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Perhaps the most important function in all mathematics is the Riemann Zeta function. For almost 150 years Mathematicians have tried to understand the behavior of the function's complex zeros. Our main aim is to investigate properties of the Riemann Zeta Function and Hurwitz Zeta Functions, which generalize the Riemann Zeta Function. The main goal of this work is to approach this problem from a traditional and computational approach. We aim to investigate derivatives of Zeta functions by exploring the behavior of its fractional derivatives and its derivatives, which has not been sufficiently examined yet.

RESULTS ABOUT FRACTIONAL DERIVATIVES OF ZETA FUNCTIONS

by

Ricky E. Farr

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Approved by

_____

Committee Chair

For my father, who taught me how to think about the world

and my mother, who taught me how to live in it.

For my wife, who taught me how to love

and my daughter, who taught me how to cherish it.

APPROVAL PAGE

This dissertation written by Ricky E. Farr has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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A major thank you is in store for Lauren, my wife, and our beautiful little girl Mackenzie. Thank you for allowing me to accomplish my dreams, and pushing me to do so. I am so lucky to have a wife and child who love me so much. Thank you, Babydoll (Lauren), for allowing me to work, when this meant long nights of not having me home or having to take care of Mackenzie alone. Thank you, Mackenzie, for allowing Daddy to work, although I know you wanted to watch Peppa Pig or play. God truly blessed me when he gave me both of you.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Perhaps the most important function in all mathematics is the Riemann Zeta function. For almost 150 years Mathematicians have tried to understand the behavior of the function's complex zeros. Our main aim is to investigate properties of the Riemann Zeta Function and Hurwitz Zeta Functions, which generalize the Riemann Zeta Function.

The main goal of this work is to approach this problem from a traditional and computational approach. We aim to investigate derivatives of Zeta functions by exploring the behavior of its fractional derivatives and its derivatives, which has not been sufficiently examined yet.

For the traditional approach, our goal is to use these data sets to visually find patterns between the zeros of higher derivatives of Zeta functions. Using this visualization to our advantage, we aim to use it to guide new theoretical results. Another goal is to generalize certain well-known results regarding the distribution of zeros of the Riemann Zeta function and its higher derivatives in the complex plane.

1.1 Fractional Derivatives

Leibniz invented the notation $\frac{d^n y}{dx^n}$ to denote the nth derivative of y. This notation prompted L'Hospital to ask Leibniz, "What if n be 1/2?". Leibniz responded in 1695 [22] with, "It will lead to a paradox, from which useful consequences will be drawn, because there are no useless paradoxes."

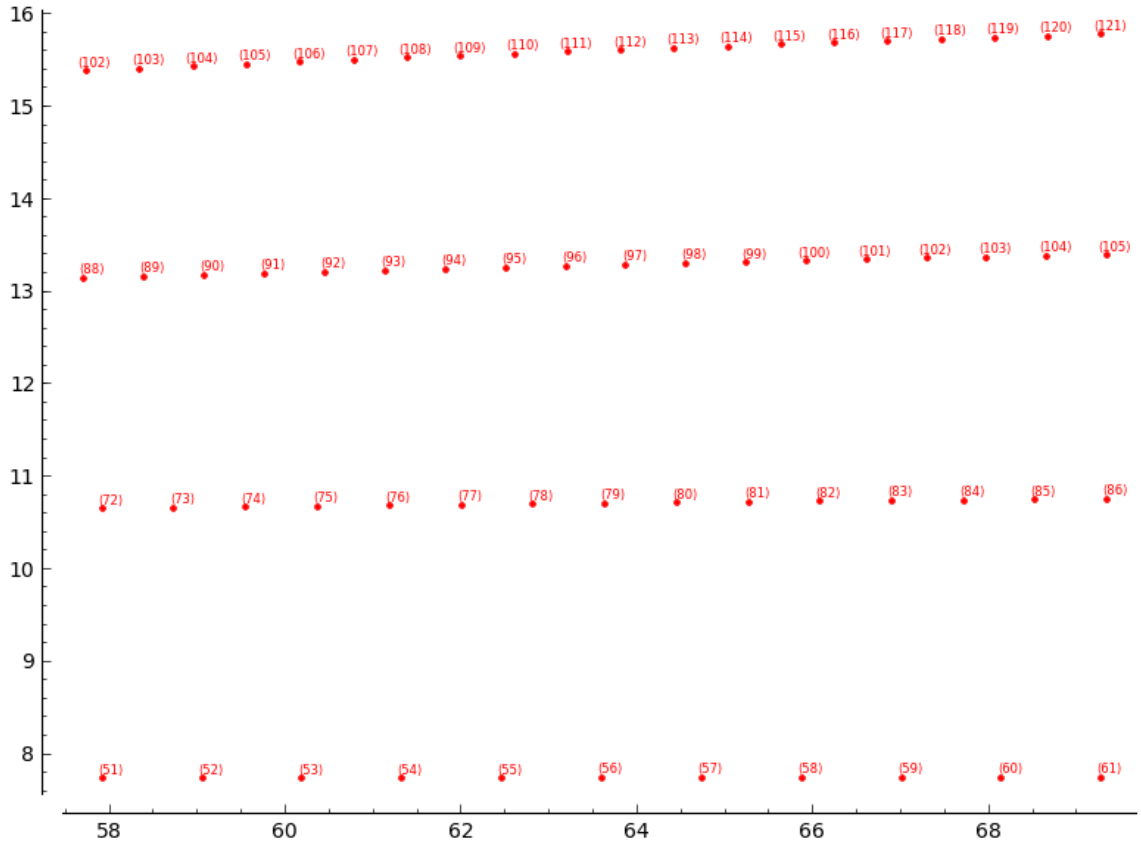

Figure 1. Zeros of derivatives of the Riemann Zeta function where $\bullet^{(k)}$ denotes a zero of the k-th derivative.

Since this time, fractional calculus has attracted the attention of many great mathematicians such as Abel [1], Riemann [31], and Liouville [25, 26].

In [27], the authors define the fractional derivative operator as any extension of the familiar differentiation operator D^n to arbitrary (integer, rational, or complex) values of n. Using this as our definition of a fractional derivative operator, we now motivate the extension of the fractional derivative operator that we seek. Intuitively, the fractional derivative of a function is well understood but not explicitly formulated.

To formulate the idea, we describe what is desired. Throughout this discussion we assume the principle branch of the complex logarithm.

For every function $f(z)$ (belonging to some class of functions) and every $\alpha \in \mathbb{C}$, we wish to assign a new function $D_z^\alpha[f(z)]$ subject to the following criteria:

(1) If $f(z)$ is an analytic function of a complex variable z, then $D_z^\alpha[f(z)]$ is an analytic function of α and z.

(2) The operation $D_z^\alpha[f(z)]$ must produce the same result as ordinary differentiation when α is a positive integer.

(3) The operation of $D_z^0[f(z)]$ leaves the f unchanged. That is, $D_z^0[f(z)] = f(z)$.

(4) The operation of $D_z^\alpha[f(z)]$ is linear. That is, for arbitrary $a, b \in \mathbb{C}$, $D_z^\alpha[af(z) + bg(z)] = aD_z^\alpha[f(z)] + bD_z^\alpha[g(z)]$.

(5) The law of exponents holds. That is, $D_z^\alpha\left[D_z^\beta\left[f(z)\right]\right] = D_z^{\alpha+\beta}\left[f(z)\right]$.

Other criteria could be added to this list, but these are generalizations of some of the most basic properties of integer order differentiation.

It should be noted that the differentiation operator that meets the above criteria is not necessarily unique. Perhaps the most natural definition of fractional differentiation was initiated by Grünwald in 1867 [16] and rigorously examined in 1868 by Letnikov [23]. We have found that the Grünwald-Letnikov fractional derivative is suited best for our purposes. We introduce it in Chapter III.

In Figures 1 and 2 we illustrate how connections between values, in this example the zeros of higher derivatives of the Riemann Zeta function, become clearer with the use of fractional derivatives. Figure 1 suggests a zero of the k-th derivative of the

Riemann Zeta function corresponds to a zero of its $(k+1)$-st derivative. Including the zeros of the fractional derivative makes the correspondence more concrete (Figure 2).

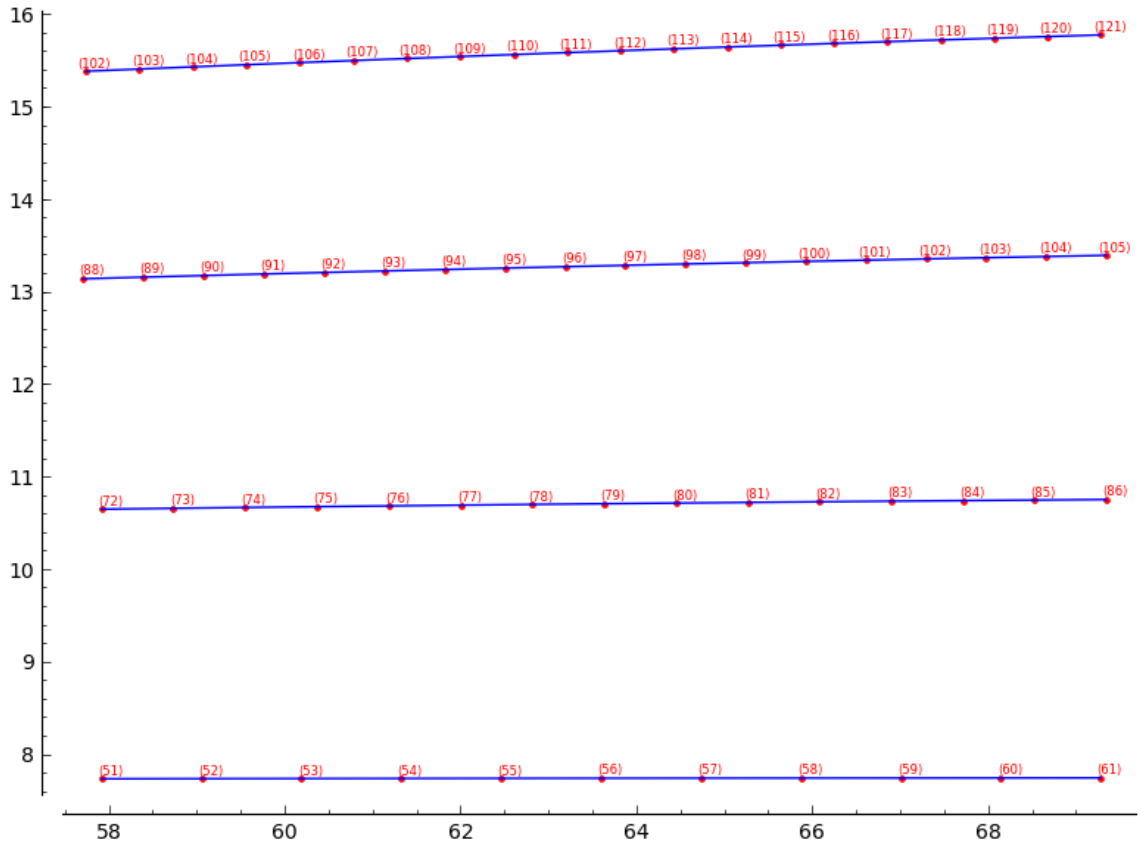

Figure 2. Zeros of derivatives and fractional derivatives of the Riemann Zeta function where $\bullet^{(k)}$ denotes a zero of the k-th derivative and the zeros of the fractional derivatives are on the curves.

1.2 Overview

In this thesis we cover four topics concerning the Riemann Zeta function or one of it's generalizations, namely the Hurwitz Zeta functions and the derivatives or fractional derivatives of these functions.

In Chapter II we present the zeros of the derivatives, $\zeta^{(k)}(\sigma + it)$, of the Riemann Zeta function for $k \leq 28$ with $-10 < \sigma < \frac{1}{2}$ and $-10 < t < 10$. Our computations show an interesting behavior of the zeros of $\zeta^{(k)}$, namely they seem to lie on curves which are extensions of certain chains of zeros of $\zeta^{(k)}$ that were observed on the right half plane. This is joint work with Sebastian Pauli appeared in [11].

The remaining chapters contain orginal results of the author that have been written up in collaboaration with Sebastian Pauli and Filip Saidak.

In Chapter III we discuss the fractional (or non-integral generalized) Stieltjes constants $\gamma_\alpha(a)$ arising naturally from the Laurent series expansions of the fractional derivatives of the Hurwitz Zeta functions $\zeta^{(\alpha)}(s, a)$. See Figures 3 and 4. We prove that if one defines $h_a(s) := \zeta(s, a) - 1/(s-1) - 1/a^s$ and $C_\alpha(a) := \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$, then $C_\alpha(a) = (-1)^{-\alpha} h_a^{(\alpha)}(1)$, for all real $\alpha \geq 0$, where $h^{(\alpha)}(x)$ denotes the α-th Grünwald-Letnikov fractional derivative of the function h at x. This result confirms the conjecture of Kreminski [21], originally stated in terms of the Weyl fractional derivatives. In article form this chapter is [12].

In Chapter IV we discuss methods of evaluation of non-integral generalized Stieltjes constants $\gamma_\alpha(a)$, arising naturally from the Laurent series expansions of the fractional derivatives of the Hurwitz Zeta functions $\zeta^{(\alpha)}(s, a)$. We give upper bounds for $C_\alpha(a) = \gamma_\alpha(a) - \log^\alpha(a)/a$ for $1 < \alpha$.

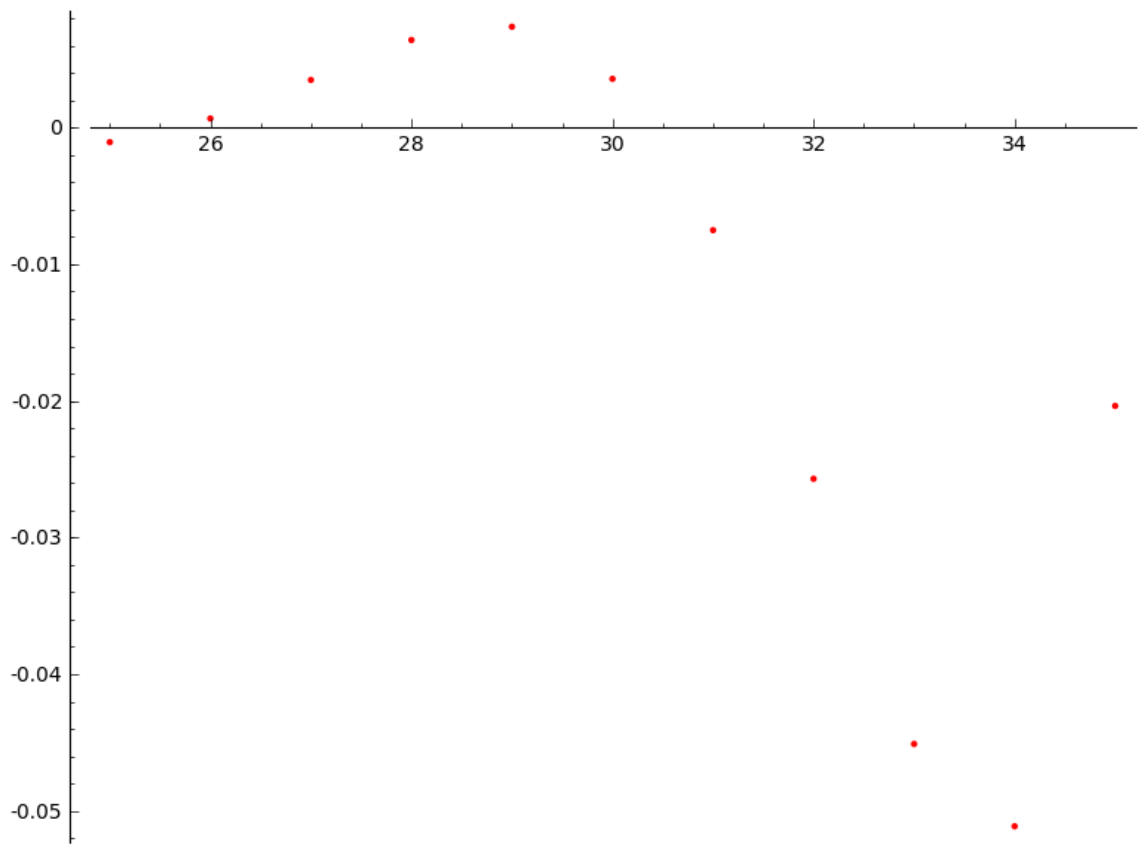

Figure 3. The (integral) Stieltjes constants $\gamma_m(1)$ for $25 \leq m \leq 35$.

Evaluation of our bound and previously known bounds for $\gamma_n(\alpha)$ for $n \in \mathbb{N}$ suggests that our upper bound is lower than known bounds for $n > 100$. In article form this chapter is [13].

In Chapter V we present a zero free region about 1 of the fracional derivarives of the Riemann Zeta function. For any $\alpha \in \mathbb{R}$, we denote by $D_s^\alpha[\zeta(s)]$ the α-th Grünwald-Letnikov fractional derivative of the Riemann Zeta function $\zeta(s)$. We prove that inside the region $|s-1| < 1$,

$$D_s^\alpha[\zeta(s)] \neq 0.$$

This result is proven by a careful analysis of integrals involving Bernoulli polynomials and bounds for fractional Stieltjes constants. In article form this chapter is [14].

