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In this thesis we apply some of the important concepts of Complex Analysis and Fractional Calculus to several special functions from Analytic Number Theory. For Euler eta, Dirichlet L-functions and polynomials we establish new zero-free regions and explore the paths of zeros of their fractional derivatives, using our algorithms for their evaluation. We also consider the number of zeros of integral derivatives for Euler eta function and the behavior of fractional Stieltjes constants of eta and Dirichlet L-functions. Most of these results extend classical theorems concerning integral derivatives of these functions and lead to a better understanding of the general theory.

ON ZEROS OF FRACTIONAL DERIVATIVES OF DIRICHLET SERIES AND POLYNOMIALS

by

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I would like to dedicate this paper to all of my friends and family that supported me throughout this journey and pushed me to strive for success.

APPROVAL PAGE

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Chapter 1: Introduction

The ultimate goal of our work is to understand the Riemann zeta function, which is perhaps the most important function in all of mathematics. For more than 150 years mathematicians have tried to understand the behavior of this function's zeros. One of the seven Millennium Prize Problems, a prize of one million dollars, is for the proof of the Riemann hypothesis, which states that its non-real zeros have real part $\frac{1}{2}$, (shown as black points on Figure 1.1), was announced in 2000.

Our aim in this thesis is not to prove the Riemann hypothesis, but to gain a better understanding of other properties of the Riemann zeta function. We will do so by investigating the zeros of its derivatives and the zeros of derivatives of several related functions, such as the Euler η function and Dirichlet L-functions.

There are a lot of results for the Riemann zeta function and its zeros as you take higher and higher integer order derivatives, see Sections 3.4, 3.6 and 3.5. However, a relatively new method for analyzing derivatives are fractional derivatives.

The notion of Fractional Calculus has been circulating around in the mathematical community for over three hundred years, but has only been heavily studied in the past sixty years. It all started when Leibniz introduced the notion of derivatives as $\frac{d^n y}{dx^n}$ and many mathematicians including Bernoulli raised the question; "Can you have a non-integer order derivative?" to which Leibniz answered "It will lead to a paradox, a paradox from which one day useful consequences will be drawn, because there are no useless paradoxes." And thus fractional Calculus was born.

Over the years many mathematicians and scientists alike have studied this notion of fractional calculus with the intent to better understand things like electromagnetism, signal processing, and other various engineering methods. This unfortunately brought rise to many different methods for approximating fractional derivatives. We will discuss these approaches in Section 2.2. We use these derivatives to explore the path the zeros take for the Riemann zeta function. In the previous work of my advisors, Sebastian Pauli and Filip Saidak, it was found that these zeros exhibit a surprisingly regular behavior and that this property also extends to fractional derivatives. For real valued functions this phenomenon has become known as the crystallization of zeros of derivatives. In this thesis we present a proof of an extension of this interesting quasi-periodic behaviour to the general theory of Dirichlet L-functions.

We start by looking at the alternating zeta function, also known as the Euler/Dirichlet eta function. As in the Riemann zeta case, an algorithm for evaluating the function is needed. We looked at the series of two functions and then applied the Euler-Maclaurin formula to get an explicit algorithm. This allows us to make fast computations and create visuals to confirm our intuition of the relation between the Dirichlet eta function and the Riemann zeta function. The Dirichlet eta function also takes on this regular behavior. We also notice that the zeros of the fractional derivative of eta have a one-to-one correspondence to the original zeros the the Riemann zeta function. That and other proofs are given in Chapter 4. We turn our attention to generalizations of these results to Dirichlet L-functions.

In Chapter 5, we present an algorithm for evaluating Dirichlet L-functions for any character χ. This allows us to extend the crystallization property to all Dirichlet series. We also formulate an algorithm for the evaluation of Stieltjes constants, see Section 5.3, and establish results about the asymptotic behaviors for these constants.

In the last chapter, we study the effect fractional derivatives have on polynomials, the most fundamental of all mathematical functions. We use two different methods to compute the fractional derivatives, namely, the Riemann-Liouville fractional derivative, introduced in Section 2.2.1, and the Caputo derivative, in Section 2.2.2. We find that the Riemann-Liouville is better suited for polynomials, since their zeros form continuous paths. These paths have very interesting properties, namely, as one continues to take higher positive fractional derivatives the path tends to the center of the fractional derivatives. These paths also exhibit asymptotic behavior when considering derivatives higher than the degree of the polynomial and negative fractional derivatives.

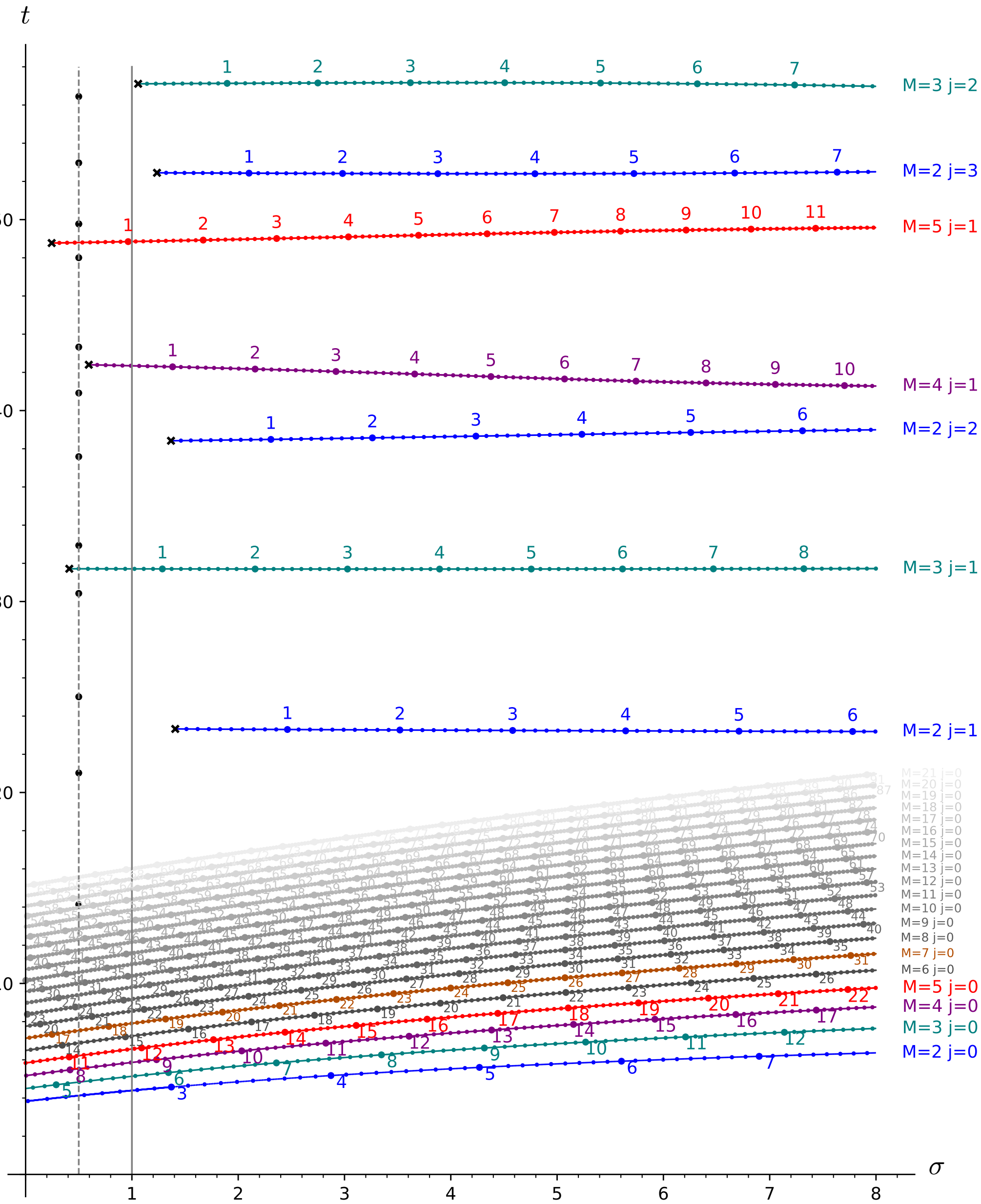

Figure 1.1. • Zeros of $\zeta(s)$, x Zero of $\zeta(s)-1$, $\bullet^{(k)}$ Zero of $\zeta^{(k)}(s)$

Chapter 2: Preliminaries

2.1 Complex Analysis

We start by recalling some results from complex analysis. Definitions and theorems can be found in [BA18], [Ahl66] and [Wri64]. We define i to be a solution to the equation

$$x^2 + 1 = 0,$$

that is, $i^2 + 1 = 0$ or $i^2 = -1$. A complex number, denoted by $s = \sigma + it$, can be broken down into its real part, $\sigma = \Re(s)$, and its imaginary part, $t = \Im(s)$. Another way to look at s is in the polar form: $s = re^{i\theta}$, where $r = |s|$ is the modulus, indicating its distance from the origin, and $\theta = \arg(s)$ is the unique argument within the interval $(-\pi, \pi]$.

Now that we have established a definition of complex numbers, we can delve into the intricacies of complex functions and their derivatives, but first we introduce some elementary topology.

Definition 2.1. Suppose G is a subset of $\mathbb{C}$.

(a) A point $a \in G$ is an interior point of G if some open disk with center a is a subset of G.

(b) A point $d \in G$ is an isolated point of G if some open disk centered at d contains no point of G other than d.

Moreover, a set G is open if all its points are interior points and is closed if it contains all it boundary points. Now we can introduce the notion of a path.

Definition 2.2. A path (or curve) in $\mathbb{C}$ is a continuous function $c : [a, b] \to \mathbb{C}$, where $[a, b]$ is a closed interval in $\mathbb{R}$. We may think of c as a parametrization of the image that is painted by the path and often write the parametrization as $c(t)$ with $a \le t \le b$. The path c is called smooth when it is differentiable and the derivative c' is continuous and nonzero.

Definition 2.3. The path $c[a, b] \to \mathbb{C}$ is simple if $c(t)$ is one-to-one, with the possible exception that $c(a) = c(b)$.

Once a path is established, we can then discuss complex functions and explore the idea of continuity.

Definition 2.4. A complex function f is a map from a subset $G \subset \mathbb{C}$ to $\mathbb{C}$; in this situation we write $f : G \to \mathbb{C}$ and call G the domain of f. This means that each element $s \in G$ gets mapped to exactly one complex number, called the image of s and usually denoted by $f(s)$.

There are various examples of complex functions such as, but not limited to polynomials, discussed in Chapter 6 and and rational functions, i.e. $f(s) = 1/n^s$ for any $n = 1, 2, 3, \ldots$. which will be the main topic of Chapters 3, 4, and 5.

Example 2.5. The exponential function is the solution of the differential equation

$$f'(s) = f(s)$$

with the initial value $f(0) = 1$. The solution is denoted by e^s or $\exp(s)$ where for a complex number $s = \sigma + it$

$$e^s = e^{\sigma+it} = e^{\sigma}e^{it} = e^{\sigma}(\cos(t) + i\sin(t)).$$

Example 2.6. The inverse of the complex exponential is the principal logarithm. The principal logarithm is the function $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined via $\log(s) := \log(|s|) + i\arg(s)$.

In the mathematical field of complex analysis, a branch point of a multi-valued function (usually referred to as a "multifunction" in the context of complex analysis is a point such that if the function is n-valued (has n values) at that point, all of its neighborhoods contain a point that has more than n values [Das11]. Another way to view a branch cut, is a curve (with ends possibly open, closed, or half-open) in the complex plane across which an analytic multivalued function is discontinuous. For convenience, branch cuts are often taken as lines or line segments. Branch cuts (even those consisting of curves) are also known as cut lines [AW85, p. 397], slits [Kah87], or branch lines.

In 1758, Lambert [Lam58] solved the trinomial equation $x = q + x^m$ by giving a series development for x in powers of q. Later in [Lam71], he extended the series to give powers of x as well. In [Eul83], Euler transformed Lambert's equation to the more symmetrical form

$$x^a - x^b = (a - b)vx^{a+b}$$

by substituting x^{-b} for x and setting $m = ab$ and $q = (a-b)v$. Euler looked at special cases, starting with $a = b$. To see what this means in the original trinomial equation to get

$$\log x = vx^a.$$

Euler noticed that if we can solve the equation above for $a = 1$, then we can solve it for any $a \neq 0$. Which then leads us to the Lambert W function [CGH+96],

$$W_k(z)e^{W_k(z)} = z$$

The Lambert W function has various different properties, but the one we will use for this thesis, Chapters 3 and 5, is its association to branch cuts.

For each integer k there is one branch, denoted $W_k(s)$, which is a complex-valued function of one complex argument. For $k = 0$, W_0 is known as the principal branch. A principal branch is a function which selects one branch ("slice") of a multi-valued function.

Now that we have some clear concrete examples of complex functions, we can look at what it means for these complex functions to be continuous. We will use the following definitions of continuity. The first being the topological definition, namely:

Definition 2.7. Suppose $f : G \to \mathbb{C}$. If $s_0 \in G$ and either s_0 is an isolated point of G or

$$\lim_{s \to s_0} f(z) = f(s_0)$$

then f is continuous at s_0, More generally, f is continuous on $E \subset G$ if f is continuous at every $s \in E$.

The second being the more classical ϵ and δ definition, which is used heavily in continuity proofs.

Definition 2.8. Suppose $f : G \to \mathbb{C}$ and $s_0 \in G$. Then f is continuous at s_0 if, for every positive real number ϵ there is a positive real number δ so that

$$|f(s) - f(s_0)| < \epsilon \quad \text{for all } s \in G \text{ satisfying} \quad |s - s_0| < \delta.$$

Having honed our foundational grasp of paths and complex functions, we turn our attention to the central theme of this paper—the introduction of the concept of a complex derivative.

Definition 2.9. Suppose $f : G \to \mathbb{C}$ is a complex function and s_0 is an interior point of G. The derivative of f at s_0 is defined as

$$f'(s_0) = \lim_{s \to s_0} \frac{f(s) - f(s_0)}{s - s_0},$$

provided this limit exists. In this case, f is called differentiable at s_0.

If f is differentiable for all points in an open disk centered at s_0 then f is called holomorphic at s_0. in the disk. Functions that are differentiable (and hence holomorphic) in the whole complex plan $\mathbb{C}$ are called entire.

Let's delve into other fundamental definitions to unveil a cornerstone in complex analysis—the Cauchy integration formula.

Definition 2.10. Suppose c_0 and c_1 are closed paths in the region $G \subset \mathbb{C}$, parameterized by $c_0(s)$, $0 \le s \le 1$, and $c_1(s)$, $0 \le s \le 1$, respectively. Then c_0 is G-homotopic to c_1 if there exist a continuous function $h : [0,1]^2 \to G$ such tat, for all $s, r \in [0,1]$

$$\begin{aligned} h(s,0) &= c_0(s), \\ h(s,1) &= c_1(s), \\ h(0,r) &= h(1,r). \end{aligned}$$

We use the notation $c_1 \sim_G c_2$ to mean c_1 is G-homotopic to c_2

Definition 2.11. Let $G \subset \mathbb{C}$ be a region. If the closed path c is $G-$homotopic to a point (that is, a constant path) then c is $G-$contractible, and we write $c \sim_G 0$.

Theorem 2.12 (Cauchy's Integral Formula [Cau23])**.** *Suppose f is holomorphic in the region G and c is a positively oriented, simple, closed, piecewise smooth path, such that s is inside c and $c \sim_G 0$. Then*

$$f(s) = \frac{1}{2\pi i} \oint_c \frac{f(\tau)}{\tau - s}\, d\tau$$

This Theorem can then be extended for higher derivatives as the following.

Theorem 2.13 (Cauchy's Differentiation Formula, compare [BA18, p.151])**.** *Suppose f is holomorphic and differentiable in the region G and c is a positively oriented, simple, close, piecewise smooth path, such that t is inside c and $c \sim_G 0$ and $n \in \mathbb{Z}$. Then*

$$f^{(n)}(s) = \frac{n!}{2\pi i} \oint_c \frac{f(\tau)}{(\tau - s)^{n+1}} d\tau$$

This theorem serves as the foundation for delving into the realm of fractional calculus, as explored in Section 2.2.

Theorem 2.14 (Rouché [Rou62])**.** *If $f(s)$ and $g(s)$ are two analytic functions within and on a simple closed curve C such that $|f(s)| > |g(s)|$ at each point on C. then both $f(s)$ and $f(s) + g(s)$ have the same number of zeros inside C.*

Theorem 2.15 (Jensen's formula [Jen99])**.** *Let f be an analytic function on a region containing the closed ball $\overline{B}(0;r)$ and suppose that $a_1, \dots, a_n$ are the zeros of f in an open ball $B(0;r)$ repeated according to multiplicity, if $f(0) \ne 0$ then*

$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta.$$

Daniel Bernoulli (1700-1782), in a noteworthy achievement, derived a function representing the factorial operation for natural numbers, laying the groundwork for our exploration. For a natural number n, $n! = \prod_{j=1}^{n} j$. The complex valued Gamma function is the "continuous factorial", see Artin [Art64].

Definition 2.16 (Compare [Her11])**.** Let $n \in \mathbb{R}$ and $z \in \mathbb{C}$,

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)(z+2)\cdots(z+n)}$$

or alternatively

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=0}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where

$$\gamma = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) = 0.57721\ldots$$

known as the Euler-Mascheroni constant [Eul40] and [Mas90].

For $\Re(z) > 0$ we can represent the Gamma function as the Legendre integral form [Leg18]

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Some basic proprieties of the Gamma function are as follows

$$\Gamma(1+z) = z\Gamma(z).$$

From the equation above, we can then deduce to the fact for all natural numbers n

$$\Gamma(1+n) = n!.$$

Corollary 2.17. *For $\alpha \in \mathbb{R}^+$ and $k \in \mathbb{N}$*

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}. \tag{2.1}$$

Lemma 2.18 (Euler Reflection Formula)**.** *Let $z \in \mathbb{C}$. We have*

1. $\Gamma(1-z)\Gamma(z) = \dfrac{\pi}{\sin \pi z}$ *for $z \notin \mathbb{Z}$*

2. $\Gamma(z-n) = (-1)^{n-1} \dfrac{\Gamma(-z)\Gamma(1+z)}{\Gamma(n+1-z)}$ *where $n \in \mathbb{Z}$*

Lemma 2.19. *The reciprocal of the Gamma function, $\frac{1}{\Gamma(z)}$, is an entire function of z with simple zeros at $z = 0, -1, -2, \dots,$ and has zeros nowhere else.*

To Euler we also owe the discovery of the Euler-MacLaurin summation formula.

Lemma 2.20 (compare [J+13])**.** *Let f be a complex function that is analytic in a given domain $N \in \mathbb{Z}$ and $U \in \mathbb{Z}$ and let M be a positive integer. Let B_n denote the n-th Bernoulli number and $P_n(t) = B_n(t - \lfloor t \rfloor)$ denote the n-th periodic Bernoulli polynomial. The Euler-Maclaurin summation formula states*

$$\sum_{k=m}^{n} g(k) = \int_{m}^{n} g(x)dx + \sum_{k=1}^{\ell} \frac{(-1)^k B_k}{k!} g^{(k-1)}(x)\Big|_{m}^{n} + (-1)^{\ell+1} \int_{m}^{n} P_\ell(x) g^{(\ell)}(x)dx. \quad (2.2)$$

2.1.1 Weierstrass and Hadamard factorization

The following section has several definitions and theorems from [Con78] that provide some important insights into the Weierstrass (2.31) and Hadamard (2.32) factorization. Revisiting the concept of a path, i.e. definition (2.2), we introduce and define some additional properties, namely:

Definition 2.21 (compare [Con78, p.81])**.** If c is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{c\}$

$$n(c; a) = \frac{1}{2\pi i} \int_c (s - a)^{-1} dz$$

is called the index of c with respect to the point a. It is also sometimes called the winding number of c around a.

Using (2.21), one can deduce the following results.

Lemma 2.22 (compare [Con78, p.82])**.** *Let c be a closed rectifiable curve in $\mathbb{C}$. Then $n(c; a)$ is constant for a belonging to a component of $G = \mathbb{C} - \{c\}$. Also, $n(c; a) = 0$ for a belonging to the unbounded component of G.*

Lemma 2.23 (Argument principle, compare [Con78, p. 123])**.** *Let f be meromorphic in G with poles $p_1, p_2, \dots, p_m$ and zeros $s_1, s_2, \dots, s_n$ counted according to multiplicity. If c is a closed rectifiable curve in G with $c \approx 0$ and not passing through $p_1, \dots, p_m$; $s_1, \dots, s_n$; then*

$$\frac{1}{2\pi i} \int_c \frac{f'(s)}{f(s)} dz = \sum_{k=1}^{n} n(c; s_k) - \sum_{j=1}^{m} n(c; p_j). \quad (2.3)$$

For notational purposes, it is important to introduce the next definition.

Definition 2.24. An elementary factor is one of the following functions $E_p(s)$ for $p = 0, 1, \cdots$:

$$E_0(s) = 1 - s,$$
$$E_p(s) = (1 - s)\exp\left(s + \frac{s^2}{2} + \cdots + \frac{s^p}{p}\right), \quad p \geq 1.$$

The function $E_p(s/a)$ has a simple zero at $z = 0$ and no other zero. We'll now delve into the connection between the rank and zeros of a complex function f.

Definition 2.25. Let f be an entire function with zeros $\{a_1, a_2, \dots\}$ repeated according to multiplicity and arranged such that $|a_1| \leq |a_2| \leq \cdots$. Then f is of finite rank if there is $r \in \mathbb{Z}$ such that

$$\sum_{n=1}^{\infty} |a_n|^{-(r+1)} < \infty.$$

If r is the least integer such that this occurs, then f is of rank r. A function with only a finite number of zeros has rank 0. A function is of infinite rank if it is not of finite rank.

We use Definition (2.25) and Definition (2.24) to produce a product representation for an entire function f.

Definition 2.26. Let f be an entire function of rank r with zeros $\{a_1, a_2, \dots\}$. Then the product

$$P(z) = \prod_{n=1}^{\infty} E_r\left(\frac{z}{a_n}\right)$$

is the standard form for f.

Now armed with this product representation, we can glean additional insights into the function and its corresponding bounds. We first define what it means for a function to have finite genus.

Definition 2.27. An entire function f has finite genus if f has finite rank and if $f(z) = z^m e^{g(z)} P(z)$ where P is in standard form for f and the resulting function g is a polynomial. If r is the rank of f and q is the degree of the polynomial g, then $\mu = max\{r, q\}$ is the genus of f.

We can use this finite genus to bound our respected function in the following way:

Lemma 2.28. *Let f be an entire function of genus μ. For each positive number β there is a number r_0 such that for $|z| > r_0$ we have $|f(z)| < \exp(\beta |z|^{\mu+1})$.*

Definition 2.29. An entire function f is of finite order if there is a positive constant a and an $r_0 > 0$ such that $|f(z)| < \exp(|z|^a)$ for $|z| > r_0$. If such a and r_0 do not exist then f is of infinite order. If f is of finite order then $\lambda = \inf\{a||f(z)| < \exp(|z|^a)$ for $|z|$ sufficiently large$\}$ is the order of f.

A direct consequence can be seen in the following lemma.

Lemma 2.30. *Let f be a non constant entire function of order λ with $f(0) \neq 0$ and let $\{a_1, a_2, \dots\}$ be the zeros of f repeated according to multiplicity and arranged so that $|a_1| \leq |a_2| \leq \cdots$. If r is an integer such that $r > \lambda - 1$ then*

$$\frac{d^r}{dz^r}\left[\frac{f'(z)}{f(z)} = -r! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{r+1}}\right]$$

for $z \in \{a_1, a_2, \dots\}$.

With a grasp of the definitions and theorems in hand, we can now proceed to introduce the Weierstrass and Hadamard Factorization theorems.

Theorem 2.31 (The Weierstrass Factorization Theorem, [Wei94])**.** *Let f be an entire function and let $\{a_n\}$ be the zeros, where $a_i \neq 0$ for $i = 1, \dots n$, of f repeated according to multiplicity; suppose f has a zero at $z = 0$ of order $m \geq 0$ (a zero of order $m = 0$ at $z = 0$ means $f(0) \neq 0$). Then there is an entire function g and a sequence of integers $\{p_n\}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Theorem 2.32 (The Hadamard's Factorization Theorem, [Had93])**.** *If f is an entire function of finite order λ then f has finite genus $\mu \leq \lambda$. Therefore, f can be factored as*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{\mu}\left(\frac{z}{a_n}\right)$$

where g is a polynomial of degree at most μ.

These definitions and theorems not only play a pivotal role in the outcomes presented in this paper but also contribute significantly to shaping a comprehensive understanding of the subject matter. Beyond serving as instrumental tools for the specific results discussed here, many of these concepts lay the foundation for delving into the intriguing realm of fractional calculus, as further explored in Section 2.2. Their multifaceted utility not only enriches the current discourse but also sets the stage for deeper explorations into the intricacies of the subject.

2.2 Fractional Calculus

There exist a multitude of different definitions of fractional derivatives, each with its own particular advantages and disadvantages. In the following section we discuss the different types of fractional derivatives and their advatages and disadvantages.

For every function $f(z)$ (belonging to some class of functions) and every $\alpha \in \mathbb{C}$, we wish to assign a new function $D^\alpha[f(z)]$ by differentiation and $I^\alpha[f(z)]$ by integration with the following criteria:

1. If $f(z)$ is an analytic function of a complex variable z, then $D^\alpha[f(z)]$ and $I^\alpha[f(z)]$is an analytic function of α and z.

2. The operation $D^\alpha[f(z)]$ must produce the same result as ordinary differentiation when α is a positive integer and $I^\alpha[f(z)]$ must produce the same result as ordinary integration when α is a positive integer.

3. The operation of $D^0[f(z)]$ and $I^0[f(z)]$ leaves the f unchanged. That is, $D^0[f(z)] = f(z) = I^0[f(z)]$.

4. The operation of $D^\alpha[f(z)]$ and $I^\alpha[f(z)]$ is linear. That is, for arbitrary $a, b \in \mathbb{C}$, $D^\alpha[af(z) + bg(z)] = aD^\alpha[f(z)] + bD^\alpha[g(z)]$.
   $I^\alpha[af(z) + bg(z)] = aI^\alpha[f(z)] + bI^\alpha[g(z)]$.

5. The composition rule. That is,
   $D^\alpha\left[D^\beta\left[f(z)\right]\right] = D^{\alpha+\beta}\left[f(z)\right]$ and $I^\alpha\left[I^\beta\left[f(z)\right]\right] = I^{\alpha+\beta}\left[f(z)\right]$.

6. $D^\alpha[f(z)]$ and $I^\alpha[f(z)]$ are inverse functions i.e., $D^\alpha\left[I^\alpha[f(z)]\right] = f(z)$

Another important property all operators must have is the generalized Leibniz Rule.

Lemma 2.33 (Leibniz Rule [Ort11, p. 19])**.** *Let $f(t) = \phi(t)\cdot\psi(t)$ where $t \in \mathbb{C}$ and assume that one of the functions is analytic in a given region, then we obtain*

$$D_\theta^\alpha[\phi(t)\psi(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} \phi^n(t)\psi^{\alpha-n}(t), \tag{2.4}$$

where $\binom{\alpha}{n}$ is defined in Corollary 2.17.

It is important to note that the formula is not commutative if only one function is analytic. However, if both functions are analytic in the given region then the formula is commutative.

2.2.1 Riemann-Liouville Fractional Derivative

The exploration of fractional derivatives begins by establishing a solid foundation through the Cauchy Integral formula (2.13), as initiated by Riemann-Liouville in their derivation process. When delving into the complexities of complex analysis, it becomes crucial to precisely define the domains in which these formulas remain valid. Therefore, for the subsequent discussion, we narrow our focus to closed intervals and Lebesgue spaces.

Definition 2.34 (Lebesque space, compare [Die10])**.** For $1 \geq p$ and $a, b \in [-\infty, \infty]$ we have the following

$$\mathcal{L}^p[a,b] = \left\{ f : [a,b] \to \mathbb{R}; \text{is measurable on } [a,b] \text{ and } \int_a^b |f(x)|^p dx < \infty \right\}.$$

Now that we have set our region we introduce the notion of the Riemann-Liouville integral and then derive the derivative.

Definition 2.35 (Riemann-Liouville fractional derivative, compare [GKP19, Definition 2])**.** Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{L}^1[t_o, t]$ and $t \in [t_0, t]$.

$${}^{RL}D_{t_0}^{(\alpha)} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} f(\tau)\, d\tau.$$

The majority of functions under consideration in our study exhibit poles, presenting significant challenges when applying the Riemann-Liouville derivative. To address this, it becomes essential to designate a specific branch cut, leading us to introduce left and right-sided integrals. When we set $t = \infty$, we refer to this as a right-sided derivative. Furthermore, if $t = -\infty$, it corresponds to a left-sided derivative.

It is important to note that a problem arises with a composition rule of functions, which is addressed by Oldham and Spanier [OS74], see Example 2.37 below for a counterexample.

For completeness, we define the derivative used by Keiper (1953-1995) [Kei75].

Remark 2.36 (Keiper)**.** Let $\alpha < -1$ and f(z) be an analytically continued function with $c \in \mathbb{R}$. Then,

$${}^{K}D_c^{(\alpha)}\left[f(z)\right] = \frac{1}{\Gamma(-\alpha)} \int_c^z \frac{f(t)}{(z-t)^{t+1}}\, dt.$$

In his master thesis on fractional derivatives, Keiper analytically continues the function above for $\alpha > 0$ and then lets $c = -\infty$ as seen in the Weyl approach.

Power Function

We only consider the special case of polynomials, composed of the simple power functions $p(x) = (x-a)^\beta$, where $\beta \in \mathbb{R}$, $a \in \mathbb{C}$. For these, the α-th Riemann-Liouville fractional derivative can be computed using the Power Rule:

$$ {}^{RL}D_a^{(\alpha)}(x-a)^\beta = \begin{cases} 0 & \text{if } \alpha-\beta-1 \in \mathbb{N} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} & \text{otherwise} \end{cases} \tag{2.5} $$

A noteworthy fact is that the Riemann-Liouville fractional derivative satisfies all properties expected of a regular derivative, with the exception of the composition rule. The following example shows why it fails:

Example 2.37. By equation (2.5), the 1.5-th derivative of $p(x) = 1$ is ${}^{RL}D_0^{(1.5)}p(x) = \frac{2}{\sqrt{\pi}}x^{-1.5}$ and ${}^{RL}D_0^{(1)}\left({}^{RL}D_0^{(0.5)}p(x)\right) = \frac{2}{\sqrt{\pi}}x^{-0.5}$. However, ${}^{RL}D_0^{(0.5)}\left({}^{RL}D_0^{(1)}p(x)\right) = {}^{RL}D_0^{(0.5)}0 = 0$.

When $\beta \in \mathbb{R} \setminus \mathbb{Z}$ we still have ${}^{RL}D_a^{(\alpha)}\left({}^{RL}D_a^{(\beta)}p(x)\right) = {}^{RL}D_a^{(\alpha+\beta)}p(x)$.

Exponential Function

For the Riemann-Liouville derivatives of the exponential function we first need to introduce the definition of the two-parameter Mittag-Leffler (ML) function.

Definition 2.38. Let β and κ be two complex parameters with $\Re(\beta)$ and $z \in \mathbb{C}$. Then the two-parameter ML function is defined by

$$ E_{\beta,\kappa}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \kappa)} $$

We can introduce the Riemann-Liouville derivatives of the exponential function.

Proposition 2.39 (compare [GKP19, Proposition 9])**.** *Let $\alpha > 0, m = \lceil \alpha \rceil$ and $t_0 \in \mathbb{R}$. For any $s \in \mathbb{C}$ and $t > t_0$ the exponential function $e^{s(t-t_0)}$ has the following fractional derivative:*

$$ {}^{RL}D_{t_0}^{(\alpha)} e^{s(t-t_0)} = (t-t_0)^{-\alpha} E_{1,1-\alpha}(s(t-t_0)) $$

Roberto Garrappa, Eva Kaslik and Marina Popolizio [GKP19] note that the main drawback arises when the Riemann-Liouville is applied to fractional differential equations (FDEs). The Riemann-Liouville derivative needs to be initialized with the same kind of values. In applications, these values are not available because they do not have a clear physical meaning and therefore the description of the initial state of a system is quite difficult. This is one of the reasons which motivated introducing the fractional Caputo derivative.

2.2.2 Caputo Fractional Derivative

In 1967, Michele Caputo [Cap67] reformulated the definition of the Riemann–Liouville fractional derivative, by switching the order of the ordinary derivative with the fractional integral. By doing so, the Laplace transform of this new derivative depends on integer order initial conditions, differently from the initial conditions when we use the Riemann–Liouville fractional derivative, which involve fractional order conditions. Therefore, we introduce the Caputo fractional derivative as the following:

Definition 2.40 (Caputo fractional derivative, [Cap67])**.** Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{L}^1[t_o, t]$ and $t \in [t_0, t]$.

$$ {}^{C}D_{t_0}^{(\alpha)} f(t) = \frac{1}{\Gamma(m-\alpha)} \left[\int_{t_0}^{t} (t-\tau)^{m-\alpha-1} f^{(n)}(\tau) \right] d\tau, $$

which is then broken up as before by left and right sided derivatives, taking $t_0 = 0$ and $t = \infty$ respectfully.

The Caputo approach is common in real world application, see [Die10] for example. Another advantage that the Caputo fractional derivative has over the Riemann-Liouville derivative is that the composition rule holds.

Power Function

As in the Riemann-Liouville case, we only consider the special case of polynomials, composed of the simple power functions $p(x) = (x-a)^{\beta}$, where $\beta \in \mathbb{R}$, $a \in \mathbb{C}$. The α-th Caputo fractional derivative is computed using the Power Rule:

$$ {}^{C}D_{a}^{(\alpha)} (x-a)^{\beta} = \begin{cases} 0 & \beta \in \{0, 1, \ldots, m-1\} \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha} & \beta > m-1 \\ \text{non existing} & \text{otherwise} \end{cases} \tag{2.6} $$

It is important to note that, for $k \in \mathbb{N}$ we have

$$ {}^{C}D^{(\alpha)} t^{k} = 0 \quad \text{for } k < \alpha $$

and diverges for $k > \alpha$.

Exponential Function

For the Caputo derivatives of the exponential function we use the two-parameter Mittag-Leffler (ML) function (2.38). The Caputo derivatives of the exponential function.

Proposition 2.41 (compare [GKP19, Proposition 9])**.** *Let $\alpha > 0, m = \lceil \alpha \rceil$ and $t_0 \in \mathbb{R}$. For any $s \in \mathbb{C}$ and $t > t_0$ the exponential function $e^{s(t-t_0)}$ has the following fractional derivative:*

$$ {}^{C}D_{t_0}^{(\alpha)} e^{s(t-t_0)} = s^m (t-t_0)^{m-\alpha} E_{1,m-\alpha+1}(s(t-t_0)). $$

2.2.3 Grünwald-Letnikov Fractional Derivative

Finally, we introduce the most commonly used method that we use for taking a fractional derivative, the Grünwald-Letnikov [Grü67, Let68a, Let68b] derivative. In order to understand the derivative, we first introduce the basic idea of an integer order derivative by taking the limit of the difference quotient,

$$ f'(z) = \lim_{h \to 0} \frac{f(z) - f(z-h)}{h} $$

Its known that you can generalize the limit above for higher order integer derivatives.

Lemma 2.42. *Let $z \in \mathbb{R}$ and $n \in \mathbb{N}$ and assume that f is n-times differentiable. Then,*

$$ f^{(n)}(z) = \lim_{h \to 0} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{f(z-jh)}{h^n}. $$

Now that we have obtained the general form of an integer order derivative, we can now extend it to the general form of the incremental ratio valid for any order as follows.

Definition 2.43. [Ort11] Let $f(z)$ be a complex valued function and define the fractional derivative by the limit of the fractional incremental ratio

$$ {}^{GL}D_{\theta}^{(\alpha)} f(z) = e^{-j\theta\alpha} \lim_{|h| \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{|h|^{\alpha}} \tag{2.7} $$

where $h = |h| e^{j\theta}$ is a complex number, with $\theta \in (-\pi, \pi]$.

We are most interested in two special values of θ. The forward derivative, indicated with a ($\rightarrow$) symbol for $\theta = 0$, and backward derivative, indicated with a ($\leftarrow$) symbol for $\theta = \pi$. The reasoning behind this is related to time. Assuming that z is time and h is real, then $\theta = 0$ deals with present and past values where $\theta = \pi$ is looking at present and future values.

In order to make some nice relationships between the Grünwald-Letnikov derivative and the other fractional derivatives we introduce the notion of the truncated Grünwald-Letnikov derivative. In essences this means that we are taking the original function f and picking a point t_0 such that everything before that point will be considered zero.

Theorem 2.44 (Truncated Grünwald-Letnikov derivative, compare [Ort11])**.** *Let $\alpha > 0$ and $f \in \mathcal{L}^n[a, b]$ and $a < x \leq b$ then the truncated Grünwald-Letnikov derivative is defined as the following*

$$ {}^{GL}\widetilde{D}_{\theta}^{(a)} f(z) = \lim_{N \to \infty} \frac{\sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} f(z - kh_N)}{h_N^{\alpha}} \tag{2.8} $$

such that $h_N = (x - a)/N$.

Power Functions

We can compute the Grünwald-Letnikov fractional derivative for causal power function defined $p(t) = t^{\beta} u(t)$ with $\beta > 0$ be defined as our power function. It is well know that the Laplace transformation for $p(t)$ is $P(a) = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$ Therefore the fractional derivative using Grünwald-Letnikov is defined as [Ort11]

$$ {}^{GL}D_f^{(\alpha)} t^{\beta} u(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} u(t). $$

Which is the same as the general order derivative if $\alpha = N$

$$ {}^{GL}D_f^{(N)} t^{\beta} u(t) = (\beta)_N t^{\beta - N} u(t). $$

However, neither the forward nor backward Grünwald-Letnikov fracional derivatives converge for polynomials.

Exponential Functions

There are many other properties that hold for Grünwald-Letnikov fractional derivatives making it seem like the ideal way to take a derivative of real order. A classic example is looking at the exponential function. Namely $f(z) = e^{sz}$. For the forward derivative we let $h > 0$, we have the series,

$$e^{sz} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{-ksh}.$$

Which converges for the main branch cut $g(s) = (1 - e^{-sh})^{\alpha}$ provided that $|e^{-sh}| < 1$, i.e $\Re(s) > 0$. That then provides us with the following,

$$\begin{aligned} {}^{GL}D_{\theta}^{(\alpha)} f(z) &= \lim_{h \to 0} \frac{e^{sz} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{-ksh}}{h^{\alpha}} \\ &= \lim_{h \to 0^+} \frac{(1 - e^{-sh})}{h^{\alpha}} e^{sz} = \lim_{h \to 0^+} \left(\frac{(1 - e^{-sh})}{h} \right)^{\alpha} e^{sz} = |s|^{\alpha} e^{j\theta\alpha} e^{sz} \end{aligned}$$

which holds if and only if $\theta \in (-\pi/2, \pi/2)$ which corresponds to the power function and assuming the branch cut line is in the left hand complex half-plane.

For the backwards derivative, we consider another binomial series:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{ksh}$$

that is convergent to the main branch of $f(s) = (1 - e^{sh})^{\alpha}$ provided $\Re(s) < 0$. We obtain,

$${}^{GL}D_{\leftarrow}^{(\alpha)} f(z) = |s|^{\alpha} e^{j\theta\alpha} e^{sz}$$

valid if and only if $\theta \in (\pi/2, 3\pi/2)$. This goes to show that e^{-z} and e^{z} derivatives cannot exists simultaneously. This gives us an idea how the fractional derivatives can still have flaws. Similar to what we saw in the Grünwald-Letnikov example. However, as we saw in the in the previous method there are flaws when taking the fractional derivative.

2.2.4 Relationships between Fractional Derivatives

Although all methods of taking the fractional derivative have drawbacks, there are some nice relationships between all of the methods.

