A note on processes with random stationary increments

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Zhang, H. and Huang, C. (2014). A Note on Processes with Random Stationary Increments. *Statistics and Probability Letters*, 94, 153 - 161. DOI: 10.1016/j.spl.2014.07.016

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Abstract:

When the correlation theory is considered for the processes with random stationary increments, Yaglom (1955) has developed the spectral representation theory. In this note, we complete this development by obtaining the inversion formula of the spectrum in terms of the structure function.

Keywords: Intrinsic stationarity | Variogram | Spectrum | Structure function | Inversion formula

Article:

1. Introduction

When the process is assumed to be with random stationary nth increments, Yaglom (1955) has studied its correlation and developed the spectral representation theory. The results in Yaglom (1955) have profound impacts on various fields including the studies of fractional Brownian motion (Mandelbrot and Van Ness, 1968), intrinsic random function (Solo, 1992 and Huang et al., 2009), and self-similar process (Unser and Blu, 2007 and Blu and Unser, 2007). It also has a strong connection to the concept of generalized random process (Itô, 1954, Gel'fand, 1955, Gel'fand and Vilenkin, 1964 and Yaglom, 1987).

The covariance function is used to characterize the second-order dependency. When the process is assumed to be stationary, its corresponding spectral density is widely used to describe the periodical components and frequencies (Brockwell and Davis, 2009). The spectrum and its covariance function are connected through the inversion of Fourier transformation. When the process is intrinsically stationary, the variogram often replaces the covariance function to model the dependency (Cressie, 1993). Furthermore, when the process is with stationary nth increments, the notion of structure equation (Yaglom, 1955) or generalized covariance function (Matheron, 1973) is developed. Such applications are essential in universal kriging (Cressie, 1993 and Chilès and Delfiner, 2012). The spectral representation of such structure equation or generalized covariance has been obtained in Yaglom (1955) and Matheron (1973). In this note, we derive the inversion formula where the spectrum can be represented by the structure function. We are not aware of such an explicit formula in literature. This enhances the understanding of the process, where our theorem offers a way to estimate the spectral function from an easily estimated structure function. In addition, by directly applying our formula, one can derive the spectral density function of commonly used power variogram.

2. Main results

In Yaglom (1955), a random process $\{X(t), t\in R\}$ is considered, where its nth difference with step $\tau>0$is defined as

$$\Delta_\tau^{(n)}X(t) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} X(t-k\tau).$$

Such a random nth increment is called stationary if its second moment

$$\mathrm{E}\{\Delta_{\tau_1}^{(n)}X(t+s)\overline{\Delta_{\tau_2}^{(n)}X(s)}\} \equiv D^{(n)}(t;\tau_1,\tau_2)$$

exists and does not depend on s. Yaglom (1955) has termed this as the structure function of such increments, and has shown that it has the following spectral representation

equation(1)

$$D^{(n)}(t;\tau_1,\tau_2) = \int_{-\infty}^{\infty} e^{it\lambda}(1-e^{-i\tau_1\lambda})^n(1-e^{i\tau_2\lambda})^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} dF(\lambda),$$

where the spectral function $F(\lambda)$ is a non-decreasing bounded measure. When $\tau_1=\tau_2$, the above structure function becomes

equation(2)

$$D_\tau^{(n)}(t) \equiv D^{(n)}(t;\tau,\tau) = \int_{-\infty}^{\infty} e^{it\lambda}2^n(1-\cos(\tau\lambda))^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} dF(\lambda).$$

In our next theorem, we obtain the inversion theorem where we express the spectral function in terms of the structure function.

Theorem 1.

Let $\lambda_1<\lambda_2$ *be the continuity points of* $F(\lambda)$. *We have*

equation(3)

$$F(\lambda_2)-F(\lambda_1)=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}D_\tau^{(n)}(t)dt\left(\int_{\lambda_1}^{\lambda_2}e^{-i\lambda t}q(\lambda;n,\tau)d\lambda\right),$$

where,

$$q(\lambda)\equiv q(\lambda;n,\tau)=\frac{\lambda^{2n}}{2^n(1-\cos(\tau\lambda))^n(1+\lambda^2)^n}.$$

Here $\tau>0$ *is selected so that the function* $(1-\cos(\lambda\tau))/\lambda^2$ *has no zeros for* $\lambda\in[\lambda_1,\lambda_2]$.

Remark 1.

When n=0, $\Delta_\tau^{(n)}X(t)=X(t)$, that is, the process is stationary itself. The structure function becomes a conventional covariance function. The inversion formula in Theorem 1 is just a regular Fourier inversion (Hannan, 1970 and Doob, 1953).

The proof of Theorem 1 follows along the lines in obtaining the inversion formula for Fourier transform in literature, for example, see Hannan (1970, Chapter 2), and Doob (1953, Chapter XI). Here we denote $\phi(\lambda)=1_{[\lambda1,\lambda2]}(\lambda)$ as the indicator function and let $h(\lambda)=\phi(\lambda)q(\lambda)$. Similar to Yaglom (1955, page 96),$\tau>0$ is selected such that for $\lambda\in[\lambda_1,\lambda_2]$, $(1-\cos(\tau\lambda))/\lambda^2$ has no zeros, and so is bounded away from zero. Therefore, there exists $C_h>0$ such that $0\le h(\lambda)\le C_h<\infty,\lambda\in R$. Moreover, $h(\lambda)$ is continuous for$\lambda\in[\lambda_1,\lambda_2]$, and is absolutely integrable. Hence, its Fourier transform exists, and is given by $\hat{h}(t)=\int_{-\infty}^{\infty}e^{-i\nu t}h(\nu)d\nu,\ t\in\mathbb{R}$. The following properties for $\hat{h}(t)$ are easily obtained: $\hat{h}(-t)=\overline{\hat{h}(t)}$, $\hat{h}(t)$ is uniformly continuous and bounded for $t\in R$.