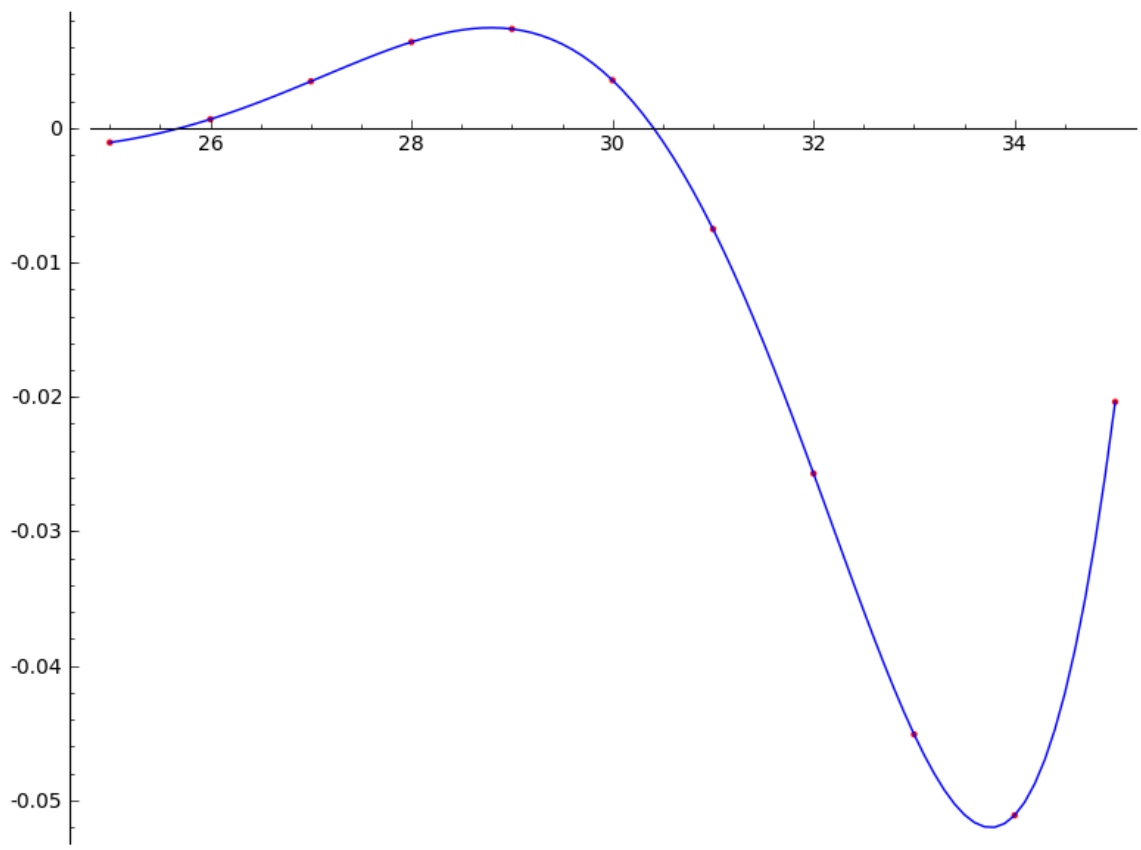

Figure 4. The fractional Stieltjes constants $\gamma_\alpha(1)$ for $25 \leq \alpha \leq 35$ with the integral Stieltjes constants (●).

CHAPTER II

MORE ZEROS OF THE RIEMANN ZETA FUNCTION ON THE LEFT HALF PLANE

Let $s \in \mathbb{C}$. We denote the real part of s by σ and the imaginary part of s by t. For $\sigma > 1$ the Riemann Zeta function ζ can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.1}$$

By analytic continuation, ζ may be extended to the whole complex plane, with the exception of the simple pole $s = 1$. This analytic continuation is characterized by the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s} \cos \frac{\pi s}{2} \Gamma(s)\zeta(s). \tag{2.2}$$

It follows directly from the functional equation (2.2) that $\zeta(-2j) = 0$ for all $j \in \mathbb{N}$. These zeros are called the real or trivial zeros of ζ. By the Riemann hypothesis, the remaining (non-trivial) zeros of ζ are of the form $\frac{1}{2} + it$.

In this paper we numerically investigate the distribution of zeros of the derivatives $\zeta^{(k)}$ of ζ on the left half plane. The results of our computations, that considerably expands the list of previously published zeros [34, 40], can be found in Tables 1, 2, and 3. For the rectangular region $-10 < \sigma < \frac{1}{2}$ and $|t| < 10$, Table 1 contains the number of zeros of $\zeta^{(k)}$, its real zeros, and its zeros with $0 < \sigma < \frac{1}{2}$. Tables 2 and 3 contain non-real zeros with $\sigma < 0$ in that region. We find that some of the conjectured

chains of zeros of the derivatives on the right half plane [6, 32] (see Figure 5) appear to continue to the left half plane which is illustrated in Figure 7.

We first recall results about the distribution of the zeros of $\zeta^{(k)}$ on the right half plane (see Section 2.1) and the left half plane (see Section 2.2). Section 2.3 contains a description of methods we used to evaluate $\zeta^{(k)}$. It is followed by a discussion of the methods that we used to find the zeros of $\zeta^{(k)}$ in Section 2.4.

2.1 Zeros on the Right Half Plane

Assuming the Riemann Hypothesis, the non-real zeros of ζ are all on the critical line $\sigma = \frac{1}{2}$, while the non-real zeros of $\zeta^{(k)}$ appear to be distributed mostly to the right of the critical line with some outliers located to its left.

Zeros with $0 < \sigma < \frac{1}{2}$

Speiser related the Riemann Hypothesis to the distribution of zeros of the first derivative.

Theorem 2.1 (Speiser [33]). *The Riemann Hypothesis is equivalent to* $\zeta'(s)$ *having no zeros in* $0 < \sigma < \frac{1}{2}$.

A simpler and more instructive proof of this result was given by Levinson and Montgomery [24]. They also proved, assuming the Riemann Hypothesis, that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros with $\sigma < \frac{1}{2}$, for $k \geq 2$.

Theorem 2.2 (Yıldırım [40]). *The Riemann Hypothesis implies that* $\zeta''(s)$ *and* $\zeta'''(s)$ *have no zeros in the strip* $0 \leq \sigma \leq \frac{1}{2}$.

The Riemann Hypothesis also implies that $\zeta^{(k)}$ for $k > 0$ has only finitely many zeros in $0 \leq \sigma \leq \frac{1}{2}$ [24].

Our computations show that higher derivatives have zeros in this strip, see Table 1. Because of the distribution of the zeros of $\zeta^{(k)}$ in Figure 6, we expect that the zeros listed in the table are the only zeros of $\zeta^{(k)}$ for $k \leq 32$.

Zeros with $\sigma > \frac{1}{2}$

The real parts of the zeros of $\zeta^{(k)}$ can be effectively bounded from above by absolute constants. For ζ' and ζ'' Skorokhodov [32] gives the bounds:

$$\zeta'(\sigma + it) \neq 0 \quad \text{for} \quad \sigma > 2.93938,$$

$$\zeta''(\sigma + it) \neq 0 \quad \text{for} \quad \sigma > 4.02853.$$

For $k \geq 3$ such general upper bounds were given by Spira [34] and later improved by Verma and Kaur [38]:

$$\zeta^{(k)}(\sigma + it) \neq 0 \quad \text{for} \quad \sigma > q_2 k + 2,$$

where q_2 is given by the formula

$$q_M = \frac{\log\left(\frac{\log M}{\log(M+1)}\right)}{\log\left(\frac{M}{M+1}\right)}.$$

Spira [34] computed zeros of the first and second derivative of $\zeta(s)$ for $0 < t < 100$ and noticed that they occur in pairs. Skorokhodov [32] went further in his computations and noticed that the zeros of derivatives of ζ seem to form chains, that is for each zero $z^{(k)}$ of $\zeta^{(k)}$ there seems to be a corresponding zero $z^{(k+1)}$ of $\zeta^{(k+1)}$. Indeed, for sufficiently large k the existence of these chains is a direct consequence of the following theorem.

Table 1. The number of zeros of $\zeta^{(k)}(\sigma+it)$ with $k \leq 32$ in $-10 < \sigma < 0$, $|t| < 10$, the number of complex conjugate pairs of non-real zeros, and the number of real zeros in this region. Furthermore, the real zeros in this region and the zeros in the strip $0 < \sigma < \frac{1}{2}$, $|t| < 10$ are given. The zeros are rounded to 4 decimal digits.

k	# of zeros of $\zeta^{(k)}(\sigma+it)$ $-10<\sigma<0$			zeros of $\zeta^{(k)}(\sigma+it)$ $-10<\sigma<0$				$0<\sigma<1/2$
	$\lvert t\rvert<10$	$0<t<10$	$t=0$	$t=0$				$\lvert t\rvert<10$
0	4	0	4	-2	-4	-6	-8	
1	3	0	3	-2.7173	-4.9368	-7.0746		
2	5	1	3	-3.5958	-6.0290	-8.2786		
3	5	2	3	-4.7157	-7.2920	-9.6047		
4	6	2	2	-6.1265	-8.7016			
5	5	2	1	-7.7119				$0.2876 \pm 4.6944i$
6	7	2	3	-4.3284	-6.6083	-9.3445		
7	8	3	2	-5.6191	-8.4425			
8	7	3	1	-7.5186				$0.4183 \pm 5.4753i$
9	9	3	3	-4.7059	-6.5553	-9.3794		
10	10	4	2	-5.7309	-8.5500			
11	9	4	1	-7.7120				$0.4106 \pm 6.1502i$
12	11	4	3	-5.1849	-6.8533	-9.6751		
13	12	5	2	-6.1124	-8.9100			
14	11	5	1	-8.1400				$0.3447 \pm 6.7636i$
15	12	5	2	-5.6697	-7.3600			
16	14	6	2	-6.6469	-9.4393			
17	13	6	1	-8.7229				$0.2494 \pm 7.3344i$
18	14	6	2	-6.1556	-8.0019			
19	15	7	1	-7.3040				
20	15	7	1	-9.4151				0.1378 ± 7.8732
21	16	7	2	-6.6561	-8.7394			
22	17	8	1	-8.0675				
23	16	8	0					$0.0163 \pm 8.3861i$
24	18	8	2	-7.1929	-9.5491			$0.4681 \pm 8.7645i$
25	19	9	1	-8.9089				
26	20	9	2	-7.3618	-8.2504			
27	19	9	1	-7.8131				$0.3116 \pm 9.244i$
28	21	10	1	-9.8049				
29	22	10	2	-7.7492	-9.1919			
30	21	10	1	-8.6103				$0.1516 \pm 9.7083i$
31	22	11	0					
32	23	11	1	-8.2087				

Theorem 2.3 (Binder, Pauli, Saidak [6]). *Let $M \geq 2$ be an integer and let u be a solution of $1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) \geq 0$, that is, $u \geq 1.1879\ldots$. If $k > \frac{u(2M+3)}{q_M - q_{M+1}}$, then for each $j \in \mathbb{Z}$ the rectangular region R, consisting of all $s = \sigma + it$ with*

$$q_M k - (M+1)u < \sigma < q_M k + (M+1)u \tag{2.3}$$

and

$$\frac{2\pi j}{\log(M+1) - \log(M)} < t < \frac{2\pi(j+1)}{\log(M+1) - \log(M)}, \tag{2.4}$$

contains exactly one zero of $\zeta^{(k)}$. This zero is simple.

So, given $M \geq 2$, $j \in \mathbb{Z}$, and $l > \frac{u(2M+3)}{q_M - q_{M+1}}$ for the zero of $\zeta^{(l)}$ in the region determined by (2.3) and (2.4) for $k = l$ there is a corresponding zero of $\zeta^{(l+1)}$ in the region determined by (2.3) and (2.4) for $k = l + 1$. Figure 5 illustrates the phenomenon of the chains of zeros of derivatives of ζ. The zeros shown in the chains labeled $M = 2, j = 0$ and $M = 2, j = 1$ are in the rectangular regions from Theorem 2.3 and the zeros in the chain labeled $M = 3, j = 1$ are in the regions for $M = 3$ and $j = 1$ starting at the 77th derivative. The other chains are labeled by the parameters M and j of the regions into which higher derivatives in the chains eventually fall farther to the right.

2.2 Zeros on the Left Half Plane

It follows immediately from the functional equation (2.2) that $\zeta(s) = 0$ for $s = -2n$ where $n \in \mathbb{N}$. The zeros of the first derivative are exactly the zeros postulated by the theorem of Rolle.

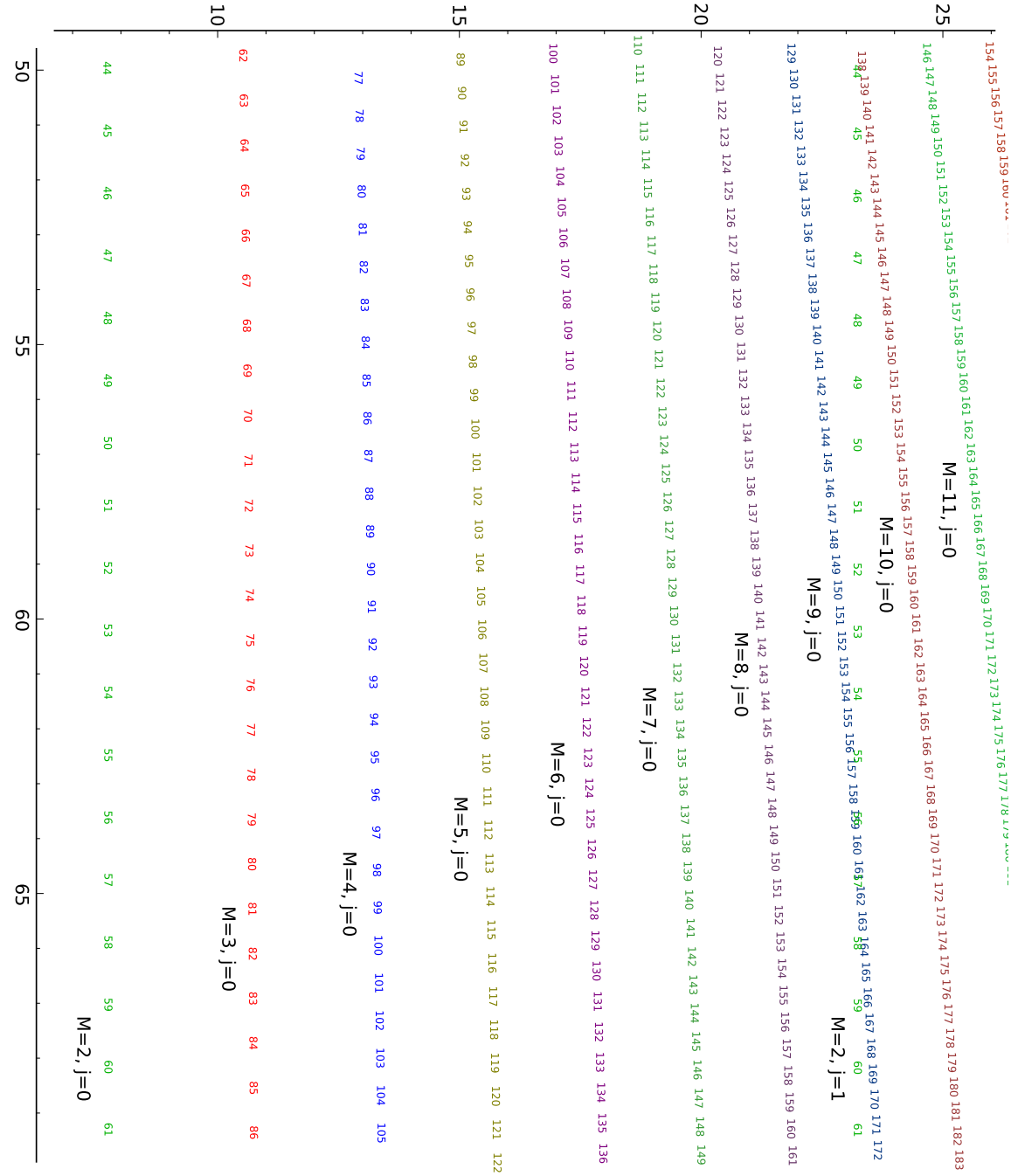

Figure 5. The zeros of $\zeta^{(k)}(\sigma+it)$ for $50 < \sigma < 70$, $0 < t < 26$, where k denotes a zero of $\zeta^{(k)}$. The conjectured chains of zeros are labeled by M and j (compare Theorem 2.3).

Theorem 2.4 (Levinson and Montgomery [24]). *For $n \geq 2$ there is exactly one zero of ζ' in the interval $(-2n, -2n+2)$ and there are no other zeros of ζ' with $\sigma \leq 0$.*

Unlike on the right half plane, on the left half plane there is no general (left) bound for the non-real zeros of $\zeta^{(k)}$. Spira showed:

Theorem 2.5 (Spira [35]). *For $k > 0$ there is an α_k so that $\zeta^{(k)}$ has only real zeros for $\sigma < \alpha_k$, and exactly one real zero in each open interval $(-1-2n, 1-2n)$ for $1-2n < \alpha_k$.*

The location of a zero of the second derivative on the left half plane shows up in [34]. For both $\zeta''(s)$ and $\zeta'''(s)$ Yıldırım [40] proved the existence of exactly one pair of conjugate non-trivial zeros with $\sigma < 0$ and gave their location.

Theorem 2.6 (Levinson and Montgomery [24]). *If $\zeta^{(k)}$ has only a finite number of non-real zeros in $\sigma < 0$ then $\zeta^{(k+1)}$ has the same property.*

Hence, the absolute value of the non-real zeros of $\zeta^{(k)}$ on the left half plane can be bounded. This can be done by iteratively generalizing Yıldırım's methods for the second and third derivatives to higher derivatives.

Table 2 contains all the zeros of $\zeta^{(k)}(\sigma + it)$ with $-10 < \sigma < 0$, $0 < |t| < 10$ for $2 \leq k \leq 29$. The patterns of the distribution of zeros in Figure 6 suggest that these are all the zeros for these derivatives on the left half plane.

