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We will investigate the number of positive solutions for nonlinear boundary value problems (BVPs) with respect to a positive parameter. The nonlinearities we consider are smooth nondecreasing functions that are eventually positive. By utilizing the so-called quadrature method, we discuss existence, nonexistence, uniqueness, and multiplicity of positive solutions depending on the behavior of the nonlinearity near the origin, its concave or convex property, and asymptotic behavior at infinity.

POSITIVE SOLUTIONS FOR A CLASS OF ONE DIMENSIONAL p-LAPLACIAN PROBLEMS

by

Adam L. Eury

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Approved by

Committee Chair

To my mother

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

Committee Chair ______________________
Maya Chhetri

Committee Members ______________________
Richard Fabiano

____
Ratnasingham Shivaji

____
Date of Acceptance by Committee

____
Date of Final Oral Examination

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CHAPTER I

INTRODUCTION

1.1 Quadrature Method

The quadrature method, first introduced by Laetsch in [Lae71], is a well-known tool for studying positive solutions of boundary value problems of the form

$$
\begin{aligned}
-u''(x) &= \lambda f(u(x)); \quad 0 < x < 1, \\
u(0) &= 0 = u(1)\,,
\end{aligned}
\tag{1.1}
$$

where $\lambda > 0$ is a parameter and $f : [0, \infty) \to \mathbb{R}$ is a C^2 function. The quadrature method is a powerful tool because one can obtain the existence, nonexistence, uniqueness, and multiplicity of positive solutions by analyzing the relationship between the parameter λ and the supremum norm of the solution $u(x)$.

This method can also be used to study positive solutions of boundary value problems of the form

$$
\begin{aligned}
-\left(|u'(x)|^{p-2}u'(x)\right)' &= \lambda f(u(x)); \quad 0 < x < 1 \\
u(0) &= 0 = u(1),
\end{aligned}
\tag{1.2}
$$

where $(|u'|^{p-2}u')'$, for $p > 1$, is called the one-dimensional p-Laplacian operator. If $p = 2$, then (1.2) reduces to (1.1).

To the best of our knowledge, [BL00] is the first paper where the quadrature method was employed to investigate the existence of positive solutions for boundary value problems like (1.2). See [LH06], [LLY07], [BL00], [KLS11], [AB99], [CS08], [FZG08], where this method was utilized to study solutions of (1.2) for different classes of nonlinearity f. In this thesis, we extend the result obtained for $p = 2$ and $f(0) < 0$ in [CS88] to the case $p > 1$. Moreover, we also establish results for the cases $f(0) = 0$ and $f(0) > 0$.

In Section 1.2 we discuss the quadrature method after establishing some preliminary results. In Chapter II, we study the case $f(0) < 0$. Next in Chapter III, we study the $f(0) > 0$ case. In Chapter IV, we investigate the case $f(0) = 0$.

1.2 Preliminary Results

Throughout the thesis, we assume $f'(s) > 0$ for $s > 0$, and by $F(s)$ we mean the primitive of $f(s)$, that is, $F(s) = \int_0^s f(t)dt$. If $f(0) \geq 0$, then since $f(s) > 0$ for $s > 0$, $F(s)$ has no positive zeros. In the case when $f(0) < 0$, since f is eventually positive, there exist unique $\beta > 0$ and $\theta > 0$ such that $f(\beta) = 0$ and $F(\theta) = 0$. To consolidate for the results that follow, we define

$$U_0 = \begin{cases} \theta & \text{if } f(0) < 0 \\ 0 & \text{if } f(0) \geq 0 \end{cases}.$$

First we gather some preliminary results related to (1.2). The following symmetry result was proven in [BL00] and [MLA13]. Here we provide a different proof.

Lemma 1.1. *For fixed $\lambda > 0$, let $u(x) > 0$ be a solution of (1.2). Let $x_0 \in (0, 1)$ and $x_1 \in (x_0, 1]$ be such that $u' > 0$ on $(0, x_0)$, $u'(x_0) = 0$, and $u' < 0$ on (x_0, x_1). Then u is symmetric about $x_0 = 1/2$ and $x_1 = 1$.*

Proof. Let u be a positive solution of (1.2) and $\rho = u(x_0)$. Then for $x \in (0, x_0)$, we have

$$-((u')^{p-1})' = \lambda f(u)\,. \tag{1.3}$$

Multiplying (1.3) by $u'(x)$ and integrating from x to x_0 yields

$$\frac{p-1}{p}[u'(x)]^p = \lambda\,[F(\rho) - F(u(x))]\,,$$

which implies

$$\left(\frac{p-1}{p}\right)^{1/p} \frac{u'(x)}{[F(\rho) - F(u(x))]^{1/p}} = \lambda^{1/p}.$$

Integrating again on (x, x_0) for $x \in (0, x_0)$, yields

$$\left(\frac{p-1}{p}\right)^{1/p} \int_{u(x)}^{\rho} \frac{1}{[F(\rho) - F(z)]^{1/p}} dz = \lambda^{1/p}(x_0 - x).$$

Now letting $x = x_0 - \epsilon$ for $\epsilon \in [0, \min\{x_0, x_1 - x_0\}]$, we obtain

$$\left(\frac{p-1}{p}\right)^{1/p} \int_{u(x_0-\epsilon)}^{\rho} \frac{1}{[F(\rho) - F(z)]^{1/p}} dz = \lambda^{1/p}\epsilon. \tag{1.4}$$

Next, since $u'(x) < 0$ on (x_0, x_1)

$$-((-u')^{p-2}u')' = \lambda f(u); \quad x \in (x_0, x_1). \tag{1.5}$$

Multiplying (1.5) by $u'(x)$ and integrating on (x_0, x) yields

$$\frac{p-1}{p}[-u'(x)]^p = \lambda[F(\rho) - F(u(x))],$$

which in turn gives

$$\left(\frac{p-1}{p}\right)^{1/p} \frac{-u'(x)}{[F(\rho) - F(u(x))]^{1/p}} = \lambda^{1/p}.$$

Integrating again on (x_0, x) for some $x \in (x_0, x_1)$, we obtain

$$\left(\frac{p-1}{p}\right)^{1/p} \int_{u(x)}^{\rho} \frac{1}{[F(\rho) - F(z)]^{1/p}} dz = \lambda^{1/p}(x - x_0).$$

For $x = x_0 + \epsilon$ with $\epsilon \in [0, \min\{x_0, x_1 - x_0\}]$, we have

$$\left(\frac{p-1}{p}\right)^{1/p} \int_{u(x_0+\epsilon)}^{\rho} \frac{1}{[F(\rho) - F(z)]^{1/p}} dz = \lambda^{1/p}\epsilon. \tag{1.6}$$

Then (1.4) and (1.6) imply $u(x_0 - \epsilon) = u(x_0 + \epsilon)$ for $\epsilon \in [0, \min\{x_0, x_1 - x_0\}]$. Thus u is symmetric about x_0.

Now suppose $x_0 \neq 1/2$ and, without loss of generality, let $x_0 < 1/2$. Then $2x_0 \in (0, 1)$ and for $\epsilon = x_0$, we get

$$0 = u(0) = u(2x_0).$$

However, this is a contradiction since $u > 0$ in $(0,1)$. Therefore $x_0 = 1/2$. Moreover, it follows from the symmetry of the positive solution u on $(0,1)$ about $x_0 = 1/2$ that $x_1 = 1$. This completes the proof. □

In this thesis, a main point of interest is a function $G : [U_0, \infty) \to (0, \infty)$ defined by

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_0^{\rho} \frac{dt}{[F(\rho) - F(t)]^{1/p}} \tag{1.7}$$

which arises from the quadrature method. We first discuss some important properties of $G(\rho)$ below, which were also discussed in [BL00]. We provide an alternate proof of Lemma 1.2.

Lemma 1.2. *The mapping $G : [U_0, \infty) \to (0, \infty)$ defined by (1.7) is continuous on (U_0, ∞) and $\lim_{\rho \to U_0^+} G(\rho)$ exists.*

Proof. Let

$$g(\rho) = \int_0^{\rho} \frac{dt}{[F(\rho) - F(t)]^{1/p}}; \quad \rho \in (U_0, \infty)$$

and consider the sequence of functions $\{g_n\}$ defined by

$$g_n(\rho) = \int_0^{\rho - \frac{1}{n}} \frac{dt}{[F(\rho) - F(t)]^{1/p}}; \quad \rho \in (U_0, \infty).$$

Then g_n is a continuous function of ρ for each $n \in \mathbb{N}$. By the Mean Value Theorem, there exists $\xi \in (t, \rho)$ such that

$$F(\rho) - F(t) = f(\xi)(\rho - t).$$

Since $f(\rho) > 0$ for $\rho \in (U_0, \infty)$, there exists $m > 0$ such that $f(t) \geq m > 0$ for $t \in [\rho - \epsilon, \rho)$. Then

$$\begin{aligned}
|g(\rho) - g_n(\rho)| &= \left| \int_{\rho - \frac{1}{n}}^{\rho} \frac{dt}{[F(\rho) - F(t)]^{1/p}} \right| \\
&\leq \frac{1}{m^{1/p}} \int_{\rho - \frac{1}{n}}^{\rho} \frac{dt}{[\rho - t]^{1/p}} \\
&= \frac{1}{m^{1/p}} \lim_{\epsilon \to 0} \int_{\rho - \frac{1}{n}}^{\rho - \epsilon} \frac{dt}{[\rho - t]^{1/p}} \\
&= \frac{1}{m^{1/p}} \frac{p}{p-1} \lim_{\epsilon \to 0} \left[\left(\frac{1}{n}\right)^{\frac{p-1}{p}} - \epsilon^{\frac{p-1}{p}} \right] \\
&= \frac{1}{m^{1/p}} \frac{p}{p-1} \left(\frac{1}{n}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Therefore $|g(\rho) - g_n(\rho)| \to 0$ uniformly as $n \to \infty$. Thus $g(\rho)$ is continuous and hence $G(\rho)$ must be continuous. Finally, $\lim_{\rho \to U_0^+} G(\rho)$ is easily shown to exist (see [BL00] and [CS88]). This completes the proof. □

Lemma 1.3. *[BL00] $G(\rho)$ is differentiable on (U_0, ∞) and*

$$G'(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \frac{H(\rho) - H(\rho z)}{[F(\rho) - F(\rho z)]^{\frac{p+1}{p}}} dz, \tag{1.8}$$

where $H(s) = F(s) - \frac{s}{p} f(s)$.

The following important result was also established in [BL00] and [MLA13].

Theorem 1.4 (Quadrature Method)**.** *$u(x)$ is a positive solution of* (1.2) *with* $\lambda > 0$ *and* $\rho = \|u\|_\infty = \sup_{x\in[0,1]} |u(x)| = u(1/2)$ *if and only if*

$$\lambda^{1/p} = G(\rho); \quad \rho \in [U_0, \infty), \tag{1.9}$$

where $G(\rho)$ is given by (1.7)*.*

Proof. Suppose $u(x)$ is a positive solution of (1.2) corresponding to $\lambda > 0$. Then Lemma 1.1 implies that $u(x)$ must be symmetric about $x = 1/2$, $\rho = u(1/2) = \|u\|_\infty$, and $u'(x) > 0$ on $(0, 1/2)$. Then

$$\left(|u'|^{p-2}u'\right)' = ((u')^{p-1})'; \quad x \in (0, 1/2). \tag{1.10}$$

Multiplying (1.10) by $u'(x)$ and integrating over $[0, x]$, for $x \in (0, 1/2)$, we obtain

$$-\frac{p-1}{p}(u'(x))^p + \frac{p-1}{p}(u'(0))^p = \lambda F(u(x)) + C; \quad x \in (0, 1/2). \tag{1.11}$$

At $x = 1/2$, $u(1/2) = \rho$ and $u'(1/2) = 0$, so

$$C = -\lambda F(\rho) + \frac{p-1}{p}(u'(0))^p.$$

Then (1.11) becomes

$$\left(\frac{p-1}{p}\right)^{1/p} u'(x) = \lambda^{1/p} \left[F(\rho) - F(u(x))\right]^{1/p}; \quad x \in (0, 1/2).$$

Note that the quantity $[F(\rho)-F(u(x))]^{1/p}>0$ for all $p>1$, $\rho\in[U_0,\infty)$, since $F(\rho)>F(u(x))$ for all $x\in[0,1/2)$. Hence,

$$\lambda^{1/p}=\left(\frac{p-1}{p}\right)^{1/p}\frac{u'(x)}{[F(\rho)-F(u(x))]^{1/p}}.$$

Integrating over $[0,x]$ for $x\in(0,1/2)$, we get

$$\lambda^{1/p}x=\left(\frac{p-1}{p}\right)^{1/p}\int_0^{u(x)}\frac{dt}{[F(\rho)-F(t)]^{1/p}}. \tag{1.12}$$

Letting $x\to 1/2$, $u(x)\to\rho$ and (1.12) becomes

$$\lambda^{1/p}=2\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\rho}\frac{dt}{[F(\rho)-F(t)]^{1/p}}.$$

Conversely, suppose

$$\lambda^{1/p}=2\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\rho}\frac{dt}{[F(\rho)-F(t)]^{1/p}}$$

and let $u:[0,1/2]\to[0,\infty)$ be defined by

$$\lambda^{1/p}x=\left(\frac{p-1}{p}\right)^{1/p}\int_0^{u(x)}\frac{dt}{[F(\rho)-F(t)]^{1/p}}. \tag{1.13}$$

Note that this function is well-defined since both sides are monotonically increasing on $[0,1/2]$ and are equal when $x=0$ and $x=1/2$.

Now define

$$K(x,u) := \left(\frac{p-1}{p}\right)^{1/p} \int_0^u \frac{dt}{[F(\rho)-F(t)]^{1/p}} - \lambda^{1/p} x.$$

Note that $K \in C^1$ at any $(x,u) \in (0,1/2) \times (0,\rho)$ and

$$\left.\frac{\partial K}{\partial u}\right|_{(x,u)} = \left(\frac{p-1}{p}\right)^{1/p} \frac{1}{[F(\rho)-F(u)]^{1/p}} \neq 0.$$

Thus if $(x,u) \in (0,1/2) \times (0,\rho)$ satisfies $K(x,u) = 0$ (i.e. $x, u(x)$ satisfies (1.13)) then the Implicit Function Theorem implies u is C^1 at x. Then differentiating (1.13) yields

$$\lambda^{1/p} = \left(\frac{p-1}{p}\right)^{1/p} \frac{u'(x)}{[F(\rho)-F(u(x))]^{1/p}}$$

which in turn gives

$$\frac{p-1}{p}(u'(x))^p = \lambda[F(\rho)-F(u(x))].$$

Differentiability of the right-hand side of the above equation implies that $u''(x)$ exists, so further differentiation yields

$$(p-1)(u'(x))^{p-2}u''(x) = -\lambda f(u(x)).$$

But $(|u'|^{p-2}u')' = (p-1)|u'|^{p-2}u''$, so we see that $u(x)$ satisfies (1.2) on $(0,1/2)$. $\square$

Remark. It follows from Theorem 1.4 that the precise study of all positive solutions of (1.2) can be achieved by analyzing the shape of the curve given by the relation

$$\lambda^{1/p} = G(\rho) := 2\left(\frac{p-1}{p}\right)^{1/p} \int_0^\rho \frac{dt}{[F(\rho)-F(t)]^{1/p}} \quad \text{for } \rho \in [U_0, \infty).$$

Also,

$$G'(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \frac{H(\rho)-H(\rho z)}{[F(\rho)-F(\rho z)]^{\frac{p+1}{p}}} dz \quad \text{for } \rho \in (U_0, \infty),$$

where $H(s) = F(s) - \frac{s}{p} f(s)$. Then

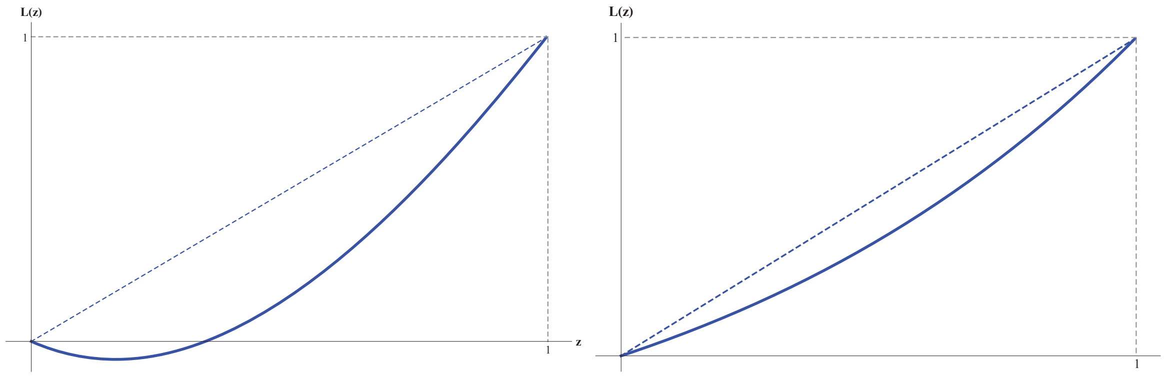
$$G'(\rho) > 0 \text{ if } H(\rho) - H(\rho z) > 0 \text{ for } z \in (0,1), \text{ and}$$

$$G'(\rho) < 0 \text{ if } H(\rho) - H(\rho z) < 0 \text{ for } z \in (0,1).$$

It was observed in [BIS81], for $p = 2$, that the sign of G' can be determined by analyzing the graph of $H(s)$. This observation remains valid for $p \neq 2$ and we utilize this to obtain our results (also see [KLS11]).

