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Over the past 40 years, much of mathematical finance has been built on the premise that stocks tend to move according to continuous-time stochastic processes, particularly geometric Brownian Motion. However, fuzzy set theory has recently been shown to hold promise as a model for financial uncertainty as well, with continuous time fuzzy processes used in place of Brownian Motion. And, like Brownian Motion, fuzzy processes also cannot be measured using a traditional Lebesque integral. This problem was solved on the stochastic side with the development of Ito's calculus. Likewise, the Liu integral has been developed to measure fuzzy processes. In this thesis we will describe and compare the theoretical underpinnings of these models, as well as "back-test" several variations of them on historical market data.

PRICING EUROPEAN STOCK OPTIONS USING STOCHASTIC AND FUZZY CONTINUOUS TIME PROCESSES

by

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CHAPTER I

INTRODUCTION

1.1 Overview and Rationale

Today's financial markets trade in a wide variety of financial instruments. These are typically divided into underlying assets (commodities, foreign currency, stocks, bonds) and derivatives whose values are based on some underlying asset. These include basic options like call options and put options (the right to buy/sell an asset at a certain price in the future), as well as more complex instruments like interest-rate swaps and the now-infamous mortgage-backed securities. Derivatives have many uses in modern financial markets, not only as speculative tools, but also as insurance against uncertainty. A manufacturer of gold watches might want to insure himself against a rise in the price of gold by purchasing a call option on gold, and a borrower could protect themselves from a rise in interest rates by purchasing an interest rate swap to lock in a particular rate. Because of their profusion and utility in today's markets, the pricing of these derivatives has become a primary concern of modern mathematical finance.

Because a derivative's price is based partly on the future price of the underlying asset, there is an element of uncertainty involved in pricing. For this reason, pricing models typically include some amount of randomness. One of the most important models, the Black Scholes Model [2], is based on the assumption that stock prices can be modeled using a stochastic process over a probability space. However, there are several other approaches to options modeling, including the one proposed by Baoding Liu [20]. Instead of the traditional probability measure, he defines the Credibility Measure and a "fuzzy process" as a possible model for the movement of stock prices. This thesis concerns itself with a theoretical and practical comparison of these two models.

1.2 Organization of Thesis

We begin with a brief overview of probability measure and an introduction to Brownian Motion and its applications in finance in Chapter 2. This will introduce the rationale and formula for the Black Scholes model. This portion of the thesis contains a review of existing literature on the topic. In Chapter 3 we develop the Credibility Measure as an alternative measure of uncertainty, and see that fuzzy processes can also be used in financial modeling. This chapter concludes with the derivation of the Liu model. While significant literature exists on the topic of fuzzy processes ([20], [14], [15], [18], [17], [16], [5]) this portion of the thesis represents a self-contained, original axiomatic compilation. In Chapter 4, a quantitative comparison between the Black Scholes and Liu model is performed, with results and implications discussed in Chapter 5. In my review of the literature on fuzzy processes in finance, we have not encountered a similar comparison. As an appendix, we have included MATLAB code used for my quantitative comparison.

CHAPTER II

BLACK SCHOLES MODEL

2.1 Probability Measure

Definition II.1 (σ-Algebra, [21]). A collection $\mathcal{F}$ of subsets of a set Ω is said to be a *σ-algebra in* Ω if $\mathcal{F}$ has the following properties:

(1) $\Omega \in \mathcal{F}$.

(2) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, where A^c is the complement of A relative to Ω.

(3) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathcal{F}$ for $n = 1, 2, 3, ...$, then $A \in \mathcal{F}$.

Definition II.2 (Probability Space, [3]). A *probability space* is an ordered triple $(\Omega, \mathcal{F}, P)$, where Ω is any non-empty set, $\mathcal{F}$ is a σ-algebra on Ω and $P : \mathcal{F} \to [0, 1]$ is a probability measure on $\mathcal{F}$ such that

(1) $P(\Omega) = 1$ and $P(\emptyset) = 0$

(2) for any countable collection $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$ with $A_i \bigcap A_j = \emptyset$, $i \neq j$:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \tag{II.1}$$

We call Ω the *sample space*, the elements of $\mathcal{F}$ *events* and each element of Ω an *elementary event*.

In defining a random variable, we wish to assign a numerical value $X(\omega) \in [0, 1]$ to every elementary element $\omega \in \Omega$.

Definition II.3 (Measurable function, [21])**.** Let E be a set and $\mathcal{F}$ be a σ-algebra on E. Then a function $X : (E, \mathcal{F}) \to \mathbb{R}$ is called *(Borel) measurable* if $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}_{\mathbb{R}}$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra generated by open intervals in $\mathbb{R}$.

Definition II.4 (Random Variable, [3])**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then a Borel-measurable mapping $X : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a called *random variable.*

This allows us to measure the probability that the random variable will take on values in any $B \in \mathcal{B}_{\mathbb{R}}$.

Definition II.5 (Independent Random Variables, [24])**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, with $\mathcal{F}_i$ be a sub-σ-algebra for i in some non-empty index set $\mathcal{I}$. Then the σ-algebras $\mathcal{F}_i$ are said to be *mutually P-independent* if for every subset $i_1, ..., i_n \subset \mathcal{I}$ and every choice $A_{i_m} \in \mathcal{F}_{i_m}$, $1 \leq m \leq n$, we have:

$$P(A_{i_1} \cap .. \cap A_{i_n}) = P(A_{i_1}) * ... * P(A_{i_n}) \tag{II.2}$$

It should be noted that the more traditional definition of independence is that two sets A and B are independent if $P(A \cap B) = P(A)P(B)$ and that these two definitions are equivalent.

For the purposes of this thesis, we will be considering only discrete time processes indexed by $t \in \mathbb{N}$ and continuous time processes indexed over $\mathbb{R}^+ = [0, \infty)$. In any case, our index set T will have a natural σ-algebra of Borel sets inherited from $\mathbb{R}$. Consequently, $T \times \Omega$ will have a product σ-algebra [21, Definition 7.1].

Definition II.6 (Stochastic Process, [10])**.** Given a probability space $(\Omega, \mathcal{F}, P)$, an index set T ($T = \mathbb{N}$ or $T = [0, \infty)$) a *stochastic process* is a measurable function $X : T \times \Omega \to \mathbb{R}$.

In particular, if X is a stochastic process, then

(i) $X(t, .) : \Omega \to \mathbb{R}$ is a random variable, denoted X_t.

(ii) $X(., \omega) : T \to \Omega$ is a measurable function.

Definition II.7 (Sample path, [3])**.** Given a stochastic process $X : T \times \Omega \to \mathbb{R}$ and $\omega \in \Omega$, the *sample path* of ω is a function $t \to X(t, \omega)$.

Definition II.8 (Brownian Motion, [10])**.** A stochastic process $B(t, \omega)$ is called a *Brownian motion* if it satisfies the following conditions:

(1) $P\{\omega | B(0, \omega) = 0\} = 1$.

(2) For any $0 \leq s < t$, $B_t - B_s$ is a normally distributed random variable with mean 0 and variance $t - s$ Here, by $B_t - B_s$ we mean a function from Ω to $\mathbb{R}$ that to each $\omega \in \Omega$ assigns a value $B(t, \omega) - B(s, \omega)$., s.t. for any $a < b$

$$P\{a \leq B_t - B_s \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b \frac{e^{-x^2}}{2(t-s)} dx \qquad \text{(II.3)}$$

(3) $B(t, \omega)$ has independent increments. This means that for any $0 \leq t_1 < t_2 < .. < t_n$, the random variables

$$B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}} \tag{II.4}$$

are independent.

(4) Almost all sample paths of $B(t, \omega)$ are continuous functions. This means that

$$P\{\omega | B(., \omega) \text{ is continuous } \} = 1 \tag{II.5}$$

Brownian motion provides us with a good framework for describing the behavior of a stock price - jumps in price occur (particularly in today's computer-driven markets) nearly instantaneously, and with a large degree of randomness. However, there are several problems with the use of standard Brownian motion to describe investments. First, standard Brownian motion can take on negative values, whereas a stock price does not. Second, financial markets do not operate completely randomly - the stock price tends to "drift" in a certain direction despite its randomness in the short-term. Thus the geometric Brownian motion was proposed, with the stock price tending to follow the stochastic differential equation [11]:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{II.6}$$

The dt term here represents the change in time over which we are observing the stock, and mimics a standard differential equation. Over time, we see a change in the price represented by the parameter μ, referred to as drift. This takes into account the fact that, over time, stocks experience a constant pull upwards, measured as the rate of return on a risk-less asset.

However, we also have our stock price affected by the term σdW_t. The dW_t here is the stochastic part of the differential equation, and can be described as the change in the path of a standard Brownian motion over an interval of length t as $t \to 0_+$. This stochastic element models the random behavior of the future stock price, with the parameter σ denoting the magnitude of expected uncertainty. The σ used is known as volatility, which is calculated as the standard deviation of the stock's rate of return, and though constant in the basic model, can also vary over time and price.

Intuitively, the stochastic differential equation provides a good way of explaining the relationship between the price of the stock and both the deterministic and Brownian elements of its movement. However this equation is really an informal way of expressing the integral

$$S_{t+s} - S_t = \int_t^{t+s} \mu(X_u, u)du + \int_t^{t+s} \sigma(X_u, u)dB_u \tag{II.7}$$

where $\int_t^{t+s} \mu(X_u, u)du$ is a traditional Lebesgue integral taken over the deterministic path of our stock's "drift" and $\int_t^{t+s} \sigma(X_u, u)dB_u$ is integrated over the path of a standard Brownian motion. Because the path of a Brownian motion is nowhere differentiable [10], standard integration cannot produce a solution to the differential equation. It was only with the work of Ito that such integrals became manageable through the process of stochastic integration [19]. The steps for constructing the Ito integral are given a very thorough treatment in Kuo, 2006 [10]. However, the basic process can be understood in parallel to the construction of the Lebesgue integral [21].

2.2 European Call Option

The most basic of stock options is the European call option, giving the bearer the right to buy a given stock at a specified price (called the "strike price" at a specific date in the future (called the "expiration date"). If the price of the stock on the expiration date (called the "spot price") is greater than the strike price, the bearer can exercise the option, buy the stock at the strike price, and then sell it at the current price, netting a gain of (price - strike) on the expiration date. An option that can be exercised for a gain is called "in-the-money." On the other hand, if the stock price is lower than the strike price, there is no gain in exercising the option, and it is called "out-of-the-money." Thus the payoff on a European call option can be defined by $\max[0,(P-K)]\}$ for $K=$strike price and $P=$price at expiration (also referred to as "spot price").

In order to compute the actual profit made on the purchase of such an option, we also have to take into account the cost of the option. And because the payoff will not occur until the exercise date at some point in the future, we must consider the potential interest that could have been earned had the price of the premium been invested in a risk-free asset for that time period. Thus the profit on a call option is given by:

$$\text{Profit} = \max[0, (P-K)] - \text{Future Value[Option Price]} \tag{II.8}$$

where the future value of money is simply the value of that amount invested at the risk-free rate for the specified period of time. It is generally calculated using continuously-compounded interest, so that $\$A$ invested at a risk free rate r for t years would have value $\$Ae^{rt}$ [19].

2.3 Black Scholes Model

Binomial Option Pricing

Consider a stock S, priced at \$55, which can take on one of two values at the end of one year, say \$65 and \$45. We wish to determine the value of an option expiring in one year, with strike price $K = 50$ and assuming a risk-free interest rate of .06. In order to do this, we will create and price a "synthetic option" by borrowing at the risk-free rate and purchasing an amount of stock so that, in one year, the payoff to the synthetic option will be equal to the payoff from the call option. The Law of One Price [19] states that positions with equal payoffs will have equal price, and so the price of our synthetic option will be the same as the option we are interested in.