Lemma 1.

For the Fourier transform function $\hat{h}(t)$ *of* $h(\nu)$, *we have*

equation(4)

$$\frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)e^{i\nu t}\hat{h}(t)dt=\int_{-\infty}^{\infty}h(\lambda)d\lambda\left(\frac{1}{2\pi}\int_{-T}^{T}e^{i(\nu-\lambda)t}\left(1-\frac{|t|}{T}\right)dt\right).$$

Moreover, the above convergence is bounded and uniformly over all $\nu\in R$.

Proof.

First, since $h(\lambda)$ is absolutely integrable, from the Fubini's theorem, one has

$$\frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)e^{i\nu t}\hat{h}(t)dt = \int_{-\infty}^{\infty} h(\lambda)d\lambda\left(\frac{1}{2\pi}\int_{-T}^{T} e^{i(\nu-\lambda)t}\left(1-\frac{|t|}{T}\right)dt\right).$$

Now the above inner integral can be simplified as follows, when λ≠ν,

$$\begin{aligned}\int_{-T}^{T} e^{i(\nu-\lambda)t}\left(1-\frac{|t|}{T}\right)dt &= 2\int_0^T \cos((\lambda-\nu)t)\left(1-\frac{t}{T}\right)dt\\ &= \frac{2}{T}\frac{1-\cos((\lambda-\nu)T)}{(\lambda-\nu)^2} = \frac{1}{T}\left[\frac{\sin^2\left(\frac{(\lambda-\nu)T}{2}\right)}{\left(\frac{\lambda-\nu}{2}\right)^2}\right] \to 0, \quad \text{as } T\to\infty.\end{aligned}$$

Note the above limit uniformly converges for any finite interval of λ not containing ν. When λ=ν,

$$\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)dt = 2\int_0^T\left(1-\frac{t}{T}\right)dt = T\to\infty, \quad \text{as } T\to\infty.$$

Note that

$$\frac{1}{2\pi T}\left[\frac{\sin^2\left(\frac{(\lambda-\nu)T}{2}\right)}{\left(\frac{\lambda-\nu}{2}\right)^2}\right] = \delta_T(\lambda-\nu),$$

with the convention that it takes the limit T/2π if λ=ν, then

$$\left|\int_{-\infty}^{\infty} h(\lambda)\delta_T(\lambda-\nu)d\lambda - h(\nu)\right| \to 0, \quad \text{as } T\to\infty,$$

In fact, the right hand side above is the so-called Dirichlet type of integral in literature. Next we will show that

$\int_{|x|>K}\delta_T(x)dx<\varepsilon/(6C_h)$.

uniformly on ν.

Note that for every ε>0, choose η=ε, such that for |x|<η, we have |h(x+ν)−h(ν)|<ε/3 by the uniform continuity of h(ν). In addition, for this ε>0, we choose K>0 such that

$\int_{|x|>K}\delta_T(x)dx<\varepsilon/(6C_h)$.

For the above selected η and K, we then have the following decomposition

$$\left|\int_{-\infty}^{\infty} h(\lambda)\delta_T(\lambda-\nu)d\lambda - h(\nu)\right| = \int_{-\infty}^{\infty} |h(\lambda)-h(\nu)|\delta_T(\lambda-\nu)d\lambda$$
$$= \int_{-\infty}^{\infty} |h(x+\nu)-h(\nu)|\delta_T(x)dx$$
$$= \left(\int_{|x|<\eta} + \int_{\eta\le|x|\le K} + \int_{|x|>K}\right)|h(x+\nu)-h(\nu)|\delta_T(x)dx$$
$$= I + II + III,$$

saying. Now we can bound each of the above integrals.

$$I < \frac{\varepsilon}{3}\int_R \delta_T(x)dx = \frac{\varepsilon}{3}, \qquad III \le 2C_h \int_{|x|>K} \delta_T(x)dx < \frac{\varepsilon}{3}.$$

Furthermore, for the given η,K>0, we choose T>L such that $4/(L\eta^2) < \frac{\varepsilon}{12C_hK}$. Therefore, if η≤|x|≤K,

$$\delta_T(x) \le \frac{4}{T\eta^2} \le \frac{4}{L\eta^2} < \frac{\varepsilon}{12C_hK}.$$

Hence,

$$II = \int_{\eta\le|x|\le K} |h(x+\nu)-h(\nu)|\delta_T(x)dx < \frac{\varepsilon}{12C_hK}\int_{\eta\le|x|\le K} |h(x+\nu)-h(\nu)|dx \le \frac{\varepsilon}{3}.$$

Putting all together, for T>L with L not depending on ν, we have

$$\left|\int_{-\infty}^{\infty} h(\lambda)\delta_T(\lambda-\nu)d\lambda - h(\nu)\right| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

Finally, taking ε=1 and for T large, we have

equation(5)

$$\left|\int_{-\infty}^{\infty} h(\lambda)\delta_T(\lambda-\nu)d\lambda\right| \le h(\nu)+1 \le C_h + 1,$$

uniformly bounded. ♢

Lemma 2.