Lemma 1.5. *Let* $L(z) := F(\rho z)/F(\rho)$ *for* $z \in [0,1]$. *Then* $L(z) \leq z$ *for all* $z \in [0,1]$.

The proof of this lemma is similar to what is shown in [CS88].



(a) $L(z) \leq z$ when $f(0) < 0$.

(b) $L(z) \leq z$ when $f(0) \geq 0$.

Figure 1. $L(z) = F(\rho z)/F(\rho)$ for $z \in [0, 1]$.

Lemma 1.6. *The function $G(\rho)$ satisfies the following inequality for all $\rho \in (U_0, \infty)$*

$$2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^{p-1}}{f(\rho)} \leq G^p(\rho) \leq 2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^p}{F(\rho)}. \tag{1.14}$$

Proof. Rewrite $G(\rho)$ as

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \rho \int_0^1 \frac{dz}{[F(\rho) - F(\rho z)]^{1/p}}$$

by making the change of variable $t = \rho z$. We will prove the left hand inequality first. Since $f'(s) > 0$ for all $s > 0$, the Mean Value Theorem gives $F(\rho) - F(\rho z) \leq f(\rho)\rho(1 - z)$.

Then we have

$$
\begin{aligned}
G(\rho) &= 2\left(\frac{p-1}{p}\right)^{1/p} \rho \int_0^1 \frac{dz}{[F(\rho)-F(\rho z)]^{1/p}} \\
&\geq 2\left(\frac{p-1}{p}\right)^{1/p} \frac{\rho}{[f(\rho)\rho]^{1/p}} \int_0^1 \frac{dz}{[1-z]^{1/p}} \\
&= 2\left(\frac{p}{p-1}\right)^{(p-1)/p} \frac{\rho^{(p-1)/p}}{[f(\rho)]^{1/p}}
\end{aligned}
$$

Hence we have the desired left hand inequality

$$
2^p\left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^{p-1}}{f(\rho)} \leq G^p(\rho).
$$

Now we prove the right hand inequality. By Lemma 1.5, we have

$$
\begin{aligned}
G(\rho) &= 2\left(\frac{p-1}{p}\right)^{1/p} \rho \int_0^1 \frac{dz}{[F(\rho)-F(\rho z)]^{1/p}} \\
&= 2\left(\frac{p-1}{p}\right)^{1/p} \frac{\rho}{[F(\rho)]^{1/p}} \int_0^1 \frac{dz}{[1-L(z)]^{1/p}} \\
&\leq 2\left(\frac{p-1}{p}\right)^{1/p} \frac{\rho}{[F(\rho)]^{1/p}} \int_0^1 \frac{dz}{[1-z]^{1/p}} \\
&= 2\left(\frac{p-1}{p}\right)^{1/p} \frac{\rho}{[F(\rho)]^{1/p}} \frac{p}{p-1} \\
&= 2\left(\frac{p-1}{p}\right)^{(1-p)/p} \frac{\rho}{[F(\rho)]^{1/p}}.
\end{aligned}
$$

Thus we have

$$G^p(\rho) \leq 2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^p}{F(\rho)}.$$

□

Lemma 1.7. *(i) If* $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$ *then* $\lim_{\rho\to\infty} G(\rho) = \infty$.

(ii) If $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$ *then* $\lim_{\rho\to\infty} G(\rho) = 0$.

(iii) If $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = M$ *for some* $M > 0$*, then* $\lim_{\rho\to\infty} G(\rho) = C(M)$.

Proof. **(i)** By Lemma 1.6 we have

$$2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^{p-1}}{f(\rho)} \leq G^p(\rho).$$

Then $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$ implies $\lim_{s\to\infty} \frac{s^{p-1}}{f(s)} = \infty$, which in turn implies $\lim_{\rho\to\infty} G^p(\rho) = \infty$.

(ii) By Lemma 1.6 we have

$$G^p(\rho) \leq 2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^p}{F(\rho)}.$$

Then $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$ implies

$$\lim_{\rho\to\infty} \frac{\rho^p}{F(\rho)} = \lim_{\rho\to\infty} \frac{p\rho^{p-1}}{f(\rho)} = 0,$$

which further implies $\lim_{\rho\to\infty} G^p(\rho) = 0$.

(iii) Lemma 1.6 gives

$$2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^{p-1}}{f(\rho)} \leq G^p(\rho) \leq 2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^p}{F(\rho)}.$$

Then $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = M$ implies $\alpha \leq \lim_{s\to\infty} G^p(\rho) \leq p\alpha$ where

$$\alpha = 2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{1}{M}.$$

Therefore, there exists $\mu \in [\alpha, p\alpha]$ such that $\lim_{\rho\to\infty} G^p(\rho) = \mu$. $\square$

CHAPTER II

SEMIPOSITONE PROBLEMS

In this chapter, we study positive solutions of (1.2) when $f : [0, \infty) \to \mathbb{R}$ is a C^2 function satisfying:

- $f(0) < 0$ (semipositone), and
- $f'(s) > 0$ for $s > 0$.

This chapter is motivated by the results obtained for the case $p = 2$ in [CS88]. In [AM05], authors partially extend the results of [CS88]. Theorem 2.1 (below) agrees with their result. See also [KLS11] where authors use the quadrature method to study positive solutions of (1.2) when f satisfies the so-called infinite semipositone structure, $\lim_{s\to 0^+} f(s) = -\infty$. In this thesis we do not deal with singular problems.

Since f is eventually positive, let $\beta > 0$ and $\theta > 0$ be the unique zeros of f and F respectively, where $F(s) := \int_0^s f(t)dt$ is the primitive of f. We recall that if u is a positive solution of (1.2) corresponding to $\lambda > 0$, then $\rho_\lambda := \|u\|_\infty = u(1/2)$ and $H(s) := F(s) - \frac{s}{p} f(s)$ for $s \geq 0$.

In Section 2.1 we will discuss the p-Superlinear case, in Section 2.2 we discuss the p-Sublinear case, and in Section 2.3 we discuss the p-Linear case.

2.1 p-Superlinear

Here we consider the case

(H1) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$. ($p$-Superlinear)

Theorem 2.1 (Convex)**.** *Let (H1) hold. Suppose*

(A1) $f''(s) > 0$ *for* $s > 0$*,*

(A2) $(p-2)f'(s) < sf''(s)$ *for* $s \geq 0$

Then there exists $\lambda^* > 0$ *such that* (1.2) *has*

(i) *a unique positive solution for* $0 < \lambda \leq \lambda^*$*, and*

(ii) *no positive solution for* $\lambda > \lambda^*$*.*

Moreover, ρ_λ *increases as* λ *decreases. In particular,* $\rho_{\lambda^*} = \theta$ *and* $\lim_{\lambda \to 0^+} \rho_\lambda = \infty$*.*

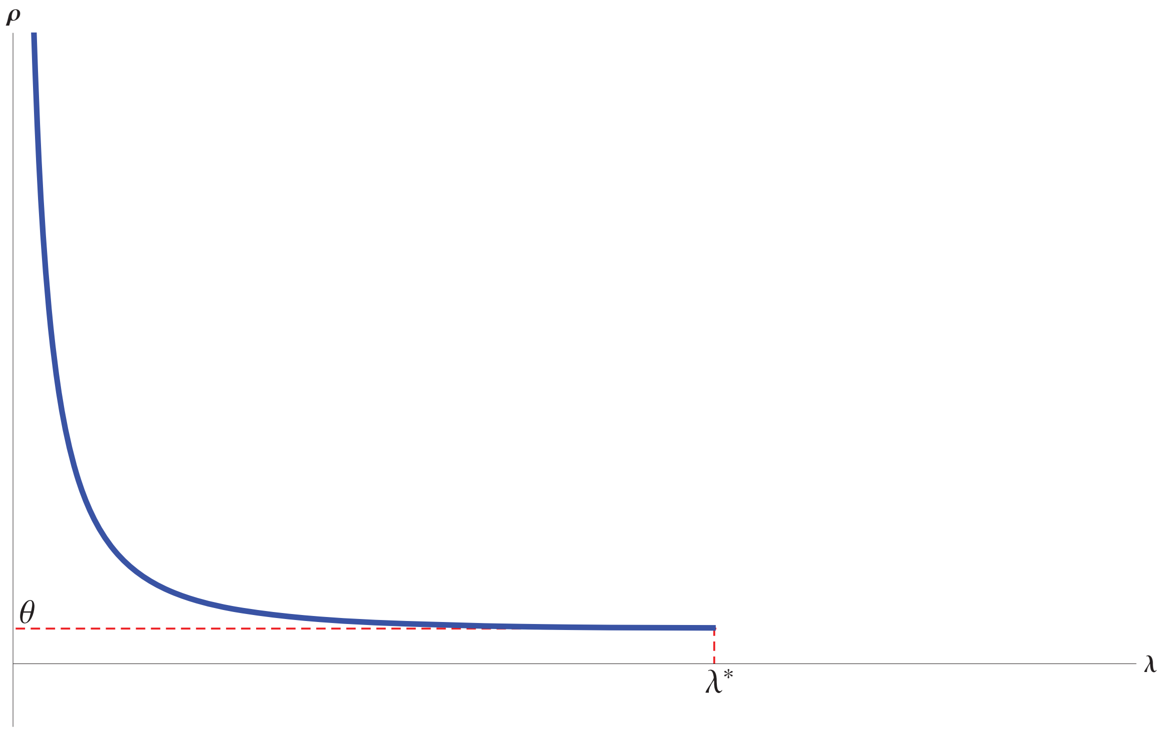


Figure 2. Theorem 2.1 Bifurcation Diagram

A prototype example satisfying the hypotheses of Theorem 2.1 is

$$f(s) = s^p + s^{p-1} - \epsilon,$$

with $\epsilon > 0$.

The concavity of the nonlinearity f influences the number of solutions of (1.2). In particular, we have the following result:

Theorem 2.2 (Concave-Convex)**.** *Let (H1) hold. Suppose*

(B1) $f''(s) < 0$ *for* $s \in (0, s_0)$ *with* $s_0 > \theta$ *and* $f''(s) > 0$ *for* $s > s_0$,

(B2) $(p-2)f'(s) - sf''(s) < 0$ *for* $s > s_0$,

(B3) $(p-1)f(\theta) < \theta f'(\theta)$,

(B4) $\lim_{s\to\infty}[(p-1)f(s) - sf'(s)] < 0$, *and*

(B5) *there exists* $\sigma > \theta$ *such that* $H(\sigma) > 0$.

Then there exist $\lambda^*, \lambda_1, \lambda_2$ *with* $0 < \lambda_1 < \lambda^* \leq \lambda_2$ *such that* (1.2) *has*

(i) a unique positive solution for $0 < \lambda < \lambda_1$,

(ii) no positive solutions for $\lambda > \lambda_2$.

(iii) There exists a range for λ *in* (λ_1, λ^*) *in which* (1.2) *has three positive solutions.*

(iv) If $\lambda_2 > \lambda^*$, (1.2) *has at least two positive solutions for* $\lambda \in [\lambda^*, \lambda_2)$.

Also $\rho_{\lambda^*} = \theta$ *and* $\lim_{\lambda\to 0}\rho_\lambda = \infty$.

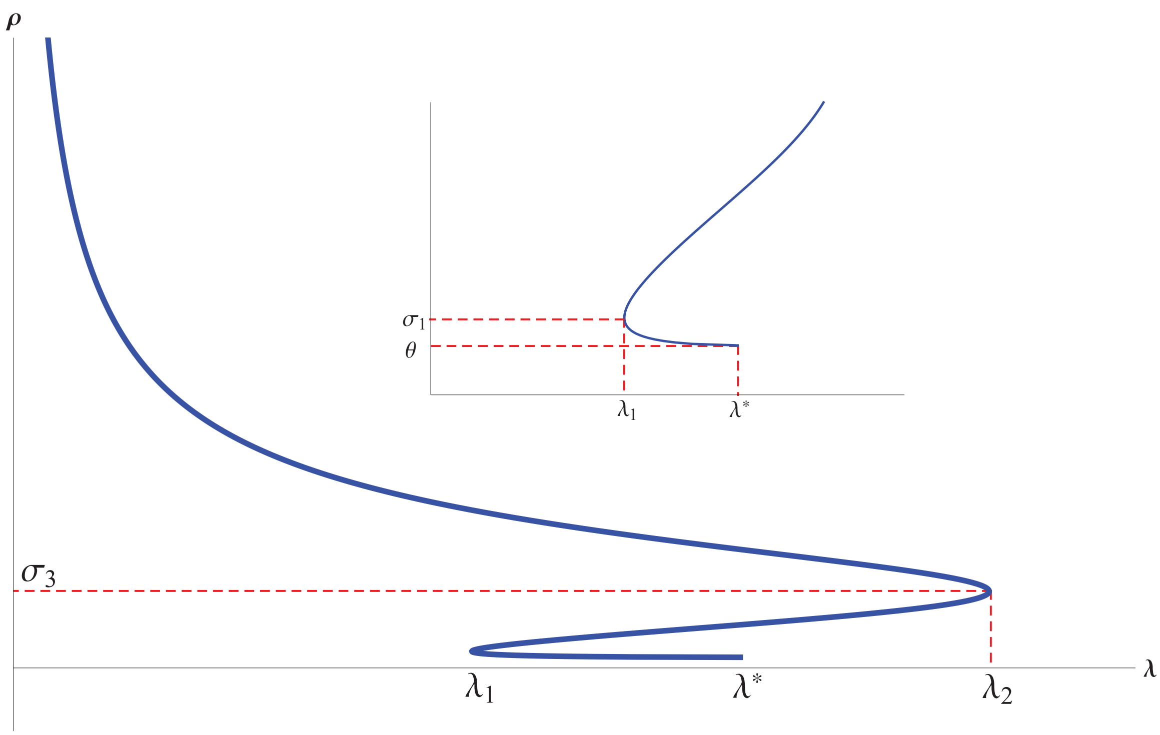


Figure 3. Theorem 2.2 Bifurcation Diagram

An example satisfying the hypotheses of Theorem 2.2 is

$$f(s) = s^3 - as^2 + bs - c,$$

where $a > 0, b > 0$, and $c > 0$ satisfy $b > \frac{8(p-4)^2a^2}{27(p-3)(p-5)}$ and $a^3 > \frac{108(p-1)c}{8-3p}$ when $p \in [2, 8/3)$. This example was discussed in [CS88] for $p = 2$. In [KLS11], authors study (1.2) with the nonlinearity

$$f(s) = \frac{s^3 - as^2 + bs - c}{s^\beta} \text{ for } \beta \in (0, 1).$$

Our result agrees with their analysis when $\beta = 0$.