Since our stock can take on only two values, we know that there are two possible payoffs for our option: $P_u = \$15$ if the stock goes up and $P_d = \$0$ if the stock goes down (since the option will not be exercised). We want to buy Δ shares of stock and "borrow" $\$B$ at the risk-free rate r such that our portfolio has equal payoff (a negative value for $\$B$ implies that we will be loaning that amount). Assuming our stock pays a dividend of $\frac{\delta}{year}$, we see that our Δ and B must satisfy the equations:

$$(\Delta * 45 * e^{\delta}) + (B * e^{r}) = 0 \tag{II.9}$$

$$(\Delta * 65 * e^{\delta}) + (B * e^{r}) = 15 \tag{II.10}$$

We can in fact solve for Δ and B for a stock with current price $\$S$ over any period h. Let S_d be the stock price after a downward move, S_u be the price after an upward move, P_d be the payoff of an option of strike price K after a downward move, P_u the payoff after an upward move. For $u = \frac{S_u}{S}$ and $d = \frac{S_d}{S}$ we have [19]:

$$\Delta = e^{-\delta h} \frac{P_u - P_d}{S(u - d)} \tag{II.11}$$

$$B = e^{-rh} \frac{uP_d - dP_c}{(u - d)} \tag{II.12}$$

$$\text{Call Price} = \Delta S + B \tag{II.13}$$

And we see that, in our case (letting $\delta = 0$), $\Delta = \frac{1}{2}$ and $B = \$ - 21.1897$. Thus our option would cost $\Delta S + B = \$6.3103$.

Needless to say, a single up/down movement per year is not realistic for any stock. However, as the period h approaches zero, we see not only that the binomial model begins to look more plausible, but also mimics the random walk on which Brownian motion is based. It can be shown that, as $h \rightarrow 0_+$, the prices predicted by the binomial model are the same as those predicted by the Black Scholes Model [7]. The interested reader should consult MacDonald's *Derivatives Markets* [19] for a more thorough treatment of the binomial stock model.

Black Scholes Formula, [2]

The solution to the stochastic differential equation (II.6) yields a formula for pricing call options and glsplput (and which can be modified to price a variety of other options). Inputs for the formula are given in the Table 1

Table 1. Notation for Black Scholes formula (II.14).

Symbol	Meaning
S	current stock price
K	option strike price
σ	volatility of stock (calculated as standard deviation of rate of return)
r	continuously compounded risk-free annual interest rate
T	time to expiration
δ	annual dividend yield for stock

The price of a call option is then given by:

$$\text{Price} = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2), \text{ where} \tag{II.14}$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{II.15}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{II.16}$$

and $N(x)$ is the cumulative distribution function for the Normal distribution, defined [19]

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-t^2}{2}\right) dt \tag{II.17}$$

which represents the probability that a standard normal random variable will take a value in the interval $(-\infty, x]$.

Assumptions of Black Scholes

The original Black-Scholes formula (II.14) is based on a set of assumptions about the financial markets in which the option is traded.

(1) There is a constant, known interest rate at which it is possible to lend and borrow cash.

(2) It is possible to buy and sell any amount, including fractions of shares, of stock.

(3) There are no transaction costs (such as brokerage fees or bid-ask spreads).

(4) The stock price follow geometric Brownian motion with a constant volatility and a drift that is both *constant* and *equal to the risk free rate.*

While these assumptions render the formula somewhat impractical, more current developments have been made to account for violations of these assumptions in the real world. Newer models, for example, include a term for continuous dividend yield, and other extensions have been made to handle discrete dividends and non-constant volatility. Particularly, it has been theorized that a stock's volatility, rather than remaining constant, behaves stochastically.

2.4 Stochastic Volatility

There are several extensions which allow for stochastic volatility within the original Black Scholes framework. Hull and White [8] and Wiggins [28] developed early extensions in which volatility behaves stochastically, but is uncorrelated to spot price. These models have shown to behave similarly to Black Scholes, though with slightly better accuracy on options with longer time to maturity [8], [28]. Another important stochastic volatility model was then developed by Heston [6] who added the assumption that volatilities are not only stochastic, but correlated with spot price.

These more sophisticated models can often provide a higher degree of predictive accuracy than the original Black-Scholes model. However, the purpose of this thesis is not forecasting, but rather to gain insight into the behavioral differences between stochastic and fuzzy processes. Thus we restrict ourselves to models with constant volatility for both simplicity and clarity.

Having now described the basic probabilistic models for option pricing, we move to those models established through fuzzy set theory.

CHAPTER III

LIU MODEL

3.1 Credibility Measure

Zadeh [29] initially introduced the possibility measure to deal with fuzzy events, assigning a value between 0 and 1 to a fuzzy event based on its "grade of membership." However, the possibility measure lacks the very useful property of self-duality (i.e. $\mathrm{Pos}\{A\} + \mathrm{Pos}\{A^c\} \neq 1$ for all $A \subset \Omega$), and so Liu and Liu [18] proposed the credibility measure as an alternative way to measure fuzzy events.

It should be noted that the possibility and credibility "measures" referred to in this thesis do not adhere to the countable additivity of traditional measure theory. For this reason, we cannot use the framework of standard measure theory, and must instead establish several useful properties axiomatically.

We construct credibility in terms of possibility, and thus some information about the possibility measure is first necessary.

Definition III.1. For a set Ω and its power set $\mathcal{P}(\Omega)$, we define a function $Pos : \mathcal{P}(\Omega) \to [0, 1]$ to be a *possibility measure* if:

(1) $\mathrm{Pos}\{\Omega\} = 1$

(2) $\mathrm{Pos}\{\emptyset\} = 0$

(3) $\mathrm{Pos}\{\bigcup_i A_i\} = \sup_i \mathrm{Pos}\{A_i\}$ for any countable collection $\{A_i\}$ in $\mathcal{P}(\Omega)$

Definition III.2. For a set Ω and its power set $\mathcal{P}(\Omega)$, we define the *necessity* of a set $A \in \mathcal{P}(\Omega)$ to be $Nec\{A\} = 1 - \text{Pos}\{A^c\}$.

In order to create a possibility measure, let us take a function $\mu : \Omega \to [0,1]$ with

$$\sup_{x \in \Omega} \mu(x) = 1. \tag{III.1}$$

For example let $\Omega = [0,2]$ and $\mu(x) = \frac{x}{2}$.

Theorem III.3 (Existence of a Possibility measure)**.** *Let $\mu : \Omega \to [0,1]$ is a function satisfying* (III.1)*. Then the function $Pos : \mathcal{P}(\Omega) \to [0,1]$ defined by*

$$Pos\{A\} = \sup_{x \in A} \mu(x) \tag{III.2}$$

is a possibility measure.

Proof. 1. $\text{Pos}\{\Omega\} = \sup_{x\in\Omega} \mu(x) = 1$.

2. $\text{Pos}\{\emptyset\} = 0$ by convention i.e. $\sup_{x\in\emptyset} \mu(x) = 0$.

3. We begin by letting

$$\alpha = \text{Pos}\left\{\bigcup_i A_i\right\} = \sup_{x \in \bigcup_i^\infty A_i} \mu(x) \tag{III.3}$$

and

$$\beta = \sup_i \text{Pos}\{A_i\} = \sup(\sup_{x \in A_i} \mu(x)). \tag{III.4}$$

Now assume, by way of contradiction, that $\alpha \neq \beta$. If $\alpha < \beta$ then there must be i_0 such that $\sup_{x \in A_{i_0}} \mu(x) > \alpha = \sup_{x \in \bigcup A_i} \mu(x)$ But this is impossible because $A_{i_0} \subset \bigcup A_i$.

We then consider the case of $\alpha > \beta$. This implies that $\alpha + \epsilon_0 = \beta$ for some $\epsilon_0 > 0$. $\alpha = \sup_{x \in \bigcup_i^\infty A_i} \mu(x)$ and so for $\frac{\epsilon_0}{2}$ there exists some $x_0 \in \bigcup_i^\infty A_i$ such that $|\alpha - \mu(x_0)| < \epsilon$ (or else α would not be the least upper bound). This implies that

$$\mu(x_0) > \alpha - \frac{\epsilon_0}{2} > \beta = \sup(\sup_{x \in A_i} \mu(x) \text{ for all } i \in \mathbb{N}). \tag{III.5}$$

But x_0 is in some A_i and so we have a contradiction. Thus we have $\alpha = \beta$. □

Theorem III.4. *For $A \subset B \subset \Omega$, $Pos\{A\} \leq Pos\{B\}$.*

Proof. Assume, by way of contradiction, that $\text{Pos}\{A\} > \text{Pos}\{B\}$, say $\text{Pos}\{A\} + \epsilon_0 = \text{Pos}\{B\}$ with $\epsilon_0 > 0$. Then for $\frac{\epsilon_0}{2}$ there must exist $x_0 \in A$ s.t $|\text{Pos}\{A\} - \mu(x_0)| < \frac{\epsilon_0}{2}$. Then for $x_0 \in A \subset B$ we have $\mu(x_0) > \sup_{x \in B} \mu(x)$, a contradiction. Thus $\text{Pos}\{A\} \leq \text{Pos}\{B\}$. □

Corollary III.5. *If $Pos\{A\} < 1$, then $Pos\{A^c\} = 1$.*

Proof. $(A \bigcup A^c) = \Omega$ and so $\text{Pos}\{A \bigcup A^c\} = \text{Pos}\{\Omega\} = 1$. But

$$\text{Pos}\left\{A \bigcup A^c\right\} = \sup(\text{Pos}\{A\}, \text{Pos}\{A^c\}) \tag{III.6}$$

and $\text{Pos}\{A\} < 1$ so $\text{Pos}\{A^c\} = 1$. □

Now that we have constructed a Possibility Measure, we will use it to create a Credibility Measure. First we outline the properties that we wish for our Credibility Measure to have.

Definition III.6 (Credibility Measure, [20]). For a set Ω and its power set $\mathcal{P}(\Omega)$, with $A, B \in \mathcal{P}(\Omega)$, we define a function $Cr : \mathcal{P}(\Omega) \to [0,1]$ to be a *credibility measure* if it satisfies the following conditions:

Axiom 1. Normality: $\mathrm{Cr}\{\Omega\} = 1$.

Axiom 2. Monotonicity: $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\} \quad \text{whenever} \;\; A \subset B \subset \Omega$.

Axiom 3. Self-Duality: $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ for any $A \subset \Omega$.

Axiom 4. Maximality: $\mathrm{Cr}\{\bigcup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any countable collection $\{A_i\}$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

We now show that a Credibility Measure exists.

Theorem III.7 (Existence of credibility Measure). *Let Ω be a set, $\mu : \Omega \to [0,1]$ satisfying* (III.1) *and Pos be a function given by* (III.2). *Then a function Cr : $\mathcal{P}(\Omega) \to [0,1]$ for $A \in \mathcal{P}(\Omega)$, defined by*

$$Cr\{A\} = \frac{1}{2}\left(Pos\{A\} + 1 - Pos\{A^c\}\right) \tag{III.7}$$

is a credibility measure on Ω.

Proof. We now show that a function, defined by (III.7), satisfies the four axioms above.

