Maximally Alpha-Like Operations

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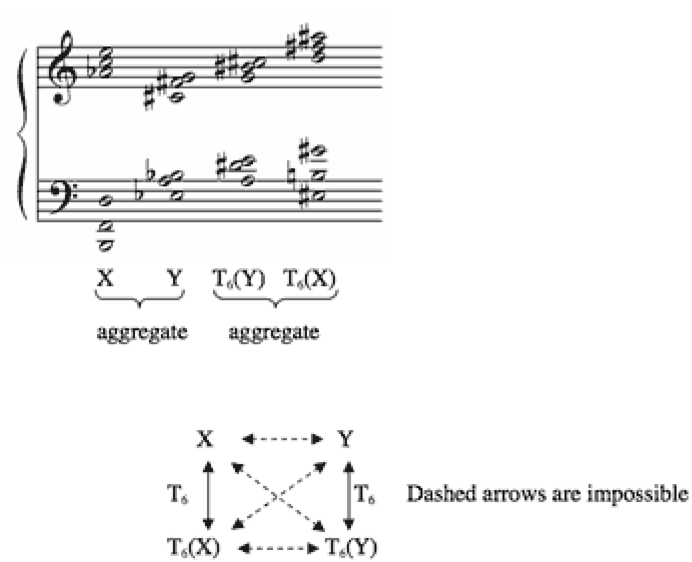
*****Note: Figures may be missing from this format of the document**

Abstract:
Any two Z-related set-classes will map onto one another under 1) TnM or TnMI, or 2) TnM or TnMI in tandem with Morris’s alpha operations, or 3) maximally alpha-like operations, the original contribution of the present paper. This brief “research notes” paper explores the theoretical formulation and analytical application of maximally alpha-like operations.
KEYWORDS: Maximally alpha-like operations, alpha, Z-relation, Z-pair, M-relation, TTOs, mappings, pcsets, transformational network

Article:

1] Example 1 shows an excerpt from a Dallapiccola song.(1) The excerpt contains four chords, labeled X, Y, T_6(Y), and T_6(X). The union of X and Y forms the pc aggregate, as does the union of T_6(Y) and T_6(X). The passage resists an overarching transformational network such as that at the bottom of Example 1 because there is no T_n, T_nI, T_nM, or T_nMI operation that will map the X and Y forms onto each other. The dashed arrows in the network represent this limitation.

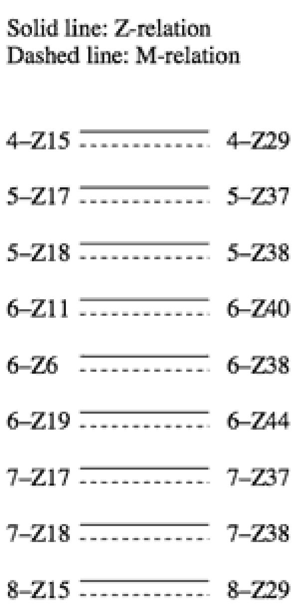
Example 1. Dallapiccola, *Quattro Liriche de Antonio Machado*, III (1948), mm. 80–85



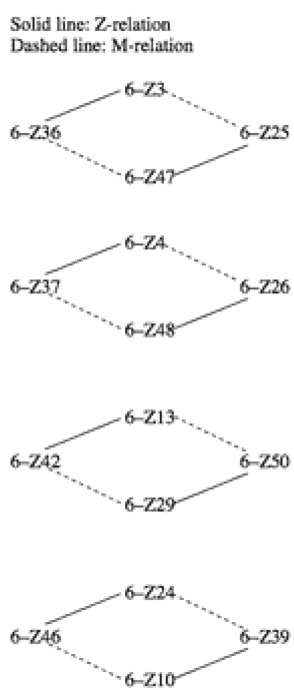
[2] The reason why X and Y cannot map onto one another is that they are *Z-related*.(2) However, not all *Z-pairs* (two Z-related scs) work this way. To explain, I shall divide the twenty-three Z-pairs (under the traditional equivalence operations T_n and T_nI) into three categories. Example 2 shows the first

