On a Product of Finite Monoids

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F. Blanchet-Sadri and F.D. Gaddis, "On a Product of Finite Monoids." Semigroup Forum, Vol. 57, 1998, pp 75-91. DOI: 10.1007/PL00005969

Abstract:
In this paper, for each positive integer m, we associate with a finite monoid S0 and m finite commutative monoids $S_1,\ldots, S_m$, a product $\lozenge_m(S_m,\ldots, S_1, S_0)$. We give a representation of the free objects in the pseudovariety $\lozenge_m(\mathbf{W}_m,\ldots, \mathbf{W}_1, \mathbf{W}_0)$ generated by these $(m + 1)$-ary products where $S_i \in W_i$ for all $0 \leq i \leq m$. We then give, in particular, a criterion to determine when an identity holds in $\lozenge_m(\mathbf{J}_1,\ldots,\mathbf{J}_1,\mathbf{J}_1)$ with the help of a version of the Ehrenfeucht-Fraïssé game ($\mathbf{J}_1$ denotes the pseudovariety of all semilattice monoids). The union $\bigcup_{m>0}\lozenge_m(\mathbf{J}_1,\ldots, \mathbf{J}_1, \mathbf{J}_1)$ turns out to be the second level of the Straubing's dot-depth hierarchy of aperiodic monoids.

Article:

1. Introduction

The theory of varieties of Eilenberg constitutes an elegant framework for discussing relationships between combinatorial descriptions of languages and algebraic properties of their recognizers. The interplay between the two points of view leads to interesting classifications of languages and finite monoids.

A variety of languages is often described as the smallest variety closed under a given class of operations, such as boolean operations, concatenation product, star, and so on. A variety of finite monoids or semigroups (or pseudovariety) is also often described with the help of operations: join, semidirect and two-sided semidirect products, and Sch utzenberger product, to name a few. In view of Eilenberg's correspondence between varieties of languages and varieties of finite monoids or semi-groups, we may expect some relationships between the operations on languages (of combinatorial nature) and the operations on monoids or semigroups (of algebraic nature).

Examples of correspondence between varieties of languages and varieties of finite monoids or semigroups include: the variety of rational (or regular) languages corresponds to the variety of all finite monoids [19]; the aperiodic or star-free languages correspond to the finite aperiodic monoids [28]; the piecewise testable languages to the finite $\mathcal{J}$-trivial monoids [29] and the locally testable languages to the finite locally idempotent and commutative semigroups [14, 20].

In this paper, we construct an $(m + 1)$-ary product of finite monoids and give a relationship between this operation on monoids and a version of the EhrenfeuchtFraïssé game corresponding to the levels of the so-called "dot-depth" hierarchy of aperiodic languages.

1.1. The Straubing hierarchy

A monoid S is said to be *aperiodic* if all groups in S are trivial. The pseudovariety of all aperiodic monoids is denoted by $\mathbf{A}$. A result of Schützenberger [28] enables us to describe the *-variety A of aperiodic languages corresponding to the pseudovariety **A**. For a finite alphabet A, the class A^*A is the least class of languages of A^* (the free monoid generated by A) satisfying the following three conditions:

- A^*A is closed under finite boolean operations,
- If $L, L' \in A^*A$, then the concatenation $LL' \in A^*A$,
- $\{u\} \in A^*A$ for all $u \in A^*$.

Cohen and Brzozowski [15] introduced the dot-depth hierarchy for the aperiodic languages of $A^+ = A^* \setminus \{1\}$ (1 denotes the empty word), and Straubing [31] defined another hierarchy for the aperiodic languages of A^*. The Straubing hierarchy is a doubly indexed hierarchy for the class A^*A. It grows out in a natural manner from the applications of concatenation and boolean operations. We proceed inductively starting with $A^*V_0 = \{\emptyset, A^*\}$. Assuming that A^*V_{k-1} is already defined for some $k > 0$, the class A^*V_k is defined as the boolean closure of all languages of the form $L_0a_1L_1\ldots a_iL_i$, with $L_0,\ldots, L_i \in A^*V_{k-1}$ and $a_1,\ldots, a_i \in A$. Clearly, $A^*V_0 \subseteq A^*V_1 \subseteq \cdots$ and the union is the class A^*A of all aperiodic languages of A^*. Within each A^*V_k ($k > 0$) one can again establish a hierarchy by defining $A^*V_{k,m}$ to be the boolean closure of the languages $L_0a_1L_1\ldots a_iL_i$ with $i \leq m$ and with $L_0,\ldots, L_i \in A^*V_{k-1}$ and $a_1,\ldots, a_i \in A$ as before. Then $A^*V_{k,1} \subseteq A^*V_{k,2} \subseteq \cdots$ and the union is A^*V_k.

The class $V_k = \{A^*V_k\}$ is a *-variety of languages for $k \geq 0$, and the classes $V_{k,m} = \{A^*V_{k,m}\}$ with $k > 0$, $m > 0$ form *-varieties of languages. For each of the *-varieties V_k ($k \geq 0$) we will denote by $\mathbf{V}_k$ the corresponding pseudovariety of monoids. Similarly, $\mathbf{V}_{k,m}$ will denote the pseudovariety of monoids corresponding to $V_{k,m}$ for $k > 0$, $m > 0$. We have the following result of Simon [29]: $\mathbf{V}_1$ is decidable or $\mathbf{V}_1 = \mathbf{J}$. In the language of the Green relations, **J** is the pseudovariety of monoids in which the $\mathcal{J}$-relation is trivial.

1.2. Games and the Straubing hierarchy

Thomas [35] gave a purely logical proof of the strictness of the Straubing hierarchy (the original proof of the strictness of the dot-depth hierarchy is due to Brzozowski and Knast [13]). Thomas' proof is based on an Ehrenfeucht-Fraïssé game which we now describe.

Let A be a finite alphabet. Consider the set of symbols $\mathcal{L} = \{<\} \cup \{Qa \mid a \in A\}$. We identify each word u of A^* (of length $|u|$) with an $\mathcal{L}$-structure

$$u = (\mathcal{U}_u, <^u, (Q_a^u)_{a\in A})$$

where $\mathcal{U}_u = \{1,\ldots, |u|\}$ represents the set of positions of letters in u, $<^u$ is the natural ordering on the integers $1,\ldots, |u|$ and, for each $a \in A$, Q_a^u is the subset of $\mathcal{U}_u$ defined by $i \in Q_a^u$ if and only if the ith letter of u is an a.

Let $u = (\mathcal{U}_u, <^u, (Q_a^u)_{a\in A})$, $v = (\mathcal{U}_v, <^v, (Q_a^v)_{a\in A})$ be $\mathcal{L}$-structures. Let $\overline{m} = (m_1,\ldots,m_k)$ be a k-tuple of positive integers, where $k \geq 0$. The Ehrenfeucht-Fraïssé game $G_{\overline{m}}(u, v)$ corresponding to $\overline{m}$ and u, v is played by two players, I and II, according to the following rules:

The letter k is the number of moves each player has to make in the course of a play of the game $G_{\overline{m}}(u,v)$. These moves are begun by Player I, and both players move alternately. The ith move consists of choosing m_i positions from $\mathcal{U}_u$ or from $\mathcal{U}_v$. If Player I chooses m_i positions from $\mathcal{U}_u$ in his ith move, then Player II must choose m_i positions from $\mathcal{U}_v$ in his ith move. If Player I chooses m_i positions from $\mathcal{U}_v$ in his ith move, then Player II must choose m_i positions from $\mathcal{U}_u$. After the kth move of Player II the play is completed. Altogether some positions $p_1,\ldots,p_n \in \mathcal{U}_u$ and $q_1,\ldots,q_n \in \mathcal{U}_v$ have been chosen where $n = m_1 + \cdots + m_k$. Player II has won the play if the following two conditions are satisfied:

- $p_i <^u p_j$ if and only if $q_i <^v q_j$ for all $1 \leq i,j \leq n$,
- $pi \in Q_a^u$ if and only if $q_i \in Q_a^v$ for all $1 < i < n$ and $a \in A$.