Definition 2.45 (Caputo fractional derivative, compare [Ort11])**.** Let f be analytic on a convex open set C let and $a \in C$. Let $\alpha > 0$ and $m = \lceil \alpha \rceil$ then the α-th Caputo fractional derivative is

$${}^{C}D_{a}^{(\alpha)} f(t) = {}^{RL}I_{a}^{(m-\alpha)} \frac{d^m}{dt^m} f(t).$$

Theorem 2.46 ([Die10, Theorem 2.25]). *Suppose that the truncated Grünwald-Letnikov derivative and the Riemann-Liouville derivative exist for a given function. We have,*

$$ {}^{GL}\widetilde{D}_{t_0}^{(\alpha)} = {}^{RL}D_{t_0}^{(\alpha)} $$

A strong result is the relationship between the Caputo method and the Riemann-Liouville method.

Theorem 2.47 ([GKP19, Proposition 5]). *Suppose that the truncated Grünwald-Letnikov derivative and Caputo derivatives exists. Then*

$$ {}^{GL}\widetilde{D}_{t_0}^{(\alpha)} \left(f(t) - T_{m-1}[f;t_0](t)\right) = {}^{C}D_{t_0}^{(\alpha)} $$

where $T_{m-1}[f;t_0](t)$ is the Taylor polynomial of f centered at t_0.

Another interesting result of the Grünwald-Letnikov derivative is the relationship it has with the generalized Cauchy derivative.

Theorem 2.48 ([OC04, Theorem 4]). *Let $f(z)$ be a complex variable function analytic in the region inside and continuous on the U shaped contour C. Then*

$$ {}^{GL}D_f^{(\alpha)} f(z) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha} = \frac{\Gamma(\alpha+1)}{2\pi i}\int_C f(w)\frac{1}{(w-z)^{\alpha+1}}. $$

Chapter 3: Riemann Zeta Function

In this chapter we present results about the Riemann zeta function to lay a base for our results in the following two Chapters 4 and 5 on the Euler-Dirichlet eta function and Dirichlet L-function. The special values of fractional derivatives in Section 3.6.2 are new.

3.1 Classical Results

The Riemann zeta function is a cornerstone in the vast landscape of mathematics, its significance echoing through the ages. Countless mathematicians, fueled by unyielding curiosity, have dedicated their lives to unraveling its intricacies and discovery of profound theorems. In the upcoming chapters, particularly Chapter 4 and Chapter 5, we will shed further light on this mathematical journey by drawing parallels to these well-known results, paving the way for new mathematical concepts.

3.1.1 Euler

The zeta function was first introduced by L. Euler who studied the distribution of prime numbers using the infinite product formula

Lemma 3.1 ([Eul37])**.** *For $s \in \mathbb{C}$ with $Re(s)$ we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where the product runs through all prime numbers p.

Already in 1734, Euler discovered a clever way using the sine product formula to compute $\zeta(2) = \frac{\pi^2}{6}$. More generally he proved for all even integers,

$$\zeta(2n) = \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = (-1)^{n+1} \frac{B_{2n} 2^{2n-1} \pi^{2n}}{(2n)!}$$

where B_n are the Bernoulli numbers (introduced by Bernoulli in [Ber13] of 1713). Building on Euler's pioneering contributions to the zeta function, subsequent mathematicians delved into its intricacies, unearthing a wealth of insights.

3.1.2 Riemann

A century after Euler looked at the zeta function, Bernhard Riemann [Rie59] considered the function for complex variable s, deriving results that eventually lead to the proof of the Prime Number Theorem by Hadamard [Had96] and de la Vallée Poussin [DLVP96].

Theorem 3.2 (Prime Number Theorem)**.** *The number of primes $p \leq x$ satisfies the asymptotic formula*

$$\pi(x) \sim \frac{x}{\log x}$$

Non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$; a fact that became crucial step to establishing the result. In 1859, Riemann [Rie76] was able to show that $\zeta(s)$ converges for $\Re(s) > 1$ and has an analytic continuation to the whole complex plane. It is also holomorphic except for a simple pole at $s = 1$ with residue 1. This analytic continuation is characterized by the functional equation

$$\zeta(1-s) = 2\Gamma(s)\zeta(s)(2\pi)^{-s}\cos(\frac{\pi s}{2}). \tag{3.1}$$

There are some other interesting properties of the Zeta function, such as zero-free regions discussed in Sections 3.4, 3.5, and 3.6.

3.1.3 Higher Derivatives

Now, for all $k \in \mathbb{N}$, the derivatives $\zeta^{(k)}(s)$ of the Riemann zeta function, for $s \in \mathbb{C}$ with $\Re(s) > 1$, are

$$\zeta^{(k)}(s) \;=\; (-1)^k \sum_{n=1}^{\infty} \frac{(\log n)^k}{n^s}, \tag{3.2}$$

since

$$\frac{d(1/n^s)}{ds} = \frac{d(e^{-s\log n})}{ds} = \frac{d(-s\log n)}{ds}e^{-s\log n} = \frac{-\log n}{n^s},$$

so that every new derivative with respect to s introduces an extra factor of $(-\log n)$. Similar to the Riemann zeta function itself, all $\zeta^{(k)}(s)$ can be extended to meromorphic functions with a single pole at $s = 1$; however, unlike $\zeta(s)$, these derivatives have neither Euler products nor functional equations. As a result, their nontrivial zeros do not lie on a line, but appear to be distributed seemingly at random, the majority of them located to the right of the critical line $\sigma = \frac{1}{2}$ (compare [Spi65b]).

However, within the apparent randomness of the distribution of zeros of $\zeta^{(k)}(s)$, certain intriguing patterns and structures can be detected. As it was shown in [BPS15], for sufficiently large values of k we have: a) an increasing number of zero-free regions in the right half-plane, with surprising vertical periodicity of the zeros located in the strips between them; and b) with the increasing integer-valued k, the zeros seem to transition (in an almost periodic fashion, see Figure 1.1) to the left, creating a lattice-like grid. There seems little doubt that this 'movement' between the zeros of high derivatives is continuous (as conjectured in [BPS15]), however that means that, in order to describe and investigate this intriguing phenomenon, the behavior of the fractional derivatives needs to be understood first. We can then apply the Grünwald-Letnikov fractional derivative to yield a fractional generalization of (3.2) to all $\alpha > 0$ for any $s \in \mathbb{C}$ with $\Re(s) > 1$:

$$\zeta^{(\alpha)}(s) = {}^{GL}D_s^{(\alpha)}\left[\zeta(s)\right] = (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n+1)}{n^s}. \tag{3.3}$$

As a direct consequence of the Laurent expansion (3.8) of the fractional derivatives we obtain:

(a) We choose the branch cut of the complex logarithm, which creates a discontinuity in ${}^{GL}D_s^{(\alpha)}[\zeta(s)]$ along $(-\infty, 1]$, for all $\alpha \notin \mathbb{N}$.

(b) ${}^{GL}D_s^{(\alpha)}[\zeta(s)]$ is analytic on $\mathbb{C} \setminus (-\infty, 1]$; it is a continuous function of both s and $\alpha > 0$.

(c) If $\sigma \in (1, \infty)$ and $\alpha \notin \mathbb{N}$, then ${}^{GL}D_s^{(\alpha)}[\zeta(\sigma)]$ is non-real.

(d) For $s \in \mathbb{C} \setminus (-\infty, 1]$, we have ${}^{GL}D_\sigma^{(\alpha)}\left[\zeta(\overline{s})\right] = (-1)^{2\alpha}\overline{{}^{GL}D_\sigma^{(\alpha)}\left[\zeta(s)\right]}$.

Properties (c) and (d) describe the symmetry of *locations* of the zeros of ${}^{GL}D_\sigma^{(\alpha)}\left[\zeta(s)\right]$ in $\mathbb{C}$, with respect to the real axis, but not the actual mirroring of properties or the related dynamics.

3.2 Evaluation

As seen in [FPS20], $\zeta^{(\alpha)}(s) = D_s^{\alpha}\left[\zeta(s)\right] = (-1)^{\alpha} \sum_{k=2}^{\infty} \frac{\log^{\alpha} k}{k^s}$ where $s \in \mathbb{C}$ with $\Re(s) > 1$. Using (2.2), let $g(x) = \frac{\log^{(\alpha)}(x)}{x^s}$. Then $\sum_{k=2}^{\infty} g(k)$ converges for $\Re(s) > 1$.

We evaluate the first summand of (2.2) as is, namely as

$$G_s^{\alpha}(m) := \sum_{k=2}^{m-1} g(k) = \sum_{k=2}^{m-1} \frac{\log^{\alpha} k}{k^s}$$

The second term of the right hand side of (2.2) can be written in terms of the Upper Incomplete Gamma function $\Gamma(\alpha, s)$ (compare [GR07, p. 346] and [AS64, 6.5.3]):

$$I_s^\alpha(m) := \int\limits_m^\infty g(x)dx = \int\limits_m^\infty \frac{\log^\alpha x}{x^s}dx = \frac{\Gamma(\alpha+1,(s-1)\log(m))}{(s-1)^{\alpha+1}}$$

For the third term we assume that v is even and get:

$$B_s^\alpha(m,v) := \sum_{k=1}^{v} \frac{(-1)^k B_k}{k!} g^{(k-1)}(x)\Big|_{x=m}^{\infty} = \frac{1}{2}\frac{\log^\alpha(m)}{m^s} - \sum_{j=1}^{\lfloor v/2 \rfloor} \frac{B_{2j}}{(2j)!}\left(\frac{\log^\alpha(m)}{m^s}\right)^{(2j-1)}$$

We use the non-central Stirling numbers $S(k,i,s)$ to evaluate the derivatives $g^{(k-1)}$. Let

$$(\alpha)_i = \alpha \cdot (\alpha-1) \cdot (\alpha-2) \cdot \cdots \cdot (\alpha-(i-1)) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)} \tag{3.4}$$

be the falling factorial and denote the Stirling numbers of the first kind by $s(j,i)$. Then

$$S(k,i,s) = \sum_{j=0}^{k-i} (-1)^{k-i+j}(-1)^k \binom{k}{j} (-\alpha)_j s(k-j,i). \tag{3.5}$$

The derivatives of g can be written as [Jan09, Theorem 1]:

$$g^{(k)}(x) = \left(\frac{\log^\alpha x}{x^s}\right)^{(k)} = \sum_{i=0}^{k} S(k,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+k}}$$

Now we determine a bound for the fourth term of (2.2). Writing $s = \sigma + it$ and

$$E_s^\alpha(m,v) := \frac{1}{v!}\int\limits_m^\infty P_v(x) g^{(v)}(x)dx$$

we get

$$\begin{aligned}
|E_s^\alpha(m,v)| &= \left|\frac{1}{v!}\int_m^\infty P_v(x)g^{(v)}(x)dx\right| \le \frac{|B_v|}{v!}\int_m^\infty |g^{(v)}(x)|dx \\
&\le \frac{|B_v|}{v!}\left(\sum_{j=0}^{v} |S(v,j,s)(\alpha)_j|\right)\left(\int_m^\infty \frac{\log^k(x)}{x^{\sigma+v}}dx\right) \\
&= \frac{|B_v|}{v!}\left(\sum_{j=0}^{v} |S(v,j,s)(\alpha)_j|\right)\frac{\Gamma(\alpha+1,(\sigma+v-1)\log(m))}{(\sigma+v-1)^{\alpha+1}}
\end{aligned}$$

The error term $E_s^\alpha(m, v)$ converges for $\sigma + v > 1$ and $m > 2$.

For all $s \in \mathbb{C} \setminus (\infty, 1]$ we can choose $m \in \mathbb{N}$ and $v \in \mathbb{N}$ such that $|E_s^\alpha(m, v)|$ becomes arbitrarily small. We can thus approximate $D_s^\alpha [\zeta(s)]$ as

$$D_s^\alpha [\zeta(s)] \approx (-1)^\alpha (G_s^\alpha(m) + I_s^\alpha(m) + B_s^\alpha(m, v))$$

where the error is $|E_s^\alpha(m, v)|$.

3.2.1 Skorokhodov connectors

The higher derivatives you take of $\zeta(s)$ the zeros start to form distinct chains, see Section 3.4. Skorokhodov also noticed that the zeros of $\zeta(s) - 1$ can be regarded as the first points in these chains, and that there are curves from some zeros of $\zeta(s)$ to these points given by the zeros of $\zeta(s) - c$ for $c \in [0, 1)$ (see Figure 3.1).

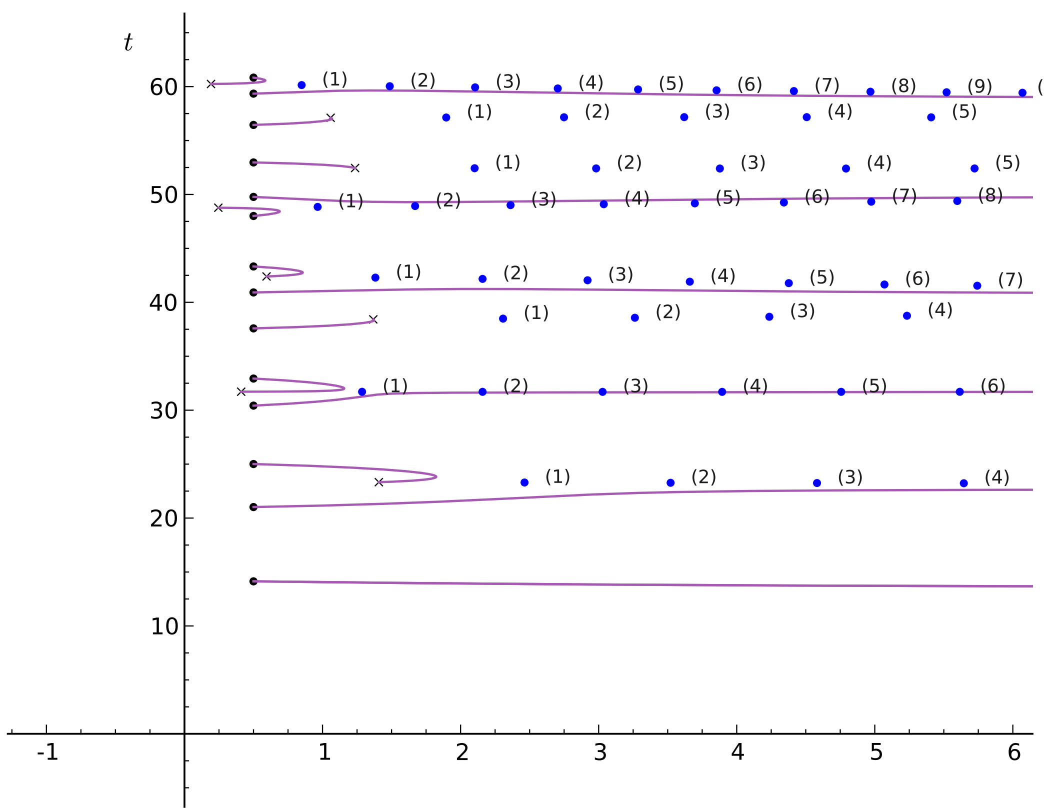

Figure 3.1. Zeros of derivatives of $\zeta^{(k)}(s)$ (denoted by $\bullet^{(k)}$) and the paths from zeros of $\zeta(s)$ (denoted by $\bullet$) to the zeros of $\zeta(s) - 1$ (denoted by $\times$).

Theorem 3.3 ([Sko03]). *The function $\zeta(s)$ is distinct from unity at $\sigma \in (\sigma_0, \infty)$, where*

$$\sigma_0 = 1.940101683745\ldots$$

is the zero of the function

$$f(\sigma) = 1 + 2^{-\sigma} - (1 - 2^{-\sigma})\zeta(\sigma), \quad \sigma > 1.$$

Skorokhodov [Sko03] noticed that the zeros of $\zeta(s) - 1$ can be regarded as the first points in these chains, and that there are curves from some zeros of $\zeta(s)$ to these points given by the zeros of $\zeta(s) - c$ for $c \in [0, 1)$. Our main utility of the connectors is finding starting point of curves of fractional derivatives.

The curves of zeros $s(c)$ of $\zeta(s) - c$ for $c \in [0, 1)$ either end at a zero of $\zeta(s) - 1$ or go off to the left approaching their asymptote $t = \Re(s) = \frac{(2m+1)\pi}{\log 2}$ for some $m \in \mathbb{Z}$ as $\sigma = \Re(s)$ approaches infinity. If each zero of $\zeta(s) - 1$ indeed corresponded to a zero of $\zeta^{(\alpha)}(s)$ for $\alpha \in \mathbb{R}$, then some zeros of $\zeta(s)$ would not correspond to zeros with derivatives, namely those from which the paths of zeros of $\zeta(s) - c$ for $c \in [0, 1)$ goes off to the right. A right bound $\sigma = 3$ for the zeros of $\zeta(s) - 1$ can be easily obtained with the triangle inequality and an estimate for $\zeta(\sigma) - \frac{1}{2^\sigma} - 1$. S. Skorokhodov was able to get a better bound by applying the triangle inequality to a real valued function that only considers terms of the zeta function with n odd:

Lemma 3.4 (Skorokhodov [Sko03]). *The function $\zeta(s)$ is distinct from unity at $\sigma \in (\sigma_0, \infty)$, where*

$$\sigma_0 = 1.940101683745\ldots$$

is the zero of the function

$$f(\sigma) = 1 + 2^{-\sigma} - (1 - 2^{-\sigma})\zeta(\sigma), \quad \sigma > 1.$$

For $c \in [0, 1)$ we find zero-free regions of $\zeta(s) - c$ that depend on t. We obtain them by considering the real and imaginary part of $\zeta(s) - c$ separately.

Lemma 3.5 ([BP13, Lemma 3]). *If $c \in [0, 1)$ and $|\sin(t \log 2)| \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1$ then $\zeta(\sigma + it) - c \neq 0$.*

Lemma 3.6 ([BP13, Lemma 4]). *If $c \in [0, 1)$ and $\cos(t \log 2) \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1$, then $\zeta(\sigma + it) - c \neq 0$.*

These regions can be extended a bit if we restrict ourselves to certain values of t.

Lemma 3.7 ([BP13, Lemma 5]). *If $c \in [0, 1)$, $m \in \mathbb{Z}$, and t is fixed at $\frac{2\pi m}{\log 2}$, then $\Re(\zeta(s) - c) \neq 0$ for $\sigma > 1.95$.*

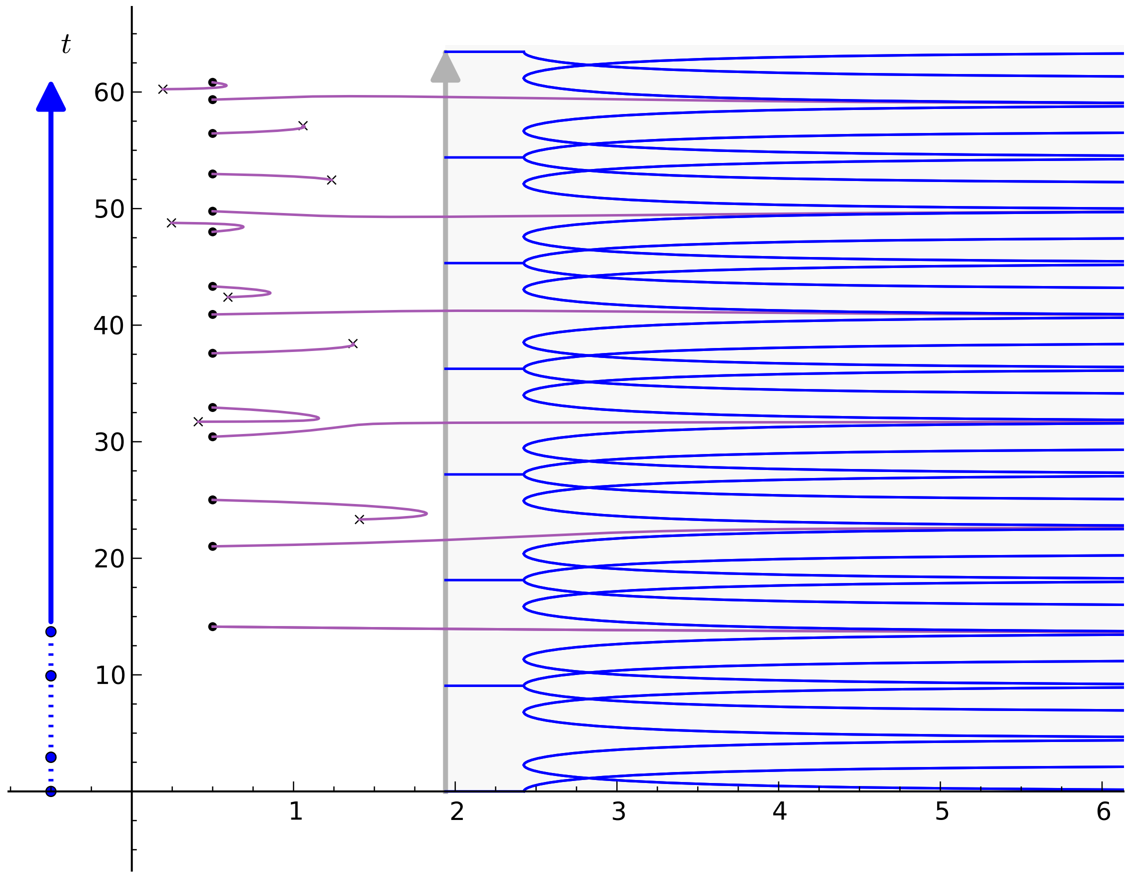

Figure 3.2. The paths from zeros of $\zeta(s)$ (denoted by $\bullet$) to the zeros of $\zeta(s) - 1$ (denoted by $\times$), the barrier on the left (denoted by $\uparrow$), the zeros of $\Im\left(\zeta\left(-\frac{1}{2} + it\right)\right)$ with $0 \leq t < 13.7$ (denoted by $\bullet$), the borders of zero-free regions of $\zeta(s) - c$ for $c \in [0, 1)$ (denoted by $-$), and the zero-free region of $\zeta(s) - 1$ on the right in grey.

3.3 Laurent Series Expansion

The Riemann zeta function is meromorphic with a single pole of order one at $s = 1$. It can therefore be expanded as a Laurent series about $s = 1$ defined as

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!}(s-1)^n \tag{3.6}$$

where the γ_n are called the Stieltjes constants [HS05]. Berndt [Ber72] shows

$$\gamma_n = \lim_{m\to\infty} \left(\left(\sum_{k=1}^{m} \frac{(\log(k))^n}{k} \right) - \frac{(\log(m))^{n+1}}{n+1} \right). \tag{3.7}$$

The constant term γ_0 is the Euler-Mascheroni constant. The Stieltjes constants were generalized to *fractional* Stieltjes constants γ_β, for $\beta \in (0, \infty)$ by Kreminski [Kre03]. They are defined by the Laurent series expansions of the fractional derivatives of the Riemann zeta function:

$$\zeta^{(\alpha)}(s) \;=\; (-1)^\alpha \left(\frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} \;+\; \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{n+\alpha}}{n!}(s-1)^n \right). \tag{3.8}$$

3.3.1 Asymptotic Behavior of Stieltjes Constants

Research on questions related to approximating and bounding Stieltjes constants, dates back to Stieltjes [HS05], Jensen [Jen87], and Ramanujan [Ram85], and more recently it has received a lot of renewed attention in the works of Adell [Ade12], Adell & Lekuona [AL17], Blagouchine [Bla16], Coffey [Cof16], Coffey & Knessl [CK11], and others.

Our Theorem 5.18 for $\gamma_\alpha = \gamma_\alpha(1,1)$ immediately yields the specialization of [FPS21, Theorem 2] to the Riemann zeta function.

Corollary 3.8 ([FPS21, Theorem 2])**.** *Let $\alpha > 0$ and set $w(\alpha) = W_0\left(\frac{\alpha i}{2\pi}\right)$ and let*

$$\widetilde{\gamma}_\alpha := \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - \Im\left(\sqrt{\frac{2\alpha}{\pi(w(\alpha)+1)}}e^{-w(\alpha)+h(w(\alpha))}\right)$$

where $h(t) = 2\pi i(e^t - 1) + \alpha \log t$. Then $\gamma_\alpha \sim \widetilde{\gamma}_\alpha$.

A similar result for discrete values can be found in [Maś22, Equation 34].

Theorem 3.9. *Let $n \in \mathbb{N}$. Then*

$$\gamma_\alpha \sim \sqrt{\frac{2}{\pi}}n! \cdot \Re\left[\frac{\Gamma(s_n)e^{-cs_n}}{(s_n)^n\sqrt{n+s_n+\frac{3}{2}}}\right]$$

where s_n is the saddle point:

$$s_n = \frac{n+\frac{3}{2}}{W_0\left(\frac{n+\frac{3}{2}}{2\pi i}\right)}$$

and $c = \log(2\pi i)$.

3.3.2 Bounds for Stieltjes Constants

In a recent paper by Pauli and Saidak, they were able to prove a new bound for the Stieltjes constants γ_α.

Theorem 3.10 ([PS24, Theorem 8]). *For $\alpha \geq 2\pi$ denote the fractional Stieltjes constants by γ_α. If we set $w_\alpha(1) := W_0\left(\frac{\alpha i}{2\pi}\right)$, where W_0 is the principal branch of the Lambert W function, then*

$$|\gamma_\alpha| < \alpha^2 + \frac{3}{4}\alpha^2 \log\alpha \cdot \left|e^{\alpha(\log w_\alpha(1) - 1/w_\alpha(1))}\right|.$$

Note that the main term of the bound in Theorem 3.10 differs only by a factor of $\alpha^2 \log\alpha$ from the conjectured bound given in [FPS21]:

$$|\gamma_\alpha| \leq 2\left|e^{\alpha(\log w_\alpha(1) - 1/w_\alpha(1))}\right|. \tag{3.9}$$

In Figure 3.3 we compare Theorem 3.10 and (3.9) with previously known bounds for

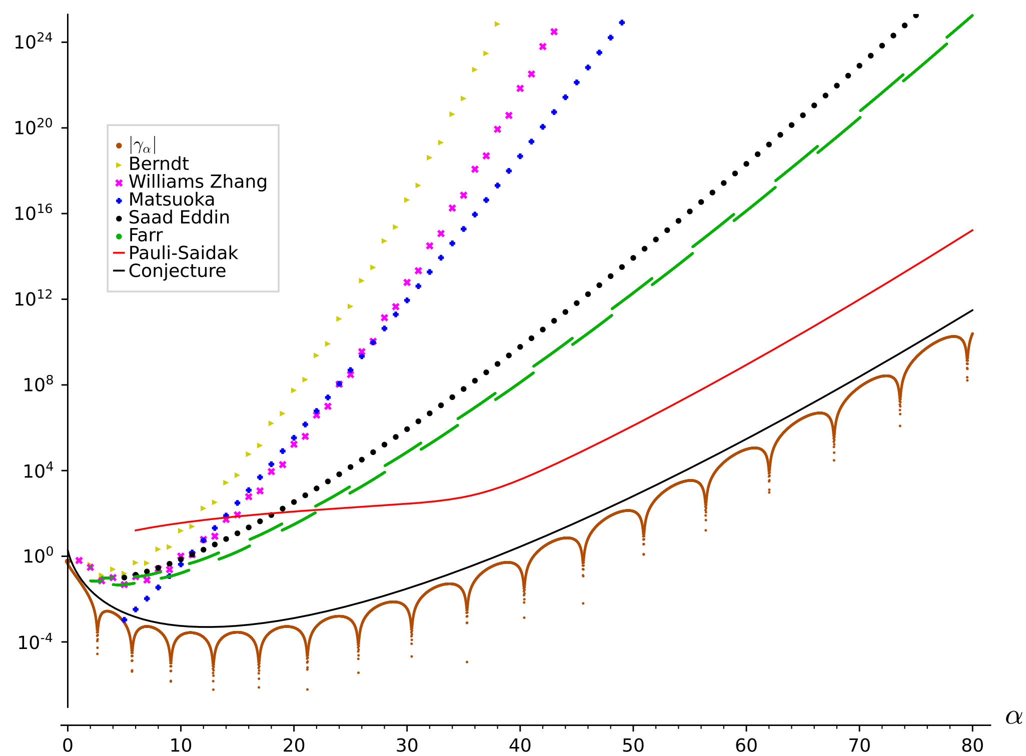

Figure 3.3. On a logarithmic scale we show the absolute values of the Stieltjes constants γ_α, along with the bounds by Berndt, Williams and Zhang, Matsuoka, and Saad Eddin, the conjecture from [FPS21] as well as the bound from Theorem 3.10.

γ_α. For $m \in \mathbb{N}$ we have:

1. the bound by Berndt [Ber72]: $|\gamma_m| \leq \frac{(3+(-1)^m)(m-1)!}{\pi^m}$
2. the bound by Williams and Zhang [ZW94]: $|\gamma_m| \leq \frac{(3+(-1)^m)(2m)!}{m^{m+1}(2\pi)^m}$
3. the bound by Matsuoka [Mat85] which holds for $m > 4$: $|\gamma_m| < 10^{-4}(\log m)^m$
4. the bound by Saad Eddin [Edd13]: Let $\theta(m) = \frac{m+1}{\log \frac{2(m+1)}{\pi}} - 1$ then

$$|\gamma_m| \leq m! \cdot 2\sqrt{2}e^{-(n+1)\log\theta(m)+\theta(m)\left(\log\theta(m)+\log\frac{2}{\pi e}\right)}\left(1 + 2^{-\theta(m)-1}\frac{\theta(m)+1}{\theta(m)-1}\right).$$

5. the bound by Farr [FPS21]: For $\alpha \in (0, \infty)$ let $x = \frac{\pi}{2}e^{W_0\left(\frac{2(\alpha+1)}{\pi}\right)}$ then

$$|\gamma_\alpha| \leq \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}}\frac{(2(n+1))!}{(n+1)!} \text{ where } n = \begin{cases} \lfloor x \rceil & \text{if } x < \alpha \\ \lceil \alpha - 1 \rceil & \text{otherwise} \end{cases}$$

3.4 Right Half-Plane

Zero-free regions on the right half-plane have been described by several authors. We start with Spira's result from 1965.

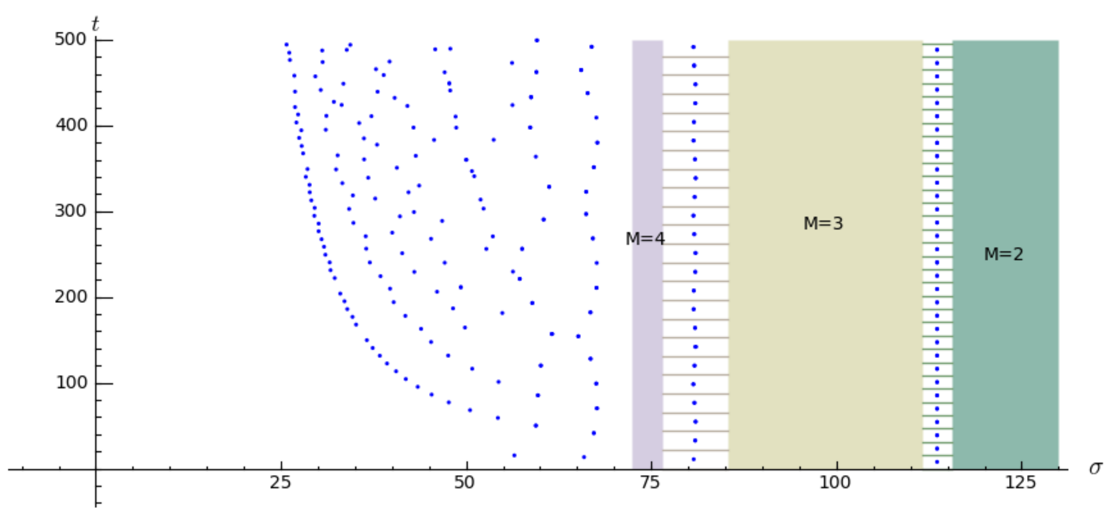

Figure 3.4. Zeros of $\zeta^{(100)}$ with zero-free regions and lines

Theorem 3.11 ([Spi65b]). *If $k \in \mathbb{N}$ and $\sigma \geq \frac{7}{4}k + 2$, then $\zeta^{(k)}(s) \neq 0$.*

Binder, Pauli and Saidak, [BPS15], were able to prove the existence of zero-free regions of integer order derivatives of $\zeta(s)$ and extended these results to fractional derivatives [PS22]. These are found where one of the terms of the Dirichlet series expansion of $\zeta^{(\alpha)}(s)$ dominates the series.

Let $\alpha \in \mathbb{R}^+$ and let $Q_n^\alpha(s) := \frac{\log^\alpha n}{n^s}$ denote the n-th term of the Dirichlet series for $(-1)^\alpha \zeta^{(\alpha)}(s)$, so that

$$(-1)^\alpha \zeta^{(\alpha)}(s) = \sum_{n=2}^{\infty} \frac{\log^\alpha n}{n^s} = \sum_{n=2}^{\infty} Q_n^\alpha(s). \tag{3.10}$$

One of the terms of (3.10), say $Q_M^\alpha(\sigma)$, dominates the rest of the series, that is, when

$$Q_M^\alpha(\sigma) > \sum_{n \neq M} Q_n^\alpha(\sigma), \tag{3.11}$$

and, in complementary fashion, we look for the zeros of $\zeta^{(\alpha)}(s)$ near the regions of the complex plane where $Q_M^\alpha(\sigma) = Q_{M+1}^\alpha(\sigma)$, in other words where no term of the series can attain dominance and, in fact, where the cancellation of terms might happen. They get:

Theorem 3.12 (Theorem 1, [BP13]). *Let $\alpha > 0$. We have:*

(a) For all $\sigma > q_2\alpha + 2.6$, we have $\zeta^{(\alpha)}(s) \neq 0$.

(b) If $Q_3\alpha + 4\log 3 < q_2\alpha - 2$, then $\zeta^{(\alpha)}(s) \neq 0$ for

$$q_3\alpha + 4\log 3 \leq \sigma \leq q_2\alpha - 2.$$

(c) If $M \in \mathbb{N}$, $M > 3$ and $q_M\alpha + (M+1)u \leq q_{M-1}\alpha - Mu$, then $\zeta^{(\alpha)}(s) \neq 0$ in the regions

$$q_M\alpha + (M+1)u \leq \sigma \leq q_{M-1}\alpha - Mu$$

where $u \in (0, \infty)$ is a solution of $1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) \geq 0$.

Note: The value $u \in (0, \infty)$ that gives us the widest zero-free regions is $u = 1.1879426249\ldots$, which is the solution of the equation

$$1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) = 0.$$

Let S_M^α be the vertical strip between the zero-free regions obtained from the dominance of $Q_M^\alpha(q_M\alpha)$ and $Q_{M+1}^\alpha(q_M\alpha)$ in (3.10), respectively, as described in Theorem 3.12. The strip S_M^α exists when α reaches

$$A_M := \begin{cases} \frac{4\log 3 + 2}{q_2 - q_3} & \text{if } M = 2 \\ \frac{(2M+3)u}{q_M - q_{M+1}} & \text{if } M > 2 \end{cases}$$

Considering the imaginary parts of the solutions of $Q_M^\alpha(q_M\alpha + it) = Q_M^\alpha(q_{M+1}\alpha + it)$ we find that $\zeta^{(\alpha)}(\sigma + it) \neq 0$ for $\sigma \in S_M^\alpha$ and

$$t = \frac{2\pi J}{\log(M+1) - \log(M)} \tag{3.12}$$

For $J \in \mathbb{Z}$. Together, the borders of the zero-free regions to the left and right of S_M^α the lines of (3.12), for $J = j$ and $J = j+!$, where $j \in \mathbb{Z}$ form a contour around the zero

$$q_M \cdot \alpha + \frac{\pi(2j+1)}{\log(M+1) - \log(M)} i. \tag{3.13}$$

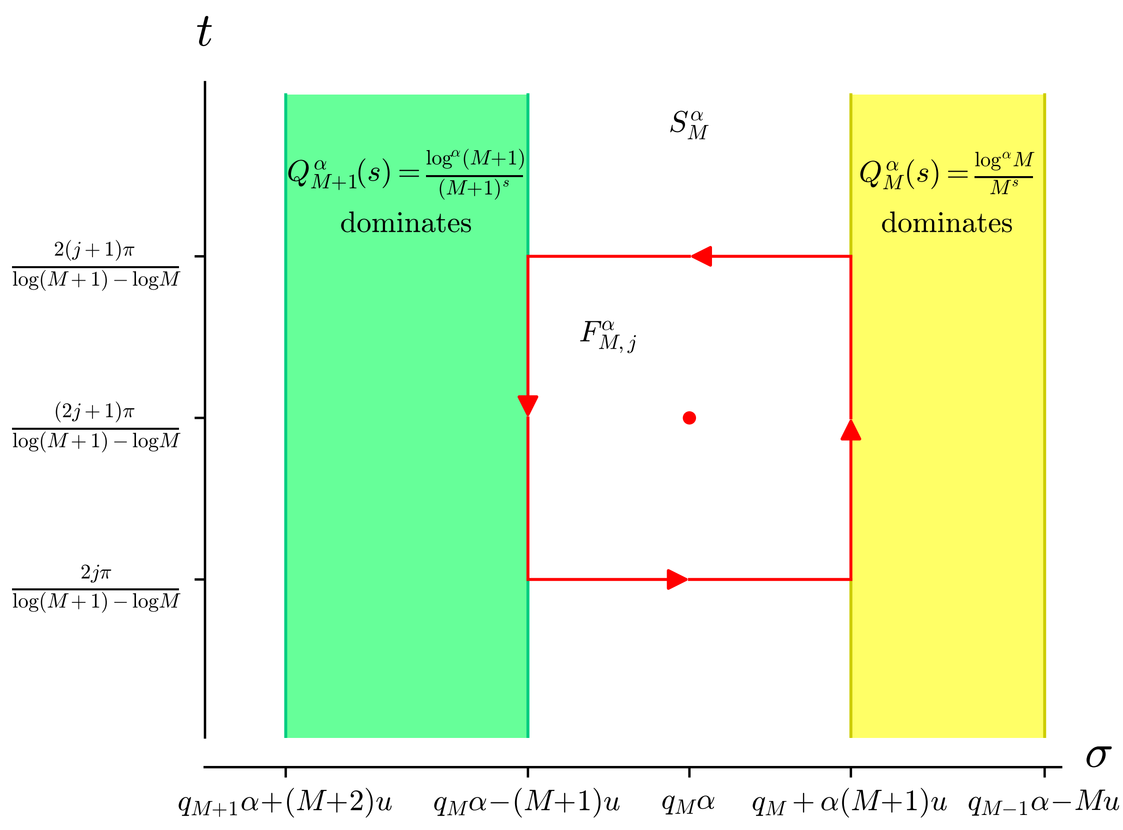

Figure 3.5. Regions $F_{M,j}^\alpha$ that contains exactly one zero of $\zeta^{(\alpha)}(\sigma+it)$. Rouché's theorem can be used to establish simplicity of the zero using the zero of $Q_M^\alpha(s) + Q_{M+1}^\alpha(s)$, see Theorem 3.12

Theorem 3.13 (Theorem 2, [BP13]). *Let $M \geq 2$ denote a natural number, $j \in \mathbb{Z}$, and $\alpha > A_M$. Let $F^{\alpha}_{M,j} \subset S^{\alpha}_{M}$ be given by*

$$\frac{2\pi j}{\log(M+1) - \log(M)} < t < \frac{2\pi(j+1)}{\log(M+1) - \log(M)} \tag{3.14}$$

Then $F^{\alpha}_{M,j}$ contains exactly one zero of $\zeta^{(\alpha)}(s)$, and the zero is simple.

3.4.1 Positive Real Axis

We can see from [BP13, Theorem 2] that there do not exist any zeros on the positive real axis for $\zeta^{(\alpha)}$ for $\sigma > 0$.

Using 3.8 and because of the branch cut of the complex logarithm, there is a discontinuity along $(-\infty, 0]$ for $\alpha \notin \mathbb{N}$. As a direct consequence we obtain the following useful property:

Proposition 3.14 (Proposition 1,[FPS20]). *If $\sigma \in (1, \infty)$ and $\alpha \notin \mathbb{N}$ then ${}^{GL}D^{(\alpha)}_{\sigma}[\zeta(\sigma)]$ is non-real*

3.5 Critical Strip

The region known as the critical strip ($0 < \sigma < 1$) shrouds itself in profound mystery, accentuated by its association with a millennium prize question.

