoring chart" (p. 45) that captures and organizes a teacher's thinking around how students might approach a problem. This chart includes the identified anticipations along with questions a teacher could use in response to the anticipations.

This conception of the practice of anticipating students' mathematics is similar to Wilson et al.'s (2024) interpretation of anticipating student responses. In their study, they distinguished between *anticipating student responses* which they defined as "the process (i.e. practice) of developing and articulating expectations of students' interpretations, approaches, and their relationships to specific mathematical learning goals" (p. 2) from a broader *teacher's anticipations* which they defined as "articulated expectations resulting from the process of anticipating student response" (p.2). While their investigation provided evidence that secondary mathematics teachers focus on students' mathematical actions and thinking, their analysis did not explore relationships among these anticipations or with teachers' purposes for anticipating.

Purpose of Anticipating Students' Mathematics

In addition to patterned actions, I consider the purposes teachers have for enacting the practice. To define the community's understanding of the purpose of anticipating, I once again draw on Smith and Sherin's (2019) key components and questions of their updated conceptualization of anticipating student responses shown in Table 2. The bottom two rows of their table suggest the purpose of the teaching practice of anticipating is to prepare to notice and respond to student thinking during instruction.

Table 2. Key Questions That Support Anticipating

What It Takes	Key Questions
Getting inside the problem	How do you solve the task? How might students approach the task? What challenges might students face as they solve the task?
Planning to respond to student thinking	What assessing questions will you ask to draw out student thinking? What advancing questions will help you move student thinking forward?
Planning to notice student thinking	What strategies do you want to be on the lookout for as students work on the task?

Note. Adapted from Smith and Sherin's (2019) *The 5 Practices in Practice: Successfully Orchestrating Mathematics Discussion in Your Middle School Classroom*, p. 38.

This elaborated view of the practice suggests that teachers' stated plans for responding to their anticipations and noticing student thinking might also be considered a part of the practice of anticipating. Additionally, Wilson et al. (2024) included planned responses as a part of what teachers anticipated in their study. However, for this study, I interpret teachers' responses to their anticipations as an indicator of their purpose for enacting the practice of anticipating.

Resources for Anticipating Students' Mathematics

Lastly, one way to think about the resources teachers use to anticipate is to consider their Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008). The MKT framework delineates the essential knowledge required for effective mathematics instruction, emphasizing specialized content knowledge that distinguishes proficient mathematics educators from subject matter experts. The framework organizes teacher knowledge into knowledge of content and pedagogical content knowledge (Shulman, 1986), each of which is comprised of three domains (Ball et al., 2008).

The MKT framework presents knowledge of content in three distinct domains. Common Content Knowledge (CCK) encompasses the foundational mathematical knowledge that is broadly shared among those proficient in mathematics and is essential for understanding and conveying basic concepts and procedures (Ball et al., 2008). Specialized Content Knowledge (SCK) is unique to educators, deepening their mathematical insight to enable the decomposing of complex ideas, identification and correction of misconceptions, and effective use of teaching strategies to enhance student comprehension (Ball et al., 2008). Knowledge of Content and Students (KCS) merges mathematical understanding with insights into how students think and learn, equipping teachers to anticipate student responses and address learning challenges effectively (Ball et al., 2008).

The framework further develops Shulman's (1986) notion of pedagogical content knowledge by specifying three distinct domains. Knowledge of Content and Teaching (KCT) combines this understanding with pedagogical strategies, focusing on the effective selection and organization of teaching resources and methods (Ball et al., 2008). Knowledge of Content and Curriculum relates to the teacher's mastery over the curriculum, emphasizing the integration of curricular materials and standards into cohesive and impactful instruction and recognizes its role in shaping instruction (Ball et al., 2008). Lastly, Horizon Content Knowledge includes the teacher's awareness of the broader mathematical landscape beyond the current curriculum, facilitating connections between current content and future mathematical concepts to ensure a coherent learning trajectory (Ball et al., 2008). Together, these domains underscore the expertise necessary for high-quality mathematics teaching and can be used to understand the resources that teachers draw upon to plan mathematics instruction and anticipate students' mathematics.

Summary

In this study, I consider the mathematics education community's understanding of the practice of anticipating students' mathematics to consist of *patterned actions* teachers use to get inside the problem (i.e. cognitively demanding tasks) by determining how a task is solved, how students might approach a task, and challenges students might face as they solve the task (Smith & Sherin, 2019). These reflections are normally organized via a monitoring chart that includes the anticipated approaches and questions that can be asked as a response (Smith & Sherin, 2019). The *purpose* of doing so includes planning to respond to student thinking by asking assessing and advancing questions and planning to notice student thinking so that it can be used to draw connections among multiple students' work and to the pre-determined learning goal (Smith & Sherin, 2019). Teachers can draw on their knowledge of content and students (Ball et al., 2008) as a *resource* to identify anticipations of students' mathematics. This conception of the practice of anticipating supports my investigation of secondary mathematics teachers' enactments of the practice in their daily professional contexts by identifying commonalities and distinctions among their *patterned actions*, *resources*, and *purposes* and their relations to the community's understanding of the practice.

In this chapter, I synthesized the current research related to mathematics teachers' practices of anticipating including connections to high-quality and equitable mathematics teaching. I also examined how previous studies have categorized the anticipations that teachers develop in professional learning settings and identified several limitations of the current knowledge base. In Chapter III, I use the review and framework developed in this chapter to specify the study's focus, including its research questions, study design, and methods.

CHAPTER III: METHODOLOGY

The previous chapter explored the existing literature related to the practice of anticipating students' mathematics and presented a framework for the study. While my framework is situated in a larger theoretical tradition of social practice theory that emphasizes shared meanings among communities, I conjecture there will be variances in teachers' practice of anticipation—specifically in their actions, resources, and purposes—that differ from the larger mathematics education community's shared meaning of the practice of anticipating. The variability in ways the practice is enacted assists in characterizing the range of ways teachers enact the practice of anticipating students' mathematics which can inform mathematics teacher educators working to support teachers to enact high-quality and equitable mathematics instruction.

The purpose of this study is to understand the complexity of teachers' practice of anticipating students' mathematics as they plan to facilitate mathematics instruction. I explore this practice with three secondary mathematics teachers by defining each case teacher's actions during, resources for, and purposes of anticipating students' mathematics. Specifically, this research was guided by the following research questions:

1. *What are the distinct teacher practices of anticipating students' mathematics of secondary mathematics teachers?*
2. *In what ways are the distinct teacher practices of anticipating students' mathematics of secondary mathematics teachers similar and different?*

In this chapter, I first define the background and context of the study. Next, I describe and justify my use of multi-case study methodology to examine the practice of anticipating students' mathematics of secondary mathematics teachers. Next, I identify the participants, describe the selection of cases, and outline the data used for my examination. After describing methods of analysis, I complete the chapter with a discussion of issues of validity and reliability.

Context

During the 2017-2018 school year, a school district partnered with a local university in the Southeastern United States with the goal of aligning a district wide vision of high-quality mathematics instruction with task-based pedagogy. The school district was a small district that served 15,000 students and supported 12 middle and high schools with a total of 32 schools. Eighty-two percent of the students were white, 4% students were Black or African American, 9% were Hispanic or Latino, and 1% were Asian. The district was looking for support to focus on mathematics instruction for two of their schools that were historically low performing.

The district utilized resources from the North Carolina Collaborative for Mathematics Learning (NC^2ML) to provide job-embedded coaching and professional learning for middle and high school mathematics teachers in the district with a focus on the two low-performing schools. NC^2ML is a statewide partnership of mathematics education leaders from school districts, universities, and the state education agency that provides large-scale professional learning resources and experiences and aims to develop a common vision for high-quality and equitable mathematics instruction among all public K-12 mathematics educators. Through the partnership, professional learning modules and curricular guidance documents that relate research on mathematics teaching and learning to practice were collaboratively designed by researchers and mathematics teacher leaders to support the implementation of the state's recently revised mathematics standards. These resources focused on the use of cognitively demanding tasks (Stein et al., 1996) that related to the new state standards, the role of the mathematics teacher as a “more knowledgeable other” (Munter, 2014), and research-based pedagogical practices (Franke, Kazemi, & Battey, 2007; Jackson et al., 2013; Jacobs & Spangler, 2017; NCTM, 2014).

As a part of this partnership, a group of high school teachers in the district completed a 30-hour professional learning series on *5 Practices for Orchestrating Mathematics Instruction* (Smith & Stein, 2011, 2018) during the 2018-2019 school year. At the request of the district, I provided professional learning and coaching for middle and high school math teachers in the district for three years. Additionally, in Summer 2019, the district adopted *Open Up Resources* (2023) for all middle schools that I supported with professional learning and coaching during the 2019-2020 and 2020-2021 school years. During the years spent as an instructional coach, I supported implementation of the *Open Up Resources* (2023) curriculum in middle schools and encouraged the use of cognitively demanding tasks using open education resources such as *Illustrative Mathematics* (2019) and *Mathematics Vision Project* in high schools. All resources identified align with the vision for high-quality and equitable mathematics instruction set by NCTM, promoted by NC^{2}ML, and feature cognitively demanding tasks necessary for utilizing the *5 Practices* framework.

As a founding member of the partnership, I worked with this rural district as an instructional coach, and this context provided an opportunity to understand the practice of anticipating students’ mathematics with grades 6-12 teachers at various points in their journey of incorporating the *5 Practices* framework into their instruction. At the time of the study, the district’s high school teachers had a deep background knowledge of the *5 Practices*. In addition, the instructional model underlying the district’s adopted middle school was based on the *5 Practices* framework. This curriculum featured cognitively demanding tasks for most lessons, and the teacher supported materials included commentary on potential ways that students might engage with lesson tasks. In both cases, my knowledge of the district's efforts and supports for high-quality and equitable mathematics instruction, as well as my relationships with the teachers, provided an opportunity to deeply investigate the practice of anticipating students' mathematics.

Research Design

This research uses a qualitative multi-case study approach to explore the practices of anticipating students' mathematics of secondary mathematics teachers working in this district. Merriam and Tisdell (2015) state that qualitative research is "interested in understanding how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences" (p. 6). Historically, research methods seeking to test hypotheses using quantitative measures have been seen as a more traditional and rigorous form of science. However, qualitative research methods have become a respected approach to deepen scientific research by focusing on the meaning and understanding of social phenomena (Merriam & Tisdell, 2015).

There are several defining characteristics of a qualitative case study; they are particularistic, descriptive, and heuristic (Merriam & Tisdell, 2015; Yin, 2014). Case studies are *particularistic* in that there is a specific focus of inquiry, examining a particular situation and creating opportunities for researchers to gain in-depth understandings of a phenomenon. Case studies are also *descriptive.* They aim to provide "thick description" (Geertz, 1973) of a social phenomenon using varied sources of information to illustrate the complexities of a situation. Finally, case studies are *heuristic* in the sense that they illuminate our understanding of a situation, reveal complexities and problems, describe the background of a situation, or explain success or failure of an intervention.

A multi-case study research design involves two or more cases defined by a common characteristic. Such studies shift the focus of inquiry from understanding a single case to the similarities and differences across cases and provide insights and enhanced understandings by comparing key aspects of various cases (Stake, 1995). Evidence compiled across multiple cases presents a more compelling and robust argument for salient features and variances of the phenomenon under investigation (Merriam & Tisdell, 2015; Yin, 2014).

Case studies are particularly useful when a depth of description is needed or when little is known about the phenomenon (Merriam & Tisdell, 2015; Yin, 2014). Because research on teachers' practice of anticipating students' mathematics is only beginning to emerge, my use of case study methodology is warranted. My aim is to describe secondary mathematics teachers' practice of anticipating in the context of teaching (Yin, 2014) to examine similarities and differences in the actions they take, the resources they use, and their purposes for anticipating. For these reasons, a qualitative multi-case study is appropriate as I seek to characterize the practice of anticipating more holistically than prior studies have accomplished.

Defining the Cases

When designing a case study, it is important to define and "bound" the case (Merriam & Tisdell, 2015; Stake, 1995; Yin, 2014). Case studies typically define an individual person as the case, but cases can also be defined as a larger entity (Yin, 2014). Whereas individual case studies focus on a critical case and are useful for corroborating, challenging, or extending theory, a multi-case study expects the outcomes of each case to either have similar results, or in some cases, contrasting ones (Yin, 2014). In this design, multiple studies are completed simultaneously, analyzing each individual case. Afterwards, conclusions can be drawn from conducting cross-case analysis, comparing differences in each case.

This study considers three cases of secondary mathematics teachers with related but contrasting experiences with the practice of anticipating students' mathematics. I selected a diverse group of teachers to represent variations in prior professional experiences and support for the practice while minimizing outside factors that might influence differences in teachers' practice of anticipating. Selecting teachers within the same district ensured policies and procedures that impact planning time or instructional decisions are similar across cases. At the same time, a purposeful selection of teachers with different backgrounds and experiences enabled me to better understand variations in teachers' actions, purposes, and resources used when enacting the practice of anticipating.

Case Teachers

My close working relationships with district teachers afforded insights into their planning and teaching, and it was evident that some teachers were more accomplished in the practice of anticipating than others. As a participant observer (Spradley, 2016), my unique relationships and prior experiences with multiple teachers informed my selection of three case teachers using purposive sampling (Miles et al., 2019). I prioritized my understanding of teachers' practice of anticipating during their planning from previous informal observations when selecting cases while also attending to a variety of curricular resources, time with me in my role as instructional coach, grade band, and teaching experience to ensure diverse enactments of the practice. Table 3 summarizes my selection criteria and related variations in case participants.

Table 3. Case Teacher Summaries

Name	Frequency of Anticipating during Planning	Curricular Resources	Time with Instructional Coach	Grade Band	Teaching Experience
Aimee	Frequent	Flexible use of multiple HQMI curricular resources	February 2018 – Time of Study (2020-2021)	9-12	7
Kathryn	Frequent	Flexible use of multiple HQMI curricular resources	February 2018 – Time of Study (2020-2021)	9-12	9
Hillary	Less Frequent	Complete adoption of IM's 6-8 Curricula	August 2019- Time of Study (2020-2021)	6-8	3

At the time of the study, Aimee and Kathryn were high school teachers with similar years of experience. I had worked extensively with each of them on using cognitively demanding tasks from multiple, high-quality curricular resources and implementing the *5 Practices* framework in their teaching. In contrast, Hillary was a beginning middle school teacher who had little previous professional learning opportunities to engage with the *5 Practices* framework. Unlike Aimee and Kathryn, Hillary had access to and used a complete curricular program with lessons organized around the *5 Practices* framework with supplemental teacher guides that included possible student responses to lesson tasks.

Data Collection and Analysis

Case study methodology requires multiple sources of qualitative data (Merriam & Tisdell, 2015; Stake, 1995; Yin, 2014). Accordingly, data for the study included multiple interviews, document analyses, field notes, and researcher reflections. Specifically, I collected these forms of data across two cycles of planning interviews with each teacher to characterize their practice of anticipating students' mathematics. In a preliminary interview (Cycle 0), I interviewed case teachers to collect data on their background and to confirm I had selected a variety of experiences with the *5 Practices* framework. In a subsequent meeting (Cycle 1), I met with case teachers to discuss the ways they planned to use cognitively demanding mathematics tasks. In this interview, I refrained from posing questions that explicitly related to the practice of anticipating and asked teachers to engage in planning to facilitating a task of their choice with a current class of students. In a final interview (Cycle 2), I specifically asked case teachers to anticipate students' mathematics as a part of planning to implement a cognitively demanding instructional task of their choice. Recordings of each interview, documents that teachers brought to the interview or produced, field notes, and reflections after the interviews were collected. Data from across interviews allowed me to identify salient features of each teachers' practice as well as similarities and differences in their enactments when asked to "plan" versus when asked to specifically anticipate students' mathematics. Throughout the following sections, I elaborate on these cycles of planning and describe the data collected to answer my research questions, including semi-structured interviews (Merrium & Tisdale, 2016) with embedded think-alouds (Ericsson & Simon, 1993), generated documents (Merriam & Tisdell, 2015), field notes, and reflection documents. These multiple sources of data provided a means for producing "thick" descriptions of each teachers' practice of anticipating students' mathematics.

Interviews

I conducted a series of semi-structured interviews (Merriam & Tisdell, 2015) with each case teacher to collect information on their background experiences, teaching philosophies, and probe their practice of planning and anticipating. All interviews were audio recorded, and verbatim transcriptions were created. In addition, all documents that case teachers brought to, or produced during, the interviews were collected.

Cycle 0 Interview

At the beginning of the study, each teacher was interviewed to better understand their background and experiences with the practice of anticipating students' mathematics (see Appendix A). This initial interview asked teachers to describe their experience with the *5 Practices* framework (Stein et al., 2008), high-quality and equitable mathematics instruction, and biographical and other information related to their teaching and individual contexts. During this interview, case teachers learned that the study was related to the *5 Practices* and their planning practices. Each case teacher was asked to identify a task for the subsequent Cycle 1 interview that they perceived to be cognitively demanding (or identified as a task to use the *5 Practices* framework). Hillary was asked to select a task of her choice identified with the "Anticipate, Monitor, Select, Sequence, Connect" instructional strategy from the *Open Up Resources* (2023) curriculum. Without a mandated high school curriculum at the district level, Aimee and Kathryn were asked to choose a task of their choice since they had both completed extensive work with identifying cognitively demanding tasks.

Cycle 1 Interview

Cycle 1 interviews focused on planning to enact their selected task with students. The interview included an embedded think-aloud (Ericsson & Simon, 1993) to better understand how the practice of planning and anticipating were enacted by each case teacher. I provided specific instructions to prompt participants to verbalize what they were thinking and feeling during the process of planning. During the think-aloud portion of the interview, I minimized my instructions to mitigate my influence on the think-aloud (Short et al., 1991). Previous research has shown that think-alouds slow down cognitive processing without altering performance during engagement of planning and anticipating (Ericsson & Simon, 1993). When the case teachers stopped talking while planning for the instructional task, I reminded them to continue to verbalize their thoughts with short and unobtrusive prompts such as “Keep talking” and “What are you thinking?” instead of being probed more thoroughly. Using a think-aloud approach during the interviews provided an opportunity to understand the actions case teachers might take organically while planning for instruction without distraction (Greene & Azevedo, 2007). After the think-aloud portion of the planning interview was complete, I probed case teachers about the actions they took and decisions they made to understand their purposes for planning and the resources they drew upon to do so. Appendix B includes the protocol for the Cycle 1 Planning Interview.

Cycle 2 Interviews

Cycle 2 interviews focused on each case teachers’ practice of anticipating by specifically using the think-aloud approach and asked teachers to anticipate the mathematics their students might use to engage with their chosen task. Like Cycle 1, teachers were asked probing questions to further explore their thoughts on anticipating once they completed the think-aloud portion of the interview (see Appendix C). Beyond their planning practices, this interview allowed me to better understand the extent to which anticipating was a distinct part of their practice.

Documents

For Cycles 1 and 2 interviews, I asked case teachers to create a hard copy of their plans (Cycle 1) and anticipation guides (Cycle 2). I collected copies of these “research generated documents” (Merriam & Tisdell, 2015) to inform my analysis of their practices of anticipating. I also collected all notes case teachers produced during the interviews, including annotations of their plans made during the interview. Table 4 gives a comprehensive overview of how each data source in this study will contribute to understanding teachers' anticipation practice informed by the conceptual framework of the teachers' actions, resources, and purposes of anticipating.