Proofs of Theorem 2.1 and Theorem 2.2

Proof of Theorem 2.1. First we will show that $G'(\rho) < 0$ for $\rho \in (\theta, \infty)$. For this, let

$$H(s) = F(s) - \frac{s}{p} f(s)$$

for $s \geq 0$. Then $H(0) = 0$. Since

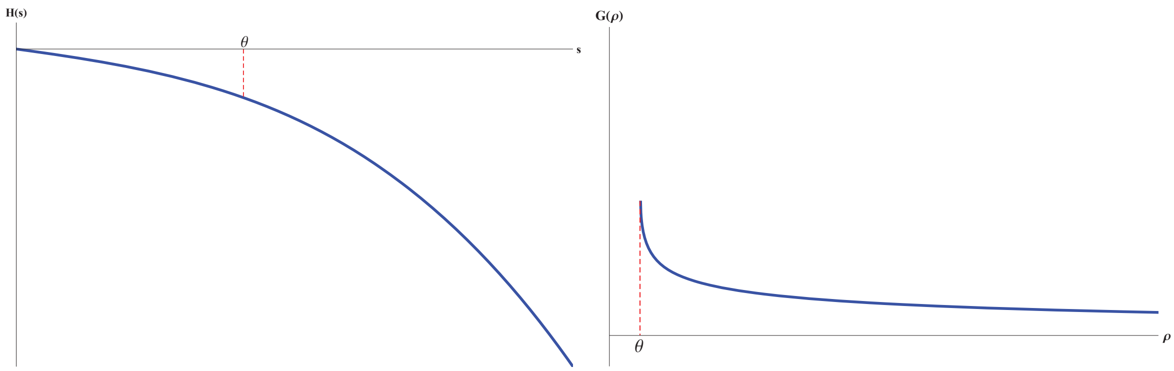
$$H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$$

and $f(0) < 0$, $H'(0) < 0$. But $(A1) - (A2)$ imply that

$$H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s) < 0 \quad \text{for } s > 0.$$

Since $H'(0) < 0$, we obtain $H'(s) < 0$ for all $s \geq 0$. Hence $H(\rho) - H(\rho z) < 0$ for $\rho \geq \theta$ and $z \in (0, 1)$. Therefore $G'(\rho) < 0$ for $\rho > \theta$.

It follows from Lemma 1.7 (ii) that $\lim_{\rho \to \infty} G(\rho) = 0$. Hence the shape of G is as depicted in Figure 3. This completes the proof of Theorem 2.1. $\square$

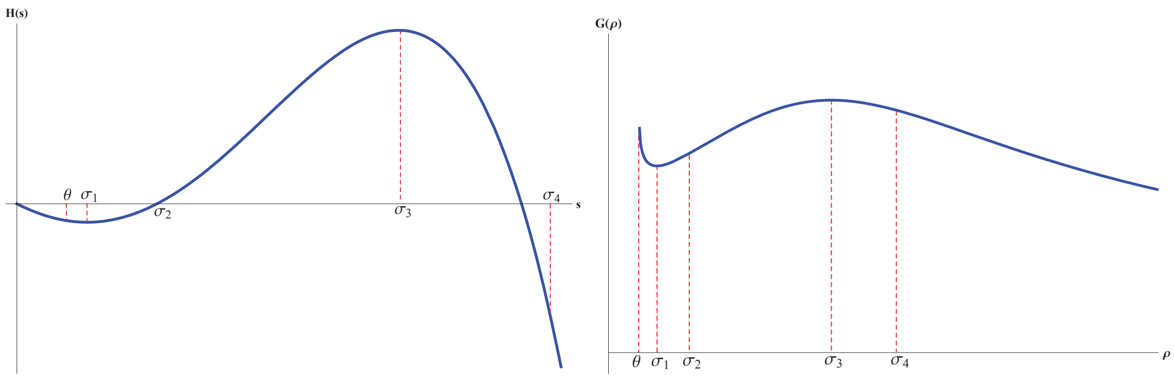


(a) H(s) (b) $G(\rho)$

Figure 4. Theorem 2.1 - $H(s)$ and $G(\rho)$

Proof of Theorem 2.2. First we will show that there exist $\sigma_i \in (\theta, \infty)$; $i = 1, 2, 3, 4$ with $\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4$ such that $G'(\rho) < 0$ for $\theta < \rho \leq \sigma_1$ and $\rho > \sigma_4$, while $G'(\rho) > 0$ for $\sigma_2 < \rho \leq \sigma_3$. To this end, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Then $H(0) = 0$ and $(B5)$ implies that $H(\sigma) > 0$ for $\sigma > \theta$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) < 0$, $H'(0) < 0$. Moreover, $(B3) - (B4)$ imply $H'(\theta) < 0$ and $\lim_{s \to \infty} H'(s) < 0$, respectively. Furthermore, since $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s)$, it follows from $(B1) - (B2)$ that $H''(s) > 0$ on $(0, s_0)$ and $H''(s) < 0$ for $s > s_0$, respectively. Therefore, there exist σ_i; $i = 1, 2, 3, 4$ such that $H(\rho) - H(\rho z) < 0$ for $\theta \leq \rho \leq \sigma_1$ and $\rho > \sigma_4$ for $z \in (0, 1)$. This implies $G'(\rho) < 0$ for $\theta < \rho \leq \sigma_1$ and $\rho > \sigma_4$. Also, $H(\rho) - H(\rho z) > 0$ for $\sigma_2 < \rho \leq \sigma_3$ on $(0, 1)$ and thus $G'(\rho) > 0$ for $\sigma_2 < \rho \leq \sigma_3$.

Hence the graphs of H and G are of the forms shown in Figure 4. Finally, $\lim_{\rho \to \infty} G(\rho) = 0$ follows from Lemma 1.7 (ii) and thus completing the proof of Theorem 2.2. □



(a) H(s) (b) $G(\rho)$

Figure 5. Theorem 2.2 - $H(s)$ and $G(\rho)$

2.2 p-Sublinear

We consider the case

(H2) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$. ($p$-Sublinear)

Theorem 2.3. *(Concave) Let (H2) hold and suppose*

(C1) $f''(s) < 0$ *for* $s > 0$,

(C2) $(p-1)f(\theta) < \theta f'(\theta)$, *and*

(C3) $\lim_{s\to\infty} [(p-1)f(s) - sf'(s)] > 0.$

Then there exists λ^*, μ_1, μ_2 *with* $0 < \mu_1 < \lambda^* \le \mu_2$ *such that* (1.2) *has*

(i) *no positive solutions for* $0 < \lambda < \mu_1$,

(ii) *at least one positive solution for* $\lambda \ge \mu_1$,

(iii) *exactly two positive solutions for* $\mu_1 < \lambda \le \lambda^*$,

(iv) *a unique positive solution for* $\lambda > \mu_2$.

Also $\rho_{\lambda^*} = \theta$ *and* $\lim\limits_{\lambda\to\infty} \rho_\lambda = \infty.$

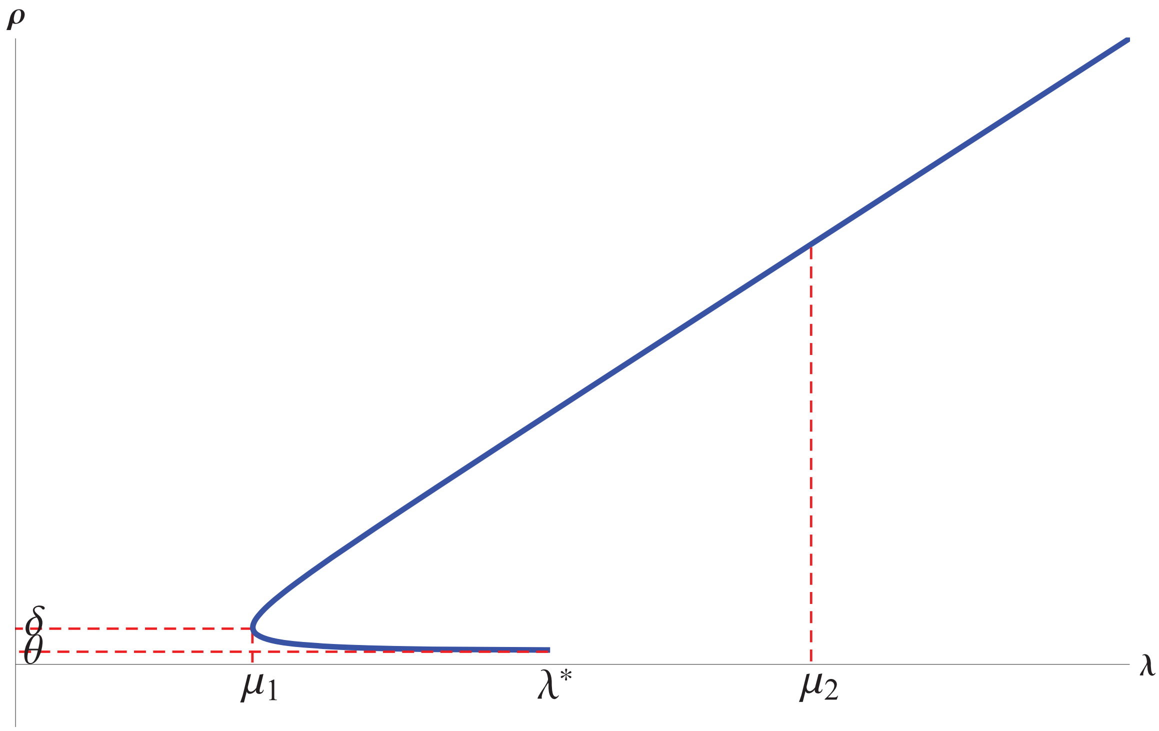


Figure 6. Theorem 2.3 Bifurcation Diagram

It can be verified that the hypotheses of Theorem 2.3 are satisfied by

$$f(s) = e^{\frac{as}{a+s}} - \eta,$$

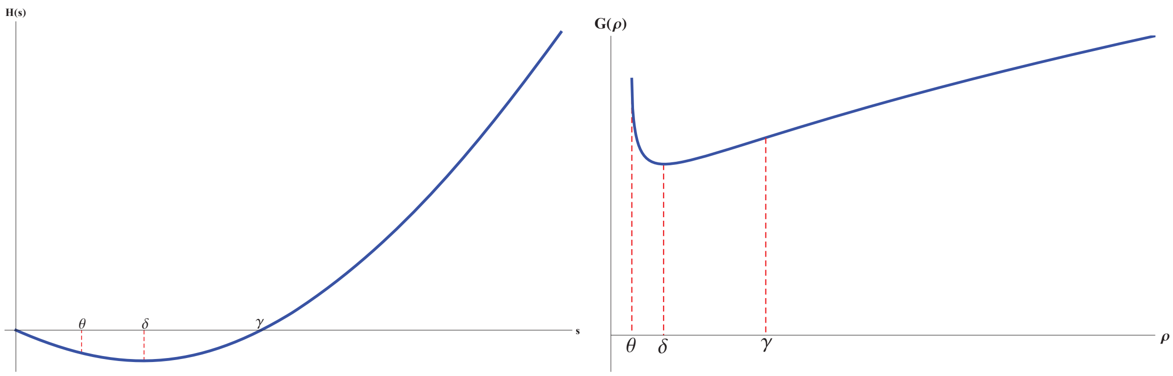
where $a << 1$ and $\eta > 1$.

Proof of Theorem 2.3

Proof of Theorem 2.3. First we will show that there exist $\delta > 0, \gamma > 0$ such that $\theta < \delta < \gamma$ and $G'(\rho) < 0$ on $(\theta, \delta]$ and $G'(\rho) > 0$ for $\rho \geq \gamma$. So let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f'(0) < 0$, $H'(0) < 0$.

Moreover, $(C2)-(C3)$ imply $H'(\theta) < 0$ and $\lim_{s\to\infty} H'(s) > 0$. Furthermore, it follows from $(C1)$ that $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s) > 0$ for $s > 0$. Therefore, $H(\rho) - H(\rho z) < 0$ for $\theta \le \rho \le \delta$ and $z \in (0,1)$. This implies $G'(\rho) < 0$ for $\theta < \rho \le \delta$. Also, $H(\rho) - H(\rho z) > 0$ for $\rho \ge \gamma$ and $z \in (0,1)$, which implies $G'(\rho) > 0$ for $\rho \ge \gamma$.

Finally, $\lim_{\rho\to\infty} G(\rho) = \infty$ follows from Lemma 1.7 (i) giving the shape of G as shown in Figure 6. This complete the proof of Theorem 2.3. $\square$



(a) H(s) (b) $G(\rho)$

Figure 7. Theorem 2.3 - $H(s)$ and $G(\rho)$

2.3 p-Linear

Here we assume

(H3) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = M$, where $0 < M \le \infty$. (p-Linear)

Theorem 2.4 (Concave)**.** *Let (H3) hold. Suppose*

(D1) $f''(s) < 0$ *for* $s > 0$*,*

(D2) $(p-1)f(\theta) < \theta f'(\theta)$*, and*

(D3) $\lim_{s\to\infty} [(p-1)f(s) - sf'(s)] > 0$.

Then there exists λ^*, μ_1, and μ_2 with $0 < \mu_1 < \mu_2 < \lambda^*$ such that (1.2) has

(i) no solutions for $\lambda < \mu_1$ and $\lambda > \lambda^*$,

(ii) at least one positive solution for $\lambda \geq \mu_1$,

(iii) exactly two positive solutions for $\mu_1 < \lambda < \mu_2$,

(iv) a unique positive solution for $\mu_2 < \lambda \leq \lambda^*$.

Further, $\rho_{\lambda^*} = \theta$ and $\lim\limits_{\lambda \to \mu_2} \rho = \infty$.

Remark. In this case, the curve bifurcating from infinity at μ_2 may exhibit different behavior depending on the conditions assumed on the nonlinearity f. In particular, it may not turn around and cross over μ_2 as λ increases.

