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Five college students' involvement in creating mathematics and the resulting effects on their perceptions of the nature of mathematics, on their perceptions of their creative ability, and on their creative behavior

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FIVE COLLEGE STUDENTS' INVOLVEMENT IN CREATING MATHEMATICS AND THE RESULTING EFFECTS ON THEIR PERCEPTIONS OF THE NATURE OF MATHEMATICS, ON THEIR PERCEPTIONS OF THEIR CREATIVE ABILITY, AND ON THEIR CREATIVE BEHAVIOR

by

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APPROVAL PAGE

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GOODMAN, BILLIE W., Ed.D. Five College Students' Involvement in Creating Mathematics and the Resulting Effects on Their Perceptions of the Nature of Mathematics, on Their Perceptions of Their Creative Ability, and on Their Creative Behavior. (1992) Directed by Dr. A. Edward Uprichard. 244 pp.

This study was concerned with the development of original material, called triometry, a variation of trigonometry, and its use as reference material in a teaching experiment in creating mathematics. The expressed purposes of triometry were to give undergraduate mathematics or mathematics education majors exposure to new mathematical ideas, to serve as a medium through which students could engage in creating their own mathematics, and to change students' personal beliefs about the nature of mathematics and their own abilities to be creative. The teaching experiment was one half of a semester course of specials topics in mathematics offered as an elective. The subjects were five undergraduates who were either majoring or minoring in mathematics or mathematics education. The materials were evaluated by four professional mathematics educators as well as the subjects themselves. Evidences of changes in the perceptions and creative behaviors of the subjects were collected from surveys, interviews, questionnaires, student journals, assignments, and observations and were analyzed qualitatively.

The materials were deemed appropriate as a creative activity by the mathematics educators as well as the participants. The subjects' perceptions of mathematics as formula and rule driven did not change, but changes in their views of mathematics as a static body of knowledge with authority derived from a textbook were altered toward the views of professional mathematicians. The perceptions of the subjects regarding their own abilities to be mathematically creative were enhanced during the teaching experiment. The creative behaviors of the subjects showed slight improvement over the course of the teaching experiment.

CHAPTER I

BACKGROUND AND INTRODUCTION

Creative experiences in mathematics are practically non-existent in high school and college classrooms across the United States. (Mathematical creativity is defined on pg. 39). This is not a recently recognized shortcoming, but rather one that has existed for many decades. In the late 1950s and early sixties, "new math," although not designed to promote creativity, did seem to offer promise in this direction. However, lack of suitable materials and inadequate teacher training prevented "new math" from accomplishing its intended objectives as well as its unintended creative possibilities (Lee, 1978).

The "back to basics" movement of the seventies was a step backwards as far as mathematical creativity was concerned. Mathematics teaching was, according to Fitzgerald (1975, p. 40), "mechanistic" and "skill oriented. "Emphasis was placed on learning facts which was accomplished through lengthy drill and practice. Classroom activity typically consisted of a "how to" demonstration by the teacher and then an assignment of endless problems just like those shown by the teacher. Manipulation of numbers and symbols was stressed while mathematical reasoning was almost totally ignored. Consequently, students developed little conceptual understanding of mathematics (Fitzgerald, 1975).

In the 1980s, mathematics educators recognized that although knowing basic facts of mathematics is necessary, it is not sufficient. To be mathematically literate, students need to know how to reason and solve problems. These themes along with using calculators and computers more in the classroom became the rallying cry of the National Council of Teachers of Mathematics (NCTM) in their Agenda for Action: School Mathematics of the 1980s. Although this was a step in the right direction, mathematics curriculums in high schools and colleges changed little. Creative curriculum and opportunities for students to be creative in mathematics still did not occur frequently if at all.

As the decade of the nineties begins, creative experiences for students are still lacking. Students' sojourns in mathematics via the back to basics has given them no realistic conception of what mathematics is all about. There is a need for students to be able to experience the true nature of mathematics as a man-made subject to be explored and discovered. The importance of this need is verified by recent research (Garofalo & Lester, 1985; McLeod, 1988) which indicates that the beliefs that students have about mathematics directly influence their mathematical performance. According to

Garofalo (1989), most students believe that solving mathematics problems is simply a matter of applying a formula, rule, or procedure that is shown by the teacher or that appears in the textbook. Furthermore, thinking mathematically means "being able to learn, remember, and apply facts, rules, formulas, and procedures" (p. 503). In view of these beliefs, it not surprising that students have a difficult time when it comes to solving problems in an unfamiliar context.

However, this situation is not entirely of the students' making since students' attitudes and beliefs are shaped by the textbooks and the teachers who deliver the lessons (Garofalo, 1989; Cooney, 1988). Unfortunately, much of mathematics teaching remains mechanistic and skill-oriented with <u>the</u> right answer being the desirable end. As for discovering or creating mathematics, teachers, knowingly or not, convey to their students that this kind of activity is possible only for the most brilliant students (Garofalo, 1989; Cooney, 1988).

Why do teachers present mathematics lessons in such a manner? Because this is how they were taught! Murray (1984) contends that mathematics teachers have learned mathematics in the reverse order of the way it should be. Beginning in elementary school and continuing through high school, students are given already proved algorithms or rules, shown that they work, and then assigned numerous examples to practice. Mathematics learned at the undergraduate level is no different. According to Walter and Brown (1969, p. 38), mathematics, as presented in secondary courses and in undergraduate courses, is "less concerned with conveying the notion that mathematics is a 'creative' man-made activity and more interested in teaching the students 'polished' mathematics." There are few opportunities for exploration or discovery or for experiencing perplexing or disorganized examples which do not fit the hypothesis (Murray, 1984). Indeed, true problem-solving opportunities are rare in college mathematics courses (Cooney, 1988). Unfortunately, these teachers go on to teach mathematics in the same manner because they know no other way (Murray, 1984). Thus, it seems, there exists a cycle of mathematics instruction and learning that perpetuates this state of affairs.

It is evident, then, that to bring about the desired changes in the manner in which mathematics should be learned, changes in curriculum and instruction must be implemented throughout the formal schooling of students. NCTM's Curriculum and Evaluation Standards for School Mathematics sets forth a realistic "vision" of what a mathematics curriculum should include, and if adopted by schools and teachers, promises to have a great impact on the mathematics learning of all students, kindergarten through high school. The Standards calls for learning experiences "that encourage and enable students to value mathematics, gain confidence in their own mathematical ability, become mathematical problem solvers, ..., and reason mathematically" (p.123). Curricular materials should emphasize conceptual understanding and mathematical problem solving rather than memorization of isolated facts and paper and pencil drills. In addition, "curricular materials should develop new topics or ideas as natural extensions or variations of ideas students already know" (p. 242).

The NCTM Standards are attainable goals, but the key to their implementation lies at the undergraduate level where future teachers get their training. In the National Research Council's Everybody Counts (1989), undergraduate mathematics is viewed as "the linchpin for revitalization of mathematics education" (p. 39). Through the undergraduate experience, future teachers acquire not only content knowledge but also attitudes about mathematics and styles of teaching. A revitalization of undergraduate mathematics must then include curriculum as well as teaching style.

The mathematics background of most mathematics/ mathematics education majors typically contains calculus, modern and linear algebra, intermediate algebra, differential equations, some analysis and numerical methods, and geometry. These are courses in which the content has been known and taught for scores of years. They are usually presented as fixed bodies of knowledge, "polished" mathematics according to Walter and Brown (1969). To add the study of additional higher mathematics courses is not the answer. Owens (1987) found that more training in abstract mathematics only contributes to preservice secondary mathematics teachers' conceptions that mathematics is essentially an exercise in manipulating symbols. What students need is innovative curriculum and instruction that "conveys the notion that mathematics is a subject to be explored and created" (Cooney, 1988, p. 359). Mathematics educators (Walter and Brown, 1971; Schoenfeld, 1987; Cooney, 1988) are in agreement that students should have the opportunity to create mathematics on their own, to make and decide on definitions, to learn to pose problems, and to experience the satisfaction of doing something original, in essence, to experience mathematics much as mathematicians do. Through such experiences, students are more likely to develop a more accurate view of the nature of mathematics and how it is done, and as teachers, to pass these ideas on to their students.

Statement of the Problem

Evidence that is available suggests a two-fold problem. First, there are few creative experiences in mathematics available at the undergraduate level. Second, students' misconceptions of the nature of mathematics deleteriously affect their doing of mathematics. Thus, there is a need to provide creative experiences for students which will give them the correct perceptions of the nature of mathematics. Addressing both parts of the problem necessitates finding suitable materials.

The investigator created an original body of material, called triometry, to be used in a teaching experiment. Questions proposed by this study are:

1. Is triometry suitable as a creative activity for mathematics/mathematics education majors?
2. Are students' perceptions of the nature of mathematics enhanced as a result of their experiences in the teaching experiment?
3. Are students' perceptions of their ability to be mathematically creative enhanced as a result of their experiences in this study?
4. Are students' creative behaviors in mathematics enhanced as a result of their experiences in this study?

Purpose of the Study

The primary purpose of this study was to develop and evaluate triometry, an original body of material that employs trigonometry as a medium. The expressed purposes of triometry are: to give students (mathematics/ mathematics education majors) exposure to new mathematical ideas that are not readily accessible in other reference texts; to serve as a springboard for engaging students in creating their own mathematics; to change students' personal beliefs about the nature of mathematics and their own ability to be creative; and to provide opportunities for students to experience mathematics in the making as mathematicians do.

Significance

The present mathematics curriculum for mathematics or mathematics education majors contains few if any opportunities for students to experience new and inventive ideas. Students are locked into the traditional study of courses in which the emphases is on learning a specified body of material. This approach reinforces the beliefs and attitudes learned from their high school teachers--that mathematics is textbook driven, is a matter of learning and applying rules and procedures, that teachers are all-knowing, and that only extremely talented students can create mathematics.

Triometry can provide the opportunity for students to create mathematics on their own. Some side benefits may be that students will gain a deeper understanding of the underlying concepts of trigonometry, will learn to appreciate the beauty of mathematical theories, and will enhance their beliefs about mathematics learning that they can then pass on to their future classrooms.

Organization of the Study

Chapter II contains a review of literature that pertains to the study. Chapter III gives definitions for the terms used in the study, describes the research methodology and design, discusses instrumentation used for data collection, and describes data analysis procedures. Chapter IV presents the results of the experiment with respect to individual students, as well as the results of evaluations of triometry. Chapter V contains discussion, conclusions, and recommendations for future research. The appendices contain all pertinent documents.

CHAPTER II

REVIEW OF RELATED LITERATURE

This chapter investigates topics that will provide background for the study. Since beliefs of students about the nature of mathematics is a central part of this study, it is relevant to explore the nature of mathematics as mathematicians view it, students' beliefs about mathematics, and factors that contribute to these beliefs. This study also involves a teaching experiment with the expressed purpose of engaging students in creative activity. Hence, creativity in the mathematics classroom is explored. This includes creative activity, content, role of the teacher, and method of instruction.

Nature of Mathematics

The nature of mathematics as it is regarded today is founded on mathematical philosophy and is evident in the work of mathematicians and in the views expressed by them. There are basically four philosophical schools of thought that have influenced the development of mathematics. Each approach is an attempt to give all of mathematics a sound foundation having consistency and without contradiction, and each has contributed to modern thought about the nature of mathematics as we know it today.

The first of these views, Platonism, was the dominant philosophy until the late nineteenth century. Platonism is the mathematics of Euclid which is based on the philosophy of Plato (Davis & Hirsh, 1981). A Platonist mathematician views all of mathematics as existing independently of human thought. Goodman (1979, p.548) describes Platonism thusly:

> Mathematics consists of truths about abstract structures existing independently of us, of the logical arguments that establish those truths, of the (mental) constructions underlying those arguments, of the formal manipulation of symbols that expresses those arguments and truths, and nothing else.

It is the mathematician's job to discover these mathematical truths. This is accomplished through rigorous proof, beginning with self-evident truths. According to Platonism, a mathematician cannot create or invent mathematics, he/she can only discover. Rene Thom, a renowned Platonist, writes (1971, p. 696),

> Everything considered, mathematicians should have the courage of their most profound convictions and thus affirm that mathematical forms indeed have an existence that is independent of the mind considering them...Yet, at any given moment, mathematicians have only an incomplete and fragmentary view of this world of ideas.

Most applied mathematicians ascribe to the Platonist philosophy.

The second view, logicism, was begun around 1884 by Gottlob Frege, a German mathematician and philosopher and was rediscovered some twenty years later by the philosophers-mathematicians Bertrand Russell and Alfred North Whitehead (Davis & Hirsh, 1981). The basic tenet of logicism is that all of mathematics is derivable from the laws of logic. Like Platonism, logicism avows the existence of abstract entities such as numbers, sets, and functions, independent of thinking which the mind can discover but not create (Snapper, 1979). To a logicist, mathematical theories have no factual content and so their (logical) truth must be established solely on the basis of their own internal structure and their relations to one another (Goodman, 1979). Theorems, then, are regarded as long and complex tautologies.

The contribution of the logicists cannot be denied since a great deal of actual mathematical practice involves the application of logic. According to Goodman (1979), logicism, more than any other philosophy, has made greater contributions to our understanding of the foundations of mathematics. He reasoned (p.547):

> The desire to reduce all of mathematics to "logic"--that is, to merely conceptual reasoning--has provided a strong impetus to simplify and unify the basic mathematical notions and to find and make explicit the fundamental principles upon which mathematics is based.

The third philosophical view of mathematics, intuitionism or constructivism, was originated by the Dutch mathematician L. E. J. Brouwer around 1908 in response to what he (Brouwer) felt was an undermining of the foundations of mathematics by the uncovering of paradoxes in Cantor's set theory by Russell and others about 1900 (Davis & Hirsh 1981). This view holds that mathematics "consists of intuitive (mental) constructions, of the formal manipulation of symbols which is their external expression, and of nothing else" (Goodman, 1979, p. 544). Intuitionists view all of mathematics as starting with the natural numbers which are intuitively known. To be considered meaningful and to exist, mathematical objects (theorems) must be constructable in a finite number of steps beginning with the natural numbers. A constructive proof is one that tells step by step how to calculate or construct the object to which a theorem refers. From the intuitionist viewpoint, the square root of 2 does not exist since it cannot be constructed from the natural numbers in a finite number of steps. Intuitionism is said to deny the "law of the excluded middle" which is to say that every proposition is either true or false. Most modern practicing mathematicians find this rather restrictive, particularly when dealing with infinite sets.

The fourth view, formalism, became the predominant mathematical philosophy in the mid-twentieth century (Davis

& Hirsh, 1981). It was conceived in its modern form by the German mathematician David Hilbert around 1910. According to formalism, mathematics may be regarded as "the rule-governed, or formal, manipulation of symbols, and nothing else" (Goodman, 1979, p. 542). Davis and Hirsh (1981, p. 319) assert that formalist mathematics "... consists of axioms, definitions and theorems--in other words, formulas...but the formulas are not about anything; they are just strings of symbols." The symbols have no meaning and a formalist merely investigates possible relationships among the symbols according to some agreed upon rules of manipulation. Sometimes these formulas have physical applications and, thus, acquire a meaning which may then be judged as being true or false. But when regarded as purely mathematical formulas, they have no meaning and, hence, no truth value. In this regard, mathematics is seen as a meaningless game. Others regard it as a game of logical deduction. Regardless of the viewpoint, formalism stresses rigorous proof in which one begins with some undefined terms, definitions, and axioms and then proceeds to prove conjectures or theorems according to some specified rules. Under formalism, mathematics is created rather than discovered.

It is interesting to note that both the logicists and formalists formalized the different branches of mathematics (Snapper, 1979). Each area of mathematics that is based on formalize an axiomatized theory, one replaces the variables, connectives, quantifiers, and undefined terms of the theory with symbols. Arithmetic with Peano's axioms is an example of an axiomatized mathematical theory. Using "0" and "+", respectively, for the undefined terms "zero" and "addition" is an example of formalization. Hilbert's formalism was an attempt to free all of mathematics from contradiction while logicism sought to prove that it belonged to logic.

The philosophical basis for most modern mathematicians is, for the most part, a composite of Platonism and formalism. The majority of writers on the subject view the typical working mathematician as

> "... a Platonist on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all (Davis & Hirsh, 1981, p. 321).

P. J. Cohen, a contemporary mathematician in set theory, views the philosophical plight of the working mathematician thusly:

> To the average mathematician who merely wants to know his work is accurately based, the most appealing choice is to avoid difficulties by means of Hilbert's program. Here one regards mathematics as a formal game and one is only concerned with the question of

> consistency...The Realist (i.e., Platonist) position is probably the one which most mathematicians would prefer to take. It is not until he becomes aware of some of the difficulties in set theory that he would even begin to question it. If these difficulties particularly upset him, he will rush to the shelter of Formalism, while his normal position will be somewhere between the two, trying to enjoy the best of two worlds (cited in Davis & Hersh, 1981, p. 321).

As for intuitionists (constructivists) they "...are a rare breed, whose status in the mathematical world sometimes seems to be that of tolerated heretics surrounded by orthodox members of an established church" (Davis & Hirsh, 1981, p. 322). Intuitionism has all but been abandoned by the mathematical community because of its restrictive nature dealing with infinite sets and because many proofs are made long and laborious.

Regardless of how others may categorize them, practicing mathematicians of today, for the most part, are concerned very little with which philosophical school they may belong.

> Probably the great majority of mathematicians have spent little, if any, time speculating on the question of possible membership in a "school of thought." They have been either too busy doing research at the higher levels of their field or disdainful of such a question (Wilder, 1965, p. 246).

Accepting what has gone before, they simply pursue their research trying to discover or create new mathematics (Crothamel, 1986).

In light of the above discussion, what is it that mathematicians do and how do they regard mathematics? According to Halmos (1968), the work that mathematicians do has very little to do with numbers, or solving a right triangle with trigonometry, or determining the rate of change by calculus. Nor are mathematicians concerned mainly with "proving theorems" which is analogous to saying the main job of a writer is "writing sentences" (Davis & Hirsh, 1981). Gian-Carlo Rota, in the introduction to The Mathematical Experience (Davis & Hirsh, 1981, p. xviii), sees the mathematician's work as

> ...mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks.

In Halmos's view, mathematicians see themselves and others as either problem-solvers or theory-creators. At work, the mathematician "...makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions" (p. 381). They are most often interested in extreme cases. What happens if some conditions are relaxed or made more stringent? What happens if some of the rules are changed just a little? Non-Euclidean geometries are examples of such wonderings. Mathematicians experience many attempts, many false starts, many discouragements, many failures, and a few successes.

There are a number of facets of the nature of mathematics on which mathematicians generally agree. Scheding (1981) summarized some of the characteristics from the writings of mathematicians (e.g. Hardy, 1967; Halmos, 1968; Poincare, 1963; Hammer, 1964; Sawyer, 1955; Lakatos, 1976) some of which are:

1) Elegance of mathematical proof is desirable. A proof is elegant if it contains the elements of unexpectedness, inevitability, and economy.
2) Mathematics is concerned with patterns and mathematicians are makers of patterns. These patterns are valued for their beauty and aesthetic value.
3) Mathematics is a creative activity. Whether the ideas are original or not does not matter.
4) Mathematics deals with ideas and relationships between ideas rather than with numbers and manipulation of numbers.
5) The role of insight and intuition in mathematics is very important. Contrary to the layman's belief, mathematical proofs are not discovered in the neat, concise, deductive form found in textbooks.
6) Mathematical thinking involves both inductive and deductive reasoning. Inductive reasoning is reasoning from the specific to the general case whereas the reverse is true of deductive reasoning.
7) Mathematics is an organized body of knowledge most of

which is axiomatized.

8) Most of what is known as "good" mathematics is generalizable which means it can be applied to various concrete or abstract situations.

Scheding used these characteristics to construct a survey to determine the perceptions of teachers and prospective teachers of mathematics about the nature of mathematics. This instrument was used in the teaching experiment and is discussed further in the Data Collection section of Chapter III.

Beliefs of Students about the Nature of Mathematics

Research in mathematics education in recent years has revealed that success or failure in solving mathematics problems often depends on much more than knowing the appropriate rules, procedures, or facts. Indeed, the beliefs that students have regarding the nature of mathematics and mathematical tasks and beliefs about themselves and others as doers of mathematics greatly influences their mathematical performance (Garofalo, 1989). Schoenfeld (1985) contends that

> belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks. One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on (p. 45).

Research shows these beliefs are not unique to particular groups of students. Rather they are embraced by students with wide ranging abilities and ages.

Schoenfeld (1983, 1985, 1987, 1989) has studied the problem solving activities of students ranging from secondary school through college. Using questionnaires, interviews, and extensive video-taping of problem solving sessions, he has extracted some notions commonly held by students which he stated as beliefs.

> Belief 1: Formal mathematics, and proof, have nothing to do with discovery or invention. When students work discovery problems they tend to ignore the results of formal mathematics.

Schoenfeld found that when students were asked to solve a geometry problem involving a construction, they were unable to do so even though all participants involved had correctly solved a related problem with a formal proof only shortly before. Thus, to many students, proofs, such as those done in geometry, are done only to verify what is already known and have no other application or purpose.

> Belief 2: If one really understands the material, all mathematics problems can be solved in ten minutes or less and should be quickly solvable in just a few steps.

If a problem is not solved within this time frame, students believe that either something is wrong with the problem or that they do not understand the material and then just give up.

> Belief 3: Only mathematically talented geniuses are capable of creating or discovering or understanding mathematics.

Students with this belief become passive recipients of mathematical knowledge dispensed by the teacher and the textbook. They accept what is presented to them at face value, memorize it, and expect to regurgitate the same without hope or expectation of understanding. The idea of deriving a formula or of producing their own mathematics is foreign to them since they are not geniuses.

Frank (1988) reported on the beliefs of mathematically talented junior high school students enrolled in a two-week problem solving with computers course. Data from her study are based on a survey of mathematical beliefs, observation of students in problem solving sessions, and a number of interviews with four students. Students in her study viewed mathematics as computation, meaning addition, subtraction, multiplication, division. To these students, "doing mathematics" meant following the rules and "learning mathematics" was mostly memorization (p. 33). Furthermore, they believed that if they did these two things well, they would accomplish the goal of doing mathematics which is to get the right answers. Work that produced a wrong answer was deemed a worthless experience by the students and was a sign to them that they did not understand the material. Like Schoenfeld's subjects, Frank's students expected quick, short solutions to all mathematics problems.

In addition to their beliefs about the nature of mathematics, students had clear expectations regarding the roles of students and teachers in mathematics. Students were the receivers of mathematical knowledge and they acknowledged their reception by producing right answers. Teachers and textbooks were the authorities in and dispensers of that knowledge and the verifiers that students had received that knowledge.

Another researcher, Garofalo (1989), formulated a set of student-held beliefs gleaned from his experiences as a mathematics teacher, as an observer of mathematics class in secondary schools, and from discussions with students, preservice teachers, and secondary school teachers. Several of his "beliefs" are similar to those already discussed. Some additional beliefs that are typically held by secondary school students (pp. 502-503) are:

> Belief 1: Almost all mathematics problems can be solved by the direct application of the facts, formulas, and procedures shown by the teacher or given in the textbook.
>
> Corollary: Mathematical thinking consists of being able to learn, remember, and apply facts, rules, formulas, and procedures.
>
> Belief 2. Mathematics textbook exercises can be solved only by the methods presented in the textbook; moreover, such exercises must be solved by the methods presented in the section in which they appear.
>
> Belief 3. Only the mathematics to be tested is important and worth knowing.

<u>Corollary</u>: Formulas are important, but their derivations are not.

Students who embrace these beliefs want their mathematics pre-packaged with explicit directions written on the outside. They approach the study of mathematics in a mechanical fashion, memorizing only those facts or formulas needed for a test and making little attempt at understanding.

<u>Factors That Contribute to Students' Beliefs</u>

How do the ideas that students have about mathematics get into their heads? "Beliefs about mathematics, like beliefs about anything else...are shaped by one's environment" (Schoenfeld, 1987, p. 36). It is disconcerting to realize that the beliefs expressed by students are shaped by practices in the classroom. From a year-long observation of a tenth grade geometry class, Schoenfeld (1987; 1988) concluded that even though the class was well taught and the students performed well on a state mandated standardized test, the students learned "some inappropriate and counterproductive conceptualizations of the nature of mathematics as a direct result of their mathematics instruction" (1988, p. 146). In lessons on constructions using compass and straightedge, emphasis was placed on speed and accuracy. It was important that the students memorize the steps of the constructions so that they would be able to do the constructions quickly and accurately on the test. There was no mention of understanding or proof. As a result of such instruction, students learned an unintended lesson--that learning mathematics is mostly memorization, and that correct answers are more important than solutions. When teachers spend as much as 70 percent of the year on computational algorithms and memorization of facts (Frank, 1985), when most mathematics word problems require only a straightforward calculation, the message to students is quite clear.

In typical geometry classes, students are taught to write proofs in a certain format called a two-column proof. First, the student writes the problem stating the "given" and the "prove." Next, students divide their paper into two columns, write numbered statements in the left column and correspondingly numbered reasons that justify each statement in the right column with the first statements being what is "given." In some cases, more class time is spent correcting the form of a student's proof than on the correctness of the proof. Schoenfeld (1988) observed one session in which 22 of the 37 minutes spent discussing one student's proof was spent on the form. Is it any wonder that students come to believe that form of expression is as important as substance?

The idea that all mathematics problems can be solved in ten minutes or less is a result of homework assignments, tests, and even such standardized tests as the SAT. It is not unusual for students to be assigned 20, 30, or even 50 problems for homework in an arithmetic or algebra class. The length of time to complete the assignment would range from 20 minutes to one hour. Schoenfeld notes that on a unit test in the geometry class which he observed, students were given 54 minutes to work 25 problems--an average of 2 minutes and 10 seconds per problem.

Everyday, in typical mathematics classrooms across the United States, students are fed an agreed-upon body of knowledge, consisting mostly of facts and procedures, in small, easily digestable pieces, and then are rehearsed so as to promote mastery. Schoenfeld contends that mathematics taught in this way causes students to regard themselves as "passive consumers of others' mathematics" (1988, p. 160). There are few opportunities for exploration, and thus, students are often denied the possibility of making sense of the mathematics on their own. As a result, students seek only to know how to use a procedure without trying to understand why it works. They perceive themselves as being incapable of understanding knowledge that has seemingly come "from on high." Besides, why bother if they do not need to know it for the test.

Research over the past few years has shown that the influence of teachers' conceptions of mathematics on their classroom instruction cannot be denied. In Thompson's (1984) case study of three junior high school mathematics teachers, one teacher conceived of mathematics as a "challenging subject whose essential processes were discovery and verification" (p. 119). She encouraged students to make conjectures, to explore, and to try to reason things out on their own. A second teacher viewed mathematics as "essentially prescriptive and deterministic in nature" (p. 119). She presented mathematical content as a static body of knowledge with emphasis on computation. Through her instruction, mathematics was portrayed as a collection of rules and procedures for finding answers. The third teacher regarded mathematics as consisting of logically interrelated topics. She emphasized mathematical meanings of concepts and the logic of mathematical procedures in her teaching, even though, like the second teacher, she presented the content as a finished product. Thompson concluded,

> teachers' beliefs...about mathematics and its teaching, regardless of whether they are consciously or unconsciously held, play a significant, albeit subtle, role in shaping the teachers' characteristic patterns of instructional behavior (p. 124).

There are other forces at work in the classroom that send hidden messages about mathematics. For teachers as well as students, the textbook is the primary source of materials and information. Textbooks that consistently place emphasis on step-by-step algorithmic procedures for solving problems, and that present "problems" that can be solved by blindly applying the procedures studied in that section convey undesirable impressions of the nature of mathematics. In addition, standardized tests as well as most teacher-made tests emphasize mechanical, algorithmic procedures and send the message that mastery of mathematical concepts is the name of the game. But then, who can blame teachers for preparing their students in the manner in which they will be tested when teachers and students alike are judged by the outcomes of scores on such tests (Schoenfeld, 1988)?

Creativity in the Mathematics Classroom

In this discussion of creativity in the mathematics classroom, there are four areas to consider: 1) creative activity, 2) the curriculum, 3) teachers' roles, and 4) method of instruction.

What is creative activity? All too often, the concept of a creative act is one that has produced something totally new or original. Hall (1978) cites a number of sources that disagree with this view. Koestler (in Hall, 1978) asserts that the creative act "selects and combines that which is already existing" (p. 22). Similarly, to Barron (in Hall, 1978) being creative is the reconstitution of something old to make something new. It is placing things in new perspectives so that one becomes aware of relationships not previously seen (Bruner in Hall, 1978); it is "the ability to "toy" with ideas" (Rogers in Hall, 1978, p. 19). Hammer (1964) states, "Creative activity can occur in many unsalable forms...the recognition of a pattern, an analogy, the smoothing over of a quarrel, the phrasing of a sentence..." (p. 518). In mathematics, creative activity may include making generalizations, discovering a relationship or proof, or solving a problem in some unique way. Hammer contends that to consider only "masterpieces of creativity" as being creative discourages students in their efforts and sends the message that only the great can create. No matter how many have done it before, students who discover a relationship or proof previously unknown to them are being creative.

As Hall (1978) points out, creative experiences do occur in most secondary school mathematics classrooms though these occasions are usually spontaneous and sporadic. However, planning for creativity is not only possible but desirable. In her model for a creative mathematics classroom, one necessary component is creative course content. This calls for material that reveals the essential nature of mathematics, but it does not necessarily have to be relevant to "the here and now" (p. 94). The author suggests giving students an arbitrary mathematical system with definitions and postulates and asking them to formulate as many theorems as possible. Such an activity would encourage students to assume a mathematician's role in searching for relationships.

Another essential feature of a creative mathematics curriculum advocated by Hall is that students be exposed to problems with many right answers. Such exposure she contends promotes divergent thinking and undermines the "fixed answer syndrome" (p. 92) that is prevalent in mathematics classrooms.

The NCTM Standards calls for curricular matherials that "develop new topics or ideas as natural extensions or variations of ideas students already know, thus making connections among topics explicit" (1989, p. 242). Suggestions for generating such materials include considering the converse of a problem, restricting or relaxing the conditions in a problem, or generalizing from a problem. The NCTM Teaching Standards (working draft, 1989) adds that the appropriateness of a particular task depends on the students' abilities and what they already know. In addition, topics do not have to relate to the familiar worlds of the students but can be "theoretical or fanciful" (p. 25).

In creative mathematics classrooms, teachers play two roles--that of risk-takers and of facilitators (Borenson, 1983). Teachers become risk-takers when they give up their roles as dispensers of knowledge and venture into unknown areas with the students. Torrance (1963) states:

> In contrast to stubbornly retaining the comfort and safety of the time-tested process and the well-travelled pathway, the teacher must be willing to permit one thing to lead to another, must be ready to break out of the mold, rather than look upon children in traditional ways, through stereotyped attitudes and thus fail to relate to them as real persons (p. 10).

Thus, teachers become learners along with their students in the search for knowledge and understanding.

When students venture into unfamiliar territory, there are no guarantees of attaining any concrete mathematical results. Hence, there is the possibility that students will become disillusioned by the experience. But the teachers, as risk-takers, are willing to take that chance knowing that the process of searching for knowledge and understanding is worthwhile in and of itself, and that they can relate this to the students (Borenson, 1983).

Throughout the NCTM Standards (1989), the role of the teacher is stressed to be one of facilitating learning rather than of dispensing information. Hall (1978) states that students need to feel comfortable in expressing their ideas and thoughts in order for creativity to occur. Teachers as facilitators establish a classroom atmosphere that is conducive to creative work by accepting all students' responses without judging them, by grouping students so that they can share ideas, by promoting a climate free of ridicule, and by encouraging students to present their ideas at the chalkboard (Borenson, 1981).

Teachers are also facilitators in that they encourage students to make conjectures and to formulate propositions and proofs. They seek to promote students' understanding of the task by asking them to clarify or simplify their observations, propositions, or proofs (Borenson, 1981).

Finally, to <u>be</u> facititators, teachers must have some understanding of a research and discovery process and be willing to try it in their classrooms. It is desirable, then, that teachers have some experiences in exploring and creating mathematics. The NCTM Professional Standards for Teaching Mathematics states:

> Teachers need to explore mathematics and to conduct their own inquires. Looking for patterns, making conjectures, constructing and evaluating arguments, and seeking generalizations should be an integral part of the mathematics content experience. Through such activities, teachers gain confidence in their ability to reason and justify their thinking and to make sense of mathematics. ...The struggles, the false starts, the informal investigations that lead to the elegant proof frequently are missing. <u>Teachers need to construct mathematics for themselves</u> [writer's emphasis] and not just experience the record of others' constructions (working draft, 1989, p. 71-72).

In a mathematics classroom that seeks to foster creativity, students must become active participants as opposed to being passive recipients of someone else's knowledge. The method of instruction that is conducive to this kind of learning is called a guided discovery

(Borenson, 1983) or inquiry-discovery approach (Hall, 1978). Learning by discovery is not a new idea. Socrates, by asking leading questions, guided students to "discover" relationships and solve problems (Hall, 1978). In fact, this technique is sometimes referred to as the "Socratic" method. The inception of "new math" created a renewed interest in discovery learning. During this period, there were a number of curriculum committees that expounded the virtues of discovery learning. The University of Illinois Committee on School Mathematics (1961, cited in Brown, 1971) states that through learning by discovery, students gain a better understanding of mathematical concepts, develop more positive attitudes toward mathematics, and are motivated to want to learn mathematics. The development of creativity and independence in students is claimed by the Cambridge Conference on School Mathematics in their report Goals for School Mathematics (1963, cited in Brown, 1971). They state:

> The discovery approach, in which the student is asked to explore a situation in his own way, is invaluable in developing creative and independent thinking in the individual. In this system, memorizing a mechanical response does not help the student to advance (in Brown, p. 233).

In summary, mathematics educators claim that through discovery, students are motivated to learn mathematics, will understand what they learn, will learn to think, and will become more creative.