For $\hat{h}(t)$ *, we have*

equation(6)

$$\lim_{T\to\infty}\frac{1}{2\pi}\frac{1}{T}\int_{-T}^{T}|t|e^{i\nu t}\hat{h}(t)dt=0,$$

and moreover, for all $T\geq 0$*, there exists* $M_0>0$ *not depending on* ν*, such that*

equation(7)

$$\left|\frac{1}{T}\int_{-T}^{T}|t|e^{i\nu t}\hat{h}(t)dt\right|\leq M_0.$$

Proof.

We first note that for $\lambda\in[\lambda_1,\lambda_2]$, $q(\lambda)$ has continuous derivatives, and hence there exists a constant $C_q>0$ such that $|q'(\lambda)|\leq C_q$. Observe that

$$\int_{-T}^{T}|t|e^{i\nu t}\hat{h}(t)dt=\int_0^T t\left(e^{i\nu t}\hat{h}(t)+e^{-i\nu t}\overline{\hat{h}(t)}\right)dt=2\int_0^T tdt\left(\int_{\lambda_1}^{\lambda_2}\cos((\lambda-\nu)t)q(\lambda)d\lambda\right),$$

and apply the integration by parts to obtain

$$\int_{\lambda_1}^{\lambda_2}\cos((\lambda-\nu)t)q(\lambda)d\lambda=\frac{1}{t}\left[(\sin((\lambda_2-\nu)t)q(\lambda_2)-\sin((\lambda_1-\nu)t)q(\lambda_1))-\int_{\lambda_1}^{\lambda_2}\sin((\lambda-\nu)t)q'(\lambda)d\lambda\right],$$

which implies

equation(8)

$$\int_0^T tdt\left(\int_{\lambda_1}^{\lambda_2}\cos((\lambda-\nu)t)q(\lambda)d\lambda\right)=q(\lambda_2)\int_0^T\sin((\lambda_2-\nu)t)dt+q(\lambda_1)\int_0^T\sin((\nu-\lambda_1)t)dt$$
$$-\int_{\lambda_1}^{\lambda_2}q'(\lambda)d\lambda\int_0^T\sin((\lambda-\nu)t)dt.$$

Again, the Fubini's theorem was used. Noticing that $|q'(\lambda)|\leq C_q$ when $\lambda\in[\lambda_1,\lambda_2]$, we have

$$\left|\frac{1}{T}\int_{-T}^{T}|t|e^{i\nu t}\hat{h}(t)dt\right|\leq 2(q(\lambda_2)+q(\lambda_1)+C_q(\lambda_2-\lambda_1)),$$

implying (7) by denoting M_0 as the right hand side in the above inequality. Moreover, on the first two terms of (8), we have, if $\nu\neq\lambda_1$ or λ_2,

$$\frac{1}{T}q(\lambda_2)\int_0^T\sin((\lambda_2-\nu)t)dt\leq q(\lambda_2)\frac{2}{(\lambda_2-\nu)T},\qquad\frac{1}{T}q(\lambda_1)\int_0^T\sin((\nu-\lambda_1)t)dt\leq q(\lambda_1)\frac{2}{(\nu-\lambda_1)T},$$

then both terms tend to zero as T tends to infinity. For the third term,

$$\int_{\lambda_1}^{\lambda_2} q'(\lambda)d\lambda \int_0^T \sin((\lambda-\nu)t)dt = \int_{\lambda_1}^{\lambda_2} q'(\lambda)d\lambda \left(\frac{\sin^2\left(\frac{(\lambda-\nu)T}{2}\right)}{\frac{\lambda-\nu}{2}}\right),$$

where the inner parenthesis takes its limit 0 if λ=ν. After divided by T, and applying the same approach used for proving (4), one can obtain that

$$\frac{1}{2\pi T}\int_{\lambda_1}^{\lambda_2} q'(\lambda)d\lambda \int_0^T \sin((\lambda-\nu)t)dt \to 0, \qquad \text{as } T\to\infty.$$

This shows the pointwise convergence of (6) for ν∈R. ♢

Lemma 3.

For h(ν), *we have*

equation(9)

$$h(\nu) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T} e^{i\nu t}\hat{h}(t)dt,$$

and the above convergence is bounded.

Proof.

Notice the following

$$\frac{1}{2\pi}\int_{-T}^{T} e^{i\nu t}\hat{h}(t)dt = \frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)e^{i\nu t}\hat{h}(t)dt + \frac{1}{2\pi T}\int_{-T}^{T}|t|e^{i\nu t}\hat{h}(t)dt.$$

The results are now obvious from Lemma 1 and Lemma 2, (5) and (7). ♢

Proof of Theorem 1.