Normality: $\mathrm{Cr}\{\Omega\} = \frac{1}{2}(\mathrm{Pos}\{\Omega\} + 1 - \mathrm{Pos}\{\emptyset\}) = 1$

Monotonicity: Take any $A, B \in \mathcal{P}(\Omega)$ with $A \subset B$. Then, by (III.4) $\mathrm{Pos}\{A\} \leq \mathrm{Pos}\{B\}$. We also have $B^c \subset A^c$ and so

$$(1 - \mathrm{Pos}\{A^c\}) \leq (1 - \mathrm{Pos}\{B^c\}). \tag{III.8}$$

$$(\mathrm{Pos}\{A\} + 1 - \mathrm{Pos}\{A^c\}) \leq (\mathrm{Pos}\{B\} + 1 - \mathrm{Pos}\{B^c\}) \tag{III.9}$$

And thus $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$

Self-Duality:

$$\begin{aligned}
&\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = \\
&\qquad \frac{1}{2}(\mathrm{Pos}\{A\} + 1 - \mathrm{Pos}\{A^c\}) + \frac{1}{2}(\mathrm{Pos}\{A^c\} + 1 - \mathrm{Pos}\{A\}) \\
&\qquad = \frac{1}{2}(2) = 1
\end{aligned}$$

Maximality($\mathrm{Cr}\{\bigcup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for countable $\{A_i\}$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$):

If $\mathrm{Cr}\{\bigcup_i A_i\} < 0.5$ then

$$\mathrm{Pos}\left\{\bigcup_i A_i\right\} < \mathrm{Pos}\left\{(\bigcup_i A_i)^c\right\}. \tag{III.10}$$

Then by Corollary 1 we have

$$\text{Pos}\left\{(\bigcup_i A_i)^c\right\} = 1 \Rightarrow \text{Cr}\left\{\bigcup_i A_i\right\} = \frac{1}{2}\text{Pos}\left\{\bigcup_i A_i\right\} \tag{III.11}$$

$$= \frac{1}{2}\sup_i \text{Pos}\{A_i\} = \sup_i \text{Cr}\{A_i\} \tag{III.12}$$

And so we have

$$\text{Cr}\left\{\bigcup_i A_i\right\} = \sup_i \text{Cr}\{A_i\} \tag{III.13}$$

□

We see that credibility is defined as the average of a subsets possibility and necessity.

Lemma III.8. *Let Pos be possibility and Cr be given by* (III.7)*. For* $A \subset \Omega$ *with* $Cr\{A\} < 0.5$*, then* $Pos\{A\} < Pos\{A^c\}$*.*

Proof. For $\text{Cr}\{A\} = \frac{1}{2}(\text{Pos}\{A\} + 1 - \text{Pos}\{A^c\}) < 0.5$ we have $(\text{Pos}\{A\} + 1 - \text{Pos}\{A^c\}) < 1$ and so $\text{Pos}\{A\} < \text{Pos}\{A^c\}$. □

Theorem III.9. *[14] Possibility measures and credibility measures are uniquely determined by each other via* (III.7)*. Furthermore:*

$$Pos\{A\} = (2Cr\{A\}) \wedge 1 \tag{III.14}$$

for any $A \subset \Omega$*.*

Proof. We have already determined a credibility measure using possibility (III.7). Now we wish to show that if two possibility measures, $\mathrm{Pos}_1\{A\}$ and $\mathrm{Pos}_2\{A\}$, both give the same credibility, then these possibility measures must be equal i.e. if for all $A \subset \Omega$

$$\mathrm{Cr}\{A\} = \frac{1}{2}(\mathrm{Pos}_1\{A\} + 1 - \mathrm{Pos}_1\{A^c\}) \text{ and} \tag{III.15}$$

$$\mathrm{Cr}\{A\} = \frac{1}{2}(\mathrm{Pos}_2\{A\} + 1 - \mathrm{Pos}_2\{A^c\}), \tag{III.16}$$

then $\mathrm{Pos}_1\{A\} = \mathrm{Pos}_2\{A\}$.

Fix $A \subset \Omega$

Case 1: ($\mathrm{Pos}_1\{A\} = 1$ and $\mathrm{Pos}_2\{A\} < 1$)

If $\mathrm{Pos}_2\{A\} < 1$ then $\mathrm{Pos}_2\{A^c\} = 1 \Rightarrow \mathrm{Pos}_2\{A\} - \mathrm{Pos}_2\{A^c\} < 0$. Then for $\mathrm{Pos}_1\{A\} = 1$ we have $\mathrm{Pos}_1\{A^c\} \leq 1 \Rightarrow \mathrm{Pos}_1\{A\} - \mathrm{Pos}_1\{A^c\} \geq 0$. Then by our assumption

$$\frac{1}{2}(\mathrm{Pos}_1\{A\}+1-\mathrm{Pos}_1\{A^c\}) = \frac{1}{2}(\mathrm{Pos}_2\{A\}+1-\mathrm{Pos}_2\{A^c\}), \text{ which implies} \tag{III.17}$$

$$\mathrm{Pos}_1\{A\} - \mathrm{Pos}_1\{A^c\} = \mathrm{Pos}_2\{A\} - \mathrm{Pos}_2\{A^c\} \tag{III.18}$$

and so we have

$$\mathrm{Pos}_1\{A\} - \mathrm{Pos}_1\{A^c\} \geq 0 > \mathrm{Pos}_2\{A\} - \mathrm{Pos}_2\{A^c\} \tag{III.19}$$

This contradicts our assumption of equality. Similar reasoning works if $\text{Pos}\{A\}_1 < 1$ and $\text{Pos}\{A\}_2 = 1$.

Case 2: ($\text{Pos}_1\{A\} < 1$ and $\text{Pos}_2\{A\} < 1$): $\text{Pos}_1\{A\} < 1$ and so by Corollary 1 we have $\text{Pos}_1\{A^c\} = 1$. By the same reasoning, $\text{Pos}_2\{A\} < 1$ implies $\text{Pos}_2\{A^c\} = 1$. By our assumption that

$$\frac{1}{2}(\text{Pos}_1\{A\} + 1 - \text{Pos}_1\{A^c\}) = \frac{1}{2}(\text{Pos}_2\{A\} + 1 - \text{Pos}_2\{A^c\}), \tag{III.20}$$

we now have $\text{Pos}_1\{A\} - 1 = \text{Pos}_2\{A\} - 1$ or $\text{Pos}_1\{A\} = \text{Pos}_2\{A\}$.

And so possibility is uniquely determined by credibility.

Now we wish to show $\text{Pos}\{A\} = (2\text{Cr}\{A\}) \wedge 1$. First consider the case $\text{Cr}\{A\} < 0.5$. This implies that $\text{Pos}\{A\} < \text{Pos}\{A^c\}$, and so by Corollary 1 $\text{Pos}\{A^c\} = 1$ and so $\text{Pos}\{A\} = 2\text{Cr}\{A\}$. For $\text{Cr}\{A\} \geq 0.5$, by (III.7) we must have $\text{Pos}\{A\} = 1$ and thus $\text{Pos}\{A\} = (2\text{Cr}\{A\}) \wedge 1$. $\square$

Definition III.10 (Fuzzy Variable [20])**.** A fuzzy variable is a function from a credibility space $(\Omega, \mathcal{P}(\Omega), Cr)$ to the set of real numbers. [16]

Definition III.11 (Membership Function [20])**.** Let ξ be a fuzzy variable defined on the credibility space $(\Omega, \mathcal{P}(\Omega), Cr)$. Then it's *membership function* is defined by

$$\mu_\xi(x) = (2\text{Cr}\{\xi = x\}) \wedge 1 \tag{III.21}$$

Theorem III.12 (Credibility Inversion Theorem, [17])**.** *Let ξ be a fuzzy variable with membership function μ_ξ. Then for any $B \subset \mathbb{R}$*

$$Cr\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu_\xi(x) + 1 - \sup_{x \in B^c} \mu_\xi(x)\right) \tag{III.22}$$

Where $Cr\{\xi \in B\}$ represents the "likelihood" that the fuzzy variable ξ will take a value in B.

We have thus far shown that there exist unique relationships between a credibility measure, possibility measure and membership function. Given one of these three, we can uniquely determine the other two.

Definition III.13. A fuzzy variable ξ is said to be normally distributed if it has associated membership function:

$$\mu_\xi(x) = 2\left(1 + \exp\left(\frac{\pi|x - \alpha|}{\sqrt{6}\sigma}\right)\right)^{-1} \tag{III.23}$$

Definition III.14 (Independent Fuzzy Variables [13])**.** Fuzzy sets $\xi_1, .., \xi_n$ are said to be independent if

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{n} \xi_i \in B_i\right\} = \min_{1 \le i \le n} \mathrm{Cr}\left\{\xi_i \in B_i\right\} \tag{III.24}$$

for all $B_i \in \mathcal{P}(\Omega)$.

Definition III.15 (Identically-Distributed Fuzzy Variables [13])**.** Two fuzzy variables ξ and ν are said to be *identically-distributed* if

$$\mathrm{Cr}\left\{\xi \in B\right\} = \mathrm{Cr}\left\{\nu \in B\right\} \tag{III.25}$$

for all $B \in \mathcal{P}(\Omega)$.

Definition III.16 (Expected Value, [18])**.** For a fuzzy variable ξ, the expected value is defined by

$$\mathrm{E}\left[\xi\right] = \int_0^{+\infty} \mathrm{Cr}\left\{\xi \geq r\right\}\, dr - \int_{-\infty}^{0} \mathrm{Cr}\left\{\xi \leq r\right\}\, dr \tag{III.26}$$

so long as at least one of the integrals is finite. If both integrals are infinite, the expected value is undefined.

Theorem III.17. *[27] Let ξ be a continuous fuzzy variable with membership function μ_ξ. If the expected value exists, and there is a point x_0 such that $\mu_\xi(x)$ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, +\infty)$, then*

$$E\left[\xi\right] = x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \mu_\xi(x)dx - \frac{1}{2}\int_{-\infty}^{x_0} \mu_\xi(x)dx \tag{III.27}$$

Proof. Consider $x_0 \geq 0$. Then by (III.12) we have

$$\mathrm{Cr}\left\{\xi \geq r\right\} = \frac{1}{2}\left(\sup_{x\in\mathbb{R}|x\geq r} \mu_\xi(x) + 1 - \sup_{x\in\mathbb{R}|x<r} \mu_\xi(x)\right) \tag{III.28}$$

By our assumption, $\mu_\xi(x_0) > \mu_\xi(x) \forall x \in \mathbb{R}$ and so $\mu_\xi(x_0) = 1$. Thus for $0 \leq r \leq x_0$ we have $\mathrm{Cr}\{\xi \geq r\} = \frac{1}{2}(1 + 1 - \mu_\xi(x))$. For $x_0 < r$ we have $\sup_{x\in\mathbb{R}|x<r} \mu_\xi(x) = 1$ which implies $\mathrm{Cr}\{\xi \geq r\} = \frac{1}{2}\mu_\xi(x)$. Likewise, $x_0 \notin (-\infty, 0)$ and so $\mathrm{Cr}\{\xi \leq r\} = \frac{1}{2}\mu_\xi(x)$ for $r < 0$. We can now re-write

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\, dx - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\, dx \tag{III.29}$$

$$\begin{aligned}
&= \int_0^{x_0} \mathrm{Cr}\{\xi \geq r\}\, dx + \int_{x_0}^{+\infty} \mathrm{Cr}\{\xi \geq r\}\, dx - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\, dx \\
&= \int_0^{x_0} (1 - \frac{1}{2}\mu_\xi(x))dx + \int_{x_0}^{\infty} \frac{1}{2}\mu_\xi(x)dx - \int_{-\infty}^{0} \frac{1}{2}\mu_\xi(x)dx \\
&= \int_0^{x_0} 1dx - \int_0^{x_0} \frac{1}{2}\mu_\xi(x)dx + \int_{x_0}^{\infty} \frac{1}{2}\mu_\xi(x)dx - \int_{-\infty}^{0} \frac{1}{2}\mu_\xi(x)dx \\
&= x_0 + \int_{x_0}^{\infty} \frac{1}{2}\mu_\xi(x)dx - \int_{-\infty}^{x_0} \frac{1}{2}\mu_\xi(x)dx
\end{aligned}$$

Similar reasoning show this to be true for $x_0 < 0$. □

Lemma III.18. *Let ξ be a continuous fuzzy variable with membership function μ_ξ. If the expected value exists, and there is a point x_0 such that $\mu_\xi(x)$ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, +\infty)$, with $\mu_\xi(x)$ symmetrical on the line $x = x_0$ we have $E[\xi] = x_0$.*

Proof. By (III.17), we know that $\mathrm{E}[\xi] = x_0 + \int_{x_0}^{\infty} \frac{1}{2}\mu_\xi(x)dx - \int_{-\infty}^{x_0} \frac{1}{2}\mu_\xi(x)dx$. Likewise, because $\mu_\xi(x)$ is symmetrical about x_0 we know that $\mu_\xi(x_0 + x) = \mu_\xi(x_0 - x) \forall x \in \mathbb{R}$.