Conjecture 3.15 (The Riemann Hypothesis,[Rie59]). *All non-trivial zeros for the Riemann zeta function lie on the critical line $\Re(s) = 1/2$.*

3.5.1 Riemann Hypothesis and Derivatives

As mathematicians began to comprehend the derivatives of the Riemann zeta function, they started drawing parallels between it and the Riemann hypothesis.

Theorem 3.16 (Speiser [Spi73]). *The Riemann Hypothesis is equivalent to the fact that the non-trivial zeros of the derivatives $\zeta'(s)$ have $\Re(s) \geq 1/2$, that is, that they are on the right of the critical line.*

The original result is due to Speiser see [LM74], however in 1974 Norman Levinson and H. L. Montgomery provide a detailed proof of:

Theorem 3.17 (Corollary to Theorem 1,[LM74]). *The Riemann Hypothesis is equivalent to $\zeta'(s)$ have no zeros in $0 < \sigma < \frac{1}{2}$.*

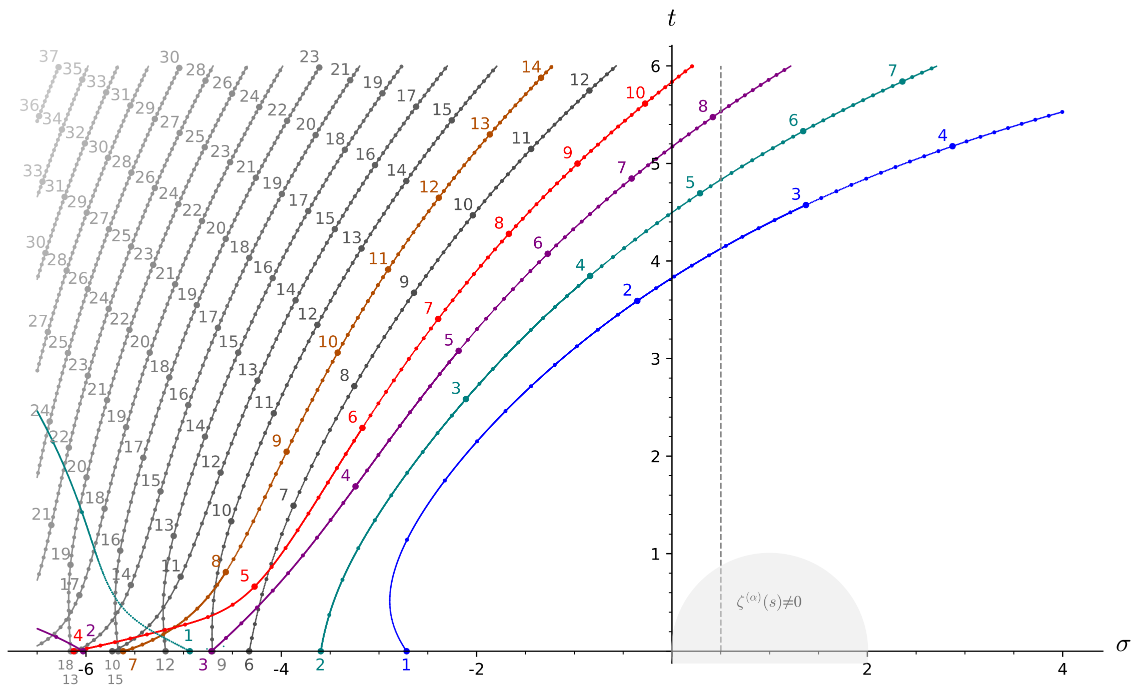

Figure 3.6. Zeros $\sigma + it$ with $t \geq 0$ of the fractional derivatives of $\zeta(s)$ on the left half-plane. For $k \in \mathbb{N}$ zeros of $\zeta^{(k)}(s)$ are labeled with k. Not all zeros on the real axis are shown. The values for α are $1/100$ apart.

3.6 Left Half-Plane

Levinson and Montgomery made the following observation about the number of zeros of integral derivatives on the left half-plane.

Theorem 3.18 ([LM74])**.** *Let $k \in \mathbb{N}$ then $\zeta^{(k)}$ has finitely many non-real zeros on the left half-plane.*

More concretely Yıldırım showed:

Theorem 3.19 ([Yıl00, Theorems 2 and 3])**.** *There is only one pair of nonreal zeros of $\zeta''(s)$ as well as $\zeta'''(s)$ in the left half-plane.*

Using an implementation of the approximation to $\zeta^{(\alpha)}(s)$, see Section 3.2, one observes, see Figure 3.6, that the zeros on the left half-plane given in [FP13] appear to be connected in a similar manner as on the right half-plane. Counting the zeros of integral derivatives obtained, one gets Table 3.1. This leads to the conjecture:

Conjecture 3.20 ([FPS20])**.** *Let $k \in \mathbb{N}$. The number of pairs of non-real zeros of $\zeta^{(k)}(s)$ with $\sigma \leq 0$ is at most $\frac{k+1}{2}$.*

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\lfloor \frac{k+1}{2} \rfloor$	0	1	1	2	2	3	3	4	4	5	5	6	6
N	0	$0^\dagger$	$1^\ddagger$	$1^\ddagger$	2	3	3	3	4	4	4	4	4

Table 3.1. The number N of pairs of non-real zeros of $\zeta^{(k)}(s)$ for $\Re(s) < 1$. † Levinson and Montgomery [LM74, Theorem 9], ‡ Yıldırım [Yıl00, Theorems 2 and 3]. The values for $k > 3$ are experimental.

3.6.1 Negative Real Axis

As a direct consequence of the functional equation (3.1) for $n \in \mathbb{N}$, we have $\zeta(-2n) = 0$. These are called the trivial zeros of $\zeta(s)$. For the first derivative, Spira noticed:

Theorem 3.21 ([Spi73]). *If $|s| > 165$, then $\zeta'(s)$ has only real zeros of $\sigma \leq 0$, and exactly one real zero in each open interval $(-1-2n, 1-2n), n = 1, 2, \cdots$.*

Rolles Theorem then yields zeros in between these intervals for all derivatives, see [LM74] for results.

Theorem 3.22 ([Spi70]). *For $k \geq 0$ there is an s_k so that $\zeta^{(k)}(s)$ has only real zeros for $\sigma \leq s_k$, and exactly one real zero in each open interval $(-1-2n, 1-2n)$ for $1 - 2n \leq \alpha_k$*

For the first derivative, these are the only zeros on the left half-plane.

Theorem 3.23 ([LM74, Theorem 9]). *For $N \geq 2$ there is a unique solution of $\zeta'(s) = 0$ in the interval $(-2n, -2n+2)$ and there are no other zeros of $\zeta'(s)$ in $\sigma \leq 0$.*

3.6.2 Special Values

As noted in the Introduction, special values of $\Gamma(s)$ and $\zeta(s)$ and their derivatives have a special place in the history of mathematics. Simple formulas that yield new insights are rare, which makes their existence for the "complicated" fractional derivatives that much more surprising, even though in their complex environment the discovered patterns only concern their real values.

We consider the Laurent series expansion of the fractional derivatives of $\zeta^{(\alpha)}$, see (3.8):

$$\zeta^{(\alpha)}(s) \;=\; (-1)^\alpha \left(\frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{n+\alpha}}{n!} (s-1)^n \right) \tag{3.15}$$

The term affected by the branch cut of the logarithm is

$$\frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}}$$

whose branches are of the form

$$\frac{\Gamma(\alpha+1)}{\exp((\log(s-1)+b\cdot 2\pi i)\cdot(\alpha+1))} \text{ where } b\in\mathbb{Z}.$$

We write $\zeta_{0-}^{(\alpha)}$ for the principal branch obtained with the principal branch of the logarithm where the branch cut is along $(-\infty,1)$. We obtain the other branches, again with branch cut along $(-\infty,1)$, as

$$\zeta_{b-}^{(\alpha)}(s) = (-1)^{\alpha}\left((-1)^{-\alpha}\zeta_{0-}^{(\alpha)} - \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \frac{\Gamma(\alpha+1)}{\exp((\log(s-1)+b\cdot 2\pi i)\cdot(\alpha+1))}\right) \tag{3.16}$$

where $b\in\mathbb{Z}$. From the above the functions $\zeta_{b+}^{(\alpha)}(s)$ with the branch cut along $(1,\infty)$ are obtained as

$$\zeta_{b+}^{(\alpha)}(s) = \begin{cases} \zeta_{b-}^{(\alpha)}(s) & \text{for } \Im(s)\geq 0 \\ \zeta_{(b+1)-}^{(\alpha)}(s) & \text{for } \Im(s)<0 \end{cases} \tag{3.17}$$

Theorem 3.24. *For all $k\in\mathbb{N}_0$, $b\in\mathbb{Z}$, and real $\sigma<1$,*

$$\Re\left(\zeta_{b+}^{(k+\frac{1}{2})}(\sigma)\right) = (-1)^b\frac{\Gamma(k+\frac{1}{2}+1)}{(1-\sigma)^{k+\frac{1}{2}+1}}. \tag{3.18}$$

Proof. We consider a special case of 3.16, with $m=2$, $v=1$, and $\alpha=k+\frac{1}{2}$:

$$\zeta_{b+}^{(k+\frac{1}{2})}(\sigma) = (-1)^{k+\frac{1}{2}}\left(\frac{\Gamma(k+\frac{1}{2}+1)}{\exp((\log(s-1)+b\cdot 2\pi i)\cdot(\frac{1}{2}+k+1))} + \sum_{n=1}^{\infty}\frac{(-1)^n\gamma_{\frac{1}{2}+k+n}}{n!}(s-1)^n\right).$$

Since $\gamma_\alpha\in\mathbb{C}\setminus\mathbb{R}$ we have

$$\Re\left(\zeta_{b+}^{(k+\frac{1}{2})}(\sigma)\right) = \Re\left((-1)^k i\frac{\Gamma(k+\frac{1}{2}+1)}{(\sigma-1)^{\frac{1}{2}+k+1}(-1)^b}\right) = (-1)^b\frac{\Gamma(k+\frac{1}{2}+1)}{(1-\sigma)^{\frac{1}{2}+k+1}}$$

which confirms the statement of the theorem, and finishes its proof. □

Remark 3.25. The special values of $\Gamma(s)$ involved in (3.18) have a closed form (see above), and imply:

$$\Re\left(\zeta^{(k+\frac{1}{2})}(\sigma)\right) \;=\; -\frac{(2k+2)!}{4^{k+1}(k+1)!(1-\sigma)^{k+\frac{1}{2}+1}}\sqrt{\pi}.$$

Since this formula is valid for all real σ, it provides a variety of new interesting special cases.

Example 3.26. Fractional half derivatives of ζ at the origin ($\sigma = 0$):

$$\Re(\zeta^{(\frac{1}{2})}(0)) = -\Gamma\left(1 + \frac{1}{2}\right) = -\frac{1}{2}\sqrt{\pi}$$
$$\Re(\zeta^{(3/2)}(0)) = -\Gamma\left(1 + \frac{3}{2}\right) = -\frac{3}{4}\sqrt{\pi}$$
$$\Re(\zeta^{(5/2)}(0)) = -\Gamma\left(1 + \frac{5}{2}\right) = -\frac{15}{8}\sqrt{\pi}$$
$$\Re(\zeta^{(7/2)}(0)) = -\Gamma\left(1 + \frac{7}{2}\right) = -\frac{105}{16}\sqrt{\pi}.$$

Example 3.27. Fractional half derivatives of ζ at the negative integers ($\sigma = -n$, with $n \in \mathbb{N}$):

$$\Re(\zeta^{(1/2)}(-1)) = -\frac{\sqrt{2\pi}}{8}$$
$$\Re(\zeta^{(1/2)}(-2)) = -\frac{\sqrt{3\pi}}{18}$$
$$\Re(\zeta^{(1/2)}(-3)) = -\frac{\sqrt{4\pi}}{32}$$
$$\Re(\zeta^{(1/2)}(-4)) = -\frac{\sqrt{5\pi}}{50}.$$

3.7 Number of Zeros

We let $N_\zeta(T)$ and $N_\zeta^k(T)$ denote the number of such zeros ρ with $0 \leq \Im(\rho) \leq T$ of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively. The classical Riemann-von Mangoldt formula (see [Lan09]) states that

$$N_\zeta(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{3.19}$$

and, according to Berndt [Ber70], we have

$$N_\zeta^k(T) = N(T) - \frac{T\log 2}{2\pi} + O(\log T). \tag{3.20}$$

So, there are about $T\log 2/2\pi$ less zeros with imaginary part less than T of $\zeta^{(k)}(s)$ than of $\zeta(s)$, which is also equal to the number of Skorokhodov changes, i.e. zeros of $\zeta(s) - c$.

Chapter 4: Euler Eta Function

4.1 Introduction

Let $s = \sigma + it$, then the Euler eta function (sometimes called Dirichlet eta function or alternating zeta function), is defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \tag{4.1}$$

which converges for $\sigma > 0$. Unlike the Riemann zeta function, the Euler eta function does not have a pole at $s = 1$. In fact, $\eta(1)$ is known as the alternating harmonic series with the value $\eta(1) = \ln(2)$. The Euler eta function satisfies the following relation with the Riemann zeta function:

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s). \tag{4.2}$$

The Euler eta function has the integral representation,

$$\eta(s)\Gamma(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t + 1} dt.$$

Godfrey Harold Hardy (1977-1947) was able to give a simple proof for the functional equation of the Euler η function [Har22],

$$\eta(-s) = 2\pi^{-s-1} s\Gamma(s) \sin\left(\frac{\pi s}{2}\right) \frac{1 - 2^{-s-1}}{1 - 2^{-s}} \eta(s+1). \tag{4.3}$$

Similar to $\zeta(s)$, the $k-$th derivative of $\eta(s)$ where $k \in \mathbb{N}$ is

$$\eta^{(k)}(s) = \sum_{n=1}^{\infty} \frac{(-1)^n \log^k(n)}{n^s} \quad \text{for } \Re(s) > 0. \tag{4.4}$$

Applying the Grünwald-Letnikov fractional derivative (Definition 2.43) we obtain

$$\eta^{(\alpha)}(s) := {}^{GL}D_{\leftarrow}^{(\alpha)}[\eta(s)] = (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n \log^{\alpha}(n)}{n^s} \tag{4.5}$$

where $s \in \mathbb{C}$ with $\Re(s) > 0$.

4.2 Evaluation

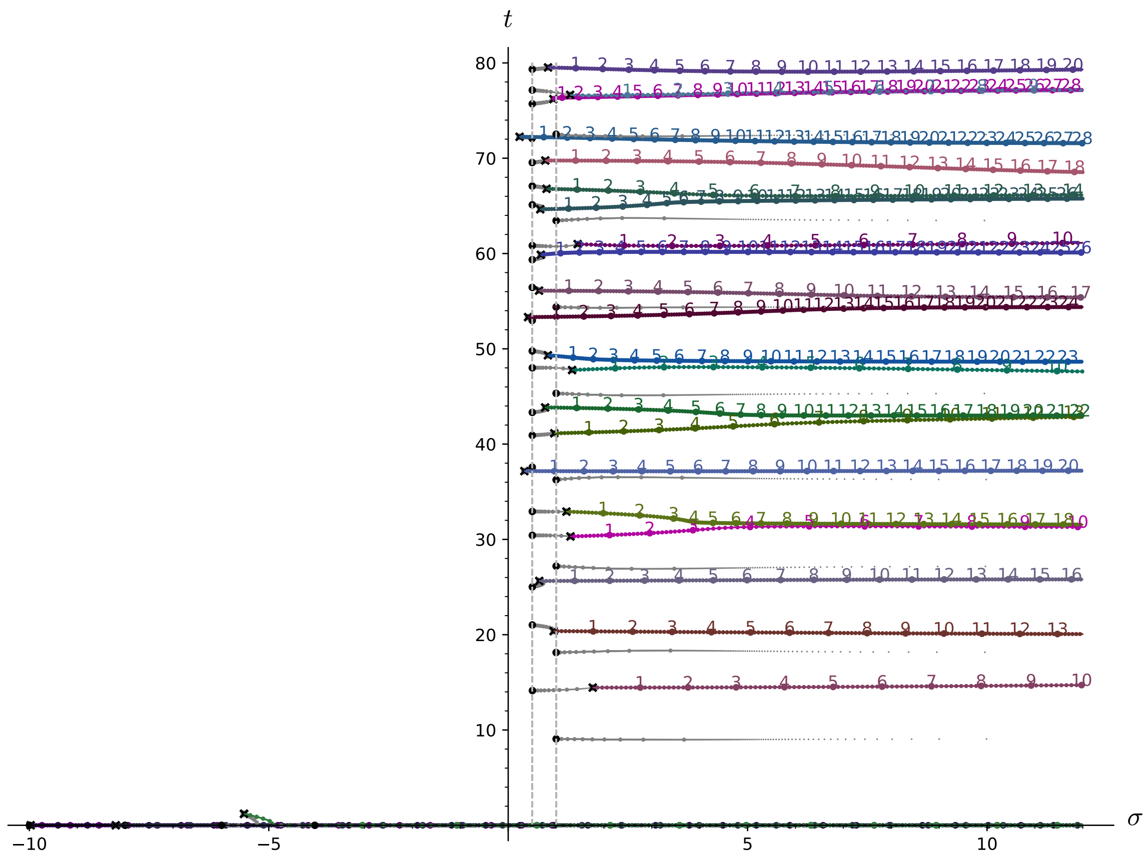

Figure 4.1. Zeros of the fractional derivatives of the Euler η function: • Zeros of $\eta(s)$, × Zero of $\eta(s)-1$, $\bullet^{(k)}$ Zero of $\eta^{(k)}(s)$, Grey lines are Skorokhodov connectors. For the nonreal zeros of the left also see Figure 4.7.

There are several methods to evaluate $\eta^{(\alpha)}(s)$. One notable method by Cohen-Villegas-Zagier is convergence acceleration of alternating series [Example 3., [CVZ00]]. This is not suitable for the left half-plane for higher derivatives because the functional equation would have to be used.

Because we also want to evaluate $\eta^{(\alpha)}(s)$ on the left half-plane we evaluate it using Euler-Maclaurin summation (2.20). Recall

$$\sum_{k=m}^{N} g(k) = \int_{m}^{N} g(x)dx + \sum_{k=1}^{v} \frac{(-1)^k B_k}{k!} g^{(k-1)}(x)\Big|_{x=m}^{N} + (-1)^{v+1} \int_{m}^{N} P_v(x) g^{(v)}(x)dx,$$

where $g(x) \in C^v[m,n]$, $v \in \mathbb{N}$, B_k denotes the k-th Bernoulli number, and $P_k(x) = \frac{B_k(x-\lfloor x \rfloor)}{k!}$ is the k^{th} periodic Bernoulli polynomial. If $g(x)$ decreases rapidly enough for $N \to \infty$, then

$$\sum_{k=2}^{\infty} g(k) = \sum_{k=2}^{m-1} g(k) + \int_m^{\infty} g(x)dx + \sum_{k=1}^{v} \frac{(-1)^k B_k}{k!} g^{(k-1)}(x)\bigg|_{x=m}^{\infty} + (-1)^{v+1} \int_m^{\infty} P_v(x) g^{(v)}(x)dx \tag{4.6}$$

For $\eta^{(\alpha)}(s)$ we have

$$\eta^{(\alpha)}(s) = \sum_{n=1}^{\infty} \frac{(-1)^n \log^{\alpha}(n)}{n^s} = \sum_{n=1}^{\infty} -\frac{\log^{\alpha}(2n)}{(2n)^s} + \frac{\log^{\alpha}(2n+1)}{(2n+1)^s}.$$

Therefore, let

$$g(x) = -\frac{\log^{\alpha}(2x)}{(2x)^s} + \frac{\log^{\alpha}(2x+1)}{(2x+1)^s}$$

Then $\sum_{n=1}^{\infty} g(n)$ converges for $\Re(s) > 0$ and $\sum_{n=1}^{\infty} g(n) = \eta^{(\alpha)}(s)$. We evaluate the first summand of (4.6) as is, namely as

$$G_s^{\alpha}(m) := \sum_{n=1}^{m-1} g(n) = \sum_{n=1}^{m-1} \left(-\frac{\log^{\alpha}(2n)}{(2n)^s} + \frac{\log^{\alpha}(2n+1)}{(2n+1)^s} \right)$$

For the second term of the right hand side of (4.6) we distinguish two cases:

Case 1: $s = 1$, we have using substitution

$$\begin{aligned}
I_1^{\alpha}(m) &:= \int_m^{\infty} g(x)dx \\
&= \int_m^{\infty} -\frac{\log^{\alpha}(2x)}{2x} + \frac{\log^{\alpha}(2x+1)}{2x+1} dx \\
&= -\frac{1}{2} \int_{2m}^{\infty} \frac{\log^{\alpha} u}{u} du + \frac{1}{2} \int_{2m+1}^{\infty} \frac{\log^{\alpha} u}{u} du \\
&= -\frac{1}{2} \int_{2m}^{2m+1} \frac{\log^{(\alpha)} u}{u} du \\
&= -\frac{1}{2(\alpha+1)} \left(\log^{\alpha+1}(2m+1) - \log^{\alpha+1}(2m) \right).
\end{aligned}$$

Case 2: for $s \neq 1$ the second term can be written in terms of the Upper Incomplete Gamma function $\Gamma(\alpha, s)$ (compare [GR07, p. 346] and [AS64, 6.5.3]):

$$\begin{aligned}
I_s^\alpha(m) &:= \int_m^\infty g(x)dx \\
&= \int_m^\infty -\frac{\log^\alpha(2x)}{(2x)^s} + \frac{\log^\alpha(2x+1)}{(2x+1)^s} dx \\
&= -\frac{1}{2}\int_{2m}^\infty \frac{\log^\alpha(u)}{(u)^s}du + \frac{1}{2}\int_{2m+1}^\infty \frac{\log^\alpha(u)}{(u)^s}du \\
&= \frac{-\Gamma(\alpha+1, (\log(2m)))}{2(s-1)^{\alpha+1}} + \frac{\Gamma(\alpha+1, (\log(2m+1)))}{2(s-1)^{\alpha+1}} \\
&= \frac{\Gamma(\alpha+1, (\log(2m+1))) - \Gamma(\alpha+1, (\log(2m)))}{2(s-1)^{\alpha+1}} \\
&= \frac{1}{2} \cdot \frac{\Gamma(\alpha+1, (\log(2m+1))) - \Gamma(\alpha+1, (\log(2m)))}{(s-1)^{\alpha+1}}.
\end{aligned}$$

Let v be an even integer. Then for the third term, we get:

$$\begin{aligned}
B_s^\alpha(m, v) &:= \sum_{n=1}^{v} \frac{(-1)^n B_n}{n!} g^{(n-1)}(x)\bigg|_{x=m}^{\infty} \\
&= \frac{1}{2}g(m) + \sum_{n=1}^{v} \frac{(-1)^n B_n}{n!} g^{(n-1)}(x)\bigg|_{x=m}^{\infty} \\
&= \frac{1}{2}g(m) + \sum_{j=1}^{v/2} \frac{B_{2j}}{(2j)!} g^{(2j-1)}(x)\bigg|_{x=m}^{\infty} \\
&= \frac{1}{2}g(m) - \sum_{j=1}^{v/2} \frac{B_{2j}}{(2j)!} g^{(2j-1)}(m).
\end{aligned}$$

As in Section 3.2, we use the non-central Stirling numbers (3.5) and the falling factorial (3.4) to evaluate the derivatives $g^{(n-1)}(x)$. Then we have,

$$\begin{aligned}
g^{(n)}(x) &= \left(-\frac{\log^\alpha(2x)}{(2x)^s} + \frac{\log^\alpha(2x+1)}{(2x+1)^s}\right)^{(n)} \\
&= \left(-\frac{\log^\alpha(2x)}{(2x)^s}\right)^{(n)} + \left(\frac{\log^\alpha(2x+1)}{(2x+1)^s}\right)^{(n)}
\end{aligned}$$

$$= 2^n \cdot \left(-\frac{\log^\alpha(2x)}{(2x)^s}\right)^{(n)} + 2^n \cdot \left(\frac{\log^\alpha(2x+1)}{(2x+1)^s}\right)^{(n)}$$

$$= 2^n \cdot \sum_{i=0}^{n} S(n,i,s)(\alpha)_i \frac{-\log^{\alpha-i}(2x)}{(2x)^{s+n}} + S(n,i,s)(\alpha)_i \frac{\log^{\alpha-i}(2x+1)}{(2x+1)^{s+n}}.$$

Now we determine a bound for the fourth term of (4.6). Writing $s = \sigma + it$ and

$$E_s^\alpha(m,v) := \frac{1}{v!} \int\limits_m^\infty P_v(x) g^{(v)}(x) dx$$

we obtain

$$\begin{aligned} |E_s^\alpha(m,v)| &= \left| \frac{1}{v!} \int_m^\infty P_v(x) g^{(v)}(x) dx \right| \le \frac{|B_v|}{v!} \int_m^\infty |g^{(v)}(x)| dx \\ &\le \frac{|B_v|}{v!} \sum_{j=0}^{v} \int_m^\infty \left| S(v,j,s)(\alpha)_j \frac{\log^{\alpha-j}(x)}{x^{s+v}} \right| dx \\ &\le \frac{|B_v|}{v!} \left(\sum_{j=0}^{v} |S(v,j,s)(\alpha)_j| \right) \left(\int_m^\infty \frac{\log^n(x)}{x^{\sigma+v}} dx \right) \\ &= \frac{|B_v|}{v!} \left(\sum_{j=0}^{v} |S(v,j,s)(\alpha)_j| \right) \frac{\Gamma(\alpha+1, (\sigma+v-1)\log(m))}{(\sigma+v-1)^{\alpha+1}}. \end{aligned}$$

The error term $E_s^\alpha(m,v)$ converges for $\sigma + v > 1$ and $m > 2$. For all $s \in \mathbb{C} \setminus (\infty, 1]$ we can choose $m \in \mathbb{N}$ and $v \in \mathbb{N}$ such that $|E_s^\alpha(m,v)|$ becomes arbitrarily small. We can thus approximate $D_s^\alpha[\zeta(s)]$ as

$$D_s^\alpha[\zeta(s)] \approx (-1)^\alpha \left(G_s^\alpha(m) + I_s^\alpha(m) + B_s^\alpha(m,v)\right)$$

where the error is $|E_s^\alpha(m,v)|$.

4.2.1 Skorokhodov connectors

As in the $\zeta(s)$ case we use the Skorokhodov connector, see Section 3.2.1, to find the starting point of the paths of the zeros of $\eta^{(\alpha)}(s)$.

Theorem 4.1. *The function $\eta(s)$ is distinct from unity at $\sigma \in (\sigma_0, \infty)$, where*

$$\sigma_0 = 1.940101683745\ldots$$

is the zero of the function

$$f(\sigma) = 1 + 2^{-\sigma} - (1 - 2^{-\sigma})\zeta(\sigma), \quad \sigma > 1.$$

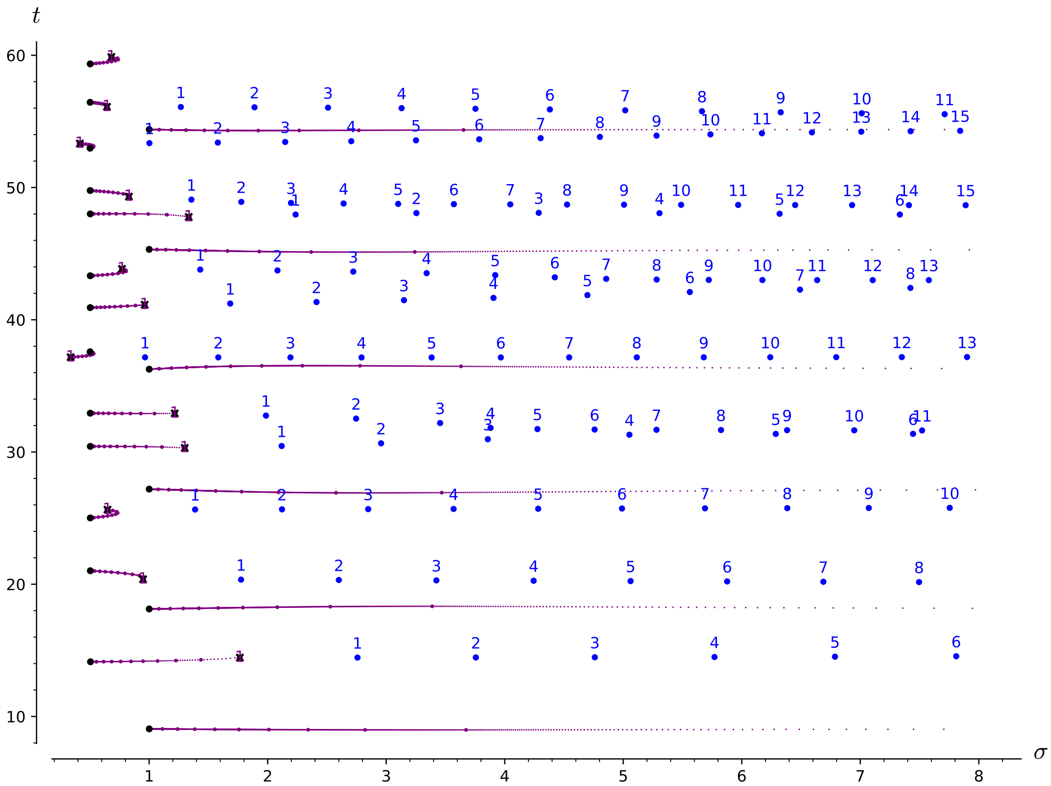

Figure 4.2. Zeros of derivatives of $\eta^{(k)}(s)$ (denoted by $\bullet^{k}$) and the paths from zeros of $\eta(s)$ (denoted by $\bullet$) to the zeros of $\eta(s)-1$ (denoted by $\times$).

Proof. Using a similar relation as in [Sko03, p.1293]

$$(1+2^{-s})\eta(s)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^s},\quad \sigma>1, \tag{4.7}$$

one obtains

$$(1+2^{-s})|\eta(s)-1|=-2^{-s}+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^s},\quad \sigma>1, \tag{4.8}$$

From (4.8), by virtue of the inequalities

$$|a|+|b|\geq|a-b|\geq|a|-|b|$$

it follows that

$$|1-2^{-s}||\eta(s)-1|\geq|-2^{-s}|-\sum_{n=1}^{\infty}\frac{1}{|(2n+1)^s|}=1+2^{-\sigma}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{\sigma}}. \tag{4.9}$$

Using (4.7), rewrite (4.9)

$$\begin{aligned}|1+2^{-s}||\eta(s)-1| &\geq 1+2^{-\sigma}-(1-2^{-\sigma})|\eta(s)| \\ &\geq 1+2^{-\sigma}-(1-2^{-\sigma})\zeta(\sigma) =: f(\sigma), \quad \sigma>1. \end{aligned}$$ □

Now that we have established a similar result to Theorem 3.3, we can find zero-free regions for $\eta(s)-c$.

Lemma 4.2. *If* $c\in[0,1)$ *and* $|\sin(t\log 2)|\geq 2^{\sigma}\zeta(s)+2^{\sigma}-1$ *then* $\eta(\sigma+it)-c\neq 0$.

Proof. We consider the imaginary part of $\eta(s)-c$ and obtain

$$\begin{aligned}|\Im(\eta(s)-c)| &\geq \left|\frac{1}{2^{\sigma}}\sin(t\log 2)\right|-\left|\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\right| \\ &= \left|\frac{1}{2^{\sigma}}\sin(t\log 2)\right|-\left|\eta(s)-1+\frac{1}{2^{\sigma}}\right| \\ &\geq \left|\frac{1}{2^{\sigma}}\sin(t\log 2)\right|-\zeta(\sigma)-1+\frac{1}{2^{\sigma}},\end{aligned}$$

which is positive when

$$|\sin(t\log 2)|\geq 2^{\sigma}\zeta(\sigma)+2^{\sigma}-1.$$ □

Lemma 4.3. *If* $c\in[0,1)$ *and* $\cos(t\log 3)\geq 3^{\sigma}\zeta(s)-3^{\sigma}-1$, *then* $\eta(\sigma+it)-c\neq 0$

Proof. For the real part of $\eta(s)-c$ we obtain

$$\begin{aligned}\Re(\eta(s)-c) &= 1-c-\frac{1}{2^{\sigma}}\cos(t\log 2)+\frac{1}{3^{\sigma}}\cos(t\log 3)-+\cdots \\ &\geq -\frac{1}{2^{\sigma}}\cos(t\log 2)+\frac{1}{3^{\sigma}}\cos(t\log 3)-\Re\left(\eta(s)-1-\frac{1}{3^{\sigma}}\right) \quad \text{assuming } c=1. \\ &\geq \frac{1}{3^{\sigma}}\cos(t\log 3)-\left(\zeta(s)-1-\frac{1}{3^{\sigma}}\right),\end{aligned}$$

which is positive when

$$\cos(t\log 3)\geq 3^{\sigma}\zeta(s)-3^{\sigma}-1.$$ □

These regions can be extended a bit if we restrict ourselves to certain values of t.

Lemma 4.4. *If* $c\in[0,1)$, $m\in\mathbb{Z}$, *then* $\Re(\eta(s)-c)\neq 0$ *for* $\sigma>1.34$, *and* $t=\frac{2m\pi}{\log 3}$

Proof. Let $\Re(\eta(s)-c) = 1 - c - \frac{1}{2^\sigma}\cos(t\log 2) + \frac{1}{3^\sigma}\cos(t\log 3) + - \cdots$ With $t\log 3 = 2m\pi$, we get:

$$\begin{aligned}
\Re(\eta(s)-c) &= 1 - c - \frac{\cos\left(\frac{\log 2}{\log 3}2m\pi\right)}{2^\sigma} + \frac{1}{3^\sigma} - + \dots \\
&\geq \sum_{\nu=1}^{\infty}\frac{1}{(3^\nu)^\sigma} - \Re\left(\eta(s) - 1 - \sum_{\nu=1}^{\infty}\frac{1}{(3^\nu)^\sigma}\right) \\
&\geq 1 + 2\sum_{\nu=1}^{\infty}\left(\frac{1}{3^\sigma}\right)^\nu - \Re(\eta(s)) \\
&\geq 1 + \frac{2}{1-\frac{1}{3^\sigma}} - \zeta(\sigma)
\end{aligned}$$

which is positive for $\sigma > 1.34$. □

Lemma 4.5. *If $c \in [0,1)$, $m \in \mathbb{Z}$, then $\Re(\eta(s)-c) \neq 0$ for $\sigma > 1.54$, and $t = \frac{2m\pi}{\log 2}$.*

Proof. Let $\Re(\eta(s)-c) = 1 - c - \frac{1}{2^\sigma}\cos(t\log 2) + \frac{1}{3^\sigma}\cos(t\log 3) + - \cdots$ With $t\log 2 = 2m\pi$, we get:

$$\begin{aligned}
\Re(\eta(s)-c) &= 1 - c - \frac{1}{2^\sigma} + \frac{\cos\left(\frac{\log 3}{\log 2}2m\pi\right)}{3^\sigma} - + \dots \\
&\geq \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{(2^\nu)^\sigma} - \Re\left(\eta(s) - 1 - \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{(2^\nu)^\sigma}\right) \\
&\geq 1 + 2\sum_{\nu=1}^{\infty}\left(-\frac{1}{2^\sigma}\right)^\nu - \Re(\eta(s)) \\
&\geq 1 + \frac{2}{1+\frac{1}{2^\sigma}} - \zeta(\sigma)
\end{aligned}$$

which is positive for $\sigma > 1.54$. □

Proposition 4.6. *If $c \in [0,1)$, $m \in \mathbb{Z}$, for any prime $p > 2$, and $t = \frac{2m\pi}{\log p}$, then $\Re(\eta(s)-c) \neq 0$ for*

$$1 + \frac{2}{1-\frac{1}{p^\sigma}} > \zeta(\sigma)$$

Proof. Let $\Re(\eta(s)-c) = 1 - c - \frac{1}{2^\sigma}\cos(t\log 2) + \frac{1}{3^\sigma}\cos(t\log 3) + - \cdots$ where $t\log 2 = 2m\pi$, we get:

$$\begin{aligned}\Re(\eta(s)-c) &= 1 - c - \frac{\cos\left(\frac{\log 2}{\log p}2m\pi\right)}{2^\sigma} + \frac{\cos\left(\frac{\log 2}{\log(9)}2m\pi\right)}{3^\sigma} - + \frac{1}{p^\sigma} - = \dots \\ &\geq \sum_{\nu=1}^{\infty}\frac{1}{(p^\nu)^\sigma} - \Re\left(\eta(s) - 1 - \sum_{\nu=1}^{\infty}\frac{1}{(p^\nu)^\sigma}\right) \\ &\geq 1 + 2\sum_{\nu=1}^{\infty}\left(\frac{1}{p^\sigma}\right)^\nu - \Re(\eta(s)) \\ &\geq 1 + \frac{2}{1-\frac{1}{p^\sigma}} - \zeta(\sigma)\end{aligned}$$

□

4.3 Power Series Expansions

The function $\eta(s)$ is entire and has a power series expansion [HK22] of the form

$$\eta(s) = \sum_{j=0}^{\infty}(-1)^j\frac{\gamma_j(\eta)}{j!}(s-1)^j \tag{4.10}$$

Repeated differentiation yields:

$$\eta^{(k)}(s) = \sum_{j=0}^{\infty}(-1)^{j+k}\frac{\gamma_{k+j}(\eta)}{j!}(s-1)^j \tag{4.11}$$

which implies that $\gamma_k(\eta) = \eta^{(k)}(1)$.

A formula similar to Williams and Zhang's formula (3.7) also exists for alternating Hurwitz zeta functions [HK22]. For the special case of $\eta(s)$ it is known

$$\gamma_j(\eta) = \lim_{m\to\infty}\sum_{n=1}^{m}(-1)^{n+1}\frac{\log^j(n)}{n}$$

Example 4.7. Some examples of $\eta^{(k)}(1)$ are the following:

$$\begin{aligned}\eta(1) &= \log 2 \approx 0.6931 \\ \eta'(1) &= \log 2\gamma - \frac{\log^2 2}{2} \approx 0.1598 \\ \eta''(1) &\approx -0.0654 \\ \eta^{(3)}(1) &\approx 0.0094\end{aligned}$$

We generalize these Taylor expansion to the $\alpha-$th derivative as the following:

$$\eta^{(\alpha)}(s) = \sum_{j=0}^{\infty}(-1)^{j+\alpha}\frac{\gamma_{\alpha+j}(\eta)}{j!}(s-1)^j \tag{4.12}$$

which implies that $\gamma_\alpha(\eta) = \eta^{(\alpha)}(1)$ for $\alpha \in [0, \infty)$. See Figure 4.3.