category, *Z-related/M-related*. Here each sc maps under T_nM or T_nMI onto the other sc in the same Z-pair; the two scs are thus Z-related and M-related.[3] Example 3 shows the second category, *Z-related/M-variant*. Here each sc maps under T_nM or T_nMI onto a sc in a different Z-pair (thus the term "variant"). Example 4 shows the third category, *Z-related/M-invariant*. Here each sc in the Z-pair maps onto *itself* under T_nM or T_nMI (thus the term "invariant"). This is perhaps the most restrictive of the three categories, in that each sc can only map onto itself. The Z-pair in Example 1, 6–Z28/6–Z49, belongs to this category.[4]

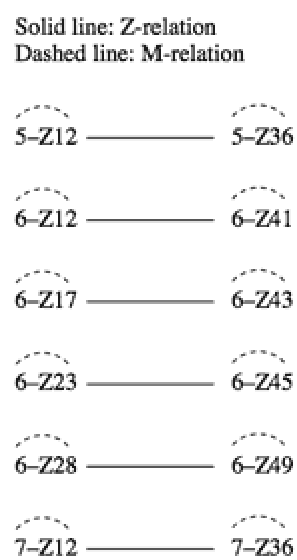
Example 2. Z-related/M-related scs



Example 3. Z-related/M-variant scs



Example 4. Z-related/M-invariant scs



[3] Robert Morris has noted that the Z-relation may appear or disappear depending on the canon of operations in use.[5] This is evident in Example 2, where scs in Z-pairs that do not relate by T_n or T_nI *do* relate by T_nM or

T_nMI. To this end, Morris develops a number of operations designed to erase the Z-relation. The most often cited of these operations is alpha (α), whose mappings are

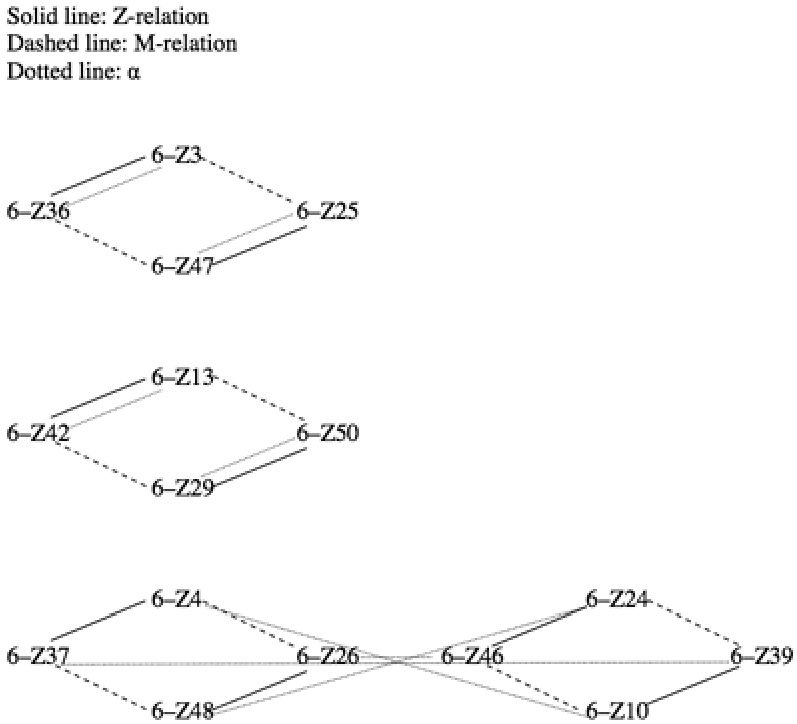
α1 = (01) (23) (45) (67) (89) (AB)
or
α2 = (12) (34) (56) (78) (9A) (B0).[6]