A large portion of the materials developed for guided discovery learning use the inductive approach wherein students reason from a number of specific examples of the attribute or concept to a generalization of the attribute or concept. Brown (1971) cautions that some of these "discovery" activities are mechanized and little different from programmed texts. For example, a series of problems designed to discover the distributive property might typically look like this:

$$3 \times 14 + 7 \times 14 = \underline{42} + \underline{98} = \underline{140} = \underline{10} \times 14$$

$$6 \times 25 + 3 \times 25 = \_\_ + \_\_ = \_\_\_ = \_\_ \times 25$$

$$5 \times 9 + 7 \times 9 = \_\_ + \_\_ = \_\_ \times 9$$

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This type of exercise, which is essentially "..filling in the blanks of someone else's digested thinking" (p. 236), gives limited opportunity for students to organize the mathematical concepts themselves. In another example, properly chosen examples may lead students to "discover" a wrong generalization such as:

$$1/2 - 1/3 = 1/2 \cdot 3$$

$$10/2 - 10/3 = 10/2 \cdot 3$$

$$1/3 - 1/4 = 1/3 \cdot 4$$

$$7/3 - 7/4 = 7/3 \cdot 4 \quad \text{(p. 237).}$$

The student is thus led to generalize, incorrectly: $a/b - a/c = a/bc$, for all real numbers a and non-zero real numbers b and c. Brown's conclusion is that discovery exercises for students should involve more than filling in the blanks and generalizing from just a few examples. Rather than just relying on the incidences given, students should be encouraged to try their own examples before attempting to generalize. Incidentally, the second example does have a "discovery" aspect. Under what conditions does $a/b - a/c = a/bc$?

Two well-known advocates of learning by discovery are Jerome Bruner and George Polya. Bruner (1966) hypothesizes that emphasis on discovery "helps the child to learn the varieties of problem solving, of transforming information for better use, helps him to learn how to go about the very task of learning" (p. 87). Through the effort of discovery, students learn the heuristics of discovery that can be generalized for solving other tasks. Bruner contends that discovery learning has motivational value in that the rewards expected by students shift from extrinsic to intrinsic.

> To the degree that one is able to approach learning as a task of discovering something rather than "looking about" it, to that degree there will be a tendency for the child to work with the autonomy of self-reward, or, more properly, to be rewarded by discovery itself (p. 88).

Finally, discovery learning contributes to conservation of memory. Bruner asserts that retrieval of information is the principal problem of human memory rather than storage, and the key to retrieval is organization which is an important aspect of the discovery process.

George Polya, in his book How to Solve It (1966), indicates how teachers can guide students to make discoveries by the skillful posing of questions. Some examples are: Can you change the unknown or data so that the new unknown and data are closer to each other? Do you know a related problem? Will relaxing some of the conditions help? Can you solve a special case? Have you used all of the pertinent data? Can you guess an answer? Such an approach, Polya believes, "alerts the student to the principles of discovery and...gives him an opportunity to practice these principles" (Davis & Hersh, 1981, p. 285).

Summary

The review of literature discussed four philosophical points of view regarding the nature of mathematics--Platonism, logicism, intuitionism, and formalism. Although most modern mathematicians do not profess allegiance to any particular philosophical school, they are typically a composite of Platonism and formalism. The work that mathematicians do and the components of the nature of mathematics with which mathematicians agree were also discussed.

The beliefs of students about the nature of mathematics and the importance of these beliefs as it relates to students' performance in the classroom also was presented. Basically, students believe that mathematics is mechanical, is mostly memorization, and that mathematics problems can be solved in ten minutes or less or else they are impossible. Furthermore, they believe that only geniuses are capable of creating mathematics.

The literature review has shown that students' beliefs are influenced by a number of factors. Classroom practices such as type of assignments, emphasis on "the" right answer and form, method of presentation, and even tests contribute to these beliefs. Other factors include beliefs of the teachers themselves, reliance on the textbook as the principal source of information, and standardized testing.

The section on creativity in the mathematics classroom included four areas of interest--creative activity, the curriculum, teachers' roles, and method of instruction. To many authorities, creative activity in mathematics does not have to be something that no one has ever done before, but rather may include discovering a relationship or proof or solving a problem in some unique fashion. The discussion of curriculum materials contained the features of a creative mathematics curriculum and suggestions for developing such a curriculum. The teachers' roles in a creative mathematics classroom were defined to be two-fold. As risktakers, teachers relinquish their roles as chief dispensers of knowledge to explore along with their students. Teachers as facilitators establish a climate in the classroom that will encourage creative activity on the part of the students. The importance of personal experiences of teachers in exploring and creating mathematics was also noted. Finally, the method of instruction most conducive to creative activity in mathematics is guided-discovery or inquiry-discovery. The essential features of this method were presented along with some cautions for effective utilization.

CHAPTER III

METHODOLOGY

As indicated in Chapter I, the primary objective of the teaching experiment was to evaluate new mathematics material, called triometry, created by the investigator. This was a study which sought to determine the suitability of triometry as a creative activity for mathematics or mathematics education majors. In addition, the experiment sought to determine whether exposure to creative activities via triometry enhanced the students' beliefs about the nature of mathematics and/or enhanced their creative behavior in mathematics as well as their confidences in their abilities to engage in creative activities.

Definition of Terms

The definitions of all of the terms, with the exception of triometry, have been derived from the research literature. Hence, the reason for their placement in this chapter.

In the present study, nature of mathematics includes the attributes that characterize mathematics in general and the nature and attributes of the work of the professional mathematician. These attributes (listed in Appendix A) are taken from Scheding (1981). They represent the views to which the mathematicians in Scheding's study generally agreed. The correct view of the nature of mathematics refers to these same attributes.

Professional mathematicians and mathematicians refer to members of the mathematical sciences faculties of colleges and universities of the rank of lecturer or above, and to other persons whose primary jobs are to engage in mathematical research.

Mathematical creativity involves any one of the following activities: selecting and combining that which is already existing (Koestler in Hall, 1978); reconstituting of something old to make something new (Barron in Hall, 1978); placing things in new perspectives so that one becomes aware of relationships not previously seen (Bruner in Hall, 1978); toying with ideas (Rogers in Hall, 1978); recognizing a pattern, making an analogy, or solving a problem in a unique way (Hammer, 1964). For example, it is a well known concept in plane geometry that the shortest distance between two points is a straight line (Figure 1). However, if the two points are corners on city streets that run east-west and north-south, then the shortest distance one could walk from point A to point B is certainly not a straight line distance (Figure 2). Thus, a new definition of "shortest distance between two points" is needed. If c represents the shortest distance from A to B, then instead of the familiar Pythagorean Theorem in which

$c = \sqrt{a^2 + b^2}$, we have $c = a + b$.

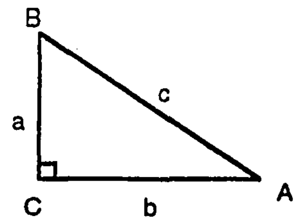


Figure 1

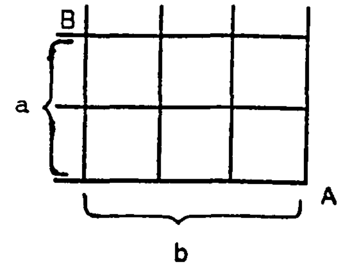


Figure 2

The term belief with respect to mathematics and mathematical tasks refers to assumptions, conceptions, perceptions, or views that one has with regard to the subject or the task. Attitude is ruled out as a descriptor since it suggests affective elements such as like/dislike, easy/difficult, exciting/frustrating, etc.

Triometry (see Appendix B) will designate the material created by this investigator for the teaching experiment.

Design of the Study

Data were collected and analyzed using a qualitative research design approach. Qualitative research, sometimes called naturalistic inquiry, is a form of descriptive, non-experimental research in which description and explanation of events and actions are sought rather than prediction based on cause and effect (Merriam, 1985; 1988). (It should be noted that although only one source is referenced, these ideas represent a composite of the works of noted authorities in qualitative research methods such as Guba &

Lincoln, Geertz, Yin, and Stake). Inquiry is carried out inductively with emphasis on process, understanding, and interpretation rather than deductively and experimentally. The results of qualitative research represent a holistic description and analysis of the situation or phenomenon and is characterized by "thick description." This involves "...literal description of the entity being evaluated, the circumstances under which it is used, the characteristics of the people involved in it, the nature of the community in which it is located, and the like..." (Geertz, in Merriam, 1985, p. 206).

As with other types of research, the issues of validity, reliability, and generalizability are of concern in qualitative research. However, there are ways of dealing with these concerns. A distinctive strength of qualitative research is the ability to use a variety of evidence such as interviews, observations, and documents like surveys and questionnaires. The use of multiple sources, called triangulation, serves to enhance validity of the findings. Reliability, as well as validity, can be addressed "through careful attention to a study's conceptualization and the way in which the data were collected, analyzed and interpreted" (Merriam, 1988, p. 165).

The generalization from qualitative research is a moot issue. When judged by the criteria for generalizability of experimental research, certainly qualitative research studies are lacking. However, most writers on the subject view generalization in qualitative studies differently from generalizing from a sample to a population. According to Stake (in Merriam, 1985), generalizing in naturalistic inquiry is

> arrived at by recognizing the similarities of objects and issues in and out of context and by sensing the natural covariations of happenings... They seldom take the form of prediction but lead regularly to expectations. They guide action, in fact they are inseparable from action (p. 212).

Edgar and Billingsley (in Merriam, 1985) propose a logical basis for generalization rather than a statistical one, and suggest that "in many cases generalization may, in fact, be more readily made from N = 1 studies than from large N studies due to the opportunity for more accurate delineation and precise control of relevant ... characteristics" (p. 212).

Finally, there are those who suggest that generalization of results be left to the reader since it is "ultimately related to what the reader is trying to learn" (Wilson in Merriam, 1985, p. 213), and "who wish to apply the findings to their own situations" (Kennedy in Merriam, 1985, p. 213).

<u>Instructional Materials and the Teaching Experiment</u>

Freudenthal (1972) believes that mathematics should be presented to students not as a "ready-made subject, entirely structured and complete" (p. 12), but as a subject axiomatized and formalized by the students themselves. The material created by this investigator, called triometry, (Appendix B) provides a basis for engaging undergraduate mathematics/mathematics education majors in creative activities wherein the students can experience first hand the processes of creating/discovering new mathematics.

Triometry began as "fun" mathematics for the investigator. In essence, it is a new way to do trigonometry. Instead of defining a function of an angle in terms of ratios of two sides of a right triangle, triometry, à la the investigator, used all three sides. For examples, (x+y)/r and (x-y)/r (Figure 3). These were

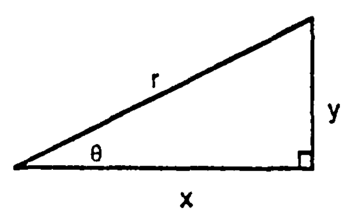


Figure 3

named S(θ) and C(θ), respectively, to parallel the sine and cosine functions, respectively, in trigonometry. From these seeds, properties analogous to trigonometric properties were developed. For example, the familiar

$\sin^2\theta + \cos^2\theta = 1$ became $S^2(\theta) + C^2(\theta) = 2$. It should be noted that other definitions are possible. The only requirement is that they satisfy the property of invariance with respect to similar right triangles just as sine and cosine do.

It is very important for the reader to understand that students did not have access to the triometry materials that were developed by the investigator. The materials, in particular the definitions, served as a back-up after the students had attempted to formulate their own definitions, make suppositions, and to verify or deny their suppositions. The students were, in effect, attempting to axiomatize and formalize a "new" trigonometry.

The teaching experiment was conducted over a period of 7 1/2 weeks. Classes met twice weekly for 15 class periods with each period being 50 minutes long. Classroom activities included discussion, working in groups, and presentations by students. The instructor did not teach triometry. Her function was to lead/guide students into creating/discovering mathematics on their own. Initially, students were asked to create definitions analogous to sine and cosine of an angle. Since none was able to produce usable definitions, it was necessary to "jump start" the class by giving them definitions formulated by the instructor. From this point, students worked individually and in groups to try to formulate structures analogous to those known in trigonometry.

The instructor was the investigator who has twenty three years experience as a mathematics teacher, sixteen at the high school level, and seven at the college level. Although the investigator had not instructed a class completely in the manner in which this experiment was conducted, she had used guided discovery as an instructional technique. Her function was in the spirit of a risk-taker as discussed in Chapter II.

The study was conducted at a comprehensive university in the sixteen-member system of the University of North Carolina. The teaching experiment was the first half of an elective course called "Exploring New Worlds of Mathematics." Students earned two semester hours credit for completion of both parts. The background needed for triometry required only trigonometry and one semester of calculus. However, the requirements for the second half of the course included linear algebra. Thus, all of the participants were either taking linear algebra or had already completed it.

Evaluation of students for grading purposes was entirely subjective, based on their participation in class activities. The syllabus for the experiment (Appendix C) delineated the evaluation criteria for the students.

Subjects

There were five undergraduates who participated in the teaching experiment. One was a junior majoring in mathematics education. There were two students majoring in applied mathematics--one a sophomore and the other a senior. A fourth student was a junior majoring in computer science and minoring in mathematics; the fifth was a sophomore who had not declared a major but was wavering between mathematics and physics. The students were volunteers in the sense that they elected to take the course of which triometry was one part. They constituted the total number who enrolled in the course. A description of the mathematical background of each student is contained in Chapter IV.

Data Collection

Evaluation of triometry. In order to evaluate the appropriateness of the developed materials, opinions of five professional mathematics educators were solicited. The five professors, who were recommended by colleagues of the investigator, were contacted by telephone to determine their willingness to evaluate the triometry materials. The criterion that was used to evaluate the materials is a composite of criteria for creative content found in Lee (1978), indicated by (*), and in NCTM's Curriculum and Evaluation Standards for School Mathematics (1989), indicated by (**). According to these sources, creative course content should:

*1) be a natural extension or variation of ideas already known by students.

**2) reflect a large background of information.

*3) reveal the essential nature of mathematics.

**4) be challenging but within capabilities of all students.

**5) allow opportunities for students to apply the mathematics already known.

**6) allow for the investigation and exploration of ideas.

**7) allow for conjecturing and the testing and verification of conjectures.

The evaluation form, which consists of the aforementioned criteria, and the accompanying cover letter are in Appendix D. In addition, at the end of the experiment, students were asked to write an evaluation of the experiment including triometry. (See Questionnaire II in Appendix E).

<u>Evaluation of changes in students' perceptions and creative behavior</u>. The teaching experiment was concerned with determining whether the experiences via triometry would (1) enhance students' perceptions of the nature of mathematics, (2) enhance their perceptions of their ability to create mathematics, or (3) enhance their mathematical

creative behavior. The methods of data collection for each of these points of investigation are outlined in Table 1. A description of the nature and content of each of the methods of data collection follows.

TABLE 1

Data Collection Methods used in the Teaching Experiment

Point Investigated	Method of Data Collection
1) Perceptions of nature of mathematics	survey, questionnaire, interviews, journals
2) Perceptions of ability to create mathematics	questionnaire, interviews, journals, observation
3) Creative behavior in mathematics	observation, assignments, subjective judgment of instructor

Questionnaire I. At the onset of the experiment, students were asked to respond to an open-ended questionnaire regarding points (1) and (2). This was administered on the second day of class. Some of the questions provided demographic information about the mathematics background of the students. Other questions were designed to allow students to give explanations for their opinions or beliefs about mathematics. These questions were constructed by this investigator and are found in Appendix E.

Questionnaire II. This was the final data collection device. It was distributed to the students on the next to last day of class and collected on the last day, a time span of ten days because of Spring Break. Students responded to questions about the nature of mathematics, their perceptions of their creative ability, triometry, and their experiences in the class (See Appendix E).

Nature of Mathematics Survey (NMS) I and II. This survey (Appendix A) was developed by Scheding (1981). The same survey was administered at the beginning of the experiment as well as at the end. NMS I denotes the survey administered at the beginning, and NMS II denotes the second time. This survey uses a five-point Likert scale and is designed to assess teachers' and prospective teachers' views about the nature of mathematics and the work of the professional mathematician. Each of the 48 items on the survey represents a position on one of seven facets regarding the nature of mathematics on which professional mathematicians seem to agree. The seven facets, together with a table indicating to which facet each item pertains, are in Appendix A. These facets were formulated from the writings of mathematicians and philosophers of science and from a pilot study in which the author surveyed a total of 107 mathematicians in four prestigious universities in the United States and five universities in New South Wales, Australia. The survey was then administered to 828 elementary teachers, secondary school mathematics teachers, elementary education seniors, and secondary mathematics education seniors in Colorado and New South Wales.

In scoring the survey, items were classified as "positive" or "negative" according to whether mathematicians in general agreed or disagreed on the item. Scores on each negative item were replaced by six minus that score. For example, "(1), disagree" was scored as 5. Thus, agreement with mathematicians' views was always scored as 5, regardless of agreeing or disagreeing with the inventory item. Total score was obtained by summing all item scores; facet scores were obtained by summing the scores of items that relate to a particular facet. A high score on a facet or on the total inventory indicated agreement with views of mathematicians in general. (See Appendix A for a list of items by facet and items that are considered negative).

The reliability coefficient for the total scale was 0.87 (or 0.83 for teachers and prospective teachers only) while facets 1 through 7 had reliabilities of 0.54, 0.60, 0.59, 0.72, 0.40, 0.49, 0.56, respectively.

Interviews. Originally, three tape recorded interviews were scheduled for each student. During the midst of the first interview session, it became apparent that an additional one would be needed. Rather than reschedule the times for the remaining interviews as stated on the syllabus, the extra one was inserted between the first and second scheduled times. The first interview was conducted during the second week of the experiment, the second, during the third week, the third, during the fourth week, and the fourth, during the seventh week. Except for the first one which took more than one hour, the interviews lasted forty-five minutes to an hour.

Responses from questionnaire I and the results of NMS I provided a basis for Interview I. The focus of the interview was to ascertain the perceptions of the students about the nature of mathematics in as much detail as possible. The interview was partly structured in that some questions were prepared ahead of time (See Appendix E). For example, students were asked to expand on their responses to the questionnaire. Other times, the interview followed the flow of the responses from the students. Even though the interviews were somewhat individualistic, some of the questions were the same for all students primarily because of similar results on NMS I.

Interview II was more structured and was aimed at determining the perceptions of students regarding creativity. The questions prepared for this interview are in Appendix E.

Interview III was conducted midway through the experiment. It, too, was semi-structured. The emphases were to determine (1) the feelings of the students about the class at that point; (2) their perceptions of their creative behavior--had they improved?; and (3) their perceptions of mathematics. Some questions were drawn from their journals and from classroom episodes. The prepared questions are in Appendix E.

The purpose of Interview IV was again directed toward determining any changes in the perceptions of the students about mathematics or their creative abilities. The interview was also semi-structured. Questions that were prepared in advance are in Appendix E.

Journal. The students were required to keep journals in which they were to record their reactions to and ideas, attitudes, emotions, opinions, etc, of all aspects of class activities as well as their attempts to work the assignments. The investigator read the journals after every three to four class periods, made appropriate comments, and returned them to the students. The contents and students' comments suggested areas for exploration in the interviews. Problems worked or attempted were used to evaluate the creative behaviors of the students.

Observations. The reactions of the students to the course were monitored throughout the experiment. This included any and all activities that might indicate nuances in their beliefs about mathematics.

Mathematical creativity. Evidence of mathematical creativity was the work that the students produced throughout the course whether written or expressed orally in class.

Data Analysis

Evaluation of the triometry material was a composite of the evaluations by the professional mathematicians and by the students.

The determination of a student's initial perceptions regarding the nature of mathematics and his/her confidence in creating mathematics was accomplished by incorporating information from NMS I, questionnaire I, and the first two interviews (triangulation). Total score on the survey indicated the degree to which the student agreed with the views of professional mathematicians in Scheding's study (1981). Total score on each facet indicated views regarding various aspects of the nature of mathematics. Responses from questionnaire I and from interview I were used to obtain a clearer and more detailed indication of how the student perceived mathematics at the onset of the teaching experiment.

Changes in each student's perceptions were monitored through the interviews, the journal entries, classroom observations, and NMS II which was administered at the end of the course. The information so gathered was used to present a history of each student's experience.

Changes in the mathematical creative behavior of students were subjectively evaluated by the instructor. These were based on a comparison of creative behaviors exhibited at the beginning of the course to those at the end. Determination of their creative behavior was based on the Criteria for Creative Behavior in Mathematics (Appendix F). These criteria were drawn from the review of literature.

Limitations

There are several limitations to this study. The results are not generalizable to any population due to a number of factors. The use of triometry was a central part of the experiment. Consequently, speculation of results using material other than triometry is not reasonable. The students and instructor were specific to the study; thus, following the same plan with another group would not necessarily produce the same outcomes even with the same instructor. Also, the students were not randomly selected and, so, could not be considered representative of mathematics/mathematics education majors.

The length of time of the experiment was a limitation. A longer time frame would be more desirable.

The Nature of Mathematics Survey was designed for groups rather than individuals. Also, the correct view of mathematics was based on a survey of mathematicians ten years ago. It is possible that a more up-to-date survey of mathematicians could produce a different correct view, although this is contrary to the investigator's perception.

The small number of students who participated in this study precluded any statistical analysis as a group. This was not a problem since the questions of interest lay primarily with the individual student. However, the use of more students--ten is a good number--would have been better in order to facilitate their working in groups.

Finally, the evaluation criteria for the purpose of grading may have presented a problem. The reasoning behind grades being based solely on students' efforts rather than on what each could produce was an attempt to reduce anxiety about entering into new approaches to learning and studying mathematics. The students loved it. But at the same time, not having the usual pressures to produce may have affected the extent to which they put forth effort.

CHAPTER IV

RESULTS

This study was concerned with the development of original material, triometry, a variation of trigonometry, and its use as reference material in a teaching experiment. Five students participated in the teaching experiment in which they were encouraged to develop their own version of trigonometry. The research questions of this study were:

1. Is the material, triometry, suitable as a creative activity for mathematics/mathematics education majors?
2. Are students' perceptions of the nature of mathematics enhanced as a result of their experiences in this study?
3. Are students' perceptions of their ability to be mathematically creative enhanced as a result of their experiences in this study?
4. Are students' creative behaviors in mathematics enhanced as a result of their experiences in this study?

This chapter is divided into four parts. The first part presents the evaluations of the triometry materials and addresses the first question. The second part recounts the teaching experiment. The third part presents results with respect to the other three questions through detailed accounts of the reactions of each of the five participants as they progressed through the teaching experiment. The fourth part details the results of the students' evaluations of the triometry materials and their opinions of the teaching experiment.

Evaluation of Triometry

Five university mathematics educators agreed to participate in the evaluation of triometry. Four completed and returned the evaluation form (See Appendix D for an outline of their credentials). Evaluators were asked to indicate the degree, using a Likert scale from 1 (Disagree) to 5 (Agree), to which they felt that triometry satisfied each of eight criteria. The criteria were derived from Hall (1978) and the NCTM Curriculum and Evaluation Standards for School Mathematics (1989). The evaluation criteria and the responses of each evaluator (coded Professor A, B, C, and D) are presented in Table 2.

There was total agreement on the first, fourth, and seventh criteria. (Professor C did not answer the latter which appeared to be an oversight). Opinions were evenly split between responses '4' and '5' on the second, fifth, and sixth criteria. The last criterion, "Is (or should be) within capabilities of students", accounted for the largest

Table 2.

Professional mathematics educators' evaluations of triometry materials.

Criteria	Professor				Mean
	A	B	C	D	
Triometry :					
Is a natural extension or variation of ideas already familiar to the student	5	5	5	5	5
Reflects a large background of information	4	4	5	5	4.5
Reveals the essential nature of mathematics	2/4	4	4	5	3.75 4.25
Will allow opportunities for students to apply mathematics already known	5	5	5	5	5
Allows for the investigation and exploration of ideas	5	5	4	4	4.5
Allows for conjecturing, testing, and verification of conjectures	5	5	4	4	4.5
Will be challenging to the student	5	5	-	5	5
Is (or should be) within capabilities of students	4	5	3	4	4

variation in responses. In Professor C's opinion, "I believe it would work better on students who had already had a proof-type course so they have a notion of the arbitrary nature of mathematics." Commenting on the wording of the criterion, Professor A said, "If your students are like my students, "should be" is the accurate wording."

The third criterion, "Reveals the essential nature of mathematics," has two means. This is because Professor A gave two responses. Clarifying these, she wrote that she tends to agree, response '4', "in the sense that students might see that developing mathematics is a constructive process. Also many mathematical ideas lead to the development of new ideas." She tended to disagree, response '2', in the sense that, "these ideas do not grow out of a particular problem that needs to be solved. Students may end up thinking that mathematics is just a "game" that isn't very useful (or interesting) to anyone but a mathematician." In Professor D's opinion, triometry alone "will not reveal the essential nature of mathematics. But in concert with other material it does." (Professor D had erased response '4' in favor of '5', hence, the comment).

The over-all response of the mathematics educators gave triometry high marks regarding its suitability as material for a teaching experiment in creative mathematics. The mean for each criterion ranged from 3.75 (or 4.25) to 5 and the general comments ranged from the polite "I liked your material" and "I found the development interesting" to the more qualitative, "It is a very neat piece of work and I was certainly impressed. I have always liked activities that require you to look outside of a topic or to get a different perspective."

The Teaching Experiment - Nature of Instruction

The teaching experiment was conducted during the first half of the spring semester, 1991 as part of a special topics course. The class met for fifty minutes per day on Tuesdays and Thursdays of each week for seven and a half weeks for a total of fifteen class periods. The first two meetings were concerned with gathering student information, administering the Nature of Mathematics Survey I (Appendix A) and questionnaire I (Appendix E), and presenting the syllabus (Appendix C). Part of the second period was spent "pumping" the class to reveal what they remembered about trigonometry. The level of expertise of each student is discussed in part three. To set the stage for having the students create their own definitions, we talked about the origins of what is presented in textbooks. Mostly, the dialogue consisted of my raising questions and their responding, "I don't know," or "I've never thought about it," or silence. At the end, I suggested to them that we would just create our own version of trigonometry.

To prepare the class for the first assignment, I first reviewed the definitions of the six basic trigonometric functions, pointing out how all six could be defined in terms of sine and/or cosine. Next, I emphasized the fact that all six functions possessed the property of invariance with regard to lengths of sides of similar right triangles. Finally, we discussed what it means to make an analogy. Following all this, the class was assigned the task of creating functions analogous to sine and cosine of an angle θ in a right triangle that would be different from any of the six basic trigonometric functions but would still have the property of invariance.

I shall not attempt to describe each of the other class meetings in detail (an outline of topics explored is in Appendix C). Rather, I should like to explain the underlying theme that guided classroom activities. This teaching experiment was conceived as a guided discovery activity in which the students were active participants. As such, a significant portion of each class period consisted of students presenting their work. Typically, a student would show the results of his/her investigation. The rest of the class would then attempt to verify the results. Or, if the results were incomplete, they would try to offer suggestions. Sometimes the class would run into a dead end so I would let them ponder a while. This might be ten minutes or a couple of days. In the latter case, we would explore other properties proposed by me or the class in the meantime. If no one was able to suggest any new approach, then I would give hints. On occasions, such as finding the Law of S and C(u-v) (triometry functions), I had to lead them through step by step.

The nature of the first assignment offers a clue to another important characteristic of the lessons that I tried to maintain. That is, flexibility. For example, if each student had produced viable definitions, I was prepared to work individually with each one to try to develop their ideas as far as possible. I must admit, however, that I anticipated the likelihood of this happening as remote. On a smaller scale, if <u>anyone</u> had produced a reasonable definition, I was also prepared to engage the entire class in the development of a "triometry" from that point.

Flexibility notwithstanding, each day's lesson was planned in the sense that the class would work on ideas pending or on a new parallel idea suggested by me or anyone in the class. On many of the assignments, in an attempt to accommodate the various levels of ability the students possessed, I would suggest several different ideas so that students could select the one on which they wanted to work. Sometimes they would choose different things, but for the most part, everyone usually worked on the same problem at the same time. Often they would be working on a couple of ideas simultaneously. The students were encouraged to work together, and I could tell from the work in their journals that some had done so.

Even though most of the students could remember some of the basic trigonometric formulas, they were unfamiliar with some of the proofs of the formulas. This was not unexpected. What was unexpected and dismaying was their lack of initiative in searching out these proofs, and once having found them, their difficulty in being able to follow and understand them. Since an understanding of these proofs was necessary in order to be able to extend to triometry, I often did the expedient thing and explained them.

The foregoing notwithstanding, throughout the experiment, I tried to maintain a position primarily as a facilitator--to keep students focused on the work, to give hints now and then, and to keep the class moving at a reasonable pace. Sometimes, this proved to be a difficult task. The students were not accustomed to an independent learning activity with no textbook or examples on which to rely. Understandably, they continually deferred to me, waiting for me to give them "the answer" or at least to acknowledge that there was "an answer." They were uncomfortable when we would sit in relative silence for even five minutes while we pondered over a problem. They were even more perplexed with the possibility that the problem on which they were laboring might not have a "nice" answer. On one occasion, the class had worked together and had come to a dead end. After twenty minutes with no headway made, one of the students asked me if there was a solution. I replied, "I don't know," which was truthful since I had not worked this particular problem before. Their response was in the sense, You mean we are trying to do a problem that may not even have a solution? The solution to that problem was not resolved until next to the last class meeting.

The Participants

In this section, I will discuss each of the five students who participated in the teaching experiment with respect to the last three research questions. Recall that the sources of information include:

1) Nature of Mathematics Surveys I and II (Appendix A), administered at the beginning and end of the teaching experiment, respectively;
2) questionnaires I and II (Appendix E), also administered at the beginning and end of the teaching experiment, respectively;
3) four interviews (Appendix E)-- interview I conducted during the first week, interview II, the second week, interview III, the third week, and interview IV, the sixth week;
4) student-kept journals, and assigned problems;
5) my own journal and observations.

Each of the students has been given a pseudonym.

Ruth

Ruth was a nineteen year-old junior majoring in computer science and minoring in mathematics. In high school, she had taken a minimum number of mathematics courses - algebra I and II and geometry - and did quite well in them. Her favorite class in high school as well as her favorite area in mathematics was algebra because it came easy to her, and as she wrote, "I like working with numbers." The mathematics courses taken in college included pre-calculus, calculus I and II, and a discrete mathematics course. She was, at the time, taking linear algebra. Her self-reported grade point average (GPA) in mathematics was approximately 2.0.

Ruth had originally planned to major in mathematics, but a calculus with computers course using Maple (a computer program) changed her mind for a somewhat bizarre (to me!) reason. She did not do well in calculus which she attributed to Maple. As a result, she had a dislike for Maple as well as calculus. She then tried a programming course and liked it! To her, there was no contradiction. "Computers are different than Maple," she said.

What Ruth liked most about mathematics was "working out equations" because "it is fun going from one step to the next, trying to decide what to do in order to get a final answer." However, she admitted having trouble when it came to deriving the equation, as in calculus with area and volume problems. Her least favorite thing about mathematics was not about mathematics per se but that some teachers require a problem to be "100% correct" or no credit at all. This, she allowed, was not fair as "it is to (sic) easy to make little mistakes, that will mess up everything else."

Her favorite mathematics teacher had been her high school algebra teacher. As Ruth explained it,

> She (the math teacher) would explain the proof, we didn't really have to know that but she would go over it. Then she would tell us what we were going to do and give us an example, then give us an example of the negatives and positives (meaning counter-examples and exceptions).

In this regard, she felt that some of her college instructors were remiss in not showing the class enough such examples before an assignment.

Ruth's background in trigonometry was minimal, having had only three to four weeks exposure in pre-calculus. Needless to say, her experiences, or lack of, created a dislike for the subject as well as a gap in her knowledge. This proved to be somewhat of a handicap for her in this class.

Nature of Mathematics. Sources of information for Ruth's perceptions of the nature of mathematics include NMS I and II, interviews I, III, and IV, and questionnaires I and II. As one can observe from Table 3 (p. 67), Ruth's

Table 3.

Scores of participants on Nature of Mathematics Survey (NMS) and mean scores of prospective secondary mathematics teachers (PSMT) and mathematicians (Math'n) from Scheding's (1981) study.

Facet*	1	2	3	4	5	6	7	Totals
PSMT	35.0	32.0	15.5	34.5	21.5	31.1	15.7	185.3
Math'n	38.8	36.3	18.4	40.6	24.4	33.8	17.7	210
Ruth	+34/37	33/33	16/15	34/35	18/21	26/34	17/17	178/192
Steve	+32/35	29/32	14/15	36/37	18/16	34/31	12/14	175/180
Nora	+30/31	23/27	13/10	27/31	18/16	33/27	14/14	158/156
Don	+37/43	33/31	17/18	31/30	23/17	26/31	14/16	181/186
Gina	+30/34	27/30	14/16	30/34	16/19	29/31	16/14	162/178
Means	32.6/36	29/30.6	14.8/14.8	31.6/33.4	18.6/17.8	29.6/30.8	14.6/15	170.8/178.4

*Key to Facets (Scheding, 1981):

1: Mathematics as an organized body of knowledge --the generalizability of mathematics is desirable.

2: Nature and attributes of proof-- deduction and induction are both important in mathematical discovery and proof.

3: Role of insight and intuition in the work of the mathematician--both are important.

4: Beauty in mathematics-- mathematics is a creative art in which elegance of proofs is sought.

5: Relative importance of massive or complex numerical calculations and abstract or symbolic thought in the work of the mathematician--their work involves more of the latter.

6: Relationship of mathematics to the real world--much of mathematics is applicable to the real world; but, application is not necessary to justify its importance or existence.

7: Existence of differing views of the nature of mathematics--mathematicians differ in their views.

+Scores on NMS I (administered at beginning of course) and NMS II (administered at end of course), respectively.

initial scores compare favorably with and in some instances are higher than the mean scores of the secondary mathematics group. Thus, her view of the nature of mathematics was similar to that of the prospective secondary mathematics teachers but not as "correct" as the mathematicians surveyed. (See Appendix A for attributes of the correct view of nature of mathematics). Her misconceptions were further borne out by questionnaire I and the subsequent first interview.

Like the subjects in Frank's study (Chapter 2. p. 20), mathematics, to Ruth, was "working with numbers and rules" with "lots of memorization of formulas." The regarding of mathematics as formula driven was reiterated throughout the first interview.

Investigator (I): Ruth, I'd like to read a few statements to you and have you express your opinion about each. First, 'Mathematics is a search for patterns.'

Ruth (R): I agree with that.

I: What do you think that means, searching for patterns?

R: You're looking for the formulas that it fits into or you're looking for the same numbers over.

I: Next. 'Mathematics is an attempt to find connections or to make connections between ideas.'

R: I think all of it goes along with formulas again because you're trying to connect it with something you already know or look for something you already know or a formula you can put it into.

I: How about, 'Mathematics deals with ideas and relationships rather than with numbers and manipulation with numbers'?

R: I always thought it deals with numbers.

I: Well, what do you think about the part 'dealing with ideas and relationships?'