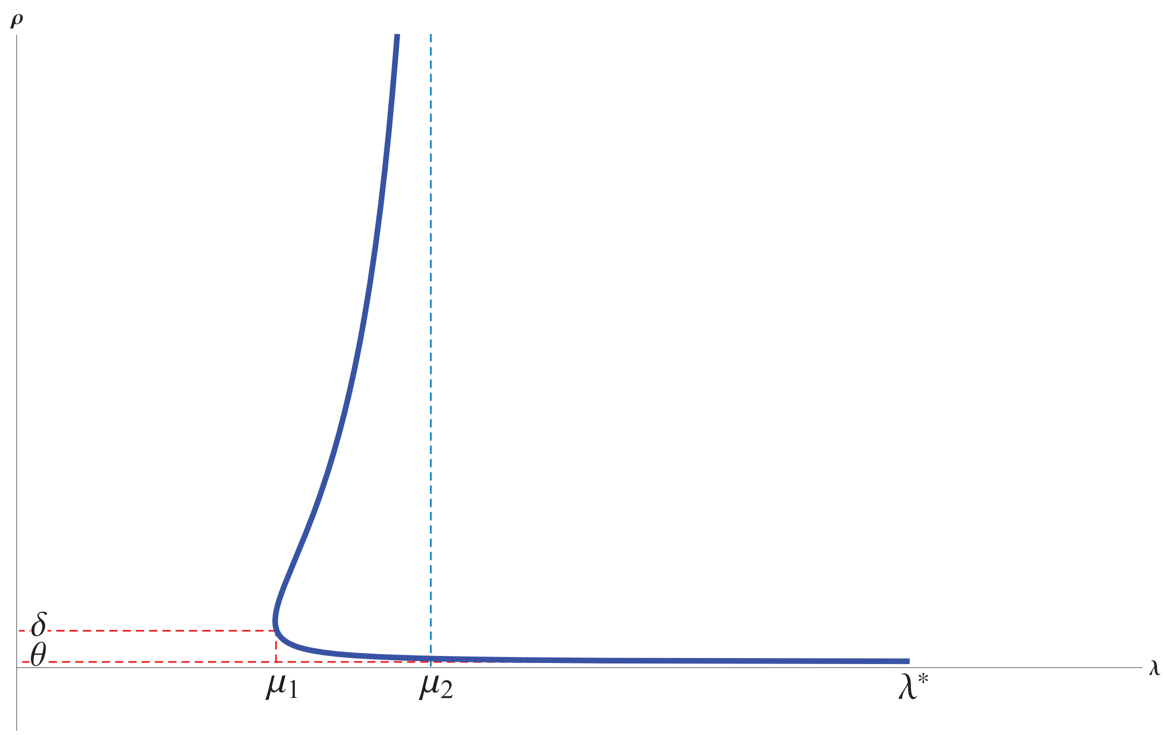


Figure 8. Theorem 2.4 Bifurcation Diagram

Theorem 2.5 (Convex)**.** *Let (H3) hold. Suppose*

(E1) $f''(s) > 0$ *for* $s > 0$, *and*

(E3) $(p-2)f'(s) < sf''(s)$ *for* $s > 0$.

Then there exists μ *with* $0 < \mu < \lambda^*$ *such that* (1.2) *has*

(1) *no positive solutions for* $\lambda < \mu$ *and* $\lambda > \lambda^*$,

(2) *a unique positive solution for* $\mu < \lambda < \lambda^*$.

Further, $\rho_{\lambda^*} = \theta$ *and* $\lim_{\lambda \to \mu} \rho = \infty$

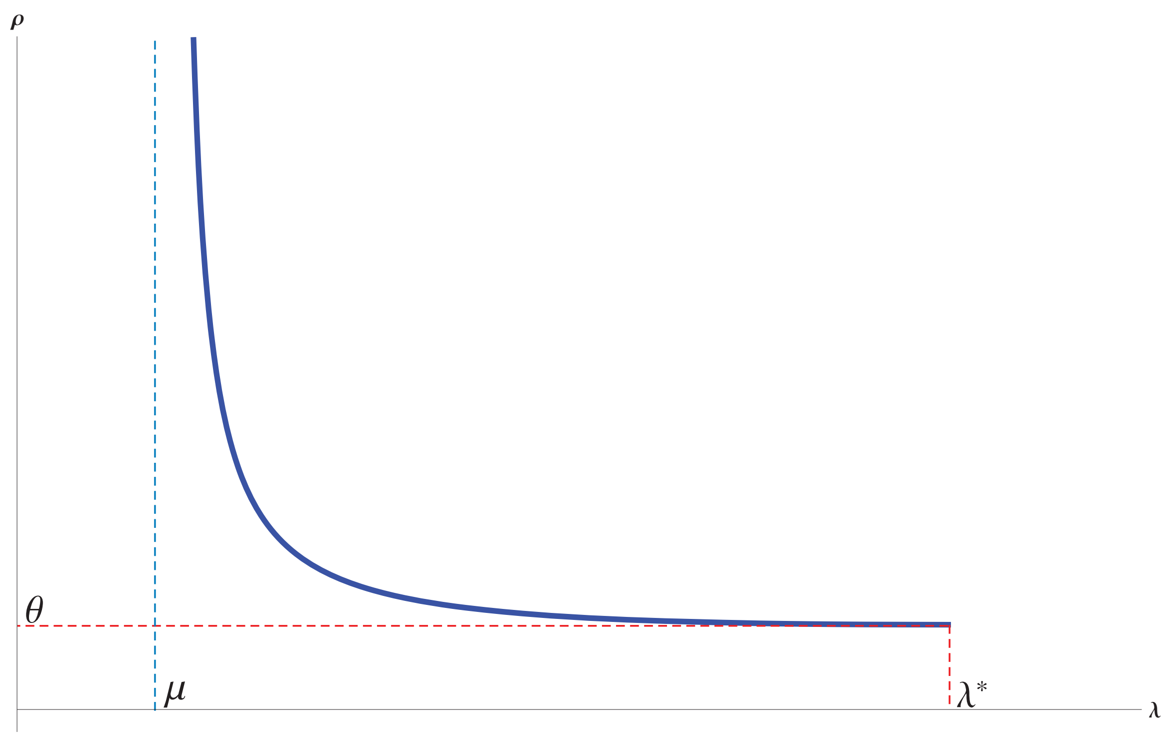
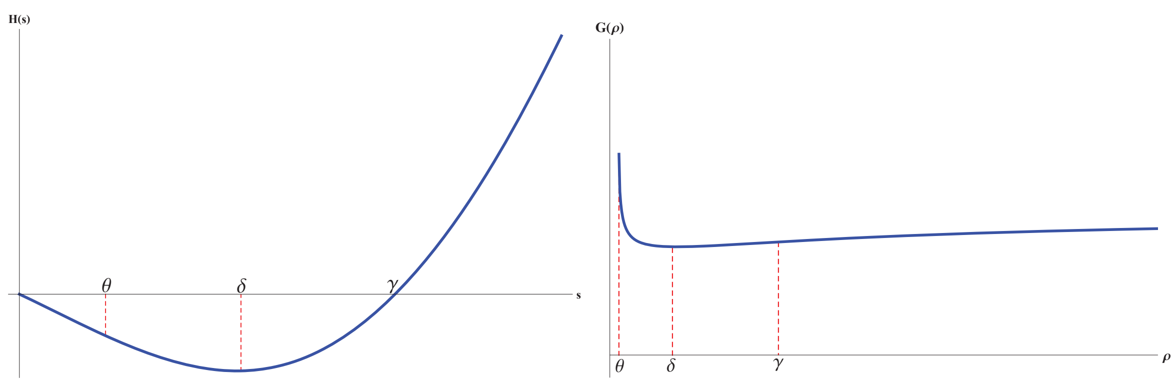


Figure 9. Theorem 2.5 Bifurcation Diagram

Proofs of Theorem 2.4 and Theorem 2.5

Proof of Theorem 2.4. First we will show that there exist $\delta > 0, \gamma > 0$ such that $\theta < \delta < \gamma$ and $G'(\rho) < 0$ for $\theta < \rho \leq \delta$, while $G'(\rho) > 0$ for $\rho \geq \gamma$. For this, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) < 0$, $H'(0) < 0$. Moreover, $(D2) - (D3)$ imply $H'(\theta) < 0$ and $\lim_{s \to \infty} H'(s) > 0$. Furthermore, since $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s)$, it follows from $(D1)$ that $H''(s) > 0$ for all $s > 0$. Therefore, there exist δ, γ such that $\theta < \delta < \gamma$ and $H'(\delta) = 0$, $H(\gamma) = 0$. Consequently, $H(\rho) - H(\rho z) < 0$ for $\theta \leq \rho \leq \delta$ and $z \in (0, 1)$. Then this implies $G'(\rho) < 0$ for $\theta < \rho \leq \delta$. Also, $H(\rho) - H(\rho z) > 0$ for $\rho \geq \gamma$ and $z \in (0, 1)$, which in turn implies $G'(\rho) > 0$ for $\rho \geq \gamma$.

Finally, $\lim_{\rho \to \infty} G^p(\rho) = \mu$ for some $\mu > 0$ follows from Lemma 1.7 (iii). Thus G has the desired shape as in Figure 9 to conclude the proof of Theorem 2.4. $\square$



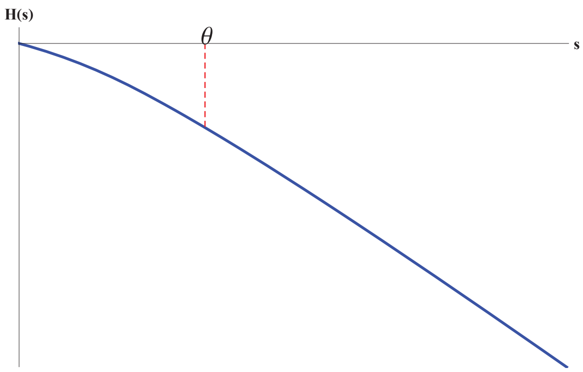
(a) H(s) (b) $G(\rho)$

Figure 10. Theorem 2.4 - $H(s)$ and $G(\rho)$

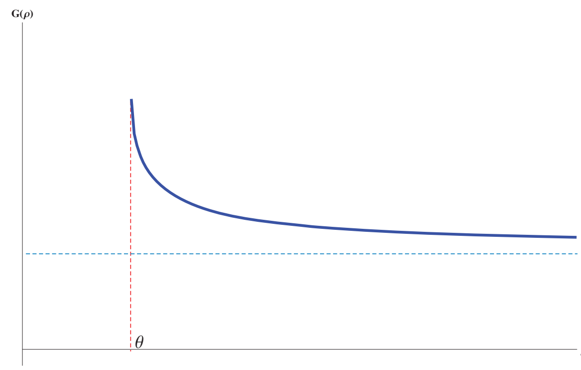
Proof of Theorem 2.5. First we will show that $G'(\rho) < 0$ for all $\rho > \theta$. So let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) < 0$, $H'(0) < 0$. Furthermore, $(E1) - (E2)$ imply that $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s) < 0$

for $s > 0$. Since $H'(0) < 0$, we obtain $H'(s) < 0$ for all $s \geq 0$. Then it follows that $H(\rho) - H(\rho z) < 0$ for $\rho \geq \theta$ and $z \in (0,1)$. Therefore, $G'(\rho) < 0$ for $\rho > \theta$.

Finally, $\lim_{\rho \to \infty} G^p(\rho) = \mu$ for some $\mu > 0$ follows from Lemma 1.7 (iii). This completes the proof of Theorem 2.5. $\square$



(a) H(s)



(b) $G(\rho)$

Figure 11. Theorem 2.5 - $H(s)$ and $G(\rho)$

CHAPTER III

POSITONE PROBLEMS

In this chapter, we study positive solutions of (1.2) when $f : [0, \infty) \to \mathbb{R}$ is a C^2 function satisfying:

- $f(0) > 0$ (positone), and
- $f'(s) > 0$ for $s > 0$.

3.1 p-Superlinear

Theorem 3.1. *(Convex) Let (H1) hold. Suppose*

(F1) $f''(s) > 0$ *for* $s > 0$*,*

(F2) $(p-2)f'(s) < sf''(s)$ *for* $s > 0$*, and*

(F3) $\lim_{s\to\infty}[(p-1)f(s) - sf'(s)] < 0.$

Then there exists $\mu > 0$ *such that* (1.2) *has*

(i) *no positive solutions for* $\lambda > \mu$*,*

(ii) *a unique positive solution at* $\lambda = \mu$*,*

(iii) *exactly two positive solutions for* $0 < \lambda < \mu$*.*

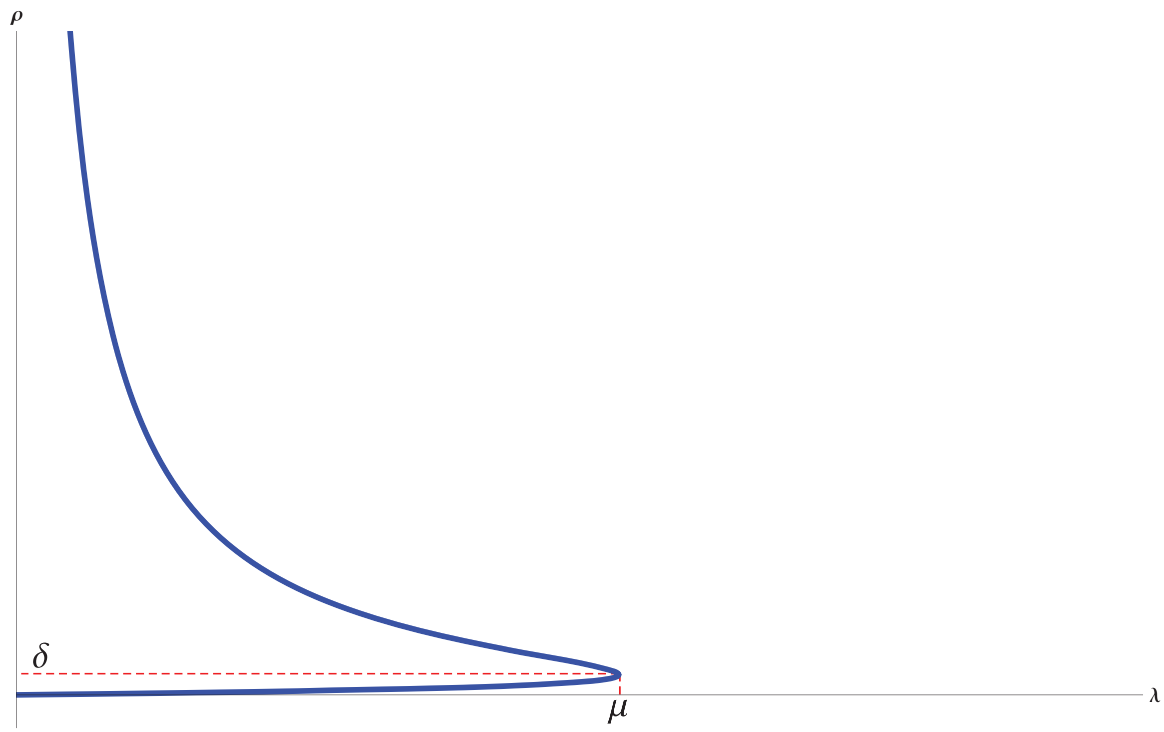
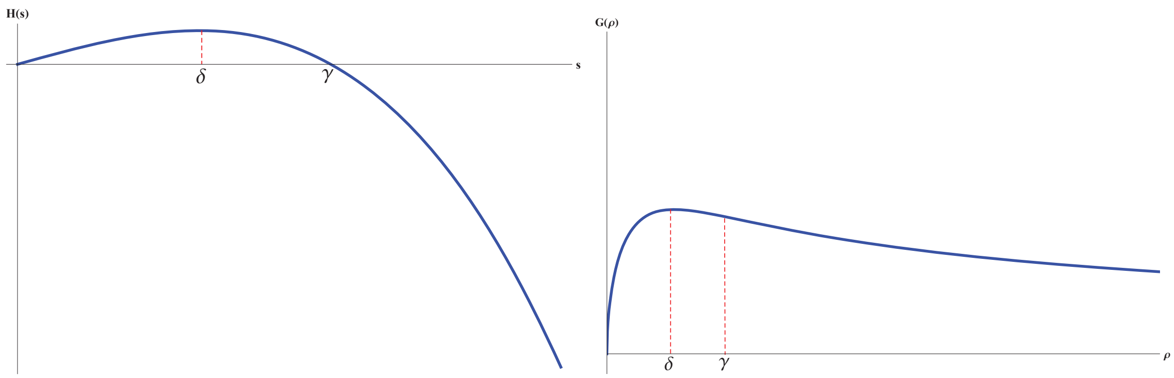


Figure 12. Theorem 3.1 Bifurcation Diagram

Proof of Theorem 3.1

Proof of Theorem 3.1. First we will show that there exist $\delta > 0$, $\gamma > 0$ such that $\delta < \gamma$ and $G'(\rho) > 0$ for $0 < \rho \le \delta$, while $G'(\rho) < 0$ for $\rho \ge \gamma$. For this, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \ge 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) > 0$, $H'(0) > 0$. Moreover, $(F3)$ implies $\lim_{s \to \infty} H'(s) < 0$. But $(F1) - (F2)$ imply $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s) < 0$ for $s > 0$. Hence there exist $\delta > 0$, $\gamma > 0$ such that $\delta < \gamma$ and $H'(\delta) = H(\gamma) = 0$. Therefore, $H(\rho) - H(\rho z) > 0$ for $0 < \rho \le \delta$ and $z \in (0, 1)$, which implies $G'(\rho) > 0$ for $0 < \rho \le \delta$. Also, $H(\rho) - H(\rho z) < 0$ for $\rho \ge \gamma$ and $z \in (0, 1)$. Thus $G'(\rho) < 0$ for $\rho \ge \gamma$.