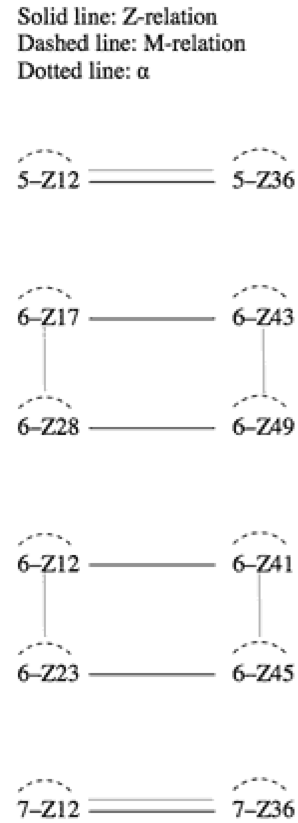
For α1, Ian Quinn notes, "each pc in the even whole-tone collection gets transposed up a semitone, and each pc in the odd whole-tone collection down a semitone."[7] For α2, each pc in the even whole-tone collection is transposed down a semitone, and each pc in the odd whole-tone collection is transposed up a semitone. Applying α1 to a pcset X may yield quite different results than applying α2 to X. For instance, if X = {012478}, a member of 6–Z17[012478], applying α1 to X yields {013569}, a member of sc 6–Z28[013569]. However, applying α2 to X yields {12378B}, another member of 6–Z17. The fact that 6–Z17 and 6–Z28 belong to the same category of Z-pairs (cf. Example 4) suggests that α may be of use in creating mappings for the Z-pairs in Examples 3 and 4.

[4] To test this hypothesis, Example 5 applies α to the scs in Example 3. The result is clear: α maps (the pcsets of) four of the eight Z-pairs onto their Z partners, thus erasing the Z-relation for these scs (6–Z3/6–Z36, 6–Z25/6–Z47, 6–Z13/6–Z42, 6–Z50/6–Z29). The four Z-pairs at the bottom of Example 5 do not map onto their Z-partners under α (6–Z4/6–Z37, 6–Z26/6–Z48, 6–Z24/6–Z46, 6–Z39/6–Z10). In like fashion, Example 6 applies α to the scs in Example 4. On the one hand, α resolves the Z-relations between 5–Z12/5–Z36, and between their abstract complements, 7–Z12/7–Z36. On the other hand, α turns the Z-related/M-*invariant* hexachords into a new set of Z-related/M-*variant* hexachords (the set is new because the variances differ from those in Examples 3 and 5). The upshot is that the Z-related/M-invariant hexachords are still unable to map onto their Z-partners.

Example 5. Adding α to the Z-related/M-variant scs

Example 6. Adding α to the Z-related/M-variant scs





[5] The success of α in resolving every Z-relation save for four Z-pairs in Example 5 and four Z-pairs in Example 6 prompts me to create *maximally α-like operations* for those Z-pairs.[8] By "maximally α-like," I am imagining operations whose cycles contain as many interval-class 1s (ic 1s) as possible, since the cycles of α consist of six ic 1s. The ic 1 cycles result in a "small" voice-leading distance between two α-related hexachords—no more than six ics of "work" are required to "move between" them.[9] As a result, maximally α-like operations will come as close as possible to six ics of work in relating hexachords. Ideally, a maximally α-like operation will contain 5 ic 1s, but we shall see that certain cases permit only 4 or even 3 ic 1s. The following sections explore maximally α-like operations in detail.

[6] Let us return to Example 1. There, X = {02458B} and Y = {13679A}. The maximally α-like operation

28 ↔ 49.1 = (01) (23) (47) (56) (89) (AB)

maps X onto Y and vice versa. The label "28 ↔ 49.1" indicates that this operation maps the 6–Z28 member X onto the 6–Z49 member Y and vice versa. ".1" indicates that this is the first of two operations that will map X onto Y and vice versa. 28 ↔ 49.1 is maximally α-like because its cycles contain five ic 1s—(01), (23), (56), (89), (AB)—and one ic 3—(47). Underlines indicate the non-ic 1 cycles.