Researcher Field Notes and Reflections

My field notes and research reflections served as secondary data for the study. For each interview, I recorded field notes of my in-the-moment thoughts related to teachers' actions, purposes, and resources used when anticipating students' mathematics. After each interview, I recorded my overall impressions and thoughts in a research journal document. These contemporaneous notes and reflective journal entries served as preliminary analytic memos (Miles et al., 2019) that guided my analysis, assisted me in interrogating early claims, and provided an audit trail of my reasoning. Table 4 summarizes the data collected in this study related to my conceptual framework.

Table 4. Conceptual Framework and Data Sources

	Initial Interview (Cycle 0)	Planning Interviews (Cycle 1 & 2)	Documents (Cycle 1 & 2)	Field Notes & Researcher Journal (Cycle 1 & 2)
Actions		X	X	X
Resources	X	X		X
Purpose		X		X

Data Analysis

Data analysis is making sense out of multiple data sources and that it includes combining and decoding what has happened throughout a study (Merriam & Tisdell, 2015; Miles et al., 2019). As they were collected, data were processed, organized, digitized (i.e., text files of interview transcriptions, scanned PDFs of documents), and loaded into the qualitative data analysis software package, ATLAS.ti (https://atlasti.com). For each case, I completed an initial round of coding using my framework and completed additional analytic memos (Miles et al., 2019) to capture my emerging thoughts about each case teachers' practice of anticipating. Using this information, I conducted a within case analysis to develop a portrait of each case teachers' practice. Lastly, I conducted a cross-case analysis using a meta-matrix (Miles et al., 2019) to examine the similarities and differences across the dimensions of my framework. The following sections detail how I conducted the within- and cross-case analyses.

Within-Case Analyses

I began by using Jacobs et al.'s (1997) approach of isolating idea units for my analysis. Idea units are segments of conversations or text that are marked by "distinct shifts in focus or change in topic" (p. 13). For this study, I defined an idea unit to be segments of transcript data where case teachers referred to specific actions, resources, or purposes for their practice of anticipating students' mathematics. In interview transcripts, I expected most of these shifts to occur in response to a new question. Though case teachers occasionally discussed ideas indirectly related to interview questions, a focus on idea units offered meaningful insights into different aspects of the case teachers' practice of anticipating by providing flexibility to capture multiple ideas in response to interview questions. For the documents collected, I used idea units to capture different anticipation actions that case teachers noted in the moment when discussing their plans for instruction and the various ways they represented their anticipations.

After identifying idea units within each case's data, I used deductive coding to identify teachers' actions, purposes, and resources for the practice of anticipating. As this study's focus was on how each teacher's practice of anticipating may vary and be distinct from the shared meaning of the community, meanings and values were not included in the analysis. Each idea unit could receive more than one code from the framework. For example, the following idea unit from Kathryn's Cycle 1 interview received both an action and purpose code because of her reflection on both how students might represent the data described in the task and how she might respond to some of their anticipations.

> And so they might start thinking … "How can I use that pattern to write some sort of rule for my 100th term?" Okay. So, at this point, um, let's talk about where I might interject some help. So, I definitely think the kids, all the groups would be counting blocks. And they would all be thinking about a pattern, and I would probably, if they weren't thinking about a pattern, I would say, can you come up with a pattern? You know, do you see a pattern? And so I would probably have to suggest a table. Or let's put these things in the case. These cases … let's think of those as maybe a way to say for case 1, how many blocks do we have? For case 2 … and I would probably structure that in a table form to just lead them in that direction a little bit for, especially for some of my kids in Standard [Math 3], that might not occur to them without my little nudge. I think once it was in the table, it would click so, "I've seen tables before," right! They're going to say, "Okay, I know what to do with tables, I look for patterns, I look for, and what's going on with y, I look to see if I can basically find any kind of predictability for how to come up with an outcome way down at the hundredth case. (Kathryn, Cycle 1 Interview)

After assigning codes to idea units, I engaged in open coding (Miles et al., 2019) with a goal of further classifying units coded as actions, purposes, and resources into categories. For example, I described the purpose of the idea unit from Kathryn's interview described above as *advancing student understanding of the mathematical goal* and the actions she took as *identifying anticipations.* I then used a constant-comparative method (Strauss & Corbin, 1998) to condense and clarify the results of my open coding procedure and identify themes (see Appendix

D for a table of themes) around the three aspects of the practice of anticipating of focus in this study. As I analyzed each case teacher's data, I examined coded units and attended to those that did not align with other categories to create descriptive themes for each case. Table 5 summarizes each *a priori* code and includes examples from the data analysis.

Table 5. Code Definitions and Examples

Code	Definition	Example
Actions	Idea units that: Identify student strategies, representations, solutions, errors, etc.	*So, the next thing would be, what my kids would do, some of them would do is they would say, "okay, for x for the first case, if I do one plus one, I get two, and if I square the two, I get four." And then they would look at the second case, and they would say, "if I do two plus one, I get three, and if I square the three, I get the nine that I need." And then they would do "three plus one is four, if I square that four, I get the 16 that I need."* (Kathryn, Cycle 1)
Purpose	Idea units that provide a rationale for engaging in the practice	*"Whenever it's like, the end of the lesson, it gives an activity towards the end of the lesson, because you want to make sure by the time that you get there you've met those needs that they need by the time they get there. So like, if they have this question, or if they're going to need to know, this method for solving, I should have already taught it by now. Before they get to that one. So I think it helps like making sure that I've set up the lesson where they can succeed on that last problem."* (Hillary, Cycle 2)
Resources	Idea units that describe: Students' mathematical skills and understandings; Teacher's knowledge of student responses, common partial understandings, and error patterns; Curricular resources that include common student responses; Prior experiences of student responses on the same or similar instructional tasks; Expertise of colleagues	*"I do remember kids, where they got stuck, and where they struggled with it. That does definitely help me anticipate because last semester was the first time I've used the exact task."* (Aimee, Cycle 2) *"And so hopefully the students will remember when you use a closed circle when you use an open circle, so if it's included or if it's not included."* (Aimee, Cycle 2)

At the conclusion of this process, I used themes that emerged from my analysis to create a case portrait describing the different aspects of each teacher's practice of anticipating. After writing each case, I used member checking (Merriam & Tisdell, 2015) to ensure each participant agreed that I accurately represented the different aspects of each teacher's anticipation practice.

Cross-Case Analysis

The individual cases served as multiple exemplars (Miles et al., 2019) that highlighted similarities and drew distinctions across cases to assist me in developing a more nuanced understanding of anticipating. To do this, I constructed a meta-matrix (Miles et al., 2019) to compare the cases with respect to aspects of their practices identified in my within-case analysis. The matrix facilitated my examination of the differences and similarities in the actions, purposes, and resources of anticipating across case teachers. From this examination, I created a categorical aggregation of ways in which secondary mathematics teachers engaged in the practice. These categories provided further insight regarding variation in secondary mathematics teachers' practice of anticipating students' mathematics.

Validity and Reliability

To conduct rigorous and credible qualitative studies in education, Creswell and Poth (2016) claim that researchers must develop and use data to search for a kind of understanding that comes only from deep and personal relationships with participants, and that such relationships require a significant amount of time. This requirement of context-rich data, as well as my goal of impacting practice through this study, necessitates a discussion of validity, reliability, and my own subjectivity as potential sources of bias that might jeopardize the credibility and trustworthiness of the study. In the following sections, I discuss strategies I used to ensure my study is trustworthy and describe how validity and reliability are defined in qualitative research.

Validity

Qualitative researchers assume that reality is not static, thus making validity relative to the purposes and context of the study (Merriam & Tisdell, 2015). For this reason, I used several strategies to ensure the findings accurately represented how case teachers understood the practice of anticipating. Creswell and Poth (2016) offer a variety of strategies for validating and exploring the credibility of qualitative research, including (a) negative case analyses, (b) reflexivity, (c) member checking, (d) prolonged engagement, (e) participant collaborative, (f) external audits, (g) rich descriptions, (h) peer reviews, and (i) triangulation. When conducting this study, I used triangulation, member checking, and reflexivity as main strategies to guard validity. First, I collected multiple and varying sources of data to investigate my research questions. From multiple interviews with participants, think-alouds, and their created planning documents, data sources were triangulated to paint a clear picture of both the individuals' practice as well as an encompassing and rich definition of each teacher's distinct anticipating practice.

Since this study's goal was to describe what anticipating students' mathematics looks like in practice as well as specify variations in individual teachers' actions, purposes, and resources, I also used member checking ensure that each case teachers' understanding of anticipating was correctly captured. Case portraits were shared with each teacher to verify that I had accurately represented their practice and were refined based on their feedback.

Finally, I provide a thick description of the teachers' context when presenting study findings. In qualitative research, transferability falls on the reader (Merriam & Tisdell, 2015), and the descriptions of case teachers' background, school, and district context are intended to support readers in understanding when and under what conditions study findings may or may not apply to other situations. The ability to use results of this study in other contexts is also enhanced by the purposeful selection of case teachers to represent variations in the practice of anticipating.

Reliability

The nature of case study research often places emphasis on understanding rather than hypothesis testing (Creswell & Poth, 2016). Therefore, rather than ensuring replicability, reliability in qualitative studies results from showing how the findings are consistent with the data. To ensure reliability, Creswell and Poth (2016) identify four strategies, including (a) triangulation, (b) peer examination, (c) clear statements of the investigator's position, and (d) an audit trail. In this investigation, I used peer examination and journaling to increase reliability in my findings. After completing my analysis, I partnered with a mathematics education researcher to explain my analytic process and emerging findings of the data. This colleague then used my coding process to code 10% of my raw data to corroborate findings. Throughout the study, I kept a researcher's journal to document my reflections on the process, emerging themes and questions about the data and phenomenon, decisions made, and various interactions with the data through the analysis and interpretation phases. This process provided an audit trail linking study data with my findings (Merriam & Tisdell, 2015). Lastly, I describe my role as the researcher in a positionality statement in the following section.

Positionality Statement

As a former high school mathematics teacher, I was first introduced to the *5 Practices* framework and the practice of anticipating through a summer institute with university partners who became colleagues and worked with me during this study. Prior to this institute, I was a relatively traditional mathematics teacher, relying on lectures and a direct instruction model. However, after experiencing this professional learning opportunity and simultaneously enrolling in a graduate program, I would describe myself using a more student-centered approach with a vision of a teacher as a “more knowledgeable other (Munter, 2014). Through continued experience in my graduate program and resulting research assistantships at the doctoral level, I began both teaching pre-service teacher education courses and facilitating in-service professional development utilizing the *5 Practices* and other practices of high-quality and equitable mathematics instruction (NCTM, 2014).

It was through these avenues that I developed a long-standing relationship with the district serving as context for this study. Not only did I start working as a mathematics coach in this district, but I also began my teaching career and attended schools within this district. Though some might argue this close relationship invokes bias, I assert that it is because of this relationship that I had earned trust and was allowed the deep access to teachers’ authentic practice, avoiding response bias that often plagues educational research.

CHAPTER IV: FINDINGS

This chapter presents findings from my analysis of three secondary mathematics teachers' practices of anticipating students' mathematics. Specifically, it answers the two research questions guiding the study:

1. *What are the distinct teacher practices of anticipating students' mathematics of secondary mathematics teachers?*
2. *In what ways are the distinct teacher practices of anticipating students' mathematics of secondary mathematics teachers similar and different?*

In this chapter, I first present each case teacher's distinct practice of anticipating students' mathematics. I then present a comparison of each case's practice of anticipating, highlighting salient similarities and distinctions.

Next, I describe the distinct practice of anticipating students' mathematics for each of the three case teachers, Hillary, Aimee, and Kathryn. For each teacher, I first provide an overview of their practice, share background information related to their practice, and present the instructional tasks they selected for their interviews. I then discuss the actions they took when anticipating, the resources they used, and their purpose of enacting the practice in their teaching.

Hillary's Practice of Anticipating

Hillary's practice of anticipating students' mathematics was focused on identifying representations promoted by the curriculum to solve problems. At the start of the anticipation interviews, she began by studying the daily learning goals and the prior representations and strategies used by the curriculum in prior lessons. She then completed the task with a student lens considering representations, student reasoning, procedural methods, and challenges, along with organizational and affective anticipations. Hillary relied on the curriculum's sequence of lessons to anticipate what strategies students would use to solve problems. She also used her one year of prior experience to remember what students may find challenging. Hillary anticipated to make plans to facilitate connections for students as laid out by the curriculum. Her purpose in anticipating was to make sense of how the curriculum connected mathematical ideas so that she would be prepared to do that for her students.

Background

Hillary was in her third year of teaching seventh grade mathematics at the time of this study. During her second year, the district adopted the *Open Up Resources* (2023) curriculum. The district had provided a very limited training on the curricular resource during adoption. They conducted a two-day training that focused on the Launch-Explore-Discuss model and the Mathematical Language Routines and Instructional Routines that were embedded in the teacher resources, one of which being "Anticipate, Monitor, Select, Sequence, Connect," modeled after the *5 Practices* (Smith & Stein, 2018). I worked with Hillary during the first two years of implementation, with this study occurring in the second year of implementation. A large focus of our work in PLCs centered on engaging in the mathematics of upcoming lessons together and making sense of how students might approach the problems for lessons. The curriculum's unit overviews also introduced strategies, models, and representations that were largely unfamiliar to the middle school mathematics teachers at this school. Hillary quickly became a teacher leader during the implementation process as she recognized how the curricular resources helped her facilitate lessons that put a greater emphasis on conceptual understanding. At the time of the study, we had not explicitly explored *5 Practices* (Smith & Stein, 2018) as a framework for student-centered instruction; however, there was an implicit connection to this framework through coaching around launching tasks (Jackson et al., 2013) in a way that maintained the cognitive load of the tasks during facilitation. During PLCs, Hillary met with the other seventh grade math teacher for one hour once a week to discuss the mathematics of upcoming lessons and informally anticipate by discussing how students may approach the problems. Teachers had been introduced briefly to a description of the Instructional Strategy "Anticipate, Monitor, Select, Sequence, and Connect" through the curricular teacher resources.

Hillary described her classroom as "discussion based, more than lecture based … I tend to ask a lot of questions because I like to get students to think outside of the box" (Cycle 0 Interview). Hillary had a limited understanding of the *5 Practices*, understanding it as a routine in the curriculum and remembering learning about it in her undergrad experience. She remembered spending the most time discussing the connecting practice from her teacher preparation program, stating "Instead of just giving them formulas, you are walking them back through and making sure kids can come up with the equations on their own because they are investigating whatever it is they are learning about. This way they can make better connections" (Cycle 0 Interview). She understood the use of *5 Practices* in her own practice sparingly, stating:

> I feel like I definitely anticipate because I've got to think about what my students are going to know and what they are not going to know. I also use sequencing a lot because of the way our curriculum sequences different concepts. That lets the kids see those connections a little bit better. For example, during our units on proportional reasoning, we do not start by throwing tables, equations, and graphs on at the same time. Instead, we start with tables, and explain what we can do with tables. And then from that we go to equations and from to graphs and then we combine all three. (Cycle 0 Interview)

Hillary's response showed that while she used language of the *5 Practices* that have become second nature in modern mathematics curriculum and professional learning experiences, she had not had an experience yet allowing her to truly understand how these practices work together to describe a way of teaching.

Hillary's Selected Tasks

Central to Hillary's practice of anticipating was her use of the curriculum. While *5 Practices* is one of the design principles upon which her curriculum is built, certain tasks that particularly lend themselves to this kind of discussion are tagged with the "Anticipate, Monitor, Select, Sequence, Connect" strategy and include specific notes in the teacher narrative around different strategies that students may use when monitoring. The first task discussed in her Cycle 1 interview "10.2 Shopping in Two Different Cities" (Appendix E) was tagged this way and included a note, "As students work, monitor for different strategies, especially students who note that they can always multiply by the same factor and students who set up and use an equation." Further, it discusses that after students have had time to work in groups, teachers should select and sequence student work to show different methods and make connections.

In her second interview, Hillary selected another task tagged with the *5 Practices* identification called, "11.2 At the Fair" (Appendix F). The curriculum's teacher resources suggest teachers monitor for students who:

- reason numerically without any diagrams or representations.
- create a tape diagram and use it to reason numerically.
- write an equation like $6(x - 1.5) = 46.5$ and solve it by using the distributive property to find the total amount saved, $6 \cdot 1.50$; and
- write an equation and solve it by first dividing it by 6 to find the cost of each discounted ticket (*Illustrative Mathematics*, 2019, Grade 7, Unit 2, Lesson 11 Teacher Resources).

Hillary requested that she was able to think through what the warm-up of the lesson that preceded the task of focus so that she could think through how the warm-up set up students for success on the task.

Hillary's Actions of Anticipating

In both interviews, Hillary first consulted the curriculum, analyzing the learning goals for the lesson and the previous lessons in the unit of study. She then worked through the tasks of the lesson thinking about how students might be able to make sense of the problems and identifying anticipations around various representations used by the curriculum, methods of solving previously taught, ways students might reason with representations, challenging points for students, how students might feel during the lesson, and organizational structures. These anticipations assisted her in planning ways to highlight the goals of the lesson as stated by the curriculum.