This means that $\int_{x_0}^{\infty} \frac{1}{2}\mu_\xi(x)dx = \int_{-\infty}^{x_0} \frac{1}{2}\mu_\xi(x)dx$ and the proof is complete. $\square$

Theorem III.19. *For a fuzzy variable ξ with normal membership function* (III.23)*, we have $E[\xi] = \alpha$.*

Proof. Using the theorem and lemma above, we need only to show that $\mu_\xi(x) = 2\left(1 + \exp\left(\frac{\pi|x-\alpha|}{\sqrt{6}\sigma}\right)\right)^{-1}$ is increasing on $(-\infty, \alpha)$, decreasing on $(\alpha, +\infty)$ and symmetric about α. We begin by calculating the derivative of $\mu_\xi(x)$.

$$\mu'_\xi(x) = \frac{\sqrt{\frac{2}{3}}\pi(\alpha - x)e^{\frac{\pi\sqrt{(\alpha-x)^2}}{\sqrt{6}\sigma}}}{\sigma\sqrt{(\alpha - x)^2}\left(e^{\frac{\pi\sqrt{(\alpha-x)^2}}{\sqrt{6}\sigma}}\right)^2} \tag{III.30}$$

This equation is positive for $x \in (-\infty, \alpha)$ and negative for $x \in (\alpha, +\infty)$ and equals zero at $x = \alpha$. Likewise, looking at $\mu_\xi(x)$, we see that $\mu(\alpha + x) = \mu(\alpha - x) \forall x \in \mathbb{R}$ and so the proof is complete. $\square$

Definition III.20 (Variance, [18])**.** Let ξ be a fuzzy variable with finite expected value α. Then the variance of ξ is defined by $\mathrm{Var}[\xi] = \mathrm{E}\left[(\xi - \alpha)^2\right].$

Example III.21. Let ξ be a fuzzy variable with normal membership function

$$\mu_\xi(x) = 2\left(1 + \exp\left(\frac{\pi|x - \alpha|}{\sqrt{6}\sigma}\right)\right)^{-1} \tag{III.31}$$

Then the variance of ξ is calculated:

$$\begin{aligned}
\mathrm{Var}\,[\xi] &= \mathrm{E}\left[(\xi - e)^2\right] \\
&= \int_0^\infty \mathrm{Cr}\left\{(\xi - \alpha)^2 \geq r\right\} dr \\
&= \int_0^\infty \mathrm{Cr}\left\{(\xi \geq \alpha + \sqrt{r}) \cup (\xi \leq \alpha - \sqrt{r})\right\} dr \\
&= \int_0^\infty \mathrm{Cr}\left\{(\xi \geq \alpha + \sqrt{r})\right\} dr \\
&= \int_0^\infty \left(1 + \exp \frac{\pi\sqrt{r}}{\sqrt{6}\sigma}\right)^{-1} dr \\
&= \frac{12\sigma^2}{\pi^2} \int_0^\infty \frac{r}{1 + \exp r} dr \\
&= \frac{12\sigma^2}{\pi^2} \frac{\pi^2}{12} = \sigma^2
\end{aligned}$$

3.2 Liu Process

Definition III.22 (Fuzzy Process, [17])**.** Let T be an index set and let $(\Omega, \mathcal{P}, Cr)$ be a credibility space. A fuzzy process is a function from T x $(\Omega, \mathcal{P}, Cr)$ to the set of real numbers.

Definition III.23 (Independent Increments, [17])**.** A fuzzy process X_t is said to have independent increments if

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_k} - X_{t_{k-1}} \tag{III.32}$$

are independent fuzzy variables for any times $t_0 < t_1 < ... < t_k$.

Definition III.24 (Stationary Increments, [17])**.** A fuzzy process X_t is said to have stationary increments if, for any given $t > 0$, the $X_{s+t} - X_s$ are identically distributed fuzzy variables for all $s > 0$.

Definition III.25 (Liu Process, [17])**.** A fuzzy process C_t is said to be a Liu process if:

(1) $C_0 = 0$,

(2) C_t has stationary and independent increments,

(3) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value αt and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x-\alpha|}{\sqrt{6}\sigma}\right)\right)^{-1} \tag{III.33}$$

The Liu process takes parameters α (called the drift co-efficient) and σ (called the diffusion coefficient), and for $\alpha = 0$ and $\sigma = 1$ we have a "standard Liu process." We see that the Liu process bears a strong resemblance to the probabilistic Brownian motion discussed earlier. Both start at 0, and have independent, stationary increments which are "normally" distributed over their respective measure spaces. The use of normally distributed variables is not arbitrary. It is based on the *Maximum Entropy Principle, [9]*, which directs us to choose the appropriate distribution (given the known parameters) which maximizes the amount of entropy. By maximizing entropy, we minimize the amount of "known" information that is built into our distribution.

For a discrete random variable with n outcomes, entropy is defined to be [22],

$$H = -\sum_{i=1}^{n} p_i \log p_i \tag{III.34}$$

Where p_i represents the probability of the i^{th} outcome. The discrete case can be extended to continuous probability distributions with probability density function $p(x)$ by the formula [22]

$$H = -\int_{-\infty}^{+\infty} p(x) \log p(x) dx \tag{III.35}$$

It can then be shown that, given a distribution with two parameters, the normal distribution maximizes entropy [25]. In a similar spirit, Li and Liu [12] define the entropy of a continuous fuzzy variable ξ to be

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mathrm{Cr}\{\xi = x\}) dx \tag{III.36}$$

where $S(t) = -t \ln t - (1-t) \ln (1-t)$.

Theorem III.26. *[15]For a continuous fuzzy variable ξ with two finite parameters, expected value α and variance σ^2*

$$H[\xi] \leq \frac{\sqrt{6}\pi\sigma}{3} \tag{III.37}$$

with equality if ξ is a normally distributed fuzzy variable with the given expected value and variance.

Theorem III.27 (Existence, [17]). *There is a Liu process.*

3.3 Liu's Stock Model

The primary assumption of Liu's model for pricing european options is that stock prices tend to follow a geometric Liu process. The geometric Liu process provides two modifications to the original Liu process which make it more suitable for the modeling of stock prices. First, the geometric process, like stock prices, will never take on a negative value. Second, the process includes a constant "drift" term, which reflects the belief that stock prices, like the market as a whole, will trend slightly upwards over time regardless of short-term fluctuations.

Definition III.28 (Geometric Liu Process, [17]). Let C_t be a standard Liu process. Then $\alpha t + \sigma C_t$ is a Liu process, and the fuzzy process

$$G_t = \exp(\alpha t + \sigma C_t) \tag{III.38}$$

is called a geometric Liu process.

In applications to stock modeling, α is the constant drift of the stock, and σ represents the diffusion or volatility of the stock, traditionally calculated as the standard deviation of the stock's rate of return [19]. Using these assumptions, we can obtain a model for the risk-free bond price X_t and stock price Y_t given by:

$$dX_t = rX_t dt \tag{III.39}$$

$$dY_t = \alpha Y_t dt + \sigma Y_t dC_t \tag{III.40}$$

These "fuzzy differential equations" mimic the stochastic differential equations on which the Black-Scholes model was built, in that they combine the deterministic element of constant drift with the uncertainty of a fuzzy/brownian process. Likewise, fuzzy processes cannot be handled using the tools of standard calculus, and so Liu developed the Liu Integral and Liu Formula to find solutions to fuzzy differential equations. We describe briefly the Liu Integral and direct the interested reader to [20].

Definition III.29 (Liu Integral)**.** Let X_t be a fuzzy process and let C_t be a standard Liu process. For any partition of $[a, b]$ with $a = t_1 < t_2 < ... < t_{k+1} = b$ we write the mesh as

$$\Delta = \max_{1 \le i \le k} |t_{k+1} - t_k|$$

We then define the Liu integral of X_t with respect to C_t as

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0_+} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}) \tag{III.41}$$

We noted earlier that the payoff of a European call option at time T is $(Y_T - K) \bigvee 0$, where Y_T is the price at time t. Thus the current value of the option (taking into account time value of money at a continuously compounded interest rate r) is $\exp(-rT)(Y_T - K) \bigvee 0$. Using Liu's model for the expected value of $(Y_T - K)$ we define the option price to be:

Definition III.30. [20] The *European call option price* f for Liu's stock model is

$$f(Y_0, K, \alpha, \sigma, r) = \exp(-rT)\mathrm{E}\left[(Y_0 \exp(\alpha T + \sigma C_T) - K)\right] \tag{III.42}$$

Solving (III.42) we get the integral form below.

Theorem III.31. *[20]The price of a European call option based on Liu's stock model is*

$$f(Y_0, K, \alpha, \sigma, r) = Y_0 \exp(-rT) \int_{K/Y_0}^{+\infty} \frac{1}{1 + \exp\left(\frac{\pi}{\sqrt{6}\sigma T}(\ln(x) - \alpha T)\right)} dx \tag{III.43}$$

Proof. Using the definition of expected value for a fuzzy variable (III.16), we see that

$$\begin{aligned}
f(Y_0, K, \alpha, \sigma, r) &= \exp(-rT)\mathrm{E}\left[(Y_0 \exp(\alpha T + \sigma C_T) - K)\right] \\
&= \exp(-rT) * [\int_0^{+\infty} \mathrm{Cr}\left\{(Y_0 \exp(\alpha T + \sigma C_T) - K)^+ \geq x\right\} dx \\
&+ \int_{-\infty}^{0} \mathrm{Cr}\left\{(Y_0 \exp(\alpha T + \sigma C_T) - K)^+ \leq x\right\} dx \\
&= \exp(-rT) * \int_0^{+\infty} \mathrm{Cr}\left\{(Y_0 \exp(\alpha T + \sigma C_T) - K)^+ \geq x\right\} dx \\
&= Y_0 \exp(-rT) * \int_0^{+\infty} \mathrm{Cr}\left\{(\exp(\alpha T + \sigma C_T) - \frac{K}{Y_0})^+ \geq x\right\} dx \\
&= Y_0 \exp(-rT) * \int_{\frac{K}{Y_0}}^{+\infty} \mathrm{Cr}\left\{(\exp(\alpha T + \sigma C_T)) \geq u\right\} du \\
&= Y_0 \exp(-rT) * \int_{\frac{K}{Y_0}}^{+\infty} \frac{\exp(\frac{\pi\alpha}{\sqrt{6}\sigma})}{\exp(\frac{\pi\alpha}{\sqrt{6}\sigma}) + \exp(\frac{\pi \ln(x)}{\sqrt{6}\sigma T})} dx \\
&= Y_0 \exp(-rT) \int_{K/Y_0}^{+\infty} \frac{1}{1 + \exp\left(\frac{\pi}{\sqrt{6}\sigma T}(\ln(x) - \alpha T)\right)} dx
\end{aligned}$$

□

As was the case with Black-Scholes, the Liu model is a somewhat simplified version of real-world market behavior. For this reason, several extensions of the Liu model have been created.