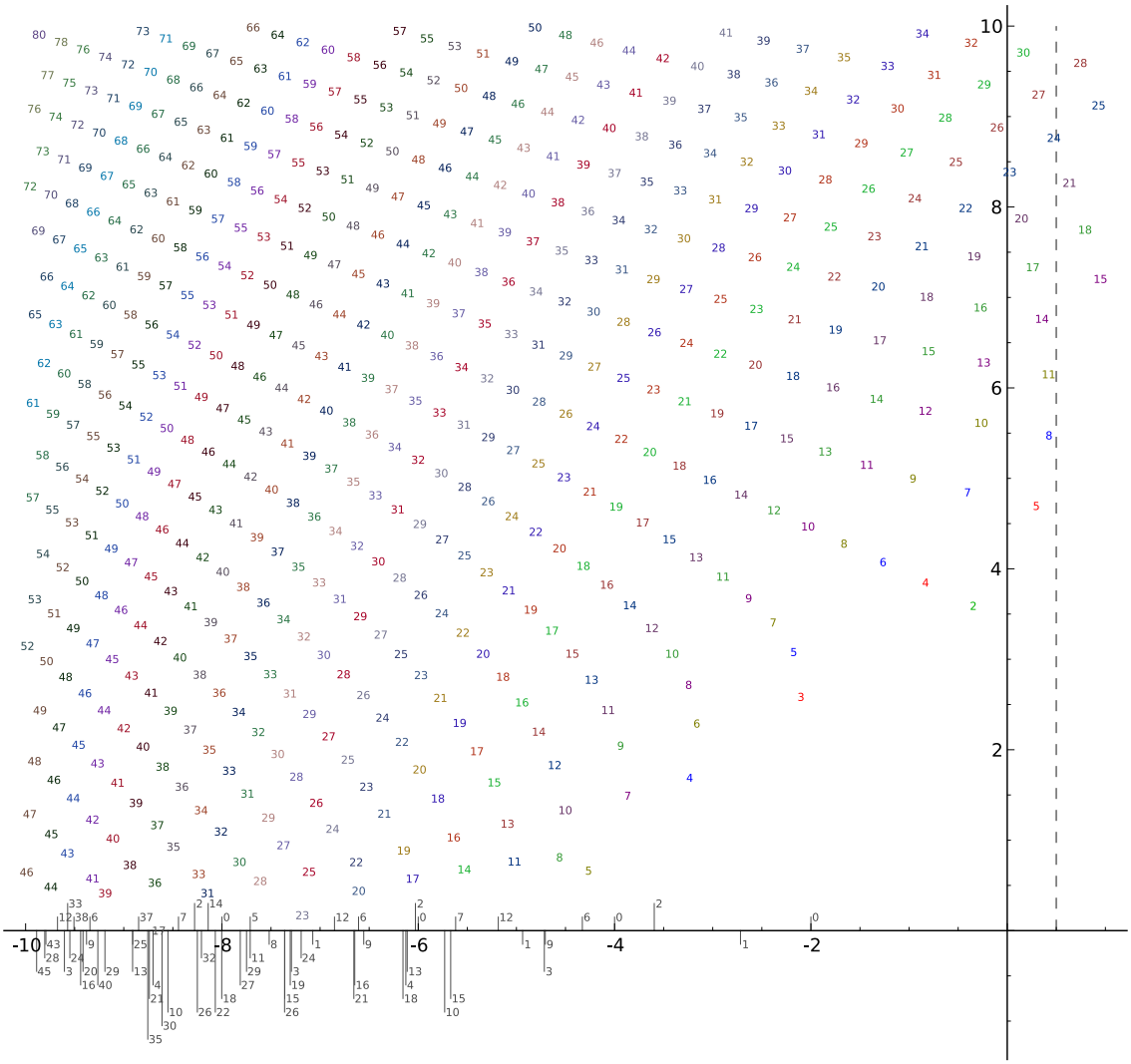

Figure 6. The zeros of $\zeta(\sigma + it)$ and its derivatives $\zeta^{(k)}(\sigma + it)$ for $k \leq 80$ in $-10 < \sigma < 1$, $0 < t < 10$, where 0 denotes a zero of ζ and k denotes a zero of $\zeta^{(k)}$. All zeros shown are simple.

2.3 Evaluating $\zeta^{(k)}$ on the Left Half Plane

Methods for evaluating ζ and $\zeta^{(k)}$ include Euler-Maclaurin summation (see, for example [10]) or convergence acceleration for alternating sums [7]. Implementations

for the evaluation of ζ can be found in various computer algebra systems. The Python library mpmath [18] contains functions for evaluating derivatives of Hurwitz Zeta functions, and thus $\zeta^{(k)}$, on the right half plane using Euler-Maclaurin summation.

We considered two different approaches for evaluating $\zeta^{(k)}$ in the left half plane. Because of speed and ease of implementation we use Euler-Maclaurin summation rather than the derivatives of the functional equation (see [4] for formulas for these). Using Euler-Maclaurin summation we obtain for $\sigma = \Re(s) > 1$ that

$$\begin{aligned}
(-1)^k \zeta^{(k)}(s) &= \sum_{n=2}^{\infty} \frac{\log^k(n)}{n^s} = \sum_{n=2}^{N-1} \frac{\log^k(n)}{n^s} + \sum_{n=N}^{\infty} \frac{\log^k(n)}{n^s} \\
&= \sum_{n=2}^{N-1} \frac{\log^k(s)}{n^s} + \int_N^{\infty} \frac{\log^k(x)}{x^s} dx + \frac{1}{2} \frac{\log^k(N)}{N^s} \\
&\qquad + \sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1}}{dx^{2j-1}} \frac{\log^k(x)}{x^s} \bigg|_{x=N}^{\infty} + R_{2v} \\
&= \sum_{n=2}^{N-1} \frac{\log^k(s)}{n^s} + \int_N^{\infty} \frac{\log^k(x)}{x^s} dx + \frac{1}{2} \frac{\log^k(N)}{N^s} \\
&\qquad - \sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1}}{dx^{2j-1}} \frac{\log^k(x)}{x^s} \bigg|_{x=N} + R_{2v},
\end{aligned}$$

where $N \in \mathbb{N}^{>2}$, $v \in \mathbb{N}^{>2}$, and R_{2v} is the error term. Repeated integration by parts yields:

$$\int_N^{\infty} \frac{\log^k(x)}{x^s} dx = \frac{\log^k(N)}{(s-1)N^{s-1}} \sum_{r=0}^{k} \frac{k!}{(k-r)!} \frac{\log^{-r}(N)}{(s-1)^r}.$$

Thus,

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{N-1} \frac{\log^k(s)}{n^s} + \frac{\log^k(N)}{(s-1)N^{s-1}} \sum_{r=0}^{k} \frac{k!}{(k-r)!} \frac{\log^{-r}(N)}{(s-1)^r} + \frac{1}{2}\frac{\log^k(N)}{N^s} \\ - \sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1}}{dx^{2j-1}} \frac{\log^k(x)}{x^s}\bigg|_{x=N} + R_{2v}, \tag{2.5}$$

The error term R_{2v} is given by

$$R_{2v} = \frac{1}{(2v)!} \int_N^\infty \hat{B}_{2v}(x) f^{(2v)}(x) dx,$$

with $f(x) = \frac{\log^k(x)}{x^s}$ as discussed in [10]. We use the non-central Stirling numbers of the first kind (see [17]), to represent the derivatives of f. The non-central Stirling numbers of the first kind $S(r, i, s)$ satisfy the recurrence:

$$S(1, 0, s) = -s$$

$$S(1, 1, s) = 1$$

$$S(r+1, 0, s) = (-s-r)S(r, 0, s)$$

$$S(r+1, i, s) = (-s-r)S(r, i, s) + S(r, i-1, s) \text{ for } 1 \leq i \leq r$$

$$S(r+1, r+1, s) = S(r, r, s).$$

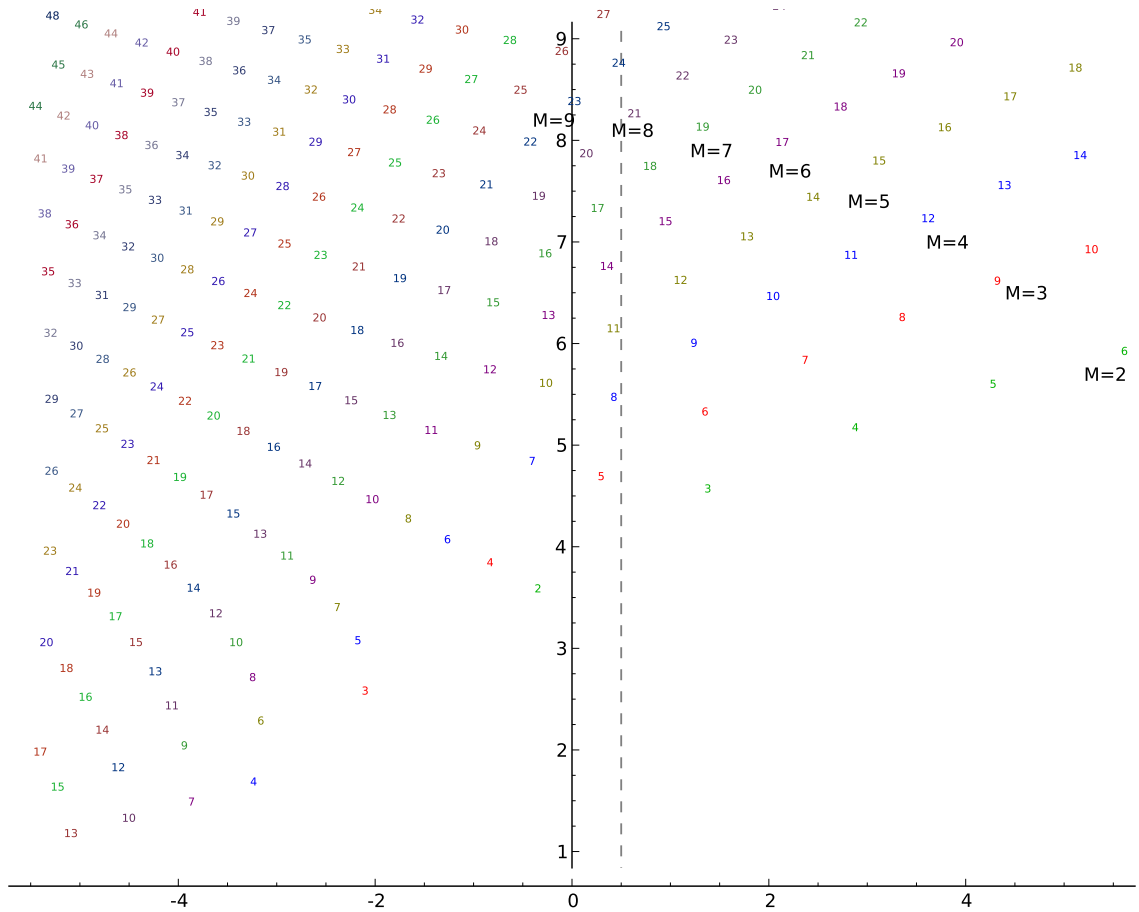

Figure 7. Zeros of $\zeta^{(k)}(\sigma + it)$. The zeros of $\zeta^{(k)}$ are at the center of the numbers k. The first eight chains of zeros that we followed from the right to the left half plane are labeled $M = 2, \ldots, M = 9$ (see Section 2.1).

With these, the derivatives of f can be written as

$$f^{(r)}(x) = x^{-s-r} \sum_{i=0}^{r} S(r, i, s)(k)_i \log^{k-i}(x)$$

where $(k)_i$ denotes the i-th falling factorial of k [17].

We now bound the error term, R_{2v}. Observe that

$$\begin{aligned}
|R_{2v}| &= \left| \frac{1}{(2v)!} \int_N^\infty \hat{B}_{2v}(x) f^{(2v)}(x) dx \right| \\
&\leq \frac{|B_{2v}|}{(2v)!} \int_N^\infty |f^{(2v)}(x)| dx \\
&= \frac{|B_{2v}|}{(2v)!} \int_N^\infty \left| x^{-s-2v} \sum_{i=0}^{2v} S(2v,i,s)(k)_i \log^{k-i}(x) \right| dx.
\end{aligned}$$

Using the triangle inequality,

$$\begin{aligned}
|R_{2v}| &\leq \frac{|B_{2v}|}{(2v)!} \sum_{i=0}^{2v} \int_N^\infty \left| S(2v,i,s)(k)_i \frac{\log^{k-i}(x)}{x^{s+2v}} \right| dx \\
&= \frac{|B_{2v}|}{(2v)!} \sum_{i=0}^{2v} |S(2v,i,s)|(k)_i \int_N^\infty \frac{\log^{k-i}(x)}{x^{\sigma+2v}} dx
\end{aligned}$$

The error term R_{2v} converges for $\sigma + 2v > 1$ and $N \in \mathbb{N}^{>2}$, thus (2.5) can be used to evaluate $\zeta^{(k)}$ for $\sigma > 1 - 2v$. Since we are evaluating $\zeta^{(k)}$ on a bounded region with $|\sigma| \leq 10$, the error can be bounded on the entire region. We set $v = 101$ which yields $\sigma + 2v > 1$ in the region and gives a good balance of the values for v and N. To determine the value N should take, we evaluate the bound given above for $N = 200, 300, \ldots$ until the error is as small as desired. For example, if $s = -10 + 10i$, $k = 100$, $v = 101$, and $N = 200$ then $|R_{2v}| < 1.769892 \cdot 10^{-100}$. If $N = 1500$ then $|R_{2v}| < 1.245704 \cdot 10^{-253}$.

Table 2. All zeros of $\zeta^{(k)}(\sigma+it)$ with $2 \le k \le 26$ in $-10 < \sigma < 0$, $0 < |t| < 10$. The column # contains the number of conjugate pairs of zeros. All zeros listed are simple and rounded to 4 decimal digits.

k	#	Zeros of $\zeta^{(k)}(\sigma+it)$ with $-10 < \sigma < 0$ and $0 < \lvert t\rvert < 10$			
2	1	$-0.3551 \pm 3.5908i$			
3	1	$-2.1101 \pm 2.5842i$			
4	2	$-0.8375 \pm 3.8477i$	$-3.2403 \pm 1.6896i$		
5	2	$-2.1841 \pm 3.0795i$	$-4.2739 \pm 0.6624i$		
6	2	$-1.2726 \pm 4.0742i$	$-3.1694 \pm 2.2894i$		
7	3	$-0.4133 \pm 4.8453i$	$-2.3934 \pm 3.4063i$	$-3.8750 \pm 1.4918i$	
8	3	$-1.6703 \pm 4.2784i$	$-3.2523 \pm 2.7170i$	$-4.5682 \pm 0.8112i$	
9	3	$-0.9672 \pm 4.9985i$	$-2.6410 \pm 3.6749i$	$-3.9459 \pm 2.0452i$	
10	4	$-0.2748 \pm 5.6133i$	$-2.0391 \pm 4.4684i$	$-3.4229 \pm 3.0609i$	$-4.5121 \pm 1.3321i$
11	4	$-1.4413 \pm 5.1493i$	$-2.9062 \pm 3.9132i$	$-4.0769 \pm 2.4384i$	$-5.0310 \pm 0.7641i$
12	4	$-0.8452 \pm 5.7473i$	$-2.3874 \pm 4.6486i$	$-3.6307 \pm 3.3459i$	$-4.6218 \pm 1.8307i$
13	5	$-0.2500 \pm 6.2811i$	$-1.8653 \pm 5.2971i$	$-3.1788 \pm 4.1283i$	$-4.2445 \pm 2.7740i$,
		$-5.1019 \pm 1.1817i$			
14	5	$-1.3402 \pm 5.8783i$	$-2.7202 \pm 4.8199i$	$-3.8543 \pm 3.5969i$	$-4.7812 \pm 2.1996i$,
		$-5.5404 \pm 0.6780i$			
15	5	$-0.8124 \pm 6.4056i$	$-2.2551 \pm 5.4415i$	$-3.4521 \pm 4.3265i$	$-4.4411 \pm 3.0614i$,
		$-5.2367 \pm 1.6383i$			
16	6	$-0.2827 \pm 6.8886i$	$-1.7845 \pm 6.0069i$	$-3.0400 \pm 4.9834i$	$-4.0887 \pm 3.8241i$,
		$-4.9528 \pm 2.5231i$	$-5.6490 \pm 1.0311i$		
17	6	$-1.3092 \pm 6.5262i$	$-2.6197 \pm 5.5821i$	$-3.7242 \pm 4.5121i$	$-4.6486 \pm 3.3161i$,
		$-5.4130 \pm 1.9836i$	$-6.0680 \pm 0.5743i$		

2.4 Finding Zeros

We found the zeros on the left half plane by following the chains of zeros of derivatives of ζ from the right half plane (see Figures 5 and 7). Given $M \geq 2$, $j \in \mathbb{Z}$, and sufficiently large k, the center

$$s = q_M k + \frac{2\pi(j+0.5)}{\log(M+1) - \log(M)}$$

of the rectangular region from Theorem 2.3 is a good approximation to the zero in this region which we improved using Newton's method.

Now assume that we know a zero $z_M^{(k)}$ of $\zeta^{(k)}$ and a zero $z_M^{(k+1)}$ of $\zeta^{(k+1)}$ in the chain given by some M and j. We used

$$s = z_M^{(k)} - \left(z_M^{(k+1)} - z_M^{(k)}\right)$$

as a first approximation for the zero of $\zeta^{(k-1)}$ in that chain, which again was improved with Newton's method.