We say that Player *II* has a winning strategy in $G_{\overline{m}}(u,v)$ and write "$u \sim_{\overline{m}} v$" if it is possible for him to win each play. The equivalence $\sim_{\overline{m}}$ naturally defines a congruence on A^* of finite index. The $\sim_{\overline{m}}$-class of u is $\{v \in A^* \mid u \sim_{\overline{m}} v\}$ and will be denoted by $[u]_{\overline{m}}$. The set of all $\sim_{\overline{m}}$-classes, $A^*/\sim_{\overline{m}}$, will be denoted by $A^*/\overline{m}$. This set becomes a monoid by considering the operation $[[u]_{\overline{m}}[v]_{\overline{m}} = [uv]_{\overline{m}}$; $[1]_{\overline{m}}$ acts as unit.

The importance of $\sim_{\overline{m}}$ lies in the fact that $\mathbf{V}_k$ can be described in terms of the congruences $\sim_{(m_1,\ldots,m_k)}$. Thomas [34, 35], and Perrin and Pin [21] infer that the monoids $A^*/(m_1,\ldots,m_k)$ (respectively $A^*/(m, m_1,\ldots,m_{k-1})$) form a family of finite monoids that generate $\mathbf{V}_k$ (respectively $\mathbf{V}_{k,m}$) in the sense that every finite aperiodic monoid in $\mathbf{V}_k$ (respectively $\mathbf{V}_{k,m}$) is a morphic image of a monoid of the form $A^*/(m_1,\ldots,m_k)$ (respectively $A^*/(m, m_1,\ldots, m_{k-1})$). In [12], we give a reduced family of generators for $\mathbf{V}_k$. In particular, we show that the monoids $A^*/(m,1)$ form a family of monoids that generate $\mathbf{V}_2$.

The problem remains open as to whether $\mathbf{V}_k$ is decidable for $k \geq 2$. Partial results have been obtained mostly for the 2nd level (Blanchet-Sadri [3, 4, 5, 6, 7, 8, 9, 10, 11, 12], Cowan [16], Pin [25], Straubing [25, 32, 33] and Weil [33, 36, 38]).

For fixed $\overline{m}$, we define the pseudovariety of monoids $\mathbf{V}_{\overline{m}}$, as follows: an A-generated monoid S is in $\mathbf{V}_{\overline{m}}$ if and only if S is a morphic image of $A^*/\overline{m}$. Note that $\mathbf{V}_{(1)} = \mathbf{J}_1$ the pseudovariety of semilattice monoids.

The congruences $\sim_{\overline{m}}$ can be defined inductively as follows: First, $u \sim_{(1)} v$ if and only if $\alpha(u) = \alpha(v)$, or u and v have the same set of letters. Then, $u \sim_{(m)} v$ if and only if u and v have the same set of subwords of length $\leq m$. Now, if $u = a_1\ldots a_n$ is a word on A, then denote the segment $a_i\ldots a_j$ by $u[i, j]$ for all $1 \leq i \leq j \leq n$.

Lemma 1.[Blanchet-Sadri [3]] *Let u and v be words on a finite alphabet A. For all positive integers m and tuples of positive integers $\overline{m}$, we have that $u \sim_{(m,\overline{m})} v$ if and only if*

- *For every $p_1,\ldots, p_m \in \mathcal{U}_u$ (where $p_1 < \cdots < p_m$), there exist $q_1,\ldots,q_m \in \mathcal{U}_v$ (where $q_1 < \cdots < q_m$) such that*

    1. *$p_i \in Q_a^u$ if and only if $q_i \in Q_a^u$ for all $1 \leq i \leq m$ and $a \in A$,*
    2. *$u[1, p_1 - 1] \sim_{\overline{m}} v[1, q_1 - 1]$,*
    3. *$u[p_i + 1, p_{i+1} - 1] \sim_{\overline{m}} v[q_i + 1, q_{i+1} - 1]$ for all $1 \leq i < m$,*
    4. *$u[p_m + 1, |u|] \sim_{\overline{m}} v[q_m + 1, |v|]$, and*

- *For every $q_1,\ldots, q_m \in \mathcal{U}_v$ (where $q_1 < \cdots < q_m$), there exist $p1,\ldots,p_m \in \mathcal{U}_u$ (where $p_1 < \cdots < p_m$) such that 1–4 hold.*■

When there is no chance of confusion, we will write u instead of $[u]_{\overline{m}}$. We note that $ab = ba$ in $\{a, b\}^*/(1)$, but ab ≠ ba in $\{a, b\}^*$.

Definition 2. Let u be a word on a finite alphabet A. For all positive integers m and tuples of positive integers $\overline{m}$, $\alpha_{(m,\overline{m})}(u)$ consists of the set

$$\{(u_0, a_1, u_1, \ldots, a_m, u_m) \in A^*/\overline{m} \times A \times (A^*/\overline{m}\ (A \cup \{1\})^{m-1} \times A^*/\overline{m} \mid u_0a_1u_1 \ldots a_mu_m = u \text{ and whenever } a_i = 1, \text{we have } u_{i-1} = 1\}.$$

For example, if $A = \{a, b\}$, then $a_{(2,1)}(abc)$ is the set

$$\{(1, a, 1,1, bc), (1, a, 1, b, c), (a, b, 1,1, c), (1, a, b, c, 1), (a, b, 1, c, 1), (ab, c, 1,1,1)\}.$$

Lemma 1 states that u $\sim_{(m,\overline{m})}$v if and only if $\alpha_{(m,\overline{m})}$ (u) = $\alpha_{(m,\overline{m})}$ (v) .

In Section 2 of this paper, some background information is presented. In Section 3, we first review the two-sided semidirect product of monoids. Next, we associate with a monoid S_0 and m commutative monoids $S_1,\ldots,S_m$, an $(m + 1)$-ary product $\Diamond_m(S_m,\ldots,S_1, S_0)$ (Proposition 5). The pseudovariety $\Diamond_m(\mathbf{W}_m,\ldots,\mathbf{W}_1, \mathbf{W}_0)$ is defined as being generated by these $(m + 1)$-ary products where $S_i \in \mathbf{W}_i$ for all $0 \leq i \leq m$.

Section 3.1 gives a criterion to determine when an identity holds in the two-sided semidirect product $\mathbf{J}_1 ** \mathbf{V}_{\overline{m}}$ with the help of $\sim_{(1,\overline{m})}$, and Section 3.2 gives such a criterion for $\Diamond_m(\mathbf{J}_1,\ldots,\mathbf{J}_1,\mathbf{V}_{\overline{m}})$ with the help of $\sim_{(m,\overline{m})}$. The essential ingredients in our proofs are a two-sided semidirect product representation of the free objects in **V** ** **W** due to Almeida and Weil [2] stated in Theorem 3, and our representation of the free objects in $\Diamond_m(\mathbf{W}_m, \ldots,\mathbf{W}_1, \mathbf{W}_0)$ stated in Theorem 6. The equalities $\mathbf{J}_1 ** \mathbf{V}_{\overline{m}} = \mathbf{V}_{(1,\overline{m})}$, $\Diamond_m(\mathbf{J}_1,\ldots,\mathbf{J}_1, \mathbf{V}_{\overline{m}}) = \mathbf{V}_{(m,\overline{m})}$ and $\mathbf{V}_2 = \bigcup_{m>0} \Diamond_m(\mathbf{J}_1,\ldots,\mathbf{J}_1,\mathbf{J}_1)$ result (Corollaries 4, 7 and 8).

2. Preliminaries

This section is devoted to reviewing basic properties of pseudovarieties of monoids and of *-varieties of recognizable languages. The reader is referred to the books of Almeida [1], Eilenberg [17] and Pin [22] for further definitions and background.

2.1. Pseudovarieties of monoids

A *pseudovariety* (of monoids) **V** is a family of finite monoids that satisfies the following two conditions:

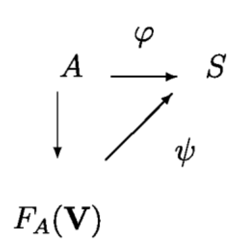
- If $S \in \mathbf{V}$ and $T < S$ (*T divides S*), then $T \in \mathbf{V}$,
- If $S, T \in \mathbf{V}$, then the cartesian product $S \times T \in \mathbf{V}$.

For any family C of finite monoids, we denote by (C) the least pseudovariety of monoids containing C. Clearly, $S \in (C)$ if and only if $S < S_1 \times \cdots \times S_i$ with $S_1,\ldots, S_i \in C$. We call (C) the pseudovariety of monoids *generated* by C.