3.4 Gao's Stock Model

One feature of stock price paths is the behavior of "mean reversion." This occurs when a stock has a tendency to move back towards a certain mean price, with the rate of movement proportional to the difference between the current price and mean. This type of behavior was first modeled by Black and Karasinski [1], and Gao presents the fuzzy counterpart [5]. The Gao model is based on the fuzzy differential equations:

$$dX_t = rX_t dt \tag{III.44}$$

$$dY_t = a(b - Y_t)dt + \sigma dC_t \tag{III.45}$$

We see, as usual, X_t, the price of a risk-less asset, moving simply according to interest rate and time. However, instead of defining the deterministic (non-fuzzy) element of the stock price simply in terms of overall drift, we instead show that it varies according to a (a type of drift) and the term $(b - Y_t)$ indicating that the stock has a tendency to move back towards some central price. We can then solve to determine the prices of call and put options using this model.

For simplicity, let $Q_1 = \frac{\sqrt{6}\pi(1-\exp(-aT)))}{(\pi a)}$ and $Q_2 = K - b - \exp(-aT)(Y_0 - b)$.

Theorem III.32. *[20] The price of a European call option for Gao's stock model is given by*

$$\exp\left(-rT\right)\left[Q_1 * \ln\left(1 + \exp\frac{Q_2}{Q_1}\right) - Q_2\right] \tag{III.46}$$

for T the expiration date and K the strike price.

CHAPTER IV

QUANTITATIVE COMPARISON

4.1 Purpose

The purpose of the quantitative comparison is two-fold. First, we are interested in testing whether or not the models provided by Liu and Gao are functional in the most basic sense. Do they predict prices which remain close to the actual values of call options? Second, we are interested in comparing the fuzzy models to the traditional stochastic models. Because the Liu model, in particular, uses the same parameters and market assumptions as Black Scholes, we hope that a comparison of the two will provide some insight into the differences and similarities between fuzzy and stochastic continuous-time processes.

4.2 Procedure

In order to evaluate the models, we set up a simple experiment using MATLAB and an on-line stock price database (Yahoo Historical Stock Data). Using interest rate data from the Federal Reserve and daily closing prices from the Yahoo Database, we pick a starting date and strike price and calculate both Black-Scholes and Liu prices for options expiring on each day following (the hypothetical expiration dates), up to a specified cut-off date. In order to account for dividend payments, we use the "Adjusted Closing" price, which adjusts for dividend payments and stock splits. Since we are using historical data, we are also able to easily calculate the actual value of these options, had they been purchased on the starting date. If the closing price is greater than the strike price, the value is simply $\$\exp(-rT) * (\text{Strike} - \text{Closing Price})$, and if the closing price is less than the strike, the option will not be exercised and so the value is \$0. What this yields (for each stock) is two lists of predicted values (one for Black-Scholes and one for Liu) and a list of actual values. On each day, the difference between predicted and actual values is calculated, and used to generate two measures of accuracy:

Definition IV.1. The *Root Mean Squared Error*(RMSE) is defined as

$$\sqrt{\frac{\sum(\text{Actual} - \text{Predicted})^2}{n}} \tag{IV.1}$$

where n is the total number of observations.

Definition IV.2. The *Mean Absolute Error*(MAE) is defined as

$$\frac{\sum|\text{Actual} - \text{Predicted}|}{n} \tag{IV.2}$$

While these two calculations both measure essentially the same thing (the average magnitude of the errors), the RMSE squares the terms before taking an average and a square root, and thus tends to give more weight to errors of high magnitude. The MAE, on the other hand, behaves linearly, placing equal weight on errors regardless of size. RMSE tends to be slightly larger than MAE. In terms of data, this yields an MAE and RMSE for both the Black Scholes and the Liu models for our stock for each trading day in our period.

Because there are many potential confounding factors that could influence a stock's behavior over a particular period, we seek to make our results more robust by using a large variety of stocks and calculating average errors over this group for each model. For this analysis, we choose the "S&P 100", a subset of the S&P 500 representing large-cap stocks across most major industries. Likewise, to account for external influences on the market as a whole, we choose multiple time periods from a 5 year interval during which the S&P 100 list remained the same. We will use time periods with lengths between 30 and 240 days (incremented by 30). Starting dates are generated randomly within the MATLAB script.

Note on Gao Model

During the preliminary analysis, it became clear that the Gao model outperformed both Liu and Black-Scholes substantially and consistently. However, since it takes more parameters than the two more simple models (along with volatility, it assumes that the stock tends to revert to some long-term mean at a specified reversion rate), we felt that its results would not provide much relevant information for the comparison and so this data was excluded.

4.3 Data Analysis

Our overall time-frame (2007-2011) is divided into periods of lengths of 30-240 days at 30 day increments (i.e. 30, 60, .. , 210, 240). For each of these periods, this yields data which includes:

(1) The length of the time period.

(2) Average RMSE (over all stocks) for the Liu and Black-Scholes models

(3) Average MAE (over all stocks) for the Liu and Black-Scholes models

The first step of our analysis will be comparing RMSE and MAE of the fuzzy models with those of Black-Scholes. This will be done by averaging RMSE and MAE for each model over each time period. We will also use this data to perform statistical tests for significant differences, using both parametric and nonparametric methods. This will be followed by a qualitative discussion of model performance and tendencies.

As an example, we have provided the following table to explain the data collected during a single $N = 30$ day trial. Regarding notation, $BS_{i,j}$ refers to the predicted Black-Scholes price for an option on Stock i expiring on Day j (likewise for $Liu_{i,j}$). We define $Actual_{i,j}$ to be the actual price of an option on Stock i expiring on Day j, and for the error terms at the bottom of the table, we have:

$$BS_{RMSE_k} = \sqrt{\frac{\sum_{j=1}^{N}(Actual_{k,j} - BS_{k,j})^2}{N}} \tag{IV.3}$$

$$Liu_{RMSE_k} = \sqrt{\frac{\sum_{j=1}^{N}(Actual_{k,j} - Liu_{k,j})^2}{N}} \tag{IV.4}$$

$$BS_{MAE_k} = \frac{\sum_{j=1}^{N}|Actual_{k,j} - BS_{k,j}|}{N} \tag{IV.5}$$

$$Liu_{MAE_k} = \frac{\sum_{j=1}^{N}|Actual_{k,j} - Liu_{k,j}|}{N} \tag{IV.6}$$

Table 2. Example 30-Day Trial

Day	Stock 1	Stock 2	...	Stock 100
Day 1	$(BS_{1,1}, Liu_{1,1}, Actual_{1,1})$	$(BS_{2,1}, Liu_{2,1}, Actual_{2,1})$	...	$(BS_{100,1}, Liu_{100,1}, Actual_{100,1})$
Day 2	$(BS_{1,2}, Liu_{1,2}, Actual_{1,2})$	$(BS_{2,2}, Liu_{2,2}, Actual_{2,2})$	...	$(BS_{100,2}, Liu_{100,2}, Actual_{100,2})$
...	...	...	...	...
Day 30	$(BS_{1,30}, Liu_{1,30}, Actual_{1,30})$	$(BS_{2,30}, Liu_{2,30}, Actual_{2,30})$	...	$(BS_{100,30}, Liu_{100,30}, Actual_{100,30})$
RMSE	$BS_{RMSE_1}, Liu_{RMSE_1}$	$BS_{RMSE_2}, Liu_{RMSE_2}$	...	$BS_{RMSE_{100}}, Liu_{RMSE_{100}}$
MAE	BS_{MAE_1}, Liu_{MAE_1}	BS_{MAE_2}, Liu_{MAE_2}	...	$BS_{MAE_{100}}, Liu_{MAE_{100}}$

The average MAE (AMAE) and RMSE (ARMSE) for this trial are then calculated as

$$BS_{ARMSE} = \frac{\sum_{i=1}^{100} BS_{RMSE_i}}{100} \tag{IV.7}$$

$$BS_{AMAE} = \frac{\sum_{i=1}^{100} BS_{MAE_i}}{100} \tag{IV.8}$$

$$Liu_{ARMSE} = \frac{\sum_{i=1}^{100} Liu_{RMSE_i}}{100} \tag{IV.9}$$

$$Liu_{AMAE} = \frac{\sum_{i=1}^{100} Liu_{MAE_i}}{100} \tag{IV.10}$$

So our single 30-day trial yields one ARMSE and one AMAE, averaging the respective error terms for each stock. We then perform 145 trials for each time period, producing, for example, 145 30-day ARMSE's and 145 30-day AMAE's.

4.4 Hypothesis Testing

In order to compare average RMSE and MAE, we will be using both parametric and nonparametric hypothesis tests. Because of the large sample size, we feel comfortable assuming normality in order to use a paired, two sample $\mathcal{T}$-test. However, some of the data does show skew (particularly in RMSE), and thus the non-parametric Wilcoxon signed-rank test was also used. In the results, we denote the p-values from the $\mathcal{T}$-test as $p_{\mathcal{T}}$ and the p-values from the signed-rank test as p_s. We will compare the average error terms for each model, and choose a one-sided test depending on the direction of this relationship. Let $\mu_{BS_R - Liu_R}$ be the mean difference in ARMSE over all 30 day periods (i.e. $BS_{ARMSE} - Liu_{ARMSE}$). For $\bar{X}_{BS_R - Liu_R} > 0$ we construct the hypotheses:

$$H_0 : \mu_{BS_R - Liu_R} = 0$$

$$H_A : \mu_{BS_R - Liu_R} > 0$$

For the sign-rank test, the hypotheses are the same, except mean (μ) is replaced by median. The inequality would be reversed if the sample mean/median of the Liu errors is greater than the sample mean/median of the Black Scholes. We will likewise perform the same test on the AMAE over all periods of each length, and each of these tests will be performed for each time period.

CHAPTER V

RESULTS

Each trial (30, 60, 90, 120, 150, 180, 210, and 240 days, respectively) contains observations from $N = 145$ random starting dates, with calculations made for hypothetical options with three different strike prices: $Price_0$, the price of the stock on the starting date, as well as $Price_0 - \$1$ and $Price_0 + \$1$. These strike prices were chosen in order to observe the behavior of these models on options of varying degrees of "moneyness" - i.e. the extent to which it is "in-the-money" or "out-of-the-money." For example, an option to buy a stock at a higher price than it is selling today will not be worth as much tomorrow as an option to buy that stock at a lower price. One consequence of this is that stocks with higher strike prices are more likely to expire "out-of-the-money" (i.e. the strike price is higher than the spot price and the option is worth \$0), particularly at the beginning of the time period. This means that there should be less difference between the Black-Scholes and Liu error terms at higher strike prices regardless of the length of the time period under consideration. While it might seem useful to alter the strike prices in proportion to the stock's price, this does not reflect reality. Stocks with very low prices are often in uncertain financial circumstances, subject to bankruptcy or buyout, and thus their volatility is not proportional to their price (likewise a high share price can often indicate a less volatile stock).

5.1 Visual Example

In order to help the reader visualize the process taking place, we have graphed some sample data from a single stock (CPB - Caraco Pharmaceutical Laborities, chosen randomly) over a 240 day trial. The first graph (1) displays the Liu and Black-Scholes option predictions against the actual value that option took on each given day. The second graph (2) shows simply the error terms ($Predicted - Actual$) for each model over the same time period.