Finally, $\lim_{\rho \to 0} G(\rho) = 0$ and $\lim_{\rho \to \infty} G(\rho) = 0$ follow from Lemma 1.6 and Lemma 1.7 (ii), respectively. $\square$



(a) H(s) (b) $G(\rho)$

Figure 13. Theorem 3.1 - $H(s)$ and $G(\rho)$

3.2 p-Sublinear

Theorem 3.2. *Let (H2) hold. Suppose*

(G1) $(p-1)f(s) > sf'(s)$ *for* $s > 0$.

Then (1.2) *has a unique positive solution for all* $\lambda > 0$. *Furthermore,* $\lim_{\lambda \to 0} \rho_\lambda = 0$ *and* $\lim_{\lambda \to \infty} \rho_\lambda = \infty$.

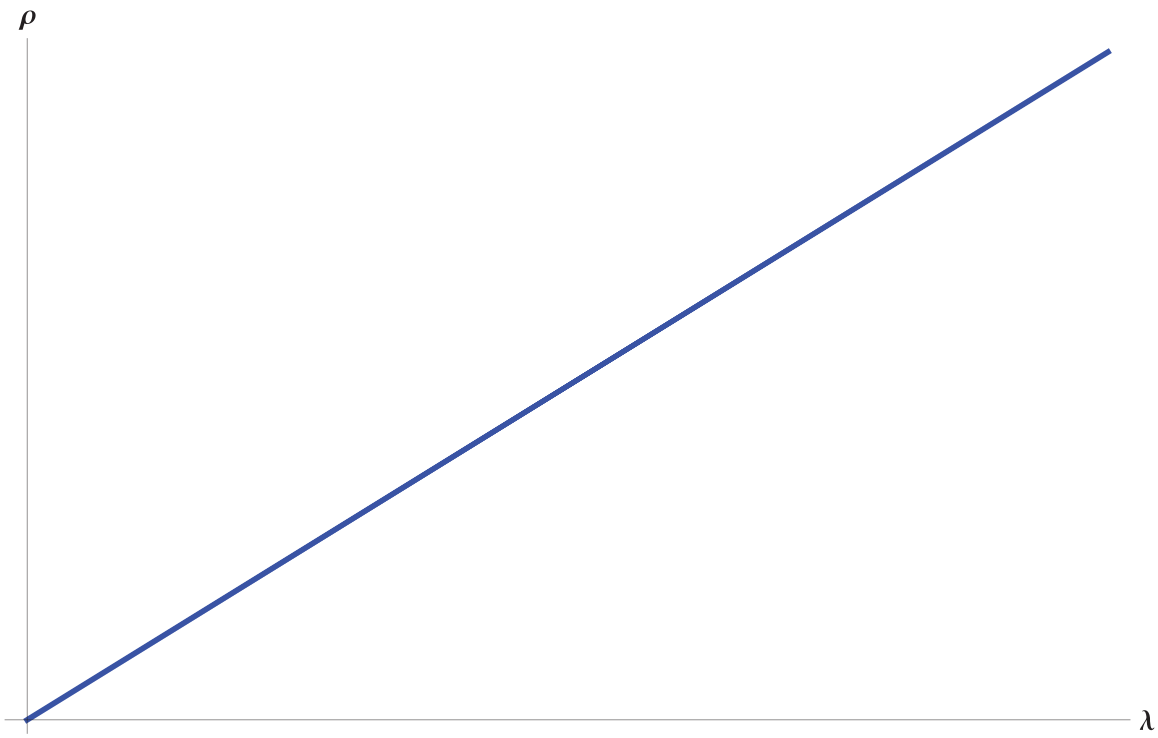


Figure 14. Theorem 3.2 Bifurcation Diagram

Theorem 3.3 (Convex-Concave)**.** *Let (H2) hold. Suppose*

(I1) $f''(s) > 0$ *for* $s \in (0, s_0)$ *and* $f''(s) < 0$ *for* $s > s_0$*,*

(I2) $(p-2)f'(s) < sf''(s)$ *for* $s \in (0, s_0)$*,*

(I3) $\lim_{s \to \infty} [(p-1)f(s) - sf'(s)] > 0$*, and*

(I4) *there exists a* $\sigma > 0$ *such that* $H(\sigma) < 0$*.*

Then there exist μ_1, μ_2 *with* $0 < \mu_1 < \mu_2$ *such that* (1.2) *has*

(i) *a unique positive solution for* $\lambda < \mu_1$ *and* $\lambda > \mu_2$*,*

(ii) *exactly three positive solutions for* $\lambda \in (\mu_1, \mu_2)$*.*

Furthermore, $\lim_{\lambda \to 0} \rho_\lambda = 0$ *and* $\lim_{\lambda \to \infty} \rho_\lambda = \infty$.

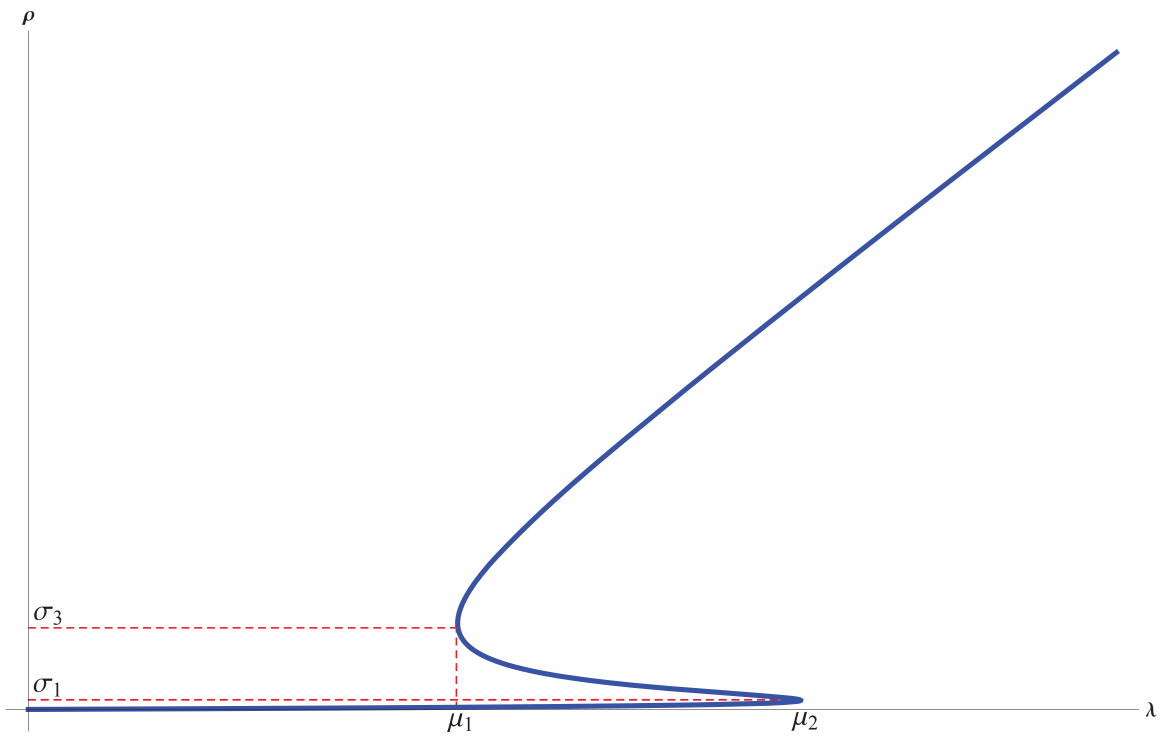
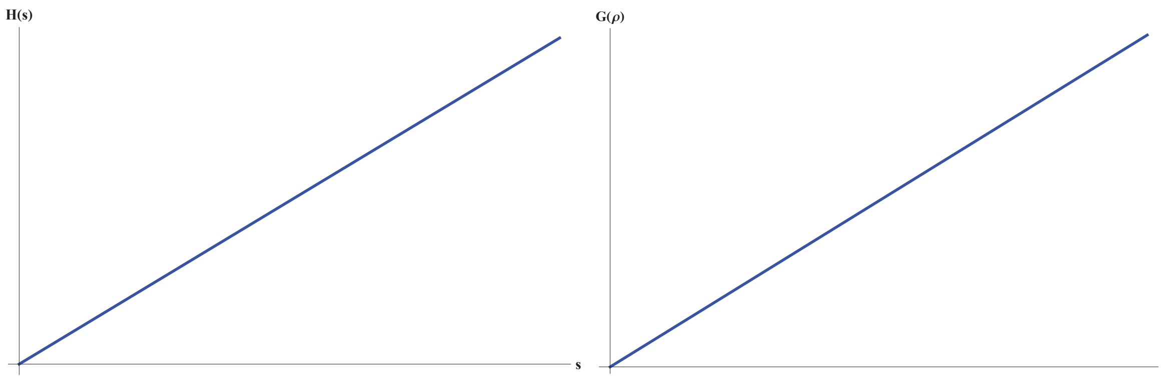


Figure 15. Theorem 3.3 Bifurcation Diagram

Proofs of Theorem 3.2 and Theorem 3.3

Proof of Theorem 3.2. First we will show that $G'(\rho) > 0$ for all $\rho > 0$. To this end, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) > 0$, $H'(0) > 0$. Moreover, $(G1)$ implies $H'(s) > 0$ for $s > 0$. Therefore, $H(\rho) - H(\rho z) > 0$ for $\rho > 0$ and $z \in (0, 1)$. Hence, $G'(\rho) > 0$ for $\rho > 0$.

Finally, $\lim_{\rho \to 0} G(\rho) = 0$ and $\lim_{\rho \to \infty} G(\rho) = \infty$ follow from Lemma 1.6 and Lemma 1.7 (i), respectively. $\square$

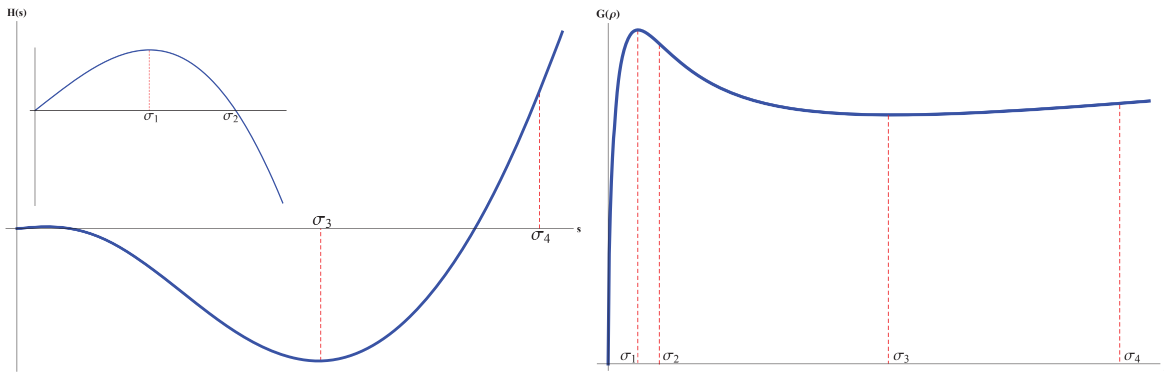


(a) H(s) (b) $G(\rho)$

Figure 16. Theorem 3.2 - $H(s)$ and $G(\rho)$

Proof of Theorem 3.3. First we will show that there exist $\sigma_i \in (0,\infty)$; $i = 1,2,3,4$ with $\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4$ such that $G'(\rho) > 0$ for $0 < \rho \le \sigma_1$ and $\rho > \sigma_4$, while $G'(\rho) < 0$ for $\sigma_2 < \rho \le \sigma_3$. For this, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \ge 0$. Then $H(0) = 0$ and $(I4)$ implies that $H(\sigma) < 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) > 0$, $H'(0) > 0$. Moreover, $(I3)$ implies $\lim_{s\to\infty} H'(s) > 0$. Furthermore, since $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s)$, it follows from $(I1) - (I2)$ that $H''(s) < 0$ on $(0, s_0)$ and $H''(s) > 0$ for $s > s_0$. Therefore, there exist $\sigma_i \in (0,\infty)$; $i = 1,2,3,4$ such that $H(\rho) - H(\rho z) > 0$ for $0 < \rho \le \sigma_1$ and $\rho > \sigma_4$ for $z \in (0,1)$. This implies $G'(\rho) > 0$ for $0 < \rho \le \sigma_1$ and $\rho > \sigma_4$. Also, $H(\rho) - H(\rho z) < 0$ for $\sigma_2 < \rho \le \sigma_3$ on $(0,1)$ and thus $G'(\rho) < 0$ for $\sigma_2 < \rho \le \sigma_3$.