Hillary's Artifacts of Anticipating

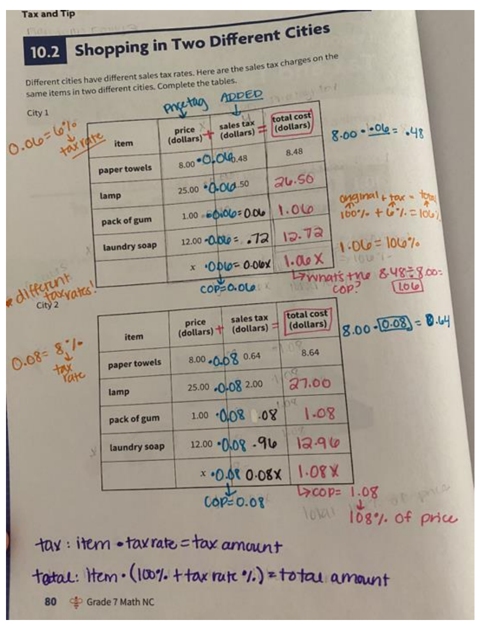
A part of Hillary's practice of anticipating students' mathematics involved creating an artifact documenting her anticipations. As seen in Figure A from her Cycle 1 interview, Hillary's artifacts included complete solutions for tasks as well as various notes and questions related to her anticipations. Hillary recorded her solution to the task in pencil before going back and rewriting her final notes in color. These notes highlighted considerations that she wanted students to see and possibly write down on their own student pages while she facilitated the task with students. Hillary explained this process in the following excerpt:

> Right now, I am planning but because I will upload a picture of these notes for students to reference at the same time, I'm using my color pens and trying to make it where my students can see the steps a little bit better with the different colors. I'm writing in pencil

right now but I'll go back probably with my colored pens so you can see the steps better. A lot of times as I'm planning, and it's my first time working through it, I'll work through it on paper, and then I'll go back with my pens and write in my book. I will star or highlight confusing things, and I'll probably just leave that as a note for myself to make sure that when we're going over that in class, I talk about it. (Cycle 1 Interview)

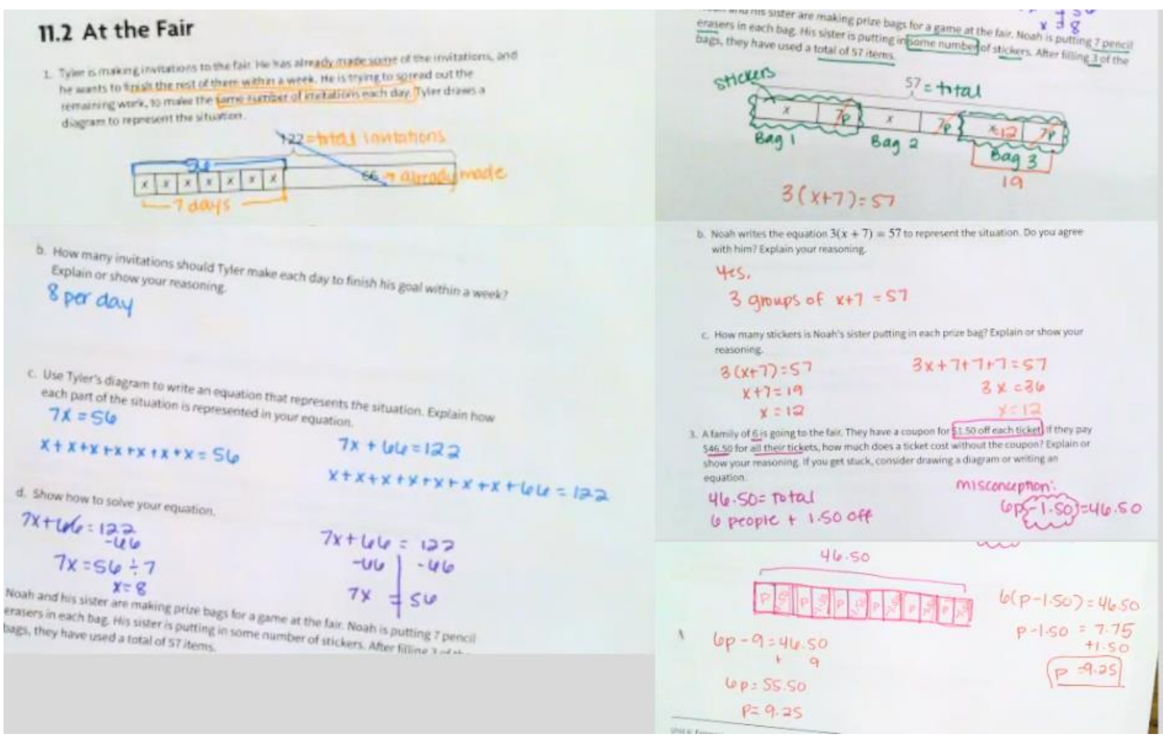
Hillary detailed how she uses her own notes as a reference for her students through her use of color to differentiate steps in a problem and to capture extra notes she has for her students. Her anticipation artifact serves two purposes. First, it is a record of her solution(s) to the problem(s) of the lesson. Second, it documents her anticipations and organizes the mathematics as she wants it displayed for students' future reference.

Figure A. Hillary's Artifact of Anticipating for the Shopping in Two Different Cities Task



Hillary's written artifacts provided very little evidence for how students might think about different strategies to solve tasks. This is particularly true in the Cycle 1 interview when she was not explicitly asked to anticipate how students would use different strategies to solve a task. In comparison, Hillary explicitly recorded two different strategies to solve an equation once while working through the entire lesson when she was specifically asked to anticipate in the Cycle 2 interview (see Figure B). In this artifact, Hillary provided evidence that she anticipated two distinct equations students might create to solve the story problem. Based on how students reasoned with the tape diagram, Hillary anticipated students might write 7x = 56 or 7x + 66 = 122. Hillary's artifacts showed little evidence of multiple anticipations she had for her students around solving the task. For example, in Figure B, Hillary's solution to 1b did not consider how students might use a table or reason with a unit rate, or any strategy at all.

Figure B. Hillary's Artifact of Anticipating for 'At the Fair' Task



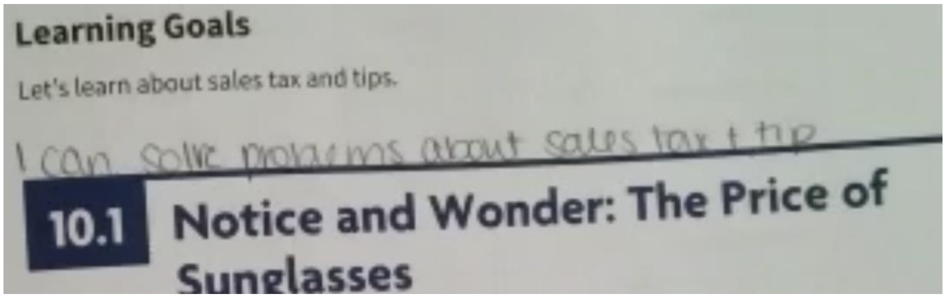
Consulting the Curriculum

Hillary's practice of anticipating students' mathematics was grounded in her curricular program. In both interviews, Hillary began anticipating by making sense of the goals for the specific lesson. To do so, she used a blank copy of the lesson, read the learning goal provided, and then identified the specific student-facing learning targets from the teacher narrative and wrote corresponding "I can" statements at the top of her student-facing lesson pages (see Figure C). She then turned to working through the tasks of the lesson and continuously referenced ideas from earlier lessons in the unit or year. This helped her situate her goals within what students had already learned and how what they had experienced might affect their work on the task. During the first interview, she stated:

> I always start with the "I can" statement [Student Facing Learning Target] and then I just start with the first thing in the lesson. I ask myself, "So, what are they going to do to begin with?" A lot of times I think about "What have they already done?" It helps me to think about what they already know, for example, in Task 10.2; as I was doing it, I knew they'd already worked with tables for proportional relationships. (Cycle 1 Interview)

Here, Hillary described how she always started with the learning target(s) before working through the task.

Figure C. Example of Hillary's Student-Facing Learning Targets



Hillary also frequently searched for connections to the mathematical representations and strategies emphasized in previous lessons in the curriculum. During her Cycle 1 interview for example, she began by reading the teacher narrative and then commented on the connections that she saw to prior units and lessons:

> In this activity, students work with tax rates with the students through reasoning repeatedly about the same percentage of different quantities [mumbling and reading to herself] … So, it's connecting back. And you can see that in the table too, it's going to connect back to what they did. And unit one and two with the tables. (Cycle 1 Interview)

By studying the lesson's learning goals, teacher narrative, and making sense of the sequences of tasks across previous lessons, Hillary use of her curricular program provided a context for her specific anticipations of students' mathematics.

Types of Anticipations

Hillary's specific anticipations of students' mathematics for a given task included the representations they might create, their mathematical reasoning, learned procedures they may employ, and specific learning, organizational, and affective challenges they might experience. As Hillary worked through the tasks in both interviews, she considered how students would experience the lesson, however Hillary increased her effort to name different ways students might approach the problem and how they would react to the mathematics of the lesson during the Cycle 2 interview when specifically asked to anticipate. Her anticipations became evident as Hillary worked through the task, created an exemplar of the work, and developed annotations of what she wanted students to consider as they engaged with the lesson.

Multiple Representations. In both interviews, Hillary considered different representations of the mathematics embedded in the tasks. She wanted students to see the different representations and know how they were connected. However, Hillary's attention to mathematical representations was always based on those explored in previous lessons. When anticipating, she often listed representations emphasized in the curriculum's previous lessons.

For example, when anticipating how students would solve #3 in “At the Fair” task (see Figure C above), she explained:

> Some people would draw the tape diagram for this problem. Some of the kids who are more visual learners do really well with the tape diagram versus some of the kids will just try and write an equation and not draw the tape diagram. I could see someone drawing hanger diagrams, too. (Cycle 2 Interview)

Hillary listed tape diagrams, equations, and hanger diagrams as specific anticipations for the task—all of which are named and emphasized representations in her curriculum.

Likewise, she discussed how the constant of proportionality showed up in a table versus an equation and how one could represent percentages with a symbol (%) or as a decimal in the “Shopping in Two Different Cities” task (see Figure A above). By highlighting the different ways percentages can be represented across different representations, she hoped to create opportunities for students to compare and contrast different representations of the mathematics on which they were working. In her discussions, she referenced the curriculum’s explicit names for representations as she thought through her anticipations.

Student Reasoning. Hillary’s anticipations often revolved around how students would reason about a specific aspect of the task. For example, when thinking through #1 from “At the Fair” task (see Figure B above), Hillary described how she thought students would make sense of the tape diagram, including what they would notice first and what initial moves they would make to solve the equation:

> So, thinking about what they’ve [students] already done [in previous lessons], we’re kind of remembering from the last unit. By this point, we’ll see that we can take that off [pointing to 66], take that chunk off. So, 122 minus 66, which would be 56. And that was

> a 66. And this part [pointing to the 7 x's] here would be 56. I can see them go into that, that these things here represent the 56 that's leftover. And so, then I try to figure out what each one of these [pointing to the 7 x's] represents. And so, there's seven of these, so 56 divided by 7, which would be 8, so this would be 8, 8, 8, 8, 8, 8, 8 [pointing at an individual x each time]. So, from that, they could see that Tyler would need to finish 8 invitations per day. (Cycle 2 Interview)

Hillary described in detail what students might notice about the tape diagram and in what order they would think through solving the diagram. When thinking through what students might notice, she referred to what mathematics had been included in previous lessons. She knew that this was not the first time that students had seen tape diagrams. Because of their previous engagement with this representation, students would likely be able to think through solving for the unknown parts.

Procedural Methods. While Hillary noted how students would solve problems, the strategies she described were often procedural in nature. For example, her description of how students might make sense of tape diagrams in the previous section focused on a process of solving the two equations she recorded in her anticipation artifact. As another example, consider Hillary's anticipation of how students might go about finding the constant rate of proportionality from the table in "Shopping in Two Different Cities."

> There are other ways to find the constant of proportionality, but typically what we've been doing before now is to take y and divide it by x. And that's what's been drilled in more, so I expect most of them to do it that way. (Cycle 1 Interview)

While Hillary noted that there is more than one way to find this rate, she also did not thoroughly anticipate other ways that students could find the rate without using the formula. This is particularly note-worthy in this task; the relationship was represented in a table, and students could have used this representation to search for patterns and find the rate instead of using a procedure.

Challenges. In both interviews, Hillary's anticipations also noted parts of tasks where she anticipates students might struggle. For the "At the Fair" task, students are asked to solve an equation that includes a coupon that will result in a negative term. Hillary stated:

> But I think the biggest struggle with this would be that the coupon is off. And that it's off each ticket. I can really see them missing this ticket there. We are going to have to have a conversation about what it means that you are taking $1.50 OFF of each ticket price. (Cycle 2 Interview)

Hillary did not think that students would not be able to overcome the challenge of representing the coupon's effect on the situation, she considered how students might overlook this information and planned to discuss what it means to take $1.50 off each ticket price when teaching the lesson.

Affective Anticipations. When anticipating, Hillary also included how students might feel when first seeing a task. After exploring the lesson warm-up in her Cycle 2 interview and beginning to discuss the "At the Fair" task, she commented:

> First off, I feel like the kids are going to look at this and say 'that's a lot of words' for this … it's a pretty long question. It just looks confusing. So, I would probably definitely be reading the first one to them. And my suggestion to them would be to go back and read it on their own and underline some of the important information. (Cycle 2 Interview)

In her initial thoughts about her students engaging with the task, Hillary discussed how students might feel discouraged or confused by the task's presentation and how she might respond to these challenges. In both interviews, Hillary described how she might respond to students who were initially frustrated with the tasks to further engage them in the lesson.

Organizational Anticipations. When explicitly asked to anticipate in her second interview, a new type of anticipation emerged concerning how students might organize their written work on the task. In her interview, Hillary included numerous references to how students might write down their notes during a lesson and on what part of the page they might record their work. When discussing her anticipations for the "At the Fair" task for example, Hillary emphasized the placement of an equation and its relation to how the students developed and understood their solution:

> Honestly, for my kids, I would see them more either writing the equation here [pointing to the diagram] like we did or writing it right here [pointing to the blank space intended for the answer]. They're probably not going to write it in both places. I need to make sure they are clear what equation goes with this model so when they look back later, they won't be confused about what they did. (Cycle 2)

In this example, Hillary discussed how her students write their solution might make it confusing when referencing their notes in the future. For Hillary, organizational anticipations were only evidenced in the second interview where the practice of anticipating was explicitly stated, but also connects to her use of task notes as references for study in the future.

Hillary's Resources for Anticipating

In carrying out actions for anticipating students' mathematics, Hillary drew upon two specific resources during the interviews. First, she relied on the coherence of her curricular program. Specifically, she used her knowledge of content and curriculum (Ball et al., 2008) to consider the representations and strategies emphasized in previous lessons as she generated anticipations for how students might engage with the tasks under consideration. For example, Hillary contemplated how some students would be able to write the equation from a story problem during the Cycle 2 interview:

> I feel like we … looking back at lesson 10 … I saw some of this with the distributive property. And I remember from lesson eight, we did stuff with distributing [flipping back through previous pages in the teacher resources book] … So yeah, they should be able to see that this is the distributive property and write the equation 3(x+7) = 57. (Cycle 2 Interview)

Here, Hillary referenced an earlier lesson and assumed that students had mastered the distributive property and therefore drew the conclusion that students should be able to write the equation as three times the quantity $x + 7$. Later in the interview, Hillary referenced earlier lessons when thinking about how students would write equations for stories dealing with negative quantities. She stated, “Looking back, I don’t remember, if we’ve dealt with negatives [flipping through teacher book] … we have. We dealt with it in lesson 9. (Cycle 2 Interview).” Hillary’s multiple references to prior lessons shows how her understanding of students’ prior mathematical knowledge was tied to specific emphases from previous lessons in her curriculum and the sequence in which they were taught.

While Hillary relied on lesson sequencing and the content in the curriculum as resources for anticipating, she rarely referenced the list of multiple anticipations of students’ mathematical approaches included in the curriculum’s teacher narrative. In both interviews, Hillary read the teacher narrative and only attended to the notes about coherence and purpose. In the interviews, there was no evidence of her using the listed anticipations of possible student strategies or notes of partial understandings to include in her plans for instruction.

To a lesser extent, Hillary also drew upon what she remembered students doing with the task the previous year when teaching with the same curriculum. In her Cycle 2 interview, she stated:

> Some people would draw the tape diagram, especially from what I remember from last year. Some of the kids who are more visual learners do really well with the tape diagram versus some of the kids will just try and write an equation and not draw the tape diagram. When I think back to last year though … looking back, I remember seeing the hanger diagrams too. I could see someone drawing hanger diagrams. (Cycle 2 Interview)

Hillary drew upon her recollections of how students engaged with the task from the previous year. In addition to relying on her curricular program's coherence, she used what she remembered from teaching the same lesson the previous year including its specific emphasis on diagrams when anticipating students' mathematics.

Hillary's Purpose of Anticipating

Hillary's actions and uses of resources when anticipating students' mathematics supported her in planning to facilitate mathematical connections represented in her curricular program for her students. Throughout Hillary's interviews, two purposes for the practice of anticipating emerged highlighting connections to the curriculum coherency through previous lessons' representations and strategies and lesson learning goals and planning for students to scribe their notes in a way that could be easily referenced in the future.

Connecting to Curriculum Coherence

Hillary responded to her anticipations of students' mathematics by planning for how she would facilitate the task in a way that made most sense to students given previous lessons and activities that students had already experienced. In her Cycle 1 interview for example, Hillary discussed how students' recent work with tables should help them be able to work through the task. She stated:

> I knew we could relate this task back to what we've already done with tables, and they would have that down. So, what do they already know? And then I'm figuring out what's the specific goal of this lesson? What's that extra piece that we are adding onto today? (Cycle 1 Interview)

In this excerpt, Hillary showed how she trusted her curricular program to constantly build from previous lessons and the importance she placed on its learning goals. Throughout the curriculum adoption process and first year of implementation, Hillary also often mentioned her appreciation of the curriculum's coherence and how she felt each lesson added one new idea to what students had learned earlier in the unit or course.

In the interviews, Hillary also noted the curriculum's coherence when describing how each task and lesson unfolds, builds, and supports her to ensure each lesson goal is met and students are prepared for the next lesson. In her second interview for example, she stated:

> The lesson gives an individual activity—towards the end of the lesson—and I want to make sure by the time that I get there, I've met those needs that they need by the time they get to that part. For example, if they have this question, or if they're gonna need to know this method for solving, I should have already taught it by now, before they get to that one. So, I think it helps like making sure that I've set up the lesson where they can succeed on that last problem. (Cycle 2 Interview)

Hillary's goal was to always stay one step ahead of her students to be able to keep adding to their mathematical understanding as outlined by the sequence of lessons provided in the curriculum.

Creating Student Reference Notes

Finally, Hillary viewed the purpose of her anticipations as a means of supporting students to best annotate their notes for later review. So that students could refer to their work from class later, she posted her anticipations on her district's digital learning platform. Like her organizational anticipations, Hillary aimed to assist students in creating a record of their learning with their notes and facilitate connections among the curriculum's local and broader learning.

Summary of Hillary's Practice of Anticipating Students' Mathematics

Hillary's practice of anticipating students' mathematics was bounded by her faith in her curricular program. When anticipating, she regularly consulted her curriculum to envision the ways students might engage with the instructional tasks that were the focus of the study interviews. For her actions, she annotated student pages to record the representations, reasoning, procedural methods, and challenges related to learning, affect, and instructional organization she anticipated. To do so, she relied on the strategies promoted by her curricular program as well as her limited experiences teaching with the curriculum. For Hillary, anticipating students' mathematics was a way to support coherence for students as they engaged with the curriculum and ensure that students had a clear record of what she wanted them to learn from a for future purposes.

Aimee's Practice of Anticipating

Aimee's practice of anticipating students' mathematics focused on taking a student's perspective and solving tasks as they might solve them while looking for points of confusion and thinking through their prior knowledge. She used these anticipations to ensure she was prepared to answer any questions students might have to avoid frustration for students. She also used her anticipations to make decisions about locating a particular task within a unit so that students would be able to successfully complete it. When anticipating students' mathematics, Aimee relied on her own mathematical knowledge as well as horizon content knowledge (Ball et al., 2008) developed from her prior experiences teaching mathematics at both the middle and high school levels.

Background

At the time of this study, Aimee was in her seventh year of teaching and was teaching Math 1 and Math 3, an integrated first and third level course of high school mathematics. Prior to the study, she had attended a four-day professional development institute focused on analyzing the cognitive demands of instructional tasks and using the *5 Practices* to facilitate mathematical discussions in the classroom. As her instructional coach, we had spent the last three years planning tasks and units together that incorporated opportunities for meaningful mathematical conversations. As a mathematics teacher leader in the district, Aimee had also participated in numerous other professional learning opportunities. One that she referenced frequently throughout the study and relied upon to plan and implement tasks involved learning to facilitate instructional tasks by using scripted lesson cycles and formative assessment tasks.

A key feature of Aimee's classroom was collaborative learning. Her classroom was organized for students to work in small groups, and students were encouraged to work together and support each other as they work on mathematics problems. At the high school level, the district did not provide a curricular program or guidance for mathematics, and Aimee worked hard to create activities that supported students in making sense of the mathematics that they were learning during each unit. Aimee focused her planning mostly at the unit level and involved gathering tasks, note-taking guides, practice problem sets from various professional learning opportunities she had attended, and open-educational resources online. She described planning as gathering instructional resources for each unit and organizing them in a way that builds students' understanding of the major work of the unit. To a lesser degree, Aimee spent a portion of her planning time considering the next day's lesson, previewing the next set of activities she had gathered for her class, and working through the mathematics embedded in activities she had not used before.