As a direct consequence of Theorem 5.18 which will be established in Chapter 5, we have

Corollary 4.8. *Let $w_\alpha(2) := W_0\left(\frac{\alpha i}{\pi}\right)$ be an extension of w_α from Theorem 3.10 and define*

$$\begin{aligned}\widetilde{\gamma}_\alpha(\eta) =& \frac{3}{2}\frac{\log^\alpha(2)}{2} + \frac{\log^{\alpha+1}(3)}{2(\alpha+1)} - \frac{\log^{\alpha+1}(4)}{4(\alpha+1)} - \Im\left(-\sqrt{\frac{2\alpha}{\pi(w_\alpha(2)+1)}}e^{-w_\alpha(2)+h_1(w_\alpha(2))}\right)\\ &+ \Im\left(-\sqrt{\frac{2\alpha}{\pi(w_\alpha(2)+1)}}e^{-w_\alpha(2)+h_2(w_\alpha(2))}\right)\end{aligned}$$

where $h_1(t) = \pi i(e^t - 1) + \alpha \log t$ and $h_2(t) = \pi i(e^t - 2) + \alpha \log t$. Then

$$\widetilde{\gamma}_\alpha(\eta) \sim \gamma_\alpha(\eta).$$

4.4 Right Half-Plane

Let $Q_n^\alpha(s) := (\log n)^\alpha / n^s$ denote the n-th term of the Dirichlet series for $(-1)^\alpha \eta^{(\alpha)}(s)$, so that

$$(-1)^\alpha \eta^{(\alpha)}(s) = (-1)^\alpha \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\log^\alpha n}{n^s} = (-1)^\alpha \sum_{n=2}^{\infty}(-1)^{n-1} Q_n^\alpha(s). \tag{4.13}$$

We prove the existence of zero-free regions where one of the terms of (4.13), say $Q_M^\alpha(\sigma)$, dominates the rest of the series, that is, when

$$Q_M^\alpha(\sigma) > \sum_{n \neq M} Q_n^\alpha(\sigma), \tag{4.14}$$

and, in a complementary fashion, we look for the zeros of $\eta^{(\alpha)}(s)$ near the regions of the complex plane where $Q_M^\alpha(s) = Q_{M+1}^\alpha(s)$, in other words where no term of the series can attain dominance and, in fact, where the cancellation of terms might happen. This occurs at

$$q_M := \frac{\log\left(\frac{\log M}{\log(M+1)}\right)}{\log\left(\frac{M}{M+1}\right)}. \tag{4.15}$$

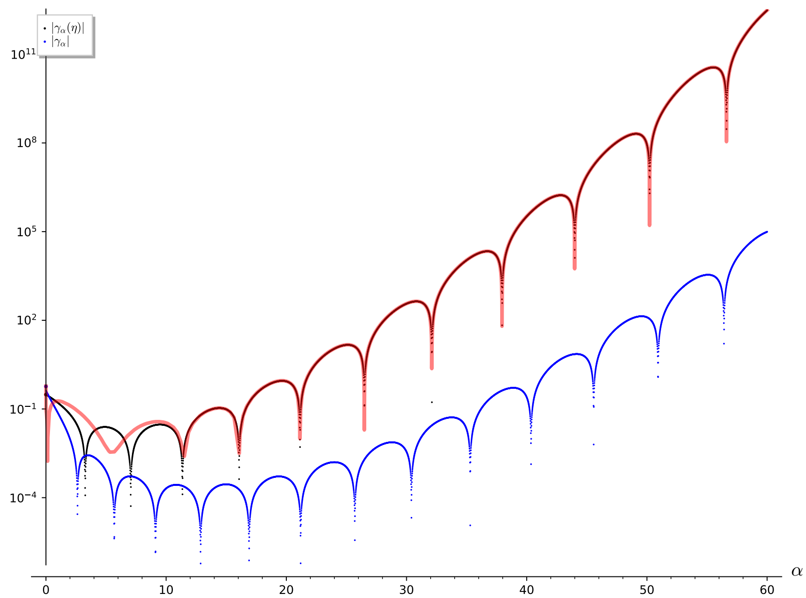

Figure 4.3. Stieljes constants γ_α for ζ in blue and $\gamma_\alpha(\eta)$ for η in black. and $\widetilde{\gamma}_\alpha(\eta)$ in red

Our main goal is to prove a generalization of [BPS15, Theorem 2.1] for $\eta^{(\alpha)}(s)$. We later generalize it for all Dirichlet L-functions in Chapter 5:

Theorem 4.9. *Let $\alpha > 0$. We have:*

(a) For all $\sigma > q_2\alpha + 2.6$, we have $\eta^{(\alpha)}(s) \neq 0$.

(b) If $q_3\alpha + 4\log 3 < q_2\alpha - 2$, then $\eta^{(\alpha)}(s) \neq 0$ for

$$q_3\alpha + 4\log 3 \leq \sigma \leq q_2\alpha - 2.$$

(c) If $M \in \mathbb{N}$, $M > 3$, and $q_M\alpha + (M+1)u \leq q_{M-1}\alpha - Mu$, then $\eta^{(\alpha)}(s) \neq 0$ in the regions

$$q_M\alpha + (M+1)u \leq \sigma \leq q_{M-1}\alpha - Mu,$$

where $u \in (0,\infty)$ is a solution of $1 - \frac{1}{e^u - 1} - \frac{1}{e^u}(1 + \frac{1}{u}) \geq 0$.

Note: The value of $u \in (0,\infty)$ that gives us the widest zero-free regions is $u = 1.1879426249\ldots$, which is the solution of the equation

$$1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) = 0. \tag{4.16}$$

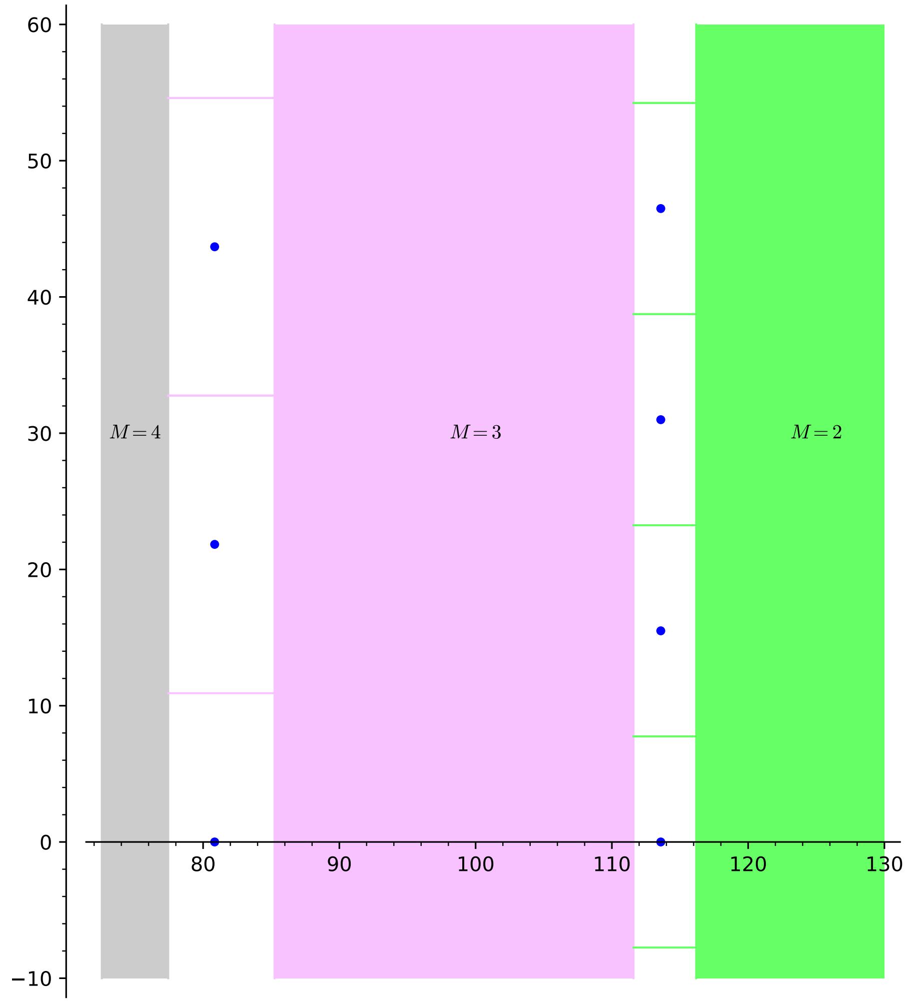

Figure 4.4. Zero-free regions and lines for $\eta^{(100)}(s)$

The proof follows from [FPS18], (see Figure 4.5). We know that $|\eta^{(\alpha)}(s)| \geq |\zeta^{(\alpha)}(s)|$. Let S_M^α be the vertical strip between the zero-free regions obtained from the dominance of $Q_M^\alpha(q_M\alpha)$ and $Q_{M+1}^\alpha(q_M\alpha)$ in (4.13), respectively, as described in Theorem 4.9. The strip S_M^α exists when α reaches

$$A_M := \begin{cases} \frac{4\log 3+2}{q_2-q_3} & \text{if } M=2 \\ \frac{(2M+3)u}{q_M-q_{M+1}} & \text{if } M>2. \end{cases}$$

Recall that $Q_M^\alpha(q_M\alpha) = Q_{M+1}^\alpha(q_M\alpha)$. Considering the imaginary parts of the solutions of $Q_M^\alpha(q_M\alpha+it) - Q_{M+1}^\alpha(q_M\alpha+it) = 0$ we find that $\eta^{(\alpha)}(\sigma+it) \neq 0$ for

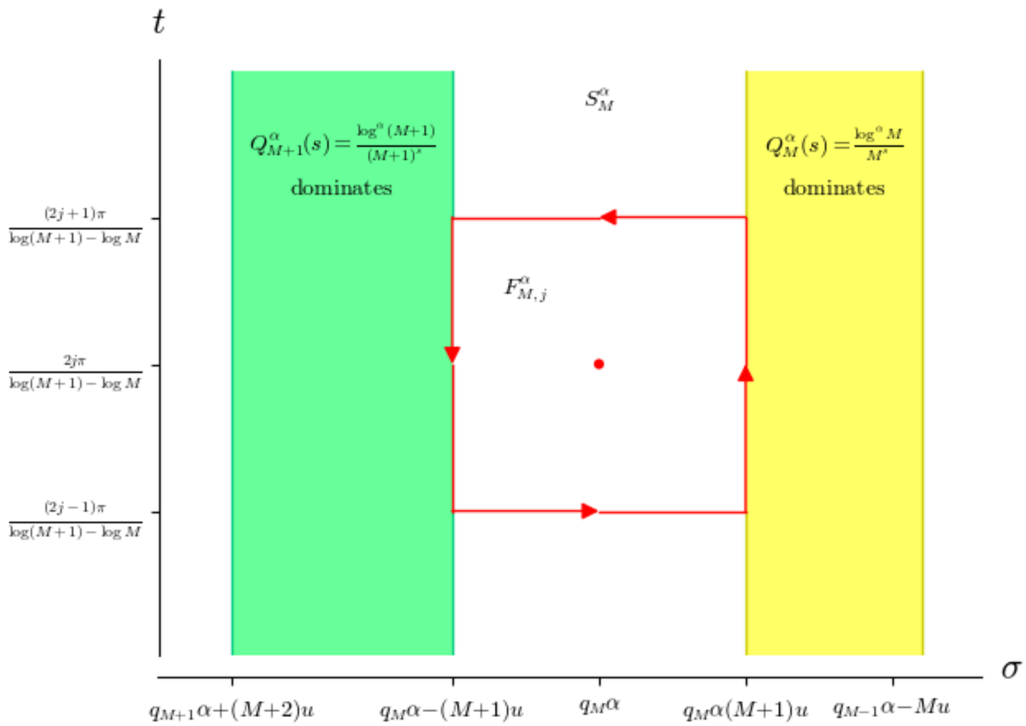

Figure 4.5. Regions $F_{M,j}^\alpha$ that contains exactly one zero of $\eta^{(\alpha)}(\sigma+it)$. Rouche's theorem can be used to establish simplicity of the zero using the zero of $Q_M^\alpha(s) - Q_{M+1}^\alpha(s)$

$\sigma \in S_M^\alpha$ and

$$t = \frac{\pi(2J+1)}{\log(M+1) - \log(M)} \tag{4.17}$$

for $J \in \mathbb{Z}$. Together with the border of the zero-free regions to the left and right of S_M^α the lines from (4.17), for $J = j$ and $J = j-1$, where $j \in \mathbb{Z}$ form a contour around the zero

$$q_M \cdot \alpha + \frac{2\pi j}{\log(M+1) - \log(M)} i \tag{4.18}$$

of $Q_M^\alpha(q_M\alpha + it) - Q_{M+1}^\alpha(q_M\alpha + it)$. Exactly as in [BPS15], Rouché's theorem immediately shows that there is exactly one zero of $\eta^{(\alpha)}(s)$ in the rectangular area shown in Figure 4.5. In other words, a natural generalization of [BPS15, Theorem 2.2] can be quickly obtained, *mutatis mutandis*, replacing integer values of k by positive real numbers α:

Theorem 4.10. *Let $M \geq 2$ denote a natural number, $j \in \mathbb{Z}$, and $\alpha > A_M$. Let*

$F^\alpha_{M,j} \subset S^\alpha_M$ *be given by*

$$\frac{\pi(2j-1)}{\log(M+1)-\log(M)} < t < \frac{(2j+1)\pi}{\log(M+1)-\log(M)}. \tag{4.19}$$

Then $F^\alpha_{M,j}$ *contains exactly one zero of* $\eta^{(\alpha)}(s)$, *and the zero is simple.*

Computations conducted with the methods from Section 4.2, suggests that the zeros in the regions $F^\alpha_{M,j}$ form continuous, mostly horizontal lines. We observe that the lines of zeros of fractional derivatives passing through the regions $F^\alpha_{M,j}$ with $j > 0$ end at a zeros of $\eta(s) - 1$ where $-\frac{1}{2} < \Re(s) < 1.9402$, see Theorem 4.1.

Far enough to the right the existence of these lines follows from Theorem 4.10: Let $M \in \mathbb{Z}$, $M \geq 2$ and $\alpha > A_M$ so that S^α_M is non empty. Then for each $j \in \mathbb{Z}$ there is $s = \sigma + it \in F^\alpha_{M,j}$ such that $\eta^{(\alpha)}(s) = 0$. As s is a simple zero of $\eta^{(\alpha)}(s)$ we have that $\eta^{(\alpha+1)}(s) \neq 0$. By the implicit function theorem there is an analytic function z defined on an open neighborhood $U \subset \mathbb{C}$ of α such that $\eta^{(\beta)}(z(\beta)) = 0$ for $\beta \in U$. As this holds for all $\alpha > A_M$ we obtain a function z that is analytic on an open neighborhood of (A_M, ∞) in $\mathbb{C}$ and thus analytic on (A_M, ∞).

Corollary 4.11. *Let* $M \in \mathbb{N}$ *with* $M \geq 2$ *and* $j \in \mathbb{Z}$. *The zeros* $s = \sigma + it$ *of* $\eta^{(\alpha)}(s)$ *for* $\alpha > A_M$ *with*

$$\frac{\pi(2j+1)}{\log(M+1)-\log(M)} < t < \frac{\pi 2j}{\log(M+1)-\log(M)}$$

are images of an analytic function $z : (A_M, \infty) \to \mathbb{C}$.

Lemma 4.12. *Let* $M \geq 2$ *and* $\alpha \in \mathbb{R}$. *If* $s \in S^\alpha_M$, *then* $\eta^{(\alpha)}(s) \neq 0$ *for*

$$s = \sigma + i \cdot \frac{\pi(2j+1)}{\log(M+1) - \log M}.$$

Proof. In the center of the strip S^α_M, that is on the line $\sigma = q_M\alpha$ we have $|Q^\alpha_M(s)| = |Q^\alpha_{M+1}(s)|$. We consider the line segments in S^α_M with

$$q_M\alpha - (M+1)u \leq \sigma \leq q_M\alpha + (M+1)u.$$

and

$$t = \frac{\pi(2j+1)}{\log(M+1) - \log M}, \text{ where } j \in \mathbb{Z},$$

see Figure 4.5. Everywhere hereafter we write $H^\alpha_M(s)$ for the "head" and $T^\alpha_M(s)$ for the "tail" of the series $\eta^{(\alpha)}(s)$ split by $Q^\alpha_M(s)$:

$$H^\alpha_M(s) := \sum_{n=2}^{M-1} |Q^\alpha_n(s)| = \sum_{n=2}^{M-1} \left|\frac{\log^\alpha(n)}{n^s}\right|$$

and

$$T_M^\alpha(s) := \sum_{n=M+1}^{\infty} |Q_n^\alpha(s)| = \sum_{n=M+1}^{\infty} \left|\frac{\log^\alpha(n)}{n^s}\right|.$$

Our choice of t gives $Q_M^\alpha(q_M\alpha + it) - Q_{M+1}^\alpha(q_M\alpha + it) = 0$ (compare equation (4.17)) and therefore $\cos(t\log M) = -\cos(t\log(M+1))$ and $\sin(t\log M) = \sin(t\log(M+1))$. We set $s = \sigma + it$, with t and σ as above, and consider the real and imaginary parts of

$$\eta^{(\alpha)}(s) = \sum_{n=2}^{\infty}(-1)^{n-1}\left(\cos(t\log n) - i\cdot\sin(t\log n)\right)Q_n^\alpha(\sigma).$$

With $|\Im(Q_n^\alpha(s)| \leq Q_n^\alpha(\sigma)$ and $|\Re(Q_n^\alpha(s)| \leq Q_n^\alpha(\sigma)$ we obtain

$$\begin{aligned}
|\Re(\eta^{(\alpha)}(s))| &\geq |\cos(t\log M)Q_M^\alpha(\sigma) + \cos(t\log(M+1))Q_{M+1}^\alpha(\sigma)| \\
&\quad - H_M^\alpha(\sigma) - T_{M+1}^\alpha(\sigma), \\
|\Im(\eta^{(\alpha)}(s))| &\geq |\sin(t\log M)Q_M^\alpha(\sigma) + \sin(t\log(M+1))Q_{M+1}^\alpha(\sigma)| \\
&\quad - H_M^\alpha(\sigma) - T_{M+1}^\alpha(\sigma).
\end{aligned}$$

If $t = 0$, the situation is trivial. If $t \neq 0$, then we either have $|\sin(t\log M)| \geq \sin(\pi/4) = 1/\sqrt{2}$ or $|\cos(t\log M)| \geq \cos(\pi/4) = 1/\sqrt{2}$. Because $|\eta^{(\alpha)}(s)| \geq |\Re(\eta^{(\alpha)}(s))|$ and $|\eta^{(\alpha)}(s)| \geq |\Im(\eta^{(\alpha)}(s))|$ we get:

$$\begin{aligned}
|\eta^{(\alpha)}(s)| &\geq \frac{1}{\sqrt{2}}\left(Q_M^\alpha(\sigma) + Q_{M+1}^\alpha(\sigma)\right) - H_M^\alpha(\sigma) - T_{M+1}^\alpha(\sigma) \\
&= Q_M^\alpha(\sigma)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{Q_{M+1}^\alpha}{Q_M^\alpha}(\sigma) - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{Q_{M+2}^\alpha}{Q_M^\alpha}(\sigma) - \frac{T_{M+2}^\alpha}{Q_M^\alpha}(\sigma)\right) \\
&= Q_M^\alpha(\sigma)\left(\frac{1}{\sqrt{2}} - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) + \frac{Q_{M+1}^\alpha}{Q_M^\alpha}(\sigma)\left(\frac{1}{\sqrt{2}} - \frac{Q_{M+2}^\alpha}{Q_{M+1}^\alpha}(\sigma) - \frac{T_{M+2}^\alpha}{Q_{M+1}^\alpha}(\sigma)\right)\right)
\end{aligned}$$

From the proof of Theorem 4.9 (b) we know that for $\sigma \geq q_{M+1}\alpha + (M+2)u$ and $u = 1.1879426249\ldots$

$$\begin{aligned}
\frac{1}{\sqrt{2}} - \frac{Q_{M+2}^\alpha}{Q_{M+1}^\alpha}(\sigma) - \frac{T_{M+2}^\alpha}{Q_{M+1}^\alpha}(\sigma) &\geq \frac{1}{\sqrt{2}} - \frac{Q_{M+2}^\alpha}{Q_{M+1}^\alpha}(\sigma)\left(1 + R_{M+2}(\sigma)\right) \\
&\geq \frac{1}{\sqrt{2}} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) > 0.
\end{aligned}$$

Similarly, since $\frac{H_M^\alpha}{Q_M^\alpha}(\sigma)$ is increasing in σ (see equation (5.21)) and because $\sigma < q_{M-1}\alpha - Mu$, we get that

$$\frac{1}{\sqrt{2}} - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) \geq \frac{1}{\sqrt{2}} - \frac{H_M^\alpha}{Q_M^\alpha}(q_{M-1}\alpha - Mu) \geq \frac{1}{\sqrt{2}} - \frac{1}{e^u - 1} > 0,$$

which concludes the proof of the lemma. ∎

Proof of Theorem 4.10. Let $Z(s) = Q_M^\alpha(s) - Q_{M+1}^\alpha(s)$. It is easy to check that the function $Z(s)$ has exactly one (simple) zero in R_j, namely

$$s = q_M\alpha + i \cdot \frac{2\pi j}{\log(M+1) - \log M}.$$

In order to be able to apply Rouché's Theorem we need to show that $|\eta^{(\alpha)}(s) - Z(s)| < |Z(s)|$ for all s on R_j.

The vertical sides of R_j are in the zero-free regions for M and $M+1$. As shown in the proof of Theorem 4.9 the term $Q_M^\alpha(s)$ dominates $\eta^{(\alpha)}(s)$ on the right vertical side of R_j and the term $Q_{M+1}^\alpha(s)$ dominates $\eta^{(\alpha)}(s)$ on the left vertical side of R_j. Thus $|\eta^{(\alpha)}(s) - Z(s)| < |Z(s)|$ on the vertical sides of R_j. Furthermore we have seen in the proof of Lemma 4.12 that $Z(s) = Q_M^\alpha(s) + Q_{M+1}^\alpha(s)$ dominates $\eta^{(\alpha)}(s)$ on the horizontal sides of R_j. Hence $|\eta^{(\alpha)}(s) - Z(s)| < |Z(s)|$ on the horizontal sides of R_j. □

4.4.1 Positive Real Axis

Recall in Section 3.4.1, $\zeta^{(\alpha)}(s)$ did not have zeros on the positive real axis. The same can not be said about the Dirichlet eta function. From the alternating property, the imaginary part of the dominating terms used in Theorem 4.10 shifts the boxes to include the real axis. This guarantees a zero of $\eta^{(\alpha)}(s)$ to lie on the real axis. Because $\eta^{(k)}(s)$ has the same number of zeros as $\zeta(s)$, see Section 4.7, we expect that these zeros on the real axis lie on paths of fractional derivatives that originate on the left half-plane, see Figure 4.6.

4.5 Critical Strip

One can quickly see from (4.2) that the zeros of $\zeta(s)$ are also the zeros of $\eta(s)$. Dirichlet eta in addition has infinitely many zeros of the form $s_n = 1 + \frac{2n\pi i}{\ln(2)}$ where n is a nonzero integer, which are the zeros of the factor $(1 - 2^{1-s})$.

4.5.1 Derivatives

There are several theorems, already discussed in Section 3.5, that relate the Riemann hypothesis to the location of the zeros of the derivatives of $\zeta(s)$. We prove a new relation of this type for $\eta(s)$.

Theorem 4.13. *The Riemann hypothesis implies $\eta'(s) \neq 0$ for $0 < \sigma < 1/2$.*

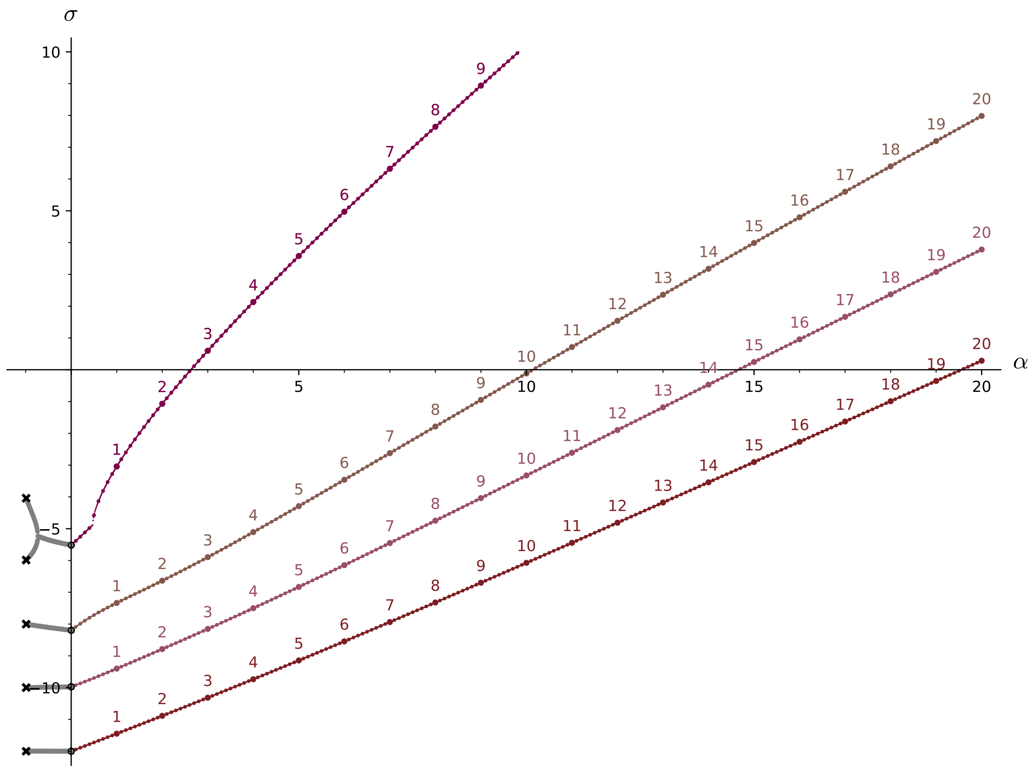

Figure 4.6. The path $\eta^{(\alpha)}(s)$ takes on the real axis in the $\sigma-\alpha$ plane: $\times$ zero of $\eta(s)$, $\bullet$ zero of $\eta(s)-1$, $\overset{k}{\bullet}$ zero of $\eta^{(k)}(s)$, Grey lines are Skorokhodov connectors

Proof. We first recall 4.2
$$\eta(s)=(1-2^{1-s})\zeta(s).$$
Therefore,
$$\eta(1-s)=(1-2^{-s})\zeta(1-s)$$
. Take $\sigma>1/2$,$|t|\neq 0$. the functional equation 3.1 of the $\zeta(s)$, and above gives

$$-\frac{\eta'(1-s)}{\eta(1-s)}=-\log(2\pi)-\frac{1}{2}\pi\tan\left(\frac{\pi s}{2}\right)+\frac{\Gamma'(s)}{\Gamma(s)}+\frac{\zeta'(s)}{\zeta(s)}+\frac{2^{-s}\log 2}{(1-2^{-s})} \tag{4.20}$$
$$=-\log(2\pi)-\frac{1}{2}\pi\tan\left(\frac{\pi s}{2}\right)+\frac{\Gamma'(s)}{\Gamma(s)}+\frac{\zeta'(s)}{\zeta(s)}+\frac{\log 2}{(2^{s}-1)}.$$

Next, using [THB86], also seen in Section 2.1.1,

$$\frac{\zeta'(s)}{\zeta(s)}=\log(2\pi)-1-\frac{\gamma}{2}-\frac{1}{s-1}-\frac{\Gamma'(\frac{s}{2}+1)}{2\Gamma(\frac{s}{2}+1)}+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{4.21}$$

where the series runs over the complex roots of the zeta function and converges absolutely, we obtain,

$$-\frac{\eta'(1-s)}{\eta(1-s)} = -\left(1+\frac{\gamma}{2}+\frac{1}{s-1}+\frac{1}{2}\pi\tan\left(\frac{\pi s}{2}\right)\right)+\frac{\log 2}{2^s-1} + \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(\frac{s}{2}+1)}{2\Gamma(\frac{s}{2}+1)} + \sum_\rho \left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \tag{4.22}$$

Since for a zero $\rho = a+bi$, we have

$$\Re\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) = \frac{\sigma-a}{(\sigma-a)^2+(t-b)^2}+\frac{a}{a^2+b^2} > 0,$$

using the Riemann hypothesis, we get

$$-\frac{\eta'(1-s)}{\eta(1-s)} > \Re\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) - \Re\left(\frac{\Gamma'(\frac{s}{2}+1)}{2\Gamma(\frac{s}{2}+1)}\right) - \Re\left(1+\frac{\gamma}{2}+\frac{1}{s-1}+\frac{1}{2}\pi\tan\left(\frac{\pi s}{2}\right)\right)+\Re\left(\frac{\log 2}{2^s-1}\right). \tag{4.23}$$

In the plane cut along the non-positive real axis,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log(s) - \frac{1}{2s} - \frac{1}{12s^2} + 6\int_0^\infty \frac{P_s(x)}{(s+x)^4}dx$$

where $P_s(x)$ is a function of period 1 which is equal to

$$x(2x^2-3x+1)/12$$

on $[0,1]$, and the log is principal. As in (4.20), $6|P_3(x)| \le \frac{1}{8}$, so

$$6\int_0^\infty \frac{P_s(x)}{(s+x)^4}dx \le \frac{1}{8}\int_0^\infty \frac{dx}{|s+x|^4} \le \frac{1}{6|s|^3}, \tag{4.24}$$

the last inequality coming from [Spi65a, equation (6)]. Thus,

$$\Re\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) \le \log|s| - \frac{1}{2}|s| - \frac{1}{12}|s|^2 - \frac{1}{6}|s|^3, \tag{4.25}$$

and

$$\Re\left(\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)}\right) \le \log|\frac{s}{2}+1| + \frac{1}{|s+2|} + \frac{1}{3}|s+2|^2 + \frac{4}{3}|s+2|^3., \tag{4.26}$$

Next,

$$\Re\left(\tan\left(\frac{\pi s}{2}\right)\right) \le |\tan\left(\frac{\pi s}{2}\right)| \le \frac{1+e^{-\pi|t|}}{1-e^{-\pi|t|}} \tag{4.27}$$

by [Spi65a, equation (14)], and this last function is monotone decreasing with increasing $|t|$. Also,

$$\log\left(\frac{2|s|^2}{|s+2|}\right) = \log 2 + \log|s| - \log|1+\frac{2}{s}|. \tag{4.28}$$

and

$$\log\left|1+\frac{2}{s}\right| \le \log\left(1+\frac{2}{|s|}\right) \le \frac{2}{|s|}. \tag{4.29}$$

Finally,

$$\Re\left(\frac{1}{(s-1)}\right) \le \left|\frac{1}{(s-1)}\right| \le \frac{1}{|s|-1}. \tag{4.30}$$

Putting (4.25),(4.26),(4.27),(4.28),(4.29), and (4.30) in (4.23), we obtain for $|s|>1$

$$\begin{aligned} -2\Re\left(\frac{\eta'(1-s)}{\eta(1-s)}\right) &> \log|s| + \log 2 - 2 - \gamma - \frac{2}{|s|} - \frac{\pi(1+e^{-\pi|t|})}{1-e^{-\pi|t|}} \\ &+ \frac{2}{|s|-1} + \frac{\log 2}{2^{|s|}-1} - \frac{1}{|s|} - \frac{1}{6}|s|^2 - \frac{1}{3}|s|^3 \\ &- \frac{1}{|s+2|} - \frac{1}{3}|s+2|^2 - \frac{4}{3}|s+2|^3. \end{aligned} \tag{4.31}$$

Using now $|t| \ge 2$, $|s| \ge 164$, we obtain easily that the left hand side of (4.31) is great than 0. Thus, on the Riemann hypothesis, we have $\eta'(s) \ne 0$ for $0, \sigma < \frac{1}{2}$, $|t| \ge 164$. $\square$

4.6 Left Half-Plane

One clear observation we can make from Figure 4.1, is that on the left, other than the double zero around 5, see Figure 4.7, there appear to be no nonreal zeros. We suspect that this is true for various different reasons. The first being in Theorem 4.10 we see that there are zero chains that are on the real axis. However, the Skorokhodov chains connect all the zeros of the Riemann zeta function to paths in the fractional chain of Dirichlet eta function. Therefore the paths along the real axis must come from the trivial zeros on the left. Path of zeros for eta can only leave the real axis or arrive on the real axis at a double zero. This observation leads us to the following conjecture by our collaborator Professor Ricky E. Farr.

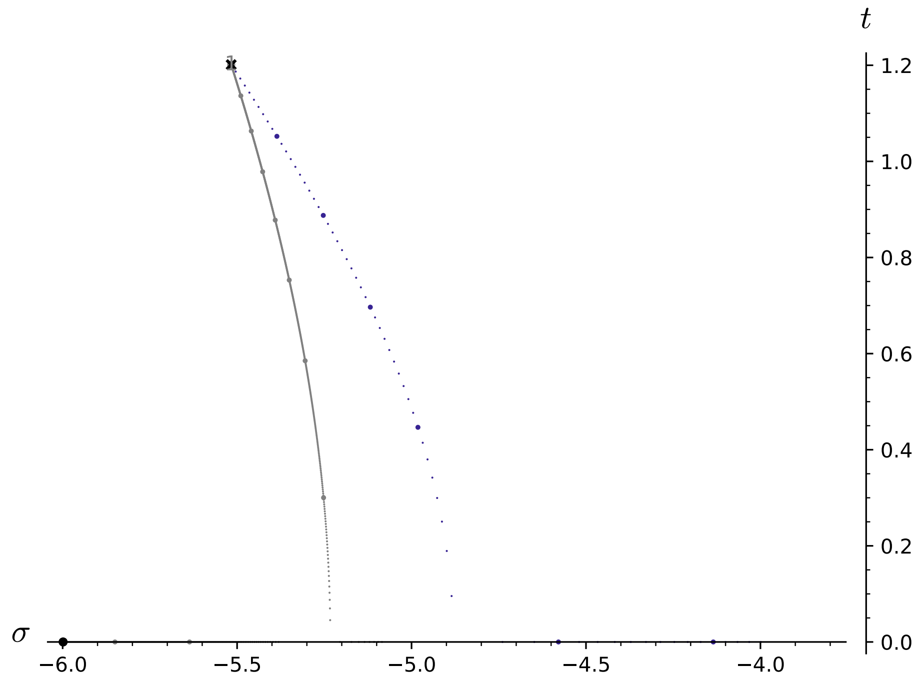

Figure 4.7. Zeros of the fractional derivatives of the Euler η function on the left half plan with a double zero for $\alpha \approx 0.514$ near $\sigma = -4.499$: • Zeros of $\eta(s)$, x Zero of $\eta(s) - 1$, $\bullet^{(k)}$ Zero of $\eta^{(k)}(s)$, Grey lines are Skorokhodov connectors.

Conjecture 4.14 (Farr [Far22]). *Let $k \in \mathbb{N}$ and $\sigma + it \in \mathbb{C}$ with $\sigma < 0$ and $t \neq 0$ then $\eta^{(k)}(\sigma + it) \neq 0$.*

4.6.1 Negative Real Axis

Since $\eta(s)$ has the same zeros of $\zeta(s)$, then $\eta(s)$ has trivial zeros at $-2n$ for $n = 1, 2, \ldots$. We were unable to prove Farr's conjecture but discovered a couple of new intriguing properties, which we state below as conjectures.

Conjecture 4.15. *If $|s| > 165$, then $\eta'(s)$ has only real zeros for $\sigma \leq 0$, and exactly one real zero in each open interval $(-1 - 2n, 1 - 2n), n = 1, 2, \ldots$.*

Conjecture 4.16. *For $k \geq 0$ there is a β_k so that $\eta^{(k)}(s)$ has only real zeros for $\sigma \leq \beta_k$, and exactly one real zero in each open interval $(-1 - 2n, 1 - 2n)$ for $1 - 2n \leq \beta_k$.*

4.6.2 Special Values

We are able to prove the following:

Theorem 4.17. *For all $k \in \mathbb{N}_0$, $b \in \mathbb{Z}$, and real $\sigma < 1$,*

$$\Re\left(\eta^{(k+\frac{1}{2})}(\sigma)\right) = 0. \tag{4.32}$$

Proof. Recall (4.12), the Taylor series expansion of $\eta^{(\alpha)}(s)$ is

$$\eta^{(\alpha)}(s) = (-1)^{\alpha} \sum_{j=1}^{\infty} (-1)^j \frac{\widetilde{\gamma}_{j+k}}{j!}(s-1)^j.$$

Since $\widetilde{\gamma} \in \mathbb{C} \setminus \mathbb{R}$, then

$$\Re\left(\eta^{(k+\frac{1}{2})}(s)\right) = \Re\left((-1)^{\alpha} \sum_{j=1}^{\infty} (-1)^j \frac{\widetilde{\gamma}_{j+k+\frac{1}{2}}}{j!}(s-1)^j\right) = 0. \qquad \square$$

4.7 Number of Zeros

As seen in Figure 4.1, the chains of the Dirichlet eta have a one to one correspondence to the zeros of the Riemann zeta function. We observe that the real part of the Skorokhodov connectors originating on the zeros of $\eta(s)$ on the line $\sigma = 1$ tend to go to infinity. This observation led us to investigate the number of zeros for the derivatives of eta. In a similar fashion as Berndt, see [Ber72], we give an explicit formula for the number of zeros for derivatives of integer order.

Theorem 4.18. *Let $k \geq 1$. Then as $T \to \infty$,*

$$N_{\eta}^{k}(T) = N_{\zeta}(T) + \mathcal{O}(\log T)$$

Proof. We choose σ_k large enough so that $\eta^{(k)}(s)$ has no zeros for $\sigma \geq \sigma_k > 1$, and β_k sufficiently negative so that $\eta^{(k)}(s)$ has no complex zeros for $\sigma \leq \beta_k$. Choose σ_k also large enough so that

$$\sum_{n=3}^{\infty} \frac{\log^k(n)}{(n/2)^{\sigma_k}} \leq \frac{1}{2}\log^k(2). \tag{4.33}$$

If follows that for $\sigma \geq \sigma_k$,

$$\begin{aligned} |\eta^{(k)}(s)| &\geq \frac{\log^k(2)}{2^{\sigma}} - \sum_{n=3}^{\infty} \frac{\log^k(n)}{n^{\sigma}} \\ &\geq \frac{\log^k(2)}{2^{\sigma}} - \frac{1}{2}\frac{\log^k(2)}{2^{\sigma}} \end{aligned}$$

$$= \frac{1}{2}\frac{\log^k(2)}{2^\sigma} \tag{4.34}$$

Choose $\tau_k > 0$ so that $\eta^{(k)}(s)$ has no zeros for $0 < t \leq \tau_k$. Lastly, choose $T_k = T$ so that the line $t = T$ is free of zeros for $\eta^{(k)}(s)$. Let C be the rectangle (described positively) with vertices

$$\beta_k + i\tau_k, \sigma_k + i\tau_k, \sigma_k + iT, \beta_k + iT.$$

By the principle of the argument,

$$\begin{aligned} N_\eta^k(T) &= \frac{1}{2\pi i}\int_C \frac{d}{ds}\log(\eta^{(\alpha)})ds \\ &= \frac{1}{2\pi i}\left\{\int_{\beta_k+i\tau_k}^{\sigma_k+i\tau_k} + \int_{\sigma_k+i\tau_k}^{\sigma_k+iT} + \int_{\sigma_k+iT}^{\beta_k+iT} + \int_{\beta_k+iT}^{\beta_k+i\tau_k}\right\}\frac{d}{ds}\log(\eta^{(\alpha)})ds \\ &= \frac{1}{2\pi i}\{I_1 + I_2 + I_3 + I_4\}, \end{aligned}$$

say. We examine I_1, I_2, I_3 and I_4 in turn.