Finally, $\lim_{\rho\to 0} G(\rho) = 0$ and $\lim_{\rho\to\infty} G(\rho) = \infty$ follow from Lemma 1.6 and Lemma 1.7 (i). □



(a) H(s) (b) $G(\rho)$

Figure 17. Theorem 3.3 - $H(s)$ and $G(\rho)$

3.3 p-Linear

Theorem 3.4 (Concave)**.** *Let (H3) hold. Suppose*

(J1) $f''(s) < 0$ *for all* $s > 0$*, and*

(J2) $(p-1)f(s) > sf'(s)$ *for all* $s > 0$*,*

Then there exists a $\mu > 0$ *such that* (1.2) *has*

(i) *no positive solutions for* $\lambda > \mu$*,*

(ii) *a unique positive solution for* $\lambda < \mu$*,*

Furthermore, $\lim_{\lambda \to 0} \rho_\lambda = 0$ *and* $\lim_{\lambda \to \mu} \rho_\lambda = \infty$*.*

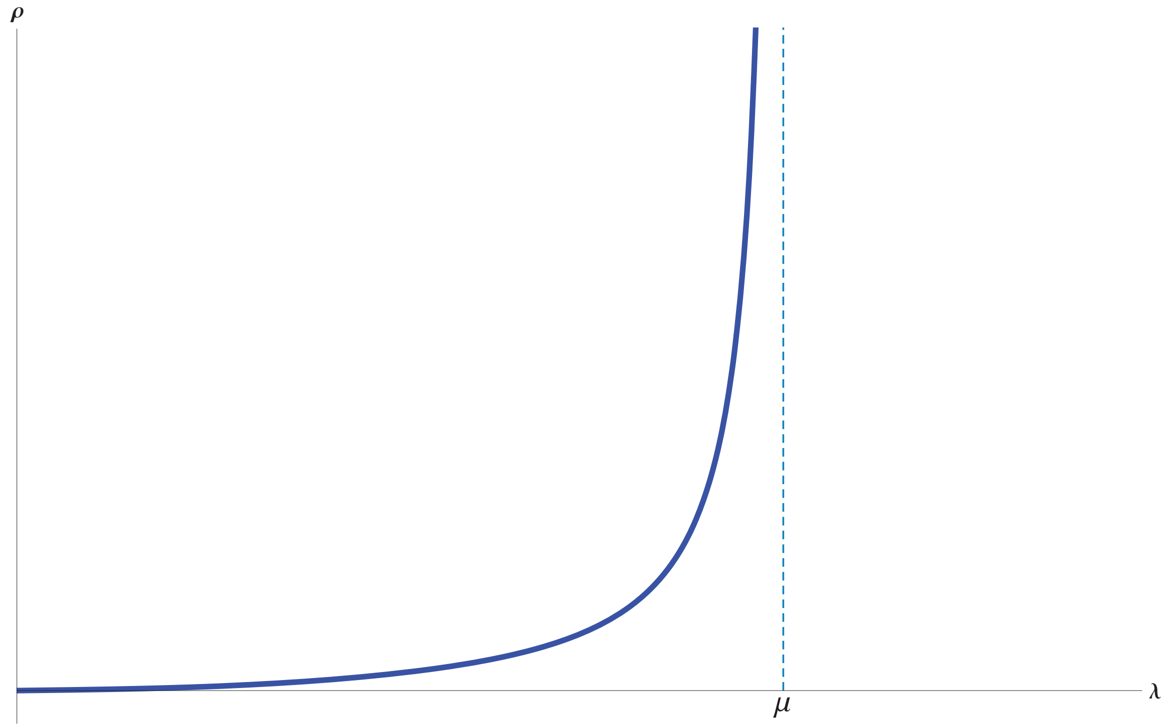


Figure 18. Theorem 3.4 Bifurcation Diagram

Theorem 3.5 (Convex)**.** *Let (H3) hold. Suppose*

(K1) $f''(s) > 0$ *for* $s > 0$*,*

(K2) $(p-2)f'(s) < sf''(s)$ *for* $s > 0$*, and*

(K3) $\lim_{s\to\infty} [(p-1)f(s) - sf'(s)] < 0.$

Then there exist $\mu_1 > 0$*,* $\mu_2 > 0$ *such that* $\mu_1 < \mu_2$ *and* (1.2) *has*

(i) *no positive solutions for* $\lambda > \mu_2$*,*

(ii) *a unique positive solution for* $\lambda < \mu_1$*,*

(iii) *exactly two positive solutions for* $\lambda \in (\mu_1, \mu_2)$*.*

Furthermore, $\lim_{\lambda \to 0} \rho = 0$ *and* $\lim_{\lambda \to \mu_1} \rho = \infty$.

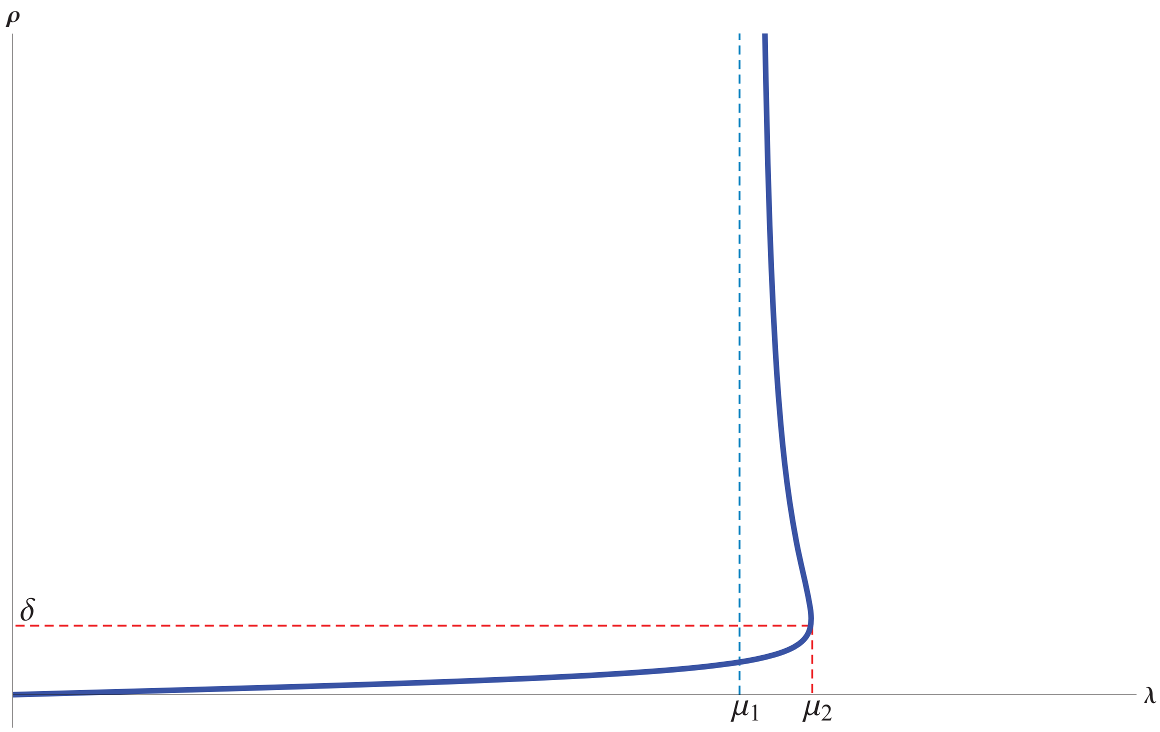
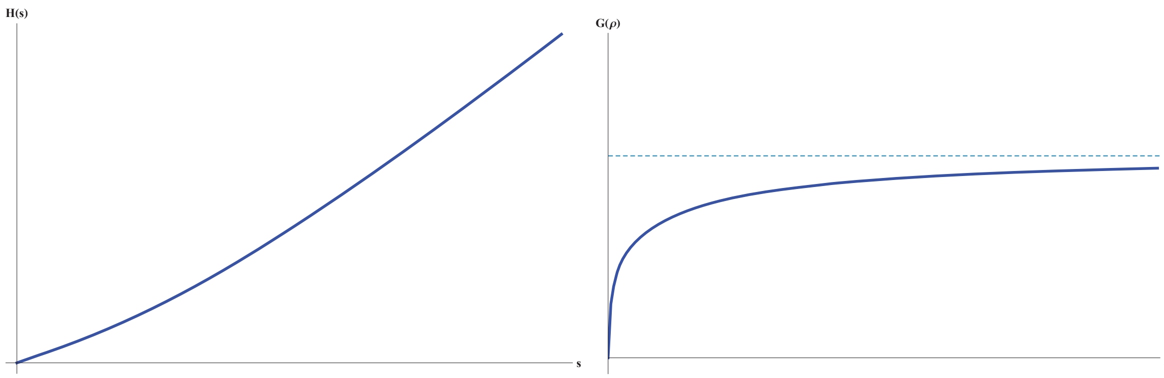


Figure 19. Theorem 3.5 Bifurcation Diagram

Proofs of Theorem 3.4 and Theorem 3.5

Proof of Theorem 3.4. First we will show that $G'(\rho) > 0$ for $\rho > 0$. For this, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) > 0$, $H'(0) > 0$. Moreover, $(J2)$ implies $H'(s) > 0$ for $s > 0$. Furthermore, since $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s)$, it follows from $(J1)$ that $H''(s) > 0$ for $s > 0$. Since $H'(0) > 0$, we obtain $H'(s) > 0$ for $s \geq 0$. Hence $H(\rho) - H(\rho z) > 0$ for $\rho > 0$ and $z \in (0, 1)$. Therefore, $G'(\rho) > 0$ for $\rho > 0$.

Finally, $\lim_{\rho \to 0} G(\rho) = 0$ and $\lim_{\rho \to \infty} G^p(\rho) = \mu$ for some $\mu > 0$ follow from Lemma 1.6 and Lemma 1.7 (iii), respectively. $\square$

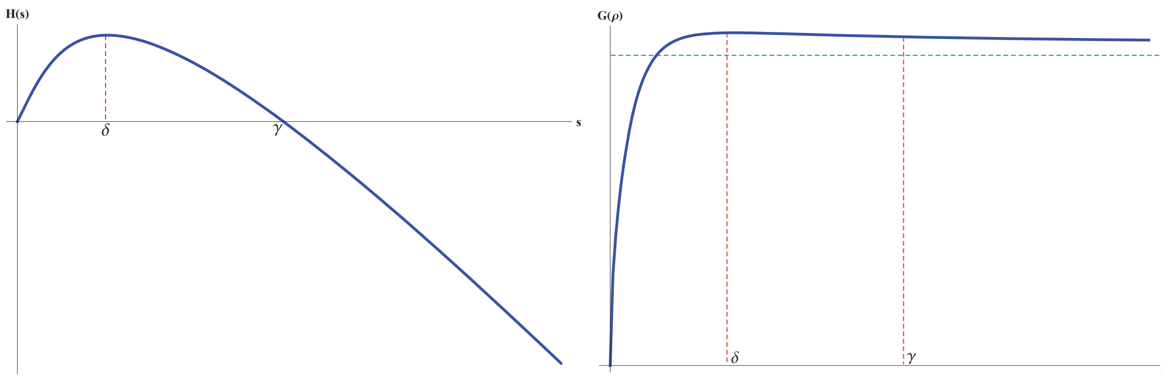


(a) H(s) (b) $G(\rho)$

Figure 20. Theorem 3.4 - $H(s)$ and $G(\rho)$

Proof of Theorem 3.5. First we will show that there exist $\delta > 0$, $\gamma > 0$ such that $\delta < \gamma$ and $G'(\rho) > 0$ for $0 < \rho \leq \delta$, while $G'(\rho) < 0$ for $\rho \geq \gamma$. To this end, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) > 0$, $H'(0) > 0$. Moreover, $(K3)$ implies $\lim\limits_{s\to\infty} H'(s) < 0$. Furthermore, since $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s)$, it follows from $(K1) - (K2)$ that $H''(s) < 0$ for $s > 0$. Hence there exist $\delta > 0$, $\gamma > 0$ such that $\delta < \gamma$ and $H'(\delta) = H(\gamma) = 0$. Then $H(\rho) - H(\rho z) > 0$ for $0 < \rho \leq \delta$ and $z \in (0, 1)$, which implies $G'(\rho) > 0$ for $0 < \rho \leq \delta$. Also, $H(\rho) - H(\rho z) < 0$ for $\rho \geq \gamma$ and $z \in (0, 1)$. Thus $G'(\rho) < 0$ for $\rho \geq \gamma$.

Finally, $\lim\limits_{\rho\to 0} G(\rho) = 0$ and $\lim\limits_{\rho\to\infty} G^p(\rho) = \mu$ for some $\mu > 0$ follow from Lemma 1.6 and Lemma 1.7 (iii), respectively. □



(a) H(s) (b) $G(\rho)$

Figure 21. Theorem 3.5 - $H(s)$ and $G(\rho)$

CHAPTER IV

$f(0) = 0$ PROBLEMS

In this chapter, we study positive solutions of (1.2) when $f : [0, \infty) \to \mathbb{R}$ is a C^2 function satisfying:

- $f(0) = 0$,
- $f'(s) > 0$ for $s > 0$.

Note that $u \equiv 0$ is a solution of (1.2) for all $\lambda > 0$.

Lemma 4.1. *If* $\lim_{s\to 0} \frac{f'(s)}{s^{p-2}} = m$ *for some* $m > 0$*, then* $\lim_{\rho\to 0} G(\rho) = C(m)$ *for some* $C(m) > 0$*.*

Proof. Lemma 1.6 gives

$$2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^{p-1}}{f(\rho)} \leq G^p(\rho) \leq 2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{\rho^p}{F(\rho)}.$$

for all $\rho \in (0, \infty)$. Then $\lim_{s\to 0} \frac{f'(s)}{s^{p-2}} = m$ implies

$$\eta \leq \lim_{\rho\to 0} G^p(\rho) \leq p\eta,$$

where

$$\eta = 2^p \left(\frac{p}{p-1}\right)^{p-1} \frac{p-1}{m}.$$

Thus there exists $\mu \in [\eta, p\eta]$ such that $\lim_{\rho \to 0} G^p(\rho) = \mu$. $\square$

4.1 p-Superlinear

Theorem 4.2. *Let (H1) hold. Suppose*

(L1) $f''(s) > 0$ *for* $s > 0$*,*

(L2) $(p-2)f'(s) < sf''(s)$ *for* $s > 0$*,*

(L3) $(p-1)f(s) < sf'(s)$ *for* $s > 0$*, and*

(L4) $\lim_{s \to 0} \frac{f'(s)}{s^{p-2}} = m$ *for some* $m > 0$*.*

Then there exists $\mu > 0$ *such that* (1.2) *has*

(i) no positive solutions for $\lambda > \mu$*,*

(ii) a unique positive solution at $0 < \lambda < \mu$*,*

Furthermore, $\lim_{\lambda \to \mu} \rho_\lambda = 0$ *and* $\lim_{\lambda \to 0} \rho_\lambda = \infty$

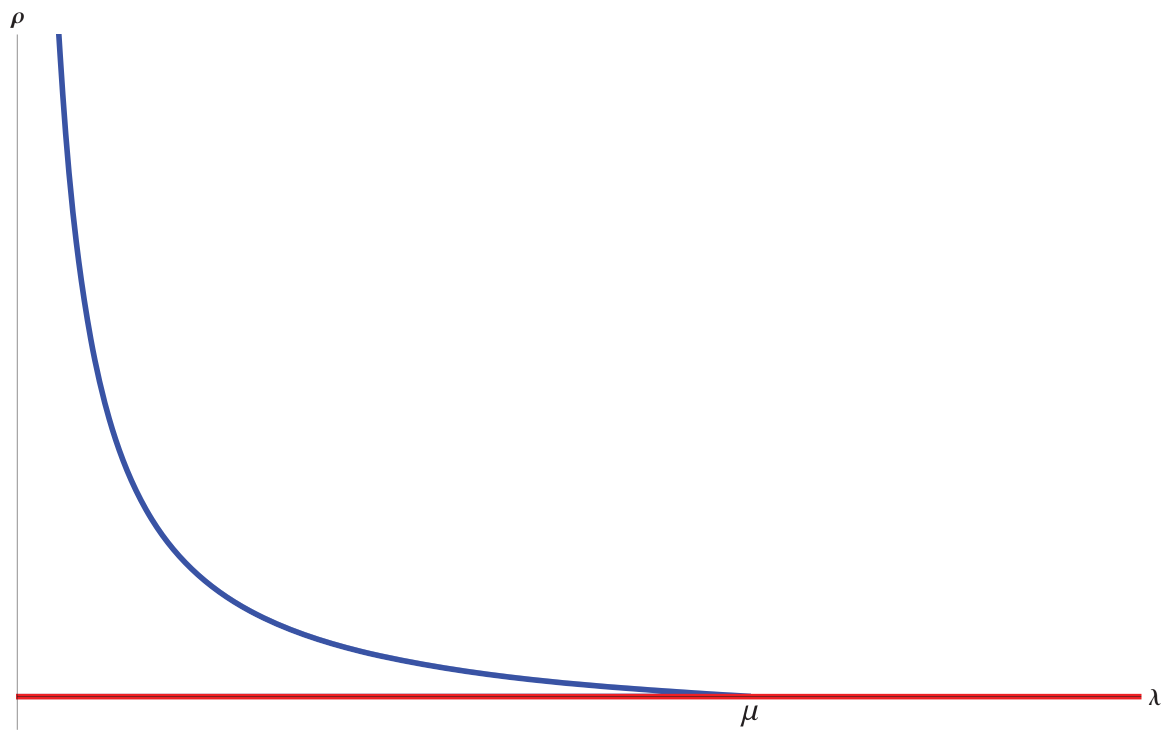
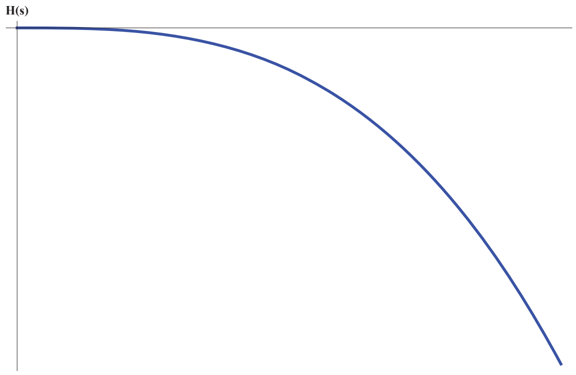