Due to a small staff and scheduling issues, Aimee and her colleagues had very limited collaborative planning time. Only one other mathematics teacher at Aimee's school was teaching Math 3 or Math 1 at the time of the study, and they did not share a planning period. As a result, co-planning with colleagues was rare for Aimee and was done on their own time outside of school hours.

Aimee's Selected Tasks

Because Aimee did not have a district-endorsed and supported curriculum, she chose to gather activities from multiple sources for her classroom. The task she selected for the Cycle 1 interview came from her work with the formative assessment lesson cycles described above. The task was entitled "Buying Chips and Candy" and was written for an eighth-grade content standard focused on writing and solving systems of linear equations (see Appendix G). Aimee intended to use this task for her Math 1 course in the Systems of Equations unit. As Aimee discussed, she chose this task because:

> In Math 1, we have to be able to build equations and do systems of equations to be able to find the values of the variables. So, I thought this was a good task to use with the kids. It's not a task that I would use at the beginning of the unit, but probably right before I start to do some systems, just to see if they could build equations... or if I kind of get a feel of the class, I might actually do it once we've talked about systems and built our own

systems a couple times, and then see if they could take this task and make it their own. (Cycle 1 Interview)

Aimee's second interview focused on two tasks taken from an *Illustrative Mathematics* (2019) Algebra 1 lesson and was selected for her Math 3 course as an introduction to piecewise functions. The two tasks were titled "Postage Stamps" and "Bike Sharing" and selected to formatively assess what students could remember about graphing inequalities before introducing piecewise functions (Appendix H). Aimee provided a rationale for selecting this task, stating:

> I will be using this right before I start piecewise functions. The students will know what a piecewise function looks like, but I want them to be able to try to read this graph on their own to see how they interpret if it is the values included or not included with a closed circle and open circles, because I think it's interesting to see if they remember, closed circle and open circle for things to know what's included and what's not included.

While this was a part of a complete lesson from *Illustrative Mathematics* (2019), Aimee only used the "Postage Stamps" and "Bike Sharing" tasks for the Cycle 2 interview and did not discuss how the warm-up was related or affected her anticipations for this task.

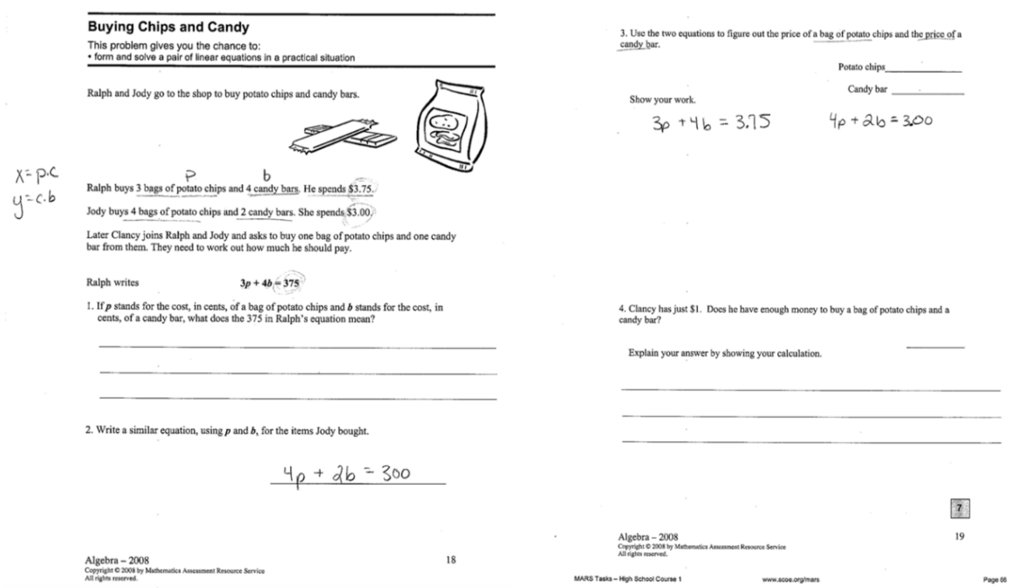
Aimee's Actions of Anticipating

During her interviews, Aimee grounded herself in the mathematical topic of the unit before thinking through how students would reason about the tasks she selected. In doing so, she identified what procedural methods they might use to engage with the problem as well as possible points of difficulty students might experience.

Aimee's Artifacts of Anticipating

Unlike Hillary, it was unclear if documenting her anticipations was a regular part of Aimee's practice of anticipating students' mathematics. In both interviews, Aimee recorded very little when thinking through her anticipations for both tasks. In her Cycle 1 interview focused on the “Buying Chips and Candy” task, she underlined phrases in the description of the problem to define variables for the systems of equations task as shown in Figure D. In her Cycle 2 interview where she selected the “Postage Stamps” task to building understandings of piecewise functions, Aimee asked if it were okay to not write anything down because she could easily think through the task aloud.

Figure D. Aimee’s Artifact of Anticipating for the Buying Chips and Candies Task



Buying Chips and Candy

This problem gives you the chance to:
• form and solve a pair of linear equations in a practical situation

Ralph and Jody go to the shop to buy potato chips and candy bars.

x = p.c
y = c.b

Ralph buys 3 bags of potato chips and 4 candy bars. He spends $3.75.

Jody buys 4 bags of potato chips and 2 candy bars. She spends $3.00.

Later Clancy joins Ralph and Jody and asks to buy one bag of potato chips and one candy bar from them. They need to work out how much he should pay.

Ralph writes $3p + 4b = 375$

1. If *p* stands for the cost, in cents, of a bag of potato chips and *b* stands for the cost, in cents, of a candy bar, what does the 375 in Ralph's equation mean?

2. Write a similar equation, using *p* and *b*, for the items Jody bought.

$4p + 2b = 300$

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3. Use the two equations to figure out the price of a bag of potato chips and the price of a candy bar.

Potato chips ______

Candy bar ______

Show your work.

$3p + 4b = 3.75$ $4p + 2b = 3.00$

4. Clancy has just $1. Does he have enough money to buy a bag of potato chips and a candy bar?

Explain your answer by showing your calculation.

7

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Consulting the Unit Topic

Aimee’s practice of anticipating students’ mathematics focused on her students in relation to the mathematical goals of her unit of instruction. When anticipating Aimee quickly delved into the task to think through the math once she had selected one that was dealing with the unit topic. In Cycle 1 she was planning for her Math 1 course and went into it looking for a task for her Systems of Equations unit. In Cycle 2, she was planning for Math 3 and looking for a task dealing with topics for her introductory unit on functions which had a heavy focus on piecewise functions in this context. In both interviews, she had a general idea of where in the unit she wanted to implement her chosen task but did not make a final decision until after thinking through her anticipations.

Types of Anticipations

Aimee's anticipations of students' mathematics for the tasks she selected considered the ways students might reason about problems they were given, strategies that they might remember when solving problems, and parts of tasks that might be challenging.

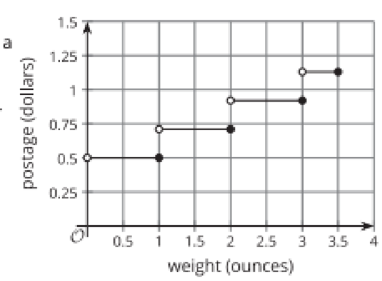
Student Reasoning. Aimee often thought through how students might reason about the task, considered what they might notice, conjectured ways they might draw from prior understandings to find a solution, or noted what they should know about the mathematics entailed in the problem. In the "Postage Stamps" task for example, students were given a piecewise graph and asked to determine how much a letter weighed if it costs $0.92 to mail (See Figure E).

Figure E. Aimee's Task for Cycle 2

12.2: Postage Stamps

The relationship between the postage rate and the weight of a letter can be defined by a **piecewise function**.

The graph shows the 2018 postage rates for using regular service to mail a letter.



1. What is the price of a letter that has the following weight?
   a. 1 ounce
   b. 1.1 ounces
   c. 0.9 ounce
2. A letter costs $0.92 to mail. How much did the letter weigh?

Aimee stated:

> I think most students will be able to figure where 92 cents is because this is the only option in this area. But wondering what they're going to pick, if they're going to put two ounces, or if they're going to put exactly three because there's a closed circle. I'm trying to anticipate what they're going to pick or if they're going to write a condition statement saying between two and three ounces is how much it's going to cost… And so hopefully, the students will remember when you use a closed circle and when you use an open circle, so if it's included or if it's not included. (Cycle 2 Interview)

In this scenario, Aimee was looking at the graph of a piecewise function and noting that she believed her students would quickly find the $0.92 value of the function represented by the graph but might differ in how they understood which domain values matched the $0.92 function value. She anticipated that some students might select one values at the end of that section of the graph while others would "remember" what closed or open circles notate and what they represent about the values of the domain and range of the function (see Figure E).

Aimee's anticipations related to student reasoning also focused on what students might notice because of what they remember from prior mathematical experiences. For example, at the end of her Cycle 2 interview, Aimee summarized how students might think about the "Postage Stamps" task:

> Well, so they should like, whenever I'm thinking about, like, what they know and what they don't know, they'll know what a piecewise function looks like. So, they won't be completely shocked. Because whenever they see new things on the graph, they're like, "What is this," or whatever. So, they'll know, I would have already shown them, like what a piecewise function looks like what a set function looks like to them. And what

> they, they should also know, when I'm thinking about planning a task like this, like, they should know the whole open circle/close circle from Math 1 of graphing inequalities, even maybe middle school standards are graphing inequalities and when to use an open circle and a closed circle. (Cycle 2 Interview)

Aimee relied on what students should remember about graphing solutions of inequalities and the relation to graphing piece-wise functions to anticipate how students might reason about the task.

Procedural Methods. At times, Aimee thought through different ways students might approach a problem in ways that were procedural. Her anticipations for the "Buying Chips and Candy" task demonstrate the procedural nature of strategies when she thinks through how students will solve a system of equations:

> They can use their skills that I've already taught them on how to do substitution and elimination and be able to work it out to find the price of a bag of potato chips and one candy bar. So, this is where they would set up their equations and be able to solve. (Cycle 1 Interview)

In this example, Aimee considered substitution and elimination, two procedures students have already learned to find solutions to the task while ignoring possible reasoning with a table or a graph of the situation.

Challenges. The most common type of Aimee's anticipations were aspects about tasks and their solutions that might pose a challenge for students' engagement or progress. For example, Aimee talked through how she expected her students to solve the "Buying Chips and Candy" task and noted students might have difficulty seeing the connection between $3.75 and how it is represented in the given equation as 375:

> So, if P stands for the cost in cents of a bag of potato chips, and B stands for cost in cents for the candy bar, what does 375 (inaudible) equation mean? So, then maybe if some kids didn't understand where that was coming from, I can maybe talk to them about if they could turn this into like, maybe cents like pennies or something? Like how many pennies would this be and see if they can recognize the relationship between the $3.75 to the 375. So, hopefully, after they have been able to see that this three cents for three bags of potato chips, and the four cents for the four candy bars, they have a total of 375 cents. And (inaudible), they can write the equations for Jodi. So, four bags of potato chips, and then two candy bars, then it's $3. So, they can convert that to 300. So, I would just know that some kids are gonna, might get confused by the 375 and the 300. And might not know how to convert that. So, I would be ready to help the kids be able to know that maybe, let's turn this into like pennies or cents. Like how many pennies make up $3? And then they would add the 75 to get the 375. (Cycle 1 Interview)

Like other instances from her interviews, Aimee identified a part of the task that she believed might be confusing for students and then thought through how they might be able to make connections with her support.

Aimee's Resources for Anticipating

Aimee relied on her horizon content knowledge (Ball et al., 2008) specifically her understanding of mathematics concepts taught in previous grades and her experience teaching Math 3 when anticipating students' mathematics. Even when extensive teacher notes that included narratives around different strategies to monitor for were provided with the instructional resource she selected (e.g., the "Postage Stamp" and "Bike Sharing" tasks from *Illustrative Mathematics*, 2019, Algebra 2), Aimee did not engage with these supports and relied on her own knowledge. As a former middle school mathematics teacher, Aimee often referenced concepts students should have learned in previous courses or grade levels. In both interviews, Aimee relied mostly on her knowledge of explicit strategies that either she had introduced to students or from previous courses to anticipate students' mathematics.

Aimee's Purposes of Anticipating

When considering Aimee's actions and the resources she drew upon, two main purposes for her practice anticipation of students' mathematics emerged from the analysis. In her interviews, Aimee seemed to anticipate to prepare for answering questions students would have and supporting them in making sense of the mathematics of the task. Her anticipations assisted her in preparing to remove barriers they might experience. Second, Aimee's anticipations of how students would go about solving the task assisted her in deciding when to use a task in her unit of instruction so that students would have the understandings, they needed to successfully complete the task.

Removing Barriers

Aimee's anticipations assisted her in questions she might ask students in response to a difficulty they might have with the task. For the "Buying Chips and Candy" task for example, she first worked through the task, making sense of what it was asking by underlining key phrases and labeling variables. As described in the previous section, she then discussed how students might be confused about the "375" in the equation:

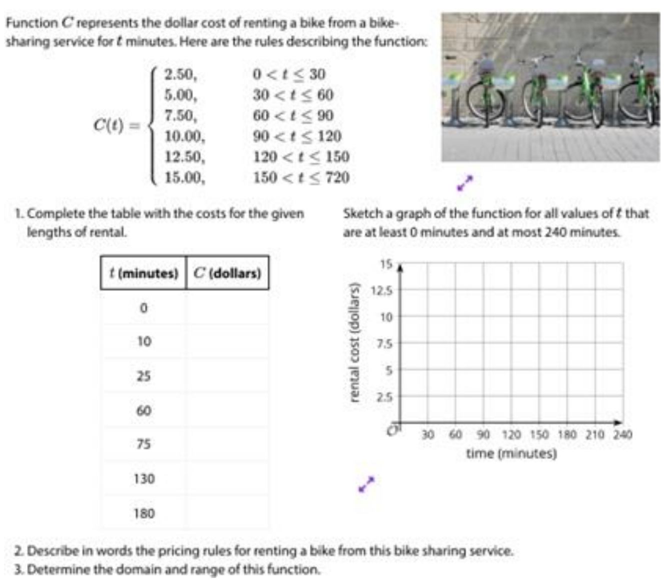
> So, I would know that the kids might be able to recognize that this 375 comes from the $3.75. Then maybe if kids didn't understand where that was coming from, I can maybe talk to them about if they could turn this into maybe cents, like pennies, or something?

How many pennies would this be and then see if they can recognize the relationship between the $3.75 to the 375. (Cycle I Interview)

After she anticipated the challenge students may have with interpreting the 375 in the equation, Aimee immediately brainstormed a question she could pose to address the conjectured confusion.

At times, Aimee also thought through ways she might guide students through difficulty. When considering the “Bike Sharing” task (see Figure F), Aimee anticipated that students might say that the cost of the rental at 60 minutes is $7.50 instead of $5.00, because they would fail to make the connection of the difference between the less than and the less than or equal to sign.

Figure F. The “Bike Sharing” Task



Function C represents the dollar cost of renting a bike from a bike-sharing service for t minutes. Here are the rules describing the function:

$$C(t) = \begin{cases} 2.50, & 0 < t \leq 30 \\ 5.00, & 30 < t \leq 60 \\ 7.50, & 60 < t \leq 90 \\ 10.00, & 90 < t \leq 120 \\ 12.50, & 120 < t \leq 150 \\ 15.00, & 150 < t \leq 720 \end{cases}$$

1. Complete the table with the costs for the given lengths of rental.

t (minutes)	C (dollars)
0	
10	
25	
60	
75	
130	
180	

Sketch a graph of the function for all values of t that are at least 0 minutes and at most 240 minutes.

2. Describe in words the pricing rules for renting a bike from this bike sharing service.
3. Determine the domain and range of this function.

She then followed this anticipation with a plan for how to respond:

It’ll be interesting to see what they pick because here’s 60 and here’s 60 [pointing to the 60’s in the piecewise function algebraic representation]. So, this statement [pointing to the second expression] says that the time is greater than 30, but less than or equal to 60 is $5. And it’s $7.50 if you rent it from the time that is greater than 60, but less than or

> equal to 90. So, I anticipate that I'll have some people put $5 and some people will put $7.50. So that will be interesting to see what they put for that. So, if I have students that pick $7.50, I can explain to them here's another 60 [in the second expression], what does this statement mean? And then talk about what the inequality signs mean, again. So, this would be $5 for 60 minutes. (Cycle 2 Interview)

Aimee's anticipated challenge served as a reference for her to plan an emphasis on carefully noting the inequality signs so that students would interpret the piecewise graph correctly.

Aimee explicitly discussed removing barriers as one of her goals when planning for instruction and anticipating students' mathematics. The following excerpt is taken from the start of her first think aloud as she described what she was about to do:

> So, what I would do first, whenever I search for a task for a certain unit or a certain standard, I would always like to work things out. So, I would definitely work this out and see if there's any hiccups and maybe ways that I can lead the students to get to the right answer, not give them the answer, but I always want to be prepared. (Cycle 1 Interview)

In Cycle 2 when asked how her anticipations would help facilitate the task with students, she responded with:

> I think it's going to help me be ready for any questions that they might have or if they're stumped on something, then they're not quite sure how to approach the problem. Hopefully, I can help them be able to figure out where to go from there. (Cycle 2 Interview)

From these interviews, my professional relationship with Aimee, and experiences in her classroom, she clearly worked hard to ensure that she was prepared to answer any questions and support students through moments of struggle so that they could experience success in mathematics.

Sequencing Tasks

Throughout both interviews, Aimee reflected on and made decisions about how to sequence tasks such that students would have what they needed to successfully complete a task. One example of this focus on ordering tasks carefully occurred when she discussed the "Buying Chips and Candy" task:

> So, then for this question, it says, use the two equations to figure out the price of a bag of potato chips and the price of a candy bar. So, this is where the systems comes in. So, I would probably have talked about systems, and we've talked about like substitution, elimination, and things like that. So, in showing the kids how to set up the equation part, like I would probably just do the, just with numbers without the word problems. So, this would, I would end, now that I think about it, I would definitely have taught like substitution, elimination, solving algebraically. And then teaching the kids like, they can use this to learn how to set up the word problems using like, the different variables and the numbers, the coefficients in front of them. And then they can use this. They can use their skills that I've already taught them on how to do substitution and elimination, and be able to work it out to find the price of a bag of potato chips and one candy bar... [talking through anticipations] … And so, for that task now that I think about, I would definitely put it, I would teach them how to do elimination and substitution. And then this would help them learn how to set up the equations. And then I would let them use the skills that I've just taught them to solve these two problems to figure out the price of one bag of chips and the price of one candy bar. (Cycle 1 Interview)

In this moment, Aimee had been unsure of when she wanted to use this task in her unit on Systems of Equations. However, she realized that she wanted students to already know the different strategies for solving systems after talking through how students would think through the third question.

Summary of Aimee's Practice of Anticipating

Aimee consulted her unit topic and sequencing when engaging in the practice of anticipation. She wrote little to no notes about the anticipated strategies that consisted mainly of searching for challenging parts of the task, how students would reason about the math they were presented with, and how they could use procedural methods previously taught to solve problems. She relied on her experience teaching students in prior semesters to determine how students would approach a task, her horizon content knowledge (Ball et al., 2008) from experience teaching at multiple grade levels, and used her anticipations to prepare for supporting students by removing barriers to understanding and intentionally placing the task in the unit at a point where they would have enough understanding of the unit to solve the task.

