A simplified representation of the covariance structure of axially symmetric processes on the sphere

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Abstract:

Spatial processes having covariance functions that depend solely on the distance between locations are known as homogeneous. Many random processes on the sphere are not homogeneous, especially in the latitudinal dimension. As a result, we study a class of statistical processes that exhibit axial symmetry, whereby their covariance function depends on differences in longitude alone. We develop a new and simplified representation for a valid axially symmetric process, reducing computational complexity considerably. In addition, we explore longitudinally reversible processes and the construction of parametric models for axially symmetric processes.

Keywords: Associated Legendre polynomials | Longitudinally reversible process | Mercer's theorem | Spherical harmonics

Article:

1. Introduction

While random processes in Euclidean space are widely studied in the literature, global-scale processes are receiving increased attention, especially in the environmental and geophysical sciences. In order to study the spatial dependency in such processes, one needs to ensure that the covariance functions are valid on the sphere. Huang et al. (2011) examine the validity of the most popularly used covariance and variogram functions, demonstrating that valid covariance functions in Euclidean space are not necessarily positive definite on the sphere.

In the study of random processes on the sphere, homogeneity is often assumed (Yadrenko, 1983 and Yaglom, 1987). However, this assumption may not always be satisfied in practice (Cressie and Johannesson, 2008, Stein, 2007, Stein, 2008, Jun and Stein, 2007, Jun and Stein, 2008 and Bolin and Lindgren, 2011). For instance, because energy and moisture gradients are strongest from equator to pole, geophysical processes are most likely to exhibit symmetry on longitude rather than latitude. To overcome these problems, Jones (1963) introduces the concept of axial symmetry where the covariance function depends on the longitudes only through their difference. Stein, 2007 and Stein, 2008 has applied this approach to model total column ozone on a global scale, while Jun and Stein, 2007 and Jun and Stein, 2008 consider the axially symmetric process by applying the differential operators to an isotropic process. In this note, we obtain a simplified representation of a valid axially symmetric covariance function on the sphere.

2. Axial symmetry

We consider a complex-valued random process X(P) on a unit sphere S^2, where $P=(\lambda,\phi)\in S^2$ with longitude $\lambda\in[0,2\pi)$ and latitude $\phi\in[0,\pi]$. Assume the process is continuous in quadratic mean with respect to the location P, and has finite second moment, then X(P) can be represented by spherical harmonics, with convergence of the series in quadratic mean, (Li and North, 1997)

$$X(P) = X(\lambda, \phi) = \sum_{\nu=0}^{\infty} \sum_{m=-\nu}^{\nu} Z_{\nu,m} e^{im\lambda} P_\nu^m(\cos\phi),$$

where $P_\nu^m(\cdot)$ is a normalized associated Legendre polynomial so that its squared integral on $[-1,1]$ is 1, and $Z_{\nu,m}$ are the coefficients satisfying

$$Z_{\nu,m} = \int_{S^2} X(P) e^{-im\lambda} P_\nu^m(\cos\phi) dP.$$

Without loss of generality, the process is assumed to have mean zero, i.e., E(X(P))=0, which impliesE($Z_{\nu,m}$)=0. Then, the covariance function of the process at two locations $P=(\lambda_P,\phi_P)$ and $Q=(\lambda_Q,\phi_Q)$ is given by,

$$R(P,Q) = E(X(P)\overline{X(Q)}) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{m=-\nu}^{\nu}\sum_{n=-\mu}^{\mu} E(Z_{\nu,m}\overline{Z}_{\mu,n}) e^{im\lambda_P} P_\nu^m(\cos\phi_P) e^{-in\lambda_Q} P_\mu^n(\cos\phi_Q),$$

where $\overline{Z}$ denotes the complex conjugate of Z. Note that the continuity of X(P) on every point P implies that R(P,Q) is continuous on all pairs (P,Q) (Cramér and Leadbetter, 1967, p. 83). If the covariance function depends solely on the spherical distance between these two locations, the process is homogeneous. That is, (Obukhov, 1947 and Yaglom, 1961)

$$R(P,Q) = R(\theta(P,Q)) = \sum_{\nu=0}^{\infty} \frac{(2\nu+1)f_\nu}{2} P_\nu(\cos\theta(P,Q)),$$