Firstly, I_1 is independent of T. Hence, $I_1 = \mathcal{O}(1)$. Secondly,

$$\begin{aligned} I_2 &= \left[\log(\eta^{(k)}(s))\right]_{\sigma_k+i\tau_k}^{\sigma_k+iT} \\ &= \left[\log\left(\frac{(-1)^{k+1}\log 2}{2^s}\right)\right]_{\sigma_k+i\tau_k}^{\sigma_k+iT} + \left[\log(1+g(s))\right]_{\sigma_k+i\tau_k}^{\sigma_k+iT} \end{aligned}$$

where,

$$g(s) = \sum_{n=3}^{\infty}\frac{(-1)^{n-2}(\log(n)/\log 2)^k}{(n/2)^s}$$

By, (4.33) we see that on the line $\sigma = \sigma_k$, $|g(s)| \leq \frac{1}{2}$. Hence, $\Re\{1+g(s)\} \geq \frac{1}{2}$ and the argument of $1+g(s)$ ranges over an interval length no greater than π as s traverses the line $\sigma = \sigma_k$. Hence,

$$I_2 = -iT\log 2 + \mathcal{O}(1).$$

To estimate I_3 we put

$$\phi_k(s) = (-1)^k e^{iT\log 2}\eta^{(k)}(s).$$

Hence, the leading term of the Dirichlet series for $\phi_k(s)$ is positive at $s = \sigma_k + iT$. Now, if q denotes the number of zeros of $\Re\{\cdot\}$ on $J = (\beta_k + iT, \sigma_k + iT)$, it is divided into at most $q+1$ subintervals in each of which $\Re\{\cdot\}$ is of constant sign. Hence, the variation of $\Im\{\log(\phi_k(s)\} = \arg\{\phi_k(s)\}$ is at most π in each subinterval.

$$\Im\{I_3\} = |\Im\{[\log(\phi_k(s)]_{\sigma_k+iT}^{\beta_k+iT}\}| \leq (q+1)\pi.$$

To estimate q we first let

$$f(z) = \frac{1}{2}\{\phi_\sigma(z+iT) + \overline{\phi_\sigma(\overline{z}+iT)}\}$$

and not that if $z=\sigma$ is real,

$$f(\sigma) = \Re\{\phi_\sigma(\sigma+iT)\}$$

. Choose T so large that

$$T > \tau_k + 2(\sigma_k - \beta_k).$$

Let $R = T - \tau_k$ and consider those z with $|z-\sigma_k| < R$. Then,

$$\Im\{z+iT\} > T - R = \tau_k > 0.$$

Thus, $\phi_\sigma(z+iT)$, and hence $f(z)$, is analytic for $|z-\sigma_k| < R$. Let $n(\rho)$ denote the number of zeros of $f(z)$ in the circle $|z-\sigma_k| \leq \rho$. If $r = 2(\sigma_k - \beta_k)$ and $r_1 = \frac{1}{2}r$, we have

$$\int_0^r \frac{n(\rho)}{\rho} d\rho \geq n(r_1) \int_{r_1}^r \frac{d\rho}{\rho} = n(r_1)\log 2$$

From Jensen's theorem 2.15, we have

$$n(r_1) \leq \frac{1}{2\pi\log 2}\int_0^{2\pi} \log|f(re^{i\theta}+\sigma_k)|d\theta - \frac{1}{\log 2}\log|f(\sigma_k)|. \tag{4.35}$$

Since $\eta^{(\alpha)}(s) = \mathcal{O}(t^A)$ as t tends to ∞ and

$$\begin{aligned} f(\sigma_k) &= \Re\{\phi_k(\sigma_k+iT)\} \\ &= \Re\left\{\frac{\log^k(2)}{2^{\sigma_k}} + \frac{-1}{2^{\sigma_k}}\sum_{n=3}^{\infty}\frac{(-1)^{n-1}\log^k(n)}{(n/2)^{\sigma_k+iT}}\right\} \\ &\geq \frac{\log^k(2)}{2^{\sigma_k}} + \frac{-1}{2^{\sigma_k}}\sum_{n=3}^{\infty}\frac{(-1)^{n-1}\log^k(n)}{(n/2)^{\sigma_k}} \\ &\geq \frac{1}{2}\frac{\log^k(2)}{2^{\sigma_k}} \end{aligned}$$

by (4.33), it follows from (4.35) that $n(r_1) = \mathcal{O}(\log T)$. Now, the zeros of $\Re\{\sigma_k+iT\}$ on J correspond to an equal number of zeros of $f(z)$ on (β_k,σ_k). since $r_1 = \sigma_k - \beta_k$, (β_k,σ_k) is contain in the disc $|z-\sigma_k| \leq r_1$. Hence $q \leq n(r_1)$ and

$$\Im\{I_3\} = \mathcal{O}(\log T).$$

Lastly,

$$I_4 = [\log(\eta^{(k)}(s)]_{\beta_k+iT}^{\beta_k+i\tau_k}$$

From the functional equation for $\eta(s)$ and Leibniz' Rule,

$$\begin{aligned}\eta^{(k)}(s) &= \left\{(2-2^s)\pi^{s-1}(-s)\sin\left(-\frac{s\pi}{2}\right)\Gamma(-s)\zeta(1-s)\right\}^{(k)} \\ &= \left\{(2-2^s)\pi^{s-1}s\sin\left(\frac{s\pi}{2}\right)\Gamma(-s)\right\}^{(k)}\zeta(1-s) \\ &+ \sum_{j=0}^{k-1}\binom{k}{j}\left\{(2-2^s)\pi^{s-1}s\sin\left(\frac{s\pi}{2}\right)\Gamma(-s)\right\}^{(j)}\zeta^{(k-j)}(1-s). \qquad (4.36)\end{aligned}$$

We set

$$\mu_k = \left\{(2-2^s)\pi^{s-1}(-s)\sin\left(-\frac{s\pi}{2}\right)\Gamma(-s)\right\}^{(k)}$$

and

$$\mu_{k-j} = \sum_{j=0}^{k-1}\binom{k}{j}\left\{(2-2^s)\pi^{s-1}s\sin\left(\frac{s\pi}{2}\right)\Gamma(-s)\right\}^{(j)}$$

Therefore from (4.36) we have for the first derivative that

$$\begin{aligned}\eta^{(k)}(s) &= \mu_k\zeta(1-s) + \mu_{k-j}\zeta'(1-s) \\ &= \mu_k + \mu_k(\zeta(1-s)-1) + \mu_{k-j}\zeta^{(k)}(1-s).\end{aligned}$$

Some important things to note are for sufficiently negative values s

$$\left|\frac{1}{2-2^s}\right| \leq 1$$

and

$$\left|\frac{1}{s}\right| \leq 1.$$

We look at the following derivatives

$$\{\pi^s\}^{(j)} = \log(\pi)^j(\pi)^s \qquad (4.37)$$

and

$$\left(\sin\left(\frac{s\pi}{2}\right)\right)^{(j)} = \pm(\frac{\pi}{2})^j\left\{\begin{smallmatrix}\sin(\frac{s\pi}{2})\\ \cos(\frac{s\pi}{2})\end{smallmatrix}\right\}, \qquad (4.38)$$

depending upon whether j is even or odd. Also, by Stirling's formula for $\Gamma(s)$, it is easily seen that

$$\Gamma^{(j)}(s) = \Gamma(s)\left\{\log^j(s) + \sum_{n=0}^{j-1} E_{nj}(s) log^n(s)\right\}, \tag{4.39}$$

where $E_{nj(s)} = \mathcal{O}(1/s)$. From (4.37)-(4.39) we see that we can write (4.36) in the form

$$\eta^{(k)}(s) = (2-2^s)(\pi)^s e^{\frac{is\pi}{2}}\Gamma(-s)\{R_1(s) + R_2(s)\}, \tag{4.40}$$

where

$$R_1(s) = \frac{\mu_k}{(2-2^s)(\pi)^s e^{\frac{is\pi}{2}}\Gamma(-s)}$$

and

$$R_2(s) = \frac{\mu_k(\zeta(1-s)-1) + \mu_{k-j}\zeta^k(s-1)}{(2-2^s)(\pi)^s e^{\frac{is\pi}{2}}\Gamma(-s)}$$

$R_1(s)$ and $R_2(s)$ are finite sums, each term of which is $\mathcal{O}(\log^k(s))$ on $J' = (\beta_k + iT, \beta_k + i\tau_k)$. Since (4.39),

$$\begin{aligned}&\left\{(2-2^s)\pi^{s-1}s\sin\left(\frac{s\pi}{2}\right)\Gamma(-s)\right\}^{(k)}\\ &= \sum_{j=0}^{k}\binom{k}{j}\left\{(2-2^s)\pi^{s-1}s\sin\left(\frac{s\pi}{2}\right)\right\}^{(k-j)}\Gamma(-s)\left\{\log^j(-s) + \sum_{n=0}^{j-1} E_{nj}(s)\log^n(-s)\right\}\end{aligned}$$

where $E_{nj(s)} = \mathcal{O}(1/s)$, we see from (4.37), (4.38), and (4.40) that for β_k sufficiently negative, $R_1(s)$ is dominated by $\log^k(-s)$ and is bounded away from zero. Also, choose β_k sufficiently negative so that (4.34) holds and so that

$$\begin{aligned}\left|\frac{R_2(s)}{R_1(s)}\right| &= \left|\frac{\mu_k(\zeta(1-s)-1) + \mu_{k-j}\zeta^k(s-1)}{(2-2^s)(\pi)^s e^{\frac{is\pi}{2}}\Gamma(-s)} \div \frac{\mu_k}{(2-2^s)(\pi)^s e^{\frac{is\pi}{2}}\Gamma(-s)}\right|\\ &= \left|\frac{\mu_k(\zeta(1-s)-1) + \mu_{k-j}\zeta^{(k)}(s-1)}{\mu_k}\right|\\ &= \left|\zeta(1-s) - 1 + \frac{\mu_{k-j}}{\mu_k}\zeta^{(k)}(1-s)\right|\\ &< 1,\end{aligned}$$

where s belongs to the segment J'. Thus $\arg\left\{1+\frac{R_2(s)}{R_1(s)}\right\}$ varies over an interval of length no greater that π as s traverses J'. Hence, from (4.40),

$$\begin{aligned}I_4 = &\left[\log(2-2^s)+\log(\pi)^s+\log(s)-\frac{is\pi}{2}+\log(\Gamma(-s))\right.\\ &\left.+\log(R_1(s))+\log\left\{1+\frac{R_2(s)}{R_1(s)}\right\}\right]_{\beta_k+iT}^{\beta_k+i\tau_k}\\ &=\mathcal{O}(1)-iT\log(\pi)+\mathcal{O}(1)+\mathcal{O}(\log T)-\frac{T\pi}{2}+\mathcal{O}(1)\\ &+\left[-(s+\frac{1}{2})\log(-s)+s+\mathcal{O}(1)\right]_{\beta_k+iT}^{\beta_k+i\tau_k}\\ &+\mathcal{O}\{\log(\log T)\}+\mathcal{O}(1),\end{aligned}$$

upon use of Stirling's formula for $\log(\Gamma(s)$. Now,

$$\begin{aligned}(\alpha_k+\frac{1}{2}+iT)\log(-\alpha_k-iT) &= (\alpha_k+\frac{1}{2}+iT)\log(-iT(1-\alpha_k)/iT)\\ &=(\alpha_k+\frac{1}{2}+iT)\log(-iT)+\mathcal{O}(1)\\ &=iT\log T+\frac{1}{2}\pi T+\mathcal{O}(log(T)).\end{aligned}$$

Thus.

$$I_4 = iT(\log T-\log(\pi)-1)+\mathcal{O}(\log T)$$

Hence,

$$\begin{aligned}N_\eta^k(T) &= \frac{1}{2\pi}\sum_{j=1}^{4}\Im\{I_j\}\\ &=\frac{1}{2\pi}\{T\log 2+T(\log T-\log(\pi)-1)\}+\mathcal{O}(\log T)\\ &=N_\zeta^k(T)+\frac{T\log 2}{2\pi}+\mathcal{O}(\log T)\\ &=N_\zeta(T)+\mathcal{O}(\log T).\end{aligned}$$

□

We arrive at the next conjecture from Figure 4.1.

Conjecture 4.19. *For $\alpha\in\mathbb{R}^+$. As $T\to\infty$, we have*

$$N_\eta^\alpha(T)=N_\zeta(T)+\mathcal{O}(\log T)$$

Chapter 5: Dirichlet L-Functions

5.1 Introduction

Our focus now shifts to Dirichlet L-functions, $L(s, \chi)$, a class of mathematical functions named after Peter Gustav Lejeune Dirichlet. Introduced in 1837, through his seminal work [Dir37], Dirichlet's Theorem 5.1 on primes in arithmetic progressions. The use of these functions marked a significant breakthrough in mathematical analysis of algebraic problems, thereby laying the foundation for this important branch of analytic number theory.

Theorem 5.1 (Dirichlet Theorem)**.** *Let $a, m \in \mathbb{Z}$, with $\gcd(a, m) = 1$. Then there are infinitely many prime numbers in the sequence of integers $a, a+m, a+2m, \ldots, a+km, \ldots$ for $k \in \mathbb{N}$.*

In the course of the proof, Dirichlet shows that $L(s, \chi)$ is non-zero at $s = 1$. Otherwise, the L function is entire.

5.1.1 Characters

In the following we give definitions and theorems, from [Apo98], that set the foundation of Dirichlet L-functions.

Definition 5.2. If $n \geq 1$ the Euler totient $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n; thus,

$$\varphi(n) = \sum_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} 1.$$

There is a simple formula for the divisor sum, namely

Lemma 5.3. *If $n \geq 1$ we have*

$$\sum_{d \mid n} \varphi(d) = n.$$

Definition 5.4. Let G be an arbitrary group. A complex-valued function f defined on G is called a character of G if f has the multiplicative property

$$f(ab) = f(a)f(b)$$

for all a, b in G, and if $f(c) \neq 0$ for some c in G.

Lemma 5.5. *If f is a character of a finite group G with identity element e, then $f(e) = 1$ and each function value $f(a)$ is a root of unity. In fact, if $a^n = e$ then $f(a)^n = 1$.*

Let G be a finite abelian group of order n. The principal character of G is denoted by f_1. The others, denoted by $f_2, f_3, \dots, f_n$ are called non-principal character. They have the property that $f(a) \neq 1$ for some $a \in G$.

Lemma 5.6. *If multiplication of characters is defined by the relation*

$$(f_i f_j)(a) = f_i(a)f_j(a)$$

for each $a \in G$, then the set of characters of G forms an abelian group of order n. We denote this group by $\hat{G}$. The identity element of $\hat{G}$ is the principal character f_1. The inverse of f_i. The inverse of f_i is the reciprocal $1/f_i$.

Definition 5.7. Let G be a finite abelian group of order n with elements $a_1, a_2, \dots, a_n$, and let $f_1, f_2, \dots, f_n$ be the characters of G, with f_1 the principal character. We denote by $A = A(G)$ the $n \times n$ matrix $[a_{ij}]$ whose element a_{ij} in the i-th row and j-th column is

$$a_{ij} = f_i(a_j).$$

Lemma 5.8. *The sum of the entries in the ith row of A is given by*

$$\sum_{r=1}^{n} f_i(a_r) = \begin{cases} n & \text{if } f_i \text{ is the principal character } (i = 1) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 5.9. *The sum of the entries in the jth column of A is given by*

$$\sum_{r=1}^{n} f_r(a_j) = \begin{cases} n & \text{if } a_j = e \\ 0 & \text{otherwise} \end{cases}$$

We want to establish that a reduced residue system modulo k is a set of $\varphi(k)$ integers $\{a_1, a_2, \dots, a_{\varphi(k)}\}$ incongruent modulo k, each of which is relatively prime to k. For each integer a the corresponding residue class $\hat{a}$ is the set of all integers congruent to a modulo k :

$$\hat{a} = \{x | x \equiv a \pmod{k}\}.$$

Multiplication of residue classes is defined by the relation

$$\widehat{a} \cdot \widehat{b} = \widehat{ab} \tag{5.1}$$

We now introduce the important notion of a character.

Definition 5.10 (Dirichlet characters)**.** Let G be the group of reduced residue classes modulo k. Corresponding to each character f of G we define an arithmetical function $\chi = \chi_f$ as follows

$$\chi(n) = f(\widehat{n}) \quad \text{if} \quad \gcd(n,k) = 1,$$
$$\chi(n) = 0 \quad \text{if} \quad \gcd(n,k) > 1$$

Where $\hat{n} = \{x | x \equiv n(\mod k)\}$. The function χ is called a Dirichlet character modulo k. The principal character χ_1 is that which has the property

$$\chi_1(n) = \begin{cases} 1 & \text{if} \gcd(n,k) = 1 \\ 0 & \text{if} \gcd(n,k) > 1 \end{cases}$$

Lemma 5.11. *There are $\varphi(k)$ distinct Dirichlet characters modulo k, each of which is completely multiplicative and periodic with period k. That is we have*

$$\chi(mn) = \chi(m)\chi(n) \quad \textit{for all } m, n \tag{5.2}$$

and

$$\chi(n + k) = \chi(n) \quad \textit{for all } n.$$

Conversely, if χ is completely multiplicative and periodic with period k, and if $\chi(n) = 0$ if $\gcd(n,k) > 1$, then χ is one of the Dirichlet characters mod k.

We use the fact that $\chi(n)^{\varphi(k)} = 1$ whenever $\gcd(n,k) = 1$, so $\chi(n)$ is a $\varphi(k)$-th root of unity. We also note that if χ is a character mod k so is the complex conjugate. This information suffice to show the following table for all Dirichlet character mod 5 and 7

Definition 5.12 ([Spi69])**.** We say a character $\chi \mod k$ is imprimitive if there is a proper divisor K of k such that if $a \equiv b \pmod{k}$ and $\gcd(a,k) = \gcd(b,k) = 1$, then $\chi(a) = \chi(b)$; otherwise, the character is call primitive. Such a number K is called a modulus of imprimitivity.

Corollary 5.13 ([Spi69])**.** *If $k \equiv 2 \pmod{4}$, then there are no primitive characters* mod *k.*

For primitive Dirichlet's characters modulus q, in other words $q \geq 3$ and $q \not\equiv 2 \pmod 4$, we define:

$$\mathfrak{a} = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1, \\ 1 & \text{if} \quad \chi(-1) = -1, \end{cases} \tag{5.3}$$

It is not difficult to list all characters modulo 3, 4, 5, 6 and 7, see Tables 5.1, 5.2, 5.3, 5.4 and 5.5.

n	1	2	3	$\mathfrak{a}$
$\chi_1(n)$	1	1	0	0
$\chi_2(n)$	1	-1	0	1

Table 5.1. All $\varphi(3) = 2$ characters modulo 3

n	1	2	3	4	$\mathfrak{a}$
$\chi_1(n)$	1	0	1	0	0
$\chi_2(n)$	1	0	-1	0	1

Table 5.2. All $\varphi(4) = 2$ characters modulo 4

n	1	2	3	4	5	$\mathfrak{a}$
$\chi_1(n)$	1	1	1	1	0	0
$\chi_2(n)$	1	-1	-1	1	0	0
$\chi_3(n)$	1	i	-i	-1	0	1
$\chi_4(n)$	1	-i	i	-1	0	1

Table 5.3. All $\varphi(5) = 4$ characters modulo 5

5.1.2 Dirichlet L-Functions

For $s = \sigma + it$ the Dirichlet L-function is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

n	1	2	3	4	5	6	$\mathfrak{a}$
$\chi_1(n)$	1	0	0	0	1	0	0
$\chi_2(n)$	1	0	0	0	-1	0	1

Table 5.4. All $\varphi(6) = 2$ characters modulo 6

n	1	2	3	4	5	6	7	$\mathfrak{a}$
$\chi_1(n)$	1	1	1	1	1	1	0	0
$\chi_2(n)$	1	1	-1	1	-1	-1	0	1
$\chi_3(n)$	1	ω^2	ω	$-\omega$	$-\omega^2$	-1	0	1
$\chi_4(n)$	1	ω^2	$-\omega$	$-\omega$	ω^2	1	0	0
$\chi_5(n)$	1	$-\omega$	ω^2	ω^2	$-\omega$	1	0	0
$\chi_6(n)$	1	$-\omega$	$-\omega^2$	ω^2	ω	-1	0	1

Table 5.5. All $\varphi(7) = 6$ characters modulo 7 where $\omega = e^{\pi i/3}$.

where χ is any Dirichlet character. The Euler product is

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad \Re(s) > 1$$

where the product is over all prime numbers p.

For primitive χ modulo $q > 1$ its functional equation

$$L(s,\chi) = \varepsilon(\chi) 2^s \pi^{s-1} q^{1/2-s} \sin\left(\frac{\pi(s+\mathfrak{a})}{2}\right) \Gamma(1-s) L(1-s,\overline{\chi})$$

where $\mathfrak{a}$ is as in (5.3) and

$$\varepsilon(\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}$$

where $\tau(\chi)$ is a Gauss sum, namely

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) \exp(2\pi i n/q).$$

It is important to note that only $L(s,\chi)$, where χ is a principal character, have a pole at $s = 1$. Now that have summarized the theory of Dirichlet L-functions we can investigate them further.

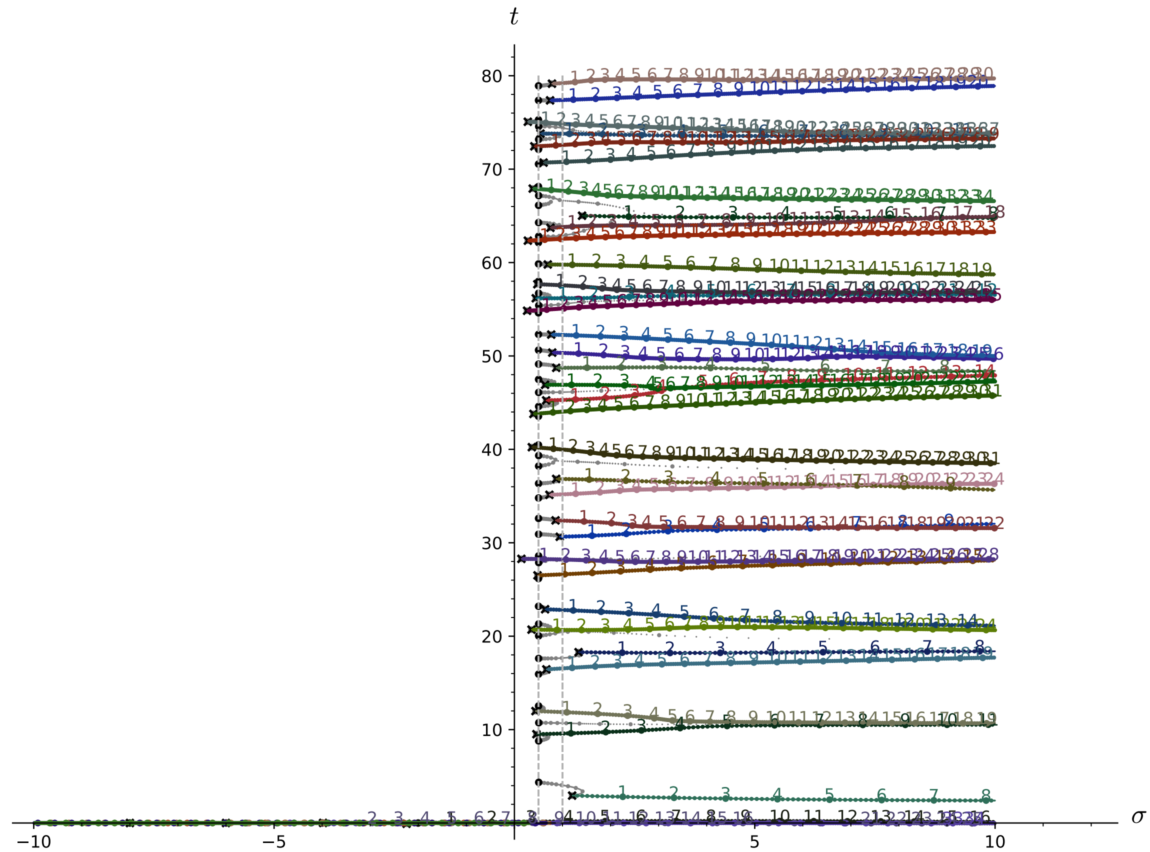

Figure 5.1. • Zero of $L(s,\chi_2)$, × Zero of $L(s,\chi_2)-1$, $\bullet^{(k)}$ Zero of $L^{(k)}(s,\chi_2)$, χ_2 : $[0,1,\omega_6^4,\omega_6^2,\omega_6^2,\omega_6^4,1]$

5.2 Evaluation

We apply Euler-Maclaurin summation (2.2) in order to evaluate Dirichlet L-functions.

Let $q\in\mathbb{N}$. For $1\le r\le q$ define

$$\zeta(s,r,q)=\sum_{n=0}^{\infty}\frac{1}{(q\cdot n+r)^s}$$

Then

$$\zeta^{(\alpha)}(s,r,q)=\sum_{n=0}^{\infty}\frac{\log^{\alpha}(n\cdot q+r)}{(n\cdot q+r)^s}$$

Let χ be a Dirichlet character of modulo q. Then

$$L(s,\chi)=\sum_{r=1}^{q}\chi(r)\zeta(s,r,q)$$

and thus

$$L^{(\alpha)}(s,\chi) = \sum_{r=1}^{q} \chi(r)\zeta^{(\alpha)}(s,r,q)$$

We apply Euler-Maclaurin formula to the double sum. Let $g(x) = \frac{\log^{\alpha}(x\cdot q+j)}{(x\cdot q+r)^s}$, then

$$\begin{aligned}
L^{(\alpha)}(s,\chi) &= \sum_{r=1}^{q} \chi(r)\zeta^{(\alpha)}(s,r,q) \\
&= \sum_{r=1}^{q} \chi(r) \sum_{n=0}^{\infty} g(k) \\
&= \sum_{r=0}^{q} \chi(r) \Bigg(\sum_{n=1}^{N-1} g(n) + \int_N^{\infty} g(x)dx + \frac{1}{2} g(N) - \sum_{j=1}^{\lfloor v/2 \rfloor} \frac{B_{2j}}{(2j)!} g^{2j-1}(N) \\
&+ (-1)^{v+1} \int_N^{\infty} P_v(x) g^{(v)}(x)dx \Bigg)
\end{aligned} \tag{5.4}$$

B_k denotes the k-th Bernoulli number, and $P_k(x) = \frac{B_k(x-\lfloor x \rfloor)}{k!}$ Now let us evaluate each term. The first term is

$$A_s^{\alpha}(N) = \sum_{r=1}^{q} \chi(r) \sum_{n=1}^{N-1} g(n) = \sum_{r=1}^{q} \chi(r) \sum_{n=1}^{N-1} \frac{\log^{\alpha}(n\cdot q+r)}{(n\cdot q+r)^s}.$$

For $s \neq 1$ the second term is

$$\begin{aligned}
I_s^{\alpha}(N) &= \sum_{r=1}^{q} \chi(r) \int_N^{\infty} g(x)dx \\
&= \sum_{r=1}^{q} \chi(j) \int_N^{\infty} \frac{\log^{\alpha}(x\cdot q+r)}{(x\cdot q+r)^s} dx \\
&= \sum_{r=1}^{q} \frac{\chi(r)}{q} \int_{N\cdot q+r}^{\infty} \frac{\log^{\alpha}(y)}{(y)^s} dy \quad \text{substitute } y = x\cdot q+r \\
&= \sum_{r=1}^{q} \frac{\chi(r)}{q} \frac{\Gamma(\alpha+1,(s-1)\log(N\cdot q+r)}{(s-1)^{\alpha+1}}.
\end{aligned}$$

Using (5.8) we see that for $s=1$ and nonprincipal χ we have with substitute $y = x\cdot q+r$:

$$
\begin{aligned}
I_1^{\alpha}(N) &= \sum_{r=1}^{q} \chi(r) \int_N^{\infty} g(x)dx \\
&= \sum_{r=1}^{q} \chi(r) \int_N^{\infty} \frac{\log^{\alpha}(x \cdot q + r)}{(x \cdot q + r)} dx \\
&= \sum_{r=1}^{q} \frac{\chi(r)}{q} \left(\int_{N \cdot q + r}^{q(N+1)} \frac{\log^{\alpha}(y)}{y} dy + \int_{q(N+1)}^{\infty} \frac{\log^{\alpha}(y)}{y} dy \right) \\
&= \sum_{r=1}^{q} \frac{\chi(r)}{q} \int_{N \cdot q + r}^{q(N+1)} \frac{\log^{\alpha}(y)}{y} dy + \sum_{r=1}^{q} \frac{\chi(r)}{q} \int_{q(N+1)}^{\infty} \frac{\log^{\alpha}(y)}{y} dy \\
&= \sum_{r=1}^{q} \frac{\chi(r)}{q} \int_{N \cdot q + r}^{q(N+1)} \frac{\log^{\alpha}(y)}{y} dy \\
&= \sum_{r=1}^{q} \frac{\chi(r)}{q} \left(\frac{\log^{\alpha+1}(q(N+1)}{\alpha+1} - \frac{\log^{\alpha+1}(N \cdot q + r)}{\alpha+1} \right) \\
&= \sum_{r=1}^{q} \frac{\chi(r)}{q} \frac{\log^{\alpha+1}(q(N+1))}{\alpha+1} - \sum_{r=1}^{q} \frac{\chi(r)}{q} \frac{\log^{\alpha+1}(N \cdot q + r)}{\alpha+1} \\
&= -\sum_{r=1}^{q} \frac{\chi(r)}{q} \frac{\log^{\alpha+1}(N \cdot q + r)}{\alpha+1}.
\end{aligned}
$$

And

$$
C_s^{\alpha}(N) = \sum_{r=1}^{q} \chi(r) \frac{1}{2} g(N) = \frac{1}{2} \sum_{r=1}^{q} \chi(r) \frac{\log^{\alpha}(N \cdot q + r)}{(N \cdot q + r)}.
$$

Then for,

$$
\begin{aligned}
B_s^{\alpha}(N, v) &= -\sum_{r=1}^{q} \chi(r) \sum_{j=1}^{\lfloor v/2 \rfloor} \frac{B_{2j}}{(2j)!} g^{(j-1)}(N) \\
&= -\sum_{r=1}^{q} \chi(r) \sum_{j=1}^{\lfloor v/2 \rfloor} \frac{B_{2j}}{(2j)!} \sum_{i=1}^{2j-1} (m)^i S(2j-1, i, s)(\alpha)_i \frac{\log^{\alpha-i}(x \cdot q + r)}{(x \cdot q + r)^{s+2j-1}}.
\end{aligned}
$$

Now we determine a bound for the fifth term of the (5.4). Writing $s = \sigma + it$ and

$$
E_s^{\alpha}(N, v) = \sum_{r=1}^{q} \chi(r)(-1)^{v+1} \int_N^{\infty} P_v(x) g^{(v)}(x) dx
$$

we obtain,

$$
\begin{aligned}
|E(N,v)| &= \left|\sum_{r=1}^{q}\chi(r)(-1)^{v+1}\int_N^\infty P_v(x)g^{(v)}(x)dx\right| \\
&= \sum_{r=1}^{q}\left|\frac{1}{v!}\int_N^\infty B_v(x-\lfloor x\rfloor)g^{(v)}(x)dx\right| \\
&\leq \sum_{r=1}^{q}\frac{|B_v|}{v!}\int_N^\infty |g^{(v)}(x)|dx \\
&\leq \sum_{r=1}^{q}\frac{|B_v|}{v!}\sum_{i=0}^{v}\int_N^\infty \left|S(v,i,s)(\alpha)_i\frac{\log^{\alpha-i}(x\cdot q+r)}{(x\cdot q+r)^{s+v}}\right|dx \\
&\leq \sum_{r=1}^{q}\frac{|B_v|}{v!}\left(\sum_{i=0}^{v}|S(v,i,s)(\alpha)_i|\right)\left(\int_N^\infty \frac{\log^{\alpha}(x\cdot q+r)}{(x\cdot q+r)^{\sigma+v}}dx\right) \\
&= \sum_{r=1}^{q}\frac{|B_v|}{v!}\left(\sum_{i=0}^{v}|S(v,i,s)(\alpha)_i|\right)\left(\frac{\Gamma(\alpha+1,(\sigma+v-1)\log(N\cdot q+r)}{(\sigma+v-1)^{\alpha+1}}\right)
\end{aligned}
$$

The error term $E_s^\alpha(N,v)$ converges for $\sigma+v>1$ and $q>2$. Therefore for all $s\in\mathbb{C}(\infty,1]$ we can choose $q\in\mathbf{N}$ and $v\in\mathbb{N}$ such that $|E_s^\alpha(N,v)|$ becomes arbitrarily small. We can thus approximate $L^{(\alpha)}(s,\chi)$ as

$$
L^{(\alpha)}(s,\chi)\approx(-1)^\alpha\left(\sum_{r=2}^{q-1}\chi(r)\frac{\log^\alpha(r)}{r^s}+A_s^\alpha(N)+I_s^\alpha(N)+C_s^\alpha(N)+E_s^\alpha(N,v)\right)
$$

5.2.1 Skorokhodov connectors

The ideas from Sections 3.2.1 and 4.2.1 can be extended to Dirichlet L-functions. The following results give us zero-free regions for $L(s,\chi)-c$ where $c\in[0,1)$.

Proposition 5.14. *If $c\in[0,1)$,$m\in\mathbb{Z}$, for $k\in[1,n]$ and $t=\frac{2m\pi}{\log k}$, then $\Re(L(s,\chi)-c)\neq 0$, when*

$$
1+\frac{1}{1+\frac{\chi(k)}{k^\sigma}}>\zeta(\sigma)
$$

Proof. $\Re(L(s,\chi)-c)=1-c+\frac{\chi(2)}{2^\sigma}\cos(t\log 2)+\frac{\chi(3)}{3^\sigma}\cos(t\log 3)+\cdots+\frac{\chi(n)}{n^\sigma}\cos(t\log n)\cdots$ Where t is fixed and $t\log k=2m\pi$, we get:

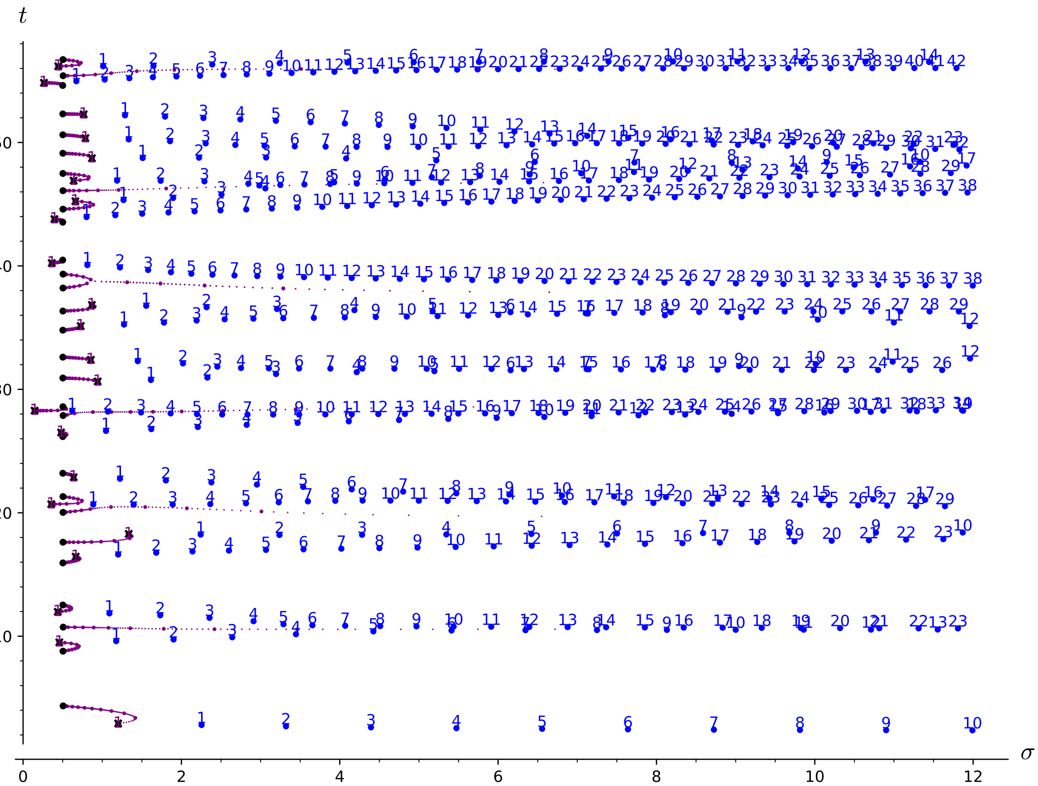

Figure 5.2. Zeros of derivatives of $L^{(\alpha)}(s,\chi)$ (denoted by $\bullet^{k}$) and the paths from zeros of $L(s,\chi)$ (denoted by $\bullet$) to the zeros of $L(s,\chi)-1$ (denoted by $\times$).

$$
\begin{aligned}
\Re(L(s,\chi)-c) &= 1-c+\frac{\chi(2)\cos\left(\frac{\log 2}{\log k}2m\pi\right)}{2^{\sigma}}+\cdots+\frac{\chi(k)}{k^{\sigma}}+\cdots \\
&\quad+\frac{\chi(n)\cos\left(\frac{\log(n)}{\log k}2m\pi\right)}{n^{\sigma}}+\cdots \\
&\geq \sum_{\nu=1}^{\infty}\frac{\chi(k)}{(k^{\nu})^{\sigma}}-\Re\left(L(s,\chi)-1-\sum_{\nu=1}^{\infty}\frac{\chi(k)}{(k^{\nu})^{\sigma}}\right) \\
&\geq 1+2\sum_{\nu=1}^{\infty}\left(\frac{\chi(k)}{k^{\sigma}}\right)^{\nu}-\Re(L(s,\chi)) \\
&\geq 1+\frac{2}{1-\frac{\chi(k)}{k^{\sigma}}}-\zeta(\sigma)
\end{aligned}
$$

□

Lemma 5.15. *If $c \in [0,1)$,$k, m \in \mathbb{Z}$ and $t = \frac{2m\pi}{\log k}$ and χ_1 is principal, then $\Re(L(s,\chi) - c) \neq 0$, when*

$$1 + \frac{2}{1 + \frac{\chi_1(k)}{k^\sigma}} \geq L(\sigma, \chi_1)$$

Proof. $\Re(L(s,\chi_1) - c) = 1 - c + \frac{\chi_1(2)}{2^\sigma}\cos(t\log 2) + \frac{\chi_1(3)}{3^\sigma}\cos(t\log 3) + \cdots + \frac{\chi_1(n)}{n^\sigma}\cos(t\log n)\cdots$ Where t is fixed and $t\log k = 2m\pi$, we get:

$$\begin{aligned}
\Re(L(s,\chi_1) - c) &= 1 - c + \frac{\chi_1(2)\cos\left(\frac{\log 2}{\log k}2m\pi\right)}{2^\sigma} + \cdots + \frac{\chi_1(k)}{k^\sigma} + \cdots \\
&+ \frac{\chi_1(n)\cos\left(\frac{\log(n)}{\log k}2m\pi\right)}{n^\sigma} + \cdots \\
&\geq \sum_{\nu=1}^{\infty} \frac{\chi_1(k)}{(k^\nu)^\sigma} - \Re\left(L(s,\chi_1) - 1 - \sum_{\nu=1}^{\infty} \frac{\chi_1(k)}{(k^\nu)^\sigma}\right) \\
&\geq 1 + 2\sum_{\nu=1}^{\infty}\left(\frac{\chi(k)}{k^\sigma}\right)^\nu - \Re(L(s,\chi_1)) \\
&\geq 1 + \frac{2}{1 - \frac{\chi_1(k)}{k^\sigma}} - L(\sigma,\chi_1),
\end{aligned}$$

where the last inequality holds because χ_1 is a principal character. □

5.3 Laurent Series Expansion

Because L-functions can be analytically continued to the whole complex plane, except for the pole at 1 for principle characters, we can express the L-function as a Laurent series. Let $\gamma_n(\chi)$ denote the n-th Laurent-Stieltjes coefficients around $s = 1$ of the associated Dirichlet L-series for a given primitive Dirichlet character χ modulo q. These constants are defined by

$$L(s,\chi) = \frac{\delta_\chi}{s-1} + \sum_{n\geq 0} \frac{(-1)^n \gamma_n(\chi)}{n!}(s-1)^n,$$

where $\delta_\chi = 1$ when χ is principal and $\delta_\chi = 0$ otherwise. We may regard $\zeta(s)$ as the Dirichlet L-functions to the principal character χ_0 modulo 1. Then, we have $\gamma_n(\chi_0) = \gamma_n$. When χ is non-principal, $(-1)^n\gamma_n(\chi) = L^{(n)}(1,\chi)$.

Recall that for $q \in \mathbb{N}$ and $1 \leq r \leq q$ and $\alpha \in \mathbb{R}^+$ we have

$$\zeta(s,r,q) = \sum_{n=0}^{\infty} \frac{1}{(q\cdot n + r)^s} \quad \text{and} \quad \zeta^{(\alpha)}(s,r,q) = \sum_{n=0}^{\infty} \frac{\log^\alpha(n\cdot q + r)}{(n\cdot q + r)^s}$$

For $\alpha \in \mathbb{R}^+$ we define the fractional Stieltjes constants $\gamma_\alpha(r,q)$ to be the coefficients of the Laurent expansion of the Grünwald-Letnikov fractional derivative of $\zeta^{(\alpha)}(s,r,q)$:

$$\zeta^{(\alpha)}(s,r,q) = \frac{\delta_\alpha(r,q)}{(s-1)^{\alpha-1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(r,q)(s-1)^n}{n!}$$

Also see Knopfmacher [Kno78].