By using the argument principle, we assured that we had found all zeros of $\zeta^{(k)}$ with $0 < k \leq 61$ in $-10 < \sigma < \frac{1}{2}$, $|t| < 10$ by counting the zeros using contour integration. The only pole of $\zeta^{(k)}$ is at one and thus outside our region of interest. So for any simple closed contour C in $-10 < \sigma < \frac{1}{2}$, $|t| < 10$, by the argument principle, the number of zeros of $\zeta^{(k)}$ inside C is

$$n = \frac{1}{2\pi i} \int_C \left(\frac{\zeta^{(k+1)}}{\zeta^{(k)}}\right)(s)\, ds.$$

For $0 < k \leq 61$ we counted the zeros of $\zeta^{(k)}$ by integrating along the border of the rectangular region $-10 < \sigma < \frac{1}{2}$, $|t| < 10$. We also integrated along the sides of

a square region with side length 10^{-6} centered around each approximation z of the zeros to make sure that this region contained exactly one zero.

All computations and plotting were conducted with the computer algebra system Sage [36]. We evaluated $\zeta^{(k)}$ with our implementation of the method described in Section 2.3 which was verified, on the right half plane, with the Hurwitz Zeta function in mpmath [18] and our implementation of $\zeta^{(k)}$ based on convergence acceleration for alternating series. For the integration we used the numerical integration function of Sage which calls the GNU Scientific Library [19] using an adaptive Gauss-Kronrod rule.

Table 3. All zeros of $\zeta^{(k)}(\sigma+it)$ with $27 \le k \le 32$ in $-10 < \sigma < 0$, $0 < |t| < 10$. The column # contains the number of conjugate pairs of zeros. All zeros listed are simple and rounded to 4 decimal digits.

k	#	Zeros of $\zeta^{(k)}(\sigma+it)$ with $-10 < \sigma < 0$ and $0 < \lvert t\rvert < 10$			
18	6	$-0.8299 \pm 7.0068i$	$-2.1924 \pm 6.1331i$	$-3.3491 \pm 5.1402i$	$-4.3279 \pm 4.0324i$,
		$-5.1468 \pm 2.8068i$	$-5.8098 \pm 1.4611i$		
19	7	$-0.3475 \pm 7.4543i$	$-1.7592 \pm 6.6440i$	$-2.9648 \pm 5.7192i$	$-3.9939 \pm 4.6871i$,
		$-4.8654 \pm 3.5483i$	$-5.5889 \pm 2.2963i$	$-6.1583 \pm 0.88585i$	
20	7	$-1.3211 \pm 7.1206i$	$-2.5729 \pm 6.2569i$	$-3.6489 \pm 5.2913i$	$-4.5694 \pm 4.2268i$,
		$-5.3472 \pm 3.0608i$	$-5.9945 \pm 1.7820i$	$-6.6140 \pm 0.43943i$	
21	7	$-0.8787 \pm 7.5677i$	$-2.1744 \pm 6.7594i$	$-3.2944 \pm 5.8530i$	$-4.2605 \pm 4.8536i$,
		$-5.0870 \pm 3.7617i$	$-5.7837 \pm 2.5734i$	$-6.3545 \pm 1.2934i$	
22	8	$-0.4328 \pm 7.9887i$	$-1.7703 \pm 7.2313i$	$-2.9319 \pm 6.3785i$	$-3.9406 \pm 5.4371i$,
		$-4.8118 \pm 4.4095i$	$-5.5554 \pm 3.2943i$	$-6.1750 \pm 2.0870i$	$-6.6413 \pm 0.7581i$
23	8	$-1.3613 \pm 7.6765i$	$-2.5625 \pm 6.8727i$	$-3.6113 \pm 5.9836i$	$-4.5240 \pm 5.0128i$,
		$-5.3115 \pm 3.9611i$	$-5.9806 \pm 2.8250i$	$-6.5366 \pm 1.5912i$	$-7.1892 \pm 0.1700i$
24	8	$-0.9481 \pm 8.0980i$	$-2.1871 \pm 7.3395i$	$-3.2737 \pm 6.4980i$	$-4.2254 \pm 5.5784i$,
		$-5.0539 \pm 4.5827i$	$-5.7671 \pm 3.5097i$	$-6.3712 \pm 2.3553i$	$-6.8798 \pm 1.1259i$
25	9	$-0.5313 \pm 8.4984i$	$-1.8064 \pm 7.7820i$	$-2.9291 \pm 6.9843i$	$-3.9174 \pm 6.1112i$,
		$-4.7841 \pm 5.1658i$	$-5.5378 \pm 4.1485i$	$-6.1844 \pm 3.0574i$	$-6.7253 \pm 1.8906i$,
		$-7.1206 \pm 0.6504i$			
26	9	$-0.1113 \pm 8.8798i$	$-1.4211 \pm 8.2028i$	$-2.5782 \pm 7.4458i$	$-3.6013 \pm 6.6153i$,
		$-4.5038 \pm 5.7155i$	$-5.2952 \pm 4.7478i$	$-5.9817 \pm 3.7117i$	$-6.5664 \pm 2.6042i$,
		$-7.0463 \pm 1.4126i$			
27	9	$-1.0318 \pm 8.6041i$	$-2.2218 \pm 7.8850i$	$-3.2780 \pm 7.0941i$	$-4.2144 \pm 6.2361i$,
		$-5.0410 \pm 5.3132i$	$-5.7647 \pm 4.3261i$	$-6.3901 \pm 3.2731i$	$-6.9206 \pm 2.1489i$,
		$-7.3814 \pm 0.9448i$			
28	10	$-0.6389 \pm 8.9878i$	$-1.8606 \pm 8.3044i$	$-2.9484 \pm 7.5503i$	$-3.9169 \pm 6.7308i$,
		$-4.7767 \pm 5.8489i$	$-5.5353 \pm 4.9061i$	$-6.1978 \pm 3.9018i$	$-6.7680 \pm 2.8338i$,
		$-7.2490 \pm 1.7019i$	$-7.6182 \pm 0.5486i$		
29	10	$-0.2428 \pm 9.3554i$	$-1.4951 \pm 8.7056i$	$-2.6132 \pm 7.9860i$	$-3.6122 \pm 7.2024i$,
		$-4.5034 \pm 6.3583i$	$-5.2947 \pm 5.4558i$	$-5.9918 \pm 4.4954i$	$-6.5986 \pm 3.4759i$,
		$-7.1165 \pm 2.3954i$	$-7.5353 \pm 1.2495i$		
30	10	$-1.1257 \pm 9.0905i$	$-2.2729 \pm 8.4034i$	$-3.3013 \pm 7.6533i$	$-4.2222 \pm 6.8443i$,
		$-5.0444 \pm 5.9789i$	$-5.7739 \pm 5.0583i$	$-6.4149 \pm 4.0822i$	$-6.9700 \pm 3.0489i$,
		$-7.4393 \pm 1.9531i$	$-7.8300 \pm 0.7596i$		
31	11	$-0.7529 \pm 9.4602i$	$-1.9282 \pm 8.8039i$	$-2.9846 \pm 8.0854i$	$-3.9340 \pm 7.3091i$,
		$-4.7854 \pm 6.4781i$	$-5.5454 \pm 5.5941i$	$-6.2186 \pm 4.6575i$	$-6.8081 \pm 3.6673i$,
		$-7.3161 \pm 2.6210i$	$-7.7489 \pm 1.5152i$	$-8.1557 \pm 0.4150i$	
32	11	$-0.3770 \pm 9.8161i$	$-1.5795 \pm 9.1891i$	$-2.6629 \pm 8.5003i$	$-3.6395 \pm 7.7548i$,
		$-4.5188 \pm 6.9560i$	$-5.3075 \pm 6.1058i$	$-6.0109 \pm 5.2053i$	$-6.6324 \pm 4.2542i$,
		$-7.1745 \pm 3.2514i$	$-7.6387 \pm 2.1955i$	$-8.0192 \pm 1.0955i$	

CHAPTER III

FRACTIONAL STIELTJES CONSTANTS

The Hurwitz Zeta function is defined, for $\Re(s) > 1$ and $0 < a \leq 1$, as $\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$. It can be extended to a meromorphic function with a simple pole at $s = 1$ with residue 1 (see [3], [8]). Moreover, the function has a Laurent series expansion about $s = 1$, given by

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)(s-1)^n}{n!}, \tag{3.1}$$

where $\gamma_n(a)$ are the generalized Stieltjes constants. The original Stieltjes constants were defined in 1885 (see [37]), but are themselves a generalization of Euler's constant γ:

$$\gamma = \gamma_0(1) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{1}{n} - \log m \right) = 0.57721\ 56649\ \cdots\ .$$

In 1972, Berndt [5] showed that for the generalized Stieltjes constants in (5.1) we have:

$$\gamma_k(a) = \lim_{m\to\infty} \left\{ \sum_{n=0}^{m} \frac{\log^k(n+a)}{n+a} - \frac{\log^{k+1}(m+a)}{k+1} \right\}. \tag{3.2}$$

Furthermore, it was established by Williams and Zhang [39] that

$$\gamma_k(a) = \sum_{r=0}^{m} \frac{\log^k(r+a)}{r+a} - \frac{\log^{k+1}(m+a)}{k+1} - \frac{\log^k(m+a)}{2(m+a)} + \int_m^{\infty} P_1(x) f_k'(x)dx, \tag{3.3}$$

where $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$ and $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

More recently, Kreminski [21] has given a generalization of $\gamma_r(a)$ for all $r > 0$, the so-called *fractional Stieltjes constants*. Kreminski found a method of computing $\gamma_r(a)$, by first computing the function

$$C_r(a) = \gamma_r(a) - \frac{\log^r(a)}{a}. \tag{3.4}$$

He also defined the function,

$$h_a(s) = \zeta(s, a) - \frac{1}{s-1} - \frac{1}{a^s}, \tag{3.5}$$

and by doing so, Kreminski conjectured that $C_r(a) = (-1)^r h_a^{(r)}(1)$ where $f^{(r)}(x)$ is interpreted as the r-th (Weyl) fractional derivative of f at x.

The aim of our paper is to first introduce the Grünwald-Letnikov fractional derivative. Using some basic properties of the Grünwald-Letnikov fractional derivative we will then generalize the results from Berndt [5] and Williams & Zhang [39] to the fractional case. We will then restate the conjecture by Kreminski [21, Conjecture IIIa] in terms of the Grünwald-Letnikov fractional derivative and prove this restatement. We end this paper by discussing the relationship between the Grünwald-Letnikov and Weyl fractional derivatives and how using this relationship also proves the original version of [21, Conjecture (IIIa)].

3.1 Fractional Derivatives

Fractional derivative operators are generalizations of the familiar differentiation operator D^n to arbitrary (integer, rational, or complex) values of n. To motivate this generalization, let $N \in \mathbb{N}$, and recall $\Delta_h^N f(z) = (-1)^N \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(z + kh)$ is the finite difference of f at z. It is known that, (see [29], for example) $f^{(n)}(z) =$

$\lim_{h\to 0} \frac{\Delta_h^n f(z)}{h^n}$ for all $n \in \mathbb{N}$. This can be naturally extended for any $\alpha \in \mathbb{C}$ (cf. [9]) via

$$\Delta_h^\alpha f(z) = (-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)$$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$. Hence, for any $\alpha \in \mathbb{C}$, the so-called reverse α^{th} Grünwald-Letnikov derivative of a function $f(z)$ is now defined as (see [16]):

$$D_z^\alpha [f(z)] = \lim_{h\to 0^+} \frac{\Delta_h^\alpha f(z)}{h^\alpha} = \lim_{h\to 0^+} \frac{(-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^\alpha} \tag{3.6}$$

whenever the limit exists. Thus defined, $D_z^\alpha[f(z)]$ coincides with the standard derivatives for all $\alpha \in \mathbb{N}$. Also, they are analytic functions of α and z (as long as $f(z)$ is analytic) and satisfy: $D_z^0[f(z)] = f(z)$ and $D_z^\alpha \left[D_z^\beta [f(z)]\right] = D_z^{\alpha+\beta} [f(z)]$.

Although the Grünwald-Letnikov derivative is defined for all $\alpha \in \mathbb{C}$, we only consider $\alpha \in \mathbb{R}$ with $\alpha \geq 0$ in this paper. The following two useful results can be found in [29].

Lemma 3.1. *Let $\alpha \in \mathbb{R}$, $\beta < 0$, and $z \in \mathbb{C}$ with $\Re(z) > 1$. Then,*

$$D_z^\alpha \left[(z-1)^\beta\right] = \frac{(-1)^\alpha \Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-1)^{\beta-\alpha}.$$

Lemma 3.2. *Let $\alpha \geq 0$, $a > 0$, and $z \in \mathbb{C}$. Then $D_z^\alpha [e^{-az}] = (-1)^\alpha a^\alpha e^{-az}$.*

For the Hurwitz Zeta function for $0 < a \leq 1$ and $\Re(s) > 1$, we have

$$\begin{aligned}\zeta(s,a) - \frac{1}{a^s} &= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} - \frac{1}{a^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n+a)^s} \\ &= \sum_{n=1}^{\infty} e^{-s\log(n+a)}.\end{aligned}$$

Since $0 < a \leq 1$, we have $\log(n+a) > 0$, applying lemma 3.2, we have

$$\begin{aligned}D_s^{\alpha}\left[\zeta(s,a) - 1/a^s\right] &= \sum_{n=1}^{\infty} (-1)^{\alpha} \log^{\alpha}(n+a) e^{-s\log(n+a)} \\ &= (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s}.\end{aligned}$$

We have thus shown the following corollary.

Corollary 3.3. *Let $0 < a \leq 1$, and $s \in \mathbb{C}$ with $\Re(s) > 1$. The Grünwald-Letnikov fractional derivative of order $\alpha \geq 0$ with respect to s of $\zeta(s,a) - 1/a^s$ is*

$$D_s^{\alpha}\left[\zeta(s,a) - 1/a^s\right] = (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s}. \tag{3.7}$$

As previously noted, Kreminski developed Corollary IIIa in [21] in terms of the Weyl fractional derivative. In the following, we let $W_z^{\alpha}\left[f(z)\right]$ denote the $\alpha - th$ Weyl fractional derivative of f at z, where $\alpha \in \mathbb{C}$. As noted in [21], the Weyl analog of lemma 3.1 for $\beta = -1$ yields $W_z^{\alpha}\left[\frac{1}{z-1}\right] = (-1)^{-\alpha} \frac{\alpha\pi\csc(\pi z)}{\Gamma(1-\alpha)(z-1)^{\alpha+1}}$. Since $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} = \pi\csc(\pi z)$ [3], we can write

$$\begin{aligned}W_z^{\alpha}\left[\frac{1}{z-1}\right] &= (-1)^{-\alpha} \frac{\alpha\pi\csc(\pi z)}{\Gamma(1-\alpha)(z-1)^{\alpha+1}} \\ &= (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(z-1)^{\alpha+1}}.\end{aligned} \tag{3.8}$$

Also, as noted in [21], the Weyl analog of lemma 3.2 states for $\alpha \geq 0$, $a > 0$, and $z \in \mathbb{C}$,

$$W_z\left[e^{-az}\right] = (-1)^{-\alpha} a^{\alpha} e^{-az}. \tag{3.9}$$

Comparing 3.8 and 3.9 to lemmas 3.1 and 3.2, we see a difference by only a constant multiple of -1. In view of this, Kreminski conjectured that $C_r(a) = (-1)^r h_a^{(r)}(1)$ whereas we will prove $C_r(a) = (-1)^{-r} h_a^{(r)}(1)$. Whether using Weyl or Grünwald-Letnikov fractional derivatives the proof will only differ by this same constant multiple of -1.

We chose to use the Grünwald-Letnikov fractional derivatives because not only is it easily motivated, but also as noted in [] the fractional derivative of a constant is undefined. Due to this difficulty, it is not clear how to apply the Weyl fractional derivative of, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\Re(s) > 1$, since the first term is constant. We can overcome this difficulty using the Grünwald-Letnikov fractional derivative, since for any $c \in \mathbb{C}$ and $\alpha > 0$, $D_z^{\alpha}[c] = 0$, as shown in [29].