We first notice that

equation(10)

$$\mathrm{var}(\Delta_\tau^{(n)}X(t)) = D_\tau^{(n)}(0) = \int_{-\infty}^{\infty} 2^n(1-\cos\tau\lambda)^n\frac{(1+\lambda^2)^n}{\lambda^{2n}}dF(\lambda) < \infty,$$

since $\Delta_\tau^{(n)}X(t)$ is second-order stationary. Now the right hand side of (3) becomes

$$\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\hat{h}(t)dt\left(\int_{-\infty}^{\infty}e^{it\lambda}q^{-1}(\lambda)dF(\lambda)\right) = \lim_{T\to\infty}\int_{-\infty}^{\infty}q^{-1}(\lambda)dF(\lambda)\left(\frac{1}{2\pi}\int_{-T}^{T}e^{it\lambda}\hat{h}(t)dt\right).$$

The interchange of integrals in the last identity is due to the Fubini's theorem through the boundedness of $\hat{h}(t)$ and (10) over $(t,\lambda)\in[-T,T]\times(-\infty,\infty)$. From Lemma 3 and the integrability condition (10), we apply the dominated convergence theorem to get

$$\begin{aligned}\lim_{T\to\infty}\int_{-\infty}^{\infty} q^{-1}(\lambda)dF(\lambda)\left(\frac{1}{2\pi}\int_{-T}^{T} e^{it\lambda}\hat{h}(t)dt\right) &= \int_{-\infty}^{\infty} q^{-1}(\lambda)dF(\lambda)\left(\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T} e^{it\lambda}\hat{h}(t)dt\right)\\ &= \int_{-\infty}^{\infty} q^{-1}(\lambda)h(\lambda)dF(\lambda) = \int_{-\infty}^{\infty}\phi(\lambda)F(d\lambda) = F(\lambda_2)-F(\lambda_1),\end{aligned}$$

since λ_1 and λ_2 are the continuity points of F.

Remark 2.

Under the assumption for Theorem 1, if $D_\tau^{(n)}(t)$ is continuous on any finite interval $[-T,T]$ for $T>0$, then we can apply the Fubini's theorem on $(t,\nu)\in[-T,T]\times[\lambda_1,\lambda_2]$,

$$F(\lambda_2)-F(\lambda_1) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{\lambda_1}^{\lambda_2} q(\nu)d\nu\int_{-T}^{T} e^{-i\nu t}D_\tau^{(n)}(t)dt.$$

Furthermore, if the following limit

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T} e^{-i\nu t}D_\tau^{(n)}(t)dt$$

exists and is denoted by $f_D(\nu)\equiv f_D(\nu;n,\tau)$, then for $\lambda_1<\lambda_2$ continuity points of F,

$$F(\lambda_2)-F(\lambda_1) = \int_{\lambda_1}^{\lambda_2} f_D(\nu)q(\nu)d\nu.$$

Remark 3.

From Remark 2, if one further assumes that the spectral function $F(\lambda)$ is absolutely continuous with spectral density $f(\lambda)$, one has

equation(11)

$$f(\lambda) = f_D(\lambda)q(\lambda), \quad \lambda\in\mathbb{R}.$$

This inversion formula furthers the development in Yaglom (1955). When the process is stationary, the spectral density plays an important role (Brockwell and Davis, 2009). Clearly, when the process is with stationary nth increments, such a spectral density can help understand the process (Mandelbrot and Van Ness, 1968 and Solo, 1992). In addition, our theorem offers a way to estimate such a function from an easily estimated structure function.

Remark 4.

From Remark 3, it is quite interesting to note that the spectral density function f(λ) given by (11) is free of τ. We notice that the structure function $D_\tau^{(n)}(t) = E[\Delta_\tau^{(n)} X(s+t)\overline{\Delta_\tau^{(n)} X(s)}]$ is positive definite. By Bochner's Theorem (for example, Brockwell and Davis, 2009), if its spectral density $f_D(\lambda)$ exists, it has the following spectral representation:

$$D_\tau^{(n)}(t) = \int_{-\infty}^{\infty} e^{it\lambda} f_D(\lambda) d\lambda.$$

Comparing the above representation with (2), we have (11), and it is clear that f(λ) is free of τ. Therefore, the spectral function exists regardless of which step τ one takes on the differences of the process.

Remark 5.

One of the most widely used real-valued processes with stationary nth random increments is the case when n=1, under which $\Delta_\tau X(t)=X(t+\tau)-X(t)$ is assumed to be stationary. This is usually termed as an intrinsically stationary process in literature. Here, the variogram function $2\gamma(\tau)=\mathrm{var}(X(t+\tau)-X(t))$is commonly used in spatial statistics (Cressie, 1993, Stein, 1999, Huang et al., 2011a and Chilès and Delfiner, 2012). One can show that (Yaglom, 1987, Section 23) $D_\tau^{(1)}(t) = \gamma(t+\tau) + \gamma(t-\tau) - 2\gamma(t)$, and $\gamma(\tau) = \frac{1}{2} D_\tau^{(1)}(0),\ \tau > 0$.