R: I don't know, I guess it could deal with ideas.

I: You tend to think of it dealing more with numbers and manipulation of numbers?

R: Right.

Ruth was uncertain about the work that mathematicians do when they "do" math. On the questionnaire, she wrote, "think out problems?" When the same question was posed in the interview, she replied,

> I don't remember what I wrote (referring to questionnaire). I know there are mathematicians that go out in the business world and more or less work for a company or there is specific ones like a calculus teacher, I didn't know which one you meant. I think they (mathematicians) try to find the best way to go about things, to help the company out. I really don't know.

Probing further,

I: Are mathematicians problem solvers?

R: I agree with that.

I: What do you think a problem solver does?

R: Try to find not just a solution but a better solution. Always looking for better.

I: What do you mean by better?

R: More economical or quicker, faster.

I: What do you think of the statement, A mathematician is a theory creator?

R: They create theories.

I: How do they create theories?

R: They try to conclude a lot of stuff, kinda like what we're trying to do (referring to our class). Like we got a problem like trying to learn how to graph it, they try to figure it out.

I: Would you say a mathematician's work involves more complex numerical calculations or proving theorems?

R: I'd say more complex.

I: When proving theorems, where do they get the theorems they prove?

R: I guess from what they already know or from a book.

It was apparent that at the beginning of the teaching experiment, Ruth's perceptions of mathematics and the role of mathematicians were rather limited. By the third interview, I could detect a few changes.

I: What do you think we are trying to do in this class?

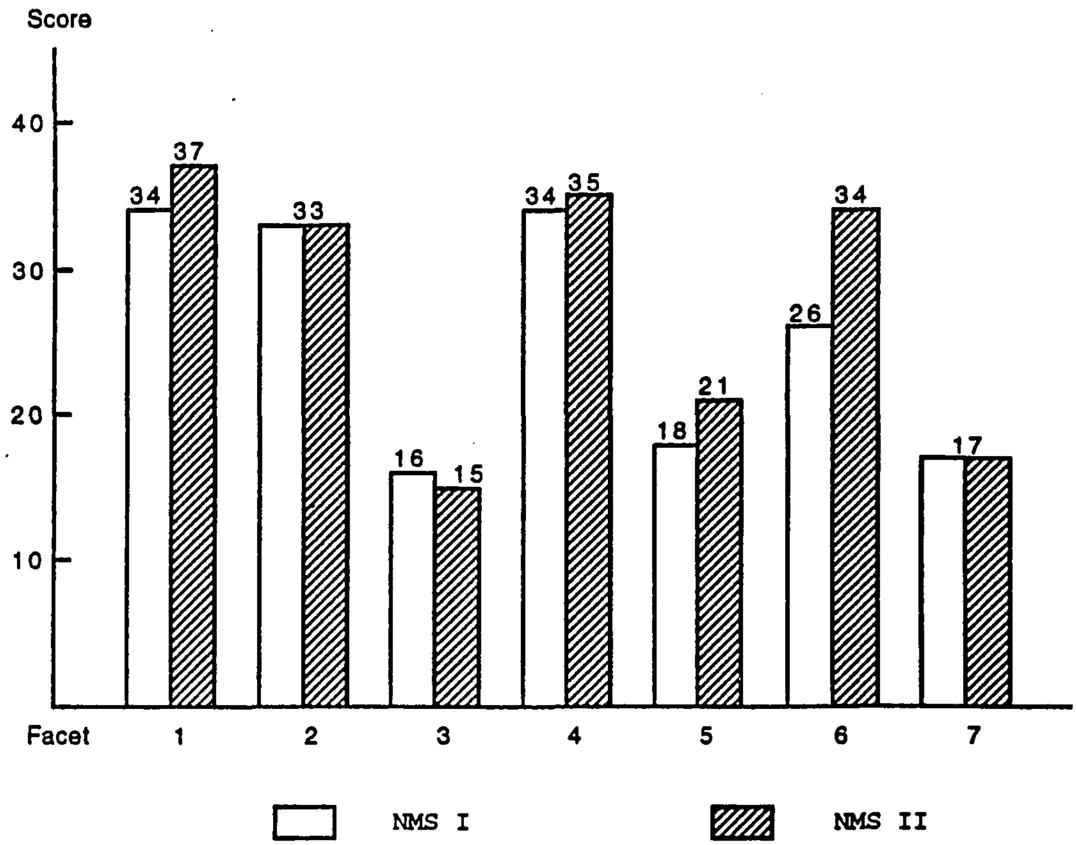
R: Just to show how people come up with stuff or prove stuff and how not to just take math for granted, that somebody had to invent it.

Here was a glimmer that she was beginning to see mathematics in a different light. What had not changed was her description of mathematics. She continued to think of it as "using formulas, multiplication, division. I think of it as using lots of formulas and calculations."

By the end of the teaching experiment, Ruth exhibited substantial growth, if scores on the second administration of the survey is any indication. Figure 4 (p. 72) compares the scores from both surveys. The most dramatic change was in Facet 6. Initially, she had scored a number of individual questions relating to this facet as 3s, indicating that she was unsure about the role of mathematics in relation to real world applications. As the second score indicates, she developed a more "correct" opinion. There was no change for Facet 2 and a one point drop in Facet 3. During the interviews, Ruth was unfamiliar with the terms of both of these facets - induction, deduction, insight and intuition - which probably accounts for this. She is right on target with the "correct" view with Facet 7. The results on Facet 4 are evidence that some ideas are hard to change. During the last interview, I asked, "Have you learned anything about mathematics?" Her answer thrilled my heart. She replied,

> I've learned that there's not just one right way to do anything. Everybody don't have to use the same formula, you can come up with new ideas. I had always thought you use whatever's in the book. You didn't question because it's been proved. But you can come up with new ways of doing it.

But my joy was tempered by her responses to the first two questions on the final questionnaire. This new way of thinking about mathematics had made no impact on her description of mathematics or of what mathematicians do. She wrote,



Key to Facets (Scheding, 1981):

1: Mathematics as an organized body of knowledge; the generality of mathematics.
2: Nature and attributes of proof; the roles of deduction and induction in mathematical discovery and proof.
3: Role of insight and intuition in the work of the mathematician.
4: Beauty in mathematics; mathematics as a creative art.
5: Relative importance of massive or complex numerical calculations and abstract or symbolic thought in the work of the mathematician.
6: Relationship of mathematics to the real world.
7: Existence of differing views of the nature of mathematics.

Figure 4. Ruth's Scores on Nature of Mathematics Survey (NMS) I and II

> Mathematics is working with numbers and formulas to solve a problem.... Mathematicians decide what formula to use in their problems. They try several different approaches looking for the best answer.

Summary. Throughout the teaching experiment, Ruth's descriptions of mathematics and the work of mathematicians did not change. She described mathematics as finding the right formula, doing a lot of computations, and a lot of memorizing. Mathematicians' work was described in similar terms in that they sought the best (easiest) answer to a problem by selecting the proper formula. Despite the constancy of these remarks, her second set of scores on the Nature of Mathematics Survey indicated changes in her perceptions toward the correct view. In particular, she gained some understanding about how mathematics is created, and that authority derived from textbooks is not absolute.

Perceptions of Ability to Be Creative. Information for this section came from interviews II, III, and IV, questionnaires I and II, and Ruth's journal. Ruth's perception of creativity in mathematics and of her own creativeness may be likened to bifocal vision in which one sees things on two different planes. Through one lens, she sees creativity as "being able to come up with ideas on your own. To be able to figure out how to see things and mostly come up with ideas, discover new ideas." Being creative in mathematics is mostly a matter of "adding on" to what you already know. As she put it, "you've already got formulas but you create something from that so you've got a little bit of help, you just add on." (This explanation may have been influenced by classroom activities at the time since we were just beginning to develop identities in triometry analogous some in trigonometry). When asked what it takes to be creative in mathematics, she replied, "I think you have to think a lot. I think it's kinda hard because I don't think of people making stuff in mathematics. It's already there, why think about it." She described something creative in mathematics as "coming up with a new formula, something like the Pythagorean Theorem." When I interjected, "something someone hadn't already done?", she replied, "yeah." By her own reckoning against this criteria, Ruth did not believe that she had ever done anything creative in mathematics. "I just try to do the problems; I don't try to get anything else from it," she said.

The second view of mathematical creativity was actually about creativity in working problems. This became apparent following a series of questions about whether she considered solving problems in different ways, like reversing the order of the operations in an equation, as being creative. Her affirmative response was accompanied by a qualitative, "I guess when I think of being creative I think of something big, but I guess if you did the least little bit you could call that creative. That [reversing order of operations] would be the smallest amount at being creative." It is with this view that she was able to merge her concept of her own creativity with mathematics.

Although Ruth did not see herself as possessing any artistic talents, she did consider herself as being creative. On a scale from one to ten, she rated herself "7." As she explained it, she tries "to look for solutions and try to figure out how things could be easier." She cited row reduction in matrices as an example:

> ...instead of just going by the formula every time, I always try to do it by not going into fractions. I think that's more creative than some people because they just keep doing the row of operations in order. I always avoid it. If I'm doing a problem and I see a row of fractions coming up, I just do something else. I still get the same answer, I just go about it a different way.

Following this I again asked:

> I: What do you think it takes to be creative in mathematics?
>
> R: I don't know; I guess just being able to look ahead, to be able to come up with easier steps, something like that.
>
> I: What kind of people are creative in mathematics?
>
> R: The people who are always trying to find an easier way out, like not working with fractions.
>
> I: Does a person have to be a genius?
>
> R: No, because I know I'm not a genius. I feel I'm somewhat creative in working problems.

As the semester progressed, Ruth's impression of her creative abilities remained high even though she expressed high frustration at not being able to come up with very much on her own. Her assessment at the middle of the teaching experiment (after three weeks) was that she was better at creating math because,

> I know I can do it now. Before I wouldn't even have tried. I would have said "I can't do it." I wouldn't even have thought about doing it. Now, at least, I can try and come up with something.

This same outlook persisted through to the end of the teaching experiment three weeks later.

> I: Tell me what you think about your experiences in this class. Has it been fun? When did the fun begin to wear off?
>
> R: I think it kinda built for me because at first I thought this is not that exciting, this is stupid. Then it got more exciting because at first I wasn't getting anywhere at all. But then each time I got a little bit further, so it built up.
>
> I: How would you react to another math class like this one?
>
> R: I would feel positive about it like, yeah, I can do this. I think I'd know where to start...and how to sit down and work with it.

On the final questionnaire and evaluation she wrote, "I am glad I took the course because I feel better about myself developing ideas."

Summary. To Ruth, creativity in mathematics had two connotations. One was the creation of something original like the Pythagorean Theorem and the other was creatively solving a problem. By the latter, she meant finding an easier way to solve a problem. She considered herself somewhat creative in solving problems with a rating of seven on a ten-point scale. Though she expressed frustration at not being able to do a lot of the problems, in the latter weeks of the teaching experiment, she maintained that her ability to be creative was improved. The basis for her judgment lay in her feeling that she knew better how to start working on problems whereas at the beginning she had had no idea. This appraisal remained unchanged at the end of the teaching experiment.

Creative Behavior. Observations, assignments, and Ruth's journal were the primary sources of information for this section. Ruth's efforts at thinking creatively did not come easily for her because, she wrote, "when you learn something one way it's hard to see other views or imagine it another way." This comment accompanied the first assignment in which the students were to create their own definitions similar to but different from the definition of sine of an angle. In spite of this being a "crazy assignment" (her words), she did try several ways to make definitions. First, she used a triangle with sides x, y, r (Figure 5) and wrote, sin ? = yx/rx. Then she re-labeled the sides (Figure 6) and tried, sin ? = xz/yz. However, she recognized that both of these were the same as the definition of sine in

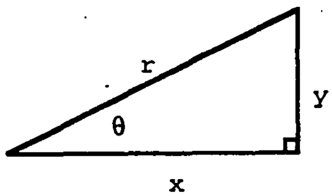


Figure 5

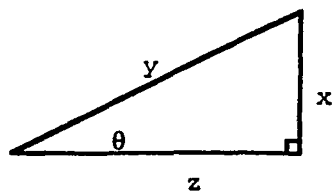


Figure 6

trigonometry. So then she became a little more creative:

$$\sin ? = x^2/y^2 \text{ and } \sin ? = (x-z)^2/(y-z)^2.$$

Using numbers for the sides of the triangle (incorrectly chosen, Figure 7), she substituted into the latter expression getting $(4-3)^2/(6-3)^2 = 1/9$. She then

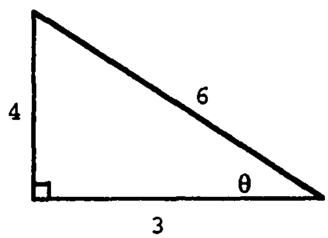


Figure 7

compared this value with $\sin \theta = 2/3$. Since they were not the same, she drew a big "X" through all the work. What she had been looking for was another way of getting the same value as the sine function which, of course, was not the objective. At one point she wrote in her journal, "I am confused. I don't know if I can plug our S in for sin[e] or not." This confusion about $S(\theta)$ and how it related to $\sin \theta$ plagued her for about half the teaching experiment.

Even though, as she often wrote, "I have no idea how to start," Ruth continued to "play around" (her words) with ideas. Unfortunately, she enjoyed few successes partly because of poor algebra skills. But, to her credit, she realized her problem. A number of times she expressed frustration at "forgetting how to do little rules." She did one exercise quite well, but it did not involve any algebra. Part of the assignment was to determine the intervals in which the newly defined S and C functions are positive. She calculated the S and C functions for all of the special angles from 0 to 2π and then indicated the signs of each around a unit circle. From this, she was able to detect a pattern and correctly identify the intervals. Admittedly, this was not a very difficult task, and some might say did not require a lot of creativity. But I felt her method of depicting the pattern was a good idea, and her success seemed to give her a boost.

I have mentioned poor algebra skills as a hindrance to Ruth's efforts. In addition, there were at least two other factors--a poor background in trigonometry coupled with an inability to understand some of the trigonometric proofs and, hence, an inability to extend beyond the ideas underlying the proofs. For example, I had given a "fun" assignment (Appendix C) that involved solving a right triangle using the new S and C functions (Figure 8).

Correctly applying the definition, she wrote $S(30^\circ) = (b + a)/4$. But this is as far as she got.

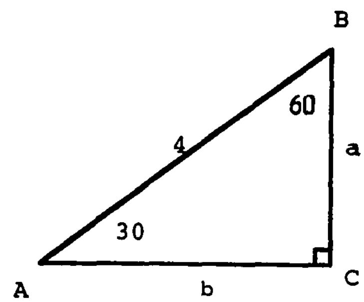


Figure 8

To complete a solution, she needed $C(30^\circ) = (b - a)/4$, substitute the values for $S(30^\circ)$ and $C(30^\circ)$, and solve the two equations simultaneously. One might ask, why should she have been expected to see this? Because previously we had derived a formula (see Law of S in Appendix B) similar to the law of sines in trigonometry in which solution by simultaneous equations was used. One explanation is that she just was not studying. Although this may have been partly the cause, the major reason was that she could not follow the proof of Law of S because, as she wrote in her journal, "I read and tried to figure out how they got the law of sines, but I just don't understand." She could have solved the problem by another method, a simple application of the Law of S formula, but I think she was unable to do so because her unfamiliarity with solving triangles in trigonometry prevented her from applying the same idea using the new functions.

Having profiled Ruth's attempts at creativity, the next question to address is, what progress did she make? In evaluating her performance, it is important to bear in mind Ruth's perception of mathematics (formula driven and a lot of memorization with authority derived from the textbook) as well as her demonstrated mathematical ability (2.0 GPA) at the time. At the beginning of the teaching experiment, the idea of making ones own definition was foreign to her since she had always regarded the mathematics that appeared in textbooks as unquestionable. By the end of the teaching experiment, she was more comfortable with the notion to the degree that she no longer considered it "crazy" and even found it exciting. In addition, she had come to understand the idea of a parallel development of trigonometry though her ability to toy with ideas was limited by her reliance on finding the right formula to "plug" into. This was evidenced by the problems she submitted for her final assignment. She was successful in deriving the derivatives of $C(\theta)$, $T(\theta)$, and the reciprocals of these functions. To get the derivative of $C(\theta)$ required following the proof of the derivative of $S(\theta)$, which had been done in class. She then combined the two to get the derivative of $T(\theta)$--a parallel to the derivative of $\tan \theta$.

Overall, her greatest advancement was in understanding the analogy being made and knowing what she was supposed to do even if she did not know how to do it. Referring to her attempts to find $S(-\theta)$ and $C(-\theta)$, parallels to $\sin(-\theta)$ and $\cos(-\theta)$, respectively, she wrote in her journal, "I tried to do what they did in the book with sin and cos but I didn't get anywhere." She had not tried to merely substitute $S(-\theta)$ for $\sin\theta$ and $C(-\theta)$ for $\cos\theta$.

Summary. Ruth's creative behavior was affected by several factors. At first, her perceptions of mathematics as textbook derived, formula driven, and a lot of memorization restricted her attempts to make the proper analogies between trigonometry and triometry. Improvement in making the analogies was noted by the end of the teaching experiment, but her poor algebra skills and a barely average ability in mathematics limited the extent to which she was able to follow through. She could play around with ideas somewhat, but the latter two deficiencies also limited her successes. A weak background in trigonometry hampered her efforts at being able to use a trigonometric concept and adapt it to its triometry counterpart.

Steve

Steve, a twenty year-old sophomore, was a rather quiet, reserved individual. All through the teaching experiment, he was reluctant to volunteer any information but would readily respond when asked. His manner was one of directness, saying what he had to say in as few words as possible. Needless to say, interviews with him were challenging. Steve's high school mathematics background was strong having taken two years of algebra, geometry, trigonometry, and calculus. Even though he had done quite well in these course, he was not confident enough to begin the calculus series his freshman year in college, electing instead to take a pre-calculus course first. He had completed the three-semester calculus series and was taking linear algebra at the time of this teaching experiment.

Steve had elected to major in applied mathematics and minor in computer science, choices no doubt motivated by his successes as well as his interests. On the initial mathematics questionnaire he wrote, "Math is something I like doing and can do fairly well." His self-reported GPA in mathematics, 3.35, attested to this. In general, he liked solving problems most because "it makes you think and be creative." In particular, he was partial to calculus since "it is the area I did the best in." The part of mathematics he liked least was geometry. He attributed his dislike to "doing all those long proofs." In particular, it was having to state reasons for each step. (A 2-column proof in geometry requires a valid reason--definition, postulate, theorem--for each statement.) Probing further during the first interview revealed that it was not just geometry proofs that Steve disliked but all formal proofs. He explained, "I don't really get how you go through a proof and everything. I don't understand some of the stuff, why this is that." He acknowledged that, to some extent, proof was an important part of mathematics but not the most important thing. It was important to be able to show how a solution to a problem was found which demonstrated that it was not a lucky guess. "But," I asked, "haven't you ever been curious as to where a property came from and how you know it's true?" He replied, "You just have to believe it."

His favorite mathematics teacher had been a college professor whom he recalled as nice and friendly. Steve liked his lectures, describing them as interesting and to the point. "He made things clear and very direct and you knew what you were supposed to do," Steve explained. This approach was how he (Steve) thought he learned best.

Steve described his attitude toward trigonometry as indifferent. Some parts interested him, like using law of sines or law of cosines to solve triangles, while other parts did not. He was unable to recall specifically which parts he did not like. Aside from this, his familiarity with trigonometry stood him in good stead during the teaching experiment.

Nature of Mathematics. Information for this section came from the Nature of Mathematics Surveys I and II, interviews I, III, and IV, and questionnaires I and II. Table 3 (page 67) reveals that on five of the facets--1, 2, 3, 5, and 7--Steve's initial scores on the Nature of Mathematics Survey were lower than the mean scores of the prospective secondary mathematics teachers but within three or fewer points on each facet. On facets 4 and 6 he was closer to the mean score of mathematicians and, in fact, higher on facet 6. Thus, Steve's views of mathematics were somewhat similar to those of the former group rather than to the "correct" view of the latter.

Basically, Steve viewed all of mathematics and the value of mathematics in terms of problem solving, which is not surprising in view of his chosen field of study. On the first questionnaire he wrote, mathematics is "the process of solving problems using methods developed over the years." He declined to expand on this during the first interview. So then I asked:

> Investigator (I): What is involved in solving problems?
>
> Steve (S): A lot of thinking. You have to look back over the stuff that you've picked up over the years and have to put it all together and use that to solve whatever you're trying to solve.

The worth of mathematics he judged by its applicability. Following a philosophical exchange about the "discovery" or "creation" of mathematics, I inquired:

I: Do you think that the math that is discovered or created is in response to some need or because some one has an idea and maybe wants to play around with it?

S: I think somebody gets an idea and then works on it and decides it's pretty good, apply it to something else.

I: What if it doesn't apply to something else?

S: Then they keep working till they find something it applies to, how to use it.

I: Suppose it doesn't apply to anything at all, at least no one knows what it applies to. Is there still any worth in it?

S: No, not really. If you can't apply it to anything, I don't think it's worth that much.

His description of the work of mathematicians was consistent with his perceptions of mathematics. "Mathematicians apply methods already developed to solve problems and create new methods." An effort to gain further clarification met with, "I think my answer is just short and to the point." Pressing on,

I: Do mathematicians prove theorems?

S: Yes

I: Do you think that is the bulk of their work?

S: No, I don't really think the bulk of it is. They've already been proven before, so I don't think the bulk of it is. They more or less apply them to something else.

I: Where do they get the theorems?

S: I think some they might get from a book or they might think up their own, I guess. I don't really know.

I: Does a mathematician's work involve mostly complex numerical calculations?

S: To a certain degree... It wouldn't always be complex.

Steve was a bit uncomfortable with the notion of mathematicians creating their own rules though he agreed it was okay as long as the rules "don't break other rules that are already there." As an example, I suggested 1+2=2, 1+5=5, 1+6=6, etc. His reaction was, "I don't think it would be perfectly right. I think it would be confusing to other people who didn't really know the symbols." Even though he acknowledged this as mathematics, he saw no value in it.

Throughout the teaching experiment, these perceptions remained constant though he was beginning to express some insights about how mathematics was developed. Midway through, he described the purpose of our class thusly:

> Basically, we're trying to do a new trig system, teach us how to come up with things on our own. It's also showing us how maybe other people came up with things in math, like how they came up with the original trig system, how things were developed. Get one equation and then get other functions from it.

I: Are we doing mathematics?

S: Yes, 'cause we're doing problem solving. I think that's mathematics.

I: Tell me some of the things you do when you do math.

S: A lot of writing, solve problems, apply formulas that you already learned.

On the final questionnaire, he claimed his ideas about mathematics had not changed since the beginning of the teaching experiment. Even so, he added a new dimension to his description of mathematics:

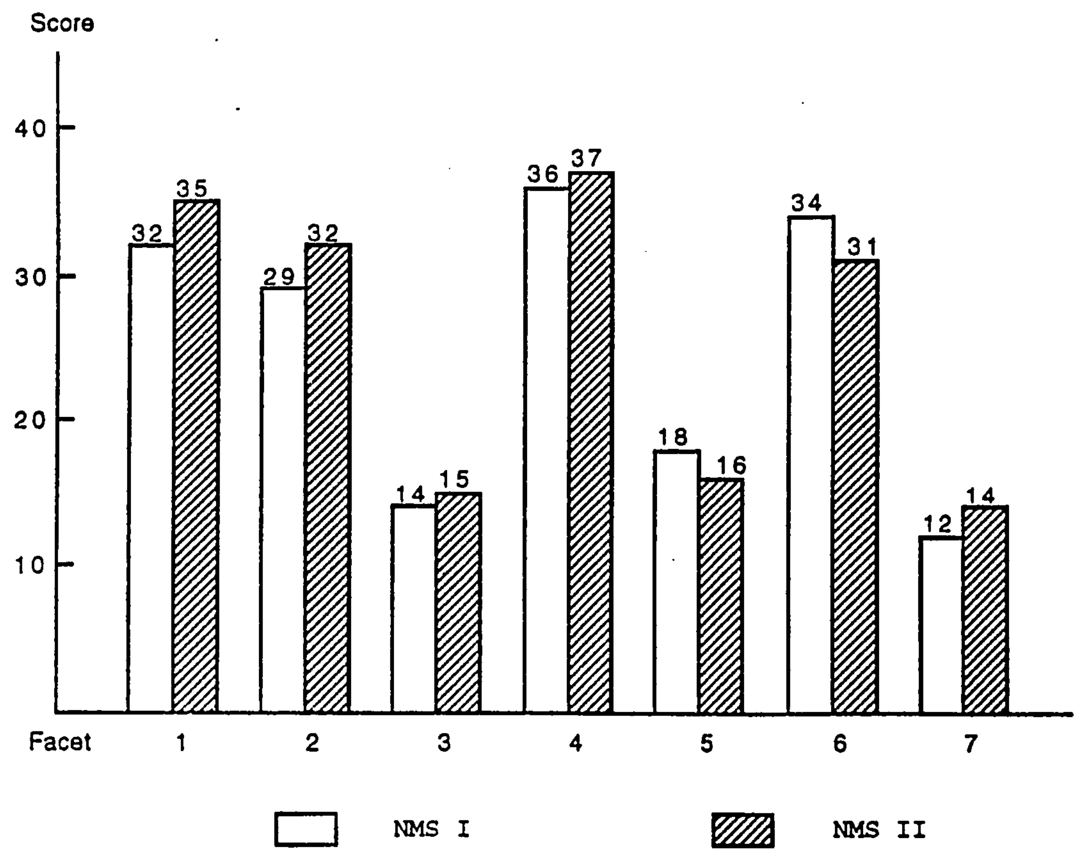
> I see mathematics as the exploration and development of new methods of problem solving and the application of these methods to solve problems.

This small change in Steve's perception of mathematics may have created some uncertainty about a view of mathematics of which he had been fairly sure at the beginning of the teaching experiment. On the first scoring of the Nature of Mathematics Survey, Steve's score on Facet 6 indicated a "correct" view of this item. This facet states in part, "...Much of mathematics is applicable to the real world but some is not. But application to the real world is not necessary to justify the importance or existence of mathematics." The second time, his score dropped three points putting him even with the prospective secondary mathematics teachers on this facet. Thus, it appears that he was less sure of the relationship of mathematics to the real world than he had been earlier in the teaching experiment. I am at a loss to explain the apparent contradiction between his first score on Facet 6 and the views of mathematical worth that he expressed during the interviews.

Comparing the first and second results of each facet of the survey (Figure 9, p. 90), one can conclude that Steve made modest improvements in his perceptions of the nature of mathematics. Scores on Facets 1 and 2 increased by three points each, Facet 7, two points, while those of Facets 3 and 4 increased by one point each. With these increases, he compared more favorably with the prospective secondary mathematics teachers. One reason that may have contributed to the original lower scores of these facets was Steve's unfamiliarity with the terms relating to these facets (e.g., inductive and deductive reasoning, intuition, conjecture, mathematical system). Upon his request during the first interview, I gave him a brief descriptions of the terms which may account for the increase in scores on those facets at the end.

Two facets show declines in scores, Facet 6, which has been discussed, and Facet 5. The "correct" view of the latter facet is that a mathematician's work involves more abstract or symbolic thought rather than complex numerical calculations. In this regard, Steve did not waver from his original description. In fact, his score would suggest a firmer commitment to this idea. He wrote a final affirmation on the last questionnaire:

> Mathematicians either develop new mathematics which can be used to solve problems or solve problems by applying mathematics already known.



Key to Facets (Scheding, 1981):
1: Mathematics as an organized body of knowledge; the generality of mathematics.
2: Nature and attributes of proof; the roles of deduction and induction in mathematical discovery and proof.
3: Role of insight and intuition in the work of the mathematician.
4: Beauty in mathematics; mathematics as a creative art.
5: Relative importance of massive or complex numerical calculations and abstract or symbolic thought in the work of the mathematician.
6: Relationship of mathematics to the real world.
7: Existence of differing views of the nature of mathematics.

Figure 9. Steve's Scores on Nature of Mathematics Survey (NMS) I and II

Summary. Scores on individual facets of the Nature of Mathematics Survey I indicate that at the beginning of the teaching experiment, Steve's perceptions of the nature of mathematics were somewhat removed from the correct views of the comparative group of mathematicians but only slightly less in harmony with the views of the prospective secondary mathematics teachers group (PSMT). The same survey at the end of the teaching experiment shows changes closer to the PSMT group and, thus, closer to the correct view. The most significant change in his perceptions was an insight into the development of mathematics. His original position had been one of mere acceptance with little or no thought as to the origins of textbook mathematics.

Two beliefs remained firmly entrenched. Basically, Steve regarded the study of mathematics as a search for techniques to solve problems. Hence, mathematics that had no application was of little value. Mathematicians, then, engaged in problem solving by applying existing mathematics or they developed new techniques.

Perception of Ability to Be Creative Information for this section was extracted from interviews I, II, III, and IV, and questionnaires I and II. Like Ruth, Steve's conception of creativity was two dimensional. To be creative, he said, is "the ability to come up with something on your own without anybody else's input." But it was okay to get an idea from another source as long as what was added was original. Creativity in mathematics was "basically the same thing." In this sense he regarded being creative as a difficult task for him. "I'm not really that creative," he said.

I: What makes you say that?

S: For example, when I write a paper that I don't know and have to think about, it takes me longer than something I can just sit down and work out.

I: Does creativity have to be something that comes easily or automatic?

S: No, I wouldn't say it comes easily, but it may come easier to other people than it comes to me. I don't come up with things easily on my own.

The second dimension of creativity concerned problem solving. On the initial questionnaire he had indicated that what he liked most about mathematics was problem solving because "it makes you think and be creative." So I inquired:

I: What do you mean by being creative? In what ways can you be creative in problem solving?

S: Sometimes you have to think of ways you haven't learned to solve something. Like being creative and putting different things you've learned together to solve one thing.

I: Are you a creative type person?

S: Somewhat. I do have problems sometimes coming up with stuff on my own. I don't really know where to start sometimes. I can usually get it if I work on it hard enough - sometimes.

I: Have you ever been creative in mathematics?

S: Yes, to solve a lot of problems I've had to solve.

I: In what ways were you creative?

S: Like some problems there were several different ways you could do it. You could figure it out and you had to decide which way you wanted to start and how to go about it. I think that's an example of being creative.

Even though Steve thought of himself as being somewhat creative in solving problems, he rated himself as "about average" based on "just the people around me, I guess."

At first he was divided in his opinion as to whether being creative was an inborn talent or one that could be enhanced through practice.

I: What does it take to be creative in mathematics?

S: I really don't know. I guess you are just born with it.

I: You mean an innate ability? Something you have or you don't?

S: For the most part, but I believe you could work on it and learn to be creative.

I: How could you learn to be creative?

S: I guess just a lot of practice of working on things. As you work you would get more confidence and you would become more creative over time.

Apparently this latter view found favor with Steve for midway through the teaching experiment I asked,

I: Do you feel that you are any better at creating mathematics than you were three weeks ago?

S: I think my ability to be creative has improved. The more practice you get, the better you should get. I think I'm getting plenty of practice.

Steve's final assessment of his creative abilities were much the same. During the last interview he expressed a feeling of being more creative because he felt an improved sense of being able to look at a problem and get ideas of how to start working on it. On the final questionnaire and evaluation he wrote, "I feel that through this course I have learned to be more creative. I have learned how to look at things in different perspectives to figure them out." At the same time, despite his expressed feeling of improvement, Steve still maintained that basically he was not very creative. He wrote, "What I liked most about this class was creating things because I believe I'm not that creative." And even though he felt his ability had improved with practice, he still rated himself as "about average."

Summary. From Steve's perspective, one could be creative in mathematics by having an original idea or by creatively solving a problem. His creative talents, which he rated as average, lay in the latter category. Possession of mathematical creativity, he stated, was basically an innate trait; but at the same time, he also believed it could be enhanced through practice. Throughout the teaching experiment, he maintained that he was not very creative while simultaneously declaring that his creative ability was improving. The latter condition he attributed to getting a lot of practice and to acquiring different perspectives on the way he looked at problems. At the end, he still perceived his mathematical creative ability as average.

Creative Behavior. The primary sources that contributed to this section were classroom observations, information from Steve's journal, and homework assignments. Steve demonstrated some creative talents from the beginning. The first assignment for the class was, given a right triangle with sides x and y and hypotenuse r, create definitions that relate the sides and angles that are different from but analogous to the sine and cosine of an angle. Steve wrote in his journal,

> When I first looked at this assignment I thought it would not be too hard to figure out but once I had worked on it for a while I saw I would have to be creative to come up with an answer.

His "answers" were indeed imaginative. With a little prompting, Steve shared the following with the class (Figure 10):

$$S(\theta) = (x^2+y^2+r^2)/4r^2$$

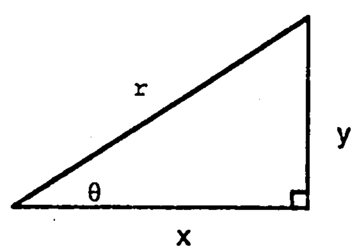


Figure 10

(The class had agreed on the symbols $S(\Theta)$ and $C(\Theta)$ for functions similar to sine and cosine, respectively, beforehand.) This did not prove to be a good definition since $S(\Theta) = 1/2$ for all right triangles. Because this one did not pan out, he declined to show us his expression for $C(\Theta)$. Upon reading his journal a week later, I discovered that his definition for $C(\Theta)$, $(C(\Theta) = x(x^2+y^2+r^2)/4yr^2)$, was better than that for $S(\Theta)$ since it possessed the property of invariance among similar right triangles. However, the definition would not have lent itself well for development of other functions, and so, I did not encourage him. Also by this time, the class had begun developing properties using definitions that I had suggested. From Steve's journal, it appeared that he had regarded $S(\Theta)$ as being the same as $\sin \Theta$ in developing his definitions. Consequently, when he used the 30° angle of a 30-60-90 triangle for Θ and got $S(\Theta) = 1/2$, he thought he had something. A similar misconception had motivated his development of his definition of $C(\Theta)$. Thus, at this stage, Steve had not understood the problem clearly.