Kathryn's Practice of Anticipating

Kathryn's practice of anticipating students' mathematics involved identifying multiple strategies her students might use to solve a problem. Each of her interviews began by considering the content standards and the specific learning goals she had established for the selected task. Subsequently, she examined the task from a student's perspective, contemplating student reasoning and exploring multiple ways of representing the mathematics, different procedural methods, and various challenges students might have in learning. Kathryn primarily relied on her own mathematical understanding and her prior experience of working with high school and undergraduate mathematics students to anticipate. This process allowed Kathryn to identify potential instructional moves to engage students with the task, advance their thinking, and determine the degree of instructional time students would have to explore the task versus more teacher-led instruction.

Background

At the time of this study, Kathryn had a decade of experience teaching high school mathematics. She held National Board certification, possessed a master's degree in mathematics, and taught select undergraduate mathematics courses at a local university. Her current assignment was teaching Math 3. She was the Math 3 lead teacher at her school and taught all but one of the Math 3 sections at the school. A novice teacher, with whom Kathryn collaborated when possible, taught the remaining section of the course. However, these two instructors did not have overlapping planning periods, and this resulted in very little collaborative planning between the two. When asked to describe her teaching style, Kathryn characterized it as "constructive" and stated that she favored a discovery-based approach. Her instructional approach typically began with a "hook" or a connection to mathematics concepts her students were already familiar with, followed by activities designed to engage their mathematical thinking. Kathryn also often pursued professional development opportunities, including the four-day workshop centered around the *5 Practices* framework (Smith & Stein, 2018), which she attended in conjunction with Aimee. During this workshop, she identified ways in which she could enhance her practice by being more deliberate in centering students thinking in her teaching. In the time since then, she had often expressed how useful and beneficial this framework was in her teaching practice.

During her interviews, Kathryn described an aversion to developing instructional materials from scratch. Rather, she stated how she had amassed a diverse collection of activities over her career which she had customized to suit her needs. During the year of this study, she came across a new Algebra 2 curriculum from *Illustrative Mathematics* (2019) that mostly aligned with the standards for her Math 3 course. Kathryn appreciated the clarity and explicitness of the learning objectives embedded in this curriculum, which also featured open-ended tasks that emphasized student thinking and would assist her students in building new concepts. Because the curriculum did not precisely match the content of her course, she described how she frequently supplemented with additional tasks or made modifications to its lessons to meet her instructional goals.

Kathryn's Selected Tasks

During her Cycle 1 interview, Kathryn was nearing the conclusion of a unit focused on polynomial functions. To synthesize the knowledge and skills her students had acquired throughout this unit, she deliberately selected the task, "Squares on Squares" (Youcubed.com, n.d.), from external curricular resources (see Appendix I). At the time of her second interview, Kathryn's classes were in the middle of a unit focused on rational functions. Having already taught key features of rational functions and presently working with students on solving rational equations, Kathryn made a deliberate choice to select a lesson from the *Illustrative Mathematics* (2019) curriculum. The lesson was comprised of three distinct tasks: an initial warm-up titled "Math Talk: Adding Rationals" and two lesson activities titled "A Rational River" and "Rational Resistance" (See Appendix J).

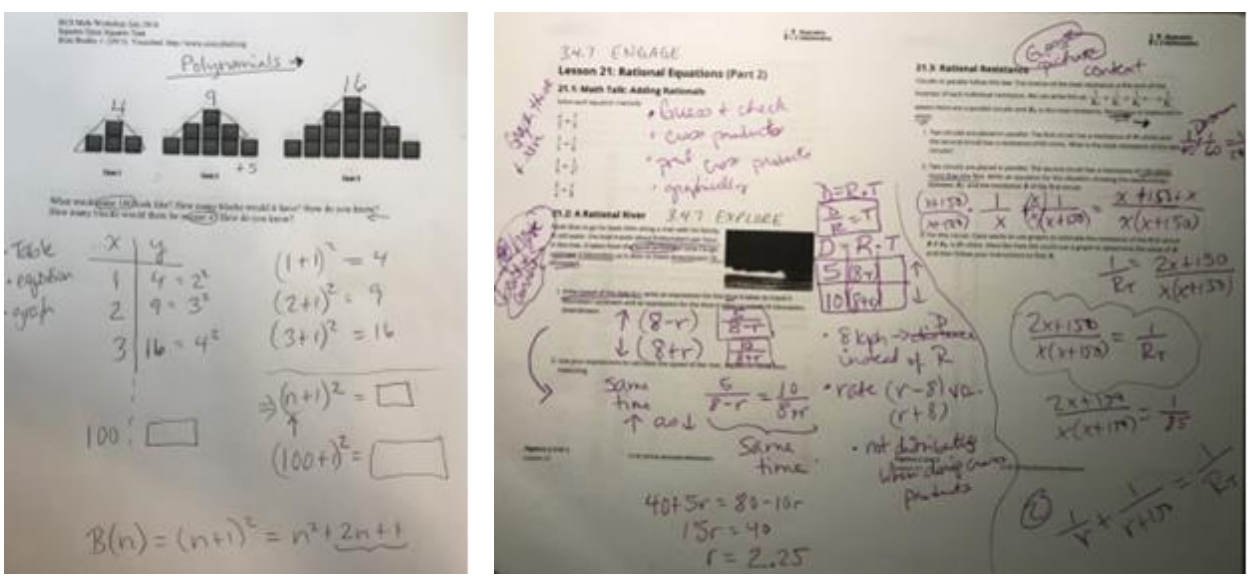
Kathryn's Actions of Anticipating

Kathryn's practice of anticipating students' mathematics involved solving the tasks she had selected on paper as she talked through them. She situated herself within the standards of the unit and her learning goal(s) for the task before thinking through how students might reason about the task, the strategies and representations they might use to solve it, and the difficulties they might encounter.

Kathryn's Artifacts of Anticipating

In both interviews, Kathryn employed a notetaking strategy to record her anticipations. Her notes served a dual purpose. First, the notes facilitated her own problem-solving process. Second, they provided a platform for creating and exploring a variety of mathematical representations her students might employ in addressing the tasks. Nearing the end of each think-aloud, Kathryn reviewed the potential problem-solving strategies she had brainstormed and created a concise bulleted list that delineated distinct strategies she envisioned her students using (see Figure G). Her lists encompassed both written and orally articulated strategies Kathryn had imagined and served as a synthesis of the multiple strategies she expected or wanted surfaced during the lesson.

Figure G. Kathryn's Artifacts of Anticipating for the "Squares on Squares" and Tasks from the Rational Equations Lesson from *Illustrative Mathematics* (2019)



For the “Squares Upon Squares” task, Kathryn enumerated three distinct representations: table, equation, and a graph. Although the list may initially seem vague and unrelated to the specifics of the task, Kathryn’s discussion during her interview evidenced a thorough understanding of how students could use these strategies to solve the problem at hand. As she shared her thinking, Kathryn considered the potential discovery of patterns within the table and pondered how students might use these patterns to find the 100^{th} term. Kathryn also acknowledged the possibility that some students might recognize and represent the quadratic growth of the pattern and derive an equation from their observations. Thus, Kathryn’s anticipations extended beyond her summary lists and represented a nuanced and more comprehensive application of these approaches she imagined her students might use. She even entertained strategies that she considered less likely for her students to employ but wanted to watch for similar thinking as a catalyst for deeper conversations during her class discussion of the task.

Grounding in Content Standards and Learning Goals

For both interviews, Kathryn arrived having already considered the standards of the unit and identified the learning goal she had for the tasks she had selected. Kathryn knew the standards and expectations for the Math 3 course well, and while she did not speak about them directly during the interviews, it was clear that she was well versed in what was and was not expected. For example, when she anticipated the level of frustration for solving the equation on “Rational Resistance” and remembered that the complexity of that equation went above the standards for the course, she made the choice to encourage the use of a graphing technology and have students focus their discussion on writing the equation. She stated:

> I know there are some things I just need to kind of move on and let technology help us with. So, in small groups, let's try to work on going from words to mathematical expressions. That's what I want them to get out of this. That idea around modeling real life circumstances with math. If they can get the model made, then let the technology help us from there. (Cycle 2 Interview)

Though she did not explicitly mention standards or the learning goal, the Math 3 standards emphasize creating and interpreting rational expressions and equations, and the expectation for solving rational equations algebraically is limited to simple equations with linear denominators. Kathryn's choice to focus on writing the equation and then using a graphing technology demonstrated her understanding of the standards and how she aligned her learning goals to them.

To determine the learning goals for the tasks, Kathryn either identified one herself or used the learning goals identified by the curricular resources when possible. Since "Squares on Squares" was a supplemental task, Kathryn assigned a learning goal:

> My end goal is to connect what we've been doing in polynomials with a real life, hands-on application. What I want to bring out in the end… well, I would make a quick little list… I want to make sure that they see multiple representations: the table, then the equation, the graph at the end. Also expanding a binomial and showing how the different algebraic representations, the factored form versus the standard form, show us different things that actually relate exactly to this pattern … (Cycle 1 Interview)

Kathryn planned to use the task to connect features of polynomial functions (in this case, a quadratic function) with their multiple representations. In her Cycle 2 interview, Kathryn used a lesson from *Illustrative Mathematics* (2019) and thus identified the same learning goals from the teacher notes:

- Create a simple equation involving two rational expressions to model a situation and use the model to answer questions.
- Use strategic multiplication to solve equations that have rational expressions on both sides. (Rational Equations Part 2 Lesson from IM)

During the interview, she spent very little time contemplating her goal for the task and focused her energy on considering how students might approach the tasks and work through them.

Types of Anticipations

Kathryn's anticipations of students' mathematics considered the ways students might reason about the mathematics embedded within the tasks she selected, the challenges students might have and partial conceptions that students may have, and the differences between her students in the standard level and honors level sections she taught. In addition, she included strategies students might use to solve while considering student thinking. When anticipating, Kathryn discussed how students might play around with the mathematics to generate new ways of reasoning, novel strategies, and difficulties.

Multiple Representations. Kathryn anticipated various representations and accompanying strategies that students might use to solve problems. As she determined different representations that students might use, she also discussed how students would use them to reason about the problem. One aspect of Kathryn's multiple representation anticipations was how she also included partial conceptions that students might have that she could leverage to make connections to her learning goal. In "Squares Upon Squares" (see Figure G) for example, she stated that her students had not had the opportunity to discuss first and second differences in their previous explorations of the rate of change of polynomial functions. Despite this, she expected some of her students might still discern a pattern among how the number of squares was growing, and this recognition would allow her to introduce the idea of finite differences and polynomial rates of change:

> So, I was trying to come up with other ways students might do it. For example, they might do 4 + 5 is 9, 9 + 7 is 16, and then have the second differences, you know, how we have first differences and second differences... that could happen. We could have a kid doing that and seeing that pattern and not knowing what second differences were but realizing they got that pattern there, and then I would say, "Hey! That's something!" (Cycle 1 Interview)

While she thought it unlikely that her students would write a system of equations to represent the pattern of first and second differences, she would value students finding this pattern and planned to use the structure they recognized as an opportunity to discuss using that pattern to write an equation with the class.

Student Reasoning. When Kathryn talked through the multiple representations she recorded in her artifact, she discussed and notated various ways students might reason about the mathematics with which they were engaging. For example, instead of just commenting that students would likely use tables to solve the "Squares on Squares" task, Kathryn described:

> So probably, then some of the students would make a table. And they would say, for cases, case 1 is 4, case 2 is 9, case 3 is 16. Okay. And so go all the way down to 100. That's where we have to go. So, I would think some students would see that 4 is a perfect square right away. That 9 is a perfect square. So, 4 is the same thing as 2^2. 9 is the same

thing as 3^2. 16 is the same as 4^2. And so, they might start thinking... "How can I use that pattern to write some sort of rule for my 100th case?" (Cycle 1 Interview)

As she anticipated, Kathryn spoke about student thinking in first-person, as if she were the student, thinking through what they might notice, the order they would notice it in, and what that might make them question about the pattern they were exploring.

Procedural Methods. Kathryn included one procedural approach when anticipating for the warm-up in the Rational Equations lesson she selected for her second interview from *Illustrative Mathematics* (2019). The warm-up was a review of prior grade-level work and provided a just-in-time scaffold for students who would be exploring algebraic approaches for solving more complex rational equations for the first time in the course. When listing ways students might solve these simple rational equations shown in Figure H, she decided that students might use "cross products" as a method that they could recall from previous classes if she reminded them of this method:

> Some students are going to think about cross products, and they're going to start umm... But that's going to take a nudge for me, I believe, to get them there because again, this is not something we've done in a while, and this is pulling back and, you know, applying what they … previous knowledge, and it's been a long time … But I always tell them, they can't use any shortcuts until they until they tell me why it works. (Cycle 2 Interview)

Figure H. Warm-Up Tasks from Kathryn's Cycle 2 Interview

21.1: Math Talk: Adding Rationals

Solve each equation mentally:

$$\frac{x}{2} = \frac{3}{4}$$

$$\frac{3}{x} = \frac{1}{6}$$

$$\frac{1}{4} = \frac{1}{x^2}$$

$$\frac{2}{x} = \frac{x}{8}$$

After Kathryn had anticipated the entire lesson, she revisited her thoughts around the warm-up at the end of the interview:

> So, I'm sitting her wondering now if I would maybe not nudge them to use cross-products here [pointing to warm-up] because if we multiply three by four, the left side would become two x and the right side would be three. If I multiplied by the common denominator in the first problem times four, I just multiply three by four, I would get four x over two, which would reduce it to x. And I would get 12 over four, which would reduce it three. So, I'd get two x equals three x equals 3/2. So that I remember that anytime I teach cross products … you know you have to be careful with these gimmicks. (Cycle 2 Interview)

When probed on why she had changed her mind about nudging students to remember to use cross-products to solve the warm-up, she explained that the learning goal included language around using strategic multiplication. She stated, "The learning goal said strategic multiplication which is not using tricks like cross-products. It's multiplying by a common denominator or using multiplication to clear out the denominators in the equation." Therefore, even though she showed evidence of including procedural methods, she also used her practice of anticipation to decide against relying on these methods without understanding, even on a quick warm-up review.

Challenges. When anticipating, Kathryn often discussed challenging aspects of the task as "stretches"—places where students may struggle on their own—and these stretches were directly related to her inclusion of partial strategies in her anticipations. In both interviews, Kathryn never thought of a portion of a task as something students could not do on their own. Rather, she noted ideas students might have more difficulty engaging with on their own, including aspects of a strategy and the mathematical connections she wanted to them to make.

Kathryn often referenced differences in her honors level courses and standard level courses when identifying difficulties. For example, when discussing how students could generalize and write an equation from the visual pattern in "Squares on Squares" task, she noted:

> For my honors kids, they're going to get to an equation without me. But for my standard kids, going from the concrete to the abstract, sometimes that is hard. It's hard for my honors kids too, but we're having them develop an equation out of data that already has numbers, and it may not occur to my standards kids to introduce a variable for that case number even though it's in the prompt. So, I might have to help them out there. Just remind them that's an option … to introduce a variable. (Cycle 2 Interview)

Kathryn anticipated students in her standard course might not readily abstract the pattern when compared to students in her honors course. However, she did not believe students would be unable to do so and planned to provide guidance as needed to move students toward her stated learning goals of modeling quantitative situations with polynomials and seeing connections among numeric, tabular, graphical, and contextual representations.

Kathryn's Resources for Anticipating

Kathryn relied on her strong mathematical background and experiences teaching at many different levels of high school and college courses when anticipating. Additionally, her time as the primary Math 3 teacher at her school for many years prior to this study provided a wealth of experiences and a deep understanding of how students would engage with the mathematics of the tasks she had selected. When asked how she generated her anticipations, Kathryn identified her own mathematical understandings as key to her ability to anticipate multiple strategies and ways students might approach a problem:

> When I'm thinking through all of these strategies, that is coming from my experience. That is … I have a master's degree in math, and I have a familiarity with numbers. I'm not the best there is or anything like that. But I've seen teachers not be able to do that because they don't think about math that way. (Cycle 1 Interview)

She reiterated this idea in her second interview, stating:

> When I'm thinking about patterns or strategies kids are going to use, that's my strength in the content. And it's because I went to grad school and swam in it for so long. I mean I can do proofs about closure, and even and odds, and other things. We talked about closure for polynomials the other day in Math 3 … they're not closed under division. Some of my colleagues don't know what that means, you know. So that strong mathematical background is so important and mine came from grad school. (Cycle 2 Interview)

Kathryn also alluded to prior experiences teaching Math 3 that affected how she anticipated students might approach problems. She would often use this prior experience when anticipating difficulties students may have with the task. In her Cycle 2 for example, when anticipating how students would work through the "Rational Resistance" task, she described, "I've taught this for umpteen years, I know where the frustration levels are and they are going to get frustrated solving this complicated equation, so I'm going to encourage them to use technology at this point" (Cycle 2 Interview). Based on her prior experiences teaching similar rational equations tasks, she knew where students might get frustrated.

Lastly, Kathryn attributed the way she could think through multiple strategies and student reasoning to a comprehensive 30-hour professional learning experience coupled with her recent use of the *Illustrative Mathematics* (2019) curricular resources:

> Well, the *5 Practices* professional development that I did … that really was great, you know, it really transformed how I taught, to look for those "out of the box" ways students can think through tasks without knowing step-by-step how to solve it. And now, I'm becoming familiar with *Illustrative Math* which has been really good because I see that same approach through their curriculum. The more I do it, I find myself talking that way. "What do you notice? What do you wonder?" It's so funny, because many times when we do our warmup, it's a notice/wonder routine. My kids are starting to respond on their own, not prompted by me when they are talking, "I noticed this… I wonder this..." (Cycle 2)

Kathryn had participated in multiple types of professional development, but she referred to her experience in the *5 Practices* professional learning as well as her own investigations into using the *Illustrative Mathematics* (2019) curriculum multiple times during her interviews. She valued both experiences as catalysts for exploring different ways students think about mathematics on their own and as support for her practice of anticipating.

Kathryn's Purposes of Anticipating

Kathryn's purpose for anticipating students' mathematics was to provide access to the tasks she selected, advance student thinking, and identify how much time she would allow for students to explore the task.

Providing Access

Kathryn responded to anticipations regarding student difficulty with a task with plans to invite students to the mathematics of the task. She did this with plans to either connect to a student's prior mathematical knowledge or connect the task to a familiar concept from students' lives. To connect to prior mathematical knowledge, Kathryn planned to offer suggestions of a representation to organize work. When discussing the "Squares on Squares" task, she described a case where students may struggle to get started and thought through what she could do to encourage students to use what they know to engage:

> So, I definitely think the kids, all the groups would be counting blocks. And they would all be thinking about a pattern. And I would probably, if they weren't thinking about a pattern, I would say, "can you come up with a pattern?" You know, "do you see any patterns you could describe to me?" And so, I may have to suggest a table to organize their work. "These cases … let's think of those as maybe a way to say for case 1. How many blocks do we have? For case 2 …" I think once it was in the table, it would click that these could so, "I've seen tables before," right, there's there, they're gonna say, "Okay, I know what to do with tables, I look for patterns, I look for, and what's going on with y, I look to see if I can basically find any kind of predictability for how to come up with an outcome way down at the hundredth case. A little bit of frustration is good. But getting stuck, not so much. You know, the goal for me with kids is to try to give them just

> enough scaffolding that they're able to move on by themselves or with their partners without getting frustrated and just quitting. So, trying to build that resiliency. (Cycle 1 Interview)

Kathryn considered a balance of productive struggle with providing support for students to work through a problem without telling them exactly how to do it when anticipating.