where the spherical distance $\theta(P,Q)=\cos^{-1}(\sin\phi_P\sin\phi_Q+\cos\phi_P\cos\phi_Q\cos(\lambda_P-\lambda_Q))$, $P_\nu(\cdot)$ is the Legendre polynomial of order ν, $f_\nu\geq 0$, and $\sum_{\nu=0}^{\infty}(2\nu+1)f_\nu<\infty$. Here, the random variable $Z_{\nu,m}$ satisfies

$$E(Z_{\nu,m}\overline{Z}_{\mu,n})=\delta_{\nu,\mu}\delta_{n,m}f_\nu,$$

where $\delta_{a,b}=1$ if $a=b$, and 0 otherwise.

Under the assumption of axial symmetry (Jones, 1963), where the covariance function depends on the longitudes only through their difference, one has

$$E(Z_{\nu,m}\overline{Z}_{\mu,n})=\delta_{n,m}f_{\nu,\mu,m}.$$

Hence, the covariance function is of the form

equation(1)

$$R(P,Q)=R(\phi_P,\phi_Q,\lambda_P-\lambda_Q)=\sum_{m=-\infty}^{\infty}\sum_{\nu=|m|}^{\infty}\sum_{\mu=|m|}^{\infty}f_{\nu,\mu,m}e^{im(\lambda_P-\lambda_Q)}P_\nu^m(\cos\phi_P)P_\mu^m(\cos\phi_Q).$$

Further conditions on $f_{\nu,\mu,m}$ are imposed in order to have the covariance function valid. In particular, $f_{\nu,\mu,m}=\overline{f}_{\mu,\nu,m}$ and for each fixed integer m, the matrix $F_m(N)=\{f_{\nu,\mu,m}\}_{\nu,\mu=|m|,|m|+1,\ldots,N}$ must be positive definite for all $N\geq|m|$. A detailed discussion of parallel conditions on $f_{\nu,\mu,m}$ under the real-valued case is given in Jones (1963).

In Eq. (1), for each $m=0,\pm1,\ldots$, we let

equation(2)

$$C_m(\phi_P,\phi_Q)=\sum_{\nu=|m|}^{\infty}\sum_{\mu=|m|}^{\infty}f_{\nu,\mu,m}P_\nu^m(\cos\phi_P)P_\mu^m(\cos\phi_Q).$$

It is obvious that $C_m(\phi_P,\phi_Q)$ is Hermitian and positive definite on $[0,\pi]\times[0,\pi]$ based on the properties of $f_{\nu,\mu,m}$ and $F_m(N)$. In addition, note that $\lambda_P-\lambda_Q\in[-2\pi,2\pi]$ and 2π is the period of the complex exponential function, we adapt the angular distance definition in Wood (1995):

$$\Delta\lambda=\begin{cases}\lambda_P-\lambda_Q+2\pi, & \lambda_P-\lambda_Q<-\pi,\\ \lambda_P-\lambda_Q, & \lambda_P-\lambda_Q\in[-\pi,\pi],\\ \lambda_P-\lambda_Q-2\pi, & \lambda_P-\lambda_Q>\pi.\end{cases}$$

Hence, we have the following proposition.

Proposition 1.

Continuous axially symmetric processes on the sphere have the covariance function

equation(3)

$$R(P,Q) = R(\phi_P, \phi_Q, \Delta\lambda) = \sum_{m=-\infty}^{\infty} e^{im\Delta\lambda} C_m(\phi_P, \phi_Q),$$

where $\Delta\lambda \in [-\pi, \pi]$*, and* $C_m(\phi_P,\phi_Q)$ *is Hermitian and positive definite with* $\sum_{m=-\infty}^{\infty} |C_m(\phi_P, \phi_Q)| < \infty$*. On the other hand, if for any integer* $m=0,\pm1,\pm2,\ldots, C_m(\phi_P,\phi_Q)$ *is Hermitian and positive definite,* $\sum_{m=-\infty}^{\infty} |C_m(\phi_P, \phi_Q)| < \infty$*,* $R(P,Q)$ *given by* (3) *is a valid covariance function of an axially symmetric process on the sphere.*