Steve's inability to separate $S(\Theta)$ and $C(\Theta)$ from $\sin \Theta$ and $\cos \Theta$ persisted into his first attempts of the next project. The assignment was to find an identity analogous to $\sin^2\Theta + \cos^2\Theta = 1$. He began

$$\begin{aligned}
S^2(\theta) + C^2(\theta) &= 1\\
\frac{(x+y)^2}{r^2} + \frac{(x-y)^2}{r^2} &=\\
\frac{x^2+2xy+y^2}{r^2} + \frac{x^2-2xy+y^2}{r^2} &=\\
\frac{2(x^2+y^2)}{r^2} &=\\
2 &= 1
\end{aligned}$$

Though he had been able to produce the work above by himself, it took the whole class talking and working together to finally conclude that $S^2(\theta) + C^2(\theta) = \underline{2}$, not 1. Steve wrote in his journal,

> It was at first hard trying to figure out a parallel identity for $\sin^2\theta + \cos^2\theta = 1$ because I was stuck on thinking it had to be equal to one (his emphasis).

This exercise seemed to be the turning point in his understanding of what we were trying to do and how to go about doing it. But it took one more exercise to clearly point the way. As a follow-up to the assignment described above, the class was to derive identities similar to the other two Pythagorean identities--$\tan^2\theta + 1 = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$. Remembering the error in his thinking on the previous problem, he wrote,

$$\begin{aligned}
T^2(\theta) + ? &= RC^2(\theta)\\
T^2(\theta) - RC^2(\theta) &= ?\\
\frac{r^2}{(x-y)^2} - \frac{(x+y)^2}{(x-y)^2} &= ?\\
&\vdots\\
\frac{-2xy}{(x-y)^2} &= ?
\end{aligned}$$

whereupon he was stuck since he could not recognize any of the basic definitions that we had established. Since no one had been able to do any better, I suggested that they consult a trigonometry book for ideas. The next class, Steve alone produced the following:

$$\begin{aligned} \text{We know}\quad S^2(\Theta) + C^2(\Theta) &= 2 \\ \frac{S^2(\Theta) + C^2(\Theta)}{C^2(\Theta)} &= \frac{2}{C^2(\Theta)} \\ \frac{S^2(\Theta)}{C^2(\Theta)} + 1 &= \frac{2}{C^2(\Theta)} \\ T^2(\Theta) + 1 &= 2RC^2(\Theta) \end{aligned}$$

He had completed a similar argument for the third identity. Both were correct.

Steve's background in trigonometry served him well since he indicated that he had not used a trigonometry book for reference. He reported similarly on the successful completion of another assignment in which he derived expressions for $S(-\Theta)$ and $C(-\Theta)$ that paralleled $\sin(-\Theta)$ and $\cos(-\Theta)$. Unfortunately this trend did not continue for very much longer. Later attempts were not nearly as productive due partly to his nemesis, proof. Proofs of parallel properties in trigonometry became more involved, and even though he did use reference materials, he had difficulty following the proofs. For example, an assignment had been given to create a property that parallels the law of sines. He had been unable to produce anything because, as he wrote in his journal, "I couldn't figure out how they derive the law of sines." After I reviewed the proof of the law of sines, the class, with a little assistance, developed the proof as far as that of law of sines. The completion of the proof required following the same patterns used in law of sines but extending a great deal beyond. Neither Steve nor any one was able to do this.

Besides difficulty in understanding trigonometric proofs, there was another reason for the decline in Steve's creative output. Like many students, Steve developed a case of the "lazies" about mid-semester (which was near the end of the teaching experiment). He readily admitted to this during the last interview. His choice of problems for the final assignment was further evidence. He chose to derive the derivatives for C(Θ) and the four other Triometry functions (See Appendix C for other possibilities). Since the class, with my assistance, had found the derivative for S(Θ) in a previous assignment, this was not a difficult task. It merely required applying what was done in the proof for derivative of S(Θ) to C(Θ) and paralleling derivatives for tangent, cotangent, secant, and cosecant to the remaining four functions. He did these quite handily.

The examples of Steve's work presented in the preceding paragraphs are evidence that he exhibited some creative behavior and that this behavior was enhanced as the teaching experiment progressed. At the onset, he was bound to the familiar definitions for sine and cosine, but his ability to make the desired analogies gradually improved. Thus, he was able to take something old, like the proof of $\sin(-\theta) = -\sin\theta$, and adapt it to make something new, like proving $S(-\theta) = C(\theta)$. There were, however, limitations on the extent to which he was able to adapt trigonometry proofs to triometry. For one, he was unable to follow some trigonometry proofs and so could not adjust them to the related function in triometry. For another, though he could toy with ideas somewhat, such as the definitions he created, when the toying involved extension of a proof beyond what was done in trigonometry, he was less successful. The work that he produced in his journal suggested that his motivation to produce anything waned near the end of the teaching experiment.

Summary. Steve demonstrated an ability to play around with ideas on the first assignment with his definitions for $S(\theta)$ and $C(\theta)$. Initially he had difficulty making the proper analogy between trigonometry and triometry, but he quickly caught on and was able to derive a number of properties. His proof of $S(-\theta) = C(\theta)$ attests to this.

Though he understood the analogies, he had limited success with reconstituting of something old to make something new. I attributed this to his difficulty in understanding some of the trigonometric proofs. Thus, he was unable to adapt those proofs to the analogous triometry function. In addition, he was stymied by some proofs that required going beyond the similar proof in trigonometry.

Nora

Nora was twelve years old when she emigrated with her family from Southeast Asia. An older brother was not allowed to leave with the family. She expressed a desire to visit him, but her father has insisted that she not go since there is no guarantee that she would be able to return. Nora could speak no English when she arrived in the United States, and although she speaks and reads English fairly well now, she indicated on the first questionnaire that, "English is my second language and is hard for me to write and think in English." This may have been a problem for her throughout the teaching experiment. More will be said about this later.

In high school Nora took the usual college preparatory courses that included a good mathematical foundation--algebra, geometry, trigonometry, and calculus. She did not have much to say about her high school experience except that it was hard not knowing the language. In college, she had elected to major in applied mathematics. At the time of this teaching experiment, Nora was a twenty-five year old senior in her final semester, consequently she had completed approximately thirty hours of mathematics. Besides this class, she was also taking a numerical methods course. She reported her GPA in mathematics as about 2.2 or 2.3, but I learned later that she had barely managed to meet the minimum of 2.0 in her major in order to graduate.

In explaining her choice of mathematics as a major, Nora had written on the mathematics questionnaire, "...because math is a good major to get in." I questioned her about this during the first interview. She answered that math presented a lot of job opportunities. Her response to my inquiry about her plans after graduation surprised me but not nearly so much as what followed. She explained that she had been working in a Japanese restaurant since being in college and that she wanted to open her own restaurant following graduation. Naturally, I asked why she had not major in restaurant management or even business instead of mathematics. She explained that she had been merely following her father's wishes.

Despite mathematics not being Nora's self-chosen field, she did like it, primarily because, "it deals with numbers and formulas so I don't have to use words," she wrote. It seems that numbers are the same in both her languages. Not surprisingly, calculus was her favorite area of mathematics. With calculus, she could "mostly just apply formula into the problems." Her least favorite part of mathematics was geometry and proof because "they deal with angles." She was referring to congruent angle and triangle proofs. Upon further inquiry, she added that she disliked all proofs because she had trouble doing them and understanding them.

Trigonometry was not one of Nora's favorite subjects but she did not dislike it. She declared, "It is easy to do and I don't have to know a lot of formula." She explained that in high school, they (the students) did not have to memorize too many trigonometry formulas. They just had to be able to use them to find answers. About two weeks later, during the third interview, she revealed that she had forgotten a lot of the basic identities of trigonometry like $\sin^2\theta + \cos^2\theta = 1$. She volunteered, "It's been a long time and when I took it I wasn't good at it." This was somewhat of a problem for her during the teaching experiment.

Nature of Mathematics. The sources of information for this section include the Nature of Mathematics Surveys I and II, interviews I, III, and IV, and mathematics questionnaires I and II. Except for Facets 6 and 7, Nora's initial scores on the Nature of Mathematics survey (Table 3, page 67) indicate that her perceptions of the nature of mathematics were far removed from the "correct" views of the mathematicians. In comparison with the PSMT group, she held somewhat similar views only on Facets 3 and 5. It is feasible that the low scores can be attributed to Nora's difficulty with the language, but there were indications from the interviews and questionnaires that this was not entirely the case.

Throughout the interviews, Nora conversed easily until I started asking questions about mathematics. Then she had difficulty expressing herself. My impression was that it was from not knowing what to say as much as not knowing how to say it. On the first questionnaire, she had not written anything in response to the questions "What is mathematics? How would you describe mathematics to someone?" So, during the first interview I asked her if she could tell me. Her response was that she did not know how, meaning she did not know what to say. But through more questioning, I was gradually able to piece together some characteristics that she associated with mathematics and mathematicians. These were:

> Mathematics deals mostly with numbers and manipulations of numbers.
>
> Mathematics is organized and beautiful.
>
> Mathematics needs an application to be of any use.
>
> Mathematicians do a lot of complex numerical calculations, prove theorems, and solve problems by applying mathematics. They can create their own symbols but not their own rules. They must use rules that are already developed.

Midway through the teaching experiment, Nora described mathematics as dealing with numbers and solving problems. Solving problems, she said, meant "creating something and trying to come up with something." The latter description may have been influenced by our classroom activities since we had been working at creating our own trigonometry. During this third interview, she expressed a change in opinion about the value of mathematics with respect to its application. We had been discussing our class work when I inquired, "Does it matter that it (triometry) may not be of any use to anybody?" She replied, "It don't matter, we're still learning. If I go out and don't use all of what we learn then that's ok." Further on, she made a statement that to me spoke of enlightenment about a way of doing mathematics that before had been more or less a rule for her.

Investigator(I): How do you feel about this problem solving experience that we've been doing in class?

Nora (N): I think it's interesting. It's different than most classes I took. <u>You don't just go home and do what the teacher tells you</u> [emphasis added]. You try to create or come up with something new. You're on your own in a way.

Two weeks later, during the final interview, she reiterated this new concept more distinctly.

I: Nora, how would you describe your experiences in this class?

N: ...It's different. It's not like anything I've ever done before. I've come up with new math. I never thought you could create math. Like I thought people before created math, so you just follow what they did. I didn't think that you could do something different from what they did.

I: What have you learned about mathematics?

N: That you can create math. Before I didn't think you could create math at all. I thought you just follow what the book said.

I: Anything else?

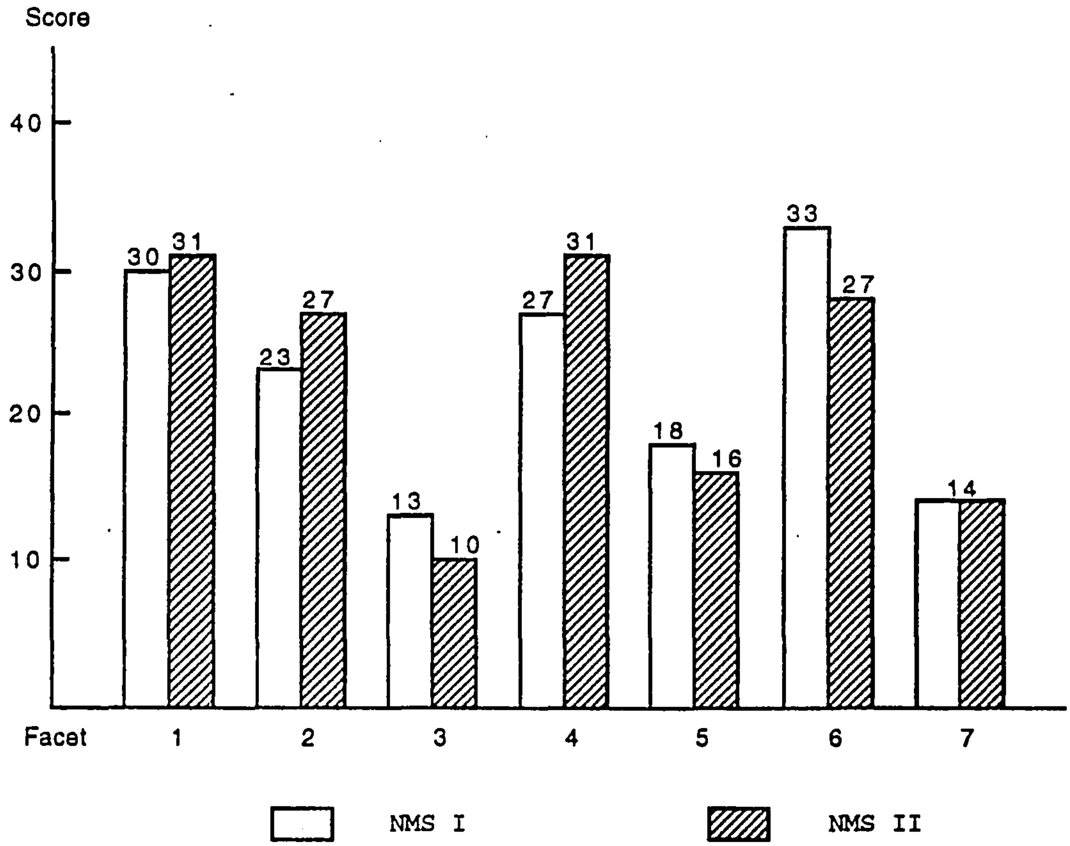
N: That you need a lot of thinking for this. Before in math you just do what the professor tells you to do. It's different now, you just look at the

> problem and try to think of something. It requires more thinking than other math classes.

Nora was "seeing" mathematics through a new window. However, her responses to related questions on the final questionnaire indicated that her basic precepts of mathematics and mathematicians were unchanged except for two additions. She made a more emphatic statement about the relationship of mathematics to numbers, and she included the concept of creativity into her description of the work of mathematicians. She wrote:

> Mathematics is a problem dealing with numbers and to solve it [you] need numbers. It cannot [be] solved by using words. [Mathematicians] do mathematics by using numbers to solve the problems that they [are] dealing with; also by applying formula and equation that they already know or create to solve the problems.

A comparison of scores on each facet from the first and second administrations of the Nature of Mathematics Survey (Figure 11, p. 107) indicates that Nora's picture of mathematics remained underexposed. Facets 1 and 7 were virtually unchanged. Her scores decreased on three of the facets--3, 5, and 6. One explanation for these results is that, like Steve and Ruth, she was unfamiliar with the terms--generality of expressions, insight, intuition, conjectures, inductive and deductive reasoning. The results of Facet 5 are consistent with her expressed views of the work of mathematicians. The drastic looking change in Facet 6 was the result of negative changes in responses to only three questions in the set.



Key to Facets (Scheding, 1981):

1: Mathematics as an organized body of knowledge; the generality of mathematics.
2: Nature and attributes of proof; the roles of deduction and induction in mathematical discovery and proof.
3: Role of insight and intuition in the work of the mathematician.
4: Beauty in mathematics; mathematics as a creative art.
5: Relative importance of massive or complex numerical calculations and abstract or symbolic thought in the work of the mathematician.
6: Relationship of mathematics to the real world.
7: Existence of differing views of the nature of mathematics.

Figure 11. Nora's Scores on Nature of Mathematics Survey (NMS) I and II

Positive changes occurred only on Facets 2 and 4. In examining individual questions relating to Facet 2, I found that on several questions Nora had tended to disagree (scored as 2) the first time. The second time, she was less sure and had scored them as 3s, hence, the higher score. Finally, the increase in the score of Facet 4 reflected her new vista of mathematics as a creative art.

Summary. The facet scores on the Nature of Mathematics Survey (NMS I) indicate that Nora's initial views of the nature of mathematics were decidedly less correct than those of either the professional mathematicians group or the prospective secondary mathematics teachers group in Scheding's study. Basically, Nora conceived of mathematics in terms of numbers, manipulation of numbers, and applications. Mathematicians used these components to solve problems and prove theorems. They could create their own symbols but not their own rules. Her ideas of doing mathematics centered around directives from textbooks or teachers.

At the end of the teaching experiment, her scores on NMS II had not improved, an indication that she held somewhat the same views as she had initially. Unfamiliarity with terms, e.g. inductive and deductive reasoning, insight, and intuition, may account for some of the low scores. Her survey scores notwithstanding, Nora did express some positive changes in her perceptions. These were:

1) Mathematics does not have to have an application to justify its existence; 2) There is more to mathematics than what is in textbooks; 3) It is permissible for mathematicians (or anyone) to create their own rules and symbols.

Perception of Ability to be Creative. This section was derived from interviews II, III, and IV, and questionnaires I and II. "Originality" characterized Nora's conception of creativity. To be creative means "you try to come up with something new, not what we already know or use," she said during interview II. In mathematics, "[you] come up with something similar to the problem...but different." Solving problems in different ways was also creative as long as one did not use a method that he/she had known previously. She described the requirements to be creative in mathematics as taking "a lot of thinking, a lot of work...You need to sit down and think and work a lot. I don't think you need to be a genius."

In Nora's judgment, she had never been creative in anything including mathematics, consequently, she was uncertain yet fatalistic about her ability. She explained, "I haven't tried before to be creative. I can try. If I don't come up with anything, then I don't think I'm creative." Being creatively uninitiated, she was predisposed in the beginning to regard her creative abilities unfavorably.

I: How would you rate your creative ability on a scale from one to ten with ten being high?

N: About three or four, very low...I just think I'm not creative.

I: Do you think you can learn to be creative?

N: Yeah, I can try but I don't know if I can do it or not.

By the middle of the teaching experiment, Nora expressed only slight improvement in her creative ability.

I: Do you feel that you are any better at creating mathematics than you were three weeks ago?

N: A little bit. When I see a problem, I try to solve it by thinking of something different, something new, than what I've already learned.

This more positive outlook not withstanding, Nora's perception of her performance in comparison with others in the class created some anxiety. She stated:

> When I couldn't come up with anything then I think I'm so dumb ... I feel bad I couldn't create anything. Maybe my brain isn't better than anybody else's...I feel like if I come up with something I'm with them, I'm not behind. If I don't come up with something, then I feel like they are better. I need to do something.

Throughout the teaching experiment, Nora continued to estimate her creative potential in mathematics guardedly. During interview IV, I inquired about how she would react to taking another class that was conducted similarly to the way this one had been and if she would have a better idea of what to do. She replied affirmatively to the idea of another class, but because she was graduating at the end of the semester, it was in the sense of "It's a good idea as long as I don't have to do it." As for the latter part of the question, she offered:

> I don't know about better idea what to do. I can try to do it as long as the professor don't expect me to come up with something, like forcing you to come up with something. If he said just try it, I would try it.

Her final self-evaluation was somewhat more optimistic. She wrote:

> I would rate my ability about average because some problem[s] you know how to do and some problem[s] you don't. My present rating is better than what I had started because I can create math now not like before I can't create anything.

Summary. Nora thought of mathematical creativity as solving problems in ways different from those one had been taught or the creating of something similar to but different from a known problem. She did not regard herself as creative in mathematics and was not sure she could be since she had never had occasion to engage in that sort of activity. She rated her creative ability as below average. Throughout most of the teaching experiment, she continued to hold reservations about her creative ability in mathematics, conceding only slight improvement by the middle of the teaching experiment. At the same time she maintained her willingness to try. Her self-evaluation at the end was more favorable. At this time, she judged her mathematical creative ability to be average since she had a better sense of what is involved in creating mathematics.

Creative Behavior. Sources of information for this section include observations, assignments, and Nora's journal. From beginning to end of the teaching experiment, Nora experienced little success in creating triometry. Her attempts were affected by several factors. First of all, she was not a very strong student as evidenced by her GPA. Secondly, though she was acquainted with the basics of trigonometry, her working knowledge and understanding of elementary trigonometric functions together with misuse of mathematical properties circumvented her efforts. For example, she interpreted $\cos(\pi/2 - \theta)$ as $\cos \pi/2 - \cos \theta$. In another instance, she divided $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2$ to get $\theta = (1 - \cos^2\theta)/\sin^2$. A third factor can be attributed to her expectations of mathematics as teacher- and textbook-directed. She could work problems if she had an example or pattern to follow such as deriving the derivative of $C(\theta)$ using the proof of derivative of $S(\theta)$ which had been done in class.

The one area of creative behavior in which Nora was able to demonstrate some improvement was in her ability to toy with ideas. At the beginning of the teaching experiment, reliance on so called "cook book" assignments made this difficult for her. For the first assignment, which was to create a definition analogous to the sine function, the most she was able to produce was $x^2 + y^2 = r^2$

(Pythagorean Theorem). She explained in her journal, "...I don't know how to go from there. I didn't really understand [what] I [am] suppose to do with the problem." The story was the same for the next assignment which was to develop an identity similar to $\sin^2\theta + \cos^2\theta = 1$. It was the third assignment before she was able to play around with the new definitions. Having seen a development of $S^2(\theta) + C^2(\theta) = 2$ (the parallel to $\sin^2\theta + \cos^2\theta = 1$), she tried $1 + T^2(\theta) = ?$, the parallel to $1 + \tan^2\theta = \sec^2\theta$. The question mark was there because the class had been conditioned by the previous proof not to expect to get results similar to its trigonometric counterpart. Even though she was not successful in deriving the identity, she had finally been able to produce <u>something</u>. Her efforts to draw parallels continued throughout the rest of the course but were unfruitful because of factors already discussed.

One last observation deserves mention. Nora's lack of creative ability notwithstanding, she was not without initiative. Once she had the basis of toying with ideas, she attempted some parallels before anyone else did. For example, she tried the derivative of $S(\theta)$ and $C(\theta)$ before they were done in class. The only thing she had done correctly was the substitution--

$$\lim_{h\to 0} \frac{S(h + \theta) - S(\theta)}{h}$$

The results she obtained, $S'(\theta) = S$, are understandable in view of the kinds of errors she was prone to commit. Finally, for the last assignment, almost every one in the class elected to work on derivatives of other triometry functions, an easy task since we had already done $S'(\Theta)$ in class. Nora started to do the same problems but then opted to try inverses of $S(\Theta)$ and $C(\Theta)$, a rather difficult task. The most positive comment I can make about what she produced is that she seemed to understand that the composition of a function and its inverse is an identity (e.g., $f[f^{-1}(x)] = x$). She was indeed playing around with ideas even if she was not in the right ballpark.

Summary. Nora's attempts to create triometry were minimally successful. Generally speaking, this was due to her being a weak mathematics student, as her GPA (2.0) in mathematics suggests. Specifically, it was due to a poor working knowledge of trigonometric functions and mis-use of mathematical properties. Nora's reliance on textbook or teacher directives also contributed to her lack of productivity. She needed an example to follow with no deviations.

The one area of creative activity in which Nora did demonstrate improvement was in toying with ideas. At the beginning of the teaching experiment, she could do none. By the end, she showed evidence in her journal of trying out ideas, but she was not successful due to the deficiencies noted.

Don

Don was a twenty-year old sophomore. He had not declared a major and was, in fact, leaning toward physics. His reasons for taking this class, he explained, were that it sounded interesting and also that one of the young ladies in the class had persuaded him to try it. The latter reason carried more weight than the former I suspect. Don lived at home and commuted about thirty miles to school each day. He was working twenty to thirty hours each week at a convenience store and, so, was taking only twelve hours of course work, the minimum for full time status. These factors combined to detract from the time and effort he put into the classwork.

Don's mathematics background was quite adequate. In high school he had taken trigonometry, geometry, and three years of algebra. His college courses included two semesters of calculus. He was taking the third semester of calculus and linear algebra concurrent with this class. He reported a 2.3 GPA in mathematics.

Trigonometry along with geometry were Don's favorite areas of mathematics. He did not mind the proofs in geometry, and trigonometry consisted of basic rules to follow. To Don, this was important because, he wrote, "I feel more comfortable when I can follow a rule or an idea instead of opinion." The latter statement seemed contradictory with what Don disliked most about mathematics

--memorization of sets of rules or functions. He explained, "I would rather learn them through their use." He liked most "the way everything is tied together and builds on itself," he wrote. His penchant for orderliness and having a set of rules to follow influenced his perceptions and expectations throughout the teaching experiment.

Don described his favorite mathematics teacher, a high school geometry teacher, as "very knowledgeable." She was adept at getting the class to learn a particular lesson by presenting the material on their level and, at the same time, making it interesting. To him, her lessons were straightforward, and he knew what was expected of him. Perhaps, he suggested, this was why he liked geometry.

Nature of Mathematics. Sources of information for this section include Nature of Mathematics Surveys (NMS) I and II, interviews I, III, and IV, and mathematics questionnaires I and II. A comparison of facet scores (Table 3, page 67) shows that, initially, Don's scores were one to two points higher than the mean scores of the prospective secondary mathematics teachers group on four of seven facets (1, 2, 3, and 5). His scores were lower on Facets 4, 6, and 7 by three, five, and one point(s), respectively. On three facets--1, 3, and 5--Don's views of mathematics were nearly like those of the professional mathematicians. Even so, over-all, his perceptions of mathematics were more like those of the prospective secondary mathematics teachers.

Basically, Don considered numbers, rules, and formulas as the components of mathematics which are used to quantify real world phenomena. "Mathematics", he wrote, "is using numbers to represent real things and then being able to describe the characteristics of these real things [such as] how many? when? where?" He acknowledged that perhaps numbers could represent things that were not real, but added, "what good is it if it's not real and something you can use." The doing of mathematics he viewed in terms of using these components. He explained, "Anything that uses the rules, patterns, number system is doing mathematics." Some of those things included solving an equation, balancing a checkbook, using a rule or formula to solve a problem, or finding derivatives of functions. In problem solving, he said at first, being able to apply a rule or formula was not all that important "as long as you know you've got the right answer, one that works." Later, in response to my question, how are you going to get the answer?, he replied, "You have to have some methods for solving for it, some rules to go by."

Don's conception of the work of mathematicians coincided with his notion of mathematics. He wrote, "They put on paper what cannot be easily seen or comprehended. They use the rules and systems of math to figure out things." As he explained later, "they (mathematicians) figure out the things that are in (text)books that ordinary people (like himself) don't know how to do--like proofs of theorems." He was not sure where the rules or the theorems that mathematicians use came from. He speculated that they were developed over time. "But," he said, "I've never really thought about it before. I've just accepted what's in the books." He reluctantly agreed that it was all right for a mathematician to create his/her own rules and symbols "as long as he is able to understand it himself." His reservation about the idea was displayed in his next statement: "If he creates his own rules and symbols, he's going to have a hard time working with anybody else if they're doing the same stuff."

Midway through the teaching experiment, Don was still describing mathematics and the work of mathematicians in the same terms. At the same time, his need for a rule to follow became more apparent. During interview III, I inquired:

> Investigator (I): Are we doing mathematics? (referring to classwork)
>
> Don (D): I suppose so. I haven't figured out what you mean by mathematics.
>
> I: It's not what I mean by it; it's what it means to you.
>
> D: I don't know other than using numbers to get results you need, figure things out.

The series of questions that followed were attempts to get him to explain his responses further. For example, the next question and response were:

I: What do you mean by 'figure things out'?

D: Like equations and stuff so we can do the problems.

I: Can you tell me a little bit more about what you mean by that?

Each time he would explain one part, I would ask him to clarify another. At the seventh question, exasperation showed in his voice and response. He said, "When I gave you the definition of mathematics, I don't know if that's right or not. I've never been told that." Quite clearly, he expected me to give him a definition of mathematics and was somewhat irritated because I would not.

The changes that did occur in Don's perceptions of mathematics should more accurately be termed additions. He continued to describe mathematics as manipulations of numbers, but he was beginning to express some different insights albeit with difficulty. During the final interview I asked:

I: Have you learned any mathematics?

D: Yes. Mathematics is not just looking at one thing and seeing what you want; it's trying new ways, methods to get there, seeing the correlations or whatever. (Here, he hesitated, pondering what to say next). Going back to my definition of mathematics, we used new ways of manipulations of numbers than I've ever seen or used before."

I: What have you learned about mathematics?

D: How the concepts can be interrelated and still be two different things. Realizing something about it that I never thought about before.

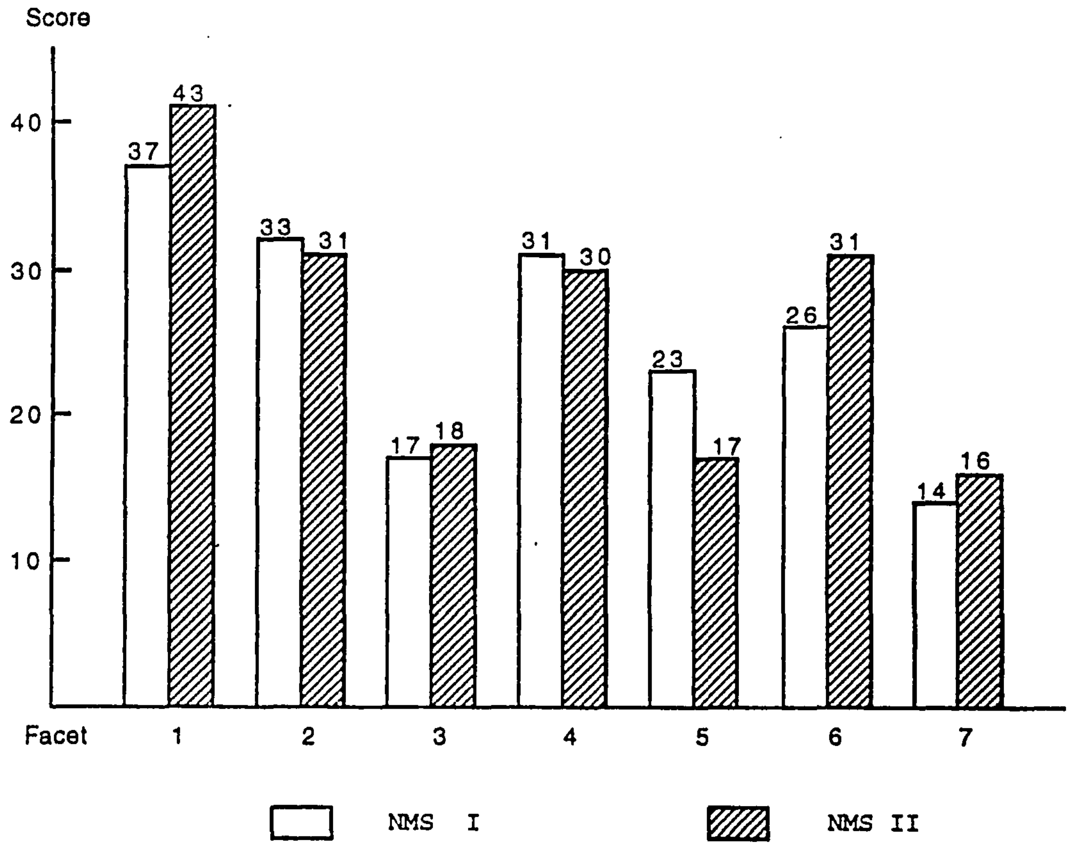
I: Like what?

D: Like how to work things together to make them turn into what you want them to be... Maybe something I thought I knew but maybe just realized how to use it this time; like thinking about going from step to step and thinking about something else that would do the same thing. I sorta knew that but just didn't really realize how to work with it.

His final stab at describing mathematics was a replay of the first time. He wrote, mathematics is "the manipulation of numbers or quantities to describe real world situations."

Comparing Don's facet scores on NMS I with NMS II (Figure 12, p. 121) shows that he improved his scores considerably on Facets 1 and 6 and declined about the same amount on Facet 5. The others remained virtually unchanged. From the beginning, he was more comfortable with the concept of mathematics as an organized body of knowledge (recall his liking of organization) and the generality of mathematics as a desirable feature (Facet 1). He was not sure of the terms "inductive reasoning" and "deductive reasoning" (Facet 2) but was familiar with "insight" and "intuition" and their roles in mathematics (Facet 3). The wide discrepancy between Don's scores and the PSMT group mean score as well as that of the professional mathematicians on Facet 4 can be attributed to his interpretation of an "elegant" proof. Don said:

> To me, the difference in a proof and an elegant proof is a proof gives the straightforward basics that you need to prove it [a problem]. An elegant proof would just add English to maybe make it sound like



Key to Facets (Scheding, 1981):
1: Mathematics as an organized body of knowledge; the generality of mathematics.
2: Nature and attributes of proof; the roles of deduction and induction in mathematical discovery and proof.
3: Role of insight and intuition in the work of the mathematician.
4: Beauty in mathematics; mathematics as a creative art.
5: Relative importance of massive or complex numerical calculations and abstract or symbolic thought in the work of the mathematician.
6: Relationship of mathematics to the real world.
7: Existence of differing views of the nature of mathematics.

Figure 12. Don's Scores on Nature of Mathematics Survey (NMS) I and II

> something interesting to someone who is not in math or doesn't know anything about it.

The improvement in Facet 6 may be that Don, being applications oriented, was not only more cognizant of the statements that contained the word "application" on the survey but was more decisive in his responses.

Facet 5 shows the most dramatic change because it seems to indicate that Don reversed his original stand. However, in retrospect, I am inclined to believe that the second score is closer to his true perceptions. This facet describes the work of mathematicians as involving more abstract or symbolic proof rather than complex numerical calculations. Don's description of the work of mathematicians throughout remained just the reverse. In fact, his last attempt read, "Mathematicians use the structure and rules of mathematics to figure out desired results."

Although Don's perceptions of the nature of mathematics remained more closely allied with those of the PSMT group, he made two statements that led me to believe that the teaching experiment had made some positive impression on him. One occurred during the last interview. I inquired about the worth of the experience. He replied, "Yes [it has been worthwhile]. It's showed me there's something out there to learn rather than just what's in the book." The second one was a response on the final questionnaire to the question, What have you learned from this class other than the content itself? Don wrote, "...how to reason through and figure out the procedure for doing things mathematically...to use thought instead of 'rules' [his emphasis] to do mathematics."

Summary. NMS I indicates that at the beginning of the teaching experiment Don's perceptions of mathematics were like those of the professional mathematicians on some facets, but in totality, they were closer to that of the prospective secondary mathematics teachers (PSMT). Don's utilitarian view of mathematics involved the manipulation of numbers to represent real world phenomena. The role of mathematicians was to use rules and formulas to solve the problems. He did not see any value in mathematicians creating their own rules and symbols since they would have difficulty communicating with others who might not be familiar with their work.

Changes in Don's views toward the correct view were small. Essentially, he had begun to see mathematics as more than the application of a rule or formula, and that there was more to mathematics than what appears in textbooks. NMS II shows that his beliefs at the end of the teaching experiment were still aligned more with the PSMT group. Don had correct views about the generality of mathematics, the role of insight and intuition, and the making of conjectures. His conceptions of inductive and deductive reasoning, mathematics as a creative art, and the work of the mathematicians were less correct than the professional mathematicians.