3.2 Fractional Stieltjes Constants

Let $\alpha > 0$ and $0 < a \leq 1$. For $s \neq 1$, we define the *fractional Stieltjes constants* to be the coefficients of the expansion

$$\sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s} = \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!}(s-1)^n. \tag{3.10}$$

To relate $\gamma_\alpha(a)$ to $C_\alpha(a)$ observe that

$$
\begin{aligned}
(-1)^{-\alpha} D_s^\alpha[\zeta(s,a) - 1/a^s] &= \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} \\
&= \sum_{n=0}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} - \frac{\log^\alpha(a)}{a^s} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_{\alpha+n}(a)}{n!}(s-1)^n - \frac{\log^\alpha(a)}{a^s} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_{\alpha+n}(a)}{n!}(s-1)^n \\
&\qquad - \log^\alpha(a) e^{-s\log(a)} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_{\alpha+n}(a)}{n!}(s-1)^n \\
&\qquad - \log^\alpha(a) e^{-s\log(a)} \frac{e^{\log(a)}}{e^{\log(a)}} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_{\alpha+n}(a)}{n!}(s-1)^n \\
&\qquad - \frac{\log^\alpha(a)}{a} e^{-(s-1)\log(a)}.
\end{aligned}
$$

Expanding the exponential about $s = 1$, we have

$$\begin{aligned}
(-1)^{-\alpha} D_s^{\alpha}[\zeta(s,a) - 1/a^s] &= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty}(-1)^n \frac{\gamma_{\alpha+n}(a)}{n!}(s-1)^n \\
&\quad - \frac{\log^{\alpha}(a)}{a}\sum_{n=0}^{\infty}(-1)^n \frac{\log^n(a)}{n!}(s-1)^n \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty}(-1)^n \frac{\gamma_{\alpha+n}(a)}{n!}(s-1)^n \\
&\quad - \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n \frac{\log^{\alpha+n}(a)}{n!}(s-1)^n \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty}(-1)^n \left(\gamma_{\alpha+n}(a) - \frac{\log^{\alpha+n}}{a}\right)\frac{(s-1)^n}{n!} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty}(-1)^n C_{\alpha+n}(a)\frac{(s-1)^n}{n!}. \qquad (3.11)
\end{aligned}$$

3.3 Kreminski's Conjecture

Now we are ready to prove the main result of this chapter, namely [21, Conjecture (IIIa)], stated in terms of the Grünwald-Letnikov fractional derivative as discussed earlier:

Theorem 3.4. *Let $h_a(s) = \zeta(s,a) - \frac{1}{s-1} - \frac{1}{a^s}$ and let $h_a^{(\alpha)}(s) = D_s^{\alpha}[h_a(s)]$ be the α-th Grünwald-Letnikov fractional derivative of h_a. Then*

$$C_{\alpha}(a) = \gamma_{\alpha}(a) - \frac{\log^{\alpha}(a)}{a} = (-1)^{-\alpha} h_a^{(\alpha)}(1).$$

Proof. We have by linearity of the fractional derivative operator:

$$\begin{aligned} h_a^{(\alpha)}(s) &= D_s^\alpha \left[h_a(s)\right] \\ &= D_s^\alpha \left[\zeta(s,a) - \frac{1}{a^s} - \frac{1}{s-1}\right] \\ &= D_s^\alpha \left[\zeta(s,a) - \frac{1}{a^s}\right] - D_s^\alpha \left[\frac{1}{s-1}\right]. \end{aligned} \tag{3.12}$$

Applying Corollary 3.3 and (3.10) to the first term of right hand side of (3.12), we have:

$$\begin{aligned} (-1)^{-\alpha} D_s^\alpha \left[\zeta(s,a) - \frac{1}{a^s}\right] &= \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} \\ &= \sum_{n=0}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} - \frac{\log^\alpha(a)}{a^s} \\ &= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!}(s-1)^n - \frac{\log^\alpha(a)}{a^s} \end{aligned} \tag{3.13}$$

Applying lemma 3.1 to the last term of (3.12) we see that

$$(-1)^{-\alpha} D_s^\alpha \left[\frac{1}{s-1}\right] = \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we see that:

$$\begin{aligned} (-1)^{-\alpha} h_a^{(\alpha)}(s) &= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!}(s-1)^n - \frac{\log^\alpha(a)}{a^s} - \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!}(s-1)^n - \frac{\log^\alpha(a)}{a^s} \end{aligned} \tag{3.15}$$

Evaluating $(-1)^{-\alpha}h_a^{(\alpha)}(s)$ at the point $s=1$ using (3.15), we obtain

$$(-1)^{-\alpha}h_a^{(\alpha)}(s) = \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a} = C_\alpha(a)$$

as desired. □

3.4 Generalization of a Theorem from Zhang-Williams

We end this paper with a generalization of (3.3). As discussed in [13], the following theorem will lead naturally to a method of evaluating the fractional Stieltjes constants.

Theorem 3.5. *Let $\alpha \in \mathbb{R}$ with $\alpha > 0$, $0 < a \leq 1$, and $m \in \mathbb{N}$. We have*

$$\gamma_\alpha(a) = \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x) f'_\alpha(x)dx, \quad (3.16)$$

where $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$ and $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

Letting $m \to \infty$ yields, for all $\alpha > 0$ and $0 < a \leq 1$, a natural generalization of (3.2):

$$C_\alpha(a) := \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a} = \lim_{m\to\infty} \left\{ \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} \right\}$$

which Kreminski [21] uses to define $\gamma_\alpha(a)$ for $\alpha \in \mathbb{R}$.

Proof. We use the following form of the Euler-Maclaurin summation formula:

$$\sum_{k=m}^{n} g(k) = \int_m^n g(x)dx + \sum_{k=1}^{v} \frac{(-1)^k B_k}{k!} g^{(k+1)}(x)\Big|_m^n + (-1)^{v+1} \int_m^n P_v(x) g^{(v)}(x)dx, \quad (3.17)$$

where $v \in \mathbb{N}$, $g(x) \in C^v[m,n]$, and $P_k(x)$ is the kth periodic Bernoulli polynomial

$$P_k(x) = \frac{B_k(x - \lfloor x \rfloor)}{k!}.$$

We take $v = 1$ in (3.17) and choose $g(x) = \frac{\log^\alpha(x+a)}{(x+a)^s}$, for $\Re(s) > 1$.

Letting $n \to \infty$, we obtain:

$$\begin{aligned}
\sum_{r=0}^{\infty} \frac{\log^\alpha(r+a)}{(r+a)^s} &= \sum_{r=0}^{m-1} \frac{\log^\alpha(r+a)}{(r+a)^s} + \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s} dx + \frac{\log^\alpha(m+a)}{2(m+a)^s} \\
&\quad + \int_m^\infty P_1(x) g'(x) dx \\
&= \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{(r+a)^s} + \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s} dx - \frac{\log^\alpha(m+a)}{2(m+a)^s} \\
&\quad + \int_m^\infty P_1(x) g'(x) dx \\
&= A(s) + B(s) - D(s) + G(s).
\end{aligned}$$

For the first term $A(s)$ we have:

$$\begin{aligned}
A(s) = \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{(r+a)^s} &= \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{r+a} e^{-(s-1)\log(r+a)} \\
&= \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{r+a} \sum_{n=0}^{\infty} \frac{(-1)^n \log^n(r+a)}{n!} (s-1)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (s-1)^n}{n!} \sum_{r=0}^{m} \frac{\log^{\alpha+n}(r+a)}{(r+a)}.
\end{aligned}$$

Now, since $\alpha \geq 0$, $m \in \mathbb{N}$, and $0 < a \leq 1$, for all $s \in \mathbb{C}$ with $\Re(s) > 1$, the second term $B(s)$ can be written in terms of the Upper Incomplete Gamma function $\Gamma(\alpha, s)$.

We obtain (compare with [15, p. 346] and [2, 6.5.3]):

$$
\begin{aligned}
B(s) &= \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s}dx = \frac{\Gamma(\alpha+1,(s-1)\log(m+a))}{(s-1)^{\alpha+1}} \\
&= \frac{1}{(s-1)^{\alpha+1}}\Bigg[\Gamma(\alpha+1) \\
&\qquad - (s-1)^{\alpha+1}\log^{\alpha+1}(m+a)\sum_{n=0}^\infty \frac{(-1)^n(s-1)^n\log^n(m+a)}{(\alpha+1+n)n!}\Bigg] \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \log^{\alpha+1}(m+a)\sum_{n=0}^\infty \frac{(-1)^n(s-1)^n\log^n(m+a)}{(\alpha+1+n)n!} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \sum_{n=0}^\infty \left(\frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1}\right)\frac{(-1)^n(s-1)^n}{n!}.
\end{aligned}
$$

For the third term $D(s)$, we write:

$$
\begin{aligned}
D(s) &= \frac{\log^\alpha(m+a)}{2(m+a)^s} = \frac{\log^\alpha(m+a)}{2(m+a)}e^{-(s-1)\log(m+a)} \\
&= \frac{\log^\alpha(m+a)}{2(m+a)}\sum_{n=0}^\infty \frac{(-1)^n\log^n(m+a)(s-1)^n}{n!} \\
&= \sum_{n=0}^\infty \left(\frac{\log^{\alpha+n}(m+a)}{2(m+a)}\right)\frac{(-1)^n(s-1)^n}{n!}.
\end{aligned}
$$

If we now define

$$
E_{\alpha,m}(n) := \sum_{r=0}^m \frac{\log^{\alpha+n}(r+a)}{r+a} - \frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1} - \frac{\log^{\alpha+n}(m+a)}{2(m+a)},
$$

then combining the above expressions for $A(s)$, $B(s)$ and $D(s)$ we get:

$$
\begin{aligned}
\sum_{r=0}^m \frac{\log^\alpha(r+a)}{(r+a)^s} &+ \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s}dx - \frac{\log^\alpha(m+a)}{2(m+a)^s} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^\infty E_{\alpha,m}(n)\frac{(-1)^n(s-1)^n}{n!}.
\end{aligned}
$$

For the last term we have:

$$G(s) = \int_m^\infty P_1(x)g'(x)dx = \int_m^\infty P_1(x)\left[-s\frac{\log^\alpha(x+a)}{(x+a)^{s+1}} + \alpha\frac{\log^{\alpha-1}(x+a)}{(x+a)^{s+1}}\right]dx.$$

From the definition of the fractional Stieltjes constants we have:

$$\sum_{n=0}^\infty E_{\alpha,m}(n)\frac{(-1)^{\alpha+n}(s-1)^n}{n!} + G(s) = \sum_{n=0}^\infty \frac{(-1)^{\alpha+n}\gamma_{\alpha+n}(s-1)^n}{n!}.$$

Taking successive derivatives with respect to s, of both sides, and then evaluating them at $s = 1$, we see that for all $n \in \mathbb{N} \cup \{0\}$,

$$\gamma_{\alpha+n}(a) = E_{\alpha,m}(n) + G^{(n)}(1). \tag{3.18}$$

Setting $n = 0$ in (3.18) and noting that $G(1) = f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$, we obtain

$$\begin{aligned}\gamma_\alpha(a) &= E_{\alpha,m}(0) + G(1)\\ &= \sum_{r=0}^m \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x)f'_\alpha(x)dx,\end{aligned}$$

which proves the result. □

3.5 Continuity of the Fractional Stieltjes Constants

We end with a theorem concerning the continuity of $\gamma_\alpha(a)$ as a function of α. Throughout this section, we let $a = 1$ and write γ_α to denote $\gamma_\alpha(1)$. The following theorem will show that γ_α as a function of α is not continuous at $\alpha = 0$. On the other hand, in view of (3.2), γ_α is a continuous on $\alpha > 0$.

Theorem 3.6. *As $\alpha \to 0^+$, $\gamma_\alpha \to \gamma - 1$ where $\gamma = \gamma_0$ is Euler's constant.*

Proof. Observe that, with $a = 1$, the left-hand sum in (3.10) becomes

$$\sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+1)}{(n+1)^s} = \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n)}{n^s} = \sum_{n=2}^{\infty} \frac{\log^{\alpha}(n)}{n^s}. \tag{3.19}$$

Letting $\alpha \to 0^+$, (3.19) becomes

$$\sum_{n=2}^{\infty} \frac{1}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) - 1 = \zeta(s) - 1. \tag{3.20}$$

Also, the right hand side of (3.10) with $a = 1$ becomes

$$\frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_{\alpha+n}}{n!} (s-1)^n. \tag{3.21}$$

Thus, in order for (3.10) to be true, we need

$$\zeta(s) - 1 = \lim_{\alpha \to 0^+} \left[\frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_{\alpha+n}}{n!} (s-1)^n \right]. \tag{3.22}$$

From the Laurent series expansion of $\zeta(s)$ about $s = 1$, (5.1), we have

$$\zeta(s) - 1 = \left[\frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{n!} (s-1)^n \right] - 1. \tag{3.23}$$

Since $\lim\limits_{\alpha \to 0^+} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} = \frac{\Gamma(1)}{s-1} = \frac{1}{s-1}$, thus (3.22) holds if and only if

$$\left[\sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{n!} (s-1)^n \right] - 1 = \lim_{\alpha \to 0^+} \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_{\alpha+n}}{n!} (s-1)^n. \tag{3.24}$$

Letting $s = 1$ in (3.24) we obtain, $\gamma_0 - 1 = \lim\limits_{\alpha \to 0^+} \gamma_\alpha$, as desired. □

CHAPTER IV

A BOUND FOR FRACTIONAL STIELTJES CONSTANTS

The Hurwitz Zeta function is defined, for $\Re(s) > 1$ and $0 < a \leq 1$, as $\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$. It can be extended to a meromorphic function with a simple pole at $s = 1$ with residue 1 (see [3], [8]). Moreover, the function has a Laurent series expansion about $s = 1$, given by

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)(s-1)^n}{n!}, \tag{4.1}$$

where $\gamma_n(a)$ are the generalized Stieltjes constants. Kreminski [21] has given a generalization of $\gamma_\alpha(a)$ to $\alpha \in \mathbb{R}^{\geq 0}$ called *fractional Stieltjes constants*. As we will see, defining $\gamma_\alpha(a)$ for $\alpha \in \mathbb{R}^{\geq 0}$ will allow us to use the power of continuity to derive a bound for $\gamma_n(a)$. In other words, later in this paper we will apply the Lambert W function to find an upper bound. Without continuity, we would be unable to apply the Lambert W function. This continuity thus gives us insight into the overall behavior of the Stieltjes constants.

These can be defined as the coefficients of the Laurent expansion of the α-th Grünwald-Letnikov fractional derivative [16] of $\zeta(s,a) - 1/a^s$ for $s \neq 1$ [12]:

$$\begin{aligned} D_s^\alpha \left[\zeta(s,a) - 1/a^s\right] &= (-1)^{-\alpha} \sum_{n=0}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} \\ &= (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!} (s-1)^n. \end{aligned}$$

The fractional Stieltjes constants generalize the Stieltjes constants.

In [12, Corollary 3.2] we show

$$\alpha \to 0^+,\ \gamma_\alpha(1) \to \gamma - 1 = -0.42278\,43350\ldots \tag{4.2}$$

where $\gamma = \gamma_0 = \gamma_0(1) = 0.57721\,46649\ldots$ is Euler's constant. In [12] we also prove a conjecture of Kreminski [21, Conjecture IIIa]:

Let $0 < \alpha \in \mathbb{R}$, $C_\alpha(a) := \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$, and $h_a(s) := \zeta(s,a) - 1/(s-1) - 1/a^s$ then

$$C_\alpha(a) = (-1)^{-\alpha} D_s^\alpha[h_a](1).$$

The goal of this paper is to compute $\gamma_\alpha(a)$ by evaluating $C_\alpha(a)$ and to find an uppper bound for $|C_\alpha(a)|$. We start by recalling and proving some results about Stirling numbers (section 4.1) that we employ in a method for evaluating $C_\alpha(a)$ (section 4.2). In section 4.3 we give an upper bound for $C_\alpha(a)$ for $\alpha > 1$ which is a generaliziation of [39, Theorem 3] to fractional Stieltjes constants and show how our bound can be minimized.

4.1 Complex Non-Central Stirling Numbers of the First Kind

For $\alpha \in \mathbb{R}$ let $(\alpha)_i = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}$, denote the falling factorial of α. We denote by $S(n,i,s)$ where $n \in \mathbb{N}_0$, $0 \le i \le n$, and $s \in \mathbb{C}$ the non-central Stirling numbers of the first kind satisfy the recurrence relations:

$$\begin{aligned}
&S(0,0,s) = 1,\ S(1,0,s) = -s,\ S(1,1,s) = 1\\
&S(n+1,0,s) = (-s-n)S(n,0,s),\ S(n+1,n+1,s) = S(n,n,s)\\
&S(n+1,i,s) = (-s-n)S(n,i,s) + S(n,i-1,s) \text{ for } 1 \le i \le n.
\end{aligned} \tag{4.3}$$

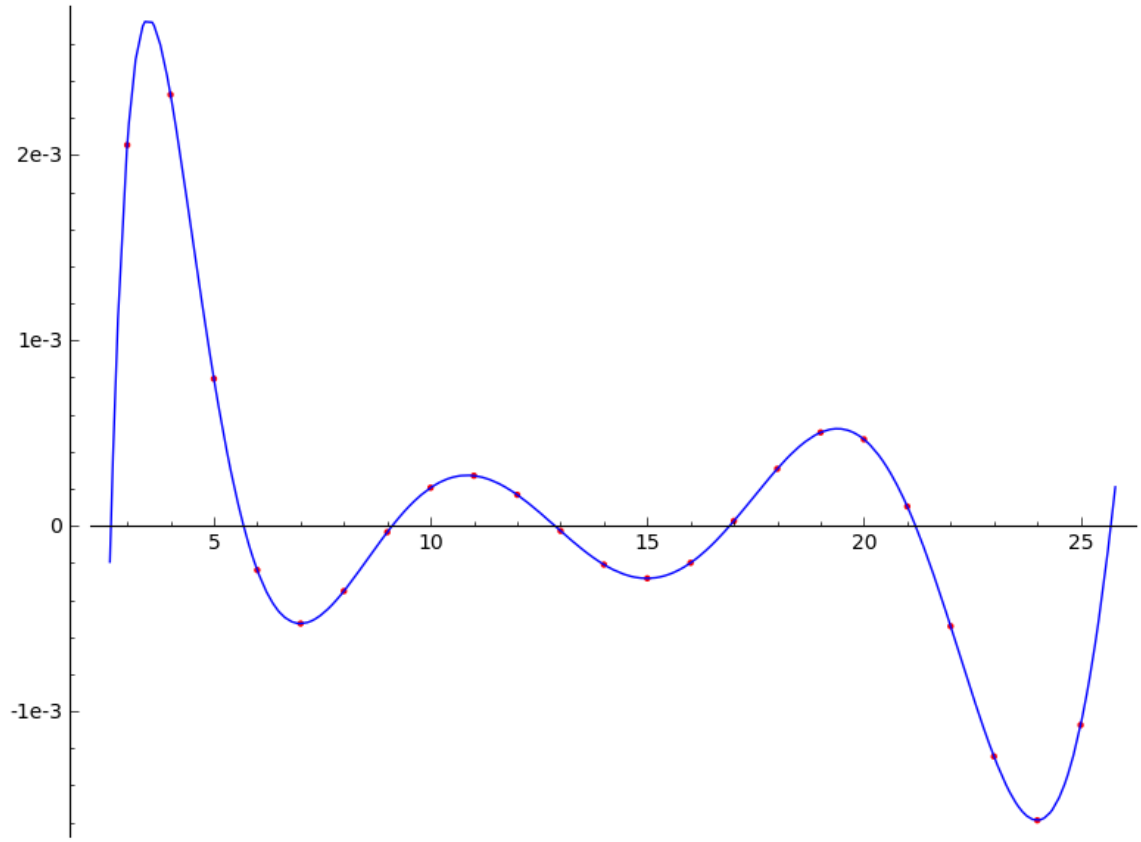

Figure 8. The fractional Stieltjes constants $\gamma_\alpha(1)$ plotted for $2.6 \leq \alpha \leq 25.8$ with (integral) Stieltjes constants (•).