There are a few things we can learn from our development. First, it is clear that $D_\tau^{(1)}(t)$ is a covariance function, which directly implies Theorem 4 in Ma (2004). Note that here we do not assume γ(·) to be bounded, as imposed by Ma (2004). Second, from (11), one has

equation(12)

$$f(\lambda) = \left(\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} D_\tau^{(1)}(u) e^{-iu\lambda} du\right) \cdot \frac{\lambda^2}{2(1+\lambda^2)(1-\cos\lambda\tau)}, \quad \lambda > 0.$$

This gives the spectral density function of a structure function directly; hence one can obtain the spectral density for the variogram as well. For example, if one considers Brownian motion X(t) which has the covariance function cov(X(s),X(t))=min(s,t), its structure function can then be computed as $D_\tau^{(1)}(u) = (\tau - |u|) I_{|u|<\tau}$. Therefore, from (12), the spectral density is given by

$$f(\lambda) = \left(\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} (\tau - |u|) I_{|u|<\tau} e^{-iu\lambda} du\right) \cdot \frac{\lambda^2}{2(1+\lambda^2)(1-\cos\lambda\tau)} = \frac{1}{2\pi(1+\lambda^2)}, \quad \lambda > 0.$$

If one uses the spectral representation of the variogram in Christakos (1984, equation 35), or Yaglom (1987, equation 4.250a), the spectral density in $\gamma(\tau) = 2\int_0^\infty (1 - \cos\tau\lambda) f_\gamma(\lambda) d\lambda$ will be of the form $f_\gamma(\lambda) = \frac{1+\lambda^2}{\lambda^2} f(\lambda) = \frac{1}{2\pi\lambda^2}$. This is commonly viewed as the spectral density function of Brownian motion.

Remark 6.

Besides the Brownian motion in Remark 5, power variogram $2\gamma(u)=|u|^\alpha, 0<\alpha<2$ is widely studied and commonly used in practice (Yaglom, 1987, Cressie, 1993 and Huang et al., 2011b). It appears that the spectral density function of the power variogram can be obtained through the variogram function representation (for examples, page 407 in Yaglom, 1987 or page 532 of Schoenberg, 1938). In this note, by directly applying (12), we can obtain its spectral density function through structure functions.

Proposition 1.

The spectral density function for the power variogram $2\gamma(u)=|u|^\alpha, 0<\alpha<2$ *is given by*

equation(13)

$$f(\lambda) = \frac{\Gamma(\alpha+1)}{2\pi(1+\lambda^2)} \left(\frac{\sin(\alpha\pi/2)}{\lambda^{\alpha-1}} \right), \quad \lambda > 0.$$

To prove this proposition, we need the following lemma.

Lemma 4.

Let $0<\alpha<2$ *and* $\lambda>0$.

1. *For each fixed* $\tau>0$, *we have*

equation(14)

$$\int_\tau^\infty ((u+\tau)^\alpha + (u-\tau)^\alpha - 2u^\alpha) \cos(u\lambda) du < \infty.$$

2. *For* $1\le\alpha<2$,

equation(15)

$$\lim_{\tau\to 0+} \int_0^\infty \left(\frac{(u+\tau)^\alpha + |u-\tau|^\alpha - 2u^\alpha}{\tau^2} \right) \cos(u\lambda) du = \alpha(\alpha-1) \int_0^\infty \frac{\cos(u\lambda)}{u^{2-\alpha}} du,$$

and for $0<\alpha<1$,

equation(16)

$$\lim_{\tau\to 0+}\int_0^\infty \left(\frac{(u+\tau)^\alpha+|u-\tau|^\alpha-2u^\alpha}{\tau^2}\right)\cos(u\lambda)du=\alpha\lambda\int_0^\infty \frac{\sin(u\lambda)}{u^{1-\alpha}}du.$$

The proof of Lemma 4 is tedious and is deferred in the Appendix.

Proof of Proposition 1.

The case when $\alpha=1$ is given in Remark 5. We now first consider the existence of $f_D(\lambda)$ in Remark 5. Notice that for $\tau>0$,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}D_\tau^{(1)}(u)e^{-iu\lambda}du &= \lim_{T\to\infty}\frac{1}{4\pi}\int_{-T}^{T}(|u+\tau|^\alpha+|u-\tau|^\alpha-2|u|^\alpha)e^{-iu\lambda}du\\ &=\frac{1}{2\pi}\int_0^\infty((u+\tau)^\alpha+|u-\tau|^\alpha-2u^\alpha)\cos(u\lambda)du<\infty\end{aligned}$$

from (14) in Lemma 4. Therefore, we apply (12) to get

$$f(\lambda)=\frac{\lambda^2}{4\pi(1+\lambda^2)(1-\cos\lambda\tau)}\int_0^\infty \cos(u\lambda)\ ((u+\tau)^\alpha+|u-\tau|^\alpha-2u^\alpha)\,du.$$

Note that $f(\lambda)$ is free of τ, and so taking the limit as $\tau\to 0+$ and applying Lemma 4, we have

$$\begin{aligned}f(\lambda)&=\lim_{\tau\to 0+}\left(\frac{\lambda^2}{4\pi(1+\lambda^2)}\int_0^\infty\left(\frac{(u+\tau)^\alpha+|u-\tau|^\alpha-2u^\alpha}{1-\cos(\lambda\tau)}\right)\cos(u\lambda)du\right)\\ &=\frac{1}{4\pi(1+\lambda^2)}\left(\lim_{\tau\to 0+}\int_0^\infty\left(\frac{(u+\tau)^\alpha+|u-\tau|^\alpha-2u^\alpha}{\tau^2}\right)\cos(u\lambda)du\right)\left(\lim_{\tau\to 0+}\frac{\lambda^2\tau^2}{1-\cos(\tau\lambda)}\right)\\ &=\begin{cases}\dfrac{\alpha(\alpha-1)}{2\pi(1+\lambda^2)}\displaystyle\int_0^\infty\frac{\cos(u\lambda)}{u^{2-\alpha}}du, & \text{if } 1\le\alpha<2,\\ \dfrac{\alpha\lambda}{2\pi(1+\lambda^2)}\displaystyle\int_0^\infty\frac{\sin(u\lambda)}{u^{1-\alpha}}du, & \text{if } 0<\alpha<1,\end{cases}\\ &=\frac{\Gamma(\alpha+1)}{2\pi(1+\lambda^2)}\left(\frac{\sin(\alpha\pi/2)}{\lambda^{\alpha-1}}\right),\end{aligned}$$

giving (13). The last equality is due to (3.761.9) and (3.761.4) of Gradshteyn and Ryzhik (2007).