Figure 1. CPB Call Option, Actual and Predicted Values

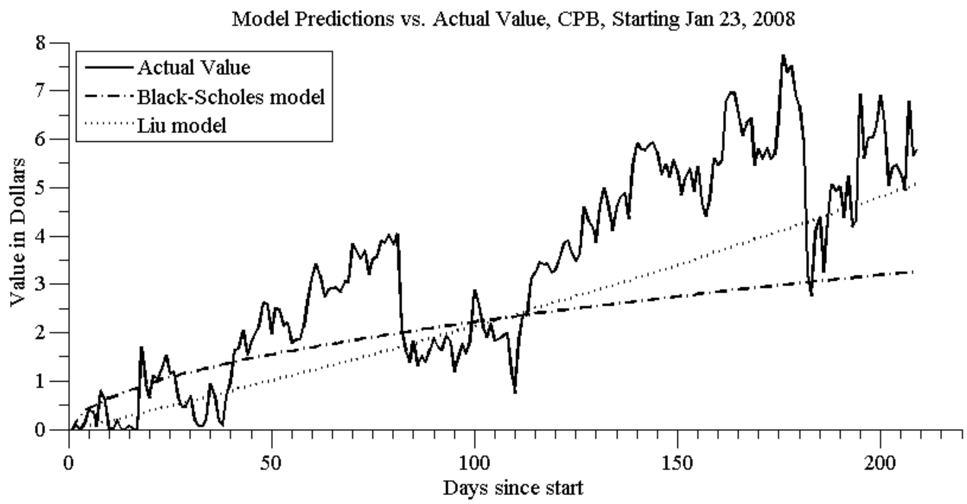
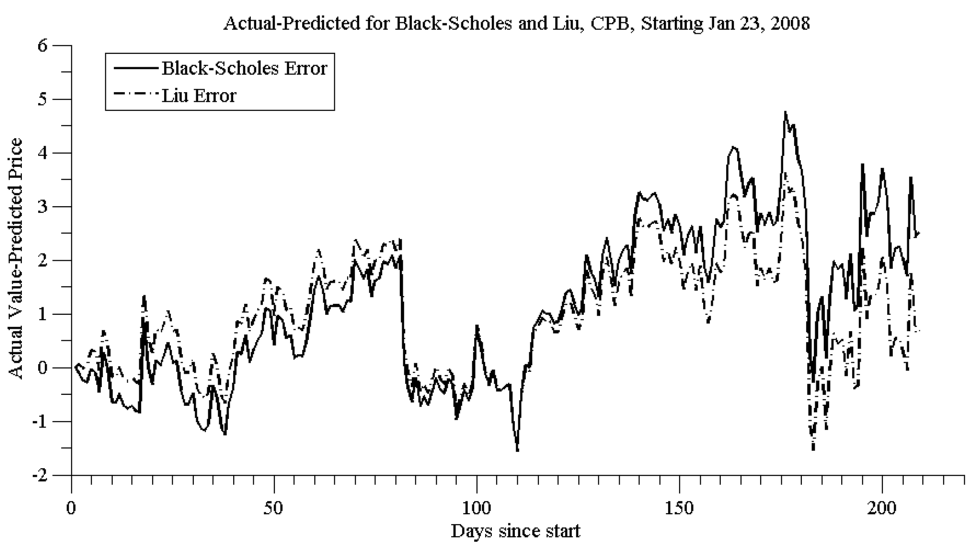


Figure 2. CPB Call Option, Error Terms over Time



5.2 Data for Each Trial

Here we report the data from each individual trial (30, 60, 90, 120, 150, 180, 210 and 240 days, respectively), including the "BS-Liu" term, which denotes the extent to which Liu outperformed Black-Scholes if positive, and vice versa if negative.

30 Day Trial

Table 3. ARMSE for 30 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	1.92486	1.67162	.25323	1.52793E-08	8.00E-06
$Price_0$	1.71410	1.56573	.14834	.001143	0.2914
$Price_0 + 1$	1.6896	1.60751	.08210	.030167	0.03016736

Table 4. AMAE for 30 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	1.76327	1.50026	.26301	9.65095E-10	1.30E-07
$Price_0$	1.50726	1.31171	.19555	.1.87E-5	0.0118
$Price_0 + 1$	1.46332	1.31489	.14842	.000288	0.0002879

The Liu model outperforms Black-Scholes over all strikes and both measures of error. This difference is lower (and less significant) at the higher strike price.

60 Day Trial

Table 5. ARMSE for 60 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	2.56435	2.25535	.28916	6E-09	3.37E-08
$Price_0$	2.38238	2.16196	.19699	5.81E-05	9.11103E-06
$Price_0 + 1$	2.33658	2.18659	.12535	.003025	0.4889

Table 6. AMAE for 60 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	2.31568	1.98591	.30964	4.09E-10	1.01E-10
$Price_0$	2.09504	1.82020	.25128	8.65E-07	2.5894E-08
$Price_0 + 1$	2.02845	1.80423	.19997	2.31E-05	0.0053

We again see Liu outperforming Black-Scholes, with a decreasing performance as strike increases. However, the differences are of a slightly greater magnitude than those from the 30 day trial.

90 Day Trial

Table 7. ARMSE for 90 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	2.95889	2.75404	.20485	2.06E-08	2.83E-06
$Price_0$	2.80372	2.68368	.12005	.00111	0.0606
$Price_0 + 1$	2.76366	2.71021	.05346	.07779	0.9745

Table 8. AMAE for 90 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	2.63206	2.39631	.23575	5.07E-10	3.35E-08
$Price_0$	2.44470	2.26261	.18209	5.22E-06	4.90E-04
$Price_0 + 1$	2.38414	2.24866	.13548	.00031	0.0208

Liu again outperforms Black-Scholes, with results very similar to those from the 30 day trial.

120 Day Trial

Table 9. ARMSE for 120 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	3.46909	3.27759	.191501	2.28E-10	3.96E-08
$Price_0$	3.31661	3.19266	.12395	5.83567E-05	0.004
$Price_0 + 1$	3.25896	3.18850	.07046	.01188	0.2146

Table 10. AMAE for 120 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	3.08833	2.83781	.25052	9.13E-13	1.66E-10
$Price_0$	2.905288	2.69522	.210064	1.74E-08	1.94E-06
$Price_0 + 1$	2.82790	2.65580	.17211	1.83E-06	2.62E-04

150 Day Trial

Table 11. ARMSE for 150 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	3.9225	3.8284	.09410	0.00014	7.81E-04
$Price_0$	3.7970	3.75662	0.04037	.06828	0.3584
$Price_0 + 1$	3.74651	3.74665	-0.00014	.49783	0.5589

Table 12. AMAE for 150 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	3.4860	3.3063	.17970	1.22E-10	5.67E-08
$Price_0$	3.32953	3.18051	.14903	8.12E-07	1.17E-04
$Price_0 + 1$	3.25826	3.13904	.11922	4.23E-05	0.0044

By this point, we have seen the $BS - Liu$ values dipping back towards 0 for both RMSE and MAE. As we will see, this trend continues as our time periods grow. It is also worth noting that the $BS_{RMSE} - Liu_{RMSE}$ decrease began at the 120-day point, whereas this decrease didn't occur until the 150-day trials for $BS_{MAE} - Liu_{MAE}$. We speculate that this is due to RMSE's higher sensitivity to large errors, which occur with greater frequency at later times (a forecast for 30 days in the future will usually be more accurate than one looking 150 days out).

180 Day Trial

Table 13. ARMSE for 180 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	4.23541	4.34606	-0.11064	.00754	0.4718
$Price_0$	4.09850	4.24894	-0.15045	.000402	5.54E-04
$Price_0 + 1$	4.03509	4.21470	-0.17958	2.52E-05	1.93E-07

Table 14. AMAE for 180 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	3.76579	3.73047	.03531	.101204	0.002
$Price_0$	3.60026	3.58468	.015581	.28848	0.136
$Price_0 + 1$	3.51812	3.52289	-0.00477	.43058	0.7166

210 Day Trial

Table 15. ARMSE for 210 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	4.67945	5.22188	-0.54243	1.43E-05	6.61E-06
$Price_0$	4.56543	5.13321	-0.56778	4.72E-06	3.24E-11
$Price_0 + 1$	4.51409	5.09813	-0.58404	2.15E-06	1.12E-15

Table 16. AMAE for 210 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	4.16009	4.41924	-0.25914	.000505	0.6736
$Price_0$	4.01559	4.28161	-0.26602	.000264	0.2166
$Price_0 + 1$	3.94428	4.22084	-0.27655	.000118	0.0209

At this point, we see that Black-Scholes is outperforming Liu over all strike prices and both measures of error.

240 Day Trial

Table 17. ARMSE for 240 Day Trial

Strike	BS ARMSE	Liu ARMSE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	4.88025	5.54879	-0.66854	1.35E-06	4.89E-13
$Price_0$	4.75448	5.43733	-0.68285	6.71E-07	8.03E-16
$Price_0 + 1$	4.69039	5.37933	-0.68895	4.59E-07	3.78E-17

Table 18. AMAE for 240 Day Trial

Strike	BS AMAE	Liu AMAE	BS-Liu	$p_{\mathcal{T}}$	p_s
$Price_0 - 1$	4.33682	4.70318	-0.36636	2.6E-05	8.24E-06
$Price_0$	4.18421	4.54965	-0.36544	1.85E-05	1.93E-07
$Price_0 + 1$	4.10179	4.46923	-0.36743	1.28E-05	6.25E-09

5.3 Discussion

Performance over Short Time Periods

Perhaps the most striking result of the data analysis is the extent to which the Liu model consistently outperforms the Black-Scholes model over short time periods. While Black-Scholes is certainly not the most sophisticated or accurate model for option prices in use today, it is it's similarity to the Liu model which is of particular interest. Both models assume that option prices are affected only by the risk-free interest rate and a constant volatility, with the interest-rate creating a deterministic movement and the volatility acting on an uncertain process. And in both cases, that uncertain process behaves according to a distribution which takes mean and variance as parameters and maximizes entropy within the respective measure. By design, the only difference between the two models is that the Black-Scholes model uses the probability measure, whereas the Liu model is based on the credibility measure. For this reason, we believe the most useful and promising outcome of this research is in investigating whether more sophisticated probability models might be improved by assuming a fuzzy process instead. For example, a model developed by Heston [6] assumes that volatility, instead of remaining constant, behaves according to a stochastic process as a function of both current and long-term mean volatility. In terms of the previously mentioned stochastic differential equation (2.1), Heston theorized that volatility could be modeled by the equation:

$$dv_t = \phi(\omega - v_t)dt + \epsilon\sqrt{v_t}dB_t \tag{V.1}$$

where v_t is volatility at time t, ω is the long-term mean volatility, ϕ is the rate at which volatility tends to revert to that mean, and ϵ is the volatility of the process used to determine volatility of the stock. It's possible, based on the results of this study, that volatility might be more accurately modeled (at least under certain circumstances) replacing the Brownian movement (dB_t) with a Liu process. Another popular model, called GARCH (generalized auto-regressive conditional heteroskedacity), assumes variance to vary according to v_t rather than it's root [4]:

$$dv_t = \phi(\omega - v_t)dt + \epsilon v_t dB_t \tag{V.2}$$

Here we see another opportunity to investigate the possibility that replacing the Brownian process with a fuzzy process might yield more accurate forecasts.

Performance over Longer Time Periods

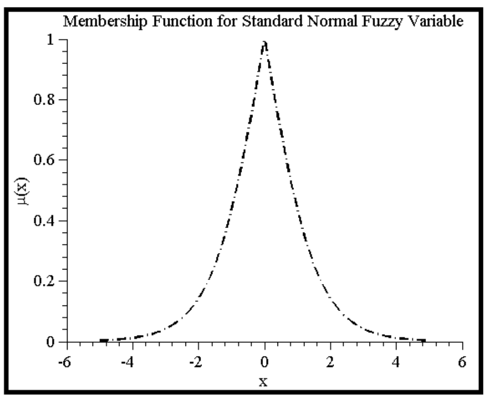
As the time periods got longer, the Liu model's accuracy began to deteriorate, with Black-Scholes outperforming Liu significantly at times over 180 days. This indicates both a potential limitation on the applicability of fuzzy processes, as well as possible evidence of a flaw in the overall rationale for using this type of process to model financial behavior. The Liu error terms tend to increase exponentially as the time period increases past 180 days.