We proceed to obtain a new presentation of the covariance structure. Since $C_m(\phi_P,\phi_Q)$ *is continuous and both Hermitian and positive definite, by Mercer's theorem (* Riesz and Sz-Nagy, 1990, *p. 245), there exists an orthonormal basis* $\{\psi_{m,\nu}(\cdot), \nu=0,1,\ldots\}$ *in* $L^2([0,\pi])$*, a complex-valued functional Hilbert space on* $[0,\pi]$*, such that*

$$C_m(\phi_P, \phi_Q) = \sum_{\nu=0}^{\infty} \eta_{m,\nu} \psi_{m,\nu}(\phi_P) \overline{\psi_{m,\nu}(\phi_Q)},$$

where $\eta_{m,\nu} \geq 0$ *are the eigenvalues,*

$$\int_0^{\pi} C_m(\phi_P, \phi_Q) \psi_{m,\nu}(\phi_Q) d\phi_Q = \eta_{m,\nu} \psi_{m,\nu}(\phi_P),$$

$\sum_{\nu=0}^{\infty} \eta_{m,\nu} < \infty$ *for each* m*, and the series expansion converges uniformly and absolutely. This leads to the following proposition.*

Proposition 2.

For each $m=0,\pm1,\ldots,$ *there exists an orthonormal basis* $\{\psi_{m,\nu}(\cdot)\} \in L^2([0,\pi])$ *such that the continuous axially symmetric covariance functions on the sphere can be expanded by*

$$R(P,Q) = \sum_{m=-\infty}^{\infty} \sum_{\nu=0}^{\infty} \eta_{m,\nu} e^{im\Delta\lambda} \psi_{m,\nu}(\phi_P) \overline{\psi_{m,\nu}(\phi_Q)},$$

where $\Delta\lambda \in [-\pi, \pi], \eta_{m,\nu} \geq 0, \sum_{m=-\infty}^{\infty} \sum_{\nu=0}^{\infty} \eta_{m,\nu} < \infty$.

Remark 2.1.

In both Jones (1963) and Stein (2007), the covariance function of the axially symmetric process on the sphere is represented in Eq. (1). That is, a triple summation is formulated in terms of associated Legendre polynomials with conditions on $f_{\nu,\mu,m}$ to ensure the validity of the covariance function. It is clear from Proposition 1 and Proposition 2 that if the eigenfunction basis $\{\psi_{m,\nu}(\cdot)\}$ is used, this representation can be reduced to a double summation. This greatly simplifies the representation of the covariance structure. In addition, Stein, 2007 and Stein, 2008 uses the truncated triple summation in estimating the covariance function, where $O(N^3)$ parameters need to be estimated with N being the number of truncated terms from(1), placing considerable limits on the choice of N. In light of Proposition 2, the number of parameters in our representation is reduced to $O(N^2)$, which reduces computational complexity greatly. In addition, the condition on $f_{\nu,\mu,m}$ in Stein's approach, where the matrix F_m has to be positive definite, further complicates the computation. However, from Proposition 2, only $\eta_{m,\nu}\geq 0$ are needed under this representation.

Remark 2.2.

The orthonormal basis $\{\psi_{m,\nu}(\cdot)\}$ in Proposition 2 depends on $C_m(\phi_P,\phi_Q)$, where

equation(4)

$$C_m(\phi_P,\phi_Q)=\frac{1}{2\pi}\int_{-\pi}^{\pi}R(\phi_P,\phi_Q,\Delta\lambda)e^{-im\Delta\lambda}d\Delta\lambda.$$

For example, if the associated Legendre polynomials happen to be the basis, the covariance function

$$R(P,Q)=\sum_{m=-\infty}^{\infty}\sum_{\nu=|m|}^{\infty}\eta_{m,\nu}e^{im\Delta\lambda}P_\nu^m(\cos\phi_P)P_\nu^m(\cos\phi_Q).$$

That is, the $f_{\nu,\mu,m}$ in Eq. (1) is reduced to $\delta_{\nu,\mu}\eta_{m,\nu}$. In practice, the choice of such a basis can be challenging, and demands further research.

Remark 2.3.