An (monoid) *identity* on an alphabet A is a pair (u, v) of words of A^*, usually indicated by a formal equality $u = v$. Given $u, v \in A^*$ and given a monoid S, we will say that S *satisfies* the identity $u = v$ (or that the identity $u = v$ *holds* in S) and we write $S \models u = v$ if $\varphi(u) = \varphi(v)$ for every morphism $\varphi: A^* \to S$ of monoids. For an identity $u = v$ and a pseudovariety **V**, the notation $\mathbf{V} \models u = v$ will abbreviate the fact that each $S \in \mathbf{V}$ satisfies $u = v$.

Work of Eilenberg and Schützenberger [18] showed that pseudovarieties of monoids are ultimately defined by sequences of identities (that is, a monoid belongs to the given pseudovariety if and only if it satisfies all but finitely many of the identities in the sequence), and that finitely generated pseudovarieties of monoids are defined by sequences of identities or are equational (that is, a monoid belongs to the given pseudovariety if and only if it satisfies all the identities in the sequence).

The free object on the alphabet A in the variety generated by a pseudovariety **V** will be denoted by $F_A(\mathbf{V})$. We say that $F_A(\mathbf{V})$ has the universal property for **V** on A in the following sense: for any $S \in \mathbf{V}$ and any function φ : $A \to S$, there exists (precisely) one morphism $\psi : F_A(\mathbf{V}) \to S$ such that the diagram



commutes, where $A \to F_A(\mathbf{V})$ is the function that sends each $a \in A$ to a (we also say that the map $A \to F_A(\mathbf{V})$ has the universal property). The morphism ψ is defined by $\psi(a_1 \ldots a_n) = \varphi(a_1)\ldots\varphi(a_n)$ and is said to be the natural extension of φ to $F_A(\mathbf{V})$.

2.2. *-Varieties of languages

Let A be a finite alphabet. Let L be a language of A^*. We define a congruence $\sim_L$ on A^* as follows: $u \sim_L v$ holds if $xuy \in L$ if and only if $xvy \in L$ for all $x, y \in A^*$. The congruence $\sim_L$ is called the *syntactic congruence* of L, and the quotient monoid $A^*/\sim_L$, which we denote by $S(L)$, is called the *syntactic monoid* of L. The language L is recognizable if and only if $S(L)$ is a finite monoid.
A *-class $V = \{A^*V\}$ consists of a family A^*V of recognizable languages of A^* defined for every finite alphabet A. We will say that a *-class V satisfying the following three conditions is a **-variety* (of languages):

- A^*V is closed under boolean operations,
- If $L \in A^*V$ and $a \in A$, then the sets $a^{-1}L = \{u\ E\ A^* \mid au \in L\}$ and $La^{-1} = \{u \in A^* \mid ua \in L\}$ are in A^*V,
- If $\varphi : B^* \to A^*$ is a morphism of monoids and if $L \in A^*V$, then $\varphi^{-1}(L) \in B^*V$.

Eilenberg [17] established a one-to-one correspondence between pseudovarieties of monoids and *-varieties of languages. For each *-variety V, define the pseudovariety

$$\boldsymbol{V} = (\{S(L) | L \in A^*V \ for \ some \ A\})$$

generated by the syntactic monoids of languages in V. For each pseudovariety of monoids $\mathbf{V}$, define the *-variety V by

$$A * V = \{L \subseteq A^* | S(L) \in V\}.$$

3. Products of monoids

We now proceed with the products mentioned in the introduction. A related product is the Schützenberger product that has been studied by several authors [23, 24, 25, 26, 28, 30].

3.1. The two-sided semidirect product

In this section, we review the definition of the two-sided semidirect product of monoids, a 2-ary product introduced by Rhodes and Tilson [27].

Let S be an additively written monoid with unit 0 (commutativity for S is not assumed). Let T be a multiplicatively written monoid with unit 1.

A *left unitary action* of T on S is a function

$$f : S \times T \to S$$
$$(x, y) \mapsto yx,$$

where for all $x_1, x_2 \in S$ and $y_1, y_2 \in T$, f satisfies the following four conditions:

- $y_1(x_1 + x_2) = y_1x_1 + y_1x_2$,
- $y_1(y_2x_1) = (y_1y_2)x_1$,
- $1x_1 = x_1$,
- $y_10 = 0$.

Let f be a left unitary action of T on S. Then associated with $S \times T$ and f is the *semidirect product* $S * T$ with operation

$$(x_1, y_1)(x_2, y_2) = (x_1 + y_1x_2, y_1y_2).$$

This operation is associative and $S * T$ is a monoid with unit (0, 1).

A *right unitary action* of T on S is a function

$$g: S \times T \to S$$
$$(x, y) \mapsto xy,$$

where for all $x_1, x_2 \in S$ and $y_1, y_2 \in T$, g satisfies the following four conditions:

- $(x_1 + x_2)y_1 = x_1y_1 + x_2y_1$,
- $(x_1y_1)y_2 = x_1(y_1y_2)$,
- $x_11 = x_1$,
- $0y_1 = 0$.

Let f be a left unitary action and g be a right unitary action of T on S. Assume for all $x \in S$ and $y_1, y_2 \in T$, that f and g satisfy the following condition:

- $y_1(xy_2) = (y_1x)y_2$.

Then associated with $S \times T$ and f, g is the *two-sided semidirect product* $S ** T$ with operation

$$(x_1, y_1)(x_2, y_2) = (x_1y_2 + y_1x_2, y_1y_2).$$

This operation is associative and $S ** T$ is a monoid with unit (0, 1). When g is trivial, then $S ** T$ is in fact a semidirect product. Neither * nor ** is associative on monoids.

For pseudovarieties of monoids **V** and **W**, their semidirect product **V** * **W** (respectively two-sided semidirect product **V** ** **W**) is defined to be the pseudovariety of monoids generated by all semidirect products $S * T$ (respectively two-sided semidirect products $S ** T$) with $S \in$ **V** and $T \in$ **W**. The operation * is associative on pseudovarieties of monoids but ** is not.

We now give the following representation of free objects for **V** ** **W** obtained by Almeida and Weil. In general, the free object on the alphabet A in the variety generated by a pseudovariety **V**, $F_A(\mathbf{V})$, does not lie in **V**. We have $F_A(\mathbf{V})$ *E* **V** if and only if $F_A(\mathbf{V})$ is finite.

Theorem 3. [Almeida and Weil [2]]
Let **V** *and* **W** *be two pseudovarieties of monoids such that* $F_A(\mathbf{V})$ *and* $F_A(\mathbf{W})$ *are finite for all finite alphabets A. Then so is* **V** ** **W**.

Moreover, if A is a finite alphabet, there exists a one-to-one morphism from $F_A(\mathbf{V} ** \mathbf{W})$ *into* $F_B(\mathbf{V}) ** F_A(\mathbf{W})$ *given by* $a \mapsto ((1, a, 1), a)$, *where* $B = F_A(\mathbf{W}) \times A \times F_A(\mathbf{W})$, *the left unitary action of* $F_A(\mathbf{W})$ *on* $F_B(\mathbf{V})$ *is defined by* $x(y, a, z) = (xy, a, z)$ *and the right unitary action by* $(y, a, z)x = (y, a, zx)$ *for all* $x, y, z \in F_A(\mathbf{W})$ *and* $a \in A$. ■

We end this section with a criterion to determine when an identity is satisfied in the two-sided semidirect product **J**1 ** $\mathbf{V}_{\overline{m}}$.

Corollary 4. *Let A be a finite alphabet and $\overline{m}$ be a tuple of positive integers. An identity u = v on A holds in* $\mathbf{J}_1$ ** $\mathbf{V}_{\overline{m}}$ *if and only if* $u \sim_{(1,\overline{m})} v$. *Consequently, an A-generated monoid S belongs to* $\mathbf{J}_1$ ** $\mathbf{V}_{\overline{m}}$ *if and only if S is a morphic image of* $A^*/(1, \overline{m})$, *or* $\mathbf{J}_1$ ** $\mathbf{V}_{\overline{m}} = \mathbf{V}_{(1,\overline{m})}$.

Proof. Let T be the multiplicatively written monoid $F_A(\mathbf{V}_{\overline{m}})$ with unit 1 ($F_A(\mathbf{V}_{\overline{m}}$ is isomorphic to $A^*/\overline{m}$) and let S be the additively written monoid $F_{T\times A\times T}(\mathbf{J}_1)$ with unit 0. Consider the left unitary action of T on S defined by $x(y, a, z) = (xy, a, z)$ and the right unitary action of T on S defined by $(y, a, z)x = (y, a, zx)$ for all $x, y, z \in T$ and $a \in A$, and the associated two-sided semidirect product S ** T.