Acknowledgments

The authors appreciate the comments from an anonymous reviewer, which greatly enhanced the quality of the paper. The authors also acknowledge the support of NSF-DMS1208847 and NSF-DMS1412343 for this work.

Appendix.

Proof of Lemma 4.

When $\alpha=1$, the results are trivial. Hence we only prove (14), (15) and (16) for $\alpha\neq1$. For this aim, we make use of Binomial series expansion for $(1+x)^{\alpha}$ (for example, Fischer, 1983, pages 406–412). In particular,

$$(1+x)^{\alpha} = \sum_{k=0}^{l} \binom{\alpha}{k} x^k + R_{l+1}(x), \quad \text{for } 0 < x < 1, \quad \text{and}$$

$$(1+x)^{\alpha} = \sum_{k=0}^{l} \binom{\alpha}{k} x^k + R^*_{l+1}(x), \quad \text{for } -1 < x < 0,$$

with

$$|R_{l+1}(x)| \le \left| \binom{\alpha}{l+1} \right| x^{l+1}, \quad \text{for } 0 < \alpha < 2, \quad \text{and}$$

$$|R^*_{l+1}(x)| \le \begin{cases} \dfrac{|(\alpha)_{l+1}|}{l!} |x|^{l+1}, & \text{if } \alpha > 1, \\ \dfrac{|(\alpha)_{l+1}|}{l!} \dfrac{|x|^{l+1}}{(1+x)^{1-\alpha}}, & \text{if } \alpha < 1 \end{cases}$$

with $(\alpha)_k=\alpha(\alpha-1)\cdots(\alpha-k+1)$. Now we use the above expansion for $\left(1+\frac{\tau}{u}\right)^{\alpha}$ and $\left(1-\frac{\tau}{u}\right)^{\alpha}$ with $u>\tau>0$ and $l=3$.

$$\left(1+\frac{\tau}{u}\right)^{\alpha} = 1 + \alpha\frac{\tau}{u} + \frac{\alpha(\alpha-1)}{2!}\frac{\tau^2}{u^2} + \frac{(\alpha)_3}{3!}\frac{\tau^3}{u^3} + R_4(\tau/u),$$

$$\left(1-\frac{\tau}{u}\right)^{\alpha} = 1 - \alpha\frac{\tau}{u} + \frac{\alpha(\alpha-1)}{2!}\frac{\tau^2}{u^2} - \frac{(\alpha)_3}{3!}\frac{\tau^3}{u^3} + R^*_4(-\tau/u),$$

with

$$|R_4(\tau/u)| \le \left| \binom{\alpha}{l+1} \right| (\tau/u)^4, \quad \text{if } 0 < \alpha < 2, \text{ and}$$

$$|R^*_4(-\tau/u)| \le \begin{cases} \dfrac{|(\alpha)_4|}{3!} (\tau/u)^4, & \text{if } \alpha > 1, \\ \dfrac{|(\alpha)_4|}{3!} \dfrac{(\tau/u)^4}{(1-\tau/u)^{1-\alpha}}, & \text{if } 0 < \alpha < 1. \end{cases}$$

Hence,

$$\left(1+\frac{\tau}{u}\right)^{\alpha} + \left(1-\frac{\tau}{u}\right)^{\alpha} - 2 = \alpha(\alpha-1)\frac{\tau^2}{u^2} + R_4(\tau/u) + R^*_4(-\tau/u).$$

That is, for u>τ,

$$\begin{aligned}\frac{(u+\tau)^\alpha+(u-\tau)^\alpha-2u^\alpha}{\tau^2}-\frac{\alpha(\alpha-1)}{u^{2-\alpha}} &= \frac{u^\alpha}{\tau^2}\left(\left(1+\frac{\tau}{u}\right)^\alpha+\left(1-\frac{\tau}{u}\right)^\alpha-2\right)-\frac{\alpha(\alpha-1)}{u^{2-\alpha}} \\ &= \frac{u^\alpha}{\tau^2}\left(\alpha(\alpha-1)\frac{\tau^2}{u^2}+R_4(\tau/u)+R_4^*(-\tau/u)\right)-\frac{\alpha(\alpha-1)}{u^{2-\alpha}} \\ &= \frac{u^\alpha}{\tau^2}R_4(\tau/u)+\frac{u^\alpha}{\tau^2}R_4^*(-\tau/u).\end{aligned}$$