Let χ be a Dirichlet character of modulo q. Then

$$L(s,\chi) = \sum_{r=1}^{q} \chi(r)\zeta(s,r,q) \quad \text{and} \quad L^{(\alpha)}(s,\chi) = \sum_{r=1}^{q} \chi(r)\zeta^{(\alpha)}(s,r,q).$$

If we denote the coefficients of the Laurent expansion $L^{(\alpha)}(s,\chi)$ by $\gamma_\alpha(\chi)$ and $\delta_\alpha(\chi)$ such that for complex numbers $s \neq 1$

$$L^{(\alpha)}(s,\chi) = \frac{\delta_\alpha(\chi)}{(s-1)^{\alpha-1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(\chi)(s-1)^n}{n!}$$

then

$$\delta_\alpha(\chi) = \sum_{r=1}^{q} \chi(r)\delta_\alpha(r,q) \quad \text{and} \quad \gamma_\alpha(\chi) = \sum_{r=1}^{q} \chi(r)\gamma_\alpha(r,q).$$

Remark 5.16. If χ is principal, then $\delta_\alpha(\chi) = 1$ and if χ is non-principal, $\delta_\alpha(\chi) = 0$.

Theorem 5.17. *Let $\alpha \in \mathbb{R}$ with $\alpha > 0$, $q \in \mathbb{N}$ and $1 \leq r \leq q$. Then*

$$\delta_\alpha(r,q) = \frac{\Gamma(\alpha+1)}{q} \quad \text{and}$$

$$\gamma_\alpha(r,q) = \sum_{k=0}^{m} \frac{\log^\alpha(k\cdot q+r)}{k\cdot q+r} - \frac{\log^{\alpha+1}(m\cdot q+r)}{q(\alpha+1)} - \frac{\log^\alpha(m\cdot q+r)}{2(m\cdot q+r)} + \int_m^\infty P_1(x)g'(x)dx,$$

where $g(x) = \frac{\log^\alpha(q\cdot x+r)}{q\cdot x+r}$ *and* $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

Proof. Recall the following form of the Euler-Maclaurin summation formula 2.2

$$\sum_{k=m}^{n} g(k) = \int_m^n g(x)dx + \sum_{k=1}^{\ell} \frac{(-1)^k B_k}{k!} g^{(k-1)}(x)\Big|_m^n + (-1)^{\ell+1}\int_m^n P_\ell(x)g^{(\ell)}(x)dx, \quad (5.5)$$

where $g(x) \in C^\ell[m,n]$, $\ell \in \mathbb{N}$ and $P_k(x)$ denotes the k^{th} periodic Bernoulli polynomial

$$P_k(x) = \frac{B_k(x - \lfloor x \rfloor)}{k!}.$$

In (2.2) we choose $\ell = 1$ and set $g(x) = \frac{\log^\alpha(x\cdot q+r)}{(x\cdot q+r)^s}$. Letting $n \to \infty$, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\log^\alpha(n\cdot q+r)}{(n\cdot q+r)^s} &= \sum_{n=0}^{m-1} \frac{\log^\alpha(n\cdot q+r)}{(n\cdot q+r)^s} + \int\limits_m^\infty \frac{\log^\alpha(x\cdot q+r)}{(x\cdot q+r)^s}\,dx + \frac{\log^\alpha(m\cdot q+r)}{2(m\cdot q+r)^s} \\
&\quad + \int\limits_m^\infty P_1(x) g'(x)\,dx \\
&= \sum_{n=0}^{m} \frac{\log^\alpha(n\cdot q+r)}{(n\cdot q+r)^s} + \int\limits_m^\infty \frac{\log^\alpha(x\cdot q+r)}{(x\cdot q+r)^s}\,dx - \frac{\log^\alpha(m\cdot q+r)}{2(m\cdot q+r)^s} \\
&\quad + \int\limits_m^\infty P_1(x) g'(x)\,dx \\
&=: A(s) + I(s) - C(s) + E(s).
\end{aligned}$$

We consider each of these four terms separately. For the first term $A(s)$ we have:

$$\begin{aligned}
A(s) &= \sum_{n=0}^{m} \frac{\log^\alpha(n\cdot q+r)}{(n\cdot q+r)^s} \\
&= \sum_{n=0}^{m} \frac{\log^\alpha(n\cdot q+r)}{n\cdot q+r} e^{-(s-1)\log(n\cdot q+r)} \\
&= \sum_{n=0}^{m} \frac{\log^\alpha(n\cdot q+r)}{n\cdot q+r} \sum_{k=0}^{\infty} \frac{(-1)^k \log^k(n\cdot q+r)}{k!}(s-1)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (s-1)^k}{k!} \sum_{n=0}^{m} \frac{\log^{\alpha+k}(n\cdot q+r)}{n\cdot q+r}.
\end{aligned}$$

The second term $I(s)$ can be written in terms of the Upper Incomplete Gamma function $\Gamma(\alpha, s)$ using [GR07, p. 346] or [AS64, 6.5.3]:

$$\begin{aligned}
I(s) &= \int_m^\infty \frac{\log^\alpha(x\cdot q+r)}{(x\cdot q+r)^s} dx \quad y = x\cdot q+r,\ \frac{dy}{dx} = q \\
&= \frac{1}{q}\int_{m\cdot q+r}^\infty \frac{\log^\alpha(y)}{e^{s\log y}} dx \quad u = \log y,\ \frac{du}{dy} = \frac{1}{y} = \frac{1}{e^u} \\
&= \frac{1}{q}\int_{\log(m\cdot q+r)}^\infty \frac{u^\alpha}{e^{u(s-1)}} dt \quad t = u(s-1),\ \frac{dt}{du} = s-1 \\
&= \frac{1}{q(s-1)^{\alpha+1}} \int_{(s-1)\log(m\cdot q+r)}^\infty \frac{t^\alpha}{e^t} dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q(s-1)^{\alpha+1}} \Gamma\left(\alpha+1, (s-1)\log(m\cdot q+r)\right) \\
&= \frac{1}{q(s-1)^{\alpha+1}} \left[\Gamma\left(\alpha+1\right) - \gamma\left(\alpha+1, (s-1)\log(m\cdot q+r)\right)\right] \\
&= \frac{1}{q(s-1)^{\alpha+1}} \Bigg[\Gamma(\alpha+1) \\
&- (s-1)^{\alpha+1}\log^{\alpha+1}(m\cdot q+r)\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n\log^n(m\cdot q+r)}{(\alpha+1+n)n!}\Bigg] \\
&= \frac{\Gamma(\alpha+1)}{q(s-1)^{\alpha+1}} - \log^{\alpha+1}(m\cdot q+r)\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n\log^n(m\cdot q+r)}{q(\alpha+1+n)n!} \\
&= \frac{\Gamma(\alpha+1)}{q(s-1)^{\alpha+1}} - \sum_{n=0}^{\infty}\left(\frac{\log^{\alpha+n+1}(m\cdot q+r)}{q(\alpha+n+1)}\right)\frac{(-1)^n(s-1)^n}{n!}.
\end{aligned}
$$

For the third term $C(s)$, we write:

$$
\begin{aligned}
C(s) &= \frac{\log^{\alpha}(m\cdot q+r)}{2(m\cdot q+r)^s} \\
&= \frac{\log^{\alpha}(m\cdot q+r)}{2(m\cdot q+r)}\sum_{n=0}^{\infty}\frac{(-1)^n\log^n(m\cdot q+r)(s-1)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{\log^{\alpha+n}(m\cdot q+r)}{2(m\cdot q+r)}\right)\frac{(-1)^n(s-1)^n}{n!}.
\end{aligned}
$$

If we define

$$
G_{\alpha,m}(n) := \sum_{k=0}^{m}\frac{\log^{\alpha+n}(k\cdot q+r)}{k\cdot q+r} - \frac{\log^{\alpha+n+1}(m\cdot q+r)}{q(\alpha+n+1)} - \frac{\log^{\alpha+n}(m\cdot q+r)}{2(m\cdot q+r)},
$$

then combining the above three expressions for $A(s)$, $I(s)$ and $C(s)$ yields:

$$
\begin{aligned}
\sum_{k=0}^{m}\frac{\log^{\alpha}(k\cdot q+r)}{(k\cdot q+r)^s} + \int_m^{\infty}\frac{\log^{\alpha}(x\cdot q+r)}{(x\cdot q+r)^s}dx - \frac{\log^{\alpha}(m\cdot q+r)}{2(m\cdot q+r)^s} \\
= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty}G_{\alpha,m}(n)\frac{(-1)^n(s-1)^n}{n!}.
\end{aligned}
$$

From the definition of the fractional Stieltjes constants it follows that:

$$
\frac{\Gamma(\alpha+1)}{q(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty}G_{\alpha,m}(n)\frac{(-1)^{\alpha+n}(s-1)^n}{n!} + E(s)
$$

$$= \frac{\delta_\alpha(r,q)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^{\alpha+n}\gamma_{\alpha+n}(q,r)(s-1)^n}{n!}.$$

Thus $\delta_\alpha(r,q) = \Gamma(\alpha+1)/q$. Subtracting the first term and taking successive derivatives with respect to s, of both sides of this equation, and then evaluating them at $s = 1$, we obtain for all $n \in \mathbb{N} \cup \{0\}$:

$$\gamma_{\alpha+n}(r,q) = G_{\alpha,m}(n) + E^{(n)}(1). \tag{5.6}$$

Setting $n = 0$ in (5.6) we obtain

$$\begin{aligned}\gamma_\alpha(r,q) &= G_{\alpha,m}(0) + E(1)\\ &= \sum_{k=0}^{m} \frac{\log^\alpha(k\cdot q+r)}{k\cdot q+r} - \frac{\log^{\alpha+1}(m\cdot q+r)}{q(\alpha+1)} - \frac{\log^\alpha(m\cdot q+r)}{2(m\cdot q+r)} + \int\limits_m^\infty P_1(x)g'(x)dx,\end{aligned}$$

where $g'(x) = \left(\frac{\log^\alpha(q\cdot x+t)}{q\cdot x+r}\right)'$. which proves the theorem.

□

5.3.1 Asymptotic Behavior of Fractional Stieltjes Constants

In the following we obtain an asymptotic formula for the $\gamma_\alpha(r,q)$ using the method of steepest decent or saddle point method, see [Erd56, Section 2.4 Laplace's method] for an overview. We have

$$\int_b^d f(t)e^{\alpha h(t)}\,dx \sim f(t_0)e^{\alpha h(t_0)}\sqrt{\frac{2\pi}{-\alpha h''(t_0)}} \quad \text{as } \alpha \to \infty \tag{5.7}$$

where t_0 is a saddle point of $h(t)$.

Recall that $\gamma_\alpha(q,r)$ are the coefficients of the Laurent expansion

$$\sum_{n=1}^{\infty} \frac{1}{(n\cdot q+r)^s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n\gamma_\alpha(r,q)}{n!}(s-1)^n$$

We generalize [FPS21, Theorem 2].

Theorem 5.18. *Let $\alpha > 0$ and set $w_\alpha(q) = W_0\left(\frac{q\alpha i}{2\pi}\right)$ and let*

$$\widetilde{\gamma}_\alpha(r,q) := \frac{1}{2}\frac{\log^\alpha(r)}{r} - \frac{\log^{\alpha+1}(q+r)}{q(\alpha+1)} + \Im\left(-\sqrt{\frac{2\alpha}{\pi(w_\alpha(q)+1)}}e^{-w_\alpha(q)+h(w_\alpha(q))}\right)$$

where $h(t) = 2\pi i(e^t - r)/q + \alpha\log t$. Then $\gamma_\alpha(r,q) \sim \widetilde{\gamma}_\alpha(r,q)$.

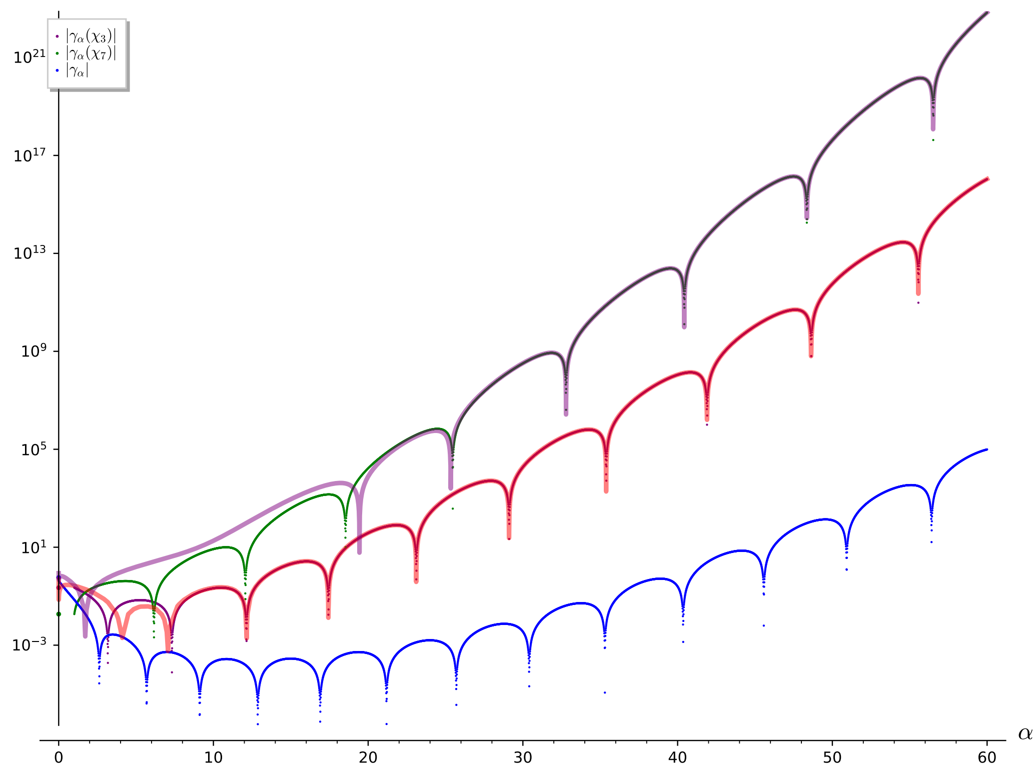

Figure 5.3. Absolute values of the fractional Stieltjes constants γ_α for $\zeta(s)$ and $\gamma_\alpha(\chi_3)$ for $L(s,\chi_3)$ where $\chi_3 = \chi_2(n)$ modulo 3 in Table 5.1 and $\gamma_\alpha(\chi_7)$ for $L(s,\chi_7)$ where $\chi_7 = \chi_2(n)$ modulo 7 in Table 5.5.

Proof. Let

$$g_\alpha(x) = \frac{\log^\alpha(x \cdot q + r)}{x \cdot q + r}.$$

Then

$$g'_\alpha(x) = \frac{\alpha \cdot \log^{\alpha-1}(x \cdot q + r) \cdot q}{(x \cdot q + r)(x \cdot q + r)} - \frac{\log^\alpha(x \cdot q + r) \cdot q}{(x \cdot q + r)^2} = \frac{q \cdot \log^{\alpha-1}(x \cdot q + r)(\alpha - \log(x \cdot q + r))}{(x \cdot q + r)^2}.$$

We set $m = 0$ in Theorem 5.17 and get

$$\gamma_\alpha(q,r) = \frac{\log^\alpha(r)}{r} - \frac{\log^{\alpha+1}(r)}{q(\alpha+1)} - \frac{\log^\alpha(r)}{2r} + \int_0^\infty P_1(x) g'_\alpha(x)\, dx$$

$$= \frac{1}{2}\frac{\log^\alpha(r)}{r} - \frac{\log^{\alpha+1}(r)}{q(\alpha+1)} + \int_0^\infty P_1(x) g'_\alpha(x)\, dx$$

for $\alpha \in \mathbb{R}$ with $\alpha > 0$ The first periodized Bernoulli polynomial P_1 has the Fourier series [AS64, page 805]

$$P_1(x) = \frac{-1}{\pi} \sum_{j=1}^{\infty} \frac{\sin(2\pi j x)}{j}.$$

With the above and the change of variable $t = \log(x \cdot q + r)$, $x = (e^t - r)/q$, $\frac{dt}{dx} = q/(x \cdot q + r) = q/e^t$. We obtain

$$\begin{aligned}
\int_0^\infty P_1(x) g'_\alpha(x)\, dx &= \sum_{j=1}^{\infty} \frac{-1}{\pi j} \int_0^\infty \sin(2\pi j x) \frac{q \cdot \log^{\alpha-1}(x \cdot q + r)(\alpha - \log(x \cdot q + r))}{(x \cdot q + r)^2}\, dx \\
&= \sum_{j=1}^{\infty} \frac{-1}{\pi j} \int_0^\infty \Im\left(e^{2\pi i j x}\right) \frac{q \cdot \log^{\alpha-1}(x \cdot q + r)(\alpha - \log(x \cdot q + r))}{(x \cdot q + r)^2}\, dx \\
&= \sum_{j=1}^{\infty} \frac{-1}{\pi j} \int_{\log(r)}^\infty \Im\left(e^{2\pi i j}\right) \frac{t^{\alpha-1}(\alpha - t)}{e^{2t}} (e^t) dt \\
&= \Im\left(\sum_{j=1}^{\infty} \frac{-1}{\pi j} \int_{\log(r)}^\infty e^{2\pi i j (e^t - r)} e^{-t + \alpha \log t} \frac{\alpha - t}{t} dt \right).
\end{aligned}$$

Comparing the Fourier series for P_1 with the Fourier series expansion of $x - [x]$ one sees that the series is dominated by the $j = 1$ term.

To find an asymptotic expression for the integral we apply the method of steepest decent (5.7). We set $h(t) = 2\pi i (e^t - r)/q + \alpha \log t$. Then $h'(t) = 2\pi i e^t/q + \alpha/t$ and $h''(t) = 2\pi i e^t/q - \alpha/t^2$. We have saddle points where

$$h'(w_\alpha(q)) = 2\pi i e^{w_\alpha(q)}/q + \alpha/w_\alpha(q) = 0. \tag{5.8}$$

The Lambert W function yields $w_\alpha(q) = W_0\left(\frac{q\alpha i}{2\pi}\right)$. We get $h''(w_\alpha(q)) = -\alpha/w_\alpha(q) - \alpha/w_\alpha(q)^2$ and thus

$$\begin{aligned}
\int_{\log(r)}^\infty e^{2\pi i (e^t - r)/q + \alpha \log t} e^{-t} \frac{\alpha - t}{t} dt &= \int_{\log(r)}^\infty e^{h(t)} e^{-t} \frac{\alpha - t}{t} dt \\
&\sim \frac{\alpha - w_\alpha(q)}{w_\alpha(q)} \frac{\sqrt{2\pi}}{\sqrt{-h''(w_\alpha(q))}} e^{h(w_\alpha(q))} e^{-w_\alpha(q)} \\
&= \left(\frac{\alpha}{w_\alpha(q)} - 1\right) \frac{\sqrt{2\pi}}{\sqrt{-h''(w_\alpha(q))}} e^{h(w_\alpha(q))} e^{-w_\alpha(q)} \\
&= \frac{1}{w_\alpha(q)} (\alpha - w_\alpha(q)) \frac{\sqrt{2\pi}}{\sqrt{\alpha/w_\alpha(q) + \alpha/w_\alpha(q)^2}} e^{h(w_\alpha(q)) - w_\alpha(q)} \\
&= \sqrt{\frac{2\pi}{\alpha(w_\alpha(q) + 1)}} e^{-w_\alpha(q) + h(w_\alpha(q))} (\alpha - w_\alpha(q))
\end{aligned}$$

$$\sim \sqrt{\frac{2\pi\alpha}{w_\alpha(q)+1}} \cdot e^{-w_\alpha(q)+h(w_\alpha(q))}.$$

Therefore

$$\int_1^\infty P_1(x) g'_\alpha(x)\,dx \sim \Im\left(\frac{-1}{\pi}\sqrt{\frac{2\pi\alpha}{w_\alpha(q)+1}} e^{-w_\alpha(q)+h(w_\alpha(q))}\right) \tag{5.9}$$

□

Corollary 5.19. *Let χ be a Dirichlet character, then*

$$\gamma_\alpha(\chi) = \sum_{r=1}^{q} \chi(r)\gamma_\alpha(r,q) \sim \sum_{r=1}^{q} \chi(r)\widetilde{\gamma}_\alpha(r,q)$$

Lemma 5.20 ([PS24, Lemma 1])**.** *Let W_0 be the principal branch of the Lambert W function. Consider $I(t) := \Im(W_0(it))$ for $t \in (0,\infty)$. We have:*

1. $\Re(W_0(it)) = I(t) \cdot \tan(I(t))$.
2. $T(y) : \left(0, \frac{\pi}{2}\right) \to (0,\infty)$ *given by* $T(y) = \frac{y}{\cos y} \cdot e^{y \cdot \tan(y)}$ *is the inverse of* $I(t)$.
3. $T'(y) > 0$ *for* $y \in \left(0, \frac{\pi}{2}\right)$, *and therefore* $I(t) > 0$, *for* $t \in (0,\infty]$.
4. $\lim_{t\to\infty} I(t) = \frac{\pi}{2}$.
5. $\Re(W_0(it)) > 0$ *for* $t \in (0,\infty)$.

By Lemma 5.20 the function $I\left(\frac{q\alpha}{2\pi}\right)$ and thus $\Re(w_\alpha(q)) = I\left(\frac{q\alpha}{2\pi}\right)\tan(I\left(\frac{q\alpha}{2\pi}\right))$ both increase with α and q. We have $w_\alpha(q) = I\left(\frac{q\alpha}{2\pi}\right)(i + \tan(I\left(\frac{q\alpha}{2\pi}\right)))$.

Lemma 5.21. *Let $w_\alpha(q) = W_0\left(\frac{q\alpha}{2\pi} i\right)$. Then*

1. $\arg(w_\alpha(q)) = \frac{\pi}{2} - \Im(w_\alpha(q))$
2. $\Re(w_\alpha(q)) < \log\left(\frac{q\alpha}{2\pi}\right)$ *for* $\alpha q > 2\pi e$

Proof. Using the notation from Lemma 5.20 we write

$$w_\alpha(q) = I\left(\frac{q\alpha}{2\pi}\right) + I\left(\frac{q\alpha}{2\pi}\right)\tan\left(I\left(\frac{q\alpha}{2\pi}\right)\right)$$

1. For the argument of $w_\alpha(q))$ we get

$$\begin{aligned}\arg(w_\alpha(q)) &= \arg\left(I\left(\frac{q\alpha}{2\pi}\right)\tan\left(I\left(\frac{q\alpha}{2\pi}\right)\right) + iI\left(\frac{q\alpha}{2\pi}\right)\right) \\ &= \tan^{-1}\left(\frac{I\left(\frac{q\alpha}{2\pi}\right)}{I\left(\frac{q\alpha}{2\pi}\right)\tan\left(I\left(\frac{q\alpha}{2\pi}\right)\right)}\right) \\ &= \tan^{-1}\frac{1}{\tan\left(I\left(\frac{q\alpha}{2\pi}\right)\right)} \\ &= \frac{\pi}{2} - I\left(\frac{q\alpha}{2\pi}\right)\end{aligned}$$

2. By Lemma 5.20 2. we have

$$\frac{\alpha q}{2\pi} = T\left(\Im(w_\alpha(q))\right) = \frac{\Im(w_\alpha(q)}{\cos\left(\Im(w_\alpha(q))\right)} \cdot e^{\Re(w_\alpha(q)}$$

Thus

$$\log\left(\frac{\alpha q}{2\pi}\right) = \log\left(\frac{\Im(w_\alpha(q))}{\cos\left(\Im(w_\alpha(q))\right)}\right) + \Re(w_\alpha(q)))$$

Because $0 \leq \Im(w_\alpha(q)) < \frac{\pi}{2}$ and because $\Im(w_\alpha(q)) > 1$ for $\alpha q > 2\pi e$ we have

$$\log\left(\frac{\alpha q}{2\pi}\right) > \Re(w_\alpha(q)))$$

□

Lemma 5.22. *Let $V_\alpha(r,q) = -\frac{1}{2}\log(w_\alpha(q)+1) - w_\alpha(q) + h(w_\alpha(q))$ and let T as in Lemma 5.20.- Then*

1. *$\Re\left(V_\alpha(r,q)\right)$ does not depend on r.*
2. *$\frac{d}{dq}\Re\left(V_\alpha(r,q)\right) > 0$ for $\alpha > \max\left\{2\pi T(\pi/4), -2W_{-1}\left(-\frac{1}{q^2}\right)\right\}$*
3. *$\frac{d}{dq}\Im\left(V_\alpha(r,q)\right) < 0$ for $\alpha > 2\pi T(\pi/4)$.*

Proof. With Equation 5.8 we get

$$\begin{aligned}V_\alpha(r,q) &= -\frac{1}{2}\log(w_\alpha(q)+1) - w_\alpha(q) + \frac{2\pi i(e^{w_\alpha(q)} - r)}{q} + \alpha\log w_\alpha(q) \\ &= -\frac{1}{2}\log(w_\alpha(q)+1) - w_\alpha(q) - \frac{2\pi i r}{q} - \frac{\alpha}{w_\alpha(q)} + \alpha\log w_\alpha(q).\end{aligned}$$

1. Because $w_\alpha(q)$ does not depend on r, also

$$\Re(V_\alpha(r,q)) = \Re\left(-\frac{1}{2}\log(w_\alpha(q)+1) - w_\alpha(q) - \frac{\alpha}{w_\alpha(q)} + \alpha\log w_\alpha(q)\right)$$

does not depend of r.

2. For the derivative with respect to q of $V_\alpha(r,q)$ we gave

$$\frac{d}{dq}V_\alpha(r,q) = -\frac{1}{2}\frac{w'_\alpha(q)}{w_\alpha(q)+1} - w'_\alpha(q) + \frac{\alpha\cdot w'_\alpha(q)}{(w_\alpha(q))^2} + \frac{\alpha\cdot w'_\alpha(q)}{w_\alpha(q)} + \frac{2\pi i r}{q^2}.$$

The derivative of the Lambert W-function is $W'(t) = \frac{W(t)}{t(W(t)+1)}$. Thus writing w for $w_\alpha(q)$ we get

$$\begin{aligned}\frac{d}{dq}V_\alpha(r,q) =& -\frac{1}{2}\frac{w}{q(w+1)^2} - \frac{w}{q(w+1)} + \frac{\alpha}{q\cdot w(w+1)} + \frac{\alpha}{q\cdot(w+1)} - \frac{2\pi i r}{q^2}\\ =&\frac{1}{q}\frac{-\frac{1}{2}w^2 - w^2(w+1) + \alpha(w+1) + \alpha\cdot w(w+1)}{w(w+1)^2} + \frac{2\pi i r}{q^2}\\ =&\frac{1}{q}\frac{-w^3 + (\alpha-\frac{3}{2})w^2 + 2\alpha\cdot w + \alpha}{w^3+2w^2+w} + \frac{2\pi i r}{q^2} \qquad (5.10)\end{aligned}$$

Considering the real part of (5.10) and expanding by the conjugate of the denominator we obtain the numerator

$$\begin{aligned}&\Re\big(-|w|^6 - 2|w|^4 w - |w|^2w^2 + (\alpha-\tfrac{3}{2})(|w|^4\overline{w} + 2|w|^4 + |w|^2 w\\ &\quad + 2\alpha(|w|^2\overline{w}^2 + 2|w|^2\overline{w} + |w|^2) + \alpha(\overline{w}^3 + 2\overline{w}^2 + \overline{w})\big)\\ =& -|w|^6 + (\alpha-\tfrac{7}{2})|w|^4\Re(w) + (2\alpha-3)|w|^4 + (2\alpha-1)|w|^2\Re(w^2)\\ &\quad + (5\alpha-\tfrac{3}{2})|w|^2\Re(w) + 2\alpha|w|^2 + \alpha\Re(\overline{w}^3) + 2\alpha\Re(w^2) + \alpha\Re(w)\end{aligned}$$

For $\frac{d}{dq}\Re(V_\alpha(r,q)) > 0$ it is sufficient that $|w|^2 < (\alpha-\frac{7}{2})\Re(w)$ if α is sufficiently large. It follows from Lemma 5.20 that for $\alpha > 2\pi T(\pi/4) = 15.3066\ldots$ we have

$$|w|^2 = \Re(w)^2 + \Im(w)^2 < 2\Re(w)^2.$$

We obtain the stronger inequality $2\Re(w) < \alpha - \frac{7}{2}$ and with $0 < \Re(w) < \log(\frac{q\alpha}{2\pi})$ for $\alpha > \frac{2\pi e}{q}$ by Lemma 5.21 we see that it is sufficient to consider

$$2\log\left(\frac{q\alpha}{2\pi}\right) < \alpha - \frac{7}{2}. \qquad (5.11)$$

The inequality holds for $q = 1$ and $\alpha > 2\pi T(\pi/4)$. Suppose $q \geq 2$. Since $2\log(2\pi)) > \frac{7}{2}$ a solution of

$$2\log(q\alpha) < \alpha.$$

is also a solution of the inequality (5.11). Because the right hand side increases faster with α than the right hand side, we only need to find $\alpha_0(q)$ such that $q^2\alpha^2 = e^\alpha$ or equivalently

$$-\frac{\alpha}{2}e^{-\alpha/2} = -\frac{1}{q^2}$$

The solution is given by the Lambert W-functions as $-2W_{-1}\left(-\frac{1}{q^2}\right)$.

3. Taking the derivative of (5.10) we get

$$\begin{aligned}\frac{d^2}{dq^2}V_\alpha(r,q) =& -\frac{1}{2}\frac{w'q(w+1)^2 - w(q2(w+1)w' + (w+1)^2)}{q^2(w+1)^4} \\ & - \frac{w'q(w+1) - w(qw'+w)}{q^2(w+1)^2}\end{aligned}$$

Considering the imaginary part of (5.10) and expanding by q^2 and the conjugate of the denominator we obtain the numerator

$$\begin{aligned}& q\cdot\Im\big(-|w|^6 - 2|w|^4 w - |w|^2w^2 + (\alpha - \tfrac{3}{2})(|w|^4\overline{w} + 2|w|^4 + |w|^2 w) \\ & \quad + 2\alpha(|w|^2\overline{w}^2 + 2|w|^2\overline{w} + |w|^2) + \alpha(\overline{w}^3 + 2\overline{w}^2 + \overline{w})\big) + 2\pi r \\ =& q\cdot\big(-2|w|^4\Im(w) - |w|^2\Im(w^2) + (\alpha - \tfrac{3}{2})(-|w|^4\Im w + |w|^2\Im(w)) \\ & \quad + 2\alpha(-|w|^2\Im(w^2) - 2|w|^2\Im w) + \alpha(-\Im(w^3) - 2\Im(w^2) - \Im(w))\big) + 2\pi r \\ =& q\cdot\big((-\tfrac{1}{2} - \alpha)|w|^4\Im(w) + (-1-2\alpha)|w|^2\Im(w^2) + (-3\alpha - \tfrac{3}{2})|w|^2\Im(w) \\ & \quad + \alpha(-\Im(w^3) - 2\Im(w^2) - \Im(w))\big) + 2\pi r\end{aligned}$$

We get that $\frac{d}{dq}\Im(V_\alpha(r,q)) < 0$ for $\alpha > 2\pi T(\pi/4)$.

□

Theorem 5.23. *For* $V_\alpha(r,q) = -\frac{1}{2}\log(w_\alpha(q)+1) - w_\alpha(q) + h(w_\alpha(q))$ *we have*

$$\widetilde{\gamma}_\alpha(r,q) = \frac{1}{2}\frac{\log^\alpha(r)}{r} - \frac{\log^{\alpha+1}(r)}{q(\alpha+1)} - \sqrt{\frac{2\alpha}{\pi}}\sin\Im(V_\alpha(r,q))\cdot e^{\Re(V_\alpha(r,q))}$$

The amplitude $e^{\Re(V_\alpha(r,q))}$ *of* $\gamma_\alpha(r,q)$ *increases with* q *for* $\alpha > \max\left\{2\pi T(\pi/4), -2W_{-1}\left(-\frac{1}{q^2}\right)\right\}$.

Proof. We have

$$\Im\left(-\sqrt{\frac{2\alpha}{\pi(w_\alpha(q)+1)}}e^{-w_\alpha(q)+h(w_\alpha(q))}\right) = -\sqrt{\frac{2\alpha}{\pi}}\cdot\Im\left(e^{V_\alpha(r,q)}\right)$$

$$= -\sqrt{\frac{2\alpha}{\pi}} \cdot \sin \Im(V_\alpha(r,q)) \cdot e^{\Re(V_\alpha(r,q))}$$

Thus with Theorem 5.18 we have

$$\widetilde{\gamma}_\alpha(r,q) = \frac{\log^\alpha(r)}{r} - \frac{\log^{\alpha+1}(r)}{q(\alpha+1)} - \frac{\log^\alpha(r)}{2r} - \sqrt{\frac{2\alpha}{\pi}} \cdot \sin \Im(V_\alpha(r,q)) \cdot e^{\Re(V_\alpha(r,q))}$$

With Lemma 5.22 2. we get that the amplitude $\Re(V_\alpha(r,q))$ increases for $\alpha > \max\left\{2\pi T(\pi/4), -2W_{-1}\left(-\frac{1}{q^2}\right)\right\}$. ☐

With the above and Theorem 5.18 we see that $\gamma_\alpha(r,q)$ increase with q for $\alpha > \max\left\{2\pi T(\pi/4), -2W_{-1}\left(-\frac{1}{q^2}\right)\right\}$. The analogous result holds for $\gamma_\alpha(\chi)$ where χ is a character modulo q.

5.4 Critical Strip

Let χ be a primitive character modulo q, with $q > 1$. There are no zeros of $L(s,\chi)$ with $\Re(s) > 1$. for $\Re(s) < 0$, there are trivial zeros for certain negative integers. Namely:

(i) If $\mathfrak{a} = 0$, then only zeros of $L(s,\chi)$ with $\Re(s) < 0$ are simple zeros at $-2, -4, -6, \ldots$ (There is also a zero at $s = 0$). These correspond to the poles of $\Gamma(\frac{s}{2})$ in the functional equation.

(ii) If $\mathfrak{a} = 1$, then only zeros of $L(s,\chi)$ with $\Re(s) < 0$ are simple zeros at $-1, -3, -5, \ldots$. These correspond to the poles of $\Gamma(\frac{s+1}{2})$ in the functional equation.

The remaining zeros lie in the critical strip $0 \leq \Re(s) \leq 1$, and are non-trivial zeros. the non-trivial zeros are symmetrical about the critical line $\Re(s) = 1/2$. That is, if $L(\rho,\chi) = 0$ then $L(1-\overline{\rho},\chi) = 0$ too, because of the functional equation. If χ is a real character, then the non-trivial zeros are also symmetrical about the real axis, but not if χ is a complex character. The generalized Riemann hypothesis is the conjecture that all the non-trivial zeros lie on the critical line $\Re(s) = 1/2$. Yıldırım was able to show an equivalence of the Generalize Riemann Hypothesis to the non-vanishing of the first derivative.

Theorem 5.24 ([Yıl96, Theorem 1]). *Assume General Riemann Hypothesis. If $\mathfrak{a} = 0$ and $q \geq 216$, then $L'(s,\chi)$ has exactly one zero ρ_1 with $0 \leq \Re(\rho_1) < \frac{1}{2}$, at $\frac{1}{\log q} + \mathcal{O}(\frac{\log\log q}{\log^2 q})$. If $\mathfrak{a} = 1$ and $q \geq 23$, then $L'(s,\chi)$ has no zeros in the left-half of the critical strip.*

Yıldırım also remarked:

Remark 5.25. (i) For small q the possibility remains that $L'(s\chi)$ has zeros $\beta_1 + i\gamma_1$ with $0 < \beta_1 < \frac{1}{2}, |\gamma_1| < 3$.

(ii) We can see that $\Re\frac{L'}{L}(\frac{1}{2} + it, \chi) < 0$ if $L(\frac{1}{2} + it, \chi) \neq 0$. So $L'(s, \chi)$ nay vanish on the critical line only at a multiple zero of $L(s, \chi)$.

Example 5.26. With the computations we have produces, we can easily verify Remark 5.25 (i). For examples, $\chi_3 : [0, 1, \omega_6^2, \omega_6, \omega_6^4, \omega_6^5, -1]$ modulo 7 has only one zero at $s \approx 0.1725 - i0.6901$ and $\chi_5 : [0, 1, \omega_6^4, \omega_6^5, \omega_6^2, \omega_6, -1]$ modulo 7 has only one zero at $s \approx 0.1725 + i0.6901$. It is clear that χ_3 and χ_5 modulo 7 are conjugate pairs, which makes sense since the respective zeros are conjugates of each other.

5.4.1 Derivatives

In 1996 Yıldırım proved that the zero-free regions only depend on the modulus of the character.

Theorem 5.27 ([Yıl96, Theorem 2])**.** *Let m be the smallest prime that doesn't divide q. Then $L^{(k)}(s, \chi) \neq 0$ for $\sigma > 1 + \frac{m}{2}\left(1 + \sqrt{1 + \frac{4k^2}{m \log m}}\right)$.*

Theorem 5.28 ([Yıl96, Theorem 5])**.** *Assuming General Riemann Hypothesis, there are at most finitely many zeros of $L^{(k)}(s, \chi)$ in the strip $-\epsilon \leq \sigma \leq \frac{1}{2}$.*

5.5 Right Half-Plane

Let $Q_n^\alpha(s) := (\log n)^\alpha / n^s$ denote the n-th term of the Dirichlet series for $(-1)^\alpha L^{(\alpha)}(s, \chi)$, so that

$$(-1)^\alpha L^{(\alpha)}(s, \chi) = (-1)^\alpha \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\chi(n) \log^\alpha n}{n^s} = (-1)^\alpha \sum_{n=2}^{\infty} (-1)^{n-1}\chi(n) Q_n^\alpha(s). \tag{5.12}$$

We prove the existence of zero-free regions where one of the terms of (5.12), say $Q_M^\alpha(\sigma)$, dominates the rest of the series, that is, when

$$Q_M^\alpha(\sigma) > \sum_{n \neq M} Q_n^\alpha(\sigma), \tag{5.13}$$

and, in a complementary fashion, we look for the zeros of $\eta^{(\alpha)}(s)$ near the regions of the complex plane where $Q_M^\alpha(s) = Q_{M+d}^\alpha(s)$, in other words where no term of the

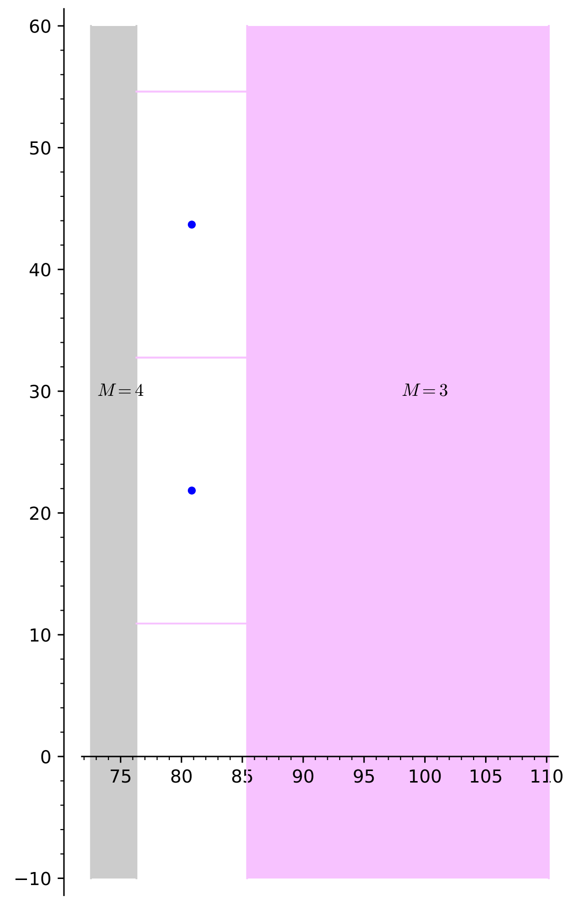

Figure 5.4. Zeros of $L^{(100)}$, where $\chi = [1, 1, -1, 1, -1, -1]$ with zero-free regions and lines

series can attain dominance and, in fact, where the cancellation of terms might happen. This occurs at

$$q^{M}_{M+d} = \frac{\log(\frac{\log M}{\log(M+d)})}{\log(\frac{M}{M+d})} \quad q^{M-c}_{M} = \frac{\log(\frac{\log M}{\log(M-c)})}{\log(\frac{M}{M-c})} \tag{5.14}$$

where we denote the gap of zeros of size d after M and the gap of zeros of size c before M.