Perception of Ability to be Creative. Sources of information include interviews II, III, and IV and questionnaires I and II. Most of the recording of interview II on creativity with Don was lost due to mechanical failure of the tape recorder. When this was discovered, I promptly wrote down the essence of his responses as best I could. Being creative, Don said, is "being able to take what you've been given and expand upon it. Use your own ideas to make it something else or fix it your own way." In mathematics, it meant, "To take the math [that you know] and use it in different ways than you've been taught. To be able to make it work for what you want it to work for, what you're trying to figure out." Originality, as in producing something that no one else had ever done before, was not a requirement. He stated that something like applying a formula or solving an equation in a unique way would be a creative activity as long as the person doing it had not seen it done before. He considered that the prerequisites for individuals to create mathematics consisted of some brains, but they did not necessarily have to be geniuses, a good background in the basics, algebra, geometry, etc., and a lot of hard work.

Don did not believe that he had ever done anything creative, mathematically or otherwise. As for as his creative capabilities in mathematics, he guessed no more than average since he had no experience. By the middle of the teaching experiment, this rating had not changed:

I: Do you feel that you are any better at creating mathematics than you were three weeks ago?

D: No, not really. I've just done it more. I had never tried it before. I didn't know what it was or how to do it.

I: What has been the hardest for you to do?

D: Just knowing where to start, what to try, where to go.

The last statement gave an indication that Don was uncomfortable with the creative activities that the class had been attempting. This was borne out at the end of the teaching experiment in interview IV.

I: How would you feel about a similar type of exploration, like we've been doing, in another course?

D: The more it would rely on me to do the creative or develop the stuff the less comfortable I would be, because I sometimes get on the wrong road and can't get off. I have a hard time knowing where to start.

I: What if someone gave you an idea of where to begin, could you strike out on your own?

D: I'm not really comfortable with that.

He added that working with a group would be different since the others in the group could compensate for his weaknesses. "Unless I'm really sure about what I'm doing, I don't really like working with it by myself," he said.

Indications of a positive change in Don's perception of his creative ability were slow in coming. It was not until the last interview that he gave one small glimmer. Aside from his difficulty with knowing where to begin a problem, Don asserted that the classroom experiences had given him the idea that he could do this kind of activity, but he added, "it still hasn't given me the confidence." He offered a brighter outlook on the final questionnaire. In response to the questions, Have your ideas about mathematics changed since the beginning of this class? If so, in what ways?, he wrote, " Just in the way I look at beginning a problem. I think I've become more analytical." In contrast, the confidence in his ability had apparently not improved, since he declined to rate the changes in his creative ability at the end.

Summary. Don perceived that creativity in mathematics involved using the mathematics that one knows in ways different than he/she has been taught. For example, solving an equation in a way that one had not seen before would be a creative activity. He had little confidence in his own creative talents and rated his ability as no more than average. He based his rating on the fact that he was not aware of having ever done anything creative.

Throughout the teaching experiment, Don expressed difficulty in knowing how or where to initiate a problem solution. Because of this, Don continued to regard his creative abilities in a dim light. The only positive outlook in his perceptions was his sense of being more analytical in the way that he approached problems.

Creative Behavior. Information for this section was taken from Don's journal, assignments, observations, and interviews. Don's creative attempts were slow to develop due primarily to his need to have a rule or formula to follow. Without such, he was "dead in the water", so to speak, as far as knowing where to begin a problem. As he stated a number of times, he didn't know "where to start, what to try, where to go." His journal held no evidence that he had even attempted the first two or three assignments. These were to make definitions analogous to the sine and cosine functions, and after we had agreed on the definitions, to use them to develop identities that paralleled the Pythagorean identities. The first sign of his playing around was some work he did on the Law of S, a parallel to the law of sines. This was about three weeks into the teaching experiment (half-way). Their work on the identities had made Don suspicious about merely substituting $S(\theta)$ for $\sin \theta$, but he gave it a try anyway getting

$$\frac{a}{S(A)} = \frac{b}{S(B)}$$

He then substituted values from a triangle that he had sketched (Figure 13) and calculated the lengths of the segments using trigonometric functions and his calculator.

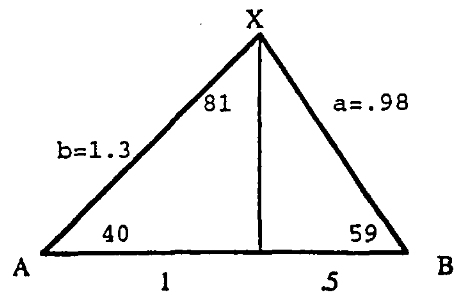


Figure 13

Correctly using the definitions for S(A) and S(B), he found that the two fractions were not equivalent. He wrote, "I tried working an example to see if the law of sines might work. It didn't. I think that this will work on an isosceles triangle. But so what." His recognition of a particular instance and the fact that he was careful not to use a triangle that might be a special case in the first place, demonstrated some insightfulness on his part.

The only other evidence of Don's playing around was a number of problems in which he had used formulas that we had derived to explore function values of S and C of various combinations of sums and differences of two angles, for examples, S(90-Θ) and C(180+Θ). He was "just curious," he said. He noted that their results were always C(Θ), S(Θ), or their negatives. He observed, "That's strange how these all tie together so neatly." He was mystified that something that we had "made up" was so well behaved like in trigonometry. This became more apparent to him later on.

To Don, having an example to follow had the same effect as having a rule to follow as long as there were no deviations required. He found a proof of $\sin(-\theta) = -\sin\theta$, which requires reflecting an angle θ and a point (x,y) on its terminal side about the x-axis and using $(x,-y)$ as the coordinates of the point symmetric to (x,y) (Figure 14), and successfully used it to find an expression for $S(-\theta)$ in terms of θ. He then went on to find equivalences for $C(-\theta)$ and $T(-\theta)$.

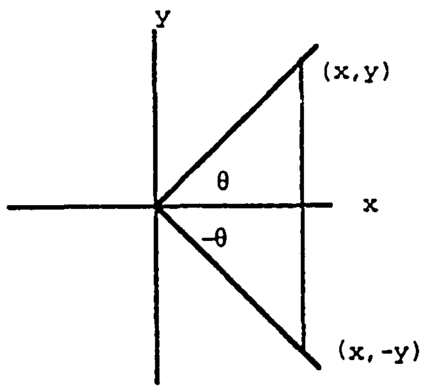


Figure 14

Although Don had difficulty knowing where or how to start a problem, he demonstrated a number of times that he was adept at analyzing information at hand. Before he found a proof for $S(-\theta)$ or $C(-\theta)$, he examined a table of function values that he and the class had calculated and observed that $S(-\theta) = C(\theta)$ and $C(-\theta) = S(\theta)$. He wrote in his journal, "I can see by looking at our tables that this is true, but how do you prove it?" He used a similar technique to speculate that $S^2(\theta) + C^2(\theta) = 2$, whereas everyone in the class who had been working on the problem (Don had nothing to show) was trying to show the sum was "1" because of its analogous identity, $\sin^2\theta + \cos^2\theta = 1$.

One more example is worthy of mention. At about the same time that Don was toying with the formulas, he discovered that the definitions of $S(\theta) = (x+y)/r$, and $C(\theta) = (x-y)/r$, are the same as $\cos\theta + \sin\theta$ and $\cos\theta - \sin\theta$, respectively, although he mistakenly identified x/r as $\sin\theta$ and y/r as $\cos\theta$. Don was the only one in class to recognize this relationship. About his discovery he wrote, "It surprised me but after thinking about it, it is a simple thing to see. Knowing this makes $S^2(\theta) + C^2(\theta) = 2$ easier to see and quicker to find." But I would not let him use this, since I wanted the class to develop the properties and proofs independent of trigonometry.

Besides Don's dependency on rules, formulas, and needing an example to follow, there were two other factors that had some bearing on his creative activity. Even though he was quite familiar with the trigonometric properties and identities, like the others in the class, Don had difficulty following some of the trigonometric proofs, especially if there were steps missing. The second factor occurred during the latter part of the teaching experiment. This was the time and effort that he devoted to the class. During the interviews midway through the teaching experiment and at the end, he admitted that he had not had enough time to work on the assignments like he should have. The final assignment that he submitted bore this out. Fifteen minutes before class, he used a computer program that he was using in another class to graph $S(\theta)$, $C(\theta)$, and $T(\theta)$. He did this by using their trigonometric equivalences that he had discovered. He hastily labeled the graphs after I pointed out there was nothing to indicate what they represented. Unfortunately, two of them were wrong.

The evidence presented in the preceding paragraphs suggests that Don's creative activity though not excessive was multi-faceted. He could recognize a pattern and use it to make conjectures as with $S(-\theta) = C(\theta)$. He was placing things in new perspectives when he discovered the equivalence of $S(\theta)$ with $\cos\theta + \sin\theta$. His toying with ideas was limited primarily to repeated applications of formulas. Though I have no evidence on paper that he came to understand the analogy of proofs in triometry with those in trigonometry (recall his slacking off at the end), my observations of his classroom participation lead me to conclude that this is so. He thought so too since he said, "It's taken till now [end of the teaching experiment] to get the general idea of what we need to be doing."

Summary. A number of factors conspired to make Don's attempts to create triometry marginally successful. Foremost was his reliance on having a rule, a formula, or an example to follow. Another factor was difficulty in understanding trigonometric proofs. Lastly, toward the end of the teaching experiment, the time and effort he devoted to the course declined.

The incidences in which Don had limited success in toying with ideas involved using formulas repeatedly to evaluate expressions. But as a result, he demonstrated an ability to recognize and analyze patterns of information. His insights allowed him to make correct conjectures even though he could not prove them. Also, he was the only student to relate the triometry definition of $S(\theta)$ with its trigonometric equivalent. Although he had difficulty in drawing the proper analogies between trigonometry and triometry at the beginning, by the end of the teaching experiment, he had made some improvement.

Gina

Gina was in her senior year in high school before she decided to go to college. Even so, she had prepared herself well mathematically by taking four years of mathematics which included pre-calculus. At the time of this teaching experiment, Gina was twenty years old and a junior. She had decided to major in mathematics education and minor in physics.

Gina's regard for mathematics and the ability to do mathematics had figured prominently in her decision to attend college as well as in her choice of major. Even if she decided not to teach, she would "have a good choice of careers." She explained,

> I told my parents [about her decision to attend college] I didn't want to take something easy like home ec [economics]. I wanted something that uses my mind. Everybody says if you have a math major and you go into different fields, that looks good because it shows you have brains...Math shows that you have the capacity to learn. It's not something everybody can do.

Judging by her reported GPA in mathematics of 3.0, Gina could do mathematics well. She had taken pre-calculus (an easy A, she said), two semesters of calculus, and a statistics course and was taking linear algebra and third semester calculus concurrently with this teaching experiment.

What Gina liked most about mathematics was "working on problems because I like figuring out how things go together, knowing the different rules and what you can do to make things work together," she stated. She was partial to algebra, calculus (calculus I more than calculus II), and trigonometry. Algebra problems, like solving equations and writing equations of lines, were her favorite kind, but she also liked finding derivatives, areas, and volumes. She was not intimidated by word problems in the least, regarding them as "a breeze" as a result of her experiences in physics. As for trigonometry, she was not sure why she liked it except that "maybe the trig I've had so far hasn't been too hard." Quite naturally, Gina's career interests lay in teaching high school algebra II, trigonometry, or calculus. But she could not explain exactly what she liked about them just that they were "different." She rejected teaching lower levels of mathematics because "it would be boring."

Her least favorite area of mathematics was geometry because she did not like having to write the two-column proofs even though she knew how to do it. Gina's attitude toward proofs in general was one of disdain. Though she recognized the importance of proof because "that shows why it works," she added, "I don't like doing it; it's excessive. I'd rather just take it for granted. Guess I'm just lazy or something."

Although at times Gina found it difficult to express herself about mathematics, she could not say enough about her favorite mathematics teacher who had taught pre-calculus in high school. She described him as very smart (he even knew Shakespeare), very imaginative in his lessons (sometimes they played games like jeopardy), and very organized (you knew what to expect daily even if you were absent). The latter characteristic particularly appealed to Gina. He had taught her a lot of math, some little tricks to help remember certain properties, but most of all "he made math fun."

Nature of Mathematics. Information for this section comes from Nature of Mathematics Surveys I and II, interviews I, III, and IV, and mathematics questionnaires I and II. At the beginning of the teaching experiment, Gina's scores on four out of seven facets--1, 2, 4, and 5 --of the Nature of Mathematics Survey I were lower by four or more points than the mean scores of the PSMT group (Table 3, p. 67). Her scores were considerably less than the mean scores on all facets of the professional mathematicians group. Thus, her over-all view of the nature of mathematics was less correct than that of the PSMT group and considerably removed from that of the professional mathematicians group.

Gina's description of mathematics was somewhat vague. At the beginning, she wrote, "Math is figuring out things. It is trying to understand how different things work." Later, during the first interview, she added, "I think about working with numbers. There's different types of math like general math where you're adding and subtracting. It just depends on what you want to do with it. You use math every day, you count money..." From the survey results and these initial responses, my first impression was that Gina's perception of mathematics was rather elementary and much like the students in Frank's or Schoenfeld's studies (Chap.

2, pp. 18-20). Further questioning revealed that this was not entirely the case. I asked her to respond to a number of statements about mathematics which were similar to or in some cases exactly the same as statements in the survey. Here are some of the statements and Gina's responses which I interpreted as having the correct view of mathematics:

> Investigator (I): Mathematics is a search for patterns.
>
> Gina (G): I'd say in a way it is because you have all the rules and things that apply to different type problems and they relate to each other and they use the same rules... A lot of times you see things in a problem you've seen in other problems so you get a feel for how to work it.
>
> I: Mathematics attempts to find connections.
>
> G: If you see something you've never seen before, you try to think of what you <u>have</u> seen that you could relate it to.
>
> I: Mathematics deals with ideas and relationships rather than with numbers and manipulation of numbers.
>
> G: I think it deals with all that. It just depends on what level you're working at.

There were a few other indications that her perceptions were closer to correct than her scores indicated. Rules, formulas, and memorization were important in mathematics, she said, but "not so much because you can always look formulas up. You need more of an understanding of how things work." She explained <u>intuition</u> as "having an idea of where to start even if it might not be the right way," and agreed that it has a place in mathematics. She recognized mathematics without application as having value although her impression was still somewhat utilitarian in that "...it gives you certain skills that you need in other things, reasoning skills and things like that," she said.

Discrepancies in scores of facet items with verbal understanding was not total. There were some terms and ideas with which Gina was unfamiliar or uncertain. She did not know the word conjecture, but did know about "educated guess" and acknowledged its importance and use in mathematics. Inductive reasoning and deductive reasoning were terms that she did not clearly understand. Following a brief explanation--deductive reasoning is reasoning from the general to the specific and inductive reasoning is reasoning from the specific to the general--she offered, "You use both because you have to break things down into steps." At first, Gina thought that mathematics was more specific in nature as opposed to general. But then she vacillated saying, "It's general because there's a lot of things you can apply to one thing. It's a little bit of both." Gina had never thought of mathematics as being beautiful. Her conception of an "elegant" proof was one that was "...fancy, not straight to the point, kinda covered up. When I think of something as being elegant, it has extras..."

Gina was uncertain about the work that mathematicians do though she suspected that there was more to their work than was evidenced by classroom activities. Initially, she wrote, they "work problems [and] reason through ideas." I encouraged her to say more during the first interview. She responded, "I'm not really sure what they do. I know they don't have to work problems and stuff." Here, she hesitated, then continued, "I guess maybe they can work problems and figure out how they apply to every day life." The problems on which they worked might be textbook type, but more likely, the kind that relate to industry she allowed. When asked about mathematicians proving theorems she said, "I don't know if they sit around and prove theorems because they've been proven before." As for mathematicians creating their own rules and symbols, Gina felt that it was better to "...stick with what's been proven or used." She went on to explain,

> I've never had a class where you try to come up with new things. It's more like this is what's been used forever and this is what we'll use. That's what we've been taught and that's how it's going to be.

Despite Gina's initial acknowledgement of some correct views of mathematics her verbal and written descriptions remained virtually the same as they had been at the beginning. During the final interview she responded that when you do mathematics you "work with numbers, work with formulas, try to come up with ways of figuring things out." On the last questionnaire, she included another dimension briefly describing mathematics as "...working problems [and] finding relationships." Her description of the work of mathematicians was similar in brevity but also contained an aspect not previously recognized. She wrote, "mathematicians solve problems, think, and try new things."

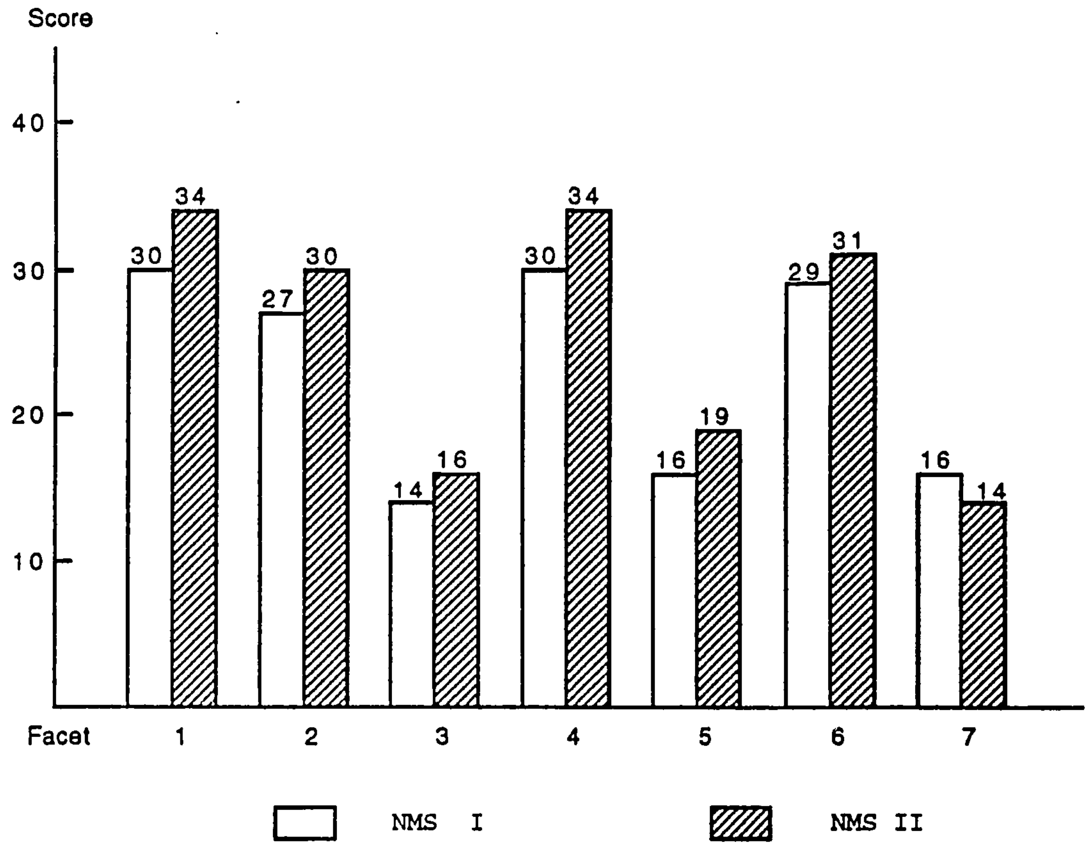
The statement "mathematicians...try new things" reflects one of the more significant changes in Gina's perceptions about mathematics. At the onset, her feelings about the development of new mathematics were mixed and reflected her lack of exposure. In a way, she felt that there was nothing else new to be done. But Gina was not one to commit herself to an idea completely. She stated:

> I haven't really been introduced to this stuff [new mathematics]. We're just using what's been used and not really exploring the new fields...I think there's more out there. I'm pretty sure people are working on it. I've never had a teacher who said this is what I came up with and we're going to use it now. It's always this is what somebody a way back came up with.

As she progressed through the teaching experiment, she gave indications that she understood that she did not have to abide by the status quo. Half way through, I asked what she liked about the course so far. She replied, "Trying to see how things work together and how you can come up with a different thing rather than what's already been accepted one way forever." This new revelation continued to impress her for she wrote at the end, "I thought no new math was being worked on. I thought the old way was the only way. Now I know that's not so."

In the final analysis of changes in Gina's perceptions of the nature of mathematics, Figure 15 (p. 141) compares scores from Nature of Mathematics Surveys I and II on each facet. Her scores increased by three or more points on the four facets (1, 2, 4, 5) that differed greatly from the PSMT group at the beginning of the teaching experiment. Facets 3 and 6 increased two points while Facet 7 decreased by two. In coinciding more closely with the PSMT group, Gina's views moved nearer to the correct view of the professional mathematicians.

One final observation bears mentioning. In trying to reconcile the initial inconsistencies between Gina's scores and some of her verbal responses, I studied her responses on the survey and noted that she had recorded 3, meaning neutral or no opinion, for more than half of the statements, a larger proportion than anyone else. Apparently, unfamiliarity with terminology or uncertainty about her position relative to that particular idea prevented her from committing herself to an opinion at this stage. On the second survey, she recorded only six 3s. But even though she was more willing to commit to an opinion, she did so conservatively, recording only two 5s, meaning "agree," and no 1s, meaning "disagree." The rest of the responses were divided evenly between "tend to agree" and "tend to disagree."



Key to Facets (Scheding, 1981):
1: Mathematics as an organized body of knowledge; the generality of mathematics.
2: Nature and attributes of proof; the roles of deduction and induction in mathematical discovery and proof.
3: Role of insight and intuition in the work of the mathematician.
4: Beauty in mathematics; mathematics as a creative art.
5: Relative importance of massive or complex numerical calculations and abstract or symbolic thought in the work of the mathematician.
6: Relationship of mathematics to the real world.
7: Existence of differing views of the nature of mathematics.

Figure 15. Gina's Scores on Nature of Mathematics Survey (NMS) I and II

Summary. Gina's first survey results (NMS I) indicated that her perceptions of the nature of mathematics on all but three of the facets were considerably less correct than the prospective secondary mathematics teachers group and distanced even more from views of the professional mathematicians group. A study of her answer patterns suggested that she had been diffident about what she believed. With perhaps the same uncertainty, Gina described mathematics as primarily working with numbers and formulas, and the work of mathematicians, as working problems, reasoning through ideas, or perhaps solving problems related to industry. She was not comfortable with the notion of mathematicians making their own rules and symbols. She was unfamiliar with or had misconceptions about conjectures, inductive and deductive reasoning, and what is meant by an elegant proof.

During the interviews, Gina revealed that her beliefs on some facets were closer than her scores seemed to indicate. She perceived correctly the ideas of searching for patterns, making connections, and intuition in mathematics.

The scores on NMS II intimate that by the end of the teaching experiment, Gina's beliefs about mathematics had become more like those of the PSMT group and, hence, moved closer to the professional mathematicians group. Her descriptions of mathematics and the work of mathematicians remained nearly the same. But she did express some new insights. She included "looking for relationships" in her description of mathematics and "try new things" for the work of mathematicians. One other significant change occured. She realized that new mathematics is being developed, and that one does not have to stick with what is done in textbooks to do mathematics.

<u>Perception of Ability to be Creative</u>. Sources of information for this section are interviews II, III, and IV and questionnaires I and II. Being creative "has a lot to do with originality, coming up with different things than the normal," Gina said. In that respect, Gina tended to think of creativity in terms of artistic output like paintings, "things that you can stand back and appreciate," she explained. Mathematics did not exactly fit this category in her mind. She could not imagine that people would stand back and admire a math problem, at least not most people. Though she did not regard mathematics as an art form, she did concede creativity in it.

I: You don't think of math as being creative?

G: I suppose it could be; but I've just never thought about it.

I: What do you think it means to be creative in mathematics?

G: Maybe coming up with different ways to approach a problem than typical.

The different ways did not have to be original ideas. By different she meant an alternate approach as opposed to something simple like reversing the order of operations in solving an equation.

Gina's conception of her own creativity was mixed. At first she claimed, "I'm not a very creative person. I'm not much on creative stuff; I don't know that much about it." A little later, she softened her appraisal. I asked:

I: You don't think you've ever created anything?

G: I suppose I have in ways. I can't think of anything off the top of my head; but I think everybody's got a certain amount of creativity in them. It's just finding it.

Gina did not think she possessed any mathematical creativity either. Even so, she rated her ability as average "because I haven't been trained to be creative. Whatever they tell me to do, I just do," she explained.

The last part of the initial interview on creativity concerned the requirements for creative activity in mathematics.

I: What do you think it takes to be creative in mathematics?

G: Brains! I guess to not be afraid to break out of what is already there, to try new things.

I: What kind of people are creative in mathematics?

G: People that are always looking for a new answer, a different way of doing things.

I: Do you think these people have to be geniuses?

G: No, not really. It takes some smarts but not really out of the ordinary. You have to have a basic understanding of math. Some people get it

> and others don't. You'd have to be one of those people who get it.

From this point on, Gina's assessment of her creative ability in mathematics was more positive based on her ability to "break away from the old way" and "try new things." Midway through, she judged her creative ability as improved "...because I've opened my mind more. There are other ways out there; it's not just the way you've learned. You have to find them." She judged her attempts at creating triometry as fairly successful, stating, "Even if they're [attempts] not actually right, I've still thought about it and tried things on my own so I've had pretty much success breaking away."

At the end of the teaching experiment, I sought her reaction to taking another class similar in exploratory nature to the one we had just completed. Her reply reflected a continued positive perception of her creative ability and what she had learned. She expressed confidence and a willingness to try since she would have a better idea of what to do and how to go about doing it. She explained, "Since I've had this class, I've learned places to start and different things to try to get a feel for things. Not just jump in. You kinda have an idea, and if that didn't work, where to try something else." Questionnaire II contained a final appraisal of her mathematical creative ability. Albeit not very definitive, it seemed less upbeat than previously. She wrote, "I think I could do some new math but I'm not sure how successful I would be. I think I could now when before I thought I probably couldn't."

Summary. Gina's original impressions of something creative were of an artistic nature like a painting. She did not regard mathematics as an art form, but she did acknowledge that one could be creative in doing mathematics. A creative activity in mathematics might involve different ways to approach a problem than usual. Originality was not a requirement. She reasoned that for someone to be creative in mathematics they would have to have a basic understanding of mathematics, some intelligence, and the courage to try new things.

At first, Gina judged her own creativity in mathematics as average since she had no experience. She identified her problem with creating triometry as an inability to break away from what was familiar to try something new. As the teaching experiment progressed, she gained confidence in her ability to do mathematics like we had been doing in class. By the end of the teaching experiment, Gina's evaluation of her ability to create mathematics was positive although guardedly so.

Creative Behavior. Information for this section came from Gina's journal, assignments, and observations. "You always want to go back to what you know. It's as though your mind is closed to [new] things. Nobody has really asked you so far to think of new things." This statement, made by Gina during the first interview, accurately described the problems she had early on in trying to create triometry. Her efforts on the first assignment mirrored this conflict. She was to create a definition of a function for an angle of a right triangle, symbolized by $S(\theta)$, that was analogous to sine, would have the same property of invariance for similar triangles, but would be different from any of the trigonometric functions. Gina drew a 3-4-5 triangle and a 6-8-10 triangle. She defined $S(\theta)$ as y/x, meaning side opposite the angle divided by side adjacent to the angle. Clearly, the property of invariance was satisfied since for both triangles she got $S(\theta) = 4/3$. Unfortunately, Gina had renamed the tangent function without realizing it. She had also tried $S(\theta) = xy$, but recognized that this would not work. At this stage, Gina's understanding was much like the others'--confused.

Her confusion continued into the next assignments which involved creating identities similar to the Pythagorean identities in trigonometry. After definitions had been agreed upon, Gina, like the others, tried substituting the new symbols, $S(\theta)$ for $\sin\theta$ and $C(\theta)$ for $\cos\theta$. At first, she had written $S^2(\theta) + C^2(\theta) = 1$ and had made the proper substitutions from the definitions. Then she thought better of it and replaced the "1" with a "?" thusly

$$\frac{(x + y)^2}{r^2} + \frac{(x - y)^2}{r^2} = ?$$

From this point, she did some algebraic manipulations like multiplying by r^2 and transposing of terms, but there was no evidence of expanding the binomials which might have helped. Later, after someone else in the class had derived the identities, she wrote in her journal, "Probably if I would have thought about where the identities come from in trig it would have been easier to come up with identities in this system."

I was encouraged by what seemed like a revelation in her understanding. But the light was only dimly lit at this stage. Another assignment involved creating an identity analogous to $\sin(-\theta) = -\sin(\theta)$. Gina wrote $S(-\theta) = -S(\theta)$ and attempted to verify it with examples from a table of values of special angles that she had made. But in the process, she became confused with finding two angles that were negatives of each other and the function values for those angles. For example, $\pi/3$ and $5\pi/3$ (the same as $-\pi/3$) are the correct "θ's", but $S[\pi/3] = 1.366$ and $S[-\pi/3] = -.366$. One example was correct, $S[\pi/2] = 1$ and $S[-\pi/2] = -1$. It was written first, and from this she probably assumed the others, a classic example of "freshman induction." Again, after seeing the correct derivation, she wrote, "Whenever I try to figure out where things come from in trig, it makes it easier to figure out things in our system."

The recognition of connections that Gina had been making but not heeding finally took effect about three-fourths of the way through the teaching experiment. The assignment was to find the derivative of $S(\Theta)$. I had reviewed the class on the derivative of sin Θ, and we had derived the necessary identities in triometry. Gina wrote:

$$S'(\Theta) = \lim_{h\to 0} \frac{S(\Theta+h) - S(\Theta)}{h}$$

$$= \lim_{h\to 0} \frac{1/2\{S(\Theta)[S(h)+C(h)] + C(\Theta)[S(h)-C(h)]\} - S(\Theta)}{h}$$

$$\vdots$$

$$= \frac{1}{2}[S(\Theta)+C(\Theta)] \lim_{h\to 0} \frac{S(h)}{h} + \frac{1}{2}[S(\Theta)-C(\Theta)] \lim_{h\to 0} \frac{C(h)-S(\Theta)}{h}$$

She then explained:

> Now I need to figure out $\lim_{h\to 0} \frac{S(h)}{h}$ and $\lim_{h\to 0} \frac{C(h)}{h}$. Also, I'm not quite sure what to do with $S(\Theta)$. I know it's not dependent on h so maybe when I find the limits, $S(\Theta)$ can be added or subtracted out. I know to try the pinching method but what do I squeeze it between? I've looked at the method the book uses for finding $\lim_{h\to 0} \frac{\sin h}{h}$.

At this point, Gina was stuck. She had made one big mistake by incorrectly separating the terms in the numerator to be divided by h; plus, she did not think of the definitions as substitutes for S(h) and C(h). But in getting this far, she had accomplished quite a bit. She had made the correct connection and recognized the direction she needed to go even if she had not been able to complete it. In further praise of her efforts, Gina was the only one to produce anything on this problem beyond the second step.

Once Gina had seen the derivative of $S(\theta)$ (finished in class as a group effort with a little help from me), she easily derived the derivatives of $C(\theta)$ and the other triometry functions for her final assignment. She also included some observations about her results which showed that she was thinking. For example, following the derivative of $T(\theta)$, she wrote, "This turned out a lot like derivative of tangent except since $S^2(\theta) + C^2(\theta) = 2$, the derivative of $T(\theta)$ has a 2 in it."

During the final interview, Gina very accurately described the difficulty she had had in her attempts to create triometry. She said the hardest thing was "breaking away from the old, to get stuck and think this is the way it was in trig so it should be the same way here."

Gina's creative endeavors were slow to develop but eventually she was able to enjoy moderate success. The cautious nature she exhibited with the Nature of Mathematics Survey I may have retarded her toying with ideas. She could make analogies but had difficulty "breaking away from the old." This slowed her efforts in the reconstituting of proofs in trigonometry to make something new in triometry.

Summary. Gina's first attempts to create triometry were unsuccessful because of her inability to break away from the familiar trigonometry. She realized this, but continued to equate the triometry functions with their parallel in trigonometry, e.g., $S(\theta)$ for $\sin \theta$ in trigonometric identities. However, by the end of the teaching experiment, she had made great improvement. She demonstrated by her work on the derivative of $S(\theta)$ that she understood the analogy with $\sin \theta$, but did not equate the two.

Her creative behavior involved primarily the making of analogies. She had limited success with the reconstituting of something old to make something new and toying with ideas.

Students' Evaluations of Triometry Materials and the Teaching experiment

Interview IV and questionnaire II provided the information for this section. The results of the students' evaluations of the triometry materials are presented, followed by their opinions regarding the teaching experiment.

During interview IV, each student concurred that the triometry material was within their capabilities even though some of it was hard. As Steve remarked, "I hit some snags along the way, but most of it was reasonable." On questionnaire IV, there were two questions that related to the triometry materials. The first, question 4, asked the students to evaluate the content by indicating which of the terms--easy, hard, challenging, trivial, interesting, boring, or any others--they thought applied to any part of triometry and to give reasons for their choices. Ruth wrote hard "to look at something in a new way"; challenging, "it involved a lot of thought and a lot of trial and error"; exciting "when you come up with something."

Steve said, "I feel that they all apply to this course except boring at one point or another because this class had a whole different approach than any other class I've had."

For Nora, triometry was challenging and interesting. I could not clearly interpret what she intended by challenging, but by interesting she meant that it was interesting to compare the identities we got in triometry, like $S^2(\theta) + C^2(\theta) = 2$, with the ones in trigonometry, like $\sin^2\theta + \cos^2\theta = 1$.

Triometry to Don was easy "to enjoy and spend time working on the material"; hard "knowing how or what to do next in relating the 'old' trig to the 'new' trig"; interesting and challenging "seeing how properties can be related and discovering how to connect them."

Gina's opinion encompassed all the terms. She wrote, easy-"parts of it was"; hard "trying to find things I didn't really understand where it came from in trig"; challenging-"it made me think and try different things"; trivial-

"plotting graphs and just plugging things in formulas"; interesting "finding new things"; boring-"everything can be boring at times."

The second question relating to the evaluation of triometry, number 6, stated: As a means of introducing students to the idea of creating their own mathematics, are the triometry materials appropriate or inappropriate? Please explain. Their responses to this question were:

Ruth: I think they are appropriate because you have something to go along with. It was also neat seeing how our new trig could get so close to the other.

Steve: I feel that it was appropriate because many similar functions and equations could be created.

Nora: Appropriate because the problems that we created in a way it related to the old trig...

Don: It is appropriate. It would be more so if it was taken by student fresh out of trig and geometry. It would be easier and quicker to develop the new trig.