Now consider the one-to-one morphism of Theorem 3, $\varphi : F_A(\mathbf{J}_1$ ** $\mathbf{V}_{\overline{m}}) \rightarrow S$ ** T, where for all $a \in A$, $\varphi(a) = ((1, a, 1), a)$. If u and v are words on A, then $u = v$ holds in $\mathbf{J}_1$ ** $\mathbf{V}_{\overline{m}}$ if and only if $u = v$ holds in $F_A(\mathbf{J}_1$ ** $\mathbf{V}_{\overline{m}})$ if and only if $\varphi(u) = \varphi(v)$ if and only if $u \sim_{(1,\overline{m})} v$. To see that "$\varphi(u) = \varphi(v)$" and "$u \sim_{(1,\overline{m})} v$" are equivalent, the morphism φ maps the word $u = a_1 \ldots a_i$ into the 2-tuple

$$\big((1, a_1, a_2 \ldots a_i) + (a_1, a_2, a_3 \ldots a_i) + \ldots + (a_1 \ldots a_{i-1}, a_i, 1), a_1 \ldots a_i\big), (1)$$

and $v = b_1 \ldots b_j$ into

$$\Big(\big(1, b_1, b_2 \ldots b_j\big) + \big(b_1, b_2, b_3 \ldots b_j\big) + \ldots + \big(b_1 \ldots b_{j-1}, b_j, 1\big), b_1 \ldots b_j\Big). (2)$$

The equality $\varphi(u) = \varphi(v)$ holds if and only if corresponding components of the 2- tuples (1) and (2) are equal. If u' (respectively v') denotes the first component of (1) (respectively (2)), then the condition "the first components of (1) and (2) are equal" is equivalent to $S \models u' = v'$ or $\alpha_{(1,\overline{m})}(u) = \alpha_{(1,\overline{m})}(v)$, and the condition "the second components of (1) and (2) are equal" is equivalent to $T \models u = v$ or $u \sim_{\overline{m}} v$. The condition $\sim_{(1,\overline{m})} (u)$ = ~(1,¯*m*) (*v*) (which is clearly equivalent to ~(1,¯m) (*u*) = ~(1,¯*m*) (*v*) by Definition 2) implies the condition *u* —*m*¯ *v*. Hence, we conclude that $W(u) = W(v)$ if and only if *u* —(1,¯*m*) *v*.

The equality $\mathbf{J}_1$ ** $\mathbf{V}_k = \mathbf{V}_{k+1,1}$ is known to Weil (this is a particular case of Proposition 2.12 in [371).

3.2. An $(m + 1)$**-ary product**

In this section, we construct an $(m + 1)$-ary product of monoids for each positive integer m.

Let S_0 be a multiplicatively written monoid with unit 1. Let S_i ($1 \le i \le m$) be an additively written *commutative* monoid with operation $+_i$ and unit 0_i. For the following discussion, we will use subscripts to indicate the monoid and different letters to indicate different members within the monoid (for instance $x_i, y_i \in Si$).

For $0 \le i, j \le m$ and $i + j \le m$, let $h_{i,j}$ be a function

$$h_{i,j} : S_i \times S_j \rightarrow S_{i+j}$$
$$(x_i, x_j) \mapsto x_i x_j.$$

Now assume that $h_{0,0}(x_0, y_0)$ is the product x_0y_0 in S_0 and that

1. $x_i(x_j +_j y_j) = x_ix_j +_{i+j} x_iy_j$ for all $0 \le i \le m$, $1 \le j \le m$ with $i + j \le m$,
2. $(x_i +_i y_i)x_j = x_ix_j +_{i+j} y_ix_j$ for all $1 \le i \le m$, $0 \le j \le m$ with $i + j \le m$,
3. $x_i(x_jx_k) = (x_ix_j)x_k$ for all $0 \le i, j, k \le m$ with $i + j + k \le m$,
4. $x_i1 = x_i = 1x_i$ for all $0 < i < m$,
5. $x_i0_j = 0_{i+j}$ for all $0 \le i \le m$, $1 \le j \le m$ with $i + j \le m$,
6. $0_ix_j = 0_{i+j}$ for all $1 \le i \le m$, $0 \le j \le m$ with $i + j \le m$.

Then associated with $S_m \times \cdots \times S_1 \times S_0$ and the functions $h_{i,j}$ is the $(m + 1)$-*ary product* $\lozenge_m(S_m, \ldots, S_1, S_0)$ with operation

$$(x_m, \ldots, x_1, x_0)(y_m, \ldots, y_1, y_0) = (z_m, \ldots, z_1, z_0),$$

where $z_0 = x_0y_0$ and for all $1 < i < m$,

$$z_i = x_iy_0 +_i x_{i-1}y_1 +_i \ldots +_i x_1y_{i-1} +_i x_0y_i.$$

Proposition 5. *The above operation is associative and* $\lozenge_m(S_m, \ldots, S_1, S_0)$ *is a monoid with unit* $(0_m, \ldots, 0_1, 1)$.

Proof. Let $x = (x_m, \ldots, x_1, x_0)$, $y = (y_m, \ldots, y_1, y_0)$, $z = (z_m, \ldots, z_1, z_0) \in S_m \times \ldots \times S_1 \times S_0$. Then for all $1 \le i \le m$, we have $((xy)z)_i$

$= (x_iy_0 +_i \ldots +_i x_0y_i)z_0 +_i (x_{i-1}y_0 +_{i-1} \ldots +_{i-1} x_0y_{i-1})z_1$
$\quad +_i \ldots +_i (x_1y_0 +_1 x_0y_1)z_{i-1}(+_i (x_0y_0)z_i,$
$= ((x_iy_0)z_0 +_i \ldots +_i (x_0y_i)z_0) +_i ((x_{i-1}y_0)z_1 +_i \ldots +_i (x_0y_{i-1})z_1)$
$\quad +_i \ldots +_i ((x_1y_0)z_{i-1} +_i (x_0y_1)z_{i-1}) +_i (x_0y_0)z_i$ (Condition 2),
$= (x_iy_0)z_0 +_i \ldots +_i (x_0y_i)z_0 +_i (x_{i-1}y_0)z_1 +_i \ldots +_i (x_0y_{i-1})z_1$
$\quad +_i \ldots +_i (x_1y_0)z_{i-1} +_i (x_0y_1)z_{i-1} +_i (x_0y_0)z_i$ (Associativity in S_i),
$= x_i(y_0z_0) +_i \ldots +_i x_0(y_iz_0) +_i x_{i-1}(y_0z_1) +_i \ldots +_i x_0(y_{i-1}z_1)$
$\quad +_i \ldots +_i x_1(y_0z_{i-1}) +_i x_0(y_1z_{i-1}) +_i x_0(y_0z_i)$ (Condition 3),
$= x_i (y_0 z_0) +_i x_{i-1}(y_1z_0) +_i x_{i-1} (y_0z_1) +_i \ldots +_i x_1 (y_{i-1}z_0)$
$\quad +_i \ldots +_i x_1(y_0z_{i-1}) +_i x_0(y_iz_0) +_i \ldots +_i x_0(y_0z_i)$ (Commutativity in S_i),
$= x_i(y_0z_0) +_i x_{i-1}(y_1z_0 +_1 y_0z_1) +_i \ldots +_i x_1(y_{i-1}z_0$
$\quad +_{i-1} \ldots +_{i-1} y_0z_{i-1}) +_i x_0(y_iz_0 +_i \ldots +_i y_0z_i)$ (Condition 1),
$= (x(yz))_i.$

Clearly $((xy)z)_0 = (x_0y_0)z_0 = x_0(y_0z_0) = (x(yz))_0$.