Another way Kathryn planned for providing access was to find ways of connecting it to students' lived experiences outside of school, even if the problem had a different context. When anticipating students might struggle to make sense of how the river's current would affect the boat's rate in the "Rational River" task, she described how she planned to connect this concept to Nascar Racing that she knew many of her students had former experiences with:

> I could talk to them about Nascar racing and the draft and all that comes into play there. Because most kids in the room can think of some examples where they see running into the wind versus running with the wind. (Cycle 2 Interview)

By discussing racing and draft, Kathryn was thinking about ways of connecting students' lived experiences with the task at hand.

Advancing Student Thinking

From our work together prior to this study, I knew that Kathryn used probing questions in her teaching regularly, and the interviews provided very little evidence of questions Kathryn planned to use to assess student understanding. However, her practice of anticipating students' mathematics helped her identify opportunities for students to make mathematical connections among their own strategies or their peers' strategies and to the learning goal(s). In her discussion of the "A Rational River" task, she pondered questions she wanted to ask students after they found the rate:

So, I may say "How does that answer you got relate to our story? What does it mean for Noah and his boat? What would happen if the rate of the current was ever bigger than the rate of the boat?" One thing students do sometimes that is not good is they mix up the signs for the $r + 8$ and $r – 8$. [It should be $8 + r$ and $8 – r$] Then the answer at the end will have the wrong sign. I could have them compare and talk out which one makes sense for our story. How are the solutions similar? Different? (Cycle 2 Interview)

By anticipating that students would have different ways of thinking about rates, Kathryn identified an instance where she might help students advance their thinking by "comparing and talking out."

Kathryn also developed advancing questions when anticipating as a means of connecting to her learning goal. During the "Squares on Squares" interview, she took a moment to reflect on the anticipations she had generated for the task and how she might bring that discussion back to the learning goal of modeling visual patterns with functions:

We would show all those patterns and representations, and if no one graphed their equation, I would ask them to graph it. And so, since we're in the polynomials unit, we would take it further at that point, we would probably expand that binomial into standard form. So, since it's n, maybe we would call it B(n). So, B stands for blocks, because we are working with function notation as well. And we would do, and some may have $n^2 + 2n + 1$. And I would ask what we notice about our rule if we wrote it out that way, in the 2n+1 part is the of blocks that are being added each time. So, if we go back up to the pattern... For example, for case 2, we added five blocks between case 1 and case 2. So, if 2 = n, 2 x 2 = 4 plus one more is 5. And that's how many blocks got added. Yeh, I would want them to see that. Now I might say, "Well, where's the –?" … I think you're thinking

> about real life. This polynomial is really representing what we're doing up here. And that connection is so huge in a math classroom with you know, with the idea of modeling. And so, at that point I would graph it and then we have our table made on our sheet and we have our explicit formula. And then when we graph it, we're going to have that negative side of the quadratic, we're going to have that positive side of the quadratic. And we're going to be able to talk about reasonable domain and why is it that sometimes math models straight out of a calculator, we have to put limits on them, so they actually model real life? And that's a very common thing. "So why is it that we have to do that?" "Does it make sense that this function is discrete or continuous?" The outputs versus the inputs. That's the kind of things I am going to ask and do. (Cycle 1 Interview)

In this part of the interview, Kathryn simultaneously planned to connect students' strategies, press on their understandings of the different representations, and discuss the key features and characteristics of the function that describes the pattern and how those features model the pattern. Through anticipating, she developed a plan to connect student strategies through discussing differences and similarities. At the same time, she planned to use those connections to advance students' understandings of how equations model growing patterns and different types of mathematical structure underlying representations. Her plan would allow her to reinforce key features of quadratics functions and how those key features tell us a story of the scenarios being modeled.

Balancing Student Exploration and Teacher-Led Instruction: "T"asks vs. "t"asks

Kathryn did not consider all cognitively demanding tasks the same. During the "Squares on Squares" task interview, she described a distinction she made between this type of task and the tasks she implements daily:

> I feel like I do tasks every day. But a task like this, this task right here could take 30 minutes. And that's a lot of time. And then I've still got to pull it down and synthesize it so the kids are clear about what they've learned and can get some practice in. And you know, tasks like these, they're a big-time investment. (Cycle 1 Interview)

In our interview discussions, we started calling these types of tasks "big T tasks" ("T"asks) to differentiate from her daily "little T tasks." For Kathryn, "T"asks took an entire block of instruction. These tasks were often supplemental to the curricular resources she was using but could also sometimes be found in the curriculum. To Kathryn, the biggest difference is the amount of time that she gave for students to work independently or in small groups without direct instruction or whole group questioning:

> So, the difference in "big T tasks" and "little t tasks" is how much I leave the kids alone on the task. I really think that's the difference. These are the tasks that I plan for us to have the luxury of really having time to just kind of explore. It's fun! But we still have standards to meet. We have a pacing guide and all that. So, I can't just let these kids go for an undetermined amount of time on a task like this. And then of course, I would lose kids, I have to manage all of that. So, for the tasks that I do daily, where we're going through the task, and then we're going to do a synthesis, and then we're going to have time to practice the cool down [daily exit ticket], and so forth. I'm pacing those more. I'm saying things like, "Take three minutes and look at this … Alright, regroup … What did you find out? … Okay, now take two minutes and let's do this. And I'm reading the task out loud to them because it's helping them [snapping in rhythm] stay right here with it, you know, the whole time. (Cycle 2 Interview)

Kathryn talked about the ongoing negotiation of time, course expectations, and pacing as considerations when judging how student-centered her implementation of different tasks would be. Her anticipations of how students might engage with a task and the connections that she could facilitate through those anticipations were weighed against the constraints of time, standards, and pacing. She felt that this balance ensured that students had opportunities to do the real work of mathematicians – playing with math, exploring ideas with their peers, and coming up with strategies to non-routine problems – while still honoring the limited time she had to teach a course with many standards. In her definition, both "T"asks and "t"asks provided students time to take ownership of the mathematics, but "t"asks were more directly facilitated by the teacher, often relying on quick "think-pair-share" routines and keeping the whole class on a relatively similar pacing. In comparison, "T"asks were less directly facilitated by the teacher, giving ample amounts of time for students to mathematize with their peers at their own pace and holding class level connections to the final discussion.

Even when anticipating a series of tasks she planned to use from the *Illustrative Mathematics* (2019) lesson, Kathryn identified "T"asks and "t"asks within one lesson. In her Cycle 2 interview, she explained:

> Well, I think with "Rational Resistance," I think I was anticipating walking them through this task really right there with them. Just because of time… I know how much time that "A Rational River" is going to take and I'm a little bit worried about saying, "Okay, now try this one [Rational Resistance] and just hand it off. I'd have to gauge at the moment and see how they feel. For "A Rational River," I can hand that off and give them a chance to work a little bit on their own and that's going to give me the opportunity to find out their own strategies to solve the equation and how they are thinking about it so we can

have a discussion when we synthesize that activity. I wouldn't jump in on this first one [A Rational River]. The second one [Rational Resistance], they can use the equation example in the problem to help them write it, so there's not going to be a lot of different responses. But it's a more complicated equation that I'm not that worried about them doing on their own because of what is expected out of them in Math 3. So, I think I would not do all that work I just did on solving the equation with them. It's going to wear them out. Even [if] I forgot what the heck I was even solving for by the time I got to the end. So, I think I'm just going to have them talk through #2 where they set the equation up … it matches exactly with the template that we're given at the top. That turns a 20-minute problem into a 10-minute problem, which gives us time to really dig into that first problem [A Rational River]. And that's good. And then for this one, I'll invite them to use Desmos. And I can show them this [looking at graph of $\frac{1}{R} + \frac{1}{R+150}$ in Desmos] is what we're looking for. "See those branches? Let's try to find where that intersects that 1/85!" And then let's go through a couple more of those problems. And let's use Desmos, so they'll set up the equation in their groups and put it in Desmos. And then we'll use it to find our solutions. Yeah, that's what I would do on that one. (Cycle 2 Interview)

By considering both main tasks in the lesson, Kathryn decided that "Rational Resistance" provided less opportunities for flexible thinking and involved a more complicated equation that she was less concerned with students solving algebraically. She knew she had two learning goals for this lesson, one relating to creating rational equations to model a situation and answer questions and another relating to solving simple rational equations. By allowing more time for students to make sense of "A Rational River," create their own equation to model the situation, and discussing s different strategies for completing the task, she could provide students opportunities for students to explore the mathematics and create rational equations that model situations. Because “Rational Resistance” was narrower than “A Rational River,” Kathryn decided it was less productive for students to spend time exploring it but believed it would still be helpful for advancing her goal of strategies for solving rational equations. As a result, she planned to facilitate more directly “A Rational Resistance” to take advantage of the opportunity it gave students to use strategies of solving rational equations from the discussion of “A Rational River.” She planned to question students on key aspects of the task’s solution (e.g., how would you write the equation for #2?) and invite them to use technology to solve.

Summary of Kathryn’s Practice of Anticipating

Kathryn’s practice as anticipating students’ mathematics allowed her to prepare to orient students’ mathematical ideas to each other in a way that explicitly connected to her predetermined learning goal. Her practice involved grounding herself in the content standards and learning goal for the task she had selected and then completing the task from the perspective of her students. In doing so, she considered multiple ways students might reason, create and use representations, and encounter challenges in finding solutions. For Kathryn, anticipating also involved planning to respond to her anticipations by identifying ways to invite students to explore the mathematics of the task, advance student thinking, and make connections. Kathryn used this practice to also gauge how much instructional time she would allocate to the task depending on her lesson goals and whether the task would lead to struggle that would be productive for her students.

Comparing Hillary’s, Aimee’s, and Kathryn’s Practices of Anticipating

My analysis comparing Hillary’s, Aimee’s, and Kathryn’s practices of anticipating revealed multiple similarities among some of the actions they took and the resources they drew upon. However, there were several key distinctions to their practice, including the referents they used to anticipate and their purposes for enacting the practice. In this section, I present these findings by comparing and contrasting each one's actions of, resources for, and purposes associated with their practice.

Actions of Anticipating

When comparing each teacher's actions associated with their practice of anticipating, there were several similarities. First, each teacher produced some record of their anticipations that included correct answers to the tasks they had selected. Whereas Aimee's record was concise and did not include much detail, both Hillary and Kathryn recorded complete solutions that documented key aspects of their approaches as well as additional annotations. For Hillary, her artifacts of anticipating included worked out solutions and reference notes for her students. For Kathryn, these artifacts documented potential representations and strategies that she anticipated students might use when engaging with the tasks.

Another similarity between the teachers' practices is related to the types of anticipations they identified. Across their interviews, each teacher anticipated ways that students might reason about the mathematics of the task, certain procedures they might use when developing their solution, and aspects of the task they might experience as challenging. Additionally, Hillary and Kathryn also anticipated multiple ways that students might represent the mathematics embedded within the tasks they selected. Only Hillary made anticipations related to how she might organize her instruction in response to her students or considered her students' affective responses to the tasks.

Despite these similarities and nuanced differences, the referent each teacher used when anticipating varied. For Hillary, the conceptual development and organization of her curricular program bounded her anticipations. In contrast, Aimee considered the mathematical topic of her unit of instruction when anticipating how her students might engage with the tasks she had selected. Kathryn's considerations focused on content standards for the course and her learning goals for the lesson as she thought about what students might think and do with the tasks that were her focus.

Resources for Anticipating

In terms of the resources used when anticipating, all three teachers relied on their previous experiences with students. Though Aimee and Kathryn had many more years of teaching experience than Hillary, each used their knowledge of how students had engaged with the mathematics of the task in the past or what they believed students would know. Each teacher recalled points of confusion with previous students or possible student strategies based on prior experiences as they engaged in the interviews.

Beyond their knowledge of students, each teacher drew upon additional resources when anticipating. Hillary frequently referenced her curriculum and its comprehensive teacher resources when anticipating student strategies (e.g., tape diagrams, hanger methods, etc.) introduced and refined in previous lessons. While all three teachers used lessons from *Illustrative Mathematics* (2019)[1]; with identical teacher resources and instructional design in their interviews, only Hillary referenced these resources in depth, and none of them used likely strategies included in the resources to think through ideas for how students might approach their selected tasks or the partial understandings they might use to engage. Aimee relied on the mathematical knowledge for teaching she had developed when teaching previous courses and grade levels to anticipate. In particular, she used these experiences to identify aspects of

[1] *Illustrative Mathematics* is called *Open Up Resources* at the middle school level.

solutions that students might find challenging or confusing. Kathryn attributed her ability to anticipate multiple strategies, representations, and partial understandings that might be useful in meeting her learning goals to her mathematical content knowledge and advanced degree in mathematics. She used these understandings to find a wide variety of strategies to solve tasks that included a robust collection of representations.

Purposes of Anticipating

When comparing each teacher's practice of anticipating, the greatest differences were in their purpose for engaging in the practice. Hillary's purpose for anticipating was to understand the mathematics of the tasks she selected and immerse herself into the mathematical story that the curriculum was weaving through the units and lessons. She referenced how the previous lessons built to the next lesson and described her responsibility as one of helping students to see how to use what they had learned previously to make sense of the current tasks. For this reason, she wanted clear notes for students to revisit solutions for each task, color-coding her solutions to guide them in recognizing the connections among the lesson's learning goal and previous strategies they had learned.

In contrast, Aimee anticipated to avoid confusion for her students and ensure that she was prepared to support them in overcoming challenges they might face. This resulted in a secondary purpose of identifying the sequence of tasks in the unit that would minimize student struggle. Her anticipations helped both decide where to teach the task and prepare her for student questions and ways she might respond to them. She carefully thought through how she might engage with students to help them make sense of the problem and scaffold them with steps to take to solve the tasks she had selected.

Where Hillary anticipated to support students in experiencing the coherence of the curriculum and Aimee to prevent frustration and struggle, Kathryn used her anticipations to plan access points for all her students to engage with the task and identify how students' discussions of their solutions might assist her in meeting her learning goals. Her anticipations often included strategies that students might not use naturally but were important ideas that she could use in instruction to meet her intended learning goals. She consistently considered the amount of instructional time she had with her students and used her anticipations to develop contingent plans that might balance student exploration with teacher-led instruction in relation to her learning goals. She selected "T"asks that were most connected to her learning goal and provided rich context for student exploration and discussion. She also chose "t"asks as contexts for students to reason mathematically but planned to facilitate their engagement more directly to save instructional time for discussions of "T"asks.

Table 6 summarizes major similarities and differences among Hillary's, Aimee's, and Kathryn's practices of anticipating students' mathematics. Common aspects of their practices are represented with bold text and distinctions among them with italics. In Chapter V, I discuss these comparisons and their implications for researchers and teacher educators in further detail.

Table 6. Comparison of Hillary’s, Aimee’s, and Kathryn’s Practices of Anticipating Students’ Mathematics

	Hillary	Aimee	Kathryn
Artifacts of Anticipating	**Solutions (Complete)** *Annotations (Notes & Questions*	**Solutions (depending on task)** *Few Annotations (depending on task)*	**Solutions (Complete)** *Annotations (Representations)*
Consultation	*Curricular Program*	*Unit Topic*	*Content Standards & Learning Goals*
Types of Anticipations	**Multiple Representations** **Student Reasoning** **Procedural Methods** **Challenges** *Organizational* *Affective*	**Student Reasoning** **Procedural Methods** **Challenges**	**Multiple Representations** **Student Reasoning** **Procedural Methods** **Challenges**
Resources for Anticipating	**Knowledge of Content and Students** *Knowledge of Content and Curriculum*	**Knowledge of Content and Students** *Horizon Content Knowledge*	**Knowledge of Content and Students** *Common Content Knowledge*
Purposes of Anticipation	*Curriculum coherence* *Student Notes*	*Removing Barriers* *Sequencing Tasks*	*Providing Access* *Advancing Thinking* *Balancing Exploration Time*

CHAPTER V: DISCUSSION

This chapter presents a final discussion from my study of three secondary mathematics teachers' practices of anticipating students' mathematics. In this chapter, I first answer the study's research questions and situate the findings within the existing literature. Next, I discuss related implications for teacher educators and researchers and outline areas of future research related to teachers' practice of anticipating students' mathematics. I conclude by identifying limitations to the study.

Secondary Mathematics Teachers' Practice of Anticipating Students' Mathematics

My first research question sought to characterize secondary mathematics teachers' practices of anticipating students' mathematics, and the results of my with-in case analyses indicated several components of the practice for teachers. First, the practice of anticipating involves consulting one or more curricular referents to understand what mathematical knowledge students might bring to bear on a particular task. Teachers in this study reviewed their curricular program, academic standards, and the mathematical topic of a unit of instruction to situate themselves and their students in the mathematics of the task prior to anticipating. Like Smith and Sherin's (2019) description of "getting inside the mathematics" (p. 38), this study's case teachers practice of anticipating first involved understanding the mathematical learning space afforded by the problem (Akyuz et al., 2013). For teachers in this study, the practice of anticipating also involved generating possible strategies, representations, ways of reasoning, mathematical procedures, conceptual challenges, and other descriptions of how students might express or experience their learning when engaging with the task. Like other studies of teachers' anticipations (Akyuz et al., 2013; Didiş Kabar & Erbaş, 2021; Hughes, 2007; Kartal et al., 2020; Krause et al., 2016; Morrissey et al., 2019; Rupe, 2019; Şen Zeytun et al., 2010; Vale et al., 2019; Wilson et al., 2024), teachers in this study described what mathematical actions they might observe. In some cases, they also discussed ways students might think mathematically about a task. This finding adds additional evidence to numerous findings that teachers' anticipations also attend to ways students might think and reason about a particular task (Didiş Kabar & Erbaş, 2021; Morrissey et al., 2019; Nickerson & Masarik, 2010; Şen Zeytun et al., 2010; Wilson et al., 2024). Finally, teachers in this study also included possible affective responses students might present when engaging with a task. Though less documented in the literature, this result is like Wilson et al.'s (2024) conclusion that teachers do consider responses focused on the emotional and physical responses of students to a task. The differences in findings of this study and Wilson et al.'s (2024) are explored in the next section.

Instructional practices, including anticipating students' mathematics, accomplish one or more goals for teachers, and their purposes for anticipating were investigated as a component of practice. Teachers in this study discussed why they anticipated and how their anticipations were helpful in their teaching, and results from my analysis suggest an association between their purposes for anticipating and the specific anticipations teachers identified for their students (also discussed in the next section). Scholars investigating instructional practices have noted that teachers have varying and sometimes multiple goals for their instructional moves and decisions (e.g., Jacobs & Spangler, 2017). While varied, these individual goals are derived from a more general orientation to content and instruction (Thompson et al., 1994) and what teachers believe it means to know and do mathematics (Ernest et al., 2016). The findings of this study suggest that teachers' purposes for anticipating students' mathematics shape how they believe students might engage with cognitively demanding tasks.

Finally, results of this study suggest that teachers draw upon knowledge and curricular resources when enacting their practice of anticipating students' mathematics. First, case teachers used different domains of their mathematical knowledge for teaching (Ball et al., 2008) to anticipate students' mathematics. Ball et al. (2008) include anticipating student's mathematics in their "knowledge of content and students" domain, and results from my analysis show that teachers drew upon their knowledge and experiences with students as a resource for their practice of anticipating. Teachers in this study also drew upon other domains of knowledge outlined in the MKT framework, including "horizon content knowledge" and "knowledge of content and curriculum" when situating their selected task and preparing to anticipate. Second, findings suggest that some teachers may use curricular resources when anticipating students' mathematics in perhaps unexpected ways. While Hillary focused on her curriculum and its previous development of strategies and representations when anticipating, she did not refer to the anticipated student strategies provided by the curriculum. These results suggest that teachers may draw upon multiple domains of knowledge as well as material resources as resources when anticipating students' mathematics.