1. When 1≤α<2, the above quantity can then be bounded by

equation(17)

$$\left|\frac{(u+\tau)^\alpha+(u-\tau)^\alpha-2u^\alpha}{\tau^2}-\frac{\alpha(\alpha-1)}{u^{2-\alpha}}\right| \le \frac{u^\alpha}{\tau^2}\left|\binom{\alpha}{4}\right|\left(\frac{\tau}{u}\right)^4+\frac{u^\alpha}{\tau^2}\frac{|(\alpha)_4|}{3!}\left(\frac{\tau}{u}\right)^4 \equiv \frac{C(\alpha)\tau^2}{u^{4-\alpha}},$$

where $C(\alpha)=\left|\binom{\alpha}{4}\right|+\frac{|(\alpha)_4|}{3!}>0$ solely depending on α. Therefore, for M≫τ,

$$\begin{aligned}&\left|\int_M^\infty \cos(u\lambda)\left(\frac{(u+\tau)^\alpha+(u-\tau)^\alpha-2u^\alpha}{\tau^2}\right)du-\alpha(\alpha-1)\int_M^\infty\frac{\cos(u\lambda)}{u^{2-\alpha}}du\right| \\ &\quad\le \int_M^\infty\left|\frac{(u+\tau)^\alpha+(u-\tau)^\alpha-2u^\alpha}{\tau^2}-\frac{\alpha(\alpha-1)}{u^{2-\alpha}}\right|du \\ &\quad\le C(\alpha)\tau^2\int_M^\infty\frac{du}{u^{4-\alpha}}=\frac{C(\alpha)\tau^2}{3-\alpha}\frac{1}{M^{3-\alpha}}\to 0,\quad \text{as } M\to\infty.\end{aligned}$$

2. When 0<α<1, we have

equation(18)

$$\left|\frac{(u+\tau)^\alpha+(u-\tau)^\alpha-2u^\alpha}{\tau^2}-\frac{\alpha(\alpha-1)}{u^{2-\alpha}}\right| \le \frac{u^\alpha}{\tau^2}\left|\binom{\alpha}{4}\right|\left(\frac{\tau}{u}\right)^4+\frac{u^\alpha}{\tau^2}\frac{|(\alpha)_4|}{3!}\left(\frac{\tau}{u}\right)^4\frac{1}{(1-\tau/u)^{1-\alpha}},$$

and so

$$\begin{aligned}&\left|\int_M^\infty \cos(u\lambda)\left(\frac{(u+\tau)^\alpha+(u-\tau)^\alpha-2u^\alpha}{\tau^2}\right)du-\alpha(\alpha-1)\int_M^\infty\frac{\cos(u\lambda)}{u^{2-\alpha}}du\right| \\ &\quad\le \left|\binom{\alpha}{4}\right|\tau^2\int_M^\infty\frac{du}{u^{4-\alpha}}+\frac{1}{\tau^2}\frac{|(\alpha)_4|}{3!}\int_M^\infty\frac{u^\alpha\left(\frac{\tau}{u}\right)^4}{(1-\tau/u)^{1-\alpha}}du \\ &\quad\le \frac{\left|\binom{\alpha}{4}\right|\tau^2}{3-\alpha}\frac{1}{M^{3-\alpha}}+\frac{|(\alpha)_4|}{3!}\frac{\tau^2}{(M-\tau)^{1-\alpha}}\int_M^\infty\frac{1}{u^3}du \\ &\quad= \frac{\left|\binom{\alpha}{4}\right|\tau^2}{3-\alpha}\frac{1}{M^{3-\alpha}}+\frac{|(\alpha)_4|}{3!}\frac{\tau^2}{2(M-\tau)^{1-\alpha}M^2}\to 0,\quad \text{as } M\to\infty.\end{aligned}$$

Note that for $0<\alpha<2$,

$$\int_M^\infty \frac{\cos(u\lambda)}{u^{2-\alpha}}du \to 0, \quad \text{as } M \to \infty \Longrightarrow \int_\tau^\infty \left(\frac{(u+\tau)^\alpha + (u-\tau)^\alpha - 2u^\alpha}{\tau^2}\right)\cos(u\lambda)du < \infty,$$

and so is (14), completing the first part. For the second part, when $1<\alpha<2$, we consider the following difference,

$$\begin{aligned}
&\left|\int_0^\infty \left(\frac{(u+\tau)^\alpha + |u-\tau|^\alpha - 2u^\alpha}{\tau^2}\right)\cos(u\lambda)du - \alpha(\alpha-1)\int_0^\infty \frac{\cos(u\lambda)}{u^{2-\alpha}}du\right| \\
&\quad \le \left|\int_0^\tau \left(\frac{(u+\tau)^\alpha + (\tau-u)^\alpha - 2u^\alpha}{\tau^2}\right)\cos(u\lambda)du\right| + \int_0^\tau \alpha(\alpha-1)\left|\frac{\cos(u\lambda)}{u^{2-\alpha}}\right|du \\
&\qquad + \left|\int_\tau^\infty \left(\frac{(u+\tau)^\alpha + (u-\tau)^\alpha - 2u^\alpha}{\tau^2}\right)\cos(u\lambda)du - \alpha(\alpha-1)\int_\tau^\infty \frac{\cos(u\lambda)}{u^{2-\alpha}}du\right| \\
&\quad \equiv I + II + III.
\end{aligned}$$