For all $1 \le i \le m$, we have

$(x(0_m, \ldots, 0_1, 1))_i$

$= x_i1 +_i x_{i-1}0_1 +_i \cdots +_i x_00_i,$
$= x_i +_i x_{i-1}0_1 +_i \cdots +_i x_00_i$ (Condition 4),
$= xi +i\ 0i +i \cdots +i\ 0i$ (Condition 5),
$= xi.$

Similarly, we have $((0_m, \ldots, 0_1, 1)x)_i = x_i$. Clearly, $(x(0_m, \ldots, 0_1, 1))_0 = x_01 = x_0 = 1x_0 = ((0_m, \ldots, 0_1, 1)x)_0$. Therefore, $(0_m, \ldots, 0_1, 1)$ acts as unit. ■

For a pseudovariety of monoids $\mathbf{W}_0$ and commutative pseudovarieties of monoids $\mathbf{W}_1, \ldots, \mathbf{W}_m$, their $(m + 1)$-ary product $\lozenge_m(\mathbf{W}_m, \ldots, \mathbf{W}_1, \mathbf{W}_0)$ is defined to be the pseudovariety of monoids generated by all $(m + 1)$-ary products $\lozenge_m(S_m, \ldots, S_1, S_0)$ with $S_i \in \mathbf{W}_i$ for all $0 \le i \le m$.

We give the following representation of free objects for $\lozenge_m(\mathbf{W}_m, \ldots, \mathbf{W}_1, \mathbf{W}_0)$. For a positive integer m, $\overline{1}_m$ will denote a sequence of $2m - 1$ 1's. For instance, $(1, a, \overline{1}_3) = (1, a, 1, 1, 1, 1, 1)$.

Theorem 6. *Let m be a positive integer. Let* $\mathbf{W}_0,\ldots,\mathbf{W}_m$ *be pseudovarieties of monoids (with* $\mathbf{W}_1,\ldots,\mathbf{W}_m$ *commutative) such that* $F_A(\mathbf{W}_i)$ *is finite for all finite alphabets A* ($0 \le i \le m$)*. Then so is* $\lozenge_m(\mathbf{W}_m,\ldots,\mathbf{W}_1,\mathbf{W}_0)$.

Moreover, if A is a finite alphabet, there exists a one-to-one morphism

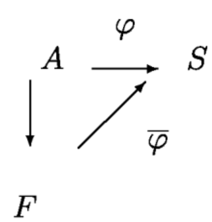
$$F_A(\lozenge_m(\mathbf{W}_m,\ldots,\mathbf{W}_1,\mathbf{W}_0)) \rightarrow \lozenge_m\left(F_{B_m}(\mathbf{W}_m),\ldots,F_{B_1}(\mathbf{W}_1),F_A(\mathbf{W}_0)\right)$$

given by $a \mapsto ((1, a, \overline{1}_m),\ldots,(1, a, \overline{1}_2),(1, a, 1), a)$, *where* $B_i = F_A(\mathbf{W}_0) \times A \times (F_A(\mathbf{W}_0) \times (A \cup \{1\}))^{i-1} \times F_A(\mathbf{W}_0)$ *for all* $1 \le i \le m$, *and the functions* $h_{i,j}$ *are defined by*

$$(x_0, a_1, x_1, \ldots, a_i, x_i)(y_0, b_1, y_1, \ldots, b_j, y_j) = (x_0, a_1, x_1, \ldots, a_i, x_i y_0, b_1, y_1, \ldots, b_j, y_j),$$

for all $x_0, \ldots, x_i, y_0, \ldots, y_j \in F_A(\mathbf{W}_0)$, $a_1, b_1 \in A$, *and* $a_2,\ldots, a_i, b_2, \ldots, b_j \in A \cup \{1\}$ ($0 \le i,j \le m$ *with* $i+j \le m$).

Proof. Let F be the submonoid of $\lozenge_m(F_{B_m}(\mathbf{W}_m),\ldots,F_{B_1}(\mathbf{W}_1), FA(\mathbf{W}_0))$ generated by A~ = $\{\overline{a} \mid a \in A\}$ where $\overline{a}$ = $((1, a, \overline{1}_m),\ldots,(1, a,$ ~12$), (1, a, 1), a)$. We show that F is isomorphic to $F_A(\lozenge_m(\mathbf{W}_m,\ldots,\mathbf{W}_1,\mathbf{W}_0))$. In order to do this, we show that the mapping from A into F given by $a \mapsto \overline{a}$ has the appropriate universal property, or for any $S \in \lozenge_m(\mathbf{W}_m, \ldots, \mathbf{W}_1, \mathbf{W}_0)$ and any function $\varphi : A \rightarrow S$, there exists (precisely) one morphism $\overline{\varphi} : F \rightarrow S$ such that the diagram



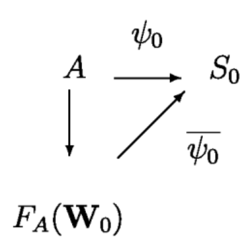
commutes (or $\overline{\varphi}(\overline{a}) = \varphi(a)$ for every $a \in$ A).

Let S E $o_m(\mathbf{W}_m, \ldots, \mathbf{W}_1, \mathbf{W}_0)$ and let $\varphi : A \rightarrow S$ be a function. Then there exist $S\ell \in \mathbf{W}_\ell$ ($0 \le \ell \le m$) and an $(m + 1)$-ary product $\lozenge_m(S_m,\ldots,S_1,S_0)$ such that $S < \lozenge_m(S_m,\ldots,S_1,S_0)$, say S is a morphic image of a submonoid T of $\lozenge_m(S_m,\ldots,S_1,S_0)$ under an onto morphism $\mathcal{X} : T \rightarrow S$. Let $\psi : A \rightarrow T$ be such that $\mathcal{X} \circ \psi = \varphi$. Thus, we have a mapping $\psi : A \rightarrow \lozenge_m(S_m,\ldots,S_1,S_0)$. The existence of a unique extension of φ to F follows from the existence of a unique extension of ψ to F, and so we proceed to work with the function ψ.

Let $\psi_0,\ldots,\psi_m$ be the components of ψ, that is the functions $\psi\ell$: $A \rightarrow S_\ell$ ($0 \le \ell \le m$) such that $\psi(a) = (\psi_m(a),\ldots,\psi_1(a), \psi_0(a))$ for any $a \in A$. By the universal property of $F_A(\mathbf{W}_0)$, let

$$\overline{\psi_0} : F_A(\mathbf{W}_0) \rightarrow S_0$$

be the unique morphism from $F_A(\mathbf{W}_0)$ to S_0 such that the diagram



commutes, where $A \longrightarrow F_A(\mathbf{W}_0)$ is the function that sends each $a \in A$ to a. For $1 \leq \ell \leq m$, letψ'_ℓ: $B_\ell \longrightarrow S_\ell$ be the function that maps $(x_0, a_1, x_1,\ldots, a_\ell, x_\ell)$ to

$$\overline{\psi_0}(x_0)\psi_{i_2-i_1}(a_{i_1})\overline{\psi_0}(x_{i_1})\ldots\overline{\psi_0}(x_{i_2-1})\ldots$$
$$(\psi_{i_r-i_{r-1}}(a_{i_{r-1}})\overline{\psi_0}(x_{i_{r-1}})\ldots\overline{\psi_0}(x_{i_{r-1}})\psi_{\ell-i_r+1}(a_{i_r})\overline{\psi_0}(x_{i_r})\ldots\overline{\psi_0}(x_\ell)$$

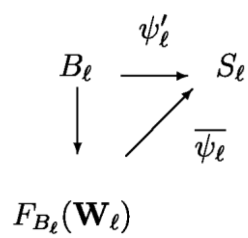
where $a_{i_1},\ldots,a_{i_r}$ are those a's that are not 1's and $1 = i_1 < \cdots < i_r \leq \ell$ (that is, the result of $\overline{\psi_0}(x_0) \in S_0$, $\psi_{i_2-i_1}(a_{i_1}) \in S_{i_2-i_1}$, $\overline{\psi_0}(x_{i_1}) \in S_0,\ldots$, $\overline{\psi_0}(x_{i_2-1}) \in S_0,\ldots$, and $\overline{\psi_0}(x_\ell) \in S_0$ for any $x_0,\ldots, x_\ell \in F_A(\mathbf{W}_0)$, $a_1 \in A$, and $a_2,\ldots, a_\ell \in (A\cup \{1\})$). For instance,

$$\psi'_3(x_0,a_1,x_1,a_2,x_2,a_3,x_3) = \overline{\psi_0}(x_0)\psi_1(a_1)\overline{\psi_0}(x_1),\psi_1(a_2)\overline{\psi_0}(x_2)\psi_1(a_3)\overline{\psi_0}(x_3),$$
$$\psi'_3(x_0,a_1,x_1,1,x_2,a_3,x_3) = \overline{\psi_0}(x_0)\psi_2(a_1)\overline{\psi_0}(x_1)\overline{\psi_0}(x_2)\psi_1(a_3)\overline{\psi_0}(x_3),$$
$$\psi'_3(x_0,a_1,x_1,a_2,x_2,1,x_3) = \overline{\psi_0}(x_0)\psi_1(a_1)\overline{\psi_0}(x_1)\psi_2(a_2)\overline{\psi_0}(x_2)\overline{\psi_0}(x_3),$$
$$\psi'_3(x_0,a_1,x_1,1,x_2,1,x_3) = \overline{\psi_0}(x_0)\psi_3(a_1)\overline{\psi_0}(x_1)\overline{\psi_0}(x_2)\overline{\psi_0}(x_3).$$