Variations in Aspects of Secondary Mathematics Teachers' Practice of Anticipating

This study also sought to identify meaningful variations in secondary mathematics teachers' practices of anticipating, and results of my cross-case analysis highlighted several commonalities and distinctions in the actions they take, purposes they have, and resources they use when enacting the practice.

Types of Anticipations

There were several similarities among the types of anticipations that the three case teachers in this study identified. To some degree, all three teachers included ways that students might reason, various procedural methods they might use, and challenges that students might encounter when anticipating. Two of the three also identified multiple representations that students might create when engaging with their selected tasks. Similar to other studies of elementary and middle grades mathematics teachers' anticipations (Kartal et al., 2020; Krause et al., 2016; Şen Zeytun et al., 2010), findings from this study indicate that secondary mathematics teachers also include a variety of solutions, representations, and strategies as well as hypotheses about students' mathematical thinking such as ways of reasoning and types of difficulties they might encounter when considering how students engage in cognitively demanding mathematics tasks. Results from this study provide additional evidence to Wilson and colleagues' (2024) findings that students' mathematical actions and students' mathematical thinking are major foci of secondary mathematics teachers' anticipations.

Beyond these similarities, findings from this study also indicate that some teachers consider students' affect when anticipating. Unlike Aimee or Kathryn, Hillary included discussions of how students might feel when first engaging with the tasks she selected. Like Wilson et al. (2014) and Janike (2019), these results suggest that beyond mathematical activity and thinking, teachers may consider students' emotional responses to cognitively demanding mathematics tasks when anticipating.

In addition, Hillary also considered how students might record and organize their thoughts as a source of confusion or difficulty. Previously unidentified in the literature, this type of anticipation may be related to the challenges teachers believe students might have when engaging with complex mathematics tasks and actions they might take during instruction to mitigate their struggle. Wilson et al. (2024) also included teachers' responses to how students might engage with a task as a significant focus of their anticipations. By imagining ways that her students might get confused if their work was not recorded in an organized way, Hillary may have been preparing to support her students in avoiding difficulties or errors. Von Glasersfeld (1998) discusses how anticipations of events can lead to changes in action prior the event to increase or decrease its likelihood, and this result adds additional evidence to Wilson and colleagues' claim that anticipations may inform instructional decisions to prevent particular difficulties students might have when engaging with cognitively demanding mathematics tasks.

Purposes of Anticipating and for Using Cognitively Demanding Mathematics Tasks

Results of my analysis identified distinct purposes for each teacher's practice of anticipating students' mathematics. Hillary's purpose for anticipating centered around connecting her instruction to the curriculum and its coherence. Aimee anticipated to remove barriers for her students and intentionally sequence to set up her students for success. Kathryn's anticipations assisted her in providing students access to the task, advancing student thinking towards her learning goal, and balancing student exploration with the constraints of instructional time. None of the investigations reviewed to inform this study focused on teachers' rationales for anticipating students' mathematics, and my findings indicate that the purposes teachers have for anticipating are varied and suggest that their purposes may explain differences in the actions they take and resources they use when enacting the practice.

The Role of Cognitively Demanding Tasks

Munter's (2014) notion of instructional vision—in particular, a teacher's vision for the role that mathematics tasks play in supporting student learning—is useful in understanding the differences in purposes identified by my analysis. Munter's (2014) rubrics allowed for an analysis of teacher's interpretation of how tasks are used and distinguishing between task form (quality and nature) versus their function (intended outcome). At the foundational level, teachers may not distinguish between high and low-quality tasks or their role in instruction as defined by Stein et al. (1996). As teachers develop a more complex vision for the role of a task, at first there is a shift from seeing tasks as mere procedural practice to recognizing them as reform-oriented activities for engagement. When a teacher's functional view of tasks begins to match their form, view tasks as opportunities to foster conceptual understanding, offer multiple solution paths, and encourage active mathematical engagement, moving beyond mere engagement to support deep learning and connections between mathematical ideas.

Anticipating students' mathematics assisted Kathryn in planning for instruction where all students could access complex mathematics tasks, develop and use mathematical ideas that were the focus of her lesson, and provide opportunities for mathematical exploration. These goals for instruction suggest that she held what Munter (2014) would describe as a "sophisticated" vision of the role mathematics tasks play in fostering learning. For Kathryn, complex mathematics tasks support students' conceptual understanding and provide opportunities for "doing mathematics" (e.g., connecting representations, forming and investigating conjectures, providing explanations and justifications). Together, her understanding of how mathematics tasks support learning and goal of preparing for instruction that creates these kinds of opportunities for students led to types of anticipations that included ways to connect mathematical representations, different forms of mathematical reasoning, and particular conceptual challenges with which students might struggle in productive ways.

For Aimee, anticipating allowed her to identify potential points of struggle for her students and order mathematics tasks in a way that would limit student struggle so they might experience success. Rather than viewing mathematics tasks as opportunities to engage in problem solving, making connections, and exploration, Aimee seemed to see mathematics tasks as an opportunity for students to practice procedures with a goal of finding the correct solution. This less sophisticated vision of the role of mathematics tasks and her desire to remove barriers may explain why she most commonly anticipated particular challenges to student engagement and progress.

Hillary anticipated to support students in experiencing her curriculum coherently, and my analysis revealed no evidence that she viewed mathematics tasks as an aspect of instruction that could be manipulated or that task quality might be related to student learning. This view of mathematics tasks as a feature of instruction prescribed by an external authority and the lack of understanding of their function in promoting learning are characteristics of less sophisticated visions of mathematics tasks and may explain Hillary's focus on the representations and strategies from her curriculum as well as her attention to students' affect and organization.

Though beyond the scope of this study, my findings suggest a relationship between a teacher's purpose for anticipating and their understanding of the role that mathematics tasks play in fostering learning. Additional research is needed to better understand the relationships between instructional vision and the practice of anticipating.

Global and Local Anticipations

For teachers in this study, the variation in their purposes for anticipating also related to the grain size of their anticipations. Kathryn's anticipations were global in nature. When anticipating, she considered the entire task and brainstormed various strategies and representations students might use to engage with and complete the task. This holistic approach to anticipating aligns with her purpose of anticipating—to support students in mathematical explorations of cognitively demanding tasks. In contrast, Aimee and Hillary's anticipations concerned smaller chunks of the tasks they had selected. Their local anticipations included specific errors or ways students might reason about specific questions embedded within the larger task. This specific and isolated approach to anticipating may have allowed them to prepare targeted responses or scaffolds to overcome barriers in Aimee's case or make connections to previous lessons in Hillary's.

Though not an explicit focus of my examination, the global-local distinction in teachers' anticipations emerged from a comparison of their purposes for enacting the practice. Additional research investigating the granularity of teachers' anticipations, its relation to their practice of anticipating, and the extent to which these approaches support teachers in maintaining the cognitive demand of tasks during implementation (Jackson et al., 2013; Stein & Lane, 1996) is needed.

Cognitive Demand

The comparison of purposes for anticipating suggests a third area for future investigation—the relationship between teachers' practices of anticipating students' mathematics when engaging with cognitively demanding tasks and the extent to which the demands of the task are maintained when implemented in instruction. For Kathryn, anticipating was a means of preparing to support students in developing conceptual understanding, making connections, and engaging in mathematical practices with a complex task. With "T"asks, she planned to introduce the tasks and then provide time for small groups of students to collaboratively make sense of the mathematics and engage by using a variety of strategies and representations. Though she planned to facilitate "t"asks with more scaffolded support, Kathryn's anticipations assisted her in planning to provide opportunities for students to make sense of the tasks and valid strategies while guarding instructional time. For Kathryn, anticipations for "t"asks helped her maintain some of the complexity of thinking afforded by the tasks even if not at the level of "T"asks.

In contrast, Aimee's anticipations helped her identify difficulties to avoid and order instructional tasks so that students experienced success. Hillary's anticipations assisted her in planning to focus students' ideas on the strategies and representations being developed by her curriculum. Though the tasks both teachers selected for the study had the potential to engage students in making connections among representations, concepts, and approaches, their anticipations and stated purposes for the practice suggest their cognitive demand would decline during implementation. Previous research has documented how teachers tend to decrease the complexity of mathematics tasks during instruction (Jackson et al., 2013; Stein & Lane, 1996), and further research is needed to understand the extent to which the practice of anticipating supports teachers in maintaining a task's cognitive demands during implementation.

Mathematical Knowledge for Teaching as a Resource

Finally, results of my analysis identified similar and distinct resources teachers use anticipating students' mathematics. As discussed in the previous section, Hillary, Aimee, and Kathryn each relied on their prior experiences teaching students mathematics, the *Knowledge of Content and Students* domain of the Mathematical Knowledge for Teaching (MKT) framework (Ball et al., 2008). Though Ball et al. (2008) locate anticipating students' mathematics in this domain, evidence from my analysis suggests that teachers also draw upon other domains of their MKT as a resource for anticipating. Hillary relied heavily upon her curricular program when anticipating students' mathematics, while Aimee used her vertical understanding of middle grades and high school mathematics courses when anticipating. Kathryn attributed her elaborate and comprehensive anticipations to her extensive mathematics preparation. In what follows, I discuss the various domains of MKT that teachers in this study used as resources when anticipating students' mathematics.

Knowledge of Content and Students

It is well documented that anticipating students' mathematics is more difficult for pre-service and beginning teachers (Asquith et al., 2007; Didiş & Erbaş, 2021; Şen-Zeytun et al., 2010) just as it was more difficult for Hillary in this study. Teachers with significant experience like Aimee and Kathryn have engaged students in mathematics and developed a more robust domain of knowledge upon which they can draw upon to anticipate how students might engage with particular mathematics tasks. While this *knowledge of content and students* (Ball et al., 2008) is necessary in learning to anticipate in ways that support equitable facilitation of mathematical tasks, it is also well documented that teachers' expectations of students impact student outcomes both in achievement and identity (Brophy & Good, 1974; Brophy, 1985; Hurwitz, Elliott, & Braden, 2007; Jussim et al., 1996; Rosenthal & Jacobson, 1968; Rubie-Davies & Peterson, 2016; Schrank, 1968), particularly for students from marginalized groups. While *knowledge of content and students* is an essential resource for teachers when anticipating students' mathematics, future research should examine the relationships among teachers' expectations, their anticipations of students' mathematics, and instruction.

Knowledge of Content and Curriculum

Just as Hillary had less experience to draw upon when anticipating compared to Aimee and Kathryn, researchers have shown that teachers with less classroom experience tend to have less knowledge of students and curricula to draw upon when anticipating (Asquith et al., 2007; Didiş & Erbaş, 2021; Şen-Zeytun et al., 2010). Hillary, a beginning teacher at the time of the study, relied on her curricular materials—her *knowledge of content and curriculum* (Ball et al., 2008)—as a primary resource for anticipating. She referred to common representations and strategies emphasized by her curricular program to anticipate different strategies students might use to solve the tasks. At the same time, Hillary spent little time examining the possible strategies students might use included in the program's teacher notes. While some curricular programs are now incorporating anticipations of common strategies used by students to support the development of the teacher's own mathematical knowledge for teaching (e.g., Drake et al., 2014) and research shows that educative curricula can assist teachers in anticipating and interpreting student thinking (Stein et al., 2007), additional research is needed to understand if and how teachers use these aspects of mathematics curricula and the extent to which learning from educative curricula leads to changes in instruction and student learning.

Horizon Content Knowledge

Horizon content knowledge—knowledge of how concepts and representations are connected over the span of mathematics, what students are likely to know and be able to do from previous courses, and what ideas will build from and extend current learning (Ball et al., 2008)—was a resource that Aimee used when anticipating how students might engage with the tasks she selected. Aimee relied on prior experiences teaching previous mathematics courses when anticipating what prior knowledge students might use, strategies they might have in their repertoire, and obstacles they might face when engaging with her selected tasks. Some researchers have documented how an understanding of the ways mathematical ideas develop over time can assist teachers in focusing on what students know and bring to instruction (Mojica, 2010; Myers, 2014; Wilson, 2009) and support them in anticipating students' mathematics (Edgington, 2012; Wilson et al., 2015). Others have noted that teachers tend to underestimate what students know and understand when anticipating (Lin, 2016; Lin & Chiu, 2010). Additional research investigating the affordances and potential limitations of horizon content knowledge as a resource for anticipating is needed.

Common and Specialized Content Knowledge

Though all teachers in this study used their mathematical knowledge as a resource for anticipating, Kathryn's explicitly and repeatedly identified her advanced mathematical background as her primary resource for anticipating. Though studies have shown a positive relationship between teachers' content knowledge and their instructional quality (Hill et al., 2008) and their students' achievement (Hill et al., 2005), the role of that *common and specialized content knowledge* plays in specific instructional practices remains under researched. Future investigations might explore the ways robust content knowledge supports teachers in developing anticipations of nonroutine approaches and mathematical connections students might make as well as the extent to which teachers' content knowledge is enhanced through the practice of anticipating.

Implications

The results of this study and the questions raised in the previous section have several implications for mathematics teacher educators and mathematics teacher education researchers. For mathematics teacher educators, one of the key implications from this study is that the practice of anticipating students' mathematics is taken up and enacted in different ways. Mathematics teacher educators who engage in the work of preparing teachers or supporting teachers in improving their practice through a focus on high-quality and equitable mathematics instructional practices in general—and anticipating students' mathematics specifically in the context of Smith and Stein's (2011) *5 Practices* framework—should be mindful of these differences. When designing professional learning experiences, mathematics teacher educators should plan opportunities for teachers to consider not just how to anticipate student responses but also what resources are available to support them and what they hope the practice will accomplish for them. When facilitating professional learning experiences or providing instructional support, mathematics teacher educators should be mindful that existing teachers' knowledge, instructional practices and goals, and professional contexts all shape the meanings teachers make of the practice of anticipating and thus the impact the practice will have on their instruction and their students' learning. Most importantly, teachers' purposes for the practice of anticipating may or may not align with those of the broader mathematics education community, and creating opportunities for teachers to grow in their teaching should include explicit attention to surfacing and problematizing instructional aims that curtail productive struggle, originate from deficit perspectives of students, or communicate low expectations. In particular, professional learning designers and facilitators should explicitly address the potential for anticipations to reduce the cognitive demands of a mathematics tasks during implementation and support teachers in developing responses to ensure that students remain engaged in ways that advance their learning.

For mathematics teacher education researchers, one implication of this study is the challenge of operationalizing an instructional practice for empirical investigation. Many contemporary theories conceptualize one or more practices in relation to professional communities from which they derive meaning and find value, and I initially framed this study to include such components to characterize secondary mathematics teachers' practices of anticipating. However, teaching in contemporary schools offers few opportunities for teachers to develop professional communities focused on instruction, and I found it difficult to identify any meanings or values that teachers derived from their communities at the level of a specific instructional practice. Focusing on the actions teachers take, their ultimate purpose for taking those actions, and the resources they use to accomplish these goals proved to be a useful way of identifying salient aspects of teachers' practices of anticipating and for drawing meaningful contrasts. Second, my results suggest that investigations of the practice of anticipating would benefit from research designs that attend to aspects of student learning beyond cognition, specifically affect, and the ways teachers plan to use their anticipations during instruction. Finally, my focus on the purposes teachers have for enacting the practice of anticipating provided a means to better understand differences in teachers' practices. Other mathematics teacher education researchers seeking to characterize instructional practices should include teachers' underlying orientations toward teaching, learning, and mathematics as objects of their investigations.

Limitations

This study's multi-case approach provided an in-depth investigation of three case teachers' practices of anticipating students' mathematics. While this methodology is invaluable for understanding the nuances and complexities of the practice of anticipating in real-world contexts, it is important to recognize the bounds of knowledge generated by this approach. The findings of this investigation should be interpreted with an understanding of the purpose and scope of this study. Multi-case studies focus on depth and detail rather than generalizability (Stake, 1995); therefore, the insights gained are best understood as specific to the three cases explored, examples of existence rather than patterns of generality (Flyvbjerg, 2004), and as advisory in nature (Plomp, 2010) for those seeking to design professional learning opportunities for teachers.

One boundary of the knowledge generated by this study stems from its exclusive focus on the practice of anticipating during teacher's planning without investigating the how teachers used their anticipations during instruction. Although this focus has yielded valuable understandings around teachers' preparation for implementing high-quality and equitable mathematics instruction, the research design did not include a focus on how teachers used their anticipations to implement cognitively demanding mathematics tasks in their classroom. Consequently, no claims regarding the impact of teachers' anticipations on maintaining the cognitive demands of tasks while facilitating student exploration and discourse can be made. Second, the findings of this study suggest a possible relationship between teachers' mathematical knowledge for teaching and their practice of anticipating. However, this study did not specifically investigate teachers' knowledge. Measures of teachers' knowledge were not included, and relations among different domains of teachers' mathematical knowledge for teaching and their practices of anticipating students' mathematics remains an empirical question.

Conclusion

In current realities of mathematics education reform where Smith and Stein's (2008) framework is prominent in both educative curricular resources and professional learning initiatives, the practice of anticipating should not be taught in silo from topics such as students' mathematical identity and agency and without considering the relationship to teachers' instructional visions. With good intentions, the mathematics teacher education community has adopted Smith and Stein's (2011) framework as a fundamental tool for improving mathematics instruction. However, an exclusive focus on discrete instructional practices in the absence of attention to creating spaces where students are seen as valued contributors of mathematics in a way that disrupts spaces of marginality inside of the mathematics classroom (Aguirre et al., 2013) runs a risk of reinforcing an idea that supporting student learning can be reduced to technical skills that respond to "Common Misconceptions" documented in curricular resources.

While contributing to the field's understanding of the practice of anticipating students' mathematics, specifically for secondary mathematics teachers, this study identifies areas in need of additional investigation. Research examining the relations among teachers' anticipations, knowledge, expectations of students, professional contexts, and implementation of cognitively demanding instructional tasks would deepen our understanding of the complexities inherent in the practice of anticipating students' mathematics. This study provides a foundation for such investigations by highlighting commonalities and distinctions of three secondary mathematics teachers' actions, purposes, and resources for anticipating students' mathematics.

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APPENDIX A: CYCLE 0 INTERVIEW PROTOCOL

Cycle 0 Interview Protocol

Introduction

Thank you for participating in this study, I know your time is valuable. This interview will be recorded so that I can pay more attention to the conversation instead of focusing on my notes. The overall total time for the interviews will last around 30 minutes. I am conducting a study on planning math tasks and am curious about your past experiences. This interview will just get a feel for your background and thoughts about planning for instruction.

Probing Questions
What courses are you currently teaching?
How many years have you been teaching?
Tell me about your teaching style.
How do you plan for math instruction?
There are a set of practices for teaching math called the “Five Practices.” They consist of five practices for orchestrating math discussions including: anticipating, monitoring, selecting, sequencing, and connecting. Are you familiar with these practices?
If so, in what ways have you engaged with the Five Practices? For example, have you read books, attended workshops, used them in your own teaching?
What’s a teaching practice that you do all the time? Describe it.