Note that from (17),

$$\begin{aligned}
II + III &\le \alpha(\alpha-1)\int_0^\tau \frac{1}{u^{2-\alpha}}du + C(\alpha)\tau^2\int_\tau^\infty \frac{1}{u^{4-\alpha}}du \\
&= \alpha\tau^{\alpha-1} + \left(\frac{C(\alpha)}{3-\alpha}\right)\tau^{\alpha-1} = \left(\alpha + \frac{C(\alpha)}{3-\alpha}\right)\tau^{\alpha-1} \to 0,
\end{aligned}$$

as $\tau\to0$. For term I, after applying L'Hôpital rule, one can easily obtain that $I\to0$ as $\tau\to0$. When$0<\alpha<1$, we cannot use (18) for our purpose. We first convert it to the case of $1<\alpha+1<2$ and then apply (15). Using integration by parts, one can obtain

$$\begin{aligned}
\int_0^\infty \cos(u\lambda)\left((u+\tau)^\alpha + |u-\tau|^\alpha - 2u^\alpha\right)du = \frac{\lambda}{1+\alpha}\Bigg[&\int_\tau^\infty \sin(u\lambda)\left((u+\tau)^{\alpha+1} + (u-\tau)^{\alpha+1} - 2u^{\alpha+1}\right)du \\
&+ \int_0^\tau \sin(u\lambda)\left((u+\tau)^{\alpha+1} - (\tau-u)^{\alpha+1} - 2u^{\alpha+1}\right)du\Bigg].
\end{aligned}$$

After divided by τ^2, one can easily apply the same reasoning as before to show that

$$\lim_{\tau\to0+}\int_\tau^\infty \sin(u\lambda)\left(\frac{(u+\tau)^{\alpha+1} + (u-\tau)^{\alpha+1} - 2u^{\alpha+1}}{\tau^2}\right)du = \alpha(\alpha+1)\int_0^\infty \frac{\sin(u\lambda)}{u^{1-\alpha}}du.$$

In addition, applying L'Hôpital rule, one easily has

$$\lim_{\tau\to0+}\int_0^\tau \sin(u\lambda)\left(\frac{(u+\tau)^{\alpha+1} - (\tau-u)^{\alpha+1} - 2u^{\alpha+1}}{\tau^2}\right)du = 0.$$

Combining the above both limits gives (16). This completes the proof of Lemma 4.

References

Blu, T., Unser, M., 2007. Self-similarity: part II - optimal estimation of fractal processes. IEEE Trans. Signal Process. 55, 1364–1378.

Brockwell, P.J., Davis, R.A., 2009. Time Series: Theory and Methods, second ed. Springer, New York.

Chilès, J., Delfiner, P., 2012. Geostatistics: Modeling Spatial Uncertainty, second ed. Wiley, New York.

Christakos, G., 1984. On the problem of permissible covariance and variogram models. Water Resour. Res. 20 (2), 251–265.

Cressie, N., 1993. Statistics for Spatial Data, revised ed. Wiley, New York.

Doob, J.L., 1953. Stochastic Processes. Wiley, New York.

Fischer, E., 1983. Intermediate Real Analysis. Springer, New York.

Gel'fand, I.M., 1955. Generalized random processes. Dok. Acad. Nauk SSSR 100, 853–856.

Gel'fand, I.M., Vilenkin, N.Y., 1964. Applications of Harmonic Analysis (Generalized Functions), Vol. 4. Academic Press, New York.

Gradshteyn, I.S., Ryzhik, I.M., 2007. Tables of Integrals, Series, and Products, seventh ed. Academic Press, New York.

Hannan, E.J., 1970. Multiple Time Series. Wiley, New York.

Huang, C., Hsing, T., Cressie, N., 2011a. Nonparametric estimation of variogram and its spectrum. Biometrika 98, 775–789.

Huang, C., Yao, Y., Cressie, N., Hsing, T., 2009. Multivariate intrinsic random functions for cokriging. Math. Geosci. 41, 887–904.

Huang, C., Zhang, H., Robeson, S., 2011b. On the validity of commonly used covariance and variogram functions on the sphere. Math. Geosci. 43, 721–733.

Itô, K., 1954. Stationary random distributions. Mem. College Sci. Univ. Kyoto, Series A 28, 209–223.

Ma, C., 2004. The use of the variogram in construction of stationary time series models. J. Appl. Probab. 41, 1093–1103.

Mandelbrot, B.B., Van Ness, J.W., 1968. Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10, 422–437.

Matheron, G., 1973. The intrinsic random functions and their applications. Adv. Appl. Probab. 5, 439–468.

Schoenberg, I., 1938. Metric spaces and positive definite functions. Trans. Amer. Math. Soc. 44, 522–536.

Solo, V., 1992. Intrinsic random functions and the paradox of 1/f noise. SIAM J. Appl. Math. 52, 270–291.

Stein, M., 1999. Interpolation of Spatial Data: Some Theory for Kriging. Springer, Berlin.

Unser, M., Blu, T., 2007. Self-similarity: Part I - splines and operators. IEEE Trans. Signal Process. 55, 1352–1363.

Yaglom, A.M., 1955. Correlation theory of processes with random stationary nth increments. Mat. Sb. 37, 141–196. (in Russian). English translation in

(1958) American Mathematical Society Translations: Series 2, 8, 87–141. American Mathematical Society, Providence, R.I., 1958.

Yaglom, A.M., 1987. Correlation Theory of Stationary and Related Random Functions, Vol. I. Springer, New York