[7] Example 7 lists a second maximally α-like operation

28 ↔ 49.2 = (09) (12) (34) (56) (78) (AB)

that also maps X onto Y and vice versa. 28 ↔ 49.2 also contains five ic 1s—(12), (34), (56), (78), (AB)—and one ic3—(09)—and is thus as α-like as 28 ↔ 49.1. In the abstract, the choice between 28 ↔ 49.1 and 28 ↔ 49.2 is essentially arbitrary, but in a specific musical context, factors such as instrumentation, register, and voicing may suggest one operation over another.

[8] Example 8 renotates the transformational network of Example 1, using 28 ↔ 49. Because the registral spacing of the piano chords does not correspond to either of the 28 ↔ 49 operations, I use the generic label 28 ↔ 49 as opposed to the more specific 28 ↔ 49.1 or 49.2. The 28 ↔ 49 operation allows us to assert the relations that were not possible in Example 1's network. By reading the network clockwise beginning from X, we follow the chronological procession of the hexachords in Example 1, <X, Y, T_6Y, T_6X>, and their respective transformations <28 ↔ 49, T_6, T_6 28 ↔ 49 T_6>.

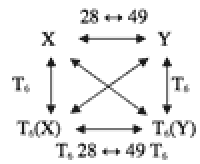
[9] A contextual factor in the definition of maximally α-like operations involves the two pcsets that will map onto one another. Up to this point, the 28 ↔ 49 operations have mapped X = {02458B} onto its *literal complement*, Y = {13679A}. However, to map X onto T1 of Y = {2478AB}, for example, it will not be possible to define a maximally α-like operation (1-to-1 and onto) since X and T1 of Y share common tones. A simple workaround involves retaining the already-defined 28 ↔ 49 operations, then transposing or inverting the resulting pcset. Because maximally α-like operations do not commute with T_n or T_nI, the initial choice of orthography must be adhered to. Throughout this paper, I use right-to-left orthography. For example,

Example 7. Two maximally α–like operations

28 ↔ 49.1	(01) (23) (47) (56) (89) (AB)	5 ic 1s, 1 ic 3
28 ↔ 49.2	(09) (12) (34) (56) (78) (AB)	5 ic 1s, 1 ic 3

(click to enlarge)

Example 8. Redo of the transformational network in Example 1 using 28 ↔ 49



Diagonal arrows:

X → T_6(Y) = T_6 28 ↔ 49 (right-to-left orthography: first 28 ↔ 49, then T_6)

T_6(Y) → X = 28 ↔ 49 T_6

Y → T_6(X) = T_6 28 ↔ 49

T_6(X) → Y = 28 ↔ 49 T_6

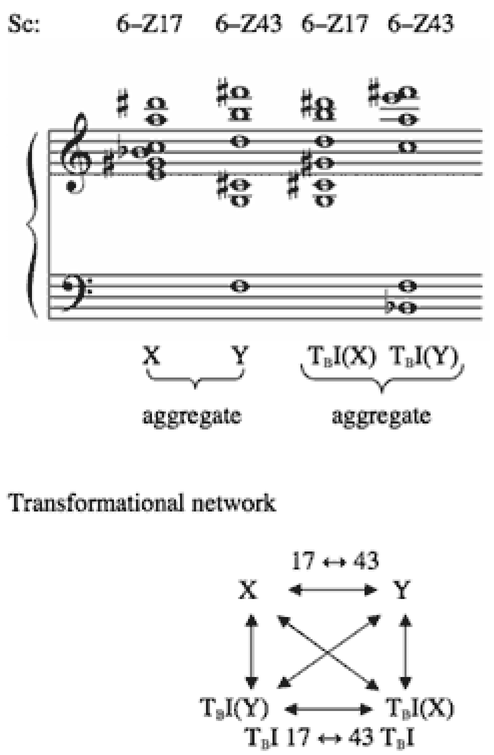
the compound operation T1 28 ↔ 49 maps X onto T1 of Y first through the application of 28 ↔ 49 to X (which maps X onto Y), and second through the application of T1 to Y.