By the universal property of $F_{B_\ell}(\mathbf{W}_\ell)$, let

$$\overline{\psi_\ell}: F_{B_\ell}(\mathbf{W}_\ell) \to S_\ell$$

be the unique morphism from $F_{B_\ell}(\mathbf{W}_\ell)$ to S_ℓ such that the diagram

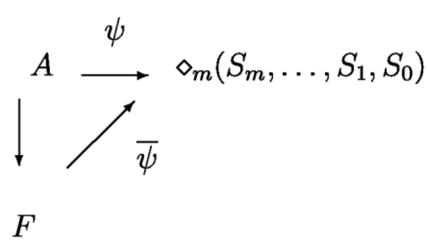


commutes, where $BtFB,(\mathbf{W}t)$ is the function that sends each $(x_0, a_1, x_1,\ldots, a_\ell, x_\ell) \in B_\ell$ to $(x_0, a_1, x_1,\ldots, a_\ell, x_\ell)$.

We wish to show that there exists a unique morphism

$$\overline{\psi}: F \to \Diamond_m (S_m,\ldots,S_1,S_0)$$

such that the diagram



commutes (or $\overline{\psi}(\overline{a}) = \psi(a)$ for any $a \in A$). The unique function that may have such a property has to satisfy the following condition:

$$\overline{\psi}(\overline{a_1}\ldots\overline{a_i}) = \psi(a_1)\ldots\psi(a_i).$$

Once we establish that the above formula defines a function, the proof will be complete, since it will then certainly define a morphism.

Suppose that $\overline{a_1} \dots \overline{a_i} = \overline{b_1} \dots \overline{b_j}$ in F ($a_1, \dots, a_i, b_1, \dots, b_j \in A$) (we also suppose that $i,j \geq m$; the other cases are simpler). In view of the definition of the operation in the $(m + 1)$-ary product $\Diamond_m(F_{B_m}(\mathbf{W}_m), \dots, F_{B_1}(\mathbf{W}_1), F_A(\mathbf{W}_0))$, the preceding equality means that we have the following equalities, respectively in $F_{B_\ell}(\mathbf{W}_\ell)$ ($1 \leq \ell \leq m$) and $F_A(\mathbf{W}_0)$:

$$sum_\ell(a_1 \dots a_i) = sum_\ell(b_1 \dots b_j), \quad (3)$$

$$a_1 \dots a_i = b_1 \dots b_j (4)$$

Here, $sum_\ell(a_1 \dots a_i)$ is the sum $+_\ell$ in S_ℓ of all expressions of the form

$$(x_0, c_1, x_1, \dots, c_\ell, x_\ell)$$

(where $c_1 \in \{a_1, \dots, a_i\}$ and $c_2, \dots, c_\ell$ E $(\{a_1, \dots, a_i\} \cup \{1\})$) satisfying: If $c_{k_1}, \dots, c_{k_r}$ are those c's that are not 1's ($1 = k_1 < \cdots < k_r \leq \ell$), all the x's are equal to 1 except possibly x_0, $x_{k_2-1}, \dots, x_{k_r-1}$ and x_ℓ. In such cases, by letting $c_{k_1} = a_{k'_1}, \dots, c_{k_r} = a_{k'_r}$, we have $x_0 = a_1 \dots a_{k'_1-1}$, $x_{k_2-1} = a_{k'_1+1} \dots a_{k'_2-1}, \dots, x_{k_r-1} = a_{k'_{r-1}+1} \dots a_{k'_1-1}$, $x_\ell = a_{k'_1+1} \dots a_i$. For instance, for $\ell = 8$, such an expression might be (for $i \geq 16$):

$$(a_1a_2, a_3, 1, 1, a_4a_5a_6a_7, a_8, 1,1,1,1, a_9a_{10}a_{11}a_{12}a_{13}a_{14}, a_{15}, 1,1,1,1, a_{16} \dots a_i).$$

To see that the calculations of (3) and (4) hold, we have

$$\overline{a_1} \dots \overline{a_i} = \Big((1, a_1, \overline{1}_m), \dots, (1, a_1, \overline{1}_2), (1, a_1, 1) a_1\Big) \dots \Big((1, a_i, \overline{1}_m), \dots, (1, a_i, \overline{1}_2), (1, a_i, 1), a_i\Big)$$

We have $\overline{a_1} \dots \overline{a_\ell} = (\dots((\overline{a_1 a_2})\,\overline{a_3})\dots)\,\overline{a_\ell}$ for each $1 \leq \ell \leq i$, and so

$(\overline{a_1a_2})_m = (1, a_1, \overline{1}_m)a_2 +_m (1, a_1, \overline{1}_{m-1})(1, a_1, 1) +_m \dots +_m (1, a_1, 1)(1, a_2, \overline{1}_{m-1}$
$\quad +_m a_1(1, a_2, \overline{1}_m)$,
$\quad = (1, a_1, \overline{1}_{m-1}, 1, a_2) +_m (1, a_1, \overline{1}_{m-1}, a_2, 1) +_m \dots +_m (1, a_1, 1, a_2, \overline{1}_{m-1})$
$\quad +_m (a_1, a_2, \overline{1}_m)$,
$\quad = sum_m(a_1a_2)$,
$\quad \vdots$
$(\overline{a_1a_2})_3 = (1, a_1, \overline{1}_3)a_2 +_3 (1, a_1, \overline{1}_2)(1, a_2, 1) +_3 (1, a_1, 1)(1, a_2, \overline{1}_2) +_3 a_1 (1, a_2, \overline{1}_3)$,
$\quad = (1, a_1, 1,1,1,1, a_2) +_3 (1, a_1, 1,1,1, a_2, 1) +_3 (1, a_1, 1, a_2, 1,1,1)$
$\quad +_3 (a_1, a_2, 1,1,1,1,1)$,
$\quad = sum_3 (a_1a_2)$,
$(\overline{a_1a_2})_2 = (1, a_1, \overline{1}_2)a_2 +_2 (1, a_1, 1)(1, a_2, 1) +_2 a_1(1, a_2, \overline{1}_2)$,
$\quad = (1, a_1, 1, 1, a_2) +2 (1, a_1, 1, a_2, 1) +2 (a_1, a_2, 1, 1, 1) = sum_2(a_1a_2)$,
$(\overline{a_1a_2})1 = (1, a_1, 1)a_2 +_1 a_1(1, a_2, 1) = (1, a_1, a_2) +1 (a_1, a_2, 1) = sum_1 (a_1a_2)$,
$(\overline{a_1a_2})_0 = a_1a_2$.

Combining all the components, we see that

$\overline{a_1a_2} = (sum_m(a_1a_2), \dots, sum_1(a_1a_2), a_1 a_2)$.