Gina did not respond to the question.

The students's reactions to the teaching experiment, which are presented next, are a composite of their statements from the interview and the questionnaire. Ruth was glad she had taken the course because "I feel better about myself developing ideas," she said, and "it made me think alot and have a different outlook about math." Furthermore she would feel positive about taking another course with a similar approach. But to carry out the same ideas in other mathematics classes "could confuse the facts from experimenting" and grading would be difficult. She liked best not having tests and working at her own pace and on what interested her. What she liked least was "never knowing if you had something or not and whether it could be simplified." She would recommend the course to others.

Steve liked "creating things" best about the class "because I'm not that creative," and "figuring out proofs" least because "I have trouble following them." He would take another class similar in nature but would not want all math courses to be like this one since it would be boring. His over-all impression was that the class was "interesting and different and one of the least stressful math classes I've had." He would recommend the course to others because "It's good to take different approaches," he said.

In Nora's opinion, it would be good for students to take a class like this one because "once in a while students need to do something different than just follow what is in the book," she wrote. She liked being evaluated on effort and not having to produce "an answer" each time. She liked least being frustrated when she would try to do a problem and not get anything. She would not mind taking another class like this one. But she was graduating so this was a remote possibility.

Don qualified his recommendation of the class to others by adding, "only if they were interested in learning how to learn and in exploring new methods of doing things." He was not keen on taking a similar class himself. He wrote, "It was a refreshing change and fun while it lasted, but it would be hard to learn new ideas and concepts by developing them from others." The tediousness in working out the relations (in triometry) was his least liked part of the class while the best part was that it was different from other classes.

For Gina the class was enjoyable because it was different from other classes. She touted the exploration as a good learning experience in "trying out things and not just worrying about getting an answer" which made one more willing to try. She would take another class similar in kind and would recommend the same to others. She liked the interviews least, but even they had not been very bad.

Summary. From the students' point of view, the triometry materials were within their capabilities. Their concepts of the material were reflected in the descriptors they chose. The most frequently chosen was hard followed by challenging, interesting, and easy. Two students each thought that parts of triometry were boring, trivial, or exciting. Four students judged the materials appropriate as a means of introducing students to the idea of creating their own mathematics. One student did not offer an opinion.

The over-all impressions of the class by the participants were favorable. One or more liked the exploration, creating things, a new and different approach to studying mathematics, and the emphasis and evaluation on effort rather than on finding "an answer." Cumulatively, they liked least the uncertainty of whether a result had been found or could be simplified, the frustration from trying to work a problem and not getting anywhere, deciphering proofs, the tediousness of working out triometry relation, and the interviews. Only two students rejected the idea of having similar experiences in other mathematics courses. Their reasons centered around the difficulty of learning new ideas and concepts and the resulting confusion with established facts. Each of the students would recommend the class to others as a class that was different and having a new approach. One student suggested that anyone taking the class should be interested in learning and exploring new methods of doing things.

CHAPTER V

CONCLUSIONS

This study involved a teaching experiment in which material developed by the investigator was used. The material, called triometry, uses definitions for relationships of angles and sides in right triangles that are different from the familiar sine, cosine, etc., in a parallel development of trigonometry. The purposes of triometry are to give mathematics/mathematics education majors exposure to new mathematical ideas and to serve as a medium through which students can engage in creating their own mathematics. The teaching experiment sought to determine the appropriateness of triometry as stated. In addition, changes in students' perceptions of the nature of mathematics and in their perceptions of their own creative abilities and changes in their creative behaviors were investigated.

The teaching experiment was the first half of a one semester course offered as an elective at a comprehensive university, one of sixteeen institutions in the University of North Carolina system. Classes met twice weekly for fifty minutes each day for seven and one half weeks, a total of fifteen class periods. Of the five participants, one was a mathematics education major, two were majoring in applied mathematics, one was minoring in mathematics, and one was contemplating mathematics or physics as a major. The prerequisites for the teaching experiment included one semester of calculus and knowledge of trigonometry.

A qualitative research design was used for data collection and analysis. Instrumentation included surveys, questionnaires, interviews, student-kept journals, and observations.

This study focused on the resolution of four questions. The first question concerned the appropriateness of the triometry materials. Conclusions are drawn with respect to the evaluations of the four professional mathematics educators and the five participants. The other three questions dealt with specific responses of the students to the teaching experiment. The summary and conclusions drawn are discussed with respect to each question rather than with respect to each student.

Question 1. Is the material, triometry, suitable as a creative activity for mathematics/mathematics education majors?

The results of the study strongly support the conclusion that the triometry materials are appropriate as a creative activity in mathematics. Both the mathematics educators and the participants gave favorable evaluations.

The mean for each criterion of the evaluation of triometry by the four professional mathematics educators ranged from 4.25 to 5 on a scale of 1 to 5. Thus, there is general agreement among the mathematics educators that triometry materials are appropriate as a creative activity. Further evidence of their approval of triometry is indicated by their comments such as "I liked your material;" "I found the development interesting;" and "It is a very neat piece of work and I was certainly impressed."

In view of the credentials of the mathematics educators, their evaluations carry a great deal of weight. However, as an old saying goes, the proof is in the pudding. The views of the five participants with respect to triometry are based on their experiences in the course. The evidence presented in Chapter IV (pp. 151-155) indicates that these experiences were meaningful and even enjoyable and the doability of the materials was reasonable.

The significance to the participants is expressed in their comments that accompanied their willingness to recommend the course to other students. Gina's "a new approach to math," Ruth's "it made me think a lot and have a different outlook about math," and Don's qualified "only if they [other students] were interested in learning how to learn and in exploring new methods of doing things" are indications that triometry was not the usual textbook oriented class to them.

Of course, to discuss the affective enjoy, one must take into account the novelty effect. Triometry being "different" and "a new approach" partly contributed to the students' expressions of enjoyment. Also "trying to please the teacher" may have been at work. But these two possibilities are not entirely the case. This last contention is based on the positive responses of three of the participants to the questions, "Would you take another course similar to this one? and Would you like other mathematics courses to contain similar aspects of exploration as part of the course? Why or why not?" The three qualified their answers with phrases "good learning experience," "good to take a different approach," and "good to do something different than just follow what is in the book." The other two students were hesitant about taking another class. They were more emphatically against the latter question because to do so might confuse the "facts." The investigator claims a majority!

The doability of the materials is confirmed in part by the participants, all of whom considered triometry to be appropriate for the intended purpose and within their capabilities. The latter aspect is a significant point in light of the abilities of the participants as indicated by their GPAs in mathematics. Another indication of the way that they perceived triometry lies in the frequency with which they selected descriptors of any part of the material. Four of them selected <u>hard</u> but not because of the mathematics. Their comments, such as Ruth's "hard to look at something in a new way," suggest that the difficulty lay in the approach rather than in the substance. A bare majority considered parts of the materials challenging, interesting, or easy. Significantly, only one person thought of any part of triometry as boring, and that was in relation to the table of values that they composed. The investigator was somewhat disappointed that only two people deemed triometry exciting.

Question 2. Are students' perceptions of the nature of mathematics enhanced as a result of their experiences in this study?

The results presented in Chapter IV for the participants on the Nature of Mathematics Survey I and II together with their written and oral responses to questions relating to the nature of mathematics support a conclusion that the students' perceptions of mathematics were enhanced, albeit not a great deal. At the onset of the teaching experiment, the five participants described mathematics in terms of working with numbers, rules, and formulas which involved a considerable amount of memorization. These components were used to solve problems and to quantify real world phenomena. For the majority, mathematics without application was of no value. Authority was derived from the textbook and the teacher. The students had never considered the origins of the mathematics found in textbooks or that there could be anything more than that. Their descriptions of the work of mathematicians was consistent with their views of mathematics. Basically, they considered mathematicians to be problem solvers. To that end, they tried to find the best (easiest) answer to a problem by selecting the proper formula. They not only applied methods already developed to solve problems but also devised new methods. They proved theorems which they got from a book or that they thought up themselves. The bulk of their work involved numerical calculations as opposed to abstract or symbolic thought. In general, mathematicians could create their own symbols but not their own rules since this could cause confusion with others who might not be familiar with them.

By the end of the course, the students' perceptions of these two topics had wavered only slightly. Mathematics still dealt primarily with manipulations of numbers and formulas to solve problems and to describe real world situations. But they also included in their descriptions phrases such as "...exploration and development of new methods of problem solving...", "...a search for techniques..", and "...finding relationships...." In addition, their concepts of a textbook as a sole authority had been challenged. As Ruth said, "(I) didn't question [what is in the book] because it's been proved. But you can come up with new ways of doing it." Too, they were made aware that new mathematics can be created and is being created. Gina's comment attests to this. She said, "I thought no new math was being worked on. I thought the old way was the only way. Now, I know that's not so." Their views of the work of mathematicians was also enhanced as evidenced by the different dimension that they included in their descriptions. In addition to mathematicians solving problems by applying methods and formulas already known, they included "...develop new math, ... create new formulas, ...think, ...try new things." Thus, by their additions, they were acknowledging that perhaps mathematicians' work involves more than numerical calculations.

On an individual facet basis, the results on NMS I and II lend meager support for enhancement of students' perceptions of the nature of mathematics. A comparison of mean scores shows that the perceptions of the class as a whole changed positively to a measurable degree on Facet 1 only. Facet 1 concerned mathematics as an organized body of knowledge in which generalizability is a desirable characteristic. There was no direct mention of this characteristic during the course of the teaching experiment. However, much of the class time was spent discussing trigonometric properties and the generalizing to triometry. Hence, a reasonable explanation is that the students grasped some significance of this characteristic as a result of their experiences in the class.

There was a slight move toward the correct view on Facet 4. The focus of this facet is on the beauty of mathematics and mathematics as a creative art. The consensus of the class at the beginning of the course was that none had ever thought of mathematics as a creative art. Paraphrasing Gina, art is something that one stands back and admires, like a painting, not a solution to a mathematics problem. One reason that a more substantial growth was not realized may be attributed to their misconceptions of an elegant proof, a term that was used in the survey statements for this facet. Steve's and Ruth's descriptions were of a proof that is precise, nice, and easy to understand. Gina described it as one that has unnecessary extras so that it is not straight to the point while Don perceived it as one that adds English to make it interesting. Nora could give no explanation. That there was any improvement for this facet is significant in view of their conceptions of mathematics already noted.

There was virtually no change in their perceptions of the remaining facets. The constancy on Facets 2 and 3 can be attributed to a lack of familiarity with terminology. These facets deal with inductive and deductive reasoning, making of conjectures, and the role of insight and intuition. The majority of the students either did not know their meanings or was not acquainted with their relationships to mathematics. None of these terms were an expressed part of the teaching experiment. The final results on facets 5 and 6 are consistent with the students' expressed views of mathematics and the work of mathematicians. Facet 5 concerns the relative importance of complex calculations and abstract thought in mathematicians' work, and Facet 6 is about the relationship of mathematics to the real world. These results were somewhat of a disappointment in that the experiment involved abstraction and symbolic thought as well as demonstrating that mathematics does not need an application to be of value. Apparently, from the students' viewpoint, the teaching experiment was perceived as another exercise in manipulating symbols and searching for formulas.

If conclusions are based on facets singly, the inclination is to conclude that no significant positive changes were accomplished. However, each participant's survey scores considered in totality does support the conclusion. With one exception, there was an increase of five or more points from NMS I to NMS II which signifies a move toward the correct view of mathematics. In view of conflicting evidence (Conroy, 1987) concerning the amenability of students' perceptions to change, this represents a small victory. This is particularly so in light of the short duration of the experiment.

Question 3. Are students' perceptions of their ability to be mathematically creative enhanced as a result of this experiment?

Evidence collected from the participants and presented in Chapter IV support an affirmative response to this question. The perceptions of one's ability to do a certain task is most certainly influenced by one's perception of what is meant by the task as well as one's experiences with that task. The students' explanations of what constitutes creativity in mathematics fell into two categories--getting ideas or producing something on one's own and using mathematics in ways different from how one had been taught as in solving problems by a method not seen before. There was total agreement that one did not have to be a genius to create mathematics, but it did require some intelligence, a great deal of thinking, and as Gina put it, "the courage to try new things." They also concurred that originality was not a requirement. Measured by these criteria, they each asserted that they had never done anything creative in mathematics.

At the onset of the course, being novices at creative activity, each expressed uncertainty. Yet all, save Nora, rated their ability as at least average. At mid-course, Ruth, Steve, and Gina felt that they had improved. The reasons they offered centered around an improved sense of knowing how and where to start working on a triometry problem. Don's and Nora's senses of not having improved were based on the negative of the reasons offered by the other three.

Except for Don, their verbal and written assessments at the end of the course were positive. They expressed confidence in being able to do a creative activity like triometry since they had a better sense of what is involved --what to do and how to do it. At the same time, they were reserved in their judgment as to how successful they would be. Even though Don did not believe that he had made any progress in creating triometry, he did feel that he had become more analytical in looking at a problem.

Question 4: Are students' creative behaviors in mathematics enhanced as a result of their experiences in this study?

In Chapter IV, the creative behavior of each student was reported, and the progress that each student made relative to the criteria for creative behavior was noted. Collectively, these results show conclusively that the creative behaviors of the students were enhanced in at least two categories, albeit not to a great degree. The basis for this statement lies in a comparison of where the students were, creatively speaking, at the beginning of the teaching experiment to where they were at the end. At the onset of the experiment, all of the students asserted that, apart from the triometry materials, they had never had a course like this one. Their struggles with the first two or three assignments give an indication of their lack of creative expertise at this point. Consequently, at this stage, their slate of demonstrated creative behavior was clean.

One of the criteria in which all the students made improvement is toying with ideas. The students' efforts to toy with ideas were retarded at the beginning of the course. This was evident in that only two students displayed significant attempts to play around with creating definitions. Two others had not been able to produce anything. The amount the fifth person did was somewhere between the two groups. The difficulty they all experienced was accurately pinpointed by Gina and Ruth later when they said (paraphrased) it's hard to take something old that you have learned and think of it in a new way.

Evidence from their journals indicates that, as the course progressed, the students were able to play around with ideas more than they had at the beginning. In the process, the toying took different forms for different students. For example, for Don, it was repeated applications of formulas. But from his experimenting, he was able to recognize patterns and make good conjectures. For Nora, Ruth, and Gina, it was searching for the right formula to "plug" into. For Steve, who enjoyed the most success, toying consisted of trying to adapt a trigonometric property to triometry. Occasionally, their efforts were fruitful in being able to derive parallel properties. For some, like Ruth and Nora, success was elusive because of poor algebra skills or poor working knowledge of trigonometry.

The making of an analogy is a second creative behavior that was enhanced. The problem that they all experienced at the beginning in making the proper analogies with trigonometry is apparent in their substituting of $S(\theta)$ and $C(\theta)$ for $\sin \theta$ and $\cos \theta$, respectively in the trigonometric identity, $\sin^2\theta + \cos^2\theta = 1$, and expecting the outcome to be "1" also. This example likewise illustrates the influence of their perceptions of the nature of mathematics as rule/formula driven. Continually, for the first two weeks, they looked at trigonometric formulas as recipes that they could follow exactly. In essence, they were keen to <u>adopt</u> rather than trying to <u>adapt</u> properties of trigonometry to triometry.

The students' grasping of the ideas surrounding the making of an analogous development of triometry was a gradual process with each catching on at various times. Steve was the most adept, understanding the idea somewhere around the third assignment. Gina and Ruth showed signs of comprehension near mid-course. Don and Nora took longer. Regardless of the length of time involved, the most important point is that at the end of the course, the students understood <u>what</u> to do as far as taking a trigonometric property and starting an analogous proof in triometry even if they did not know <u>how</u> to do it. For example, following the proof of derivative of sin θ, the students correctly set up the problem for the derivative of $S(\theta)$ thusly:

$$S'(\theta) = \lim_{h \to 0} \frac{S(\theta+h) - S(\theta)}{h}$$

They used the definition for $S(u+v)$ to substitute for $S(\theta+h)$ and simplified as much as they could. They did not merely substitute $S(\theta)$ for sin θ as they had earlier.

Some students improved in other categories. But there are a number of factors that prevented the students from progressing more than they did. First, poor algebra skills was one. The consequence was that, being unsure of what they were doing in the first place, even when they had begun a proof correctly, their errors subverted their efforts. Second, a poor working knowledge of trigonometry was another. Not knowing how to solve a particular problem in trigonometry made it impossible to solve an analogous problem in triometry. A third factor that afflicted all the students to some degree was difficulty in following and understanding a proof in trigonometry, particularly if some of the steps were missing. It is for this reason that the students did not show more progress in the reconstituting of something old to make something new, another creative criteria. The fourth, and perhaps the most significant, reason goes back to their perceptions of mathematics as formula-rule driven. These perceptions guided their initial approach to triometry. When the usual formula-rule application did not work, they were stymied. But, as the discussion in the preceding paragraphs has pointed out, they were able to override some of their inclinations and have some success in creating triometry.

Comments and Recommendations

The nature and design of this experiment was such that to try to draw generalizations from it would be risky indeed. However, it is not only proper but worthwhile to talk about some implications. The most significant implication is that even though the students' perceptions of doing mathematics did not appear to change, their thinking about mathematics was altered. The indications from their comments are that, to them, mathematics is no longer an immutable collection of knowledge found in textbooks. They mentioned repeatedly that they were unaware that new mathematics could be created. This shift in perception is important for two reasons. First, given conflicting evidence (Conroy, 1987) regarding responsiveness of students to changing their beliefs, that there was a shift at all is noteworthy in view of the short duration of the experiment. Second, this small breakthrough suggests that perhaps given time and exposure to the right kind of activities, their other perceptions of mathematics can also be changed. This is particularly important for students who may become public school mathematics teachers since their views of mathematics will be imparted to the students they teach.

Another implication of this study is that isolated incidences such as this teaching experiment are not going to be sufficient to significantly change students' beliefs about mathematics. Students need experiences that allow them to explore mathematics throughout their schooling. Ideally, mathematics educators should aim at instilling the correct view of mathematics in students from the beginning of their mathematical education. This will involve not only creative content but also creative approaches to instruction.

A third implication concerns proof. A number of university mathematics professors with whom this investigator is acquainted have recently been stressing the need for students to have more experiences in just reading and understanding elementary proofs like those found in most trigonometry or calculus books. The students' reports of problems with proofs as well as the difficulties they had during the course of the experiment confirm that this is a deficiency that needs to be addressed. Professor C, in his evaluation of the triometry materials, noted that proof might be a problem for the students. He wrote, "I believe it would work better on students who had already had a proof-type course so they have a notion of the arbitrary nature of mathematics." He was right.

Now a few words about triometry are in order. Aside from the favorable evaluations from the students and the mathematics educators, from the point of view of this investigator, the triometry materials served the purpose for which they were intended quite well. It was different from anything the students had seen, yet it contained ideas and concepts with which the students were familiar. The level of difficulty of the material was variable which accommodated the various background experiences of the students. Also, given the mathematical ability of the students as measured by their GPA's, the fact that they were able to enjoy some successes is an indication that students do not have to be exceptionally talented to handle triometry.

The experience with triometry in the teaching experiment has convinced the investigator that activities which allow mathematics/mathematics education students to experience the true meaning of creating mathematics can be developed. There is no doubt that they are worth doing. Two questions to be resolved are: How can such creative experiences be worked into an already crowded curriculum, and do mathematics educators have the desire and commitment to do it?

In the course of the analyzing and reflecting that have transpired in the writing of this paper, a few questions have occurred to the investigator that could serve as ideas for further research.

1) Would a longer exposure to a similar type of exploration alter students' concepts of <u>doing</u> mathematics?

2) Assuming the answer to question 1 is yes, how long? Is college soon enough to try to introduce students to creating mathematics or should it begin sooner?

3) What is the constancy of students' changes in perceptions? Do they return to their old way of thinking if reinforcement is not provided along the way? (A long-term study would be called for here.)

4) In using triometry, students did not have a reference source in which they could look for solutions to problems. As a creative activity, does the use of material for which there are no references available have an advantage over the use of material for which problem solutions can be found?

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Appendix A

Scheding's Nature of Mathematics Survey*

Facets of the Nature of Mathematics--the Correct View

Facet 1: Mathematics is an organized body of knowledge in which generalizability is a desirable characteristic.

Facet 2: Induction and deduction are both important in mathematical discovery and proof.

Facet 3: Insight, intuition, and the making of conjectures are important in the work of the mathematician.

Facet 4: Mathematics is a creative art in which elegance of proofs is sought.

Facet 5: The work of the mathematician involves more abstract or symbolic thought rather than massive or complex numerical calculations.

Facet 6: Mathematics is a way in which mankind/wonamkind has tried to make sense of his/her world. Much of mathematics is applicable to the real world but some is not. But application to the real world is not necessary to justify the importance or existence of mathematics.

*Scheding, 1981.

<u>Facet 7</u>: There are differing philosophical views of the foundations of mathematics each of which has/is influencing, in varying degrees, the mathematics of today. Mathematicians differ in their views of the nature of mathematics.

Item Numbers That Relate to Each Facet

Facet	Item Numbers
1	4, 5, 9, 18, 19, 28, 38, 43, 48
2	14, 17, 26, 34, 35, 36, 42, 47
3	8, 12, 25, 27
4	20, 21, 22, 24, 29, 33, 37, 41, 45
5	1, 3, 13, 15, 31, 40
6	2, 6, 7, 10, 11, 23, 32, 46
7	16, 30, 39, 44

Note: Underscored items are *negative* items. These are scored by subtracting item score from 6.

NATURE OF MATHEMATICS SURVEY

This survey seeks to find out YOUR views of the nature of mathematics and the work of the professional mathematician. For each statement below, circle the number that best indicates your opinions about it, from 1 ("Disagree") up through 5 ("Agree").

	Disagree	Tend to Disagree	Neutral or no Opinion	Tend to Agree	Agree
1. Mathematics can be viewed as the study of patterns.	1	2	3	4	5
2. If you use calculus to solve an engineering problem, you are doing mathematics.	1	2	3	4	5
3. Accountants do mathematics.	1	2	3	4	5
4. Mathematicians strive to build their results into deductive systems.	1	2	3	4	5
5. Mathematics is an unsystematized collection of facts, techniques, and results.	1	2	3	4	5
6. High school algebra, geometry, and arithmetic are mathematics,	1	2	3	4	5
7. Mathematical theories and systems need not be related to real objects.	1	2	3	4	5
8. Intuition plays an important part in the creation or discovery of mathematics.	1	2	3	4	5
9. Mathematical systems often consist of axioms, definitions, theorems, and proofs.	1	2	3	4	5

10.	One part of mathematics is the application of known mathematical results in order to solve real-world problems.	1	2	3	4	5
11.	You are doing mathematics if you solve a mathematical problem by applying a method you have found in a textbook.	1	2	3	4	5
12.	A mathematician working on a new theory usually doesn't feel it is true until he is able to prove it deductively.	1	2	3	4	5
13.	It is quite acceptable for a mathematician to create his own symbols and his own rules for manipulating them.	1	2	3	4	5
14.	A mathematical proof which has only one mistake in it is more valid than a similar proof containing several mistakes.	1	2	3	4	5
15.	Mathematics can be viewed more as the study of patterns than as the performing of complex calculations.	1	2	3	4	5
16.	Although discoveries are still being made on the frontiers of mathematics, the foundations of mathematics were completely worked out years ago.	1	2	3	4	5
17.	The development of mathematics seldom occurs as the result of a conjecture.	1	2	3	4	5
18.	Mathematicians are keen to develop mathematics which applies in a wide variety of situations.	1	2	3	4	5

19.	Mathematicians mainly use undefined terms because of laziness.	1	2	3	4	5
20.	The purpose of seeking a new version of a correct proof is often to make the proof more elegant.	1	2	3	4	5
21.	There is a lot of beauty in mathematics.	1	2	3	4	5
22.	Mathematics restricts one's thinking to the use of rules and formulae.	1	2	3	4	5
23.	Mathematics has been of great assistance in helping man to understand his physical environment.	1	2	3	4	5
24.	Mathematicians often view their work as exciting.	1	2	3	4	5
25.	Trial and error methods have no place in mathematics.	1	2	3	4	5
26.	Mathematics is usually created in just the form in which it later appears in the textbooks.	1	2	3	4	5
27.	In the development of mathematics, intuition is more of a hindrance than a help.	1	2	3	4	5
28.	Mathematics is a very precise language.	1	2	3	4	5
29.	Memorizing rules and formulae is extremely important for success in solving mathematical problems.	1	2	3	4	5

30.	There is only one correct view of the nature of mathematics.	1	2	3	4	5
31.	Accountants do the same sort of mathematics that mathematicians do.	1	2	3	4	5
32.	Providing models of physical phenomena is a basic goal of mathematics.	1	2	3	4	5
33.	Problem solving using mathematics amounts to finding a rule or formulae which fits the situation, and then applying it.	1	2	3	4	5
34.	Very little new mathematics is developed using the process of examining many specific cases.	1	2	3	4	5
35.	Many mathematicians are interested in the style of a proof as well as in its validity.	1	2	3	4	5
36.	A proof considered correct by mathematicians of one era can be considered incorrect by those of another era.	1	2	3	4	5
37.	Very little new mathematics is being developed today.	1	2	3	4	5
38.	Given a particular mathematical system, its properties are the same on the moon or Mars as they are on Earth.	1	2	3	4	5
39.	There are several schools of thought among mathematicians as to what mathematics really is.	1	2	3	4	5

40.	Mathematics is primarily the study of numbers and the way they combine.	1	2	3	4	5
41.	Mathematics can be exciting.	1	2	3	4	5
42.	Deductive reasoning is the only type of reasoning which can be used in developing new mathematics.	1	2	3	4	5
43.	One of the characteristics of mathematics is its generality.	1	2	3	4	5
44.	Almost all mathematicians have the same view of the nature of mathematics.	1	2	3	4	5
45.	The work of professional mathematicians is mostly routine and repetitive.	1	2	3	4	5
46.	A mathematical system is of little importance if it has no current application.	1	2	3	4	5
47.	Just as novels can differ in style, so can proofs of mathematical theorems.	1	2	3	4	5
48.	Undefined terms are necessary in mathematics.	1	2	3	4	5

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SCHOOL OF PRIMARY AND COMPUTER EDUCATION

April 26, 1990

Professor Billie Goodman
Appalachian State University
North Carolina

Dear Billie,

Thank you for your request for permission to use the survey from my dissertation. I am very happy to grant you that permission, and indeed very pleased to see the survey instrument being used.

I wish you well with your research.

Regards,

John Scheding

John Scheding
Co-ordinator of Computing Courses

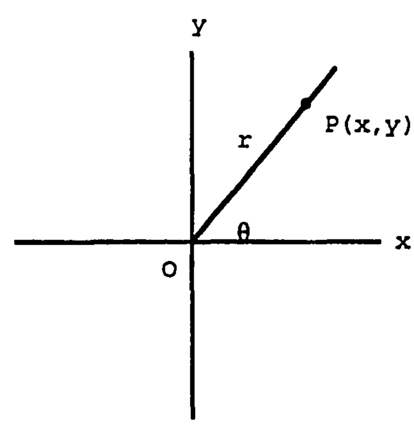
Appendix B

Triometry

Trigonometry is a well known branch of mathematics that is studied by students in high school as well as college. The familiar functions of sine, cosine, tangent, etc. are ratios of two sides of a right triangle or ratios of two of the components of (x, y, r) which define an angle of rotation. In the material presented here, a variation on these definitions is used to develop a body of material which we call Triometry. By making new definitions for the basic functions, Triometry presents an essentially parallel development of trigonometry without using the usual trigonometric functions.

The functions of Triometry are defined using all three components (x, y, r) of an angle of rotation.

<ins>Definition</ins>. Let P(x,y) be any point on the terminal side of θ, an angle of rotation in standard position (Figure 1) where $x^2 + y^2 = r^2$. Then we define the following:



<ins>Figure 1</ins>

(a) $S(\theta) = \frac{x + y}{r}$, and (b) $C(\theta) = \frac{x - y}{r}$.

From these two, four other functions are defined:

(c) $T(\theta) = \frac{S(\theta)}{C(\theta)} = \frac{x + y}{x - y}$

(d) $RT(\theta) = \frac{1}{T(\theta)} = \frac{x - y}{x + y}$

(e) $RC(\theta) = \frac{1}{C(\theta)} = \frac{r}{x - y}$

(f) $RS(\theta) = \frac{1}{S(\theta)} = \frac{r}{x + y}$

If $r = 1$, then:

$$S(\theta) = x + y \qquad C(\theta) = x - y$$

$$RS(\theta) = \frac{1}{x + y} \qquad RC(\theta) = \frac{1}{x - y}.$$

Proposition 1. These definitions are independent of the choice of a point $P(x,y)$ selected on the terminal side.

Proof. Let $P(x_1,y_1)$ and $Q(x_2,y_2)$ be any two points on the terminal side of θ (Figure 2).

We know by definition that

$S(\theta) = \frac{x_1 + y_1}{r_1}$. Also

$S(\theta) = \frac{x_2 + y_2}{r_2}$. Draw

perpendiculars from P and Q to the x-axis. The two right triangles formed are similar and so

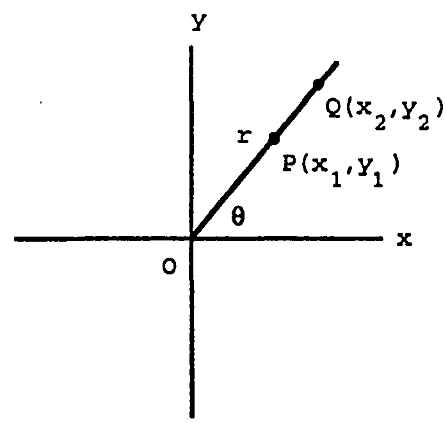


Figure 2

(1) $\frac{x_1}{r_1} = \frac{x_2}{r_2}$ or $x_1 r_2 = r_1 x_2$.

Likewise,

(2) $\frac{y_1}{r_1} = \frac{y_2}{r_2}$ or $y_1 r_2 = r_1 y_2$.

Adding (1) and (2) we have $x_1 r_2 + y_1 r_2 = r_1 x_2 + r_1 y_2$ or $r_2(x_1 + y_1) = r_1(x_2 + y_2)$. Dividing by $r_1 r_2$ gives,

$$\frac{x_1 + y_1}{r_1} = \frac{x_2 + y_2}{r_2}$$

which is the desired result.

Similarly it can be shown that $C(\theta)$ is invariant with respect to the point selected on the terminal side of an angle.■

Function Values of Special Angles.

For certain special angles, the values of the Triometry functions can be computed directly from the definitions. For example in Figure 1, if $\theta = 0$, then $P(x,y)$ lies on the x-axis and so $y = 0$ and $x = r$. Thus,

$$S(0) = \frac{r + 0}{r} = 1,$$

$$C(0) = \frac{r - 0}{r} = 1, \text{ and}$$

$$T(0) = \frac{r - 0}{r + 0} = 1.$$

Since the other three functions are reciprocals of these three, we have that all six Triometry functions have the value 1 when $\theta = 0$.

The angle $\theta = \frac{\pi}{6}$ is another special angle. Referring to Figure 1; if $\theta = \frac{\pi}{6}$, then, selecting r = 2 gives $P(x,y) = P(\sqrt{3}, 1)$. Now, applying the definitions gives

$$S\left[\frac{\pi}{6}\right] = \frac{\sqrt{3} + 1}{2},$$

$$C\left[\frac{\pi}{6}\right] = \frac{\sqrt{3} - 1}{2},$$

$$T\left[\frac{\pi}{6}\right] = \frac{\sqrt{3} + 1}{\sqrt{3} - 1},$$

$$RS\left[\frac{\pi}{6}\right] = \frac{2}{\sqrt{3} + 1},$$

$$RC\left[\frac{\pi}{6}\right] = \frac{2}{\sqrt{3} - 1}, \text{ and}$$

$$RT\left[\frac{\pi}{6}\right] = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}.$$

Other special angles and the corresponding values of the Triometry functions are contained in Table 1.

Graphs of the Triometry Functions.

Having obtained some values for the Triometry functions, we are now in a position to consider the graphs of these functions. We first consider the graph of $y = S(x)$. In Figure 3, we have $S(x) = s + t$. In order to determine the shape of the graph, we examine the behavior of S(x) for x in specific intervals. As x increases through the interval 0 to $\frac{\pi}{4}$, then S(x) increase from 1 to $\sqrt{2}$. As x increases through the interval $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, S(x) decreases from $\sqrt{2}$ to 0. As x increases through the interval $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$, S(x) decreases from 0 to $-\sqrt{2}$. In the interval $\frac{5\pi}{4}$ to 2π, S(x) increases from $-\sqrt{2}$ at $\frac{5\pi}{4}$ to 0 at $\frac{7\pi}{4}$ and to 1 at 2π. It is clear that the values of S(x) will begin to repeat at 2π and so S(x) is periodic with period 2π. It should be noted that S(x) is continuous over any interval. Using these facts together with the values from Table 1, we can construct the graph of y = S(x) which is shown in Figure 4.

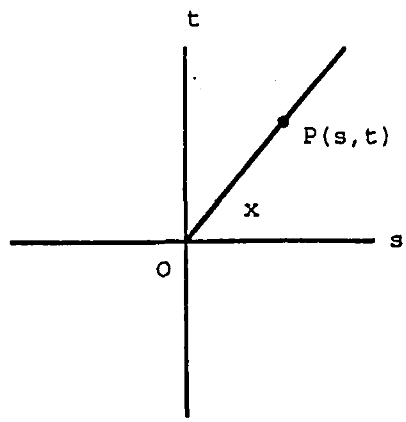


Figure 3

A similar type of analysis together with the values in Table 1 results in the graphs of the other five functions as shown in Figures 5 through 9. The functions S, C, RS, and RC all have period 2π while T and RT have period π.