In the estimation procedure of Stein, 2007 and Stein, 2008, the associated Legendre polynomials play an essential role. Solutions involving such polynomials complicate the procedure. In light of Proposition 1, one may replace such polynomials by another complete basis, for example, the $\{\cos(\nu\phi),\nu=0,1,\ldots\}$, which leads to

$$R(P,Q)=\sum_{m=-\infty}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}f_{\nu,\mu,m}e^{im\Delta\lambda}\cos\nu\phi_P\cos\mu\phi_Q,$$

where $f_{\nu,\mu,m}$ satisfies the same condition to ensure the validity. This can eliminate the computational complexities of associated Legendre polynomials.

Remark 2.4.

If the process is real-valued, so is the covariance function R(P,Q).
Therefore, $C_m(\phi_P,\phi_Q)=\overline{C_{-m}(\phi_P,\phi_Q)}$, which leads to

$$R(P,Q)=C_{0,R}(\phi_P,\phi_Q)+2\sum_{m=1}^{\infty}[\cos(m\Delta\lambda)C_{m,R}(\phi_P,\phi_Q)-\sin(m\Delta\lambda)C_{m,I}(\phi_P,\phi_Q)],$$

where $C_m(\phi_P,\phi_Q)=C_{m,R}(\phi_P,\phi_Q)+iC_{m,I}(\phi_P,\phi_Q)$. In Proposition 2, denoting $\psi_{m,\nu}(\phi)=\psi_{m,\nu,R}(\phi)+i\psi_{m,\nu,I}(\phi)$, with $\eta'_{m,\nu}=2\eta_{m,\nu},\,m\neq 0$ and $\eta'_{0,\nu}=\eta_{0,\nu}$, we obtain

$$R(P,Q) = \sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}\eta'_{m,\nu}(\psi_{m,\nu,R}(\phi_P)\psi_{m,\nu,R}(\phi_Q)+\psi_{m,\nu,I}(\phi_P)\psi_{m,\nu,I}(\phi_Q))\cos(m\Delta\lambda)$$
$$+\eta'_{m,\nu}(\psi_{m,\nu,R}(\phi_P)\psi_{m,\nu,I}(\phi_Q)-\psi_{m,\nu,I}(\phi_P)\psi_{m,\nu,R}(\phi_Q))\sin(m\Delta\lambda).$$

We arrive at a simpler expansion parallel to the Eq. (12) in Jones (1963) but with just a double summation representation.

Remark 2.5.

Given the conditions in Proposition 1 and Proposition 2, a continuous axially symmetric process X(P) can be represented

equation(5)

$$X(P) = X(\lambda,\phi) = \sum_{m=-\infty}^{\infty}\sum_{\nu=0}^{\infty} W_{m,\nu}e^{im\lambda}\psi_{m,\nu}(\phi),$$

with the convergence of the series in quadratic mean. In this representation,

$$W_{m,\nu} = \frac{1}{2\pi}\int_{S^2} X(P)e^{-im\lambda}\overline{\psi_{m,\nu}(\phi)}dP,$$

with $E(W_{m,\nu}\overline{W_{n,\mu}}) = \delta_{m,n}\delta_{\nu,\mu}\eta_{m,\nu}$. When the process is real and Gaussian, $W_{m,\nu}$ are independent normal random variables. In addition, this process can be viewed as a homogeneous random process on the circle with angular distance given by $\Delta\lambda$. That is, for each ϕ, one can expand X(P) in a Fourier series that is convergent in quadratic mean (Roy, 1972):

$$X(\lambda,\phi) = \sum_{m=-\infty}^{\infty} W_m(\phi)e^{im\lambda},$$

where

$$W_m(\phi) = \frac{1}{2\pi}\int_0^{2\pi} X(\lambda,\phi)e^{-im\lambda}d\lambda,$$

with $E(W_m(\phi_P)\overline{W_n(\phi_Q)}) = \delta_{m,n}C_m(\phi_P,\phi_Q)$.