Figure 22. Theorem 4.2 Bifurcation Diagram

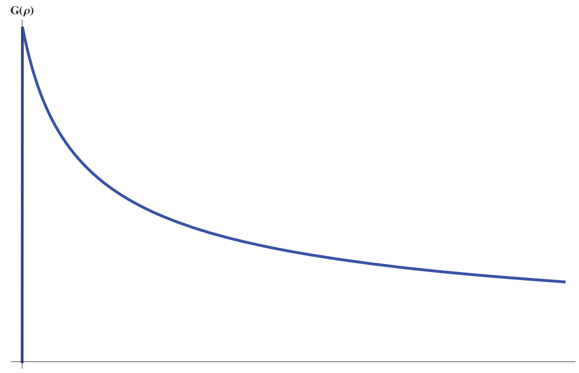
Proof of Theorem 4.2

Proof of Theorem 4.2. First we will show that $G'(\rho) < 0$ for all $\rho > 0$. For this, let $H(s) = F(s) - \frac{s}{p}f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p}f(s) - \frac{s}{p}f'(s)$ and $f(0) = 0$, $H'(0) = 0$. Moreover, $(L3)$ implies $H'(s) < 0$ for $s > 0$. Furthermore, since $H''(s) = \frac{p-2}{p}f'(s) - \frac{s}{p}f''(s)$, it follows from $(L1) - (L2)$ that $H''(s) < 0$ for $s > 0$. Hence $H(\rho) - H(\rho z) < 0$ for $\rho > 0$ and $z \in (0,1)$, which implies $G'(\rho) < 0$ for $\rho > 0$.

Finally, $\lim\limits_{\rho\to\infty} G(\rho) = 0$ follows from Lemma 1.7 (ii), which implies $\lim\limits_{\lambda\to 0} \rho_\lambda = \infty$. By Lemma 4.1 and $(L4)$ we get that $\lim\limits_{\rho\to 0} G(\rho) = \mu$ for some $\mu > 0$. This shows $\lim\limits_{\lambda\to\mu} \rho_\lambda = 0$. $\square$



(a) H(s)



(b) $G(\rho)$

Figure 23. Theorem 4.2 - $H(s)$ and $G(\rho)$

4.2 p-Sublinear

Theorem 4.3 (Concave)**.** *Let (H2) hold. Suppose*

(M1) $f''(s) < 0$ *for* $s > 0$*,*

(M2) $(p-1)f(s) > sf'(s)$ *for* $s > 0$*, and*

(M3) $\lim\limits_{s \to 0} \frac{f'(s)}{s^{p-2}} = m$*;* $m > 0$

Then there exists a $\mu > 0$ *such that* (1.2) *has*

(i) *no positive solutions for* $\lambda < \mu$*,*

(ii) *a unique positive solution for* $\lambda > \mu$*,*

Furthermore, $\lim\limits_{\lambda \to \mu} \rho_\lambda = 0$ *and* $\lim\limits_{\lambda \to \infty} \rho_\lambda = \infty$*.*

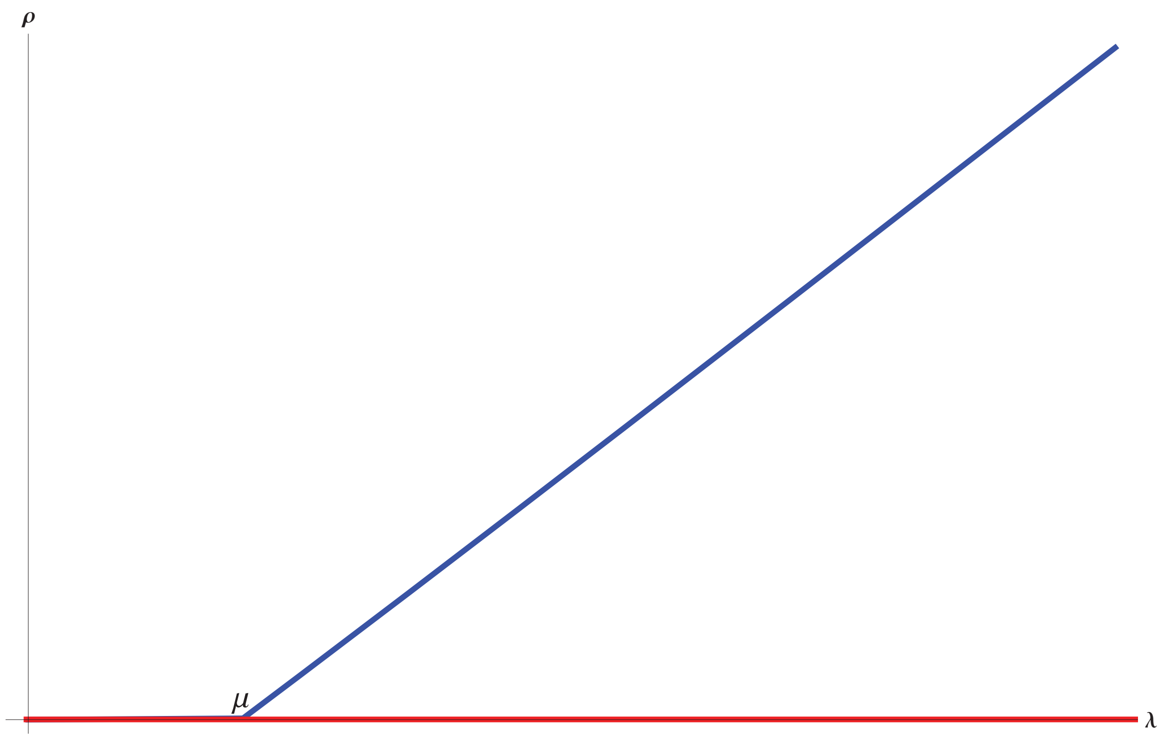


Figure 24. Theorem 4.3 Bifurcation Diagram

Theorem 4.4 (Convex-Concave). *Let (H2) hold. Suppose*

(N1) $f''(s) > 0$ *for* $s \in (0, s_0)$ *and* $f''(s) < 0$ *for* $s > s_0$,

(N2) $(p-2)f'(s) < sf''(s)$ *for* $s \in (0, s_0)$,

(N3) $(p-1)f(s) < sf'(s)$ *for* $s \in (0, s_0)$,

(N4) $\lim_{s\to\infty} [(p-1)f(s) - sf'(s)] > 0$,

(N5) *there exists a* $\sigma > 0$ *such that* $H(\sigma) < 0$, *and*

(N6) $\lim_{s\to 0} \frac{f'(s)}{s^{p-2}} = m$ *for some* $m > 0$.

Then there exist $\mu_1 > 0, \mu_2 > 0$ with $\mu_1 < \mu_2$ such that (1.2) has

(i) no positive solutions for $\lambda < \mu_1$,

(ii) a unique positive solution for $\lambda > \mu_2$,

(iii) exactly two positive solutions for $\lambda \in (\mu_1, \mu_2)$.

Furthermore, $\lim\limits_{\lambda \to \mu} \rho_\lambda = 0$ and $\lim\limits_{\lambda \to \infty} \rho_\lambda = \infty$.

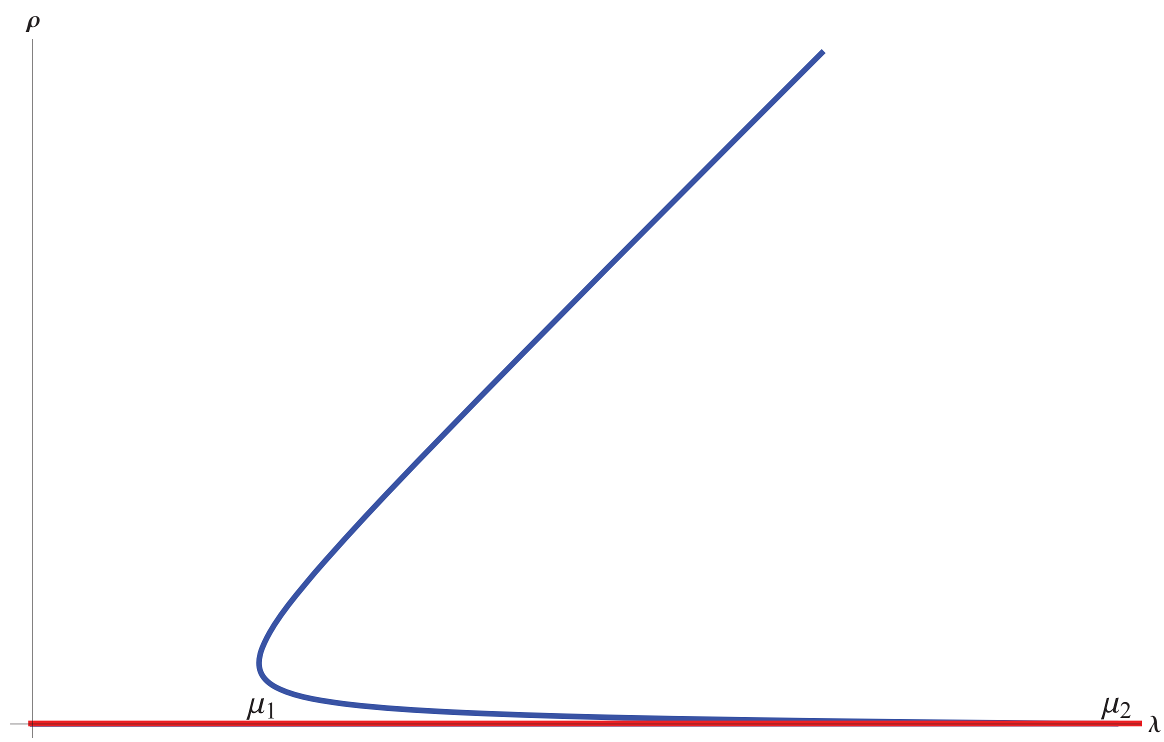


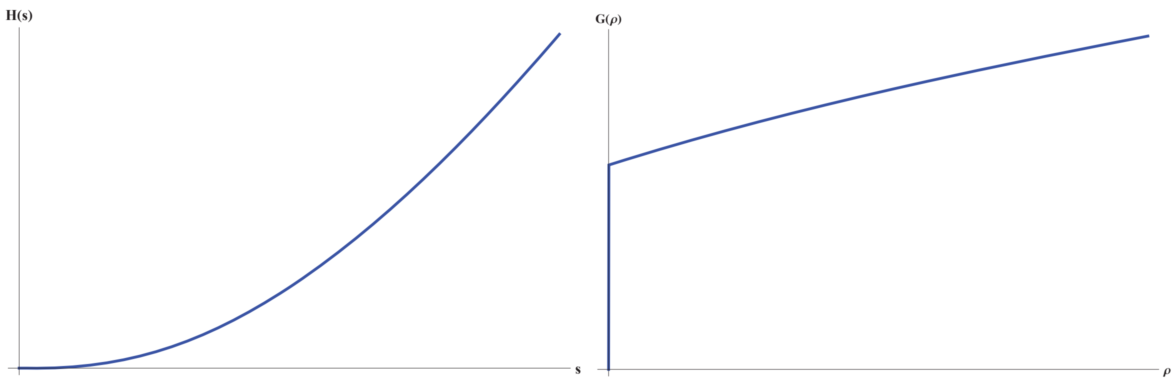
Figure 25. Theorem 4.4 Bifurcation Diagram

Proofs of Theorem 4.3 and Theorem 4.4

Proof of Theorem 4.3. First we will show that $G'(\rho) > 0$ for $\rho > 0$. For this, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) = 0$, $H'(0) = 0$. Moreover, $(M2)$ implies $H'(s) > 0$ for $s > 0$. Furthermore, since $H''(s) =$

$\frac{p-2}{p}f'(s) - \frac{s}{p}f''(s)$, it follows from $(M1)$ that $H''(s) > 0$ for $s > 0$. Hence $H(\rho) - H(\rho z) > 0$ for $\rho > 0$ and $z \in (0,1)$, which implies $G'(\rho) > 0$ for $\rho > 0$.

Finally, $\lim_{\rho\to\infty} G(\rho) = \infty$ follows from Lemma 1.7 (i), which implies $\lim_{\lambda\to\infty} \rho_\lambda = \infty$. By Lemma 4.1 and $(M3)$ we get that $\lim_{\rho\to 0} G(\rho) = \mu$ for some $\mu > 0$. This shows $\lim_{\lambda\to\mu} \rho_\lambda = 0$. □

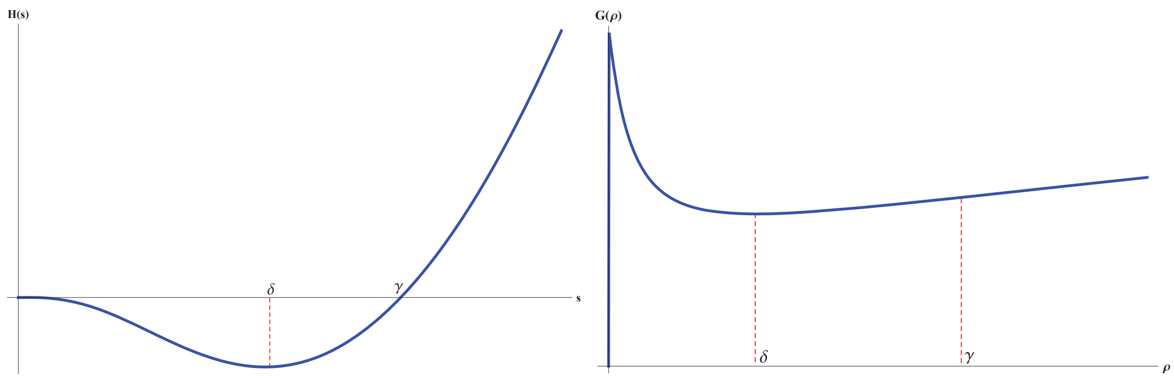


(a) H(s) (b) $G(\rho)$

Figure 26. Theorem 4.3 - $H(s)$ and $G(\rho)$

Proof of Theorem 4.4. First we will show that there exist $\delta > 0$, $\gamma > 0$ such that $\delta < \gamma$ and $G'(\rho) < 0$ for $0 < \rho \leq \delta$, while $G'(\rho) > 0$ for $\rho > \gamma$. To this end, let $H(s) = F(s) - \frac{s}{p}f(s)$ for $s \geq 0$. Then $H(0) = 0$ and $(N5)$ implies $H(\sigma) < 0$. Since $H'(s) = \frac{p-1}{p}f(s) - \frac{s}{p}f'(s)$ and $f(0) = 0$, $H'(0) = 0$. Moreover, $(N3) - (N4)$ imply $H'(s) < 0$ on $(0, s_0)$ and $\lim_{s\to\infty} H'(s) > 0$, respectively. Furthermore, since $H''(s) = \frac{p-2}{p}f'(s) - \frac{s}{p}f''(s)$, it follows from $(N1) - (N2)$ that $H''(s) < 0$ on $(0, s_0)$ and $H''(s) > 0$ for $s > s_0$. Hence there exist $\delta > 0$, $\gamma > 0$ such that $\delta < \gamma$ and $H'(\delta) = H(\gamma) = 0$. Then $H(\rho) - H(\rho z) < 0$ for $0 < \rho \leq \delta$ and $z \in (0,1)$, which implies $G'(\rho) < 0$ for $0 < \rho \leq \delta$. Also, $H(\rho) - H(\rho z) > 0$ for $\rho > \gamma$. Thus $G'(\rho) > 0$ for $\rho > \gamma$.