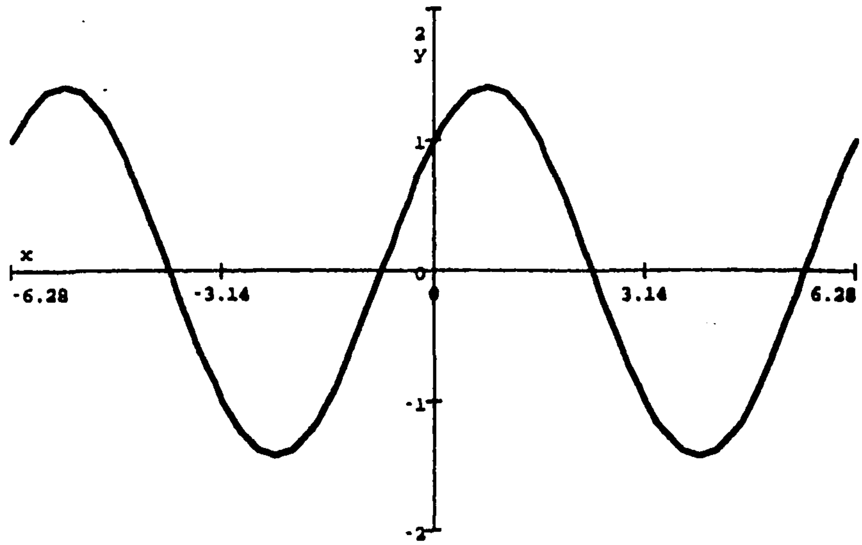


Figure 4. Graph of y = S(x)

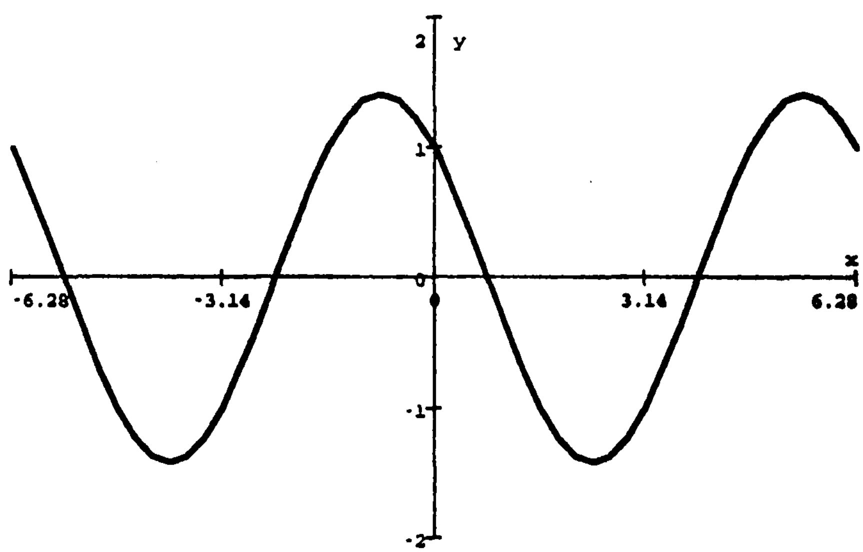


Figure 5. Graph of y = C(x)

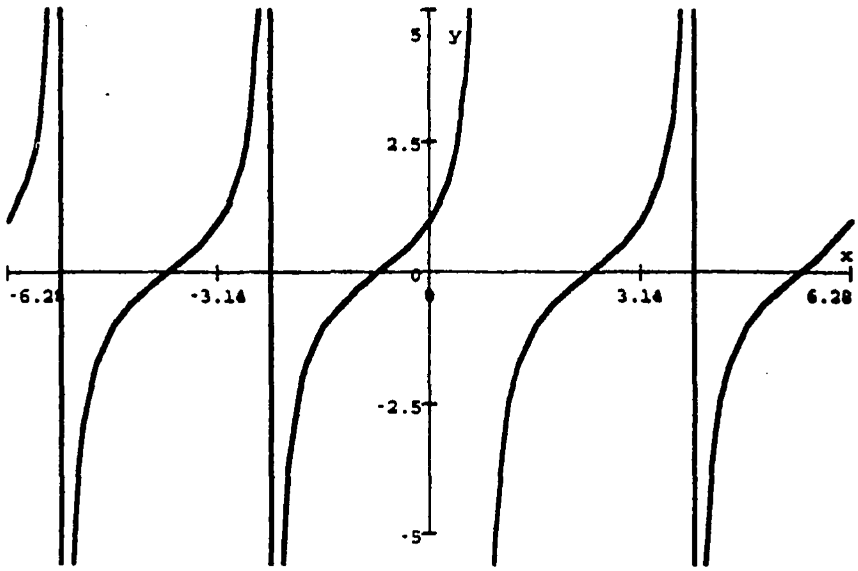


Figure 6. Graph of y = T(x)

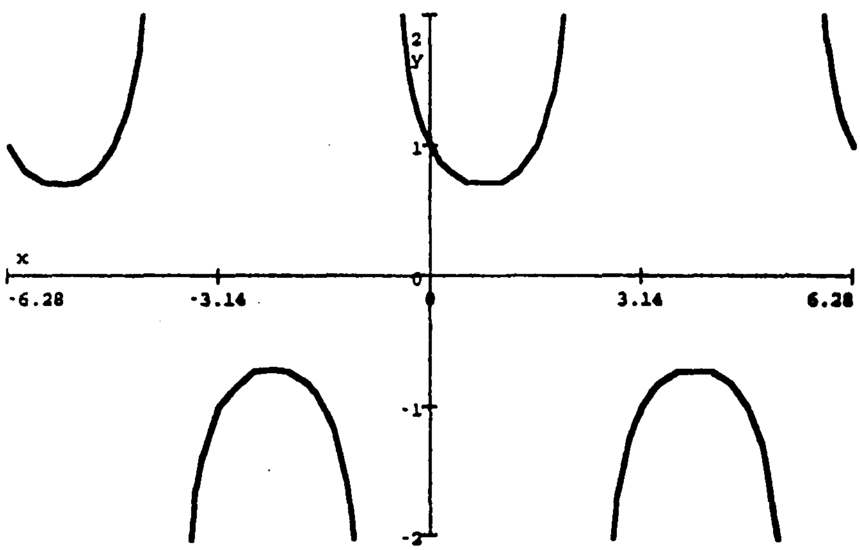


Figure 7. Graph of y = RS(x)

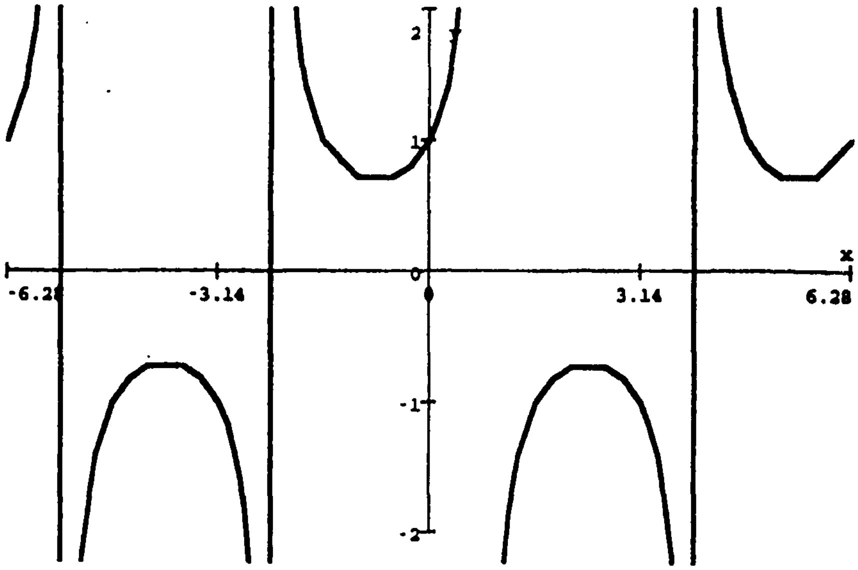


Figure 8. Graph of y = RC(x)

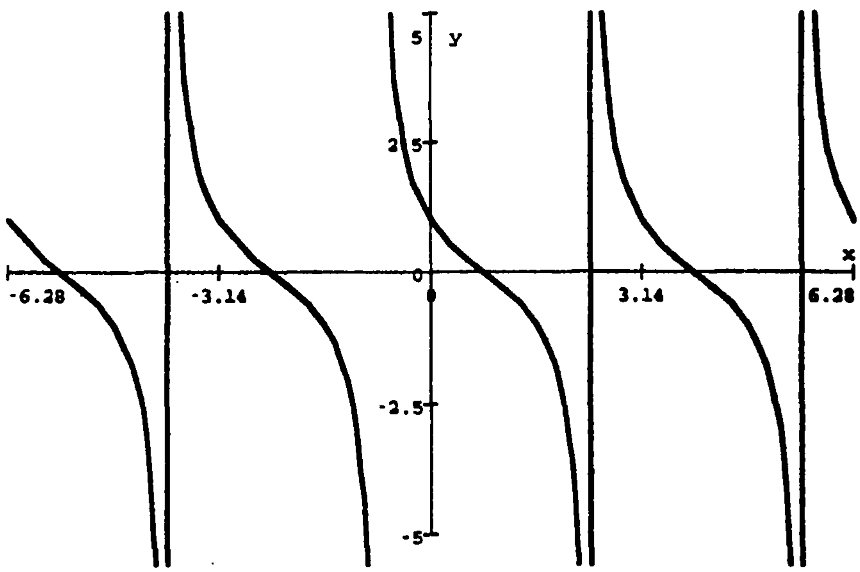


Figure 9. Graph of y = RT(x)

Table 1.

Values of Triometry Functions

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$S(\theta)$	1	$\frac{\sqrt{3}+1}{2}$	$\sqrt{2}$	$\frac{1+\sqrt{3}}{2}$	1	$\frac{-1+\sqrt{3}}{2}$	0	$\frac{-\sqrt{3}+1}{2}$	-1
$C(\theta)$	1	$\frac{\sqrt{3}-1}{2}$	0	$\frac{1-\sqrt{3}}{2}$	-1	$\frac{-1-\sqrt{3}}{2}$	$-\sqrt{2}$	$\frac{-\sqrt{3}-1}{2}$	-1
$T(\theta)$	1	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	undef	$\frac{1+\sqrt{3}}{1-\sqrt{3}}$	-1	$\frac{-1+\sqrt{3}}{-1-\sqrt{3}}$	0	$\frac{-\sqrt{3}+1}{-\sqrt{3}-1}$	1
$RS(\theta)$	1	$\frac{2}{\sqrt{3}+1}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{1+\sqrt{3}}$	1	$\frac{2}{-1+\sqrt{3}}$	undef	$\frac{2}{-\sqrt{3}+1}$	-1
$RC(\theta)$	1	$\frac{2}{\sqrt{3}-1}$	undef	$\frac{2}{1-\sqrt{3}}$	-1	$\frac{2}{-1-\sqrt{3}}$	$-\frac{1}{\sqrt{2}}$	$\frac{2}{-\sqrt{3}-1}$	-1
$RT(\theta)$	1	$\frac{\sqrt{3}-1}{\sqrt{3+1}}$	0	$\frac{1-\sqrt{3}}{1+\sqrt{3}}$	-1	$\frac{-1-\sqrt{3}}{-1+\sqrt{3}}$	undef	$\frac{-\sqrt{3}-1}{-\sqrt{3}+1}$	1

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$S(\theta)$	$\frac{-\sqrt{3}-1}{2}$	$-\sqrt{2}$	$\frac{-1-\sqrt{3}}{2}$	-1	$\frac{1+\sqrt{3}}{2}$	0	$\frac{\sqrt{3}-1}{2}$	1
$C(\theta)$	$\frac{-\sqrt{3}+1}{2}$	0	$\frac{-1+\sqrt{3}}{2}$	1	$\frac{1+\sqrt{3}}{2}$	$\sqrt{2}$	$\frac{\sqrt{3}+1}{2}$	1
$T(\theta)$	$\frac{-\sqrt{3}-1}{-\sqrt{3}+1}$	undef	$\frac{-1-\sqrt{3}}{-1+\sqrt{3}}$	-1	$\frac{-1-\sqrt{3}}{-1+\sqrt{3}}$	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	1
$RS(\theta)$	$\frac{2}{-\sqrt{3}-1}$	$-\frac{1}{\sqrt{2}}$	$\frac{2}{-1-\sqrt{3}}$	-1	$\frac{2}{1-\sqrt{3}}$	undef	$\frac{2}{\sqrt{3}-1}$	1
$RC(\theta)$	$\frac{2}{-\sqrt{3}+1}$	undef	$\frac{2}{-1+\sqrt{3}}$	1	$\frac{2}{1+\sqrt{3}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{\sqrt{3}+1}$	1
$RT(\theta)$	$\frac{-\sqrt{3}+1}{-\sqrt{3}-1}$	0	$\frac{-1-\sqrt{3}}{-1+\sqrt{3}}$	-1	$\frac{1-\sqrt{3}}{1+\sqrt{3}}$	undef	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	1

IDENTITIES

Negative Angle Identities.

Proposition 2. For any angle θ, we have the following:

(a) $S(-\theta) = C(\theta)$ and $C(-\theta) = S(\theta)$

(b) $T(-\theta) = RT(\theta)$ and $RT(-\theta) = T(\theta)$

(c) $RC(-\theta) = RS(\theta)$ and $RS(-\theta) = RC(\theta)$

Proof of $S(-\theta) = C(\theta)$. In Figure 10, since θ and $-\theta$ are reflections of each other about the x-axis, the point $P'(x,-y)$ is symmetric to $P(x,y)$. Thus,

$$S(-\theta) = \frac{x + (-y)}{r} = \frac{x - y}{r} = C(\theta).$$

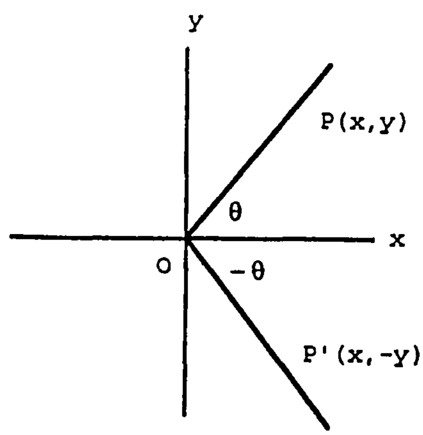


Figure 10

The other identities are proved in a similar fashion.■

Pythagorean Type Identities.

Proposition 3: $S^2(\theta) + C^2(\theta) = 2$.

Proof: $$S^2(\theta) + C^2(\theta) = \left[\frac{x + y}{r}\right]^2 + \left[\frac{x - y}{r}\right]^2$$

$$= \frac{x^2 + 2xy + y^2}{r^2} + \frac{x^2 - 2xy + y^2}{r^2}$$

$$= 2\left[\frac{x^2 + y^2}{r^2}\right]$$

$$= 2\left[\frac{r^2}{r^2}\right]$$

$$= 2. \blacksquare$$

Proposition 4: $T^2(\theta) + 1 = 2RC^2(\theta)$.

Proof: $$T^2(\theta) + 1 = \frac{S^2(\theta)}{C^2(\theta)} + 1$$

$$= \frac{S^2(\theta) + C^2(\theta)}{C^2(\theta)}$$

$$= \frac{2}{C^2(\theta)}$$

$$= 2RC^2(\theta). \blacksquare$$

Proposition 5: $1 + RT^2(\theta) = 2RS^2(\theta)$.

Proof: $$1 + RT^2(\theta) = 1 + \frac{C^2(\theta)}{S^2(\theta)}$$

$$= \frac{S^2(\theta) + C^2(\theta)}{S^2(\theta)}$$

$$= \frac{2}{S^2(\theta)}$$

$$=2RS^2(\theta). \blacksquare$$

Sum and Difference Formulas.

Proposition 6:

(a) $$C(u - v) = \frac{1}{2}\Big[C(u)\big(S(v) + C(v)\big) + S(u)\big(S(v) - C(v)\big)\Big].$$

(b) $$C(u + v) = \frac{1}{2}\Big[C(u)\big(C(v) + S(v)\big) + S(u)\big(C(v) - S(v)\big)\Big]$$

(c) $S(u - v) = \frac{1}{2}\Big[S(u)\big[C(v) + S(v)\big] + C(u)\big[C(v) - S(v)\big]\Big]$

(d) $S(u + v) = \frac{1}{2}\Big[S(u)\big[S(v) + C(v)\big] + C(u)\big[S(v) - C(v)\big]\Big]$

(e) $T(u - v) = \dfrac{T(u) + T(u)\cdot T(v) - T(v) + 1}{T(v) + T(u)\cdot T(v) - T(u) + 1}$

(f) $T(u + v) = \dfrac{T(u) + T(u)\cdot T(v) + T(v) - 1}{T(v) - T(u)\cdot T(v) + T(u) + 1}$.

Proof: We first consider (a). In Figure 11, we have (1) $C(u - v) = x_2 - y_2$. Since chord BD is equal to chord AC, (equal central angles), we have by the distance

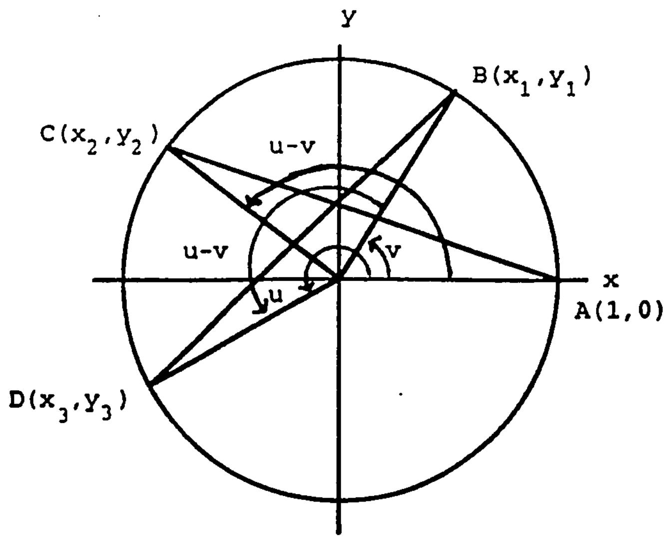


Figure 11

formula, $(x_1 - x_3)^2 + (y_1 - y_3)^2 = (x_2 -1) + (y_2 - 0)^2$ Squaring and simplifying we have

$(x_1^2 + y_1^2) + (x_3^2 + y_3^2) - 2(x_1x_3 + y_1y_3) =$
$(x_2^2 + y_2^2) - 2x_2 + 1$. By the Pythagorean Theorem,
$x_1^2 + y_1^2 = 1$, $x_3^2 + y_3^2 = 1$, and $x_2^2 + y_2^2 = 1$.
So $1 + 1 - 2(x_1x_3 + y_1y_3) = 1 - 2x_2 + 1$ or

(2) $x_2 = x_1x_3 + y_1y_3$.

To find y_2, rotate v 90^o counter-clockwise (this is equivalent to reflecting v about y = x and then reflecting about the y-axis). The new coordinates corresponding to (x_1,y_1) are $(-y_1,x_1)$ (Figure 12). Chord

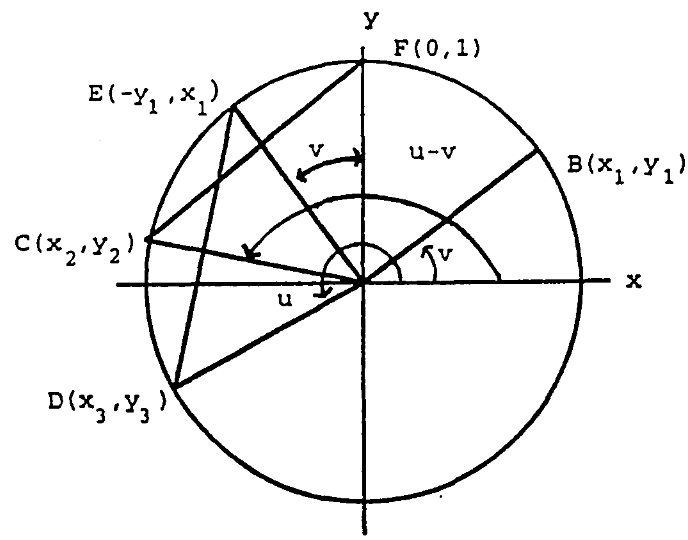


Figure 12

DE = chord CF (central angles = $u - v - \pi/2$) so

$(x_2 - 0)^2 + (y_2 - 1)^2 = (x_3 + y_1)^2 + (y_3 - x_1)^2$

which simplifies to

(3) $y_2 = x_1y_3 - x_3y_1$

Substituting into (1), we have

(4) $C(u - v) = x_1x_3 + y_1y_3 - (x_1y_3 - x_3y_1)$.

By definition

$$S(v) = x_1 + y_1 \qquad S(u) = x_3 + y_3$$

$$C(v) = x_1 - y_1 \qquad C(u) = x_3 - y_3$$

Solving both pairs simultaneously produces:

(5) $$x_1 = \frac{1}{2}\left(S(v) + C(v)\right) \qquad x_3 = \frac{1}{2}\left(S(u) + C(u)\right)$$

$$y_1 = \frac{1}{2}\left(S(v) - C(v)\right) \qquad y_3 = \frac{1}{2}\left(S(u) - C(u)\right)$$

Substituting each of (5) into (4), we have

$$C(u - v) = \left[\frac{1}{2}\left(S(v) + C(v)\right)\right]\left[\frac{1}{2}\left(S(u) - C(u)\right)\right] -$$

$$\left[\frac{1}{2}\left(S(u) + C(u)\right)\right]\left[\frac{1}{2}\left(S(v) - C(v)\right)\right] \quad \text{or}$$

$$C(u - v) = \frac{1}{2}\left[C(u)\left(S(v) + C(v)\right) + S(u)\left(S(v) - C(v)\right)\right].$$

We now prove (b).

$$C(u + v) = C(u - (-v))$$

$$= \frac{1}{2}\left[C(u)\left(C(v) + S(v)\right) + S(u)\left(C(v) - S(v)\right)\right].$$

The proofs of (c) and (d) are similar to the proofs of (a) and (b).

Next, we prove (e).

$$T(u - v) = \frac{S(u - v)}{C(u - v)}$$

$$= \frac{\frac{1}{2}\Big[S(u)\{C(v) - S(v)\} + C(u)\{C(v) - S(v)\}\Big]}{\frac{1}{2}\Big[C(u)\{S(v) + C(v)\} + S(u)\{S(v) - C(v)\}\Big]}$$

Now, dividing both numerator and denominator by C(u)C(v), we get

$$T(u - v) = \frac{\frac{S(u)}{C(u)}\left[1 + \frac{S(v)}{C(v)}\right] + 1\left[1 - \frac{S(v)}{C(v)}\right]}{1\left[\frac{S(v)}{C(v)} + 1\right] + \frac{S(u)}{C(u)}\left[\frac{S(v)}{C(v)} - 1\right]}$$

Which simplifies into

$$T(u - v) = \frac{T(u) + T(u)T(v) - T(v) + 1}{T(v) + T(u)T(v) - T(u) + 1}.$$

The proof of (f) is similar to the proof of (e).■

Function-Product Identities.

The following are presented without proof but are easily derived from the sum/difference formulas.

Propostion 7. For any angles u and v,

(a) $S(u + v) - C(u + v) = S(u)\cdot S(v) - C(u)\cdot C(v)$

(b) $S(u + v) + C(u + v) = S(u)\cdot C(v) + C(u)\cdot S(v)$

(c) $S(u - v) + C(u - v) = S(u)\cdot S(v) + C(u)\cdot C(v)$

(d) $S(u - v) - C(u - v) = S(u)\cdot C(v) - C(u)\cdot S(v)$.

Double Angle Formulas.

Proposition 8. For any angle u,

(a) $S(2u) = \frac{1}{2}\left[S^2(u) + 2S(u)\cdot C(u) - C^2(u)\right]$;

(b) $C(2u) = \frac{1}{2}\left[C^2(u) + 2C(u)\cdot S(u) - S^2\cdot(u)\right]$;

(c) $T(2u) = \dfrac{T^2(u) + 2T(u) - 1}{1 + 2T(u) - T^2(u)}$.

Proof. Each of these formulas follows immediately from the corresponding sum formula. We demonstrate by proving (c). From (f) of Proposition 6, we have

$$T(2u) = T(u + u) = \frac{T(u) + T(u)T(u) + T(u) - 1}{T(u) - T(u)T(u) + T(u) + 1}$$

which simplifies into (c).■

<u>Other Useful Identities</u>.

<u>Proposition 9</u>. For any angle θ,

(a) $S(\pi/2 - \theta) = S(\theta)$.

(b) $C(\pi/2 - \theta) = -C(\theta)$.

(c) $T(\pi/2 - \theta) = -T(\theta)$.

(d) $RS(\pi/2 - \theta) = RS(\theta)$.

(e) $RC(\pi/2 - \theta) = -RC(\theta)$.

(f) $RT(\pi/2 - \theta) = -RT(\theta)$.

Proof: We prove only (a). The others are proved in a similar fashion.

$$S(\pi/2 - \theta) = \frac{1}{2}\Big[S(\pi/2)\left[C(\theta) + S(\theta)\right] + C(\pi/2)\left[C(\theta) - S(\theta)\right]\Big]$$

$$= \frac{1}{2}\Big[1\left[C(\theta) + S(\theta)\right] + (-1)\left[C(\theta) - S(\theta)\right]\Big]$$

$= S(\theta)$. ∎

The following identities allow a triometry function of any angle to be expressed in terms of an angle between 0 and $\pi/2$. They follow immediately from the sum and difference identities.

If $0 < \theta < \frac{\pi}{2}$, then

$$S(\pi - \theta) = -C(\theta) \qquad S(\pi + \theta) = -S(\theta) \qquad S(2\pi - \theta) = C(\theta)$$

$$C(\pi - \theta) = -S(\theta) \qquad C(\pi + \theta) = -C(\theta) \qquad C(2\pi - \theta) = S(\theta)$$

$$T(\pi - \theta) = T(\theta) \qquad T(\pi + \theta) = T(\theta) \qquad T(2\pi - \theta) = T(\theta)$$

Examples: $S\left[\frac{5\pi}{6}\right] = S\left[\pi - \frac{\pi}{6}\right] = -\,C\left[\frac{\pi}{6}\right].$

DERIVATIVES

In order to determine the derivatives of $S(\theta)$ and $C(\theta)$ using the limit definition of the derivative, it is necessary to find the following limits.

Proposition 10.

(a) $\lim_{h \to 0} \dfrac{S(h) + C(h) - 2}{h} = 0$ (b) $\lim_{h \to 0} \dfrac{S(h) - C(h)}{h} = 2.$

Proof: We first prove (a). In Figure 13, circle O is a unit circle and angle h is in radians.

$$\lim_{h \to 0} \frac{S(h) + C(h) - 2}{h} = \lim_{h \to 0} \frac{x + y + x - y - 2}{h}$$

$$= \lim_{h\to 0} \frac{2x - 2}{h}$$

$$= -2\cdot\lim_{h\to 0} \frac{1 - x}{h}$$

From Figure 13 we see that 0 < chord AB < length of arc AB.

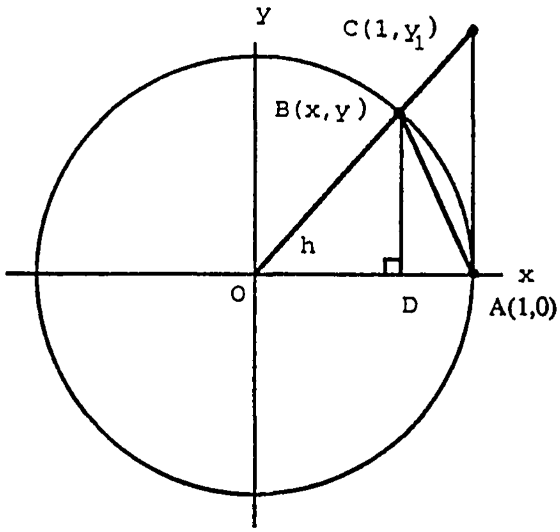


Figure 13

So $0 < \sqrt{(x - 1)^2 + y^2} < |h|$. Since all quantities are positive, we can square each one giving

$$0 < (x - 1)^2 + y^2 < h^2 \text{ or } 0 < x^2 - 2x + 1 + y^2 < h^2.$$

But $x^2 + y^2 = 1$; hence $0 < 2 - 2x < h^2$. If $h > 0$, then

$$0 < \frac{1 - x}{h} < \frac{h}{2}\ .$$

Taking the limit as $h\to 0^+$ we have

$$\lim_{h\to 0^+} 0 \le \lim_{h\to 0^+} \frac{1 - x}{h} \le \lim_{h\to 0^+} \frac{h}{2}$$

or $$\lim_{h\to 0^+} \frac{1-x}{h} = 0.$$

Similarly, if $h < 0$, then $0 > \frac{1-x}{h} > \frac{h}{2}$, and

$$\lim_{h\to 0^-} 0 \geq \lim_{h\to 0^-} \frac{1-x}{h} \geq \lim_{h\to 0^-} \frac{h}{2}$$

or

$$\lim_{h\to 0^-} \frac{1-x}{h} = 0.$$

Hence, $\lim_{h\to 0} \frac{1-x}{h} = 0$. Therefore,

$$\lim_{h\to 0}\frac{S(h) + C(h) - 2}{h} = -2\cdot\lim_{h\to 0} \frac{1-x}{h} = -2\cdot 0 = 0$$

which completes the proof of (a).

We now prove (b). Refering to Figure 13, we have

$$\lim_{h\to 0} \frac{S(h) - C(h)}{h} = \lim_{h\to 0} \frac{x + y - (x - y)}{h} = \lim_{h\to 0} \frac{2y}{h}.$$

From the Figure,

0 < area ΔAOB < area sector AOB < area ΔAOC or

$0 < \frac{1}{2}(1)(y) < \frac{h}{2\pi}(\pi)(1)^2 < \frac{1}{2}(1)(y_1)$.

Now ΔDOB is similar to ΔAOC and so $\frac{y}{x} = \frac{y_1}{1}$ or $y_1 = \frac{y}{x}$. Thus

$$0 < \frac{1}{2}\, y < \frac{1}{2}\, h < \frac{1}{2} \cdot \frac{y}{x} \quad \text{or}$$

$$x < \frac{y}{h} < 1.$$

Now,

$$\lim_{h\to 0} x \leq \lim_{h\to 0} \frac{y}{h} \leq \lim_{h\to 0} 1 \quad \text{or}$$

$$1 \le \lim_{h \to 0} \frac{y}{h} \le 1.$$

We now have $\lim_{h \to 0} \frac{y}{h} = 1$

Therefore, $\lim_{h \to 0} \frac{S(h) - C(h)}{h} = \lim_{h \to 0} \frac{2y}{h} = 2 \cdot 1 = 2.$ ■

Propostion 11. $S'(\theta) = C(\theta)$.

Proof: From the definition of the derivative and the limits in Proposition 10, we get

$$S'(\theta) = \lim_{h \to 0} \frac{S(\theta+h) - S(\theta)}{h}$$

$$= \lim_{h \to 0} \frac{1/2\left[S(\theta)\left[S(h)+C(h)\right] + C(\theta)\left[S(h)-C(h)\right]\right] - S(\theta)}{h}$$

$$= \lim_{h \to 0} \frac{1/2\left[S(\theta)\left[S(h)+C(h)\right] + C(\theta)\left[S(h)-C(h)\right] - 2S(\theta)\right]}{h}$$

$$= \frac{1}{2} \lim_{h \to 0} \left[\frac{S(\theta)\left[S(h)+C(h)-2\right]}{h} + \frac{C(\theta)\left[S(h)-C(h)\right]}{h}\right]$$

$$= \frac{1}{2} \left[S(\theta) \cdot 0 + C(\theta) \cdot 2\right]$$

$$= C(\theta).$$ ■

The following derivatives can now be easily proved.

Proposition 12.

(1) $C'(\theta) = -S(\theta)$.

(2) $T'(\theta) = 2RC^2(\theta)$.

(3) $RT'(\theta) = -2RS^2(\theta)$.

(4) $RS'(\theta) = -RT(\theta) \cdot RS(\theta)$.

(5) $RC'(\theta) = T(\theta) \cdot RC(\theta)$.

INTEGRALS

From Propositions 11 and 12 and the definition of the antiderivative, we have the following:

Proposition 13.

(1) $\int S(\theta)d\theta = -C(\theta) + k$

(2) $\int C(\theta)d\theta = S(\theta) + k$

(3) $\int RC^2(\theta)d\theta = 1/2T(\theta) + k$

(4) $\int RS^2(\theta)d\theta = -1/2RT(\theta) + k$

(5) $\int RT(\theta) \cdot RS(\theta)d\theta = -RS(\theta) + k$

(6) $\int T(\theta) \cdot RC(\theta)d\theta = RC(\theta) + k$

(7) $\int T(\theta)d\theta = \int \frac{S(\theta)}{C(\theta)}d\theta = -\ln|C(\theta)| + k$

(8) $\int RT(\theta)d\theta = \int \frac{C(\theta)}{S(\theta)}d\theta = \ln|S(\theta)| + k$

INVERSE FUNCTIONS

Definition. If t is a real number and θ is an angle of rotation,

(1) $S^{-1}(t) = \theta$ if and only if $S(\theta) = t$ where

$-\sqrt{2} \le t \le \sqrt{2}$ and $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$.

(2) $C^{-1}(t) = \theta$ if and only if $C(\theta) = t$ where

$-\sqrt{2} \le t \le \sqrt{2}$ and $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$.

(3) $T^{-1}(t) = \theta$ if and only if $T(\theta) = t$ where $-\infty < t < +\infty$

and $-\frac{3\pi}{4} < \theta < \frac{\pi}{4}$.

(4) $RS^{-1}(t) = \theta$ if and only if $RS(\theta) = t$ where $|t| \ge \frac{\sqrt{2}}{2}$

and $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$, $\theta \ne -\frac{\pi}{4}$.

(5) $RC^{-1}3(t) = \theta$ if and only if $RC(\theta) = t$ where $|t| \ge \frac{\sqrt{2}}{2}$

and $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$, $\theta \ne \frac{\pi}{4}$.

(6) $RT^{-1}(t) = \theta$ if and only if $RT(\theta) = t$ where $-\infty < t < +\infty$

and $-\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

<u>Proposition 14</u>. If t is a real number and θ is an angle of rotation,

(1) $S(S^{-1}(t)) = t$ if $-\sqrt{2} \le t \le \sqrt{2}$.

(2) $S^{-1}(S(\theta)) = \theta$ if $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$.

(3) $C(C^{-1}(t)) = t$ if $-\sqrt{2} \le t \le \sqrt{2}$.

(4) $C^{-1}(C(\theta)) = \theta$ if $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$.

(5) $T(T^{-1}(t)) = t$ for all t.

(6) $T^{-1}(T(\theta)) = \theta$ if $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$.

Proof. If $-\sqrt{2} \le t \le \sqrt{2}$, then, by definition of S, there is a θ, $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$, for which $S(\theta) = t$. Then, by definition of S^{-1}, we must have $S^{-1}(t) = \theta$. It now follows that $S(S^{-1}(t)) = S(\theta) = t$ which completes the proof of (1).