[10] Having defined maximally α-like operations for 6–Z28/6–Z49, I now proceed to the Z-pair 6–Z17/6–Z43. Example 9 grounds the discussion with a passage from Carter's *Retrouvailles*. Like the Dallapiccola excerpt in Example 1, *Retrouvailles* features an opening chord X with its literal complement Y, followed by transformations of X and Y that form a second aggregate. Here X = {03489A} and Y = {12567B}, and the lone maximally α-like operation that maps X onto Y (and vice versa) is

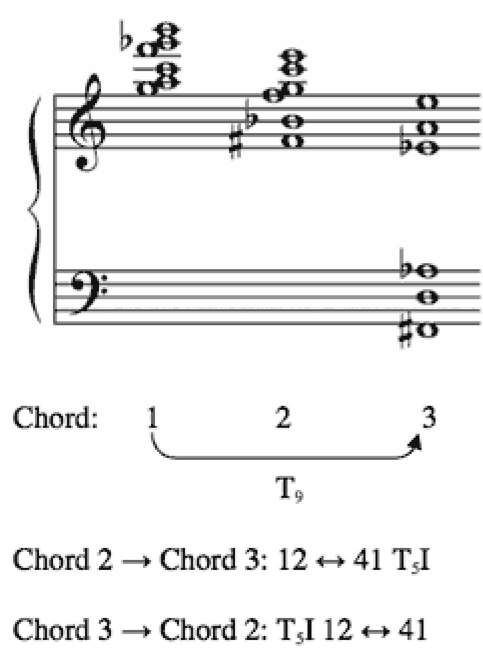
17 ↔ 43 = (01) (23) (45) (69) (78) (AB) (5 ic 1s, 1 ic 3)

This operation permits the transformational network at the bottom of Example 9, which strongly recalls the network in Example 8. By reading the Example 9 network clockwise beginning from X, we follow the chronological procession of the hexachords, <X, Y, T_BI(X), T_BI(Y)>.

Example 9. Carter, *Retrouvailles* (2000), mm. 5–10



Example 10. Webern, Op. 7, No. 2 (1910) (Forte 1990, 249)



[11] I now define the single maximally α-like operation for the Z-pair 6–Z12/6–Z41. Example 10 provides a musical context for the discussion, reproducing a passage that Allen Forte discusses in detail.[(10)] Forte observes

two transformational relations among the chords in Example 10: first, that chord 3 is T_9 of chord 1, and second, that chord 3 is T_5I of the literal complement of chord 2. The following operation formalizes Forte's second observation:

12 ↔ 41 = (03) (12) (45) (67) (8B) (9A) (4 ic 1s, 2 ic 3s).

Chord 2 is the 6–Z41 member {04567A} and chord 3 is the 6–Z12 member {234689}. 12 ↔ 41 maps {234689} onto its literal complement {0157AB} and vice versa. The arrows at the bottom of Example 10 indicate the T_9 relation from chord 1 to chord 3, and the T_5I/12 ↔ 41 relations between chords 2 and 3.[11]

[12] Example 11 grounds the discussion of the final pair of Z-related/M-invariant hexachords, 6–Z23/6–Z45, with a second passage discussed by Forte.[12] The passage contains an opening chord X = {02359B} followed by T_2 of X's literal complement, {03689A}. Because the chords share pcs, a 1-to-1 operation from one to the other is not possible. For this reason, I shall list the two maximally α-like operations that map X = {02359B} onto its literal complement {14678A}:

23 ↔ 45.1 = (07) (12) (34) (56) (89) (AB) (5 ic 1s, 1 ic 5)

and

23 ↔ 45.2 = (01) (27) (34) (56) (89) (AB) (5 ic 1s, 1 ic 5).

Example 12 lists maximally α-like operations for the remaining hexachords in Example 5.