The following is a generalization of results found in [11] and [17].

Lemma 4.1. *Let $\alpha \in \mathbb{R}$, $s \in \mathbb{C}$, and $g_\alpha(x) = \frac{\log^\alpha(x)}{x^s}$. For any $n \in \mathbb{N}_0$*

$$g_\alpha^{(n)}(x) = \sum_{i=0}^{n} S(n,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+n}}.$$

Proof. We proceed by way of induction on n. When $n = 0$, the result is trivially true. For $n = 1$ we have

$$g'_\alpha(x) = -s\frac{\log^\alpha(x)}{x^{s+1}} + \alpha\frac{\log^{\alpha-1}(x)}{x^{s+1}} = S(1,0,s)(\alpha)_0\frac{\log^\alpha(x)}{x^{s+1}} + S(1,1,s)(\alpha)_1\frac{\log^{\alpha-1}(x)}{x^{s+1}}.$$

Thus, the induction has been anchored. Now assume the result holds for some $k \in \mathbb{N}$. Differentiating $g_\alpha^{(k)}(x)$ we get

$$\begin{aligned} g_\alpha^{(k+1)}(x) &= \sum_{i=0}^{k}(-s-k)S(k,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+k+1}} + \sum_{i=0}^{k} S(k,i,s)(\alpha)_i(\alpha-i)\frac{\log^{\alpha-i-1}(x)}{x^{s+k+1}} \\ &= \sum_{i=0}^{k}(-s-k)S(k,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+k+1}} + \sum_{i=0}^{k} S(k,i,s)(\alpha)_{i+1}\frac{\log^{\alpha-i-1}(x)}{x^{s+k+1}}. \end{aligned}$$

Making a change of variables in the second sum yields

$$\begin{aligned} g_\alpha^{(k+1)}(x) &= \sum_{i=0}^{k}(-s-k)S(k,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+k+1}} + \sum_{i=1}^{k+1} S(k,i-1,s)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} \\ &= (-s-k)S(k,0,s)(\alpha)_0\frac{\log^{\alpha}(x)}{x^{s+k+1}} \\ &\quad + \sum_{i=1}^{k+1}\left((-s-k)S(k,i,s) + S(k,i-1,s)\right)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} \end{aligned}$$

With the recurrence relation (4.3) we obtain $S(k+1,0,s) = (-s-k)S(k,0,s)$ and $S(k+1,i,s) = (-s-k)S(k,i,s) + S(k,i-1,s)$. Hence, we have

$$\begin{aligned} g_\alpha^{(k+1)}(x) &= S(k+1,0,s)(\alpha)_0\frac{\log^{\alpha}(x)}{x^{s+k+1}} + \sum_{i=1}^{k+1} S(k+1,i,s)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} \\ &= \sum_{i=0}^{k+1} S(k+1,i,s)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}}. \end{aligned}$$

Thus the relation holds for $g_\alpha^{(k+1)}(x)$. Hence, by induction, the lemma is proven. $\square$

Recall that the (signed) Stirling numbers $s(i,j)$ of the first kind are generated by the recurrence:

$$s(0,0) = 1,\ s(n,0) = s(0,n) = 0 \text{ for } n \in \mathbb{N} \tag{4.4}$$

$$s(n+1,i) = -ns(n,i) + s(n,i-1) \text{ for } n \in \mathbb{N}_0 \text{ and } i \in \mathbb{N}_0.$$

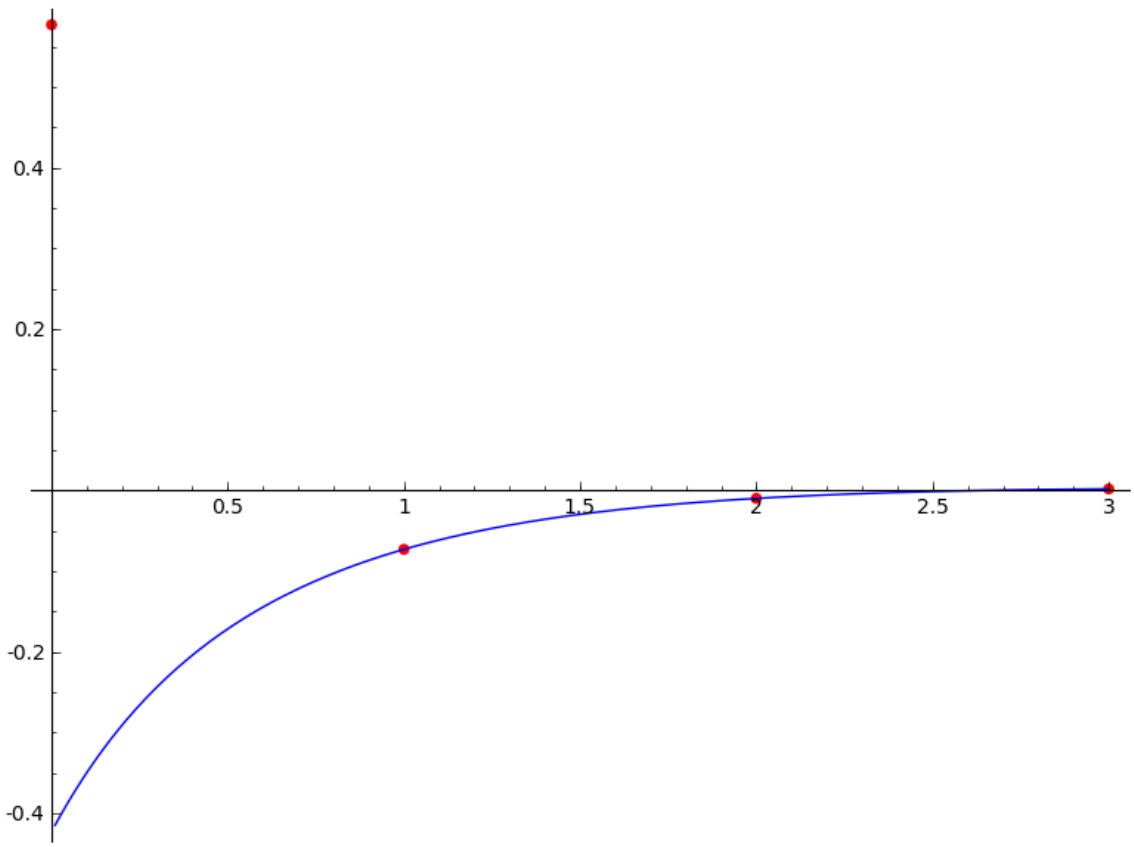

Figure 9. The fractional Stieltjes Constants $\gamma_\alpha(1)$ plotted for $0 \leq \alpha \leq 3$ with (integral) Stieltjes constants (•). This plot illustrates the discontinuity of $\gamma_\alpha(1)$ at $\alpha = 0$, compare (4.2).

Proposition 4.2. *Let $\alpha \geq 0$, $0 < a \leq 1$, and $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$. Then for any $n \in \mathbb{N}_0$,*

$$f_\alpha^{(n)}(x) = \sum_{i=0}^{n} s(n+1, i+1)(\alpha)_i \frac{\log^{\alpha-i}(x+a)}{(x+a)^{n+1}}. \tag{4.5}$$

Proof. In view of Lemma 4.1, the result is proven if we show that for all $n \in \mathbb{N}_0$ and all integers $0 \leq i \leq n$, we have $S(n, i, 1) = s(n+1, i+1)$. We prove this equality by induction on n.

For $n = 0$ we get from the recurrence relation (4.3) that $S(0, 0, 1) = 1$. From (4.4) we get $s(1, 1) = 1$. Thus, the induction is anchored. Now let $n \in \mathbb{N}$ and assume for all integers $0 \leq r \leq n$, $S(r, i, 1) = s(r+1, i+1)$ for $i = 0, 1, \ldots, r$.

Next we show $S(n+1, i, 1) = s(n+2, i+1)$. With the recurrence relations (4.3) and (4.4), and the induction hypothesis we obtain

$$S(n+1, 0, 1) = (-n-1)S(n, 0, 1) = -(n+1)s(n+1, 1) = s(n+2, 1)$$

$$S(n+1, n+1, 1) = S(n, n, 1) = s(n+1, n+1).$$

For $1 \leq i \leq n$ we have

$$\begin{aligned} S(n+1, i, 1) &= (-n-1)S(n, i, 1) + S(n, i-1, 1) \\ &= -(n+1)s(n+1, i+1) + s(n+1, i) = s(n+2, i+1). \end{aligned}$$

Thus, by induction the result has been proven. □

4.2 Evaluation Of $\gamma_\alpha(a)$

To evaluate $\gamma_\alpha(a)$ we approximate $C_\alpha(a)$ and then use that $\gamma_\alpha(a) = C_\alpha(a) + \frac{\log^\alpha(a)}{a}$. Let $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$. By [12, Theorem 3.1] for $\alpha \in \mathbb{R}$ with $\alpha > 0$, $0 < a \leq 1$,

and $m \in \mathbb{N}$ we have

$$\gamma_\alpha(a) = \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x) f'_\alpha(x) dx, \quad (4.6)$$

where $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$. Thus,

$$C_\alpha(a) = \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x) f'_\alpha(x) dx. \quad (4.7)$$

Integrating by parts $v \in \mathbb{N}$ times yields

$$\int_m^\infty P_1(x) f'_\alpha(x) dx = \sum_{j=1}^{v} \left[P_j(x) f_\alpha^{(j-1)}(x) \right]_{x=m}^{\infty} + (-1)^{v-1} \int_m^\infty P_v(x) f_\alpha^{(v)}(x) dx \quad (4.8)$$

where $P_k(x) = \frac{B_k(x - \lfloor x \rfloor)}{k!}$ is the kth periodic Bernoulli polynomial and B_j is the jth Bernoulli number. For computational purposes, it is useful to recall that $B_j = 0$ for j odd.

As we will soon see, letting $m > 0$ forces the integral on the right hand side of (4.8) to converge for any $v \in \mathbb{N}$. With Proposition 4.2, we see that as $x \to \infty$, $f_\alpha^{(n)}(x) \to 0$ for any $n \in \mathbb{N}$. Thus, we can write (4.8) as

$$\int_m^\infty P_1(x) f'_\alpha(x) dx = -\sum_{j=1}^{v} P_j(m) f_\alpha^{(j-1)}(m) + (-1)^{v-1} \int_m^\infty P_v(x) f_\alpha^{(v)}(x) dx. \quad (4.9)$$

For any $j \in \mathbb{N}$ and $m \in \mathbb{N}$ we have $P_j(m) = \frac{B_j}{j!}$. We now approximate $C_\alpha(a)$ by

$$C_\alpha(a) \approx \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} - \sum_{j=1}^{v} \frac{B_j}{j!} f_\alpha^{(j-1)}(m). \quad (4.10)$$

The error in approximating $C_\alpha(a)$ by (4.10) is given by $R_v = (-1)^{v-1}\int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x)dx$. We now show that we can choose m and v so that the error is arbitrarily small. We choose $v > 1$. As $|P_n(x)| \leq \frac{3+(-1)^n}{(2\pi)^n}$ for any $n > 1$ (see [39], [5], or [30]) we have

$$|R_v| = \left|(-1)^{v-1}\int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x)\right| \leq \frac{3+(-1)^n}{(2\pi)^n}\int\limits_m^\infty \left|f_\alpha^{(v)}(x)\right| dx. \tag{4.11}$$

With Corollary 4.2 and the triangle inequality in (4.11) we get

$$|R_v| \leq \frac{3+(-1)^v}{(2\pi)^v}\sum_{i=0}^{v}|s(v+1,i+1)|\frac{\Gamma(\alpha+1)}{|\Gamma(\alpha-i+1)|}\int\limits_m^\infty \frac{\log^{\alpha-i}(x+a)}{(x+a)^{v+1}}dx. \tag{4.12}$$

We now write the integral in terms of the Upper Incomplete Gamma function (see [15, p. 346] and [2, 6.5.3])

$$\int\limits_m^\infty \frac{\log^{\alpha-i}(x+a)}{(x+a)^{v+1}}dx = \frac{\Gamma(\alpha-i+1, v\log(m+a))}{v^{\alpha-i+1}}. \tag{4.13}$$

Applying (4.13) in (4.12) we find an upper bound for the error:

$$|R_v| \leq \frac{(3+(-1)^v)\Gamma(\alpha+1)}{(2\pi)^v v^{\alpha+1}}\sum_{i=0}^{v}|s(v+1,i+1)|\frac{\Gamma(\alpha-i+1, v\log(m+a))v^i}{|\Gamma(\alpha-i+1)|}. \tag{4.14}$$

The error term, R_{2v}, in (4.12) converges for all v. To find suitable parameters v and m so that R_{2v} we follow a similar method to that used to evaluate $\zeta^{(k)}$ discussed in [11]. We first let v be large and then iteratively increase the value of m until the error is small as desired. To illustrate the method, letting $v = 101$ (this value was also used in [11]), we evaluate the bound (4.12) for $N = 200, 300, \ldots$ until the error is as small as desired. For example, if $\alpha = 100$, $v = 101$, and $N = 200$, then $|R_{2v}| < 1.769892 \cdot 10^{-100}$. If $N = 1500$ then $|R_{2v}| < 1.245704 \cdot 10^{-253}$.

We have shown:

Theorem 4.3. *Let $\alpha \in \mathbb{R}$ with $\alpha > 0$, $0 < a \leq 1$, $m \in \mathbb{N}$, and $v > 1$. Let*

$$C'_\alpha(a) := \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} - \sum_{j=1}^{v} \frac{B_j}{j!} f_\alpha^{(j-1)}(m).$$

Then

$$|C'_\alpha(a) - C_\alpha(a)| \leq \frac{(3+(-1)^v)\Gamma(\alpha+1)}{(2\pi)^v v^{\alpha+1}} \sum_{i=0}^{v} |s(v+1, i+1)| \frac{\Gamma(\alpha-i+1, v\log(m+a))v^i}{|\Gamma(\alpha-i+1)|}.$$

The method described was implemented in the C library, Arb. At a later date, this method will be included in the Arb library. The values for $\gamma_\alpha(a)$ in Figures 8 and 9 were computed using the method described.

4.3 An Upper Bound For $C_\alpha(a)$

Using $m = 1$ in (4.6), we have after making some minor simplifications

$$\gamma_\alpha(a) = \frac{\log^\alpha(a)}{a} + \frac{\log^\alpha(1+a)}{2(1+a)} - \frac{\log^{\alpha+1}(1+a)}{\alpha+1} + \int_1^\infty P_1(x) f'_\alpha(x)dx. \tag{4.15}$$

Since $0 < a \leq 1$ and $P_1(x) = x - \frac{1}{2}$ on $(0,1)$, integration by parts yields

$$\int_{1-a}^{1} P_1(x) f'_\alpha(x)dx = \int_{1-a}^{1} \left(x - \frac{1}{2}\right) f'_\alpha(x)dx = \frac{\log^\alpha(1+a)}{2(1+a)} - \frac{\log^{\alpha+1}(1+a)}{\alpha+1}$$

Using this in (4.15), allows us to see that

$$\gamma_\alpha(a) = \frac{\log^\alpha(a)}{a} + \int_{1-a}^{\infty} P_1(x) f'_\alpha(x)dx = \frac{\log^\alpha(a)}{a} + C_\alpha(a).$$

Using corollary 4.2 we have for any positive integer n,

$$f_\alpha^{(n)}(x) = \sum_{i=0}^{n} s(n+1, i+1)(\alpha)_i \frac{\log^{\alpha-i}(x+a)}{(x+a)^{n+1}}. \tag{4.16}$$

Assume $\alpha > 1$, let n be any arbitrary integer satisfying $1 \leq n < \alpha$, and let k be any positive integer so that $1 \leq k \leq n$. From these assumptions, we see that $f_\alpha^{(k)}(x-a)$ is a combination of positive powers of $\log(x)$ and hence, $f_\alpha^{(k)}(1-a) = 0$. Also, $f_\alpha^{(k)}(x-a) \to 0$ as $x \to \infty$. These observations and integrating by parts n times yield

$$\begin{aligned} C_\alpha(a) &= P_2(x) f'_\alpha(x)|_{x=1-a}^{\infty} - P_3(x) f''_\alpha(x)|_{x=1-a}^{\infty} + \ldots + (-1)^{n+1} P_{n+1}(x) f_\alpha^{(n)}(x)|_{x=1-a}^{\infty} \\ &\quad + (-1)^n \int_{1-a}^{\infty} P_{n+1}(x) f_\alpha^{(n+1)}(x) dx \\ &= (-1)^n \int_{1-a}^{\infty} P_{n+1}(x) f_\alpha^{(n+1)}(x) dx. \end{aligned}$$

Making a change of variable we get

$$C_\alpha(a) = (-1)^n \int_{1}^{\infty} P_{n+1}(x-a) f_\alpha^{(n+1)}(x-a) dx.$$

Knopp showed in [20] that $|P_n(x)| \leq \frac{4}{(2\pi)^n}$ for all integers $n > 1$. Ostrowski observed in [30] that for odd $n > 1$, $|P_n(x)| < \frac{2}{(2\pi)^n}$. Thus we can write $|P_n(x)| \leq \frac{3+(-1)^n}{(2\pi)^n}$ for all $n > 1$. Making use of this inequality, we now derive an upper bound for $C_\alpha(a)$.