To prove (2), let θ be such that $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$. Then, there is a t for which $S(\theta) = t$. Then, by definition of S^{-1}, $S^{-1}(S(\theta)) = S^{-1}(t) = \theta$.

The proofs of the others follow in a similar fashion.■

Reciprocal Identities.

Proposition 15.

(1) $RC^{-1}(t) = C^{-1}\left[\frac{1}{t}\right]$ if $|t| \ge \frac{\sqrt{2}}{2}$.

(2) $RS^{-1}(t) = S^{-1}\left[\frac{1}{t}\right]$ if $|t| \ge \frac{\sqrt{2}}{2}$.

(3) $RT^{-1}(t) = T^{-1}\left[\frac{1}{t}\right]$ if $t > 0$.

Proof: We prove only (1). The others are proved in a similar fashion.

Let $|t| \ge \frac{\sqrt{2}}{2}$ and suppose that $RC^{-1}(t) = \theta$. Then, by definition, $RC(\theta) = t$. But $RC(\theta) = \frac{1}{C(\theta)}$; so, $\frac{1}{C(\theta)} = t$

or $C(\theta) = \frac{1}{t}$ which means that $C^{-1}\left[\frac{1}{t}\right] = \theta$.■

<u>Derivatives of Inverses</u>.

<u>Proposition 16</u>.

$$D_t\left[S^{-1}(t)\right] = \frac{1}{\sqrt{2 - t^2}}, \quad -\sqrt{2} < t < \sqrt{2}$$

Proof: If $-\sqrt{2} < t < \sqrt{2}$, then $S^{-1}(t) = \theta$ implies $S(\theta) = t$ where $-\frac{3\pi}{4} < \theta < \frac{\pi}{4}$. Taking the derivative with respect to θ produces, $C(\theta)\frac{d\theta}{dt} = 1$ or $\frac{d\theta}{dt} = \frac{1}{C(\theta)}$ since $C(\theta) > 0$. From the Pythagorean identities, we have $C(\theta) = \pm\sqrt{2 - S^2(\theta)}$ or $C(\theta) = \pm\sqrt{2 - t^2}$. Since $C(\theta)$ is positive for $-\frac{3\pi}{4} < \theta < \frac{\pi}{4}$, we have

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{2 - t^2}}.$$■

The following proposition is proved in a fashion similar to that of Proposition 16 and so is stated without proof.

<u>Proposition 17</u>.

$$D_t\left[C^{-1}(t)\right] = -\frac{1}{\sqrt{2 - t^2}}, \quad -\sqrt{2} < t < \sqrt{2}$$

<u>Proposition 18</u>.

$$D_t\left[T^{-1}(t)\right] = \frac{1}{1 + t^2}, \quad -\infty < t < +\infty$$

Proof: For any t, $-\infty < t < t$, there is a θ, $-\frac{3\pi}{4} < \theta < \frac{\pi}{4}$, for which $T^{-1}(t) = \theta$. Then $T(\theta) = t$ and taking the derivative with respect to t gives $2RC^2(\theta)\frac{d\theta}{dt} = 1$. Since $RC(\theta) \neq 0$ for $-\frac{3\pi}{4} < \theta < \frac{\pi}{4}$, we have $\frac{d\theta}{dt} = \frac{1}{2RC^2(\theta)}$ or $\frac{d\theta}{dt} = \frac{C^2(\theta)}{2}$. Since $T(\theta) = t$, then $\frac{S(\theta)}{C(\theta)} = t$ and $C^2(\theta)\cdot t^2 = S^2(\theta)$. Also $S^2(\theta) = 2 - C^2(\theta)$ so $C^2(\theta)\cdot t^2 = 2 - C^2(\theta)$. Solving for $C^2(\theta)$ we get $C^2(\theta) = \frac{2}{1 + t^2}$. Now substituting into $\frac{d\theta}{dt} = \frac{C^2(\theta)}{2}$ produces $\frac{d\theta}{dt} = \frac{1}{1 + t^2}$ which is the desired result.■

The proof of the following proposition is similar to that of Proposition 18 and so is stated without proof.

<u>Proposition 19</u>.

$$D_t\left[RT^{-1}(t)\right] = -\frac{1}{1 + t^2}, \quad -\infty < t < +\infty$$

<u>Proposition 20</u>. $D_t\left[RC^{-1}(t)\right] = \frac{1}{|t|\sqrt{2t^2 - 1}}$, $|t| > \frac{\sqrt{2}}{2}$.

Proof: For any t, $|t| > \frac{\sqrt{2}}{2}$, there is a $\theta \neq \frac{\pi}{4}$ in the interval $-\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, for which $RC^{-1}(t) = \theta$ or $RC(\theta) = t$. The derivative with respect to t is $T(\theta)\cdot RC(\theta)\frac{d\theta}{dt} = 1$.

Since $T(\theta)$ and $RC(\theta)$ are not zero,

$\frac{d\theta}{dt} = \frac{1}{T(\theta)\cdot RC(\theta)}$. Using identity substitutions, we have

$\frac{d\theta}{dt} = \frac{1}{\frac{S(\theta)}{C(\theta)}\cdot\frac{1}{C(\theta)}}$ or $\frac{d\theta}{dt} = \frac{C^2(\theta)}{S(\theta)}$. Now, we can express $C^2(\theta)$

and $S(\theta)$ in terms of t. $C^2(\theta) = \frac{1}{t^2}$ because $RC(\theta) = t$ and

$S(\theta) = \pm\sqrt{2 - \frac{1}{t^2}}$ because $S^2(\theta) = 2 - C^2(\theta)$. Substituting

into $\frac{d\theta}{dt}$ produces $\frac{d\theta}{dt} = \frac{\frac{1}{t^2}}{\frac{\pm\sqrt{2t^2 - 1}}{t}}$ or $\frac{1}{\pm t\sqrt{2t^2 - 1}}$ where

the sign is determined by $T(\theta)$. If $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, then

$T(\theta) > 0$ and $t > 0$. If $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, then $T(\theta) < 0$ and $t < 0$.

So,

$$\frac{d\theta}{dt} = \begin{cases} \dfrac{1}{t\sqrt{2t^2 - 1}} & \text{if } t > \dfrac{\sqrt{2}}{2} \\[2ex] \dfrac{1}{-t\sqrt{2t^2 - 1}} & \text{if } t < -\dfrac{\sqrt{2}}{2} \end{cases}$$

or $$\frac{d\theta}{dt} = \frac{1}{|t|\sqrt{2t^2 - 1}}\ .\blacksquare$$

Our next proposition is proved in a fashion similar to the proof of Proposition 20 and so is stated without proof.

Proposition 21. $D_t(t) = -\dfrac{1}{|t|\sqrt{2t^2 - 1}}$, $|t| > \dfrac{\sqrt{2}}{2}$

INTEGRAL OF INVERSES

Proposition 22.

(1) $\displaystyle\int \frac{1}{\sqrt{2 - t^2}}\, dt = S^{-1}(t) + k, \; -\sqrt{2} < t < \sqrt{2}.$

(2) $\displaystyle\int \frac{1}{1 + t^2}\, dt = T^{-1}(t) + k, \; -\infty < t < +\infty.$

(3) $\displaystyle\int \frac{1}{|t|\sqrt{2t^2 - 1}}\, dt = RC^{-1}(|t|) + k, \; |t| > \frac{\sqrt{2}}{2}$

RIGHT TRIANGLE TRIOMETRY

Given right triangle ABD (see Figure 14), with D the right angle, we may superimpose a coordinate system so as to have angle A at the origin and to have side b lie along the x-axis. We can then use the definition of the triometry functions to obtain:

(1) $S(A) = \dfrac{b + a}{d}$ $\left[\dfrac{\text{adjacent + opposite}}{\text{hypotenuse}}\right]$

(2) $C(A) = \dfrac{b - a}{d}$ $\left[\dfrac{\text{adjacent - opposite}}{\text{hypotenuse}}\right]$

$$(3)\quad RT(A) = \frac{C(A)}{S(A)} = \frac{b - a}{b + a} \qquad \left[\frac{\text{adjacent - opposite}}{\text{adjacent + opposite}}\right]$$

These formulas can be used to solve right triangles given an angle and a side (including the hypotenuse.) We consider three cases.

Case I. If A and d are given, then use both (1) and and (2) to solve for either a or b. For example, to solve for a, we have from (1) and (2) $d \cdot S(A) = a + b$ and $d \cdot C(A) = a - b$ which we may solve simultaneously to obtain $d \cdot (S(A) + C(A)) = 2a$ or

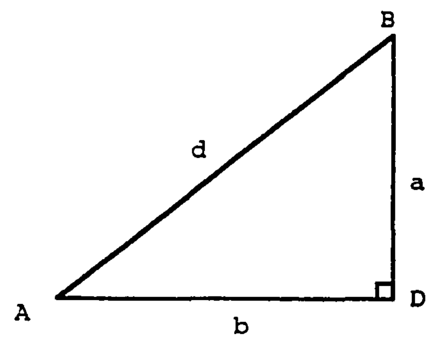


Figure 14

$$a = \frac{1}{2}\,(S(A) + C(A)).$$

Case II. If A and a (or b) is given, use (3) to solve for b (or a). To see this, suppose that A and a are given. Then, by (3) we have $(b + a) \cdot RT(A) = b - a$ which simplifies to $(RT(A) - 1) \cdot b = -a \cdot (RT(A) + 1)$. Since we may assume that $90^\circ > A > 0^\circ$, $RT(A) \neq 1$, and so we may solve for b to obtain

$$b = \frac{a \cdot (RT(A) + 1)}{1 - RT(A)}.$$

Case III. If A and a (or b), use both (1) and (2) to solve for d. Suppose that A and a are given. Then, from (1) and (2) we have $d \cdot S(A) = a + b$ and $d \cdot C(A) = a - b$.

subtracting the second equation from the first gives $d \cdot (S(A) - C(A)) = 2b$. Since we may assume that $90^\circ > A > 0^\circ$, $S(A) \neq C(A)$ and so we may solve for d to obtain

$$d = \frac{2b}{S(A) - C(A)}.$$

Example. In triangle ABD, suppose that $A = 30^\circ$ and $d = 4$. To find side a, we use (1) and (2).

$$S(30^\circ) = \frac{b+a}{4} \quad \text{or} \quad 4 \cdot S(30^\circ) = b + a, \text{ and}$$

$$C(30^\circ) = \frac{b - a}{4} \quad \text{or} \quad 4 \cdot C(30^\circ) = b - a$$

Using Table 1 to obtain values for $S(30^\circ)$ and $C(30^\circ)$, we get the two equations

$$4\left[\frac{\sqrt{3} + 1}{2}\right] = b + a \quad \text{and} \quad 4\left[\frac{\sqrt{3} - 1}{2}\right] = b - a.$$

Solving simultaneously for a yields

$$a = \frac{1}{2}\left[4\left[\frac{\sqrt{3} + 1}{2}\right] - 4\left[\frac{\sqrt{3} - 1}{2}\right]\right]$$

which simplifies into $a = 2$.

The triometry function RT can be used to find an angle in a right triangle ABD given two sides. We consider two cases.

Case I. If sides a and b are given, we have that

$RT(A) = \frac{b - a}{b + a}$. Since A is in the interval $(0^\circ, 90^\circ)$, we may use RT^{-1} to obtain $A = RT^{-1}\left[\frac{b - a}{b + a}\right]$.

Case II. If sides a (or b) and d are given, use the Pythagorean Theorem to find the missing side, then the solution in Case I applies.

Example. In triangle ABD, suppose that $a = \sqrt{3}$ and $b = 1$ and that we wish to find A. Then, we have that

$$RT(A) = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \quad \text{or} \quad A = RT^{-1}\left[\frac{1 - \sqrt{3}}{1 + \sqrt{3}}\right].$$

From Table 1 we see that $A = 30^\circ$.

Formulas for Solving Oblique Triangles

Proposition 23: (The Law of S and C.) For any triangle ABD

$$\frac{S(A) - C(A)}{a} = \frac{S(B) - C(B)}{b} = \frac{S(D) - C(D)}{d}.$$

Proof: The proof is in two cases.

Case 1. The altitude falls inside of (or on) the triangle (Figure 15). By definition,

$S(A) = \frac{x + h}{b}$, $S(B) = \frac{d - x + h}{a}$, $C(A) = \frac{x - h}{b}$, and $C(B) = \frac{d - x - h}{a}$. Solving each equation for h, respectively, gives

(1) $h = b \cdot S(A) - x$,

(2) $h = a \cdot S(B) - d + x$,

(3) $h = x - b \cdot C(A)$, and

(4) $h = d - x - a \cdot C(B)$.

Solving (1) and (2) simultaneously produces

(i) $2x - d = b \cdot S(A) - a \cdot S(B)$.

Similarly, (3) and (4) produces

(ii) $2x - d = b \cdot C(A) - a \cdot C(B)$.

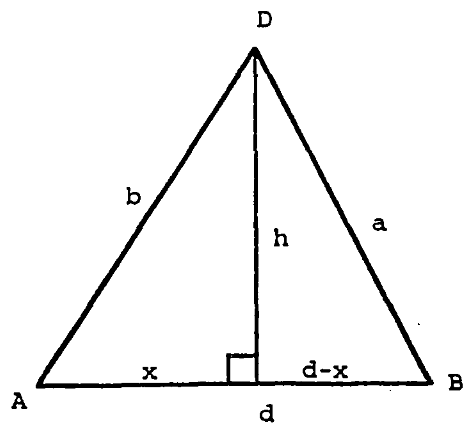


Figure 15

From (i) and (ii) we have

$$b \cdot S(A) - a \cdot S(B) = b \cdot C(A) - a \cdot C(B)$$

or

$$b\left[S(A) - C(A)\right] = a\left[S(B) - C(B)\right].$$

Dividing by ab gives $\dfrac{S(A) - C(A)}{a} = \dfrac{S(B) - C(B)}{b}$.

Case 2. The altitude falls outside the triangle.

Referring to Figure 16, we have by definition

$$S(A) = \frac{d + x + h}{b} \quad \text{and} \quad S(B) = -C(180-B) = -\frac{x - h}{a},$$

$$C(A) = \frac{d + x - h}{b} \quad \text{and} \quad C(B) = -S(180-B) = -\frac{x + h}{a}.$$

Solving each of these equations for h, we get

(1) $h = b \cdot S(A) - d - x$

(2) $h = a \cdot S(B) + x$

(3) $h = d + x - b \cdot C(A)$

(4) $h = -x - a \cdot C(B)$

Using (1) and (4) we have,

$b \cdot S(A) - d - x = -x - a \cdot C(B)$

or

(i) $d = b \cdot S(A) + a \cdot C(B)$.

Using (2) and (3) we get

$a \cdot S(B) + x = d + x - b \cdot C(A)$

which gives

(ii) $d = a \cdot S(B) + b \cdot C(A)$.

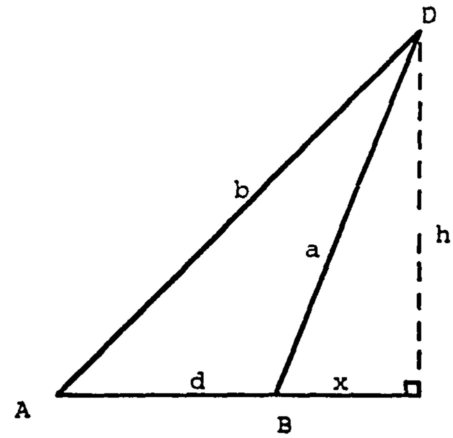


Figure 16

Now, using (i) and (ii), we get

$$b \cdot S(A) + a \cdot C(B) = a \cdot S(B) + b \cdot C(A).$$

or $\quad b[S(A) - C(A)] = a[S(B) - C(B)]$

Dividing by ab gives

$$\frac{S(A) - C(A)}{a} = \frac{S(B) - C(B)}{b}. \blacksquare$$

Area of Triangle.

Proposition 24: Given any two sides and the included angle of a triangle, the area is

$\frac{1}{4}$[product of two sides][S(included angle)-C(included angle)].

Proof: Given b, d, and $\angle A$. the area of $\triangle ABD$ (Figure 17) is

(1) $\quad \text{Area} = \frac{1}{2}dh.$

To find h in terms of $\angle A$,

$$S(A) = \frac{x + h}{b} \quad \text{and}$$

$C(A) = \frac{x - h}{b}$; so,

(2) $h = b \cdot S(A) - x$ and

(3) $x = b \cdot C(A) + h$.

Substituting (3) into (2) and simplifying gives

(4) $h = \frac{b}{2}\left[S(A) - C(A)\right]$.

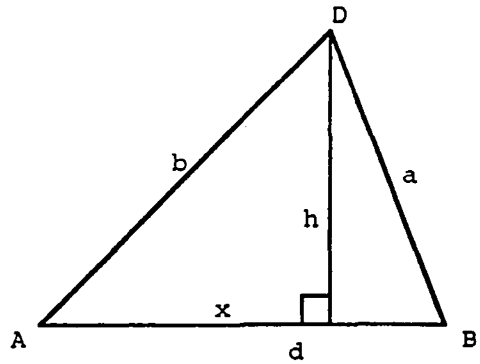


Figure 17

Finally, substituting (4) into (1) we have $A = \frac{1}{4}bd\left[S(A) - C(A)\right]$.

The proof for the other two cases is similar.■

Proposition 25. In any triangle ABF having sides of length a, b, and f, the following relationships are true.

i) $a^2 = f^2 + b^2 - fb\left[S(A) + C(A)\right]$

ii) $b^2 = a^2 + f^2 - af\left[S(B)+ C(B)\right]$

iii) $f^2 = a^2 + b^2 - ab\left[S(F) + C(F)\right]$

Proof of (i) with $\angle A$ acute:

Place $\triangle ABF$ in a rectangular coordinate plane with A at the origin and AB an the x-axis (see Figure 18). By the Pythagorean Theorem,

(1) $a^2 = (f - x)^2 + y^2$.

By definition, $S(A) = \frac{x + y}{b}$ so $x + y = b \cdot S(A)$. Likewise

$C(A) = \frac{x - y}{b}$ so $x - y = b \cdot C(A)$. Solving simultaneously produces

$$x = \frac{1}{2}b\left[S(A) + C(A)\right] \text{ and}$$

$$y = \frac{1}{2}b\left[S(A) - C(A)\right].$$

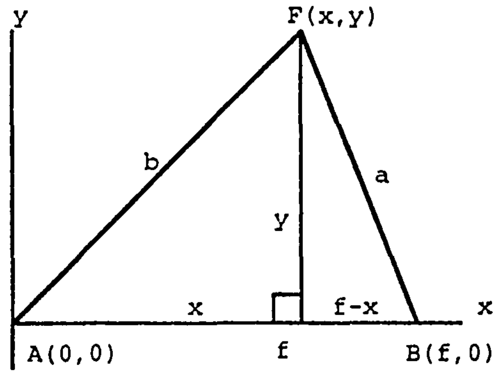


Figure 18

Substituting into (1) and simplifying,

$$a^2 = \left[f - \frac{1}{2}b\left[S(A) + C(A)\right]\right]^2 + \left[\frac{1}{2}b\left[S(A) - C(A)\right]\right]^2.$$

$$= f^2 - fb\left[S(A) + C(A)\right] + \frac{1}{4}b^2\left[S^2(A) + 2S(A)\cdot C(A) + C^2(A)\right] +$$

$$\frac{1}{4}b^2\left[S^2(A) - 2S(A)\cdot C(A) + C^2(A)\right]$$

$$= f^2 - fb\left[S(A) + C(A)\right] +$$

$$\frac{1}{4}b^2\left[2 + 2S(A)\cdot C(A) + 2 - 2S(A)\cdot C(A)\right]$$

$$= f^2 - fb\left[S(A) + C(A)\right] + \frac{1}{4}b^2(4)$$

$$= f^2 + b^2 - fb\left[S(A) + C(A)\right].$$

(ii) and (iii) can be proved in a similar fashion.

Appendix C

Course Content and Assignments

Syllabus

MAT 3530 PART I Spring 1991

Instructor: Mrs. Billie Goodman 262-2610 (office)
335 Walker 262-0798 (home)

Text: None. However, you may find any math text with a good coverage of trigonometry a handy reference.

Goal: To create some mathematics, specifically, a new trigonometry

Objectives: To get a feel for what mathematics is all about and what mathematicians do

Grading: Grading will be based exclusively on participation. There are five ways of participating that are expected of you.

1) Fill out survey and questionnaire forms, and write an evaluation at the end of the course.
2) Take part in classroom discussions.
3) Attempt to do the assignments.
4) Be interviewed three times.
5) Keep a journal.

Interviews:
I would like to interview each of you three times during this course. Each interview will take approximately a half hour to an hour and requires no preparation on your part. I will interview on the following schedule:

1. Week of Jan 23
2. Week of Feb. 6
3. Week of Feb. 27

I will work out the exact times with each of you to fit into your schedule.

Journals:
Each of you is to maintain a weekly journal. Please use loose-leaf paper. These will be returned to you at a

later date with comments. The journal should include:

1) your attempts to work the problems that have been assigned;
2) any ideas or intuitions you had about the problems even if you were not able to follow through on them;
3) your comments (favorable, unfavorable, or otherwise) about the exercise itself;
4) your feelings (frustration, elation, etc.) while working on the assignment;
5) anything else you want to say.

As much as possible, do all work in the journal. However, you may want to do your planning and scratch work on scrap paper. Please write legibly and try to have some organization so that I can follow your work. Let me emphasize that the quality or quantity of your work will not be judged. Likewise what you say. So speak your mind! You are encouraged to work together.

Distribution of Topics by Sessions

Topics	Sessions														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Basic trigonometric functions	x														
Definitions for triometry functions		x	x												
Pythagorean type identities		x	x	x											
Graph S(θ) and C(θ)					x										
Property that parallels law of sines						x	x								
Properties for negative angles (S(-θ), C(-θ), etc.)						x		x							
Formulas that parallel sum/difference formulas in trigonometry							x	x	x	x				x	
Derivatives of triometry functions											x	x	x		
Wrap-up session															x

Sample Assignments

1. Define a function of an angle θ in a right triangle that is different from any of the usual trigonometric definitions but has the property of invariance with respect to similar right triangles.

2. Using the definitions for S(θ) and C(θ), find an identity similar to $\sin^2\theta + \cos^2\theta = 1$.

3. Using the triometry definitions, find identities that parallel the other Pythagorean identities in trigonometry. ($1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$).

4. Make a table of values for S(Θ) and C(Θ) for the special angles. Determine the intervals where each are positive and negative. What do they graphs look like?

5. Use the limit definition of derivative together with the proof of the derivative of sin Θ to find the derivative of S(Θ).

Other Topics For Exploration

1) Inverse functions for the triometry functions
2) Formulas that parallel the law of cosines
3) Formula for the area of any triangle that parallels area of $\triangle$ ABC = bc·sin A
4) Integrals of triometry functions

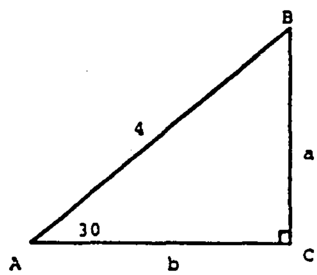
<u>Fun Assignment</u>

2/7/91

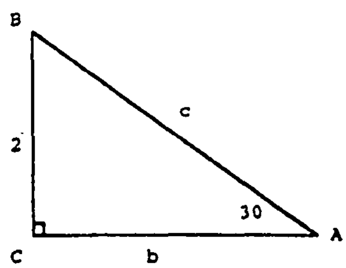
Can you use triometry to solve right triangles? Try it!

1. Solve the triangles using triometry.

a)



b)



2. Find the co-function identities.

a) $S(\pi/2 - \theta)$

b) $C(\pi/2 - \theta)$

c) $T(\pi/2 - \theta)$

Assignment

2/21/91

Choose one of the following and see what you can do with it. I want you to turn in your efforts and results next Thursday, Feb. 28. I will give hints and assistance when asked.

1) Find derivatives of $C(\theta)$, $T(\theta)$, $RT(\theta)$, $RS(\theta)$, $RC(\theta)$.

2) Find a formula for the area of any triangle (other than 1/2bh).

3) Define inverse functions of $S(\theta)$ and $C(\theta)$.

4) Wild card! - work on some other concept that interests you, but check with me first.

Appendix D

Triometry Evaluators and Evaluation Form

Cover Letter to Evaluators

Dear :

First of all, let me thank you for your willingness to assist me in this project. As I explained to you in our telephone conversation on (date), I am a doctoral student at UNC-Greensboro majoring in curriculum and teaching with emphasis on mathematics education. The focus of my dissertation is to develop and field test an original body of materials that I have created and that I call triometry.

A necessary part of the dissertation requires that the materials be evaluated by professional mathematics educators. This is where you come in. Please look over the triometry materials and then complete the evaluation form. The items in the criteria for evaluation are adapted primarily from NCTM's Curriculum and Evaluation Standards.

Enclosed is a copy of triometry, the evaluation form, a self-addressed, stamped envelope, and a cover sheet that will provide some background and the context in which triometry will be used.

Please return the completed evaluation in the enclosed envelope. It is not necessary to return the NT materials.

Thank you again for your help.

Sincerely yours,

Billie W. Goodman

Cover Sheet Sent to Evaluators

Background and Context in Which Triometry Will Be Used

For years on end, mathematics has been presented to students as a fixed body of knowledge to be absorbed and then regurgitated in like manner. There have been few if any opportunities for students to create their own mathematics and, thus, to experience mathematics as mathematicians do-- as a subject to be explored and discovered. NCTM's Curriculum and Evaluation Standards has proposed that students have more opportunities of this type. But the key to successful implementation of such curricular changes lies with the teachers. Hence, it is important that teachers have some experiences in exploring and creating mathematics. Quoting from the NCTM Professional Standards for Teaching Mathematics:

> Teachers need to explore mathematics and to conduct their own inquiries. Looking for patterns, making conjectures, constructing and evaluating arguments, and seeking generalizations should be an integral part of the mathematics content experience. Through such activities, teachers gain confidence in their ability to reason and justify their thinking and to make sense of mathematics. . . The struggles, the false starts, the informal investigations that lead to the elegant proof frequently are missing. Teachers need to construct mathematics for them-selves and not just experience the record of others' constructions (Working draft, 1989, pp. 71-72).

For a part of my dissertation, I have created a body of materials called triometry. Using trigonometry as a basis, I have formed definitions analogous to sine and cosine which are then used to propose and prove identities and properties paralleling those in trigonometry. The intent is not to teach this material but rather to use it as a springboard for engaging students in creating their own mathematics. Initially, students will be asked to create their own definitions of trigonometric functions. Students who come up with definitions that look promising even though they are different from those in the triometry materials will be encouraged to develop their ideas as far as they can. The intent will be to encourage and direct students into a parallel development of trigonometry without reference to the standard trigonometric functions.

The students involved in the study will be undergraduate mathematics/mathematics education majors who have at least had first semester calculus. The class will meet twice weekly for seven weeks and will carry one semester hour credit.

Evaluation Form

Evaluation of Triometry

This survey seeks to determine your views as to the degree to which the material New Trigonometry satisfies each of the following criteria. For each criteria circle the number that best indicates your opinions about it, from 1 (Disagree) up through 5 (Agree). Your comments and/or suggestions are also solicited.

	Disagree	Tend to Disagree	Neutral or no Opinion	Tend to Agree	Agree
1. Is a natural extension or variation of ideas already familiar to the student.	1	2	3	4	5
2. Reflects a large background of information.	1	2	3	4	5
3. Reveals the essential nature of mathematics.	1	2	3	4	5
4. Will allow opportunities for students to apply mathematics already known.	1	2	3	4	5
5. Allows for the investigation and exploration of ideas.	1	2	3	4	5
6. Allows for conjecturing, testing, and verification of conjectures.	1	2	3	4	5
7. Will be challenging to the student.	1	2	3	4	5
8. Is (or should be!) within capabilities of students.	1	2	3	4	5

Comments and/or suggestions:

Qualifications of Evaluators

The credentials of the mathematics educators who evaluated the triometry materials are outlined.

1. Professor A has an Ed. D. degree in mathematics education. She has seven years teaching experience at the college level. Her teaching duties are primarily in mathematics education.
2. Professor B has a Ph. D. in mathematics education and thirty years teaching experience at the university level. He is currently working with the State Department of Public Instruction in Georgia.
3. Professor C has a Ph. D. degree in mathematics education. He also has thirty years experience at the university level. He teaches courses in both mathematics and mathematics education.
4. Professor D has an Ed. D. degree in mathematics education and has eighteen years teaching experience at the secondary and university level. He teaches mathematics courses aimed at the prospective secondary mathematics teacher and conducts workshops for inservice mathematics teachers.

Appendix E

Questionnaires and Interviews

Questionnaire I

MATHEMATICS QUESTIONNAIRE

Name____

Age_____________ Classification_______________________

Major_______________________ Minor____________________

Approximate GPA in mathematics__________________________

1. Mathematics courses taken in high school:

2. Mathematics courses taken in college:

3. I selected a major/minor in mathematics/mathematics education because

4. What I like most about mathematics is _______ because

5. What I like least about mathematics is ________ because

6. What is mathematics? How would you describe mathematics to someone?

7. What is it that mathematicians do when they "do" math?

8. Think of your favorite math teacher. Why do you consider this person special as a teacher of mathematics?

9. My favorite area of mathematics is

10. How much trigonometry have you had?

11. I (circle one) like / dislike / indifferent about trigonometry because

12. If you could teach any area of mathematics, what area would you choose and why?

13. Have you ever "discovered" or "invented" a mathematical idea even though it may not have been original? If so, briefly describe.

Questionnaire II

Feb. '91 Questionnaire and Evaluation

Please respond to each of the following questions. Your honest opinions will be appreciated. I encourage you to refer to specific examples to illustrate your point whenever appropriate.

1. Describe mathematics as you see it and understand it.

2. What do mathematicians do when they do mathematics?

3. What have you learned from this course other than the content itself?

4. I would like for you to evaluate the content of this course by indicating which of the terms that are listed that you think are descriptive of all or part of triometry. Give reasons (and examples if possible) for your opinions.

   easy
   hard
   challenging
   trivial
   interesting
   boring
   other adjectives that you can think of

5. What did you like best about this course and why? What did you like the least and why?

6. As a means of introducing students to the idea of creating their own mathematics, are the triometry materials appropriate or inappropriate? Please explain.

7. Would you recommend this course to others? Why or why not?

8. Would you like other mathematics courses to contain similar aspects of exploration as part of the course? Why or why not?

9. Have your ideas about mathematics changed since the beginning of this course. If so, in what ways?

10. How would you rate your ability to do some creative mathematics as we have attempted to do and why? Is your present rating different from what it was at the beginning of this course? If so, in what ways?

11. Please give your over-all impression of this course.

12. Finally, what suggestions do you have about any aspect of this course for future use?

Interview I

These questions were the same for all of the students but were not necessarily asked in the order presented. Other questions were specific to the student being interviewed.

1. How would you describe mathematics to someone?
2. Tell me some of the things you do when you do mathematics?
3. What do mathematicians do when they do mathematics?
4. What do you think it means to make a conjecture?
5. Tell me what you know about inductive/deductive reasoning.
6. Tell me what you think about a mathematician creating his/her own rules and symbols.
7. Have you ever thought of proofs as being elegant? What does that mean to you?
8. Does the work of mathematicians involve more complex numerical calculations or abstract or symbolic thought?
9. Tell me what you think about insight or intuition in mathematics.

Interview II

The following questions, not necessarily asked in the order presented below, were used as a basis to explore students' perceptions of creativity. Other questions were posed from the student's responses.

1. What does it mean to be creative?
2. Are you a creative person? Why do you think so or think not?
3. What does it mean to be creative in mathematics?
4. What would be something creative in mathematics?
5. Are you creative in mathematics? Have you ever done anything creative in mathematics? Why do you think so or think not?
6. How would you rate your creative ability in mathematics?
7. What does it take to be creative in mathematics? What kind of people are creative in mathematics?
8. If two different people come up with the same idea, are they both being creative? Why or why not?
9. What if someone creates something that was actually done years earlier by another person? Was the second person just as creative as the first one? Why or why not?
10. Is solving an equation in a way different from what someone else might do a creative act? For example, a

simple equation like $7x + 9 = 11$. Subtract 9, then divide by 7 or divide by 7, then subtract 9?

Interview III

Forms of these questions were asked of all the students. The order varied.

1. How do you perceive what we are doing in class?
2. Are we doing mathematics?
3. What are some things you do when you do mathematics?
4. What are your feelings about the experience so far? Has it been enjoyable? Frustrating? Challenging, etc?
5. Do you feel that you are any better at creating mathematics than you were three weeks ago? Explain.
6. What has been the easiest for you? Hardest?
7. How do you feel about not having a textbook and precise exercises and examples to follow?

Interview IV

These questions were asked in various forms of all the participants. Again, the order sometimes varied.

1. Describe your experiences in the class so far. Has it been a worthwhile experience? Why or why not?
2. What has been a problem for you in trying to develop the ideas of triometry? What, if anything, has hampered your efforts? e.g.,
   a) not being familiar with trigonometry?
   b) not being familiar with algebra manipulations like splitting fractions?
   c) following proofs in trigonometry or calculus?
   d) being able to go beyond the proofs in trig or calculus to our system (triometry)?
3. Was the material within your capabilities? Explain.
4. Could you have done more with more effort or time? Explain.
5. When did the "fun" begin to wear off? What might we have done to overcome this?
6. What if you got into another math class and the professor says, "Class, we're going to develop our own _____ (e.g., geometry)." How do you think you would react? Would you take another class that was done similar to the way we have done triometry? That is, voluntarily?

7. If someone gave you an idea and some guidelines, like definitions, do you think you could strike out on your own? Why?
8. Have you learned any mathematics? Explain.
9. What have you learned <u>about</u> mathematics?
10. What have you learned about <u>doing</u> mathematics?

Appendix F

Criteria for Creative Behavior in Mathematics

Creative behavior in mathematics involves any of the following:

1. Selecting and combining that which is already existing (Koestler in Hall, 1978);
2. Reconstituting of something old to make something new (Barron in Hall, 1978);
3. Placing things in new perspectives so that one becomes aware of relationships not previously seen (Bruner in Hall, 1978);
4. Ability to toy with ideas (Rogers in Hall, 1978);
5. Recognizing a pattern (Hammer, 1964);
6. Making an analogy (Hammer, 1964);
7. Solving a problem in a unique way (Hammer, 1964).