3. Longitudinally reversible process

If one further assumes that the covariance function R(P,Q) given by (1) satisfies

equation(6)

$R(\phi_P,\phi_Q,\Delta\lambda)=R(\phi_P,\phi_Q,-\Delta\lambda)$,

this is termed as longitudinal reversibility (Stein, 2007). The reversibility (6) yields $C_{-m}(\phi_P,\phi_Q)=C_m(\phi_P,\phi_Q)$, which leads to the following representation

$$R(\phi_P,\phi_Q,\Delta\lambda)=C_0(\phi_P,\phi_Q)+\sum_{m=1}^{\infty}C_m(\phi_P,\phi_Q)(e^{-im\Delta\lambda}+e^{im\Delta\lambda})=\sum_{m=0}^{\infty}C_m(\phi_P,\phi_Q)\cos(m\Delta\lambda),$$

where the last C_m has been relabeled to absorb a constant 2 for $m\geq 1$. Therefore, we have the following proposition.

Proposition 3.

The covariance function of a continuous axially symmetry process that is longitudinally reversible can be written as

$$R(\phi_P,\phi_Q,\Delta\lambda_L)=\sum_{m=0}^{\infty}C_m(\phi_P,\phi_Q)\cos(m\Delta\lambda_L),$$

for $\Delta\lambda_L=|\Delta\lambda|\in[0,\pi]$, *where*

$$C_m(\phi_P,\phi_Q)=\frac{1}{\pi}\int_0^{\pi}R(\phi_P,\phi_Q,\Delta\lambda_L)\cos(m\Delta\lambda_L)d\Delta\lambda_L,\quad m=0,1,2,\ldots,$$

is Hermitian and positive definite, $\sum_{m=0}^{\infty}|C_m(\phi_P,\phi_Q)|<\infty$ *. In addition, for each* m*, there exists an orthonormal basis* $\{\psi_{m,\nu}(\cdot)\}\in L^2([0,\pi])$,

$$R(\phi_P,\phi_Q,\Delta\lambda_L)=\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}\eta_{m,\nu}\cos(m\Delta\lambda_L)\psi_{m,\nu}(\phi_P)\overline{\psi_{m,\nu}(\phi_Q)},$$

where $\eta_{m,\nu}\geq 0,\sum_m\sum_\nu\eta_{m,\nu}<\infty$.

Remark 3.1.

In Proposition 1 and Proposition 2, $\Delta\lambda$ is in the interval $[-\pi,\pi]$. Under the assumption of longitudinal reversibility, we can simplify the angular distance definition to be $\Delta\lambda_L=|\Delta\lambda|\in[0,\pi]$, since the cosine function is even and one has $\cos(m\Delta\lambda)=\cos(m\Delta\lambda_L)$. In light of Remark 2.5, a longitudinally reversible process $X(\lambda,\phi)$, for each $\phi\in[0,\pi)$, can be viewed as a homogeneous random process on the circle with the angular distance given by $\Delta\lambda_L$. The process adapts the spectral representation (5), with the further conditions $\eta_{-m,\nu}=\eta_{m,\nu}$ and $\psi_{-m,\nu}(\cdot)=\psi_{m,\nu}(\cdot)$.

Remark 3.2.

If $C_m(\phi_P,\phi_Q)$ is further assumed to have the associated Legendre polynomials as basis with the corresponding eigenvalues $\eta_{\nu,m}=f_\nu\geq 0$ for all m, then

$$R(P,Q)=\sum_{m=0}^{\infty}\sum_{\nu=m}^{\infty}f_\nu\cos(m(\lambda_P-\lambda_Q))P_\nu^m(\cos\phi_P)P_\nu^m(\cos\phi_Q)=\sum_{\nu=0}^{\infty}\frac{(2\nu+1)f_\nu}{2}P_\nu(\cos\theta(P,Q))$$

by the addition theorem of Legendre polynomials. Hence, R(P,Q) is the covariance function of the homogeneous process on the sphere.

4. Further discussion

We have explored the structure of the axially symmetric covariance function introduced by Jones (1963). The orthonormal basis representation can help us understand the axially symmetric process and greatly improve the estimation procedure in Stein (2007). While the nonparametric estimation procedure offers flexibility, Stein (2007) indicates that such procedure needs further investigation. The alternative would be to use parametric modeling techniques. In this section, we construct several parametric models.