We have

$$
\begin{aligned}
|C_\alpha(a)| &= \left|(-1)^n \int_1^\infty P_{n+1}(x-a) f_\alpha^{(n+1)}(x-a)dx\right| \\
&\leq \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \int_1^\infty \left|f_\alpha^{(n+1)}(x-a)\right| dx \\
&\leq \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \sum_{i=0}^{n+1} |s(n+2,i+1)|(\alpha)_i \int_1^\infty \frac{\log^{\alpha-i}(x)}{x^{n+2}} dx. \qquad (4.17)
\end{aligned}
$$

We now evaluate the integral in (4.17). After a change of variables we have

$$
\int_1^\infty \frac{\log^{\alpha-i}(x)}{x^{n+2}} dx = \frac{1}{(n+1)^{\alpha-i+1}} \int_0^\infty x^{\alpha-i} e^{-x} dx = \frac{\Gamma(\alpha-i+1)}{(n+1)^{\alpha-i+1}}, \qquad (4.18)
$$

since $\alpha - i \geq \alpha - n > 0$, and the integral converges for all $0 \leq i \leq n+1$. Using this in (4.17),

$$
|C_\alpha(a)| \leq \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \sum_{i=0}^{n+1} |s(n+2,i+1)|(\alpha)_i \frac{\Gamma(\alpha-i+1)}{(n+1)^{\alpha-i+1}}. \qquad (4.19)
$$

Since $1 \leq n < \alpha$, we can write $(\alpha)_i = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}$ for each $0 \leq i \leq n+1$. From (4.19) we get

$$
\begin{aligned}
|C_\alpha(a)| &\leq \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \sum_{i=0}^{n+1} |s(n+2,i+1)| \frac{\Gamma(\alpha+1)}{(n+1)^{\alpha-i+1}} \\
&= \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \sum_{i=0}^{n+1} |s(n+2,i+1)|(n+1)^i \\
&= \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+2}} \sum_{j=1}^{n+2} |s(n+2,j)|(n+1)^j.
\end{aligned}
$$

By [39, 6.14] we have $\sum_{i=1}^{n+2} |s(n+2,j)|(n+1)^j = \frac{(2n+2)!}{n!}$. Using this identity, we arrive at

$$|C_\alpha(a)| \leq \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+2}} \frac{(2n+2)!}{n!} = \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!}.$$

We have proven:

Theorem 4.4. *Let $0 < a \leq 1$, $\alpha > 1$, and $C_\alpha(a) = \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$. Then,*

$$|C_\alpha(a)| \leq \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!} \tag{4.20}$$

where n is any positive integer satisfying $1 \leq n < \alpha$.

We now improve Theorem 5.7. The first step is to notice that the inequality holds for any positive integer n with $1 \leq n < \alpha$. It is natural to wonder what value of n minimizes the upper bound. The Lambert-W function – the complex values $W(z)$ for which $W(z)e^{W(z)} = z$ – will help us establish this, together with the following bound: For all $n \geq 1$,

$$\frac{(2n)!}{n!} \leq \sqrt{2}\left(\frac{4n}{e}\right)^n e^{\frac{1}{24n} - \frac{1}{12n+1}} < \sqrt{2}\left(\frac{4n}{e}\right)^n. \tag{4.21}$$

This follows directly from the following sharp version of Stirling's formula:

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\frac{1}{12n+1}} \leq n! \leq \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\frac{1}{12n}}.$$

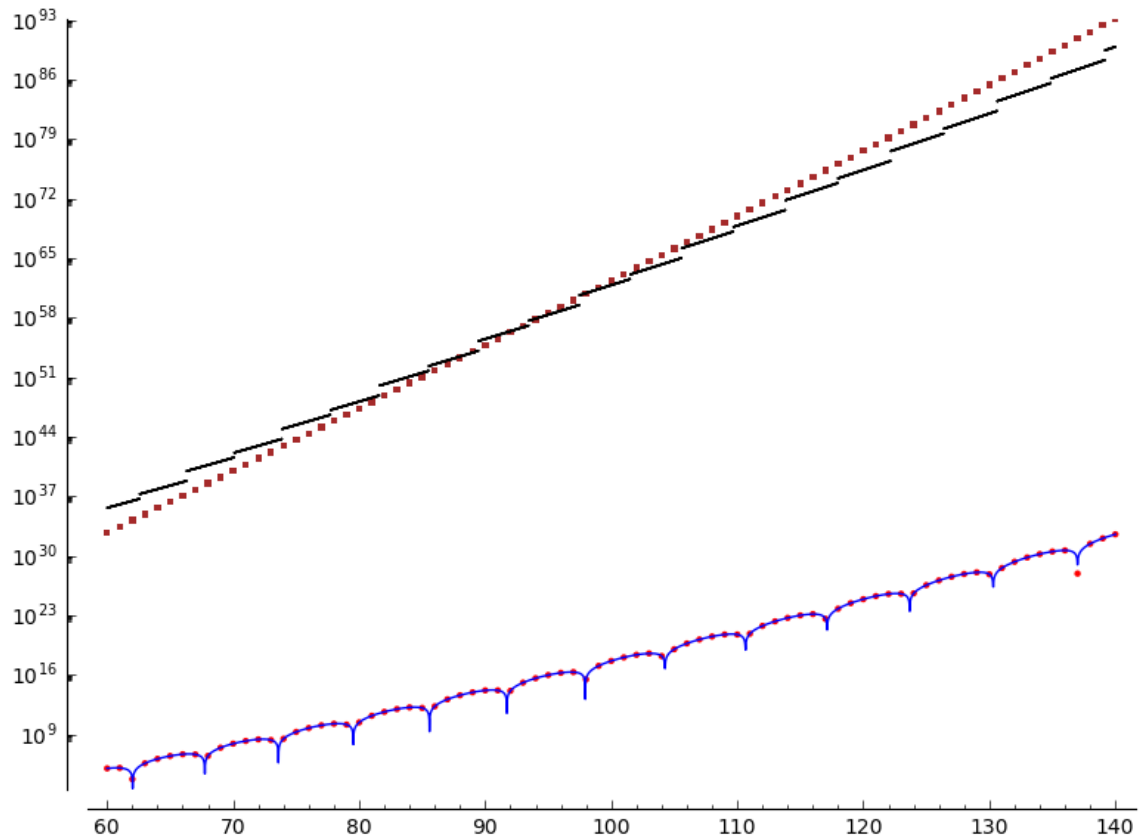

Figure 10. The absolute value of the fractional Stieltjes constants (-) $\gamma_\alpha(a)$ for $60 \leq \alpha \leq 140$; with integral Stieltjes constants (●); the bound (-) for the fractional Stieltjes constants from Theorem 4.5; the bound (■) by Matsuoka [28].

In particular, we get:

Theorem 4.5. *Let $0 < a \leq 1$ and $\alpha > 0$. Let n be chosen in the following manner: if $\frac{\pi}{2}e^{W(\frac{2(\alpha+1)}{\pi})} < \alpha$, then let n be the nearest integer to $\frac{\pi}{2}e^{W(\frac{2(\alpha+1)}{\pi})}$. Otherwise, let n be the greatest integer not exceeding α. Choosing n in this way makes the right hand side of the inequality in theorem 5.7 smallest of all the possible choices.*

Proof. We apply (4.21) to the right hand side of the inequality in theorem 5.7, and take $g(x) = \frac{4\sqrt{2}\Gamma(\alpha+1)}{x^{\alpha+1}}\left(\frac{2n}{e\pi}\right)^x$. It is our goal to find x on the closed interval $[1, \alpha]$ that

minimizes $g(x)$. Once x is found, we let n be the nearest integer to x so that $g(n)$ is smallest. Let $\tilde{C}_\alpha = 4\sqrt{2}\Gamma(\alpha+1)$. Since we are working on a closed interval and g is continuous on $[1,\alpha]$, g must attain a minimum on $[1,\alpha]$. We first find the derivative of $g(x)$ by observing

$$g(x) = \frac{\tilde{C}_\alpha}{x^{\alpha+1}}\left[\frac{2x}{\pi e}\right]^x = \tilde{C}_\alpha \exp\left[-(\alpha+1)\log(x) + x\log\left(\frac{2x}{\pi e}\right)\right].$$

Differentiating, we find

$$g'(x) = \tilde{C}_\alpha\left[\frac{-(\alpha+1)}{x} + 1 + \log\left(\frac{2x}{\pi e}\right)\right]\exp\left[-(\alpha+1)\log(x) + x\log\left(\frac{2x}{\pi e}\right)\right].$$

Setting $g'(x) = 0$ dividing both sides by the constant term and the exponential term, we get

$$\frac{-(\alpha+1)}{x} + 1 + \log\left(\frac{2x}{\pi e}\right) = \frac{-(\alpha+1)}{x} + \log\left(\frac{2x}{\pi}\right) = 0.$$

This implies that $\frac{2x}{\pi}\log\left(\frac{2x}{\pi}\right) = \frac{2(\alpha+1)}{\pi}$, and if we let $y = \log\left(\frac{2x}{\pi}\right)$, then the previous equation becomes $ye^y = \frac{2(\alpha+1)}{\pi}$. Applying the Lambert-W function, we see that we must have $y = e^{W\left(\frac{2(\alpha+1)}{\pi}\right)}$. Solving for x, using this relation we then have $x = \frac{\pi}{2}e^{W\left(\frac{2(\alpha+1)}{\pi}\right)}$. If $x \leq \alpha$, then naturally we should pick n to be the greatest integer not exceeding α. This is because this would imply that $g(x)$ is monotonically decreasing on the interval $[1,\alpha]$. If x falls within the closed interval $[1,\alpha]$, then we pick the closest integer to x. This proves the result. $\square$

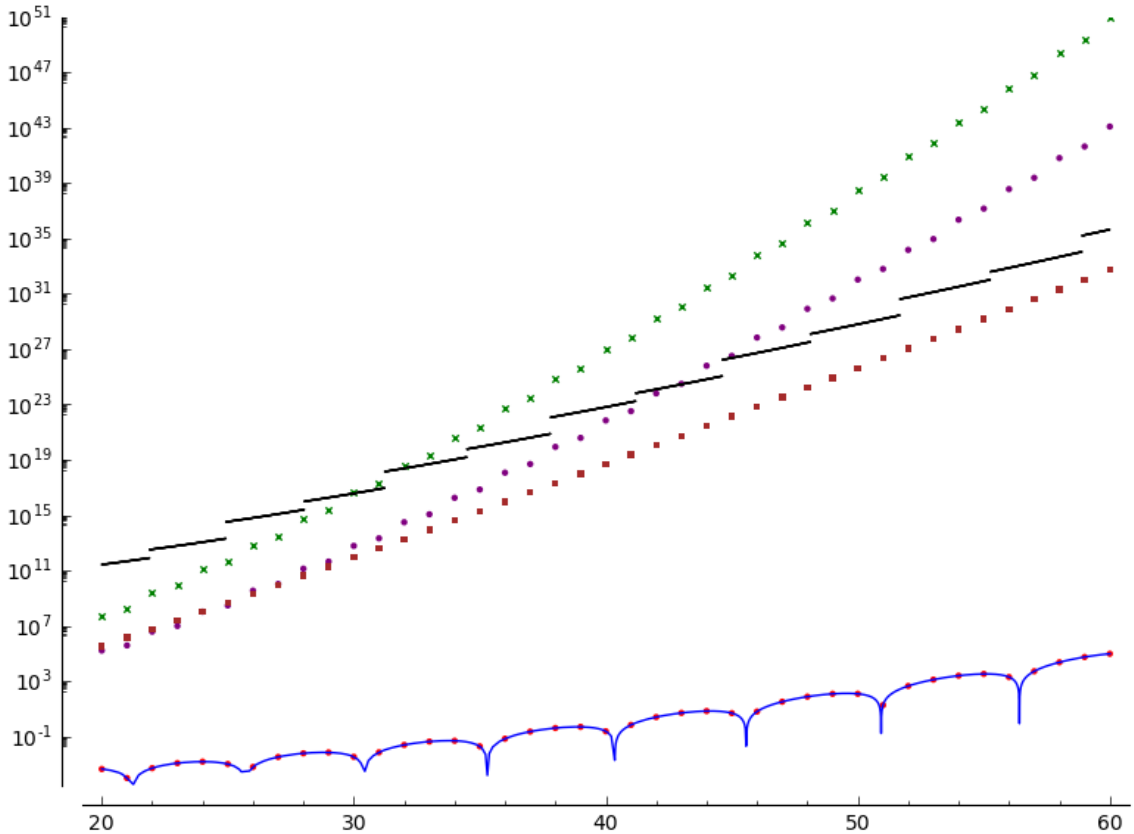

Figure 11. The absolute value of the fractional Stieltjes constants (-) $\gamma_\alpha(a)$ for $20 \leq \alpha \leq 60$; with integral Stieltjes constants (•); the bound (-) for the fractional Stieltjes constants from Theorem 4.5; the bound (×) by Berndt [5]; the bound (•) by William and Zhang [39]; the bound (■) by Matsuoka [28].

The upper bound for the fractional Stietltjes constants yields a bound for the integral Stieltjes constants. In Figures 11 and 10 we compare our bound to previously known bounds for integral Stieltjes Constants.

Namely, the bound by Berndt [5]

$$\gamma_m = \gamma_m(1) \leq \frac{(3 + (-1)^m)(m-1)!}{\pi^m}$$

and the bound by Williams and Zhang [39]

$$\gamma_m = \gamma_m(1) \leq \frac{(3 + (-1)^m)(2m)!}{m^{m+1}(2\pi)^m}$$

and the bound by Matsuoka [28]

$$\gamma_m = \gamma_m(1) < 10^{-4}(\log m)^m \text{ for } m > 1.$$

Remark. Theorem 5.7 with $n + 1 = m$ and $\alpha = n$ yields the bound by William and Zhang.

CHAPTER V

A ZERO FREE REGION FOR THE FRACTIONAL DERIVATIVES OF THE RIEMANN ZETA FUNCTION

The Riemann zeta function $\zeta(s)$ and its derivatives $\zeta^{(k)}(s)$ are

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{(\log n)^k}{n^s},$$

for all $k \in \mathbb{N}$, everywhere in the complex half-plane where $\Re(s) > 1$.

In [6], the authors have investigated the zero-free regions of higher derivatives $\zeta^{(k)}(s)$, and have discovered not only that, for all $k \in \mathbb{N}$, all of these derivatives have *identical* counts of zeros in $\Re(s) > 1/2$, but that there exists a dynamics that, with discretely increasing k, moves the non-trivial zeros of $\zeta^{(k)}(s)$, in a one-to-one fashion, to the right, in a virtually periodic manner. Due to increasing density of the zeros in vertical direction, this simple bijective idea is difficult to state quantitatively; however, the observed "flow" suggests that *fractional* derivatives (the Grünwald-Letnikov derivatives $D_s^{\alpha}[\zeta(s)]$, in particular) could provide the missing link needed to establish this property. Despite the incredible amount of research concerning $\zeta(s)$ and its derivatives, the fractional derivatives have been largely neglected.

We will not try to prove the audacious one-to-one conjecture in this paper, but we will establish a zero-free region for fractional derivatives of $\zeta(s)$, which – although modest and far from optimal – is proved in an elementary way, and seems to be the first of its kind.

We start by recalling some basics. First, note that $\zeta(s)$ can be extended to a meromorphic function with a simple pole at $s = 1$, with residue 1, and has a Laurent series expansion:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n, \tag{5.1}$$

where γ_n are the Stieltjes constants [37]. Bounds for *fractional* Stieltjes constants will be needed in the proof of our zero-free region. Before we define them, let us note that for any $\alpha \in \mathbb{C}$, the so-called "reverse α^{th} Grünwald-Letnikov derivative" of $f(z)$ is (see [16]):

$$D_z^{\alpha}\left[f(z)\right] = \lim_{h \to 0^+} \frac{\Delta_h^{\alpha} f(z)}{h^{\alpha}} = \lim_{h \to 0^+} \frac{(-1)^{\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^{\alpha}},$$

whenever the limit exists. Thus defined, $D_z^{\alpha}[f(z)]$ coincides with the standard derivatives for all $\alpha \in \mathbb{N}$. Also, they satisfy: $D_z^0[f(z)] = f(z)$ and $D_z^{\alpha}\left[D_z^{\beta}\left[f(z)\right]\right] = D_z^{\alpha+\beta}\left[f(z)\right]$. And if $f(z)$ is analytic, then $D_z^{\alpha}[f(z)]$ is an analytic function of both α and z. (Note: although the Grünwald-Letnikov derivative is defined for all $\alpha \in \mathbb{C}$, in this paper we only consider $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, since these cases are most useful in the theory of the Riemann zeta function.)