Theorem 5.29. *Let $\alpha > 0$. If $M \in \mathbb{N}$, $M \geq 3$, and $q^{m}_{M+d}\alpha + (M+d)u \leq q_{M-c}\alpha - Mu$, then $L^{(\alpha)}(s, \chi)(s) \neq 0$ in the regions*

$$q^{M}_{M+d}\alpha + (M+d)u \leq \sigma \leq q^{M-c}_{M}\alpha - (M-c)u,$$

where $u \in (0, \infty)$ is a solution of $1 - \frac{1}{e^{du}-1} - \frac{1}{e^{cu}}(1 + \frac{1}{u}) \geq 0$.

Let S_M^α be the vertical strip between the zero-free regions obtained from the dominance of $Q_M^\alpha(q_{M+d}^M\alpha)$ and $Q_{M+d}^\alpha(q_{M+d}^M\alpha)$ in (4.13), respectively, as described in Theorem 4.9. The strip S_M^α exists when α reaches

$$A_M := \left\{ \frac{(2M+d+c)u}{q_M - q_{M+1}} \quad \text{if } M \geq 2. \right.$$

Recall that $Q_M^\alpha(q_{M+d}^M\alpha) = Q_{M+d}^\alpha(q_{M+d}^M\alpha)$. Considering the imaginary parts of the solutions of $\chi(M)Q_M^\alpha(q_{M+d}^M\alpha + it) = \chi(M+d)Q_{M+d}^\alpha(q_{M+d}^M\alpha + it)$ we find that $L^{(\alpha)}(s,\chi)(\sigma + it) \neq 0$ for $\sigma \in S_M^\alpha$ and

$$it = \frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right) + 2\pi i J}{\log(M+d) - \log(M)} \tag{5.15}$$

for $J \in \mathbb{Z}$. Together with the border of the zero-free regions to the left and right of S_M^α the lines from (4.17), for $J = j$ and $J = j+1$, where $j \in \mathbb{Z}$ form a contour around the zero

$$q_{M+d}^M \cdot \alpha + \frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right) + i\pi(2J+1)}{\log(M+d) - \log(M)} \tag{5.16}$$

of $\chi(M)Q_M^\alpha(q_{M+d}^M\alpha + it) + \chi(M+d)Q_{M+d}^\alpha(q_{M+d}^M\alpha + it) = 0$. Exactly as in [BPS15], Rouché's theorem immediately shows that there is exactly one zero of $L^{(\alpha)}(s,\chi)$ in the rectangular area shown in Figure 5.5. In other words, a natural generalization of [BPS15, Theorem 2.2] can be quickly obtained, *mutatis mutandis*, replacing integer values of k by positive real numbers α:

Theorem 5.30. *Let $M \geq 3$ denote a natural number, $j \in \mathbb{Z}$, and $\alpha > A_M$. Let $F_{M,j}^\alpha \subset S_M^\alpha$ be given by*

$$\frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right) + 2i\pi j}{\log(M+d) - \log(M)} < t < \frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right) + 2i\pi(j+1)}{\log(M+d) - \log(M)}. \tag{5.17}$$

Then $F_{M,j}^\alpha$ contains exactly one zero of $L^{(\alpha)}(s,\chi)$, and the zero is simple.

In our proof of Theorem 5.29 we follow, with some modifications, the general approach developed in order to establish [BPS15, Theorem 2.1]. We show that $L^{(\alpha)}(s,\chi)(s)$ has no zeros if (α,σ) in the $\alpha\sigma$-plane lies in one of the wedges given by

$$q_{M+d}^M\alpha + b_1 \leq \sigma \leq q_M^{M-c}\alpha + b_2$$

for constants $b_1, b_2 \in \mathbb{R}$, chosen in a way that guarantees the dominance (in the modulus) of the term $|Q_M^\alpha(s)| = \left|\frac{\chi(M)\log^\alpha M}{M^s}\right|$ of the series for $L^{(\alpha)}(s,\chi)(s,\chi)$. We call

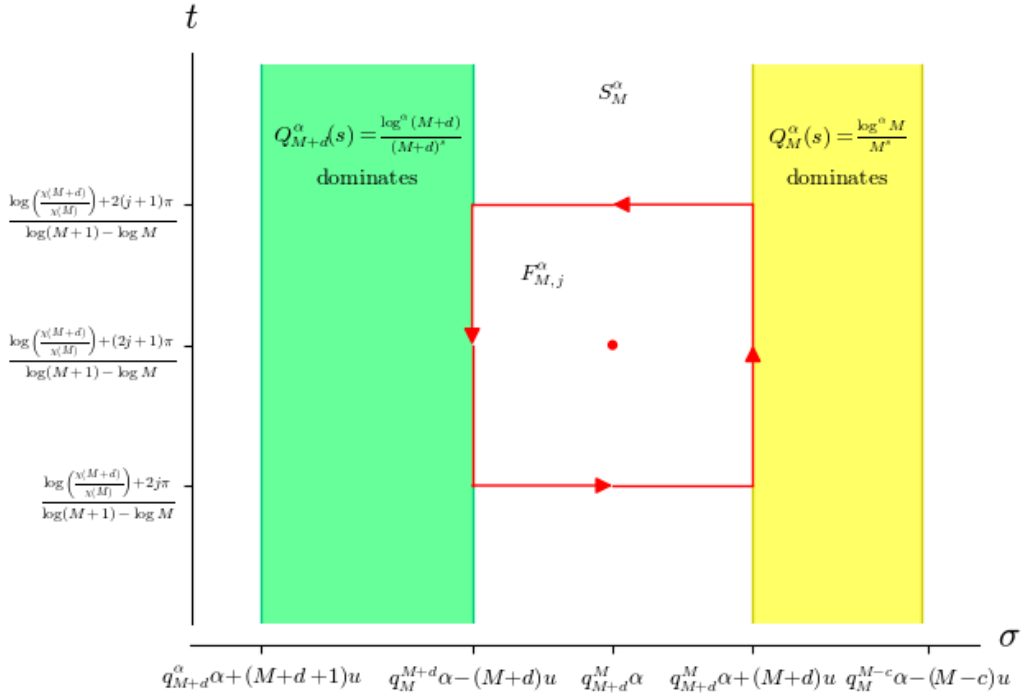

Figure 5.5. Regions $F_{M,j}^\alpha$ that contains exactly one zero of $L^{(\alpha)}(s,\chi)$. Rouche's theorem can be used to establish simplicity of the zero using the zero of $\chi(M)Q_M^\alpha(s)+\chi(M+d)Q_{M+d}^\alpha(s)$

the remaining terms of the series the 'head'

$$H_M^\alpha(s) := \sum_{n=2}^{M-c} |Q_n^\alpha(s)| = \sum_{n=c}^{M-c} \left|\frac{\chi(n)\log^\alpha n}{n^s}\right|$$

and the 'tail'

$$T_M^\alpha(s) := \sum_{n=M+d}^{\infty} |Q_n^\alpha(s)| = \sum_{n=M+d}^{\infty} \left|\frac{\chi(n)\log^\alpha n}{n^s}\right|.$$

The key idea is to show that in our well-defined regions

$$\begin{aligned} |L^{(\alpha)}(s,\chi)(s)| &\geq Q_M^\alpha(\sigma) - H_M^\alpha(\sigma) - T_M^\alpha(\sigma) \\ &= Q_M^\alpha(\sigma)\left(1 - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{T_M^\alpha}{Q_M^\alpha}(\sigma)\right) > 0, \end{aligned} \tag{5.18}$$

thus proving that $L^{(\alpha)}(s,\chi)(s)$ does not vanish. In order to find suitable upper bounds to the tails $T_M^\alpha(\sigma)$, a couple of preliminary bounds are needed. For the case of Dirichlet L-functions [PS24, Lemma 1] yields

Lemma 5.31. *Fix $2 \le M \in \mathbb{N}$, and assume $\alpha < (\sigma-1)\log M$. Then for any modulus and $d=1$*

$$T_M^\alpha(\sigma) = \sum_{n=M+1}^{\infty} \frac{\chi(n)\log^\alpha n}{n^\sigma} \le \sum_{n=M+1}^{\infty} \frac{\log^\alpha n}{n^\sigma} \le \int_M^\infty \frac{\log^\alpha x}{x^\sigma} dx \le Q_M^\alpha(\sigma) R_M^\alpha(\sigma), \tag{5.19}$$

where

$$R_M^\alpha(\sigma) = \frac{M}{\sigma-1}\left(1 + \frac{\alpha}{(\sigma-1)\log M - \alpha}\right).$$

Next, we find a bound for $R_M^\alpha(\sigma)$. We have:

Lemma 5.32 (Lemma 2[PS24])**.** *If $a_1\alpha + b_1 \le \sigma$ and $A \le \alpha$ and $a_1 > \frac{1}{\log M}$, then*

$$R_M^\alpha(\sigma) \le R_M^\alpha(a_1\alpha + b_1) \le R_M^A(a_1\alpha + b_1) \le R_M^A(a_1 A + b_1), \tag{5.20}$$

Note: In what follows, we apply the estimates for $T_M^\alpha(\sigma)$ from Lemma 5.31 in the proof of Theorem 5.29 via the useful separation

$$\begin{aligned} T_M^\alpha(\sigma) &= Q_{M+1}^\alpha(\sigma) + T_{M+1}^\alpha(\sigma) \\ &\le Q_{M+2}^\alpha(\sigma)(1 + R_{M+2}^\alpha(\sigma)) \\ &\le Q_M^\alpha(q_M\alpha + b_1)(1 + R_{M+2}^\alpha(q_M\alpha + b_1)), \end{aligned}$$

which holds since $Q_{M+2}^\alpha(\sigma) \le Q_M^\alpha(\sigma)$. The series with these $R_{M+2}^\alpha(q_M\alpha + b_1)$ converges because, by [BPS15, Lemma 3.1], $q_M > 1/\log(M+1)$.

Lemma 5.33. *Let $c \in \mathbb{R}$ be positive. Then $y(n) = \left(\frac{n-x}{n}\right)^{cn}$ is monotonously increasing with asymptote $1/e^{xc}$.*

Proof. As $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^{cn} = e^{xc}$, we evidently have $\lim_{n\to\infty}\left(\frac{n-x}{n}\right)^{cn} = 1/e^{xc}$. Finally,

$$y'(n) = c \cdot y(n)\left(\log\left(1 - \frac{x}{n}\right) + \frac{1}{n-x}\right) > 0$$

proves the monotonicity assertion. □

Before we get to the main argument of the proof of Theorem 5.29 (c), let us perform a technical transformation. We rewrite the series (5.12) as

$$\begin{aligned} H_M^\alpha(\sigma) &= Q_M^\alpha(\sigma)\left(\frac{Q_{M-c}^\alpha}{Q_M^\alpha}(\sigma) + \frac{Q_{M-c-1}^\alpha}{Q_M^\alpha}(\sigma) + \cdots + \frac{Q_2^\alpha}{Q_M^\alpha}(\sigma)\right) \\ &= Q_M^\alpha(\sigma)\left(\frac{Q_{M-c}^\alpha}{Q_M^\alpha}(\sigma)\left(1 + \frac{Q_{M-c-1}^\alpha}{Q_{M-c}^\alpha}(\sigma)\left(1 + \ldots\left(1 + \frac{Q_2^\alpha}{Q_3^\alpha}(\sigma)\right)\ldots\right)\right)\right). \end{aligned} \tag{5.21}$$

with the hope of finding bounds for $\frac{Q^\alpha_{n-c}}{Q^\alpha_n}(\sigma)$. Observe that because

$$\frac{Q^\alpha_{n-c}}{Q^\alpha_n}(\sigma) = \left(\frac{\log(n-c)}{\log n}\right)^\alpha \left(\frac{n}{n-c}\right)^\sigma$$

the quotient $\frac{H^\alpha_M}{Q^\alpha_M}(\sigma)$ increases with σ. That means that, for $2 \le n \le M$ and $\sigma \le q_M^{M-c}\alpha + b_2$, we can write

$$\frac{Q^\alpha_{n-c}}{Q^\alpha_n}(\sigma) \le \frac{Q^\alpha_{n-c}}{Q^\alpha_n}(q_M^{M-c}\alpha + b_2) \le \frac{Q^\alpha_{n-c}}{Q^\alpha_n}(q_n^{n-c}\alpha + b_2) = \left(\frac{n}{n-c}\right)^{b_2},$$

where the second inequality holds since $q_M^{M-c} < q_n$ for $n \le M$ and the equality holds because $\sigma = q_n^{n-c}\alpha$ is the solution of $Q^\alpha_n(\sigma) = Q^\alpha_{n-c}(\sigma)$. Thus, in order for $\frac{H^\alpha_M}{Q^\alpha_M}(\sigma)$ to stay bounded, we must choose $b_2 < 0$.

By [BPS15, Lemma 4.4] we have, for $3 \le n \le M$ and $\sigma \le q_M^{M-c}\alpha - u(M-c)$,

$$\frac{Q^\alpha_{n-c}}{Q^\alpha_n}(\sigma) \le \left(\frac{n}{n-c}\right)^{-u(M-c)} \le \left(\frac{M}{M-c}\right)^{-u(M-c)} \le \frac{1}{e^{cu}}.$$

Combined with the equation (5.21), this yields

$$\frac{H^\alpha_M}{Q^\alpha_M}(\sigma) \le \sum_{n=1}^{\infty} \frac{1}{(e^{cu})^n} = \frac{1}{1-\frac{1}{e^{cu}}} - 1 = \frac{1}{e^{cu}-1}. \tag{5.22}$$

We are now ready to prove the final part (c) of Theorem 5.29.

Proof. of Theorem 5.29 Let $\alpha > 0$. We show that if $M \in \mathbb{N}$, $M \ge 3$, and $q^M_{M+d}\alpha + (M+d)u \le q_M^{M-c}\alpha - Mu$ then $L^{(\alpha)}(s,\chi) \ne 0$ for

$$q^M_{M+d}\alpha + (M+d)u \le \sigma \le q_d^{M-c}\alpha - (M-c)u.$$

where $u \in (0,\infty)$ is a solution of $1 - \frac{1}{e^{cu}-1} - \frac{1}{e^{du}}(1+\frac{1}{u}) \ge 0$. Similar to the proof of Theorem 5.29 (b) we write

$$\begin{aligned}\left|L^{(\alpha)}(s,\chi)\right| &\ge Q^\alpha_M(\sigma) - H^\alpha_M(\sigma) - T^\alpha_M(\sigma)\\ &\ge Q^\alpha_M(\sigma)\left(1 - \frac{H^\alpha_M}{Q^\alpha_M}(\sigma) - \frac{Q^\alpha_{M+d}}{Q^\alpha_M}(\sigma)\left(1 + R^\alpha_{M+d}(\sigma)\right)\right).\end{aligned}$$

Now, notice that

$$R^\alpha_{M+d}(\sigma) := \frac{M+d}{\sigma-1}\left(1 + \frac{\alpha}{(\sigma-1)\log(M+d)-\alpha}\right) < \frac{1}{u}$$

is equivalent to $(\sigma-1)^2\log(M+d)-(\sigma-1)(u(M+d)\log(M+d)+\alpha)>0$ and this quadratic inequality is satisfied whenever $\sigma>1+u(M+d)+\frac{\alpha}{\log(M+d)}$. Thus, by Lemma 5.32, for $\sigma\ge q_{M+d}^M\alpha+u(M+d)$, $\alpha\ge\alpha_M:=\frac{(2M+d+c)u}{q_M^{M-c}-q_{M+d}^M}$, and $M\ge 3$, we have

$$R_{M+d}^{\alpha}(\sigma)\le R_{M+d}^{\alpha_M}(q_{M+d}^M\alpha_M+u(M+d))<\frac{1}{u}.$$

But by [BPS15, Lemma 4.4] $\left(\frac{n-1}{n}\right)^{cn}$ is monotonously increasing with the asymptote $1/e^c$. And therefore

$$\frac{Q_{M+d}^{\alpha}}{Q_M^{\alpha}}(q_{M+d}^M\alpha+u(M+d))=\left(\frac{M}{M+d}\right)^{u(M+d)}<\frac{1}{e^{du}}.$$

Finally, with the help of the bound (5.22), we can see, that for $M\ge 4$ and $q_{M+d}^M\alpha+u(M+d)\le\sigma\le q_c^{M-c}\alpha+uM$, we have

$$1-\frac{H_M^{\alpha}}{Q_M^{\alpha}}(\sigma)-\frac{Q_{M+d}^{\alpha}}{Q_M^{\alpha}}(\sigma)\left(1+R_{M+d}^{\alpha}(\sigma)\right)>1-\frac{1}{e^{cu}-1}-\frac{1}{e^{du}}\left(1+\frac{1}{u}\right)\ge 0,$$

which completes the proof of the theorem. □

Corollary 5.34. *Let $M\in\mathbb{N}$ with $M\ge 2$ and $j\in\mathbb{Z}$. The zeros $s=\sigma+it$ of $L^{(\alpha)}(s,\chi)$ for $\alpha>A_M$ with*

$$\frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right)+2i\pi j}{\log(M+d)-\log(M)}<it<\frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right)+2i\pi(j+1)}{\log(M+d)-\log(M)}$$

are images of an analytic function $z:(A_M,\infty)\to\mathbb{C}$.

Lemma 5.35. *Let $M\ge 2$ and $\alpha\in\mathbb{R}$. If $s\in S_M^{\alpha}$, then $L^{(\alpha)}(s,\chi)\ne 0$ for*

$$s=\sigma+\frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right)+2i\pi j}{\log(M+d)-\log M}.$$

Proof. In the center of the strip S_M^{α}, that is on the line $\sigma=q_{M+d}^M\alpha$ we have $|Q_M^{\alpha}(s)|=|Q_{M+d}^{\alpha}(s)|$. We consider the line segments in S_M^{α} with

$$q_{M+d}^M\alpha-(M+d)u\le\sigma\le q_{M+d}^M\alpha+(M+d)u.$$

and

$$it=\frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right)+i\pi(2j+1)}{\log(M+d)-\log M},\ \text{where } j\in\mathbb{Z},$$

see Figure 5.5. Our choice of t gives $\chi(M)Q_M^\alpha(q_{M+d}^M\alpha+it)+\chi(M+d)Q_{M+d}^\alpha(q_{M+d}^M\alpha+it)=0$ (compare equation (5.15)) and therefore $\cos(t\log M)=\cos(t\log(M+d))$ and $\sin(t\log M)=-\sin(t\log(M+d))$. We set $s=\sigma+it$, with t and σ as above, and consider the real and imaginary parts of

$$L^{(\alpha)}(s,\chi)=\sum_{n=2}^{\infty}\chi(n)\left(\cos(t\log n)+i\cdot\sin(t\log n)\right)Q_n^\alpha(\sigma).$$

With $|\chi(n)\Im(Q_n^\alpha(s))|\le Q_n^\alpha(\sigma)$ and $|\chi(n)\Re(Q_n^\alpha(s))|\le Q_n^\alpha(\sigma)$ we obtain

$$\begin{aligned}|\Re(L^{(\alpha)}(s,\chi))| &\ge |\cos(t\log M)Q_M^\alpha(\sigma)+\cos(t\log(M+d))Q_{M+d}^\alpha(\sigma)|\\ &\quad -H_M^\alpha(\sigma)-T_{M+d}^\alpha(\sigma),\\ |\Im(L^{(\alpha)}(s,\chi))| &\ge |\sin(t\log M)Q_M^\alpha(\sigma)+\sin(t\log(M+d))Q_{M+d}^\alpha(\sigma)|\\ &\quad -H_M^\alpha(\sigma)-T_{M+d}^\alpha(\sigma).\end{aligned}$$

If $t=0$, the situation is trivial. If $t\neq 0$, then we either have $|\sin(t\log M)|\ge \sin(\pi/4)=1/\sqrt{2}$ or $|\cos(t\log M)|\ge\cos(\pi/4)=1/\sqrt{2}$. Because $|L^{(\alpha)}(s,\chi)|\ge |\Re(L^{(\alpha)}(s,\chi))|$ and $|L^{(\alpha)}(s,\chi)|\ge|\Im(L^{(\alpha)}(s,\chi))|$ we get:

$$\begin{aligned}|L^{(\alpha)}(s,\chi)| &\ge \frac{1}{\sqrt{2}}\left(Q_M^\alpha(\sigma)+Q_{M+d}^\alpha(\sigma)\right)-H_M^\alpha(\sigma)-T_{M+d}^\alpha(\sigma)\\ &= Q_M^\alpha(\sigma)\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\frac{Q_{M+d}^\alpha}{Q_M^\alpha}(\sigma)-\frac{H_M^\alpha}{Q_M^\alpha}(\sigma)-\frac{Q_{M+d+1}^\alpha}{Q_M^\alpha}(\sigma)-\frac{T_{M+d+1}^\alpha}{Q_M^\alpha}(\sigma)\right)\\ &= Q_M^\alpha(\sigma)\left(\frac{1}{\sqrt{2}}-\frac{H_M^\alpha}{Q_M^\alpha}(\sigma)+\frac{Q_{M+d}^\alpha}{Q_M^\alpha}(\sigma)\left(\frac{1}{\sqrt{2}}-\frac{Q_{M+d+1}^\alpha}{Q_{M+d}^\alpha}(\sigma)-\frac{T_{M+d+1}^\alpha}{Q_{M+d}^\alpha}(\sigma)\right)\right)\end{aligned}$$

From the proof of Theorem 5.29 we know that for $\sigma\ge q_{M+d}^M\alpha+(M+d)u$

$$\begin{aligned}\frac{1}{\sqrt{2}}-\frac{Q_{M+d+1}^\alpha}{Q_{M+d}^\alpha}(\sigma)-\frac{T_{M+d+1}^\alpha}{Q_{M+d}^\alpha}(\sigma) &\ge \frac{1}{\sqrt{2}}-\frac{Q_{M+d+1}^\alpha}{Q_{M+d}^\alpha}(\sigma)\left(1+R_{M+d+1}(\sigma)\right)\\ &\ge \frac{1}{\sqrt{2}}-\frac{1}{e^{du}}\left(1+\frac{1}{u}\right)>0.\end{aligned}$$

Similarly, since $\frac{H_M^\alpha}{Q_M^\alpha}(\sigma)$ is increasing in σ (see equation (5.21)) and because $\sigma< q_M^{M-c}\alpha-(M-c)u$, we get with (5.22) that

$$\frac{1}{\sqrt{2}}-\frac{H_M^\alpha}{Q_M^\alpha}(\sigma)\ge\frac{1}{\sqrt{2}}-\frac{H_M^\alpha}{Q_M^\alpha}(q_M^{M-c}\alpha-(M-c)u)\ge\frac{1}{\sqrt{2}}-\frac{1}{e^{cu}-1}>0,$$

which concludes the proof of the lemma. □

Proof of Theorem 5.30. Let $Z(s) = \chi(M)Q_M^\alpha(s) + \chi(M+d)Q_{M+d}^\alpha(s)$. It is easy to check that the function $Z(s)$ has exactly one (simple) zero in R_j, namely

$$s = q_{M+d}^M \alpha + \frac{\log\left(\frac{\chi(M+d)}{\chi(M)}\right) + i\pi(2j+1)}{\log(M+d) - \log M}.$$

In order to be able to apply Rouché's Theorem we need to show that $|L^{(\alpha)}(s,\chi) - Z(s)| < |Z(s)|$ for all s on R_j.

The vertical sides of R_j are in the zero-free regions for M and $M+d$. As shown in the proof of Theorem 5.29 the term $Q_M^\alpha(s)$ dominates $L^{(\alpha)}(s,\chi)(s)$ on the right vertical side of R_j and the term $Q_{M+d}^\alpha(s)$ dominates $L^{(\alpha)}(s,\chi)$ on the left vertical side of R_j. Thus $|L^{(\alpha)}(s,\chi) - Z(s)| < |Z(s)|$ on the vertical sides of R_j. Furthermore we have seen in the proof of Lemma 5.35 that $Z(s) = Q_M^\alpha(s) + Q_{M+d}^\alpha(s)$ dominates $L^{(\alpha)}(s,\chi)$ on the horizontal sides of R_j. Hence $|L^{(\alpha)}(s,\chi) - Z(s)| < |Z(s)|$ on the horizontal sides of R_j. □

5.5.1 Special Cases

Corollary 5.36. *Let $\alpha > 0$ and any modulo k that has 2 as a divsor. We have :*

(a) For moduli 2^r where $r \in \mathbb{N}$, then for all $\sigma > q_3\alpha + 2.364$, we have $L^{(\alpha)}(s,\chi)(s) \neq 0$.

(b) If $M \in \mathbb{N}$, $M \geq 3$, and $q_{M+d}^M\alpha + (M+d)u \leq q_M^{M-c}\alpha - (M-c)u$, then $L^{(\alpha)}(s,\chi)(s) \neq 0$ in the regions

$$q_{M+d}^M\alpha + (M+d)u \leq \sigma \leq q_M^{M-c}\alpha - (M-c)u,$$

where $u \in (0,\infty)$ is a solution of $1 - \frac{1}{e^{du}-1} - \frac{1}{e^{cu}}(1 + \frac{1}{2u}) \geq 0$.

Note: The smallest u with $c = d = 2$ is $u = 0.594$

In our proof of Theorem 5.36 we follow, with some modifications, the general approach developed in order to establish 5.29. Thus the idea is to show that in our well-defined regions with a better bound

$$\begin{aligned} |L^{(\alpha)}(s,\chi)(s)| &\geq Q_M^\alpha(\sigma) - H_M^\alpha(\sigma) - T_M^\alpha(\sigma) \\ &= Q_M^\alpha(\sigma)\left(1 - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{T_M^\alpha}{Q_M^\alpha}(\sigma)\right) > 0, \end{aligned} \tag{5.23}$$

thus proving that $L^{(\alpha)}(s,\chi)(s)$ does not vanish.

In order to find suitable upper bounds to the tails $T_M^\alpha(\sigma)$, a couple of preliminary bounds are needed. We begin with the following lemma to better bound the tail:

Lemma 5.37. *Fix $2 \leq M \in \mathbb{N}$, and assume $\alpha < (\sigma - 1)\log M$. Let any modulo of χ be a divisor of 2. Then with $d = 2$ and odd M we have,*

$$T_M^\alpha(\sigma) = \sum_{n=M+2}^{\infty} \frac{|\chi(n)| \log^\alpha n}{n^\sigma} \leq Q_{M+2}^\alpha(\sigma) \left(1 + \frac{1}{2} R_{M+3}^\alpha(\sigma)\right), \tag{5.24}$$

where

$$R_M^\alpha(\sigma) = \frac{M}{\sigma - 1} \left(1 + \frac{\alpha}{(\sigma - 1)\log M - \alpha}\right).$$

Proof. First, for the upper incomplete Gamma function we have the bound (see [NP00, equation (3.2)]): $\Gamma(a, x) < Bx^{a-1}e^{-x}$, valid for all $B > 1, a > 1$ and $x > \frac{B(1-a)}{1-B}$. This means that we can write:

$$\begin{aligned}
T_M^\alpha(\sigma) = \sum_{n=M+2}^{\infty} \frac{\chi(n) \log^\alpha n}{n^\sigma} &\leq \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \sum_{n=M+3}^{\infty} \frac{\log^\alpha n}{n^\sigma} - \sum_{n=\frac{M+3}{2}}^{\infty} \frac{\log^\alpha 2n}{(2n)^\sigma} \\
&\leq \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \int_{M+3}^{\infty} \frac{\log^\alpha x}{x^\sigma} dx - \frac{1}{2} \int_{M+3}^{\infty} \frac{\log^\alpha x}{x^\sigma} dx \\
&= \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \left(1 - \frac{1}{2}\right) \int_{M+3}^{\infty} \frac{\log^\alpha x}{x^\sigma} dx \\
&= \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \frac{1}{2} \frac{\Gamma(\alpha + 1, (\sigma - 1)\log(M+3))}{(\sigma - 1)^{\alpha+1}} \\
&< \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \frac{1}{2} \frac{B((\sigma - 1)\log(M+3))^{\alpha+1-1} e^{-(\sigma-1)\log(M+3)}}{(\sigma - 1)^{\alpha+1}} \\
&= \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \frac{1}{2} \frac{\log^\alpha(M+3)}{(M+3)^\sigma} \frac{M+3}{\sigma - 1} B.
\end{aligned}$$

Here, with the choice of $x = (\sigma - 1)\log(M+3)$ and $a = \alpha + 1$ in $x > \frac{B(1-a)}{1-B}$, we can obtain a lower bound for B:

$$B > \frac{(\sigma - 1)\log(M+3)}{(\sigma - 1)\log(M+3) - \alpha} = 1 + \frac{\alpha}{(\sigma - 1)\log(M+3) - \alpha},$$

and if we set $B := 1 + \epsilon + \frac{\alpha}{(\sigma-1)\log M - \alpha}$, for any $\epsilon > 0$, then we get:

$$T_M^\alpha(\sigma) < \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \frac{1}{2} \left(\frac{\log^\alpha(M+1)}{(M+1)^\sigma} \frac{M+1}{\sigma - 1}\right) \left(1 + \epsilon + \frac{\alpha}{(\sigma - 1)\log(M+3) - \alpha}\right).$$

Letting $\epsilon \to 0$ this bound becomes

$$T_M^\alpha(\sigma) \leq \frac{\log^\alpha(M+2)}{(M+2)^\sigma} + \frac{1}{2} \left(\frac{\log^\alpha(M+3)}{(M+3)^\sigma} \frac{M+3}{\sigma - 1}\right) \left(1 + \frac{\alpha}{(\sigma - 1)\log(M+3) - \alpha}\right)$$

$$\begin{aligned}
&= Q^\alpha_{M+2}(\sigma) + \frac{1}{2} Q^\alpha_{M+3}(\sigma) R^\alpha_{M+3}(\sigma) \\
&\leq Q^\alpha_{M+2}(\sigma) + \frac{1}{2} Q^\alpha_{M+2}(\sigma) R^\alpha_{M+3}(\sigma) \\
&= Q^\alpha_{M+2}(\sigma) \left(1 + \frac{1}{2} R^\alpha_{M+3}(\sigma)\right)
\end{aligned}$$

since $Q^\alpha_{M+2}(\sigma) \geq Q^\alpha_{M+3}(\sigma)$ which proves the lemma. □

We conclude with the proof of Corrolary 5.36 and some immediate consequences.

of Corollary 5.36 (a). We consider the case where $Q^\alpha_3(\sigma) = \frac{\log^\alpha(3)}{3^\sigma}$ is the dominant term of $L^{(\alpha)}(s,\chi)$, that is in (5.18) we have $M = 3$. We show that, for all real $\alpha > 0$ and all $\sigma > q_3^3\alpha + 2.364$, we have $L^{(\alpha)}(s,\chi)(s) \neq 0$. First, write

$$\begin{aligned}
|L^{(\alpha)}(s,\chi)(s)| &\geq \frac{\log^\alpha 3}{3^\sigma} - T^\alpha_3(\sigma) \\
&\geq Q^\alpha_3(\sigma) - Q^\alpha_5(\sigma)\left(1 + \frac{1}{2} R^\alpha_6(\sigma)\right) \\
&\geq Q^\alpha_3(\sigma)\left(1 - \frac{Q^\alpha_5(\sigma)}{Q^\alpha_3(\sigma)}\left(1 + \frac{1}{2} R^\alpha_6(\sigma)\right)\right)
\end{aligned}$$

By Lemma 5.32 for $A \geq \alpha$ we have

$$\begin{aligned}
R^\alpha_6(\sigma) &\leq R^\alpha_6(q_3^3 A + b) \\
&= R^\alpha_6(q_3^3\alpha + b) \\
&= \frac{6}{q_3\alpha + b - 1}\left(1 + \frac{\alpha}{(q_3^3\alpha + b - 1)\log 6 - \alpha}\right) \\
&= \frac{6}{q_3^3\alpha + b - 1}\left(1 + \frac{\alpha}{(q_3^3\log 6 - 1)\alpha + (b-1)\log 6}\right) \\
&\leq R^A_6(q_3^3 A + b) = \frac{6}{q_3^3 A + b - 1}\left(1 + \frac{A}{(q_3^3 A + b - 1)\log 6 - A}\right)
\end{aligned}$$

Now, the quotient $\frac{Q^\alpha_5}{Q^\alpha_3}(\sigma)$ is decreasing in σ, and as one can easily verify

$$\frac{Q^\alpha_{M+2}}{Q^\alpha_M}(q_M\alpha + b_1) = \left(\frac{M}{M+2}\right)^{b_1}$$

for all $M \geq 3$ and real numbers b_1 and b_2. Therefore,

$$\frac{Q^\alpha_5}{Q^\alpha_3}(\sigma) \leq \frac{Q^\alpha_5}{Q^\alpha_3}(q_M\alpha + b) = \left(\frac{3}{5}\right)^b.$$

For $A = 0$ and $\alpha > A$ and $b = 2.32$ and $\sigma \geq q_3^3\alpha + b$ we get

$$1 - \frac{Q_5^\alpha(\sigma)}{Q_3^\alpha(\sigma)}\left(1 + \frac{1}{2}R_6^\alpha(\sigma)\right) \geq 1 - 0.3057(1 + 2.27) > 0.$$

Thus for all real $\alpha > 0$ and all $\sigma \geq q_3^3\alpha + 2.32$ we have $L^{(\alpha)}(s,\chi)(s) \neq 0$. □ □

Theorem 5.36 (a) generalizes Verma & Kaur's bound [VK82] to fractional derivatives. Our bound is a bit weaker than theirs, as we consider any $\alpha > 0$ instead of $\alpha \geq 3$. Smaller values of b in the proof of Theorem 5.36 (a) yield tighter bounds that hold for greater α. In particular, any $b > 0$ yields a bound that holds for all sufficiently large values of α. With $b = 2$ we obtain the bound proved in [VK82] for $\alpha \geq 3$.

Corollary 5.38. *For any $b > 0$ there is an $A \in \mathbb{R}$ such that for all $\alpha > 0$ we have $L^{(\alpha)}(s,\chi)(s) \neq 0$, for all $s = \sigma + it$ with $\sigma \geq q_3^3\alpha + b$.*

Proof. Let $b > 0$. For estimating $R_6^\alpha(q_3^3\alpha + b)$ we set $A := 0$ and $\alpha = 1/q_3^3$. We obtain $R_6^\alpha(q_3^3\alpha + b) \leq \frac{3}{b}$. We use the bounds from the proof of Theorem 5.36 (a). We have $L^{(\alpha)}(s,\chi)(s) \neq 0$ for $\sigma \geq q_3^3\alpha + b$ when

$$\left(\frac{3}{5}\right)^b\left(1 + \frac{3}{b}\right) < 1$$

Which is not dependent on α. □

By [BPS15, Lemma 4.4] we have, for $3 \leq n \leq M$ and $\sigma \leq q_{M-c}\alpha - u(M-c)$,

$$\frac{Q_{n-c}^\alpha}{Q_n^\alpha}(\sigma) \leq \frac{1}{e^{cu}}.$$

Combined with equation (5.21), this yields

$$\frac{H_M^\alpha}{Q_M^\alpha}(\sigma) \leq \frac{1}{e^{cu} - 1}. \tag{5.25}$$

We are now ready to prove the final part (b) of Theorem 5.36.

of Theorem 5.36 (b). Let $\alpha > 0$. We show that if $M \in \mathbb{N}$, $M \geq 3$, and $q_{M+d}^M\alpha + (M + d)u \leq q_M^{M-c}\alpha - (M-c)u$ then $L^{(\alpha)}(s,\chi) \neq 0$ for

$$q_{M+d}^M\alpha + (M+d)u \leq \sigma \leq q_M^{M-c}\alpha - (M-c)u.$$

where $u \in (0, \infty)$ is a solution of $1 - \frac{1}{e^{cu}-1} - \frac{1}{e^{du}}(1 + \frac{1}{2u}) \geq 0$. Similar to the proof of Theorem 5.36 (b) we write

$$\begin{aligned}\left|L^{(\alpha)}(s, \chi)\right| &\geq Q_M^\alpha(\sigma) - H_M^\alpha(\sigma) - T_M^\alpha(\sigma) \\ &\geq Q_M^\alpha(\sigma)\left(1 - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{Q_{M+d}^\alpha}{Q_M^\alpha}(\sigma)\left(1 + \frac{1}{2}R_{M+d+1}^\alpha(\sigma)\right)\right).\end{aligned}$$

Now, we know

$$R_{M+d+1}^\alpha(\sigma) := \frac{M+d+1}{\sigma - 1}\left(1 + \frac{\alpha}{(\sigma-1)\log(M+d+1) - \alpha}\right) < \frac{1}{u}$$

But by [BPS15, Lemma 4.4] $\left(\frac{n-1}{n}\right)^{cn}$ is monotonously increasing with the asymptote $1/e^c$. And therefore

$$\frac{Q_{M+d}^\alpha}{Q_M^\alpha}(q_{M+d}^M\alpha + u(M+d)) = \left(\frac{M}{M+d}\right)^{u(M+d)} < \frac{1}{e^{du}}.$$

Finally, with the help of the bound (5.25), we can see, that for $M \geq 4$ and $q_{M+d}^M\alpha + u(M+d) \leq \sigma \leq q_M^{M-c}\alpha + (M-c)u$, we have

$$1 - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{Q_{M+d}^\alpha}{Q_M^\alpha}(\sigma)\left(1 + \frac{1}{2}R_{M+d+1}^\alpha(\sigma)\right) > 1 - \frac{1}{e^{cu}-1} - \frac{1}{e^{du}}\left(1 + \frac{1}{2u}\right) \geq 0,$$

which completes the proof of the theorem. □

5.6 Left Half-Plane

Yıldırım was able to show zero-free regions on the left half-plane for Dirichlet L-functions.

Theorem 5.39 ([Yıl96, Theorem 3]). *Given any $\epsilon > 0$, there exists a $K = K(k, \epsilon, \boldsymbol{a})$ such that there is no zero of $L^{(k)}(s, \chi)$ is the region $|s| > q^K, \sigma < -\epsilon, |t| > \epsilon$.*

5.7 Number of Zeros

Yıldırım then calculates $N_k(T, \chi)$. For $k \geq 1$, the number of zeros of $L^{(k)}(s, \chi)$ in a region $-q^K < \sigma < \sigma_k$, where q and K are the same as in Theorem 5.39, $|t| \leq T$ as $T \to \infty$. T is chosen so that there are no zeros of $L^{(k)}(s, \chi)$ on the lines $t = \pm T$ and σ_k is taken large enough so that

(i) $L^{(k)}(s, \chi)$ has no zeros in $\sigma \geq \sigma_k$.

(ii) $\sum_{n=m+1}^{\infty} \frac{(\log n)^k}{n^{\sigma_k}} \leq \frac{1}{2} \frac{(\log m)^k}{m^{\sigma_k}}$. Where m satisfies Theorem 5.27.

Theorem 5.40 ([Yıl96, Theorem 4.])**.** *Let χ be a primitive character modulo q and K is the same as in Theorem 5.39. Then*

$$N_k(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi e m} + \mathcal{O}(q^K \log T),$$

as $T \to \infty$.

He then states the connection of the theorem of Bohr and Landau [BL14], a Dirichlet series which converges for $\sigma > 0$ (in particular $L^{(k)}(s, \chi)$, $k \geq 0$) has $\mathcal{O}(T)$ zeros in $\sigma > \sigma_0 > \frac{1}{2}$, $|t| \leq T$.