Finally, $\lim_{\rho\to\infty} G(\rho) = \infty$ follows from Lemma 1.7 (i), which implies $\lim_{\lambda\to\infty} \rho_\lambda = \infty$. By Lemma 4.1 and $(N6)$ we get that $\lim_{\rho\to 0} G(\rho) = \mu_2$ for some $\mu_2 > 0$. This shows $\lim_{\lambda\to\mu} \rho_\lambda = 0$. $\square$



(a) H(s) (b) $G(\rho)$

Figure 27. Theorem 4.4 - $H(s)$ and $G(\rho)$

4.3 p-Linear

Theorem 4.5 (Concave)**.** *Let* $(H3)$ *hold. Suppose*

(O1) $f''(s) < 0$ *for* $s > 0$,

(O2) $(p-1)f(s) > sf'(s)$ *for* $s > 0$,

(O3) $\lim_{s\to 0} \frac{f'(s)}{s^{p-2}} = m$ *for some* $m > 0$.

Then there exist $\mu_1 > 0, \mu_2 > 0$, *with* $\mu_1 < \mu_2$, *such that* (1.2) *has*

(i) no positive solutions for $\lambda \leq \mu_1$ *and* $\lambda > \mu_2$,

(ii) a unique positive solution for $\lambda \in (\mu_1, \mu_2)$.

Furthermore, $\lim_{\lambda\to\mu_1} \rho_\lambda = 0$ *and* $\lim_{\lambda\to\mu_2} \rho_\lambda = \infty$.

Remark. It would be interesting to analyze the behavior of the curve bifurcating from infinity at μ_2 depending on the location of μ_1 with respect to μ_2.

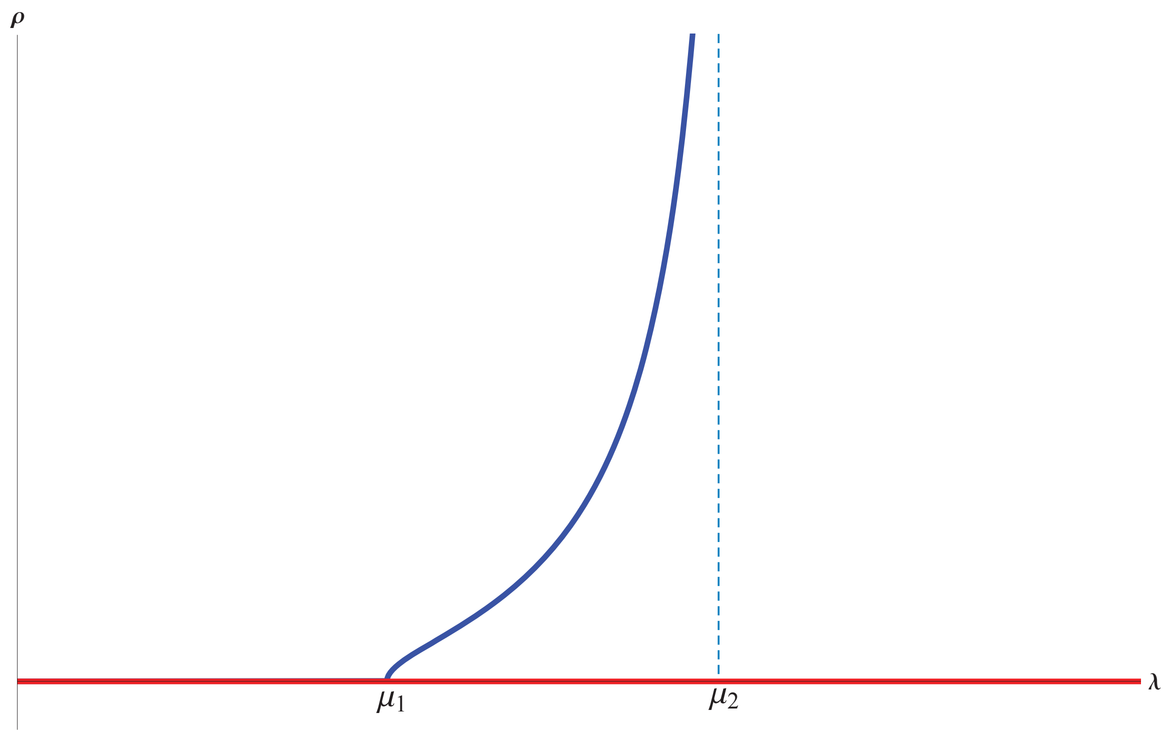


Figure 28. Theorem 4.5 Bifurcation Diagram

Theorem 4.6 (Convex)**.** *Let* $(H3)$ *hold. Suppose*

(P1) $f''(s) > 0$ *for* $s > 0$*,*

(P2) $(p-2)f'(s) < sf''(s)$ *for* $s > 0$*,*

(P3) $(p-1)f(s) < sf'(s)$ *for* $s > 0$*, and*

(P4) $\lim_{s \to 0} \frac{f'(s)}{s^{p-2}} = m$ *for some* $m > 0$*.*

Then there exist μ_1, μ_2, with $0 < \mu_1 < \mu_2$, such that (1.2) has

(i) no positive solutions for $\lambda < \mu_1$ and $\lambda > \mu_2$,

(ii) a unique positive solution for $\lambda \in (\mu_1, \mu_2)$.

Furthermore, $\lim_{\lambda \to \mu_1} \rho_\lambda = \infty$ *and* $\lim_{\lambda \to \mu_2} \rho_\lambda = 0$.

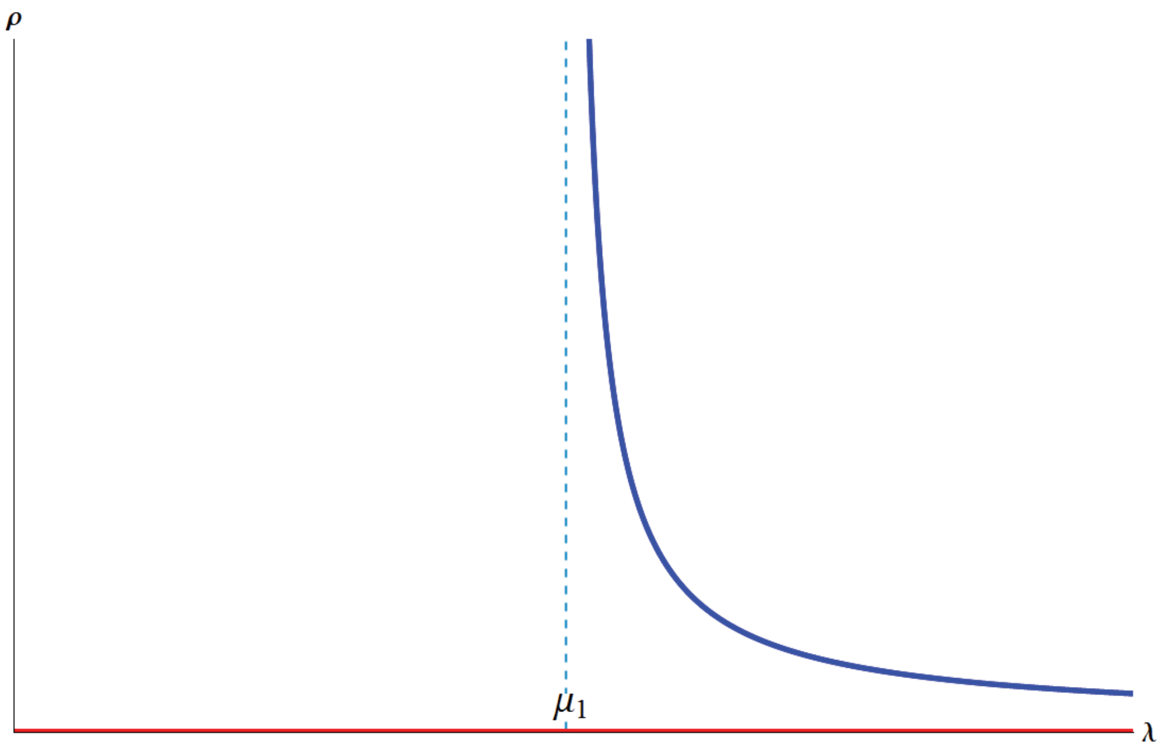
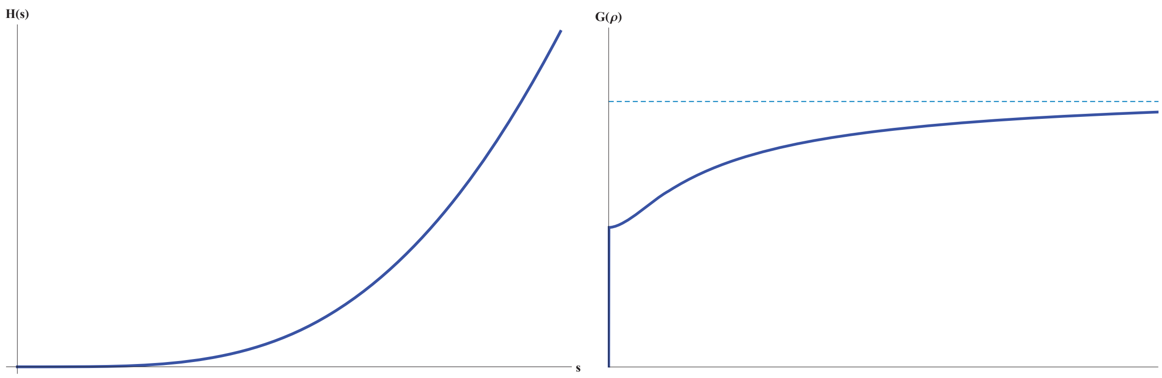


Figure 29. Theorem 4.6 Bifurcation Diagram

Proofs of Theorem 4.5 and Theorem 4.6

Proof of Theorem 4.5. First we will show that $G'(\rho) > 0$ for $\rho > 0$. For this, let $H(s) = F(s) - \frac{s}{p}f(s)$ for $s \geq 0$. Then $H(0) = 0$. Since $H'(s) = \frac{p-1}{p}f(s) - \frac{s}{p}f'(s)$ and $f(0) = 0$, $H'(0) = 0$. Moreover, $(O2)$ implies $H'(s) > 0$ for $s > 0$. Furthermore, since $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s)$, it follows from $(O1)$ that $H''(s) > 0$. Therefore $H(\rho) - H(\rho z) > 0$ for $\rho > 0$ and $z \in (0,1)$, which implies $G'(\rho) > 0$ for $\rho > 0$.

Lemma 1.7 (iii) implies that there exists a $\mu_2 > 0$ such that $\lim\limits_{\rho\to\infty} G^p(\rho) = \mu_2$. This shows $\lim\limits_{\lambda\to\mu_2} \rho_\lambda = \infty$. By Lemma 4.1 and $(O3)$ we get that $\lim\limits_{\rho\to 0} G(\rho) = \mu_1$ for some $\mu_1 > 0$. This shows $\lim\limits_{\lambda\to\mu_1} \rho_\lambda = 0$. $\square$

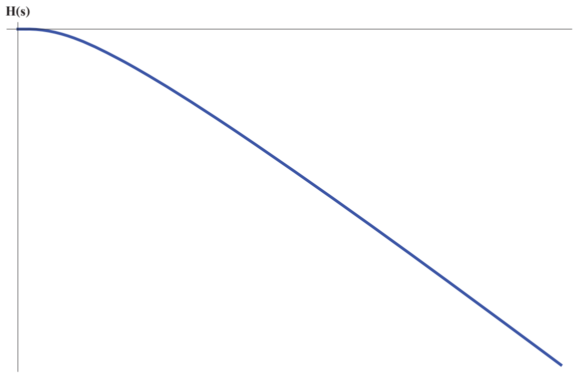


(a) H(s) (b) $G(\rho)$

Figure 30. Theorem 4.5 - $H(s)$ and $G(\rho)$

Proof of Theorem 4.6. First we will show that $G'(rho) < 0$ for $\rho > 0$. For this, let $H(s) = F(s) - \frac{s}{p} f(s)$ for $s \geq 0$. Since $H'(s) = \frac{p-1}{p} f(s) - \frac{s}{p} f'(s)$ and $f(0) = 0$, $H'(0) = 0$. Moreover, $(P3)$ implies $H'(s) < 0$ for $s > 0$. Furthermore, since $H''(s) = \frac{p-2}{p} f'(s) - \frac{s}{p} f''(s)$, it follows from $(P1) - (P2)$ that $H''(s) < 0$ for $s > 0$. Hence $H(\rho) - H(\rho z) < 0$ for $\rho > 0$ and $z \in (0,1)$. Therefore, $G'(\rho) < 0$ for $\rho > 0$.

Lemma 1.7 (iii) implies that there exists a $\mu_1 > 0$ such that $\lim\limits_{\rho\to\infty} G^p(\rho) = \mu_1$. This shows $\lim\limits_{\lambda\to\mu_1} \rho_\lambda = \infty$. By Lemma 4.1 and $(P4)$ we get that $\lim\limits_{\rho\to 0} G(\rho) = \mu_2$ for some $\mu_2 > 0$. This shows $\lim\limits_{\lambda\to\mu_2} \rho_\lambda = 0$. Therefore, the shape of the bifurcation diagram is as depicted in Figure 29. However the figure does not show μ_2, at which $\lim\limits_{\lambda\to\mu_2} \rho_\lambda = 0$. $\square$



(a) H(s)

(b) $G(\rho)$

Figure 31. Theorem 4.6 - $H(s)$ and $G(\rho)$

CHAPTER V

SUMMARY AND FUTURE DIRECTIONS

By way of the quadrature method we were able to construct bifurcation diagrams that allowed us to observe how varying the behavior of f near the origin, near infinity, and its concavity affected the existence and mulitplicity of positive solutions to (1.2). During our investigation into the number of positive solutions for (1.2) we observed, in the semipositone case, that the existence of multiple positive solutions only occurred when f exhibited concavity. In comparison, the positone problems we studied required convexity to guarantee multiple solutions. In the $f(0) = 0$ case, only when f was concave and convex, did we see the existence of multiple solutions. For future research, we are interested in utilizing the quadrature method to study (1.2) with boundary conditions other than Dirichlet.

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