Finally, let us note that, in [29] it was shown that for $z \in \mathbb{C}$ we have $D_z^{\alpha}\left[e^{-az}\right] = (-1)^{\alpha} a^{\alpha} e^{-az}$, which for $\zeta(s)$ implies the following: For all $s \in \mathbb{C}$ with $\Re(s) > 1$, we have

$$D_s^{\alpha}\left[\zeta(s)\right] = (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n+1)}{(n+1)^s}. \tag{5.2}$$

5.1 Fractional Stieltjes Constants

The *fractional Stieltjes constants* γ_α where $\alpha \in \mathbb{R}^{>0}$ were introduced by Kreminski [21] and can be defined as the coefficients of the Laurent expansion of the α-th Grünwald-Letnikov fractional derivative of $\zeta(s)$ for $s \neq 1$ III:

$$D_s^\alpha [\zeta(s)] = (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}}{n!} (s-1)^n. \tag{5.3}$$

In view of this, it becomes clear that in order to establish regions of non-vanishing of these derivatives (which is the main objective of this paper), one needs to investigate behavior of the fractional Stieltjes constants in more detail. In III (in the process of proving a conjecture of Kerminski concerning the special values of the derivatives of Hurwitz zeta functions), we have proved the following useful generalization of a result of Williams & Zhang [39]:

For $\alpha > 0$ and $m \in \mathbb{N}$,

$$\gamma_\alpha = \sum_{r=1}^{m} \frac{\log^\alpha(r)}{r} - \frac{\log^{\alpha+1} m}{\alpha+1} - \frac{\log^\alpha(m)}{2m} + \int_m^\infty P_1(x) f'_\alpha(x) dx, \tag{5.4}$$

where $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$ and $f_\alpha(x) = \dfrac{\log^\alpha x + 1}{x+1}$. Integrating (5.4) by parts m times yields

$$\begin{aligned} \int_m^\infty P_1(x) f'_\alpha(x) dx &= \sum_{j=1}^{v} \left[P_j(x) f_\alpha^{(j-1)}(x) \right]_{x=m}^{\infty} + (-1)^{v-1} \int_m^\infty P_v(x) f_\alpha^{(v)}(x) dx \\ &= -\sum_{j=1}^{v} P_j(m) f_\alpha^{(j-1)}(m) + (-1)^{v-1} \int_m^\infty P_v(x) f_\alpha^{(v)}(x) dx \end{aligned} \tag{5.5}$$

where for $k \in \mathbb{N}$, $P_k(x) = \frac{B_k(x - \lfloor x \rfloor)}{k!}$ is the kth periodic Bernoulli polynomial and B_k is the kth Bernoulli number. Furthermore, the derivatives of f_α can be written in terms

of the (signed) Stirling numbers (see Proposition 4.2) as follows:

$$f_\alpha^{(n)}(x) = \sum_{i=0}^{n} s(n+1, i+1)(\alpha)_i \frac{\log^{\alpha-i}(x+1)}{(x+1)^{n+1}}, \tag{5.6}$$

where $(\alpha)_i = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}$ is the falling factorial. This particular result was applied (see Theorem 5.7) in the proof of an upper bound of the fractional Stieltjes constants:

$$|\gamma_\alpha| \leq \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!}, \tag{5.7}$$

where $n \in \mathbb{N}$, such that $1 \leq n < \alpha$. These estimates present a natural generalization of the bounds for the so-called generalized Stieltjes constants, see [39, Theorem 3].

5.2 Three Lemmas

We begin the construction of our proof with the following three lemmas.

Lemma 5.1. *Let* $0 < \alpha \leq 1$ *and* $f_\alpha(x) = \frac{\log^\alpha(x+1)}{x+1}$. *Then* $\left| \int_1^\infty P_3(x) f_\alpha'''(x) dx \right| < 0.013$.

Note: Ostrowski observed, in [30], that for odd $n > 1$ one has: $|P_n(x)| < \frac{2}{(2\pi)^n}$.

Proof. Let us consider the expression (5.6). With the help of the triangle inequality, and the change of variables for the integral, we are able to write:

$$\begin{aligned} \left| \int_1^\infty P_3(x) f_\alpha'''(x) dx \right| &< \frac{2}{(2\pi)^3} \sum_{i=0}^{3} |s(4, i+1)(\alpha)_i| \int_1^\infty \frac{\log^{\alpha-i}(x+1)}{(x+1)^4} dx \\ &< \frac{2}{(2\pi)^3} \sum_{i=0}^{3} \frac{|s(4, i+1)(\alpha)_i|}{3^{\alpha-i+1}} \int_{3\log(2)}^\infty x^{\alpha-i} e^{-x} dx. \end{aligned} \tag{5.8}$$

We will estimate each of the four summands on the right side of the inequality separately.

We start with $i = 0$. Since $x^\alpha \leq x$ in the interval $[3\log(2), \infty)$, we can write

$$\frac{|s(4,1)(\alpha)_0|}{3^{\alpha+1}} \int\limits_{3\log(2)}^{\infty} x^\alpha e^{-x} dx \leq \frac{6}{3^{\alpha+1}} \int\limits_{3\log(2)}^{\infty} x e^{-x} dx = \frac{1}{4} \frac{3\log(2)+1}{3^\alpha}. \tag{5.9}$$

For $i = 1$, in the interval $[3\log(2), \infty)$ we have $x^{\alpha-1} \leq 3^{\alpha-1}\log^{\alpha-1}(2)$, for all $\alpha \leq 1$; thus

$$\frac{|s(4,2)(\alpha)_1|}{3^\alpha} \int\limits_{3\log(2)}^{\infty} x^{\alpha-1} e^{-x} dx \leq \frac{11\alpha}{3^\alpha} 3^{\alpha-1}\log^{\alpha-1}(2) \int\limits_{3\log(2)}^{\infty} e^{-x} dx \leq \frac{11\log^{\alpha-1}(2)}{24}. \tag{5.10}$$

Now, for the summand corresponding to $i = 2$ we have

$$\begin{aligned} \frac{|s(4,3)(\alpha)_2|}{3^{\alpha-1}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-2} e^{-x} dx &= \frac{6|\alpha(\alpha-1)|}{3^{\alpha-1}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-2} e^{-x} dx \\ &\leq \frac{3}{2} \frac{1}{3^{\alpha-1}} 3^{\alpha-2}\log^{\alpha-2}(2) \int\limits_{3\log(2)}^{\infty} e^{-x} dx = \frac{\log^{\alpha-2}(2)}{16}, \end{aligned} \tag{5.11}$$

since for $0 < \alpha \leq 1$ we have $|\alpha(\alpha-1)| \leq \frac{1}{4}$ and for $x \in [3\log(2), \infty)$: $x^{\alpha-2} \leq 3^{\alpha-2}\log^{\alpha-2}(2)$.

Finally, for $i = 3$ we can write

$$\begin{aligned} \frac{|s(4,4)(\alpha)_3|}{3^{\alpha-2}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-3} e^{-x} dx &= \frac{|\alpha(\alpha-1)(\alpha-2)|}{3^{\alpha-2}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-3} e^{-x} dx \\ &\leq \frac{2\sqrt{3}}{9} \frac{3^{\alpha-3}\log^{\alpha-3}(2)}{3^{\alpha-2}} \int\limits_{3\log(2)}^{\infty} e^{-x} dx = \frac{\sqrt{3}\log^{\alpha-3}(2)}{108}, \end{aligned} \tag{5.12}$$

since $|\alpha(\alpha-1)(\alpha-2)| \leq \frac{2}{9}\sqrt{3}$ for $\alpha \in (0,1]$ and $x^{\alpha-3} \leq 3^{\alpha-3}\log^{\alpha-3}(2)$ for $x \in [3\log(2), \infty)$.

Combining these four bounds, we conclude:

$$\left|\int_1^\infty P_3(x)f_\alpha'''(x)\right| < \tfrac{2}{(2\pi)^3}\left[\tfrac{1}{4}\tfrac{3\log(2)+1}{3^\alpha} + \tfrac{11\log^{\alpha-1}(2)}{24} + \tfrac{\log^{\alpha-2}(2)}{16} + \tfrac{\sqrt{3}\log^{\alpha-3}(2)}{108}\right] < 0.013, \tag{5.13}$$

as desired. □

Lemma 5.2. *If* $0 < \alpha < 1$, *then* $|\gamma_\alpha| < 0.436$.

Proof. Taking $m = 2$ in the representation (5.4), we get

$$\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} + \int_2^\infty P_1(x)f_\alpha'(x)dx.$$

But from (5.5) we know that

$$\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - P_2(1)f_\alpha'(1) + P_3(1)f_\alpha''(1) + \int_2^\infty P_3(x)f_\alpha'''(x)dx.$$

So, with $P_2(1) = \frac{B_2}{2!} = \frac{1}{12}$ and $P_3(1) = \frac{B_3}{3!} = 0$ and $f_\alpha'(x) = \alpha\frac{\log^{\alpha-1}(2)}{4} - \frac{\log^\alpha(2)}{4}$ we obtain

$$\begin{aligned}\gamma_\alpha &= \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - \frac{1}{12}\left[\alpha\frac{\log^{\alpha-1}(2)}{4} - \frac{\log^\alpha(2)}{4}\right] + \int_1^\infty P_3(x)f_\alpha'''(x)dx \\ &= \frac{13\log^\alpha(2)}{48} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - \frac{\alpha\log^{\alpha-1}(2)}{48} + \int_1^\infty P_3(x)f_\alpha'''(x)dx.\end{aligned}$$

Now, note that the maxima of the first three terms are attained when $\alpha = 0$. Since the bound obtained in Lemma 5.1 also holds for the absolute value of the integral $\int_2^\infty P_1(x)f_\alpha'(x)dx$, we immediately obtain the wanted bound: $|\gamma_\alpha| \le 0.436$. □

Lemma 5.3. *For all $\alpha > 0$, we have*

$$(i)\ \frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} < 0.348 \quad \text{and} \quad (ii)\ \frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} \leq 0.323.$$

Proof. Combining the bound for $|\gamma_\alpha|$ proved in Lemma 5.2 and the fact that $\Gamma(\alpha+1) \geq \Gamma(3/2) = \frac{\sqrt{2\pi}}{2}$, for $0 < \alpha \leq 1$, we deduce that $\frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} < \frac{0.436}{\frac{\sqrt{2\pi}}{2}} < 0.348$ in the region $0 < \alpha \leq 1$.

Now, in the complementary region $\alpha > 1$, by (5.7), for all $1 \leq n < \alpha$, we have

$$\begin{aligned}
\frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} &\leq \frac{4}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!} \\
&\leq \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \left(\frac{4(n+1)}{e}\right)^{n+1} \\
&= \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)^{\alpha-n}} \left(\frac{4}{e}\right)^{n+1} \\
&\leq 4\sqrt{2}\left(\frac{2}{\pi e}\right)^{n+1}.
\end{aligned}$$

Letting $n = 1$ we have

$$\frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} \leq 4\sqrt{2}\left(\frac{2}{\pi e}\right)^2 \leq 0.311.$$

which is an even sharper bound. Together, these two bounds prove (i) for all $\alpha > 0$.

Similarly, to justify (ii), note that since $\alpha + 1 > 1$, the equation (5.7) with $n = 1$ yields

$$\frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} \leq \frac{4\Gamma(\alpha+2)4!}{(2\pi)^2 2^{\alpha+2} 2!\Gamma(\alpha+1)} = \frac{12(\alpha+1)}{(2\pi)^2 2^\alpha}. \tag{5.14}$$

The maximum of $g(\alpha) = \frac{\alpha+1}{2^\alpha}$ is at $\alpha = \frac{1}{\log(2)} - 1$. This immediately yields the result (ii). $\square$

5.3 A Zero Free Region

We need one more technical lemma before we can prove our main theorem.

Lemma 5.4. *For all* $\alpha > 0$ *and* $n \in \mathbb{N} \cup \{0\}$,

$$\frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^\alpha} \;<\; \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2} \;<\; 1.036,$$

where

$$\alpha_1 = \frac{\sqrt{5\log^2(3)+4}}{2\log(3)} + \frac{1}{\log(3)} - \frac{3}{2}.$$

Proof. We proceed by induction on n. For $n = 0$ we have

$$\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)2!3^\alpha} = \frac{\alpha^2+3\alpha+2}{3^\alpha 2}.$$

The maximum of $g(\alpha) = \frac{\alpha^2+3\alpha+2}{3^\alpha 2} = \frac{(\alpha^2+3\alpha+2)e^{-\alpha\log(3)}}{2}$ is at $\alpha_1 = \frac{\sqrt{5\log^2(3)+4}}{2\log(3)} + \frac{1}{\log(3)} - \frac{3}{2}$, with $g(\alpha_1) = 1.0356$. Now, let us assume that, for all integers j with $1 \le j \le n$, we have

$$\frac{\Gamma(\alpha+j+3)}{\Gamma(\alpha+1)(j+2)!2^j(j+3)^\alpha} \;\le\; \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}.$$

We will show the assertion is true for $j = n+1$. Applying the induction hypothesis gives

$$\begin{aligned}\frac{\Gamma(\alpha+j+3)}{\Gamma(\alpha+1)(j+2)!2^j(j+3)^\alpha} &= \frac{\Gamma(\alpha+n+4)}{\Gamma(\alpha+1)(n+3)!2^{n+1}(n+4)^\alpha} \\ &= \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}\frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^\alpha} \\ &\le \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}\frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}.\end{aligned} \tag{5.15}$$

Hence, all we need to show is that $\frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3} \le 1$. However, notice that the function $g(\alpha) = \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}$ is positive for all $\alpha > 0$; and taking the logarithmic derivative we get

$$\frac{g'(\alpha)}{g(\alpha)} = \log\left(\frac{n+3}{n+4}\right) + \frac{1}{\alpha+n+3} \le -\frac{1}{n+4} - \frac{1}{2}\left(\frac{1}{n+4}\right)^2 + \frac{1}{\alpha+n+3},$$

since, from the Taylor's Theorem, we know that $\log(1-x) \le -x - \frac{1}{2}x^2$, in the range $0 \le x < 1$. Moreover, $\frac{1}{\alpha+n+3} \le \frac{1}{n+4}$, and since $g(\alpha) > 0$, we can conclude that $g'(\alpha) < 0$. Therefore $g(\alpha)$ is decreasing in the interval $[1, \infty)$, with the maximum at $g(1) = \frac{1}{2}$.

On the other hand, if $0 < \alpha < 1$, the maximum of $\left(\frac{n+3}{n+4}\right)^\alpha$ is attained at $\alpha = 0$. And since $\frac{\alpha+n+3}{n+3} < \frac{n+4}{n+3} = 1 + \frac{1}{n+3} \le \frac{4}{3}$, we have $g(\alpha) < \frac{1}{2}\frac{4}{3} = \frac{2}{3}$, for $\alpha \in (0,1)$. Combining these two results in (5.15), we deduce the bound for $j = n+1$. This completes the inductive proof. □

Now we are ready to prove our main result.

Theorem 5.5. *For all $\alpha \ge 0$, $D_s^\alpha[\zeta(s)] \ne 0$ in the region $|s-1| < 1$.*

Proof. For $\alpha = 0$, the reader is referred to [5]. We prove that $\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)} D_s^{\alpha}[\zeta(s)] \neq 0$ in the region $|s-1| < 1$. Starting with (5.3), we are able to write

$$\begin{aligned}\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| &= \left|1 + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(s-1)^{\alpha+n+1}}{\Gamma(\alpha+1)n!}\right| \\ &\geq 1 - \frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} - \frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} - \sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}.\end{aligned}$$

Applying Lemma 5.3, we see that

$$\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| \; > \; 1 - 0.492 - 0.323 - \sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}. \tag{5.16}$$

We can focus now on finding an upper bound for $\sum\limits_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}$. By (5.7) we have

$$\frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!} \; \leq \; \frac{4\Gamma(\alpha+n+1)(2(n+1))!}{(2\pi)^{n+1}(n+1)^{\alpha+n+1}(n+1)^{\alpha+n+1}(n+1)!n!\Gamma(\alpha+1)}.$$

It follows from Stirling's formula that $\frac{(2n)!}{n!} \leq \sqrt{2}\left(\frac{4n}{e}\right)^n$ for all integers $n \geq 1$. Therefore

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!} \; &\leq \; \sum_{n=2}^{\infty} \frac{4\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)^{\alpha+n+1}n!\Gamma(\alpha+1)}\sqrt{2}\left(\frac{4(n+1)}{e}\right)^{n+1} \\ &= \; \sum_{n=2}^{\infty} \frac{4\sqrt{2}\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)^{\alpha}n!\Gamma(\alpha+1)}\left(\frac{4}{e}\right)^{n+1} \\ &= \; 4\sqrt{2}\left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^{\alpha}}\left(\frac{4}{\pi e}\right)^n \\ &\leq \; 4\sqrt{2}\left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}\left(\frac{4}{\pi e}\right)^n \; < \; 0.142,\end{aligned}$$

by Lemma 5.4 (and with the same notation). Using this bound in (5.16), we obtain

$$\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| \; > \; 1 - 0.492 - 0.323 - 0.142 > 0.$$

We conclude that $D_s^\alpha[\zeta(s)] \neq 0$, for all $\alpha > 0$, in the region $|s-1| < 1$. $\square$

CHAPTER VI

CONCLUSION

In this work we covered four topics concerning the Riemann Zeta function or one of it's generalizations, namely the Hurwitz Zeta functions and the derivatives or fractional derivatives of these functions. In Chapter II we presented zeros of the derivatives, $\zeta^{(k)}(\sigma + it)$, of the Riemann Zeta function for $k \leq 28$ within the complex rectangular region defined by $-10 < \sigma < \frac{1}{2}$ and $-10 < t < 10$. Our computations show an interesting behavior of the zeros of $\zeta^{(k)}$, in the sense that they seem to lie on curves that extend certain chains of zeros of $\zeta^{(k)}$ observed on the right half plane.

In Chapter III, we discussed the fractional (or non-integral generalized) Stieltjes constants, $\gamma_\alpha(a)$. We showed that these constants arose naturally from the Laurent series expansions of the fractional derivatives of the Hurwitz Zeta functions, $\zeta^{(k)}(s, a)$. We showed that by using the Grünwald-Letnikov fractional derivative, one could prove a conjecture put forth by Kreminski in [21].

We discussed methods of evaluating the fractional Stieltjes constants in chapter IV. We also found a new upper bound for $|\gamma_\alpha(a)|$ that is sharper, for $n > 100$, than the previously known bounds given by Berndt [5], Williams and Zhang [39], and Matsuoka [28].

In Chapter V, we found a zero free region about 1 for all fractional derivatives of the Riemann Zeta function. That is, we showed that for any $\alpha \in \mathbb{R}$, $D_s^\alpha[\zeta(s)] \neq 0$ inside the region $|s - 1| < 1$.

6.1 Future Work

We have shown that fractional derivatives can be employed to give insight into the behavior of the Riemann Zeta function and its derivatives. We also showed that using fractional differentiation, one can derive new and exciting results connected to the Riemann Zeta function.

One direction of future work is to prove that zeros of $\zeta^{(k)}$ lie on curves which extend from chains of zeros of $\zeta^{(k)}$ observed on the right half plane. Using fractional differentiation may lead to further insights into this observed pattern. This method may also be a direction that one could use to prove any such insights.

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