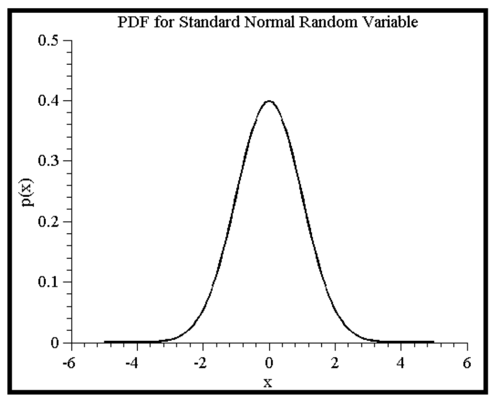
Other Relationships

Another direction for future research is an investigation of the relationship between model performance and variables other than time, such as interest rate and the difference between current price and strike price (referred to as the "moneyness" of the option - i.e. the extent to which it is "in-the-money" or "out-of-the-money"). While the options in my research do vary in interest rate and moneyness in order to provide more robust overall results, the particular relationships between these variables and overall accuracy was not closely analyzed.

The other major question that arises is "WHY does the Liu model behave the way it does?" By looking at Figure 4 (3) we can compare the distributions of normally distributed fuzzy and random variables (both random and fuzzy variables are "standardized" with mean= 0 and standard deviation/diffusion= 1). We see that the fuzzy distribution places less weight on values close to the mean, which could contribute to the difference in performance we see between the two.

Figure 3. Normal Fuzzy vs. Normal Random





We feel that a deeper analysis of this property, as well as the fact that the credibility measure averages two different values for a particular set (both possibility (III.1) and necessity (III.2)), might yield insight into the differences in behavior between random and fuzzy processes.

CHAPTER VI

CONCLUSION

In this thesis, we have provided a review of existing literature on continuous-time stochastic processes and their application in creating the Black-Scholes formula for pricing European call options. Likewise, we have compiled information on fuzzy processes into a stand-alone, axiomatic foundation for Liu's fuzzy model for option prices. This provides not only insight into the model, but also a comparison of the fundamental similarities and differences between fuzzy and stochastic continuous-time processes. We then performed a comparison between the actual results of these two models using recent historical stock data. While the average error terms of both models are significantly larger than those produced by more sophisticated models, the Liu model tended to outperform Black-Scholes over shorter time intervals. Though this does not suggest the Liu model as a replacement for more modern techniques, it is significant in that the primary difference between the two models is the use of a stochastic vs. fuzzy continuous-time process. Since many modern models are built on stochastic processes, it seems promising to consider adapting some of these models to fuzzy processes as well.

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APPENDIX A

MATLAB CODE

Single Iteration

The first program listed does the computations for a single starting date over a given list of stocks. As parameters, it takes a starting date, the length of the period, the interest rate on the starting date, a "reversion" parameter for the Gao model (which was not included in the results), and a single stock or list of stocks.

```
function [stocks, diagnostic] = date_offset(start, date_offset, strike_offset, reversion, interest_rate, varargin)
%%THE PROGRAM IS ORGANIZED INTO FIVE STEPS AS FOLLOWS:
%
%STEP 1 - BRING IN THE NECESSARY STOCK DATA FOR THE SPECIFIED STARTING DATE
%AND TIME PERIOD FOR ALL STOCKS IN LIST
%Data is brought in using the Yahoo Historical Stock Database and MATLAB's
%urlread command.
%
%STEP 2 - CALCULATE MODEL PRICES FOR HYPOTHETICAL OPTIONS EXPIRING ON DAY 1, 2,
%.., N FOR EACH STOCK
%The first part of this step is to estimate the volatility of each stock
%using the first 20 days of the time period under consideration. Volatility
%is defined as the standard deviation of the rate of return over this
%period.
%Once we have volatility, we use volatility, interest rate, current price
%and strike price to estimate option values using the Liu and Black-Scholes
%models.
%
%STEP 3 - CALCULATE ACTUAL VALUE OF THESE HYPOTHETICAL OPTIONS
%The value of any given option at time T is simply max[e^(-rT)*(Price -
%Strike), 0]. The actual value for these options is calculated for days 1,
%2, ... , N.

%STEP 4 - CALCULATE ERROR TERMS (ROOT MEAN SQUARED ERROR (RMSE) AND MEAN ABSOLUTE ERROR (MAE)
%FOR EACH STOCK

%STEP 5 - AVERAGE THESE ERROR TERMS FOR THE WHOLE LIST OF STOCKS

function f=gao_call(price,strike,rate, time,volatility,drift,reversion,level)
Q=(sqrt(6)*pi.*(1-exp(-reversion.*time)))/(pi.*reversion);
P=strike-level-((exp(-reversion.*time))*(price-level));
f=exp(-rate*time)*(Q*log(1+exp(P/Q))-P);


%This function outputs the price of a European Call using Gao's Fuzzy Mean Reversion Model
%Inputs are similar to Liu/Black Scholes, but constant, linear "drift" has been replaced by a rate of reversion ...
      and a "level" to which the price tends to revert
%Q and P variables are simply used for simplicity
end

function f=liu_call(price, strike, rate, time, volatility, drift)
y=@(x) price*exp(-rate.*time)./(1+exp((log(x)-drift*time)*(pi/(sqrt(6)*volatility*time))));
f=quad(y,strike/price,100);

%Function outputs price of a European Call based on Liu's model
```

```
%second line states function for the expected value of a Call with the given parameters
%third line integrates function from lower bound of (strike/price) to upper bound of positive infinity ...
     (although a real number is used to approximate)
end

stocks = struct([]);        %initialize data structure
diagnostic=struct('Gao_RMSE', {0}, 'BlackScholes_RMSE', {0}, 'Liu_RMSE', {0}, 'Gao_AbsErr', {0}, ...
     'BlackScholes_AbsErr', {0}, 'Liu_AbsErr', {0}, 'Ave_Gao_RMSE', {0}, 'Ave_BlackScholes_RMSE', {0}, ...
     'Ave_Liu_RMSE', {0}, 'StDev_Gao_RMSE', {0}, 'StDev_BlackScholes_RMSE', {0}, 'StDev_Liu_RMSE', {0}, ...
     'Ave_Gao_AbsErr', {0}, 'Ave_BlackScholes_AbsErr', {0}, 'Ave_Liu_AbsErr', {0}, 'Gao_Over_Percentage', {0}, ...
     'BS_Over_Percentage', {0}, 'Liu_Over_Percentage', {0});      %initialize diagnostic structure
N=20;                       %counter for iterations in volatility calculation
OFFSET=strike_offset;       %fix strike offset based on input parameters
RATE=interest_rate;         %fix interest rate based on input parameters
REVERSION=reversion;        %rate of reversion for Gao mean-reversion

%%INITIALIZE DATES
%For the starting date, we convert the date string into date vector format,
%then isolate the beginning day (bd), beginning month (bm) and beginning
%year (by) so that they can be input using the url for the stock data.

start_D=datevec(start);
bd=num2str(start_D(3));
bm=num2str(start_D(2)-1);
by=num2str(start_D(1));

%Ending date is calculated as the beginning date plus the number of days
%specified for the length of the time period. The date is broken apart into
%ending day (ed), ending month (em) and ending year (ey) similar to the
%beginning date.

end_date=datenum(start_D)+date_offset;
end_D=datevec(end_date);
ed=num2str(end_D(3));
em=num2str(end_D(2)-1);
ey=num2str(end_D(1));


% this portion of code is used to determine if a frequency of data
% reporting was requested other than "daily" - since my analysis uses daily
% value, this is not changed throughout

temp = find(strcmp(varargin,'frequency') == 1); % search for frequency
if isempty(temp)                                % if not given
    freq = 'd';                                 % default is daily
else                                            % if user supplies frequency
```

```
    freq = varargin{temp+1};                    % assign to user input
    varargin(temp:temp+1) = [];                 % remove from varargin
end
clear temp

%here we determine the type of input used to specifiy the stock(s) to be
%analyzed. for the purposes of this analysis, a .txt file containing a list
%of ticker symbols is used

if isempty(strfind(varargin{1},'.txt'))     % If individual tickers
    tickers = varargin;                     % obtain ticker symbols
else                                        % If text file supplied
    tickers = textread(varargin{1},'%s');   % obtain ticker symbols
end

h = waitbar(0, 'Please Wait...');           % create waitbar
idx = 1;                                    % idx for current stock data

% cycle through each ticker symbol and retrieve historical data
for i = 1:length(tickers)

    % update waitbar to display current ticker
    waitbar((i-1)/length(tickers),h,sprintf('%s %s %s%0.2f%s', ...
        'Retrieving stock data for',tickers{i},'(',(i-1)*100/length(tickers),'%)'))

    % download historical data using the Yahoo! Finance website
    [temp, status] = urlread(strcat('http://ichart.finance.yahoo.com/table.csv?s='...
        ,tickers{i},'&a=',bm,'&b=',bd,'&c=',by,'&d=',em,'&e=',ed,'&f=',...
        ey,'&g=',freq,'&ignore=.csv'));

    if status
        % organize data by using the comma delimiter
        [date, op, high, low, cl, volume, adj_close] = ...
            strread(temp(43:end),'%s%s%s%s%s%s%s','delimiter',',');


        stocks(idx).Ticker = tickers{i};              % obtain ticker symbol
        stocks(idx).Date = date;                      % save date data
        %disp(date);
        stocks(idx).Open = str2double(op);            % save opening price data
        stocks(idx).High = str2double(high);          % save high price data
        stocks(idx).Low = str2double(low);            % save low price data
        stocks(idx).Close = str2double(cl);           % save closing price data
        stocks(idx).Volume = str2double(volume);      % save volume data
        stocks(idx).AdjClose = str2double(adj_close); % save adjustied close data
        stocks(idx).Days=(datenum(date)-datenum(start)); %converted dates into days since start
```

```
        stocks(idx).Years=((datenum(date)−datenum(start))/365); %convert days into years for interest calculations

        %===============================================================
        %Now we flip the vectors so that Day 1 corresponds to index 1
        %===============================================================

        stocks(idx).Date  = flipud(stocks(idx).Date);
        stocks(idx).Open  = flipud(stocks(idx).Open);
        stocks(idx).High  = flipud(stocks(idx).High);
        stocks(idx).Low   = flipud(stocks(idx).Low);
        stocks(idx).Close   = flipud(stocks(idx).Close);
        stocks(idx).Volume   = flipud(stocks(idx).Volume);
        stocks(idx).AdjClose   = flipud(stocks(idx).AdjClose);
        stocks(idx).Days   = flipud(stocks(idx).Days);
        stocks(idx).Years   = flipud(stocks(idx).Years);

        %==============================================================
        %Volatility is calculated for use in modeling calculations
        %==============================================================

        %this portion of code protects the program should an error in data
        %collection force an error in the volatility calculation. should
        %this occur, all values for that particular cycle are set to 0 and
        %we move on to the next starting date in our analysis

        if numel(stocks(idx).Close)<N+1
            disp('BROKE LOOP − VOLATILITY');
            diagnostic.Ave_Gao_RMSE=0;
            diagnostic.Ave_BlackScholes_RMSE=0;
            diagnostic.Ave_Liu_RMSE=0;
            diagnostic.Ave_Gao_AbsErr=0;
            diagnostic.Ave_Liu_AbsErr=0;
            diagnostic.Ave_BlackScholes_AbsErr=0;
            close(h)
            return;
        end

        %this step calculates the volatility using the first 20 days of
        %data for the particular stock

        closing=stocks(idx).AdjClose;                      %create vector of closing prices for volatility ...
             calculation
        log_change = log(closing(2:N+1)./closing(1:N));    %create rate of return vectors for volatility
        stdev = std(log_change);                           %std dev of rate of return
        vol = stdev*sqrt(252);                             %normalize for yearly volatility based on 252 ...
             trading days in years
```

```
        stocks(idx).Volatility=vol;                         %save volatility in stock structure
        idx = idx + 1;                                      % increment stock index

        clear log_change stdev vol
    end
    % clear variables made in for loop for next iteration
    clear date op high low cl volume adj_close temp status


    % update waitbar
    waitbar(i/length(tickers),h)
end

%========================================================================
%Now we calculate predicted values of options, using a loop to cycle
%through stock tickers, then an inner loop that cycles through days,
%pricing options for each day
%========================================================================

    k=1;
    l=numel(stocks(k).Days);
    s=length(tickers)+1;
    while k < s
        j=1;                                  %initialize counter for number of days in period
        PRICE=stocks(k).AdjClose(1);          %set the stock's price to the price on day 1
        STRIKE=PRICE + OFFSET;                %set the option's strike price based on user input
        VOLATILITY=stocks(k).Volatility;      %retrieve stock's volatility from data structure
        l=numel(stocks(k).Years);

        %the "over" counters are used to keep track of times that the model
        %over-priced an option

        LIU_OVER=0;
        GAO_OVER=0;
        BS_OVER=0;
        while j < l

        %prices are calculated for an option expiring on day(j)

        [call, put]=blsprice(PRICE, STRIKE, RATE, stocks(k).Years(j), VOLATILITY, 0);
        liu=liu_call(PRICE, STRIKE, RATE, stocks(k).Years(j), VOLATILITY, RATE);
        gao=gao_call(PRICE, STRIKE, RATE, stocks(k).Years(j), VOLATILITY, RATE, REVERSION, PRICE);

        %actual value of option, had it been purchased, is calculated

        actual=(exp(-.06*stocks(k).Years(j)))*(stocks(k).AdjClose(j)-stocks(k).AdjClose(1));
```

```
        if actual>0
           stocks(k).Actual(j)=actual;
        else
            actual=0;
            stocks(k).Actual(j)=actual;
        end

        %temporary values are stored in data structure

        stocks(k).BlackScholes(j)=call;
        stocks(k).Liu(j)=liu;
        stocks(k).Gao(j)=gao;
        stocks(k).BlackScholes_Error(j)=actual-call;
        stocks(k).Liu_Error(j)=actual-liu;
        stocks(k).Gao_Error(j)=actual-gao;
        stocks(k).BlackScholes_AbsError(j)=abs(actual-call);
        stocks(k).Liu_AbsError(j)=abs(actual-liu);
        stocks(k).Gao_AbsError(j)=abs(actual-gao);
        if (actual-call)<0
            BS_OVER=BS_OVER+1;
        end
        if (actual-liu)<0
            LIU_OVER=LIU_OVER+1;
        end
        if (actual-gao)<0
            GAO_OVER=GAO_OVER+1;
        end
        j=j+1;
        end

        stocks(k).Gao(1)=0;                 %otherwise the price on day 0 (which should be 0) returns NaN
        stocks(k).Gao_Error(1)=0;           %prevents NaN values in Error, RMSE

        %===============================================================
        %Here we calculate the Root Mean Squared Error for each model
        %-This is done by summing the square of each day's error term, then
        %-taking an average, and then the square root of this average.
        %==============================================================

        BS_sum=sum((stocks(k).BlackScholes_Error).^2)/numel(stocks(k).Days);
        BS_root=sqrt(BS_sum);
        stocks(k).BlackScholes_RMSE=BS_root;
        diagnostic.BlackScholes_RMSE(k)=BS_root;
        L_sum=sum((stocks(k).Liu_Error).^2)/numel(stocks(k).Days);
        L_root=sqrt(L_sum);
        stocks(k).Liu_RMSE=L_root;
```

```
        diagnostic.Liu_RMSE(k)=L_root;
        G_sum=sum((stocks(k).Gao_Error).^2)/numel(stocks(k).Days);
        G_root=sqrt(G_sum);
        stocks(k).Gao_RMSE=G_root;
        diagnostic.Gao_RMSE(k)=G_root;
        diagnostic.Liu_Over_Percentage(k)=LIU_OVER/numel(stocks(k).Days);
        diagnostic.Gao_Over_Percentage(k)=GAO_OVER/numel(stocks(k).Days);
        diagnostic.BS_Over_Percentage(k)=BS_OVER/numel(stocks(k).Days);


        stocks(k).Gao(1)=0;                 %otherwise the price on day 0 (which should be 0) returns NaN
        stocks(k).Gao_Error(1)=0;           %prevents NaN values in Error, RMSE

        %=================================================================
        %Now we calculate Mean Absolute Error for each model
        %=================================================================

        diagnostic.Gao_AbsErr(k)=mean(stocks(k).Gao_AbsError);
        diagnostic.Liu_AbsErr(k)=mean(stocks(k).Liu_AbsError);
        diagnostic.BlackScholes_AbsErr(k)=mean(stocks(k).BlackScholes_AbsError);



        %these counters keep track of the model with lowest RMSE for each
        %stock

        if (G_root < BS_root) && (G_root < L_root)
            stocks(k).Winner='GAO!';
        end

        if (L_root < BS_root) && (L_root < G_root)
            stocks(k).Winner='Liu!';
        end

        if (BS_root < L_root) && (BS_root < G_root)
                stocks(k).Winner='Black-Scholes';
        end
    k=k+1;
    end

%=========================================================================
%Now we take an average (and standard deviation) of the RMSE and MAE over
%all stocks and store it in the diagnostic data structure for use in
%analysis
%==========================================================================
diagnostic.Ave_Gao_RMSE=mean(diagnostic.Gao_RMSE);
```

```
diagnostic.Ave_BlackScholes_RMSE=mean(diagnostic.BlackScholes_RMSE);
diagnostic.Ave_Liu_RMSE=mean(diagnostic.Liu_RMSE);
diagnostic.StDev_Gao_RMSE=std(diagnostic.Gao_RMSE);
diagnostic.StDev_Liu_RMSE=std(diagnostic.Liu_RMSE);
diagnostic.StDev_BlackScholes_RMSE=std(diagnostic.BlackScholes_RMSE);
diagnostic.Ave_Gao_AbsErr=mean(diagnostic.Gao_AbsErr);
diagnostic.Ave_Liu_AbsErr=mean(diagnostic.Liu_AbsErr);
diagnostic.Ave_BlackScholes_AbsErr=mean(diagnostic.BlackScholes_AbsErr);

diagnostic.Ave_Liu_Over_Percentage=mean(diagnostic.Liu_Over_Percentage);
diagnostic.Ave_Gao_Over_Percentage=mean(diagnostic.Gao_Over_Percentage);
diagnostic.Ave_BS_Over_Percentage=mean(diagnostic.BS_Over_Percentage);

close(h)    % close waitbar
end
```