Example 11. Stravinsky, "Sacrificial Dance" from *The Rite of Spring* (1921 edition), R3

[13] In this brief "research notes" paper, I have explored ways of mapping any Z sc onto its Z partner. For Z-related/M-related scs (Example 2), this is accomplished by T_nM or T_nMI. For four of the eight Z-related/M-variant Z-pairs (Examples 3 and 5) and two of the six Z-related/M-invariant Z-pairs (Examples 4 and 6), this is accomplished by a combination of α, T_nM, and/or T_nMI. Finally, for the remaining Z-related/M-variant hexachords (Example 5) and Z-related/M-invariant hexachords (Example 6), this is accomplished by the primary contribution of this paper, maximally α-like operations.

Example 12. Maximally α-like operations

Example 13. Maximally α-like operations for the remaining Z-pairs in Example 5

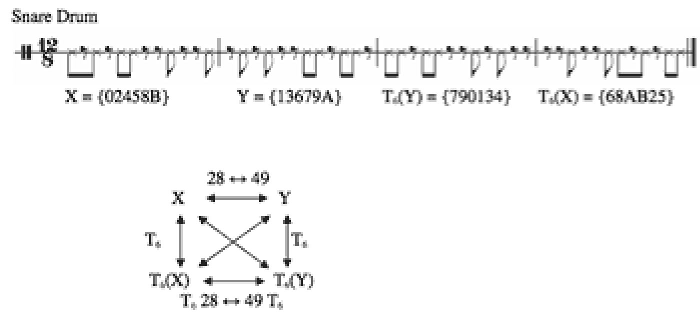
6–Z24/6–Z46
{012469} ∈ 6–Z46
{3578AB} ∈ 6–Z24
24 ↔ 46   (B0) (A1) (23) (45) (67) (89)   5 ic 1s, 1 ic3

6–Z26/6–Z48
{013578} ∈ 6–Z26
{2469AB} ∈ 6–Z48
26 ↔ 48.1   (B0) (12) (34) (56) (7A) (89)   5 ic 1s, 1 ic3
26 ↔ 48.2   (B0) (A1) (23) (45) (67) (89)   5 ic 1s, 1 ic 3
N.B.: 24 ↔ 46 and 26 ↔ 48.2 are identical

6–Z10/6–Z39
{013457} ∈ 6–Z10
{2689AB} ∈ 6–Z39
10 ↔ 39   (B0) (A1) (23) (49) (56) (78)   4 ic 1s, 1 ic 3, 1 ic 5

6–Z4/6–Z37
{012456} ∈ 6–Z4
{3789AB} ∈ 6–Z37
4 ↔ 37.1   (B0) (91) (A2) (34) (58) (67)   3 ic 1s, 2 ic 4s, 1 ic 3
4 ↔ 37.2   (B0) (A1) (23) (48) (59) (67)   3 ic 1s, 2 ic 4s, 1 ic 3

in beat-class space



Diagonal arrows:

X → T_6(Y) = T_6 28 ↔ 49 (right-to-left orthography)

T_6(Y) → X = 28 ↔ 49 T_6

Y → T_6(X) = T_6 28 ↔ 49

T_6(X) → Y = 28 ↔ 49 T_6

[14] There exist a number of avenues for future work with maximally α-like operations. I begin with spaces other than pc-space. First, maximally α-like operations can be defined for pitches in pitch-space, or beats in beat-class (bc) space. Bc-space is particularly fertile ground for the development of new operations since, to date, theorists have defined bcsets primarily in terms of T_n and T_nI.[(13)] Example 13 illustrates one such application, modeled on the 28 ↔ 49 operation (cf. §6 and Examples 7–8). The snare drum projects two mod-12 bc aggregates. First, X = {02458B} precedes its 28 ↔ 49 transformation, Y = {13679A}. Second, T_6 of Y = {790134} precedes T_6 of X = {68AB25}. The network in Example 13 is isographic with that in Example 8, and the passage in Example 13 is isographic in bc-space to the passage in Example 1 in pc-space.

[15] Returning to traditional pc-space, maximally α-like operations bear a number of similarities to models of fuzzy T_n and T_nI.[(14)] For the latter models, the benchmarks are the traditional "crisp" T_n and T_nI operations, and offset ("degrees of divergence") is measured from those cycles. In like fashion, maximally α-like operations measure offset from α by specifying the number and "size" of non-ic 1 ics.[(15)]

Appendix: Definitions

DEF 1: *Z-relation*: Two pcsets or scs are Z-related if they share an ic vector but do not relate by T_n and/or T_nI. The standard gauge of T_n/T_nI equivalence is assumed.