Next we have

$\left((\overline{a_1 a_2})\overline{a_3}\right)_m$ = $sum_m(a_1a_2)a_3 +_m sum_{m-1}(a_1a_2)(1, a_3, 1)$
$+_m \ldots +_m sum_1(a_1a_2)(1, a_3, \overline{1}_{m-1})$
$+_m a_1a_2(1, a_3, \overline{1}_m) = sum_m(a_1a_2a_3)$,

⋮

$\left((\overline{a_1 a_2})\overline{a_3}\right)_4 = sum_4(a_1a_2)a_3 +_4 sum_3(a_1a_2)(1, a_3, 1) +_4 sum_2(a_1a_2)(1, a_3, \overline{1}_2)$
$+_4 sum_1(a_1a_2)(1, a_3, \overline{1}_3) +_4 a_1a_2(1, a_3, \overline{1}_4) = sum_4(a_1a_2a_3)$,

$\left((\overline{a_1 a_2})\overline{a_3}\right)_3 = sum_3(a_1a_2)a_3 +_3 sum_2(a_1a_2)(1, a_3, 1) +_3 sum_1(a_1a_2)(1, a_3, \overline{1}_2)$
$+_3 a_1 a_2(1, a_3, \overline{1}_3)$,
$= (1, a_1, 1, 1, 1, 1, a_2a_3) +_3 (1, a_1, 1, 1, 1, a_2, a_3) +_3 (1, a_1, 1, a_2, 1, 1, a_3)$
$+_3 (a_1, a_2, 1, 1, 1, 1, a_3) +_3 (1, a_1, 1, 1, a_2, a_3, 1) +_3 (1, a_1, 1, a_2, 1, a_3, 1)$
$+_3 (a_1, a_2, 1, 1, 1, a_3, 1) +_3 (1, a_1, a_2, a_3, 1, 1), +_3 (a_1, a_2, 1, a_3, 1, 1, 1)$
$+_3 (a_1a_2, a_3, 1, 1, 1, 1, 1) = sum_3(a_1a_2a_3)$,

$\left((\overline{a_1 a_2})\overline{a_3}\right)_2 = sum_2(a_1a_2)a_3 +_2 sum_1(a_1a_2)(1, a_3, 1) +_2 a_1a_2(1, a_3, \overline{1}_2)$,
$= (1, a_1, 1, 1, a_2a_3) +_2 (1, a_1, 1, a_2, a_3) +_2 (a_1, a_2, 1, 1, a_3)$
$+_2 (1, a_1, a_2, a_3, 1) +_2 (a_1, a_2, 1, a_3, 1) +_2 (a_1 a_2, a_3, 1, 1, 1)$
$= sum_2(a_1a_2a_3)$,

$\left((\overline{a_1 a_2})\overline{a_3}\right)_1 = sum_1(a_1a_2)a_3 +_1 a_1a_2(1, a_3, 1)$,
$= (1, a_1, a_2a_3) +_1 (a_1, a_2, a_3) +_1 (a_1a_2, a_3, 1) = sum_1(a_1a_2a_3)$,

$\left((\overline{a_1 a_2})\overline{a_3}\right)_0 = a_1a_2a_3$.

Combining all the components, we see that

¯*a*1¯*a*2¯*a*3 = (*sum*$_m$(*a*1*a*2*a*3), ... , *sum*1 (*a*1 *a*2*a*3), *a*1 *a*2*a*3).

Now by a similar process we produce $(\ldots\left((\overline{a_1 a_2})\overline{a_3}\right)\ldots)\overline{a_\ell}$ for each $4 \le \ell < i$. Finally, we get

$((\ldots\left((\overline{a_1 a_2})\overline{a_3}\right)\ldots)\overline{a_i})_m = sum_m(a_1\ldots a_{i-1})a_i +_m sum_{m-1}(a_1\ldots a_{i-1})(1, ai, 1)$
$+_m \ldots +_m sum_1(a_1\ldots a_{i-1})(1, a_i, \overline{1}_{m-1}) +_m a_1\ldots a_{i-1}(1, a_i, \overline{1}_m)$,
$= sum_m(a_1\ldots a_i)$,

⋮

$((\ldots\left((\overline{a_1 a_2})\overline{a_3}\right)\ldots)\overline{a_i})_1 = sum_1(a_1\ldots a_{i-1})a_i +_1 a_1\ldots a_{i-1}(1, a_i, 1)$,
$= (1, a_1, a_2\ldots a_i) +1 \cdots +_1 (a_1\ldots a_{i-1}, a_i, 1) = sum_1(a_1\ldots a_i)$,
$((\ldots\left((\overline{a_1 a_2})\overline{a_3}\right)\ldots)\, \overline{a_i})_0 = a_1 \ldots a_i$.

Combining all the components, we see that
$\overline{a_1 a_2}\ldots\overline{a_i} = (sum_m(a_1\ldots a_i),\ldots, sum_1(a_1\ldots a_i), a_1\ldots a_i)$.

Applying to both members of the equalities (3) and (4) respectively the morphisms $\overline{\psi_\ell}$ and $\overline{\psi_0}$, we obtain that the sum $+_\ell$ in S_ℓ of all expressions of the form $\psi_{k_1}(a_1)\ldots \psi_{k_i}(a_i)$ where $k_1 + \cdots + k_i = \ell$ and $k_1,\ldots, k_i \ge 0$ is equal to the sum $+_\ell$ in S_ℓ of all expressions of the form $\psi_{k'_1}(b_1)\ldots \psi_{k'_j}(b_j)$ where $k'_1 + \cdots + k'_j = \ell$ and $k'_1,\ldots,k'_j \ge 0$, and that $\psi_0(a_1)\ldots\psi_0(a_i) = \psi_0(b_1)\ldots\psi_0(b_j)$. These conditions, in turn, by definition of the $(m + 1)$-ary product in $\Diamond_m(S_m,\ldots, S_1, S_0)$, are equivalent to the desired equality

$$\psi(a_1)\ldots\psi(a_i)$$
$$= \left(\psi_m(a_1),\ldots,\psi_1(a_1),\psi_0(a_1)\right)\ldots\left(\psi_m(a_i),\ldots,\psi_1(a_i),\psi_0(a_i)\right)$$

$$= \left(\psi_m(b_1), \ldots, \psi_1(b_1), \psi_0(b_1)\right) \ldots \left(\psi_m(b_j), \ldots, \psi_1(b_j), \psi_0(b_j)\right)$$
$$= \psi(b_1) \ldots \psi(b_j). \blacksquare$$

We end this section with a criterion to determine when an identity is satisfied in the $(m + 1)$-ary product $\Diamond_m(\mathbf{J}_1, \ldots, \mathbf{J}_1, \mathbf{V}_{\overline{m}})$.

Corollary 7. *Let A be a finite alphabet, m be a positive integer and $\overline{m}$ be a tuple of positive integers. An identity $u = v$ on A holds in $\Diamond_m(\mathbf{J}_1, \ldots, \mathbf{J}_1, \mathbf{V}_{\overline{m}})$ if and only if $u \sim_{(m,\overline{m})} v$. Consequently, an A- generated monoid S belongs to $\Diamond_m(\mathbf{J}_1, \ldots, \mathbf{J}_1, \mathbf{V}_{\overline{m}})$ if and only if S is a morphic image of $A^*/(m, \overline{m})$, or $\Diamond m(\mathbf{J}_1, \ldots, \mathbf{J}_1, \mathbf{V}_{\overline{m}}) = \mathbf{V}_{(m,\overline{m})}$.*

Proof. Let $S0$ be the multiplicatively written monoid $F_A(\mathbf{V}_{\overline{m}})$ with unit 1. Let S_i $(1 \leq i \leq m)$ be the additively written commutative monoid $F_{B_i}(\mathbf{J}_1)$ with operation $+_i$ and unit 0_i. Here B_i denotes $F_A(\mathbf{V}_{\overline{m}}) \times A \times (F_A(\mathbf{V}_{\overline{m}}) \times (A \cup \{1\}))^{i-1} \times F_A(\mathbf{V}_{\overline{m}})$. Consider for $0 \leq i,j \leq m$ with $i + j \leq m$, the function $h_{i,j} : S_i \times S_j\ S_{i+j}$ defined by

$$(x_0, a_1, x_1, \ldots, a_i, x_i)(y_0, b_1, y_1, \ldots, b_j, y_j) = (x_0, a_1, x_1, \ldots, a_i, x_i y_0, b_1, y_1, \ldots, b_j, y_j)$$

for all $x_0,\ldots,x_i, y_0,\ldots,y_j \in S_0$, $a_1, b_1 \in A$, and $a_2,\ldots,a_i, b_2,\ldots, bj \in (A \cup \{1\})$, and the associated $(m + 1)$-ary product $\Diamond_m(S_m,\ldots,S_1, S_0)$.