Aggregated Data over Multiple Iterations

The second program is used for running multiple 30 day trials at different (randomized) starting dates. It takes only the number of trials as a parameter, produces N random starting dates within the time-frame specified, runs the "date offset" program for each one, and then aggregates the information and outputs it to an Excel spreadsheet for analysis. Modifications of this were used for trials of different lengths.

```
function [M30, DATA_30, INTEREST] = MACRO_30(numtrials)

%%BRINGING IN INTEREST RATE DATA
%Data is read from an excel file, and stored in a vector indexed with the
%first cell containing the interest rate from "Day 1" of our analysis

int_rate_vector =xlsread('interest_rate_2008_2012.xlsx');
INTEREST=int_rate_vector;

%%RANDOMIZING DATES
%Using their MATLAB serial numbers, I randomly select n "dates" (integers
%in the correct range of serial numbers) from which to use as starting
%points of my test periods.
n=numtrials;
date=randi([733409,734720],1,n);

%%OTHER INPUTS
%The number of trials is taken as a parameter of the function, while the
%strike offset vector is (currently) a fixed vector running from -1 to 1.
strike=[-1:1];

%%THE LOOP
%The goal is run my model analysis function n times for each strike price,
%and then write this data to an excel workbook, with a seperate worksheet
%for each strike price.
for k=1:numel(strike)
        for i=1:n
        activeDate=date(i);
        dateTest=date(i);
        if weekday(dateTest) == 7                       %Here I'm making sure that I don't start on a Saturday ...
              (7) or Sunday (1)
            activeDate=activeDate+2;                    %for my interest rate calculation by moving the ...
                  starting date off
        end                                             %the weekends
        if weekday(dateTest) == 1
            activeDate=activeDate+1;
        end

        int_rate_date=(activeDate-733408);                %My interest rate data has data point 1 set to ...
              January 2, 2008

        while int_rate_vector(int_rate_date)> 998        %Since interest rate data is not available on ...
              weekends/market holidays
            int_rate_date=int_rate_date+1;               %their value in the vector is NaN. All NaN's were ...
                  converted to the number
```

```
        end                                              %999 so that I can bump the date up a day when I land ...
             on these.
        temp_rate=int_rate_vector(int_rate_date)/100;
        [STOCKS, SUMMARY]=date_offset(activeDate, 30, strike(k), .1,temp_rate, 'SP100.txt'); %stock loop is run ...
             for each strike

        %%DATA STORAGE
        %Data from each iteration of the stock loop is stored in the
        %DATA_30 structure

        DATA_30(k).BS_RMSE(i)=SUMMARY.Ave_BlackScholes_RMSE;
        DATA_30(k).Liu_RMSE(i)=SUMMARY.Ave_Liu_RMSE;
        DATA_30(k).BS_MAE(i)=SUMMARY.Ave_BlackScholes_AbsErr;
        DATA_30(k).Liu_MAE(i)=SUMMARY.Ave_Liu_AbsErr;
        end

        %Now that I have done all the calculations for this strike, I'd
        %like to store all this data on a worksheet labeled with that
        %particular strike.

        strikestr=num2str(strike(k));
        sheetname = genvarname(strikestr);
        xlswrite('30_day_data',  rot90(DATA_30(k).BS_RMSE,3), sheetname, 'A2');
        xlswrite('30_day_data', rot90(DATA_30(k).Liu_RMSE,3), sheetname, 'B2');
        xlswrite('30_day_data', rot90(DATA_30(k).BS_MAE,3), sheetname, 'C2');
        xlswrite('30_day_data', rot90(DATA_30(k).Liu_MAE,3), sheetname, 'D2');

end
```

APPENDIX B

GLOSSARY

GLOSSARY

call option An option to *buy* a stock, on a specified date, at a specified price. Put options have value when the true price is higher than the strike price at maturity. 1, 11, 34

continuously-compounded interest Normal compound interest re-invests interest earned during each compounding period, so that interest may be earned on it during the next period. Continuously-compounded interest occurs in the special case where the compounding period becomes infinitesimally small. 9

expiration date For a European option, the expiration date is the date on which the option can be exercised. 8, 34

grade of membership The "grade of membership" represents our belief that a certain event or outcome is possible. 14

in-the-money An option that is "in-the-money" can be exercised for a net gain. For a call option, this means that the spot price is greater than the strike price, and vice versa for a put option. 8, 40

out-of-the-money An option that is "out-of-the-money" can not be exercised for a net gain, and is thus not exercised. An "out-of-the-money" option is worth nothing. 8, 40

put option An option to *sell* a stock, on a specified date, at a specified price. Put options have value when the true price is lower than the strike price at maturity. 1

spot price The price of an option's underlying stock on the date of expiration. 8, 13

strike price The price listed on an option, at which the underlying asset can be bought/sold at expiration. 8, 34, 40

synthetic option A synthetic option is a combination of financial instruments which has the same payoff profile of another instrument. For our purposes, we consider the special case of synthetic puts and calls, which are created by financing a position in the underlying asset through borrowing at the risk-free rate. 9