The simplest model is the separable model, in which we assume $C_m(\phi_P,\phi_Q)=b_m C(\phi_P,\phi_Q)$. Here $b_m \geq 0, \sum_m b_m < \infty$, and $C(\phi_P,\phi_Q)$ is Hermitian and positive definite. Then,

$$R(\phi_P, \phi_Q, \Delta\lambda) = \tilde{C}(\Delta\lambda) C(\phi_P, \phi_Q),$$

where $\tilde{C}(\Delta\lambda) = \sum_m b_m e^{im\Delta\lambda}$. Valid covariance functions for $\tilde{C}(\Delta\lambda)$ and $C(\phi_P,\phi_Q)$ can be candidates to construct parametric models. For example, when both covariance functions are exponential (Huang et al., 2011), we have

$$R(P,Q)=c_0 e^{-a|\Delta\lambda|} e^{-b|\phi_P-\phi_Q|},$$

where a and b can be viewed as the decay parameters in longitude and latitude, respectively.

For a general covariance function that is non-separable, in view of Remark 2.4, we consider a real-valued process with

$$C_m(\phi_P, \phi_Q) = c_m e^{-a_m|\phi_P-\phi_Q|}(\cos\omega_m(\phi_P-\phi_Q) + i\sin\omega_m(\phi_P-\phi_Q)), \quad c_m \geq 0, a_m \geq 0, \omega_m \in R.$$

It is clear that $C_m(\phi_P,\phi_Q)$ is Hermitian and positive definite (Yaglom, 1987). Then,

$$R(P,Q) = c_0 e^{-a_0|\phi_P-\phi_Q|}\cos\omega_0(\phi_P-\phi_Q) + 2\sum_{m=1}^{\infty} c_m e^{-a_m|\phi_P-\phi_Q|}\cos[m\Delta\lambda + \omega_m(\phi_P-\phi_Q)].$$

If one takes $a_m=a, \omega_m=mu$, we have

$$R(P,Q) = c_0 e^{-a|\phi_P-\phi_Q|} + 2e^{-a|\phi_P-\phi_Q|}\sum_{m=1}^{\infty} c_m \cos[m\theta(P,Q,u)],$$

where $\theta(P,Q,u)=\Delta\lambda+u(\phi_P-\phi_Q)-2k\pi$, and k is chosen such that $\theta(P,Q,u)\in[0,2\pi]$. This covariance function is longitudinally irreversible. Various choices of c_m lead to different parametric models. For example, by formulas (1.447.2),(1.441,2),(1.443,3),(1.443.6) and (1.448.2)of Gradshteyn and Ryzhik (2007), one would have the following parametric models, respectively.

1. If $c_m=Cp^m$,

$$R(P,Q) = Ce^{-a|\phi_P-\phi_Q|}\frac{1-p^2}{1-2p\cos\theta(P,Q,u)+p^2}.$$

2. If $c_m = \frac{C}{2m}, c_0 = 0$,

$$R(P,Q) = Ce^{-a|\phi_P - \phi_Q|} \log \frac{1}{2(1-\cos\theta(P,Q,u))}.$$

3. If $c_m = \frac{C}{m^2}, c_0 = 0$,

$$R(P,Q) = Ce^{-a|\phi_P - \phi_Q|} \left(\frac{\pi^2}{6} - \frac{\pi\theta(P,Q,u)}{2} + \frac{\theta^2(P,Q,u)}{4}\right).$$

4. If $c_m = \frac{C}{m^4}, c_0 = 0$,

$$R(P,Q) = Ce^{-a|\phi_P - \phi_Q|} \left(\frac{\pi^4}{90} - \frac{\pi^2\theta^2(P,Q,u)}{12} + \frac{\pi\theta^3(P,Q,u)}{12} - \frac{\theta^4(P,Q,u)}{48}\right).$$

5. If $c_m=Cp^m/m, c_0=0$,

$$R(P,Q) = Ce^{-a|\phi_P - \phi_Q|} \log \frac{1}{\sqrt{1-2p\cos\theta(P,Q,u)+p^2}}.$$

When the parameter u=0,R(P,Q) is separable in latitude and longitude. Therefore, u can be viewed as separability indicator. Among above models 1, 2 and 5, the parameter value u=0 yields that the process is longitudinally reversible. Hence, in these three models, the parameter u can be viewed as reversibility indicator as well.

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