Now consider the one-to-one morphism of Theorem 6,

$$\psi : F_A\left(\Diamond_m (\mathbf{J}_1, \ldots, \mathbf{J}_1, \mathbf{V}_{\overline{m}})\right) \ \ \Diamond_m (S_m, \ldots, S_1, S_0),$$

where for all $a \in A$, $\psi(a) = ((1, a,\overline{1}_m),\ldots, (1, a,\overline{1}_2), (1, a, 1), a)$. If u and v are words on A, then $\Diamond_m(\mathbf{J}_1,\ldots,\mathbf{J}_1, \mathbf{V}_{\overline{m}}) \models u = v$ if and only if $F_A(\Diamond_m(\mathbf{J}_1,\ldots, \mathbf{J}_1, \mathbf{V}_{\overline{m}})) \models u = v$ if and only if $\psi(u) = \psi(v)$ if and only if $u \sim_{(m,\overline{m})} v$. To see that "$\psi(u) = \psi(v)$" and "$u \sim_{(m,\overline{m})} v$" are equivalent, the morphism ψ maps the word $u = a_1 \ldots a_i$ into an $(m + 1)$-tuple

$$(u_m, \ldots, u_1, u_0), \quad (5)$$

and ψ maps $v = b_1 \ldots b_j$ into an $(m + 1)$-tuple

$$(v_m, \ldots, v_1, v_0), \quad (6)$$

where u_ℓ, v_ℓ $(1 \leq \ell \leq m)$ and u_0, v_0 are written in (3) and (4) respectively. As in the proof of Theorem 6, we suppose that $i, j \geq m$; the other cases are simpler. The equality $\psi(u) = \psi(v)$ holds if and only if corresponding components of the $(m + 1)$-tuples (5) and (6) are equal. The condition "the ℓth components of (5) and (6) are equal" is equivalent to $S_\ell \models u_\ell = v_\ell$ or $\alpha_{(\ell,\overline{m})}(u) = \alpha_{(\ell,\overline{m})}(v)$ for all $1 \leq \ell \leq m$, and the condition "the last components of (5) and (6) are equal" is equivalent to $S_0 \models u_0 = v_0$ or $u \sim_{\overline{m}} v$. The set of conditions $\alpha_{(m,\overline{m})}(u) = \alpha_{(m,\overline{m})}(v),\ldots, \alpha_{(1,\overline{m})}(u) = \alpha_{(1,\overline{m})}(v)$ (which is clearly equivalent to $\alpha_{(m,\overline{m})}(u) = \alpha_{(m,\overline{m})}(v)$ by Definition 2 implies the condition $u \sim_{\overline{m}} v$. Hence, by Lemma 1 we conclude that $\psi(u) = \psi(v)$ if and only if $u \sim_{(m,\overline{m})} v$.

Corollary 8. We have $\mathbf{V}_2 = \bigcup_{m>0} \Diamond_m(\mathbf{J}_1,\ldots,\mathbf{J}_1, \mathbf{J}_1)$.

Proof. By Corollary 7 and the fact that $\mathbf{V}_2 = \bigcup_{m>0} \mathbf{V}_{(m,1)}$.

References:

[1] Almeida, J., "Finite Semigroups and Universal Algebra", World Scientific, Singapore, 1994.

[2] Almeida, J., and P. Weil, *Free profinite semigroups over semidirect products*, Izvestiya Vysshikh U~cebnykh Zavedenii" Matematica **1** (1995), 3–31.

[3] Blanchet-Sadri, F., *Some logical characterizations of the dot-depth hierarchy and applications*, Ph.D. Thesis, McGill University, 1989.

[4] Blanchet-Sadri, F., *Games, equations and the dot-depth hierarchy*, Comput. and Math. with Applic. **18** (1989), 809–822.

[5] Blanchet-Sadri, F., *On dot-depth two*, RAIRO Inform. Théor. et Applic. **24** (1990),521–529.

[6] Blanchet-Sadri, F., *Games, equations and dot-depth two monoids*, Discr. Appl. Math. **39** (1992), 99–111.

[7] Blanchet-Sadri, F., *The dot-depth of a generating class of aperiodic monoids is computable*, Intern. J. of Foundat. of Comput. Sci. **3** (1992), 419–442.

[8] Blanchet-Sadri, F., *Equations and dot-depth one*, Semigroup Forum **47** (1993) 305–317.

[9] Blanchet-Sadri, F., *Equations and monoid varieties of dot-depth one and two*, Théor. Comput. Sci. **123** (1994), 239–258.

[10] Blanchet-Sadri, F., *On a complete set of generators for dot-depth two*, Discr. Appl. Math. **50** (1994), 1-25.

[11] Blanchet-Sadri, F., *Some logical characterizations of the dot-depth hierarchy and applications*, J. of Comput. Syst. Sci. **51** (1995), 324-337.

[12] Blanchet-Sadri, F., *Inclusion relations between some congruences related to the dot-depth hierarchy*, Discr. Appl. Math. **68** (1996), 33-71.

[13] Brzozowski, J. A., and R. Knast, *The dot-depth hierarchy of star-free languages is infinite*, J. Comput. Syst. Sci. **16** (1978), 37-55.

[14] Brzozowski, J. A., and I. Simon, *Characterizations of locally testable events*, Discr. Math. **4** (1973), 243-271.

[15] Cohen, R. S., and J. A. Brzozowski, *Dot-depth of star -free events*, J. Comput. Syst. Sci. **5** (1971), 1-15.

[16] Cowan, D., *Inverse monoids of dot-depth two*, Intern. J. Alg. and Comput. **3** (1993), 411-424.

[17] Eilenberg, S., "Automata, Languages, and Machines, Vol. B", Academic Press, New York, 1976.

[18] Eilenberg, S., and M. P. Schützenberger, *On pseudovarieties*, Adv. in Math. **19** (1976), 413-418.

[19] Kleene, S. C., *Representation of events in nerve nets and finite automata*, Automata Studies, Princeton University Press, Princeton, NJ, 1956 3-42.

[20] McNaughton, R., *Algebraic decision procedures for local testability*, Math. Syst. Theory **8** (1974), 60-76.

[21] Perrin, D., and J. E. Pin, *First order logic and star-free sets*, J. of Comput. Syst. Sci. **32** (1986), 393-406.

[22] Pin, J. E., "Variétés de Langages Formels", Masson, Paris, 1984.

[23] Pin, J. E., *Hi´erarchies de concaténation*, RAIRO Inform. Théor. et Applic. **18** (1984), 23-46.

[24] Pin, J. E., *A property of the Sch¨utzenberger product*, Semigroup Forum **35** (1987),53-62.

[25] Pin, J. E., and H. Straubing, *Monoids of upper triangular matrices*, Colloquia Mathematica Societatis Janos Bolyai **39** (1981), 259{272.

[26] Reutenauer, C., *Sur les variétés de langages et de monoides*, Lect. Notes in Comput. Sci. **67** Springer-Verlag, Berlin (1979), 260-265.

[27] Rhodes, J., and B. Tilson, *The kernel of monoid morphisms*, J. Pure and Appl. Alg. **62** (1989), 227-268.

[28] Sch utzenberger, M. P., *On finite monoids having only trivial subgroups*, Inform. Control **8** (1965), 190-194.

[29] Simon, I., *Piecewise testable events*, Lect. Notes in Comput. Sci. **33** Springer-Verlag, Berlin (1975), 214-222.

[30] [30]Straubing, H., *A generalization of the Schützenberger product of finite monoids*, Th~eor. Comput. Sci. **13** (1981), 137-150.

[31] Straubing, H., *Finite semigroup varieties of the form* **V** * **D**, J. Pure and Appl. Alg. **36** (1985), 53–94.

[32] Straubing, H., *Semigroups and languages of dot-depth two*, Théor. Comput. Sci. **58** (1988), 361–378.

[33] Straubing, H., and P. Weil, *On a conjecture concerning dot-depth two languages*, Théor. Comput. Sci. **104** (1992), 161–183.

[34] Thomas, W., *Classifying regular events in symbolic logic*, J. Comput. Syst. Sci. **25** (1982), 360–376.

[35] Thomas, W., *An application of the Ehrenfeucht-Fraïssé game in formal language theory*, Mémoires de la Soc. Math. de France **16** (1984), 11–21.
[36] Weil, P., *Inverse monoids of dot-depth two*, Théor. Comput. Sci. **66** (1989), 233–245.
[37] Weil, P., *Closure of varieties of languages under products with counter*, J. Comput. Syst. Sci. **45** (1992), 316–339.
[38] Weil, P., *Some results on the dot-depth hierarchy*, Semigroup Forum **46** (1993),352–370.