DEF 2: *Z-pair*: Two Z-related pcsets or scs ("Z-partners").

DEF 3: The two scs in a Z-pair are one of the following:

Z-related/M-related (M maps each sc in the Z-pair onto the other sc in the same Z-pair);

Z-related/M-variant (M maps each sc in the Z-pair onto a sc in a different Z-pair);

Z-related/M-invariant (M maps each sc in the Z-pair onto itself).

DEF 4: An *operation* is a mapping that is 1-to-1 and onto.

DEF 5: *Alpha* (α) is an operation whose cycles are α1 = (01) (23) (45) (67) (89) (AB) or α2 = (B0) (12) (34) (56) (78) (9A) (Morris 1982).

DEF 6: A *maximally α-like operation* is an operation whose cycles mimic those of α as closely as possible by containing the maximal number of ic 1 cycles. An example is (01) (23) (<u>47</u>) (56) (89) (AB). Underlines indicate non-ic 1 cycles.

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References

1. Buchler 2000, 52–3 discusses the chords in Example 1 in connection with interval cycles.

2. They are also *ZC-related* (Morris 1982, 103), but the ZC-relation is not required for the present paper’s agenda.

3. The status of T_nM and T_nMI as equivalence operators *on par with* T_n and T_nI is controversial since T_nM or T_nMI exchanges ic 1 and ic 5 content (Winham 1970, 281–2, Morris 1987, 148, Morris 2001, 52). This can lead to drastically different “equivalent” pcsets, such as {012345} and {024579} (chromatic to diatonic). In the present paper, however, the sets under discussion are Z-pairs, whose ic vectors are identical, thereby rendering this criticism moot. Winham 1970, 282, defends T_nM and T_nMI, stating, “it would not even be correct to say

without qualification that I is a 'closer' relation than M5 or M7. For while M5 preserves the intervals 3 and 9 while complementing 2, 4, 8, and 10, and M7 does the opposite, I complements all of these and preserves none; so in that one sense I is the least 'close'." Nonetheless, while T_nM and T_nMI do not change the ic content of a Z-pair, they do change the larger subsets embedded in each sc.

4. Morris 1982, 102–9, provides pertinent commentary.

5. Again see Morris 1982, 102–9.

6. Morris 1982, 115. Morris provides further applications of α in Morris 1990, 223–30 and Morris 1997, 304–6. Applications of α by other scholars include Lewin 1995, 103 ff., Mead 1989, 224 ff., and Quinn 2004, 36–8.

7. Quinn 2004, 36.

8. Morris 1982 develops operations other than α that change the mappings among Z-partners, but notes that the only way to address 6–Z17/6–Z43 and 6–Z28/6–Z49 is to create a system of equivalence in which the fifty T_n/T_nI hexachordal scs collapse into three scs (129–31). This system is not in widespread use.

9. The notion of "ics of work" comes from Lewin 1998 and Alegant 2001, 11.

10. Forte 1990, 247–9.

11. No maximally α-like operation whose cycles contain five ic 1s exists for 12 ↔ 41.

12. Forte 1973, 148.

13. Babbitt 1962, Lewin 1987, 23, Morris 1987, 299–305, Cohn 1992. Such "isomorphisms" between pitch and rhythm have long been controversial; a recent critique appears in London 2002.

14. See, most recently, Straus 2005, 45–50.

15. Thanks to Jonathan Salter for writing a computer program to calculate maximally α-like operations, Igor Erovenko (Department of Mathematics and Statistics, UNCG) for help with matters mathematical, and Clifton Callender, J. Daniel Jenkins, Evan Jones, Rachel Mitchell, Robert Morris, Robert Peck, Jonathan Pieslak, Adam Ricci, Caleb Smith, and the anonymous *MTO* readers for their suggestions.