APPENDIX B: CYCLE 1 THINK-ALOUD AND INTERVIEW PROTOCOL

Cycle 1 Think-Aloud and Interview Protocol

Interview Guide:
This is a semi structured interview. I have prepared focus questions, but some questions may be omitted, and some questions may be added to elicit a deeper understanding, as per the interview protocol. During the think-aloud I will also take notes of different aspects of the planning process on which I would like to probe the teacher's thinking. During the interview I will ask: "I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?"

- Participants will be asked to choose a cognitively demanding task to teach to their students.
- After selecting, participants will be asked to engage in a recorded think aloud to plan for implementing the task.

Thank you for participating in this study and choosing a task to plan with me!

Probing Questions
Can you tell me about the task you have chosen to implement?
Why did you select this task?
Have you already contemplated this task and how you might implement it before meeting today? If so, what are your initial thoughts?

"I am going to ask you to think aloud as you plan to implement this task. I would like for you to plan as you normally would for this task. If you would like to type or write anything down as you plan, please do so. Whatever you feel is most helpful for you. As you plan for this task, just talk me through what you are thinking about as you plan for this task. I will not ask many questions during this process but may prompt you to keep talking or explain something I see you write down. Any questions?"

- I will take note of "noticings" during the think aloud that I would like to explore in more depth during the interview.
- If participants get quiet, I will prompt them with phrases such as "Keep talking!"

"Thank you for talking me through your process of planning today! I have a few follow up questions…"

Probing Questions
"I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?"
"I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?"
"I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?"
"I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?"
"I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?"
How would you summarize the process of planning you just talked me through? In other words, how would you describe the different aspects you just thought about in terms of implementing this task?
What aspects of implementing tasks do you focus on when planning (either informally or formally) to teach a task? In other words, are there some things you always think about before implementing the task?
How do you think your planning today may affect your implementation of this task?
Why did you choose this task to implement with your students?
What does "planning for a math task" mean to you?
What are your goals during planning for mathematics tasks?
What are any past or current experiences or resources that you use to plan for implementing a task?
What importance do you place on planning for implementing math tasks?
How typical is this type of planning in your daily practice?

APPENDIX C: PHASE 2 THINK ALOUD/INTERVIEW

Phase 2 Think Aloud/Interview

Interview Guide

This is a semi structured interview. I have prepared focus questions, but some questions may be omitted, and some questions may be added to elicit a deeper understanding, as per the interview protocol. During the think-aloud I will also take notes of different aspects of the planning process and practice of anticipating on which I would like to probe the teacher's thinking. During the interview I will ask: "I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?"

1. Participants will be asked to choose a cognitively demanding task to teach to their students.
2. After selecting, participants will be asked to engage in a recorded think aloud to anticipate student thinking on the math task and plan for implementation.

   Thank you for participating in this study and choosing a task to plan with me!

Can you tell me about the task you have chosen to implement?
Why did you select this task?
Have you already contemplated this task and how you might implement it before meeting today? If so, what are your initial thoughts?

"Today I am going to ask you to think aloud as you plan to implement this task. Today we are focusing on anticipating students' mathematics. I would like for you to sketch out your anticipations however you choose while you plan today. If you would rather do that on the computer or on paper it is up to you. As you anticipate students' mathematics ~~student thinking~~ about this task, just talk me through what you are thinking about as you record your anticipations. I will not ask many questions during this process but may prompt you to keep talking or explain something I see you write down. Any questions?"

3. I will take note of "noticings" during the think aloud that I would like to explore in more depth during the interview.
4. If participants get quiet, I will prompt them with phrases such as "Keep talking!" or "Can you explain why you are doing that?"
5. An interview will follow the think aloud.

“Thank you for talking me through your process of anticipating student thinking today! I have a few follow up questions…”

Probing Questions
“I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?”
“I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?”
“I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?”
“I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?”
“I noticed that you…, can you explain how thinking about this now will help you implement this task in real time with your students?”
How would you summarize the process of anticipating student thinking on this math task that you just talked me through? In other words, what all did you think about in order to anticipate student thinking on the task?
How do you think your anticipations today may affect your implementation of this task?
Why did you choose this task to implement with your students?
What does the practice of anticipating student mathematics mean to you?
What are your goals when anticipating kids’ thinking? Why would you take the time to do it?
What are any past or current experiences or tools that you use to anticipate student thinking?
What importance do you place on anticipating student thinking when implementing math tasks?
Are there ways that you anticipate students’ math in the past without going through this process that we went through today?
Today we sat down and spent a good amount of time thinking hard about anticipations of students’ mathematics. How does this practice rank on your priority list when planning for tasks?
How typical is this type of planning?

APPENDIX D: THEMES, DEFINITIONS, AND EXAMPLES

Themes, Definitions, and Examples

Thematic Codes for Actions	Definition	Example
Multiple Representations	Different representations of modeling mathematical situations (i.e. tape diagrams, hanger diagrams, algebraic equations, graphs, tables)	*"Some people would draw the tape diagram for this problem. Some of the kids who are more visual learners do really well with the tape diagram versus some of the kids will just try and write an equation and not draw the tape diagram. I could see someone drawing hanger diagrams too."* (Hillary, Cycle 2)
Student Reasoning	What students notice about the task and how they will make sense of the mathematics they are presented with throughout the exploration and discussion.	*"So probably, then some of the students would make a table. And they would say, for cases, case 1 is 4, case 2 is 9, case 3 is 16. Okay. And so go all the way down to 100. That's where we have to go. So I would think some students would see that 4 is a perfect square right away. That 9 is a perfect square. So 4 is the same thing as. 9 is the same thing as 16 is the same thing as. And so they might start thinking... "How can I use that pattern to write some sort of rule for my 100th case?"* (Kathryn, Cycle 1)
Procedural Methods	Procedural methods that students have learned from a teacher or another resrouce that could be used to help solve a task .	*"They can use their skills that I've already taught them on how to do substitution and elimination, and be able to work it out to find the price of a bag of potato chips and one candy bar. So this is where they would set up their equations and be able to solve."* (Aimee, Cycle 1)
Difficulty	Difficulties students will have of making sense of the mathematics of the task or solution	*"I was looking at all the different patterns that I see in this table. Because doing n+1 and squaring the quantity is a little bit of a stretch for my students I think. It's not linear, it's a little bit harder to visualize."* (Kathryn, Cycle 1)

Thematic Codes for Actions	Definition	Example
Organizational	Ways students will organize their written work on a task.	*"Honestly, for my kids, I would see them more either writing the equation here [pointing to the diagram] like we did or writing it right here [pointing to the blank space intended for the answer]. They're probably not going to write it in both places. I need to make sure they are clear what equation goes with this model so when they look back later they won't be confused about what they did."* (Hillary, Cycle 2)
Affect	Feelings students may have while working through the task.	*"First off, I feel like the kids are going to look at this and say that's a lot of words for this. It looks hard... it's a pretty long question. So I would probably definitely be reading the first one to them. And my suggestion to them would be to go back and read it on their own and underline some of the important information…"* (Hillary, Cycle 2)

APPENDIX E: HILLARY'S CYCLE 1 TASK

Hillary's Cycle 1 Task

10.2 Shopping in Two Different Cities

Different cities have different sales tax rates. Here are the sales tax charges on the same items in two different cities. Complete the tables.

City 1

item	price (dollars)	sales tax (dollars)	total cost (dollars)
paper towels	8.00	0.48	8.48
lamp	25.00	1.50	
pack of gum	1.00		
laundry soap	12.00		
	x		

City 2

item	price (dollars)	sales tax (dollars)	total cost (dollars)
paper towels	8.00	0.64	8.64
lamp	25.00	2.00	
pack of gum	1.00		
laundry soap	12.00		
	x		

APPENDIX F: HILLARY'S CYCLE 2 TASKS

Hillary's Cycle 2 Tasks

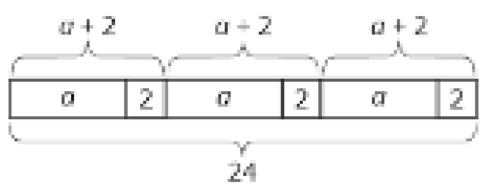
Lesson 11
Using Equations to Solve Problems

Let's use tape diagrams, equations, and reasoning to solve problems.

Learning Targets:

- I can solve story problems by drawing and reasoning about a tape diagram or by writing and solving an equation.

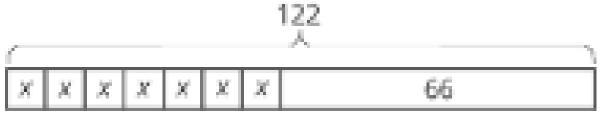
11.1 Remember Tape Diagrams



1. Write a story that could be represented by this tape diagram.
2. Write an equation that could be represented by this tape diagram.

11.2 At the Fair

1. Tyler is making invitations to the fair. He has already made some of the invitations, and he wants to finish the rest of them within a week. He is trying to spread out the remaining work, to make the same number of invitations each day. Tyler draws a diagram to represent the situation.



a. Explain how each part of the situation is represented in Tyler's diagram:

How many total invitations Tyler is trying to make.

How many invitations he has made already.

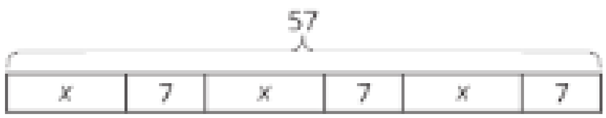
How many days he has to finish the invitations.

b. How many invitations should Tyler make each day to finish his goal within a week? Explain or show your reasoning.

c. Use Tyler's diagram to write an equation that represents the situation. Explain how each part of the situation is represented in your equation.

d. Show how to solve your equation.

2. Noah and his sister are making prize bags for a game at the fair. Noah is putting 7 pencil erasers in each bag. His sister is putting in some number of stickers. After filling 3 of the bags, they have used a total of 57 items.



a. Explain how the diagram represents the situation.

b. Noah writes the equation $3(x+7)=57$ to represent the situation. Do you agree with him? Explain your reasoning.

c. How many stickers is Noah's sister putting in each prize bag? Explain or show your reasoning.

3. A family of 6 is going to the fair. They have a coupon for $1.50 off each ticket. If they pay $46.50 for all their tickets, how much does a ticket cost without the coupon? Explain or show your reasoning. If you get stuck, consider drawing a diagram or writing an equation.

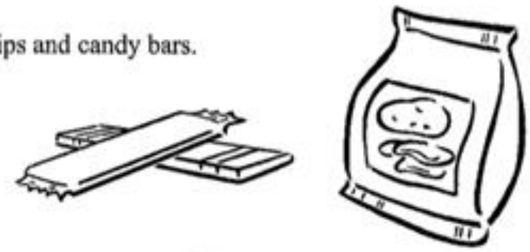
APPENDIX G: AIMEE'S CYCLE 1 TASK

Aimee's Cycle 1 Task

Buying Chips and Candy

This problem gives you the chance to:
- form and solve a pair of linear equations in a practical situation

Ralph and Jody go to the shop to buy potato chips and candy bars.



x = p.c
y = c.b

p b

Ralph buys 3 bags of potato chips and 4 candy bars. He spends $3.75.

Jody buys 4 bags of potato chips and 2 candy bars. She spends $3.00.

Later Clancy joins Ralph and Jody and asks to buy one bag of potato chips and one candy bar from them. They need to work out how much he should pay.

Ralph writes $3p + 4b = 375$

1. If p stands for the cost, in cents, of a bag of potato chips and b stands for the cost, in cents, of a candy bar, what does the 375 in Ralph's equation mean?

2. Write a similar equation, using p and b, for the items Jody bought.

$4p + 2b = 300$

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3. Use the two equations to figure out the price of a bag of potato chips and the price of a candy bar.

Potato chips______________

Candy bar ______________

Show your work.

$3p + 4b = 3.75$ $4p + 2b = 3.00$

4. Clancy has just $1. Does he have enough money to buy a bag of potato chips and a candy bar?

____

Explain your answer by showing your calculation.

____

____

____

7

APPENDIX H: AIMEE'S CYCLE 2 TASKS

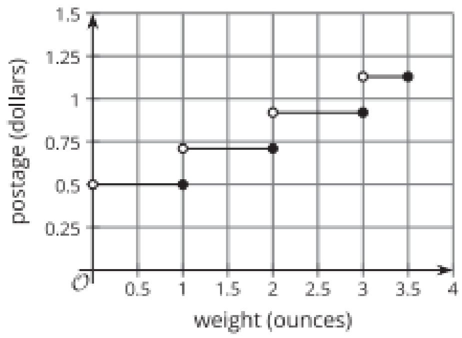
Aimee's Cycle 2 Tasks

Illustrative Mathematics

12.2: Postage Stamps

The relationship between the postage rate and the weight of a letter can be defined by a **piecewise function**.

The graph shows the 2018 postage rates for using regular service to mail a letter.



1. What is the price of a letter that has the following weight?

   a. 1 ounce

   b. 1.1 ounces

   c. 0.9 ounce

2. A letter costs \$0.92 to mail. How much did the letter weigh?

3. Kiran and Mai wrote some rules to represent the postage function, but each of them made some errors.

$$K(w) = \begin{cases} 0.50, & 0 \leq w \leq 1 \\ 0.71, & 1 \leq w \leq 2 \\ 0.92, & 2 \leq w \leq 3 \\ 1.13, & 3 \leq w \leq 3.5 \end{cases} \qquad M(w) = \begin{cases} 0.50, & 0 < w < 1 \\ 0.71, & 1 < w < 2 \\ 0.92, & 2 < w < 3 \\ 1.13, & 3 < w < 3.5 \end{cases}$$

   Identify the error in each person's work and write a corrected set of rules.

12.3: Bike Sharing

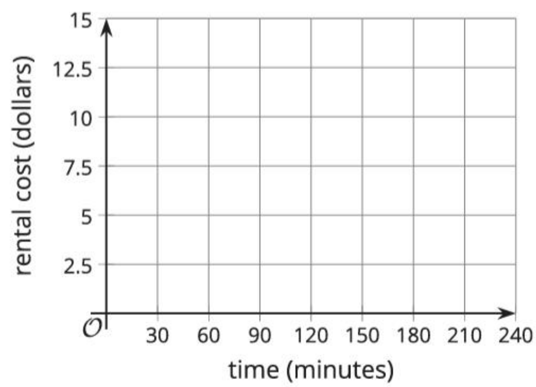
Function C represents the dollar cost of renting a bike from a bike-sharing service for t minutes. Here are the rules describing the function:

$$C(t) = \begin{cases} 2.50, & 0 < t \le 30 \\ 5.00, & 30 < t \le 60 \\ 7.50, & 60 < t \le 90 \\ 10.00, & 90 < t \le 120 \\ 12.50, & 120 < t \le 150 \\ 15.00, & 150 < t \le 720 \end{cases}$$

1. Complete the table with the costs for the given lengths of rental.

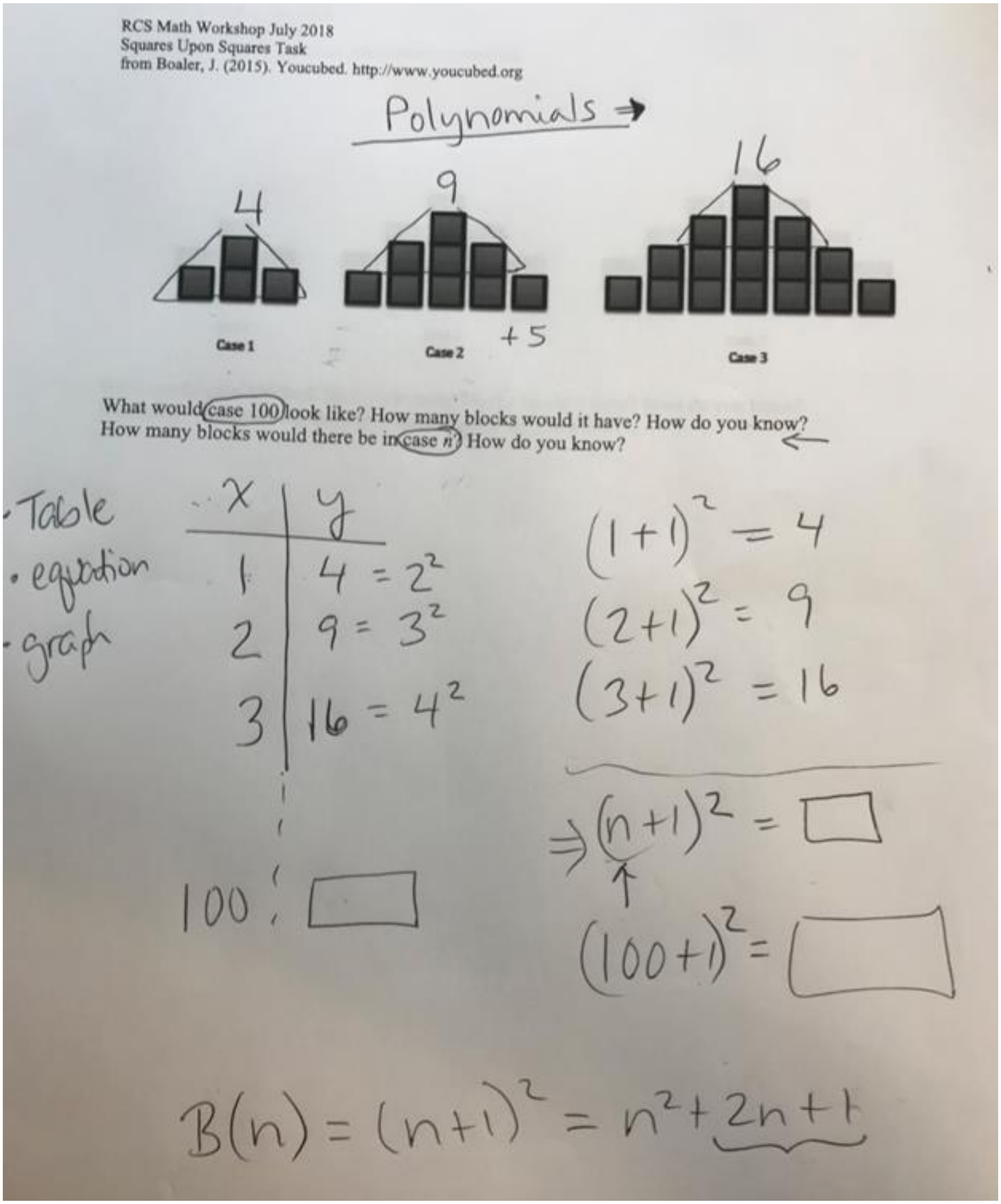
t (minutes)	C (dollars)
0	
10	
25	
60	
75	
130	
180	

Sketch a graph of the function for all values of t that are at least 0 minutes and at most 240 minutes.



APPENDIX I: KATHRYN'S CYCLE 1 TASK

Kathryn's Cycle 1 Task



RCS Math Workshop July 2018
Squares Upon Squares Task
from Boaler, J. (2015). Youcubed. http://www.youcubed.org

Polynomials →

4 9 16

Case 1 Case 2 +5 Case 3

What would case 100 look like? How many blocks would it have? How do you know?
How many blocks would there be in case n? How do you know?

- Table
- equation
- graph

x	y
1	$4 = 2^2$
2	$9 = 3^2$
3	$16 = 4^2$
100	☐

$(1+1)^2 = 4$

$(2+1)^2 = 9$

$(3+1)^2 = 16$

$\Rightarrow (n+1)^2 = $ ☐

$(100+1)^2 = $ ☐

$B(n) = (n+1)^2 = n^2 + 2n + 1$

APPENDIX J: KATHRYN'S CYCLE 2 TASKS

Kathryn's Cycle 2 Tasks