Chapter 6: Polynomials

6.1 Introduction

Questions concerning finding exact or approximate values of the zeros of polynomial functions $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ are classical, and (for the case of real coefficients $c_0, c_1, \cdots c_n$) properties of the distribution of these zeros have been studied since at least 1637, when Descartes established his fundamental Rule of Signs (in *La Géométrie* [Des27]). This important result was refined to finite intervals by Budan in 1807 and by Fourier in 1820 (see [Fou92] or [Tie36]). By then, thanks to the work of Euler, Gauss and Argand (see [FR97] for more details), the Fundamental Theorem of Algebra had been established, guaranteeing that a polynomial of degree n has exactly n complex zeros (counted with multiplicity). Not much later, in 1829, Cauchy [Cau09] was able to prove that all zeros of a monic polynomial $p(x)$, with complex coefficients, must lie inside the disk $|z| < 1 + \max_{0 \le k \le n-1} |c_k|$, the bounds that were eventually generalized by Landau [Lan07], Fejér [Féj08] and others.

In another direction, one could ask about the relation between locations of the zeros of a polynomial $p(x)$ and the zeros of its derivative $p'(x)$, as Rolle has done in his *Traité d'algèbre* of 1690 (see [Sha37]); the well-known theorem bearing his name – that states that between any two zeros of a real polynomial there lies at least one zero of the derivative – was proved rigorously by Cauchy [Cau12] in 1823. In the complex plane, the situation becomes even more interesting. As Gauss noted in 1836, all zeros of $p'(x)$ lie in the convex hull of the zeros of $p(x)$ (see Figure 6.1). The first proof of this proposition was published by Lucas [Luc74] in 1874; it is now known as the Gauss-Lucas Theorem (also see [Mar66]). At the beginning of the 20th century it was refined by Bôcher [Bôc04], Jensen [Jen12] and Walsh [Wal20], and in more recent times, several related extensions and generalizations of it have been considered by Dimitrov [Dim98], Brown & Xiang [BX99], Sendov [Sen21], Tao [Tao22] and others.

The main aim of our work is to investigate connections between these two central themes. We show that their key ideas can be combined in a very natural way, but to quite surprising effects, if one considers the fractional derivatives $p^{(\alpha)}(x)$, where $\alpha \in \mathbb{R}$ is a variable $0 \le \alpha \le n = \deg p(x)$. Our main goal in this paper is to answer one of the most intriguing questions that arises as soon as one begins to study these

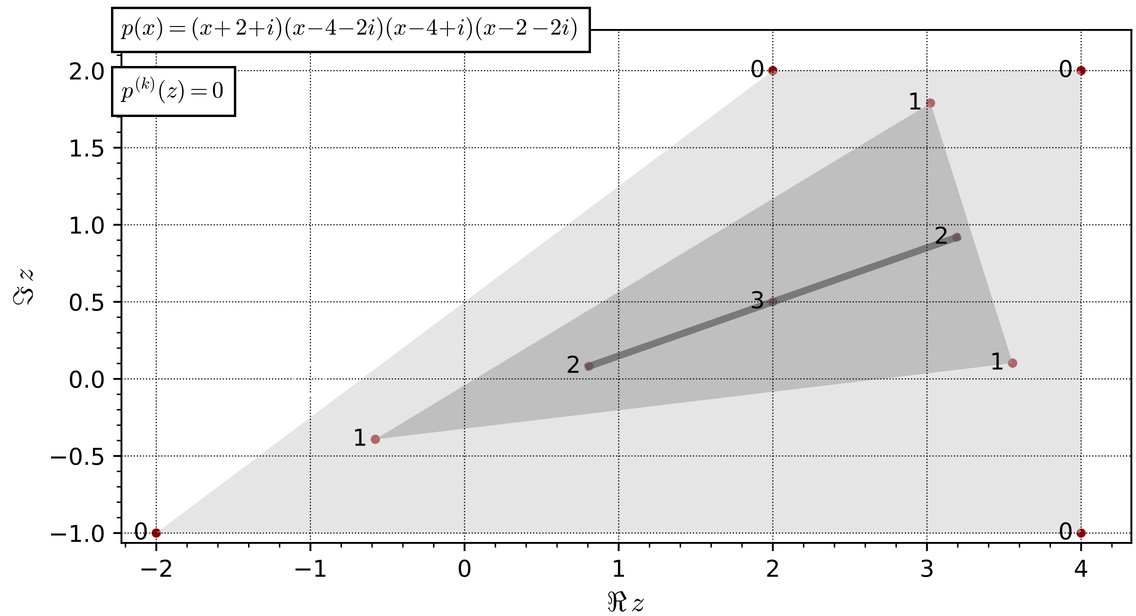

Figure 6.1. The roots of the polynomial $p(x) = (x+2+i)(x-4-2i)(x-4+i)(x-2-2i)$ and its derivatives $p^{(k)}$, illustrating that the convex hull of the roots 0• of $p(x)$ contains the roots 1• of $p'(x)$ and that the convex hull of the roots 1• of $p'(x)$ contains the roots 2• of $p''(x)$ and that that the convex hull of the roots 2• of $p''(x)$ contains the root 3• of $p'''(x)$.

topics: since obviously $\deg p'(x) = \deg p(x) - 1$, and the Fundamental Theorem asserts than the same reduction must occur for the total number of zeros, what happens to the zeros of fractional derivatives, as the real α increases continuously from 0 to n? How do the zeros of polynomials vanish, and why? As it turns out, these questions have remarkably simple and elegant answers. Namely, for a polynomial $p(x)$ of degree n, each of its n zeros belong to a path of unique length that connects it to the origin, where the "length" of the path can be measured by the number of zeros of its derivatives it contains; in other words, for each $0 \leq k \leq n-1$ there is a unique path (originating at one of the zeros of $p(x)$) that contains exactly k zeros of its higher derivatives. (Figure 6.2 shows this general property for a generic cubic polynomial).

Another goal of this chapter is to try to understand some of the particulars of the paths the fractional zeros take, their dynamical properties. In order to state our results concerning this general flow of polynomial zeros more precisely, first we need to recall some basic definitions and properties of fractional derivatives.

The remainder of this chapter is structured as follows. In Section 6.2 we recall results about Riemann-Liouville fractional derivatives of polynomials. In Section 6.3 this theory is applied to the two simplest cases: polynomials of degree one and

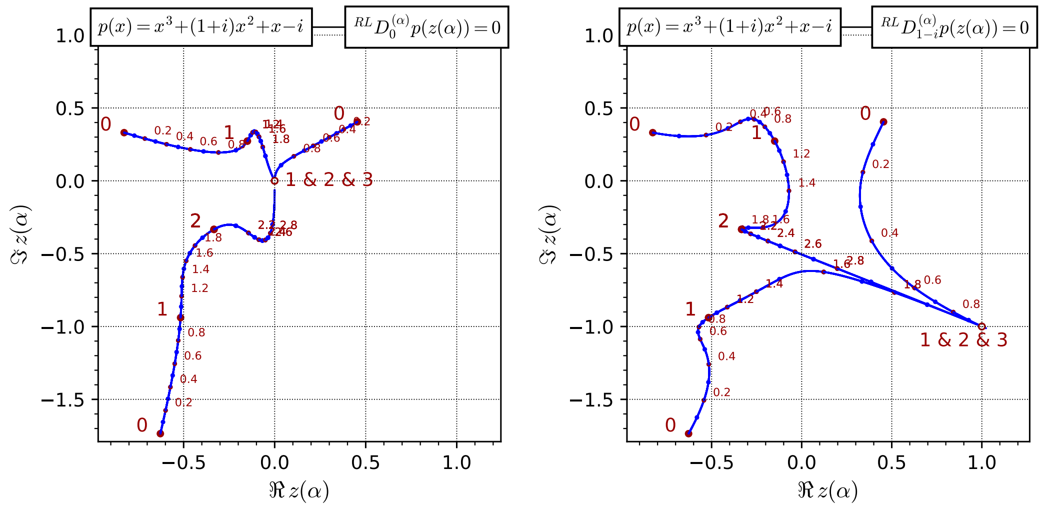

Figure 6.2. Paths $z(\alpha)$ of zeros of the Riemann-Liouville fractional derivatives ${}^{RL}D_0^{(\alpha)}p$ and ${}^{RL}D_{1-i}^{(\alpha)}p$ of the polynomial $p(x) = x^3 + (1+i)x^2 + x - i$. We end the paths when $z(\alpha)$ reaches the "origin" ($a = 0$ in the first case, and $a = 1 - i$ in the second).

two. These are the two cases where, thanks to the manageable classical formulas for the zeros, all the main questions can be conclusively answered. With the cubic polynomials things become somewhat murky, but general convergence trends can still be established. In Section 6.4 we do just that: we examine how the zeros of integral derivatives are connected to the zeros of fractional derivatives in the most general setting, and we look at the paths of zeros and investigate their convergence and the overall flow. In Section 6.5 we consider the behaviour of the zeros on a larger scale and we prove bounds for the Mahler measure of the fractional derivatives, which are then also established for the Caputo fractional derivative in Section 6.5.1 Finally, in Section 6.6, we discuss some intriguing open problems and unsolved questions.

6.2 Path of Zeros

Recall that for the simple power functions $p(x) = (x-a)^\beta$, where $\beta \in \mathbb{R}$, $a \in \mathbb{C}$, the α-th Riemann-Liouville fractional derivative can be computed using the Power Rule (Section 2.2.1 equation (2.5)):

$$ {}^{RL}D_a^{(\alpha)}(x-a)^\beta = \begin{cases} 0 & \text{if } \alpha - \beta - 1 \in \mathbb{N} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} & \text{otherwise} \end{cases} \tag{6.1} $$

Remark 6.1. The Riemann-Liouville fractional derivative of a monomial $f(x) = x^n$ is multivalued. When changing the branch of the complex logarithm in the computation of the fractional derivative all coefficients of the derivative are changed by the same factor. So choosing a different branch of the complex logarithm does not change the zeros of the derivative, which means that we can fix the branch in our consideration of zeros of derivatives of polynomials. We use the principal branch of the complex logarithm.

It should be noted that the constant a that centers the expansion (2.5) plays a key role in all our computations below, as the "origin," or the limit of convergence, of the flow of zeros of derivatives (Figure 6.2 illustrates its role).

Remark 6.2. It is possible to go beyond the standard values of $0 \leq \alpha$, and consider what happens for $\alpha < 0$. Here, there are $n+1$ complex roots, because the first term in Equation 6.2 below has the root 0. Just like in the standard case, the extended curves $z(\alpha)$ of zeros of the differintegral ${}^{RL}D_a^{(\alpha)}p$ are continuous for $\alpha < 0$ unless ${}^{RL}D_a^{(\alpha)}p$ has a double root; however, they are not be smooth at integral $\alpha > 0$. More on this will be said in Section 6.4 below.

In what follows, we consider the zeros of the the fractional derivatives of polynomials $p \in \mathbb{C}[x]$ of degree n, and we investigate the implicit functions $z : (0, n) \to \mathbb{C}$ given by

$$ {}^{RL}D_a^{(\alpha)} f(z(\alpha)) = 0. $$

If $\left({}^{RL}D_a^{(\alpha)}p(x)\right)' \neq 0$, for $\alpha \in (0, n)$, then $z(\alpha)$ is differentiable on $(0, n)$. We denote the roots of the polynomial $p(x)$ by $z_1, z_2, \ldots, z_n$ and for $1 \leq k \leq n$ we define the the implicit function $z_k : [0, n) \to \mathbb{C}$ by $z_k(0) = z_k$ and ${}^{RL}D_a^{(\alpha)}p(z(\alpha)) = 0$.

The following representation of the fractional derivatives of a general monic polynomial will be most useful.

Lemma 6.3. *Let* $p(x) = (x-a)^n + \sum_{j=0}^{n-1} c_j(x-a)^j \in \mathbb{C}[x]$ *and* $\alpha \notin \mathbb{N}$. *Then*

$$ {}^{RL}D_a^{(\alpha)}p(x) = \frac{n!}{\Gamma(n+1-\alpha)}(x-a)^{-\alpha}\left[(x-a)^n + \sum_{j=0}^{n-1}\left(\prod_{k=j+1}^{n}(k-\alpha)\right)\cdot\frac{j!}{n!}c_j(x-a)^j\right]. \tag{6.2} $$

Proof. Applying the Power Rule (2.5), we obtain

$$ \begin{aligned} {}^{RL}D_a^{(\alpha)}p(x) &= \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(x-a)^{n-\alpha} + \sum_{j=0}^{n-1}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}c_j(x-a)^{j-\alpha} \\ &= \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(x-a)^{-\alpha}\left[(x-a)^n + \sum_{j=0}^{n-1}\frac{\Gamma(n+1-\alpha)}{\Gamma(j+1-\alpha)}\frac{\Gamma(j+1)}{\Gamma(n+1)}c_j(x-a)^j\right] \end{aligned} $$

$$= \frac{n!}{\Gamma(n+1-\alpha)}(x-a)^{-\alpha}\left[(x-a)^n + \sum_{j=0}^{n-1}\left(\prod_{k=j+1}^{n}(k-\alpha)\right)\cdot\frac{j!}{n!}c_j(x-a)^j\right],$$

as wanted. □

Remark 6.4. The representation of the fractional derivatives of polynomials given in the above Lemma 6.3 has the property that their roots only depend on the factors in square brackets, which in turn implies the useful fact that the branch cut of the complex logarithm does not affect the paths $z(\alpha)$ of zeros of these fractional derivatives.

We also see that for $p(x) = (x-a)^n$ we have:

$$^{RL}D_a^{(\alpha)}p(a) = \begin{cases} 0 & \text{if } \alpha < n \\ \Gamma(n+1) & \text{if } \alpha = n \\ \text{undefined} & \text{if } \alpha > n \end{cases} \tag{6.3}$$

Remark 6.5. A few words should be said about our plots of the implicit functions $z : \mathbb{R} \to \mathbb{C}$ with $z(\alpha)$ given by $^{RL}D_a^{(\alpha)}p(z(\alpha)) = 0$. The dots labelled '•k' represent zeros of the kth Riemann-Liouville differintegral (thus '•0' represent the zeros of the polynomial $p(x)$ itself), while circles '∘k' represent points that are limits of $z(\alpha)$ as $\alpha \to k$ but are not zeros of $^{RL}D_a^{(k)}p$. These occur for integral k with $k \geq \deg p(x)$, where $^{RL}D_a^{(k)}p(x)$ is constant, for example at $x = a$, see Equation (6.3). Moreover, when a point is either a zero or a limit point of zeros of both the jth and the kth differintegrals, then it is represented by '• j & k' or '∘ j & k' respectively. In Figures 6.2, 6.4, and 6.5 we let all paths of zeros of $^{RL}D_a^{(\alpha)}p(x)$ end when they reach the origin a. In Figures 6.3, 6.4, 6.6, 6.7, and 6.8 we continue the paths past a. In Figures 6.3 and 6.6 we display the path of zeros $z(\alpha)$ for $\alpha < 0$ and $\alpha > n$ in lighter colors than for $0 \leq \alpha < n$ where n is the degree of the polynomial. The point a is only labeled with values for α for $\alpha \geq 0$.

6.3 Low Degrees

Formulas for finding zeros of polynomials of low degrees have been known for centuries. Applying the Riemann-Liouville derivative to these low-degree cases has proved to be a simple but informative exercise. In this section we summarize some of these results, stating the most useful ones as lemmas. They are examples of a dynamic that shares certain key characteristic with most high-degree cases, but some aspects of which are often unique. For example, in the linear case, the path the zeros take is also linear, while already in the quadratic case one observes a considerably more complex behavior.

Let us start with the linear polynomials. Here the situation is simple. The paths of zeros is always linear and they can be completely described. We have:

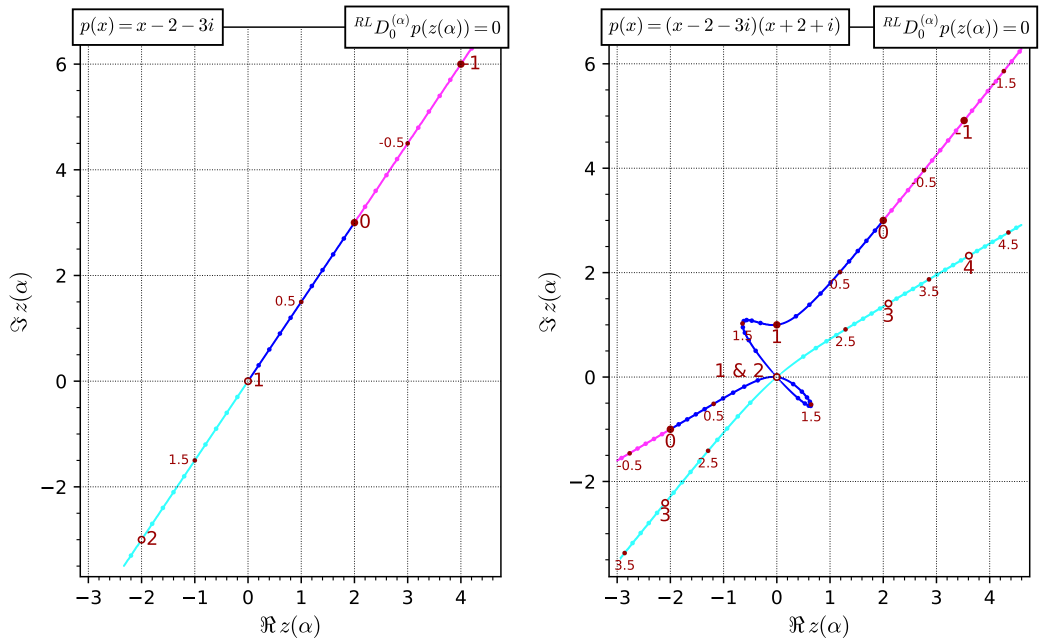

Figure 6.3. Path $z(\alpha)$ of zeros of differintegrals of the polynomials $p(x) = x - 2 - 3i$ and $p(x) = (x - 2 - 3i)(x + 2 + i)$.

Lemma 6.6 (Linear Polynomials)**.** *Let* $p(x) = (x - a) + c_0$*. As* α *increases from* 0 *to* 1*, the path of zeros of* ${}^{RL}D_a^{(\alpha)}p(x)$ *is given by* $z(\alpha) = (\alpha - 1)c_0 + a$*.*

Proof. From (2.5), with $\beta = 1$, we get: ${}^{RL}D_a^{(\alpha)}p(x) = (x - a)^{-\alpha}\left((x - a) + (1 - \alpha)c_0\right)$. □

Remark 6.7. In addition to considering the α-th derivatives in the usual range $0 \leq \alpha \leq n$, one could also look at what happens when $\alpha < 0$ and when $\alpha > n$. In the linear case this is, again, simple. From Lemma 6.6 we can deduce that, as with $\alpha < 0$, the line of roots of the derivatives continues. Similarly, for $\alpha > 1$, $\alpha \notin \mathbb{Z}$ lie on the same line, see Figure 6.3 below.

Let us now consider the quadratic case. This case is considerably more interesting, since there are now two paths of zeros of the fractional derivatives, and they exhibit a much more complex and intricate behavior. We first notice that the path of the zero closest to a directly connects with a, while the path of the zero farthest from a in the process of reaching a passes through the zero of the first derivative.

Lemma 6.8. *Let* $p(x) = x^2 + bx + c \in \mathbb{C}[x]$ *with roots* s_1 *and* s_2 *and let* $d = 1 - \frac{4c}{b^2}$*.*

For $d \in \mathbb{C} \setminus \mathbb{R}^{<0}$*, denote by* $\sqrt{d}$ *the complex number* r *with* $r^2 = d$ *and* $\Re(r) > 0$ *and for* $x \in \mathbb{R}$ *let* $\operatorname{sgn}(x) = \frac{x}{|x|}$*.*

1. *If* $d \in \mathbb{C} \setminus \mathbb{R}^{<0}$ *and* $b \neq 0$*, then for the roots* $s_1 = \frac{-b+\operatorname{sgn}(\Re(b))\sqrt{b^2-4c}}{2}$ *and* $s_2 = \frac{-b-\operatorname{sgn}(\Re(b))\sqrt{b^2-4c}}{2}$ *of* $p(x)$ *we have* $|s_2| \geq |s_1|$*.*

2. *If* $b = 0$*, then* $|s_1| = |s_2|$*.*

3. *If* $d \in \mathbb{R}^{<0}$*, then* $|s_1| = |s_2|$*.*

Proof. 1. The roots of $p(x)$ are

$$\begin{aligned}\frac{-b \pm \sqrt{b^2-4c}}{2} &= \frac{b}{2}\left(-1 \pm \frac{\sqrt{b^2-4c}}{b}\right) \\ &= \frac{b}{2}\left(-1 \pm \frac{\sqrt{b^2-4c}}{\operatorname{sgn}(\Re(b))\sqrt{b^2}}\right) \\ &= \frac{b}{2}\left(-1 \pm \operatorname{sgn}(\Re(b))\sqrt{1-\frac{4c}{b^2}}\right) \\ &= \frac{b}{2}\left(-1 \pm \operatorname{sgn}(\Re(b))\sqrt{d}\right)\end{aligned}$$

Considering the absolute value of the last term we get:

$$\begin{aligned}\left|-1 \pm \operatorname{sgn}(\Re(b))\sqrt{d}\right|^2 &= \left(-1 \pm \operatorname{sgn}(\Re(b))\sqrt{d}\right)\left(-1 \pm \operatorname{sgn}(\Re(b))\overline{\sqrt{d}}\right) \\ &= 1 \mp \operatorname{sgn}(\Re(b))\sqrt{d} \mp \operatorname{sgn}(\Re(b))\overline{\sqrt{d}} + |d| \\ &= 1 \mp 2\operatorname{sgn}(\Re(b))\Re(\sqrt{d}) + |d|\end{aligned}$$

Because $\Re\left(\sqrt{d}\right) > 0$ we have

$$\begin{aligned}|-b+\sqrt{b^2-4c}| &> |-b-\sqrt{b^2-4c}| &\quad \text{when } \operatorname{sgn}(\Re(b)) = -1 \\ |-b-\sqrt{b^2-4c}| &> |-b+\sqrt{b^2-4c}| &\quad \text{when } \operatorname{sgn}(\Re(b)) = +1\end{aligned}$$

and

$$|-b-\operatorname{sgn}(\Re(b))\sqrt{b^2-4c}| > |-b+\operatorname{sgn}(\Re(b))\sqrt{b^2-4c}|$$

which implies $|s_2| > |s_1|$.

2. When $b = 0$ the roots of $p(x)$ are $s_{1,2} = \frac{\pm\sqrt{-4c}}{2}$. Hence $|s_1| = |s_2|$.

3. Here $\Re\sqrt{d} = 0$ and thus $\left|-1+\sqrt{d}\right|^2 = 1 + |d| = \left|-1-\sqrt{d}\right|^2$, which implies $|s_1| = |s_2|$. □

Remark 6.9. As of yet, there is no known reliable ordering of the zeros for any of the higher degree polynomials. In fact, there exist examples of cubic polynomials for which the standard Euclidean distance (which works so well for the linear and the quadratic cases) can be shown to fail: see Figure 6.5.

In addition to the natural ordering on the quadratic roots, another question that seems to be of interest is the one that concerns the trends of descent of their paths, especially since it had such a nice answer in the linear case. As it turns out, the asymptotes of the two quadratic paths exist, and the quadratic formula alone is enough to help us find them.

Theorem 6.10. *For the quadratic polynomial* $p(x) = (x-a)^2 + c_1(x-a) + c_0$*, the paths of zeros of the fractional derivatives* ${}^{RL}D_a^{(\alpha)} p(x)$ *are given as*

$$z_{1,2}(\alpha) = a + \frac{-(2-\alpha)c_1 \pm \operatorname{sgn}(\Re(c_1))\sqrt{(2-\alpha)^2 \cdot (c_1)^2 - 8(2-\alpha)(1-\alpha)\cdot c_0}}{4}, \tag{6.4}$$

with $|z_1(\alpha)| \geq |z_2(\alpha)|$*, for* $\alpha \in [0,2]$*, and* $\lim_{\alpha\to 1} z_1(\alpha) = a$ *and* $\lim_{\alpha \to 2} z_{1,2}(\alpha) = a$*.*

Proof. With the help of (2.5), the fractional derivatives of $p(x)$ can be written as

$${}^{RL}D_a^{(\alpha)} p(x) = \frac{2(x-a)^{-\alpha}}{\Gamma(3-\alpha)} \left((x-a)^2 + \frac{(2-\alpha)\cdot c_1}{2}(x-a) + \frac{(2-\alpha)(1-\alpha)\cdot c_0}{2} \right).$$

Now, set $y = x - a$. Then the roots of $y^2 + \frac{(2-\alpha)\cdot c_1}{2} y + \frac{(2-\alpha)(1-\alpha)\cdot c_0}{2}$ are

$$z_{1,2}(\alpha) = a + \frac{-(2-\alpha)c_1 \pm \operatorname{sgn}(\Re(c_1))\sqrt{(2-\alpha)^2 \cdot (c_1)^2 - 8(2-\alpha)(1-\alpha)\cdot c_0}}{4}$$

The ordering of the roots $|z_1(\alpha)| \geq |z_2(\alpha)|$, for $\alpha \in [0,2]$, follows with Lemma 6.8. Furthermore, we have

$$\lim_{\alpha \to 1} z_{1,2}(\alpha) = a + \frac{-c_1 \pm \sqrt{(c_1)^2}}{4} = a + \frac{-c_1 \pm c_1}{4}.$$

Thus $\lim_{\alpha\to 1} z_1(\alpha) = a$ and $\lim_{\alpha \to 1} z_2(\alpha) = a + \frac{c_1}{2}$. Similarly

$$\lim_{\alpha \to 2} \left[a + \frac{-(2-\alpha)c_1 \pm \sqrt{(2-\alpha)^2 \cdot (c_1)^2 - 8(2-\alpha)(1-\alpha)\cdot c_0}}{4} \right] = a. \qquad \square$$

Corollary 6.11. *For* $\alpha \to \pm\infty$*, the asymptotes of the quadratic paths are*

$$z_{1,2}(\alpha) \approx \frac{(2-\alpha)c_1}{4} \left[-1 \pm \sqrt{1 - \frac{8c_0}{c_1}} \right].$$

Proof. By (6.4) we have,

$$
\begin{aligned}
z_{1,2}(\alpha) &= \frac{-(2-\alpha)c_1 \pm (2-\alpha)c_1\sqrt{1-8\frac{(1-\alpha)}{(2-\alpha)}\cdot\frac{c_0}{c_1}}}{4} \\
&= \frac{(2-\alpha)c_1}{4}\left[-1 \pm \sqrt{1-\frac{8c_0(1-\alpha)}{c_1^2(2-\alpha)}}\right],
\end{aligned}
$$

and since $\frac{1-\alpha}{2-\alpha} \to 1$, as $\alpha \to \pm\infty$, this yields linear asymptotes for $z_1(\alpha)$ and $z_2(\alpha)$. □

An noteworthy special case occurs when the polynomial has a double root. Then the paths display an interesting symmetry, see Figure 6.4. In fact, it is easy to see that specializing our Proposition 6.10 to the case of a double zero of the polynomial itself yields:

Corollary 6.12. *If* $p(x) = (x - z_0)^2$ *then the zeros of* ${}^{RL}D_0^{(\alpha)}p$ *are*

$$
z_{1,2}(\alpha) = \frac{z_0\left(-(2-\alpha) \pm \sqrt{\alpha(2-\alpha)}\right)}{2}.
$$

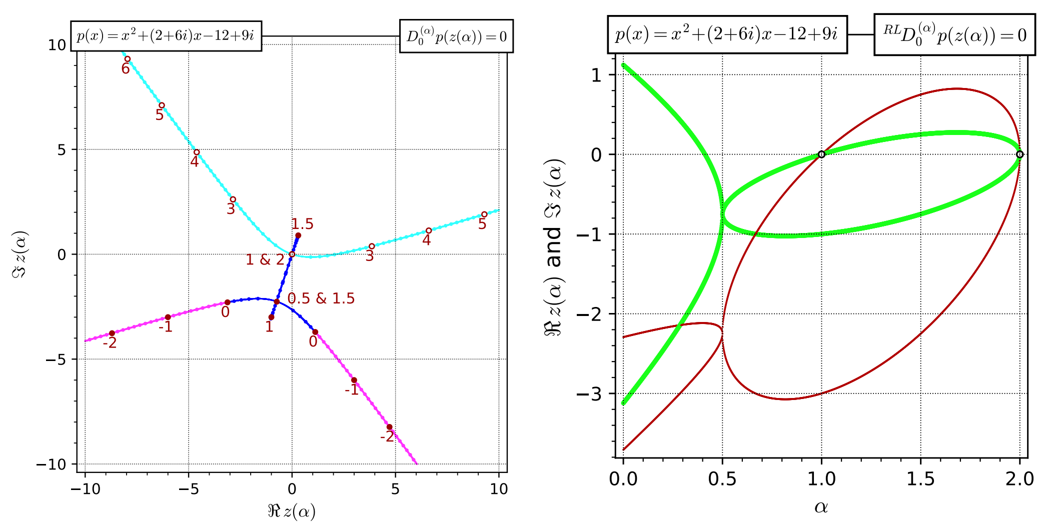

Figure 6.4. Paths $z(\alpha)$ of zeros of the fractional derivatives of $p(x) = x^2 + (2+6i)x - 12 + 9i$ where ${}^{RL}D_0^{(1/2)}p$ has a double root. In the plot on the right the wide, light green graph represents the real part of $z(\alpha)$, while the thin, dark red graph represents its imaginary part.

Another natural question to ask is whether, given that a quadratic polynomial has distinct zeros, can its fractional derivative have a double zero. Setting $z_1(\alpha) = z_2(\alpha)$ one gets:

Corollary 6.13. *Let $p(x) = (x-a)^2 + c_1(x-a)^1 + c_0(x-a)^0$. Then the fractional derivative ${}^{RL}D_a^{(\alpha)}(p(x))$ has a double zero precisely for one $\alpha \in \mathbb{R} \setminus \mathbb{N}$, namely: $\alpha = 1 - \frac{c_1^2}{8c_0 - c_1^2}$.*

6.4 Flow of Zeros

As stated above, one of our main goals was to consider the paths of zeros of the fractional derivatives of polynomials $p(x) \in \mathbb{C}[x]$ of arbitrary degrees. Unfortunately, unlike in the linear and quadratic cases, already for the cubics we find that the situation becomes considerably more complicated. This can be seen from the fact that one of the nicest properties – the natural ordering of zeros – fails already for degree 3: in other words, it is not true in general that zeros furthest away from the origin yields the longest paths of zeros of fractional derivatives on its way to the origin. Figure 6.5 shows a notable counterexample.

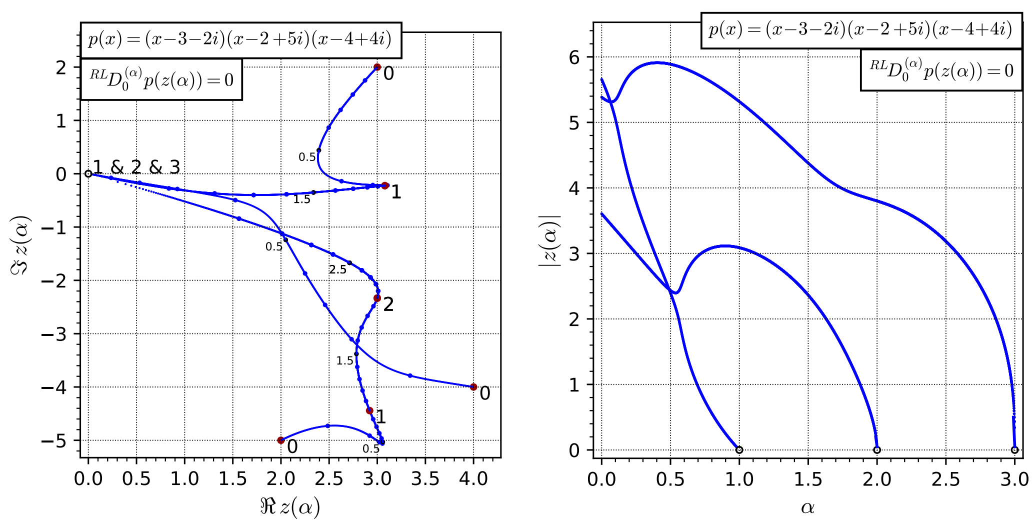

Figure 6.5. The paths $z(\alpha)$ of zeros of the fractional derivatives of the cubic $p(x) = (x-3-2i)(x-2+5i)(x-4+4i)$ and the absolute values $|z(\alpha)|$. In the latter case, the zero of $p(x)$ with the greatest absolute value is not the starting point of the longest path.

However, certain convergence properties of the paths can be established in general. For example, the following theorem shows that all the paths terminate in the origin a.

Theorem 6.14. *Let $p(x) \in \mathbb{C}[x]$ of degree n such that for all $\alpha \in [0, n]$ the fractional derivative $p^{(\alpha)}(x)$ has no double zeros. Then there is an ordering of the roots z_1, z_2, ..., z_n or p such that for $p^{(\alpha)}(z_j(\alpha)) = 0$ and $z_j(0) = z_j$ for $j \in \{1, ..., n\}$ we have $\lim_{\alpha \to j} z_j(\alpha) = a$*

Proof. We denote the coefficients in the expansion proved in Lemma 6.3 by

$$d_j^\alpha := \prod_{k=j+1}^{n} (k - \alpha) \cdot \frac{j!}{n!}. \tag{6.5}$$

Let $0 \leq j \leq n$ and $m > j$. Here, clearly

$$\lim_{\alpha \to m} d_j^\alpha = \lim_{\alpha \to m} \prod_{k=j+1}^{n} (k - \alpha) \cdot \frac{j!}{n!} = 0.$$

Write the coefficients of the derivatives ${}^{RL}D_a^{(\alpha)} p(x)$ by $d_j^\alpha \, c_j$ as symmetric functions of the roots $z_1(\alpha), z_2(\alpha), \ldots, z_n(\alpha)$ of ${}^{RL}D_a^{(\alpha)} p(x)$. Because

$$0 = \lim_{\alpha \to m} d_0^\alpha \, c_0 = \lim_{\alpha \to m} \prod_{k=1}^{n} z_k(\alpha)$$

we have $\lim_{\alpha \to m} z_{k_1}(\alpha) = 0$, for at least one $1 \leq k_1 \leq n$. Inductively, continuing with the next coefficient we get:

$$0 = \lim_{\alpha \to m} d_1^\alpha \, c_1 = \lim_{\alpha \to m} \sum_{l=1}^{n} \prod_{k \neq l} z_k(\alpha). \tag{6.6}$$

There is one summand in (6.6) that does not contain $z_{k_1}(\alpha)$. So we need to have $\lim_{\alpha \to m} z_{k_2}(\alpha) = 0$ for at least one $k_2 \neq k_1$. This argument also holds for all $d_j^\alpha \, c_j$ with $j < m$. Therefore, we get $\lim_{\alpha \to m} z_k(\alpha) = 0$ for at least m distinct $k \in \{1, \ldots, n\}$. □

6.5 Bounds

As stated in the introduction, for $f \in \mathbb{C}[x]$ the Gauss-Lucas theorem states that all zeros of f' lie in the convex hull of the set of zeros of f, see [Mar66, Theorem 6.1]. By induction this generalizes to all integral derivatives. Unfortunately, although all roots of the fractional derivatives converge to the origin, by our Theorem 6.14, the analogue of the Gauss-Lucas theorem does not hold for the fractional derivatives. This is an immediate consequence of a result by Genchev and Sendov [GS58], which is also stated as [NS14, Theorem 2]:

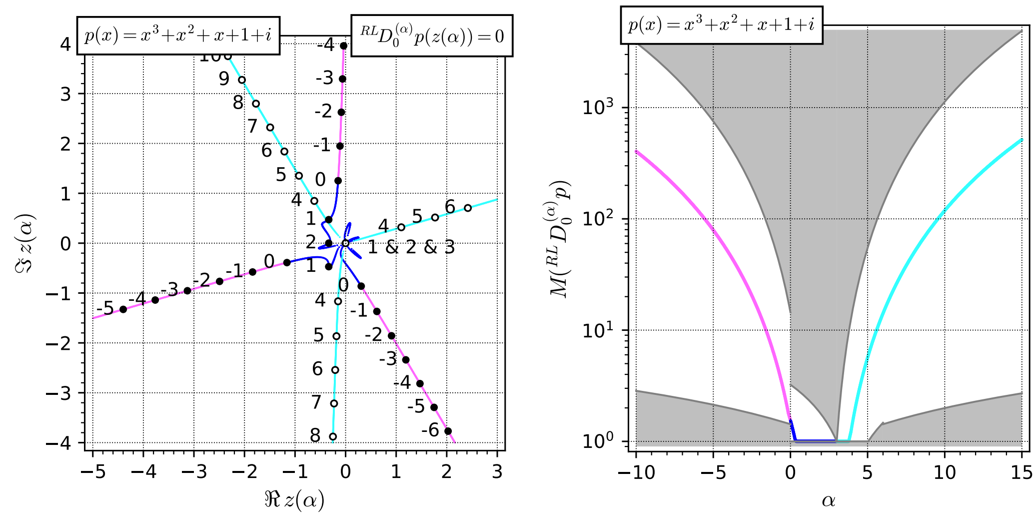

Figure 6.6. Paths $z(\alpha)$ of zeros of $^{RL}D_0^{(\alpha)}(x^3+x^2+x+1+i)$. illustrating the growth of the absolute value of $z(\alpha)$ as $\alpha \to \infty$ and $\alpha \to -\infty$ and the Mahler measure of $^{RL}D_0^{(\alpha)}p$ along with the bounds from Theorems 6.17 and 6.18.

Theorem 6.15. *Let $L : \mathbb{C}[x] \to \mathbb{C}[x]$ be a linear operator, such that $L(p) \neq 0$ implies that the convex hull of the set of roots of p contains the roots of $L(p)$. Then L is a linear functional or there are $c \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$ such that $L(p) = cp^{(k)}$.*

Figure 6.5 illustrates this result by giving a specific counterexample to the Gauss-Lucas property for the case of the Riemann-Liouville fractional derivatives. Nevertheless, it is possible to make some useful statements about how the absolute values of zeros $z_k(\alpha)$ of the fractional derivatives $^{RL}D_0^{(\alpha)}f$ decrease as α increases in terms of the Mahler measure of f.

Let $p(x) = x^n + \sum_{j=1}^{n-1} c_j x^j = \prod_{j=1}^{n}(x - z_j) \in \mathbb{C}[x]$. For the Mahler measure $M(f)$ [Mah61] we have

$$M(p) = \exp\left(\int_0^1 \log(|f(e^{2\pi i\theta})|)\, d\theta\right) = \prod_{j=1}^{n} \max\{1, |z_j|\}.$$

Denote the height of f by $\|p\|_\infty = \max\{c_0, \dots, c_n\}$ and the length of f by $\|p\|_1 = |c_0| + \cdots + |c_n|$. Recall that Mahler was able to prove the bounds

$$\binom{n}{\lfloor n/2 \rfloor}^{-1} \|p\|_\infty \leq M(p) \leq \|p\|_\infty \sqrt{n+1} \tag{6.7}$$

and

$$2^{-n}\|p\|_1 \leq M(p) \leq \|p\|_1. \tag{6.8}$$

For $\|p\|_2 = \left(\sum_{j=1}^{n} |c_j|^2\right)^{\frac{1}{2}}$ we have Landau's inequality [Lan05]

$$M(p) \leq ||p||_2. \tag{6.9}$$

We generalize the definition of the Mahler measure to fractional derivatives. Let $Z(\alpha)$ be the set of zeros of ${}^{RL}D_0^{(\alpha)}p$. We set

$$M\left({}^{RL}D_0^{(\alpha)}p\right) = \prod_{z \in Z(\alpha)} \max\{1, |z|\}.$$

and prove bounds similar to Equations 6.7, 6.8, and 6.9 for the fractional cases. We first estimate the coefficients d_j^α from Proposition 6.14 and then use them to derive bounds for $M\left({}^{RL}D_0^{(\alpha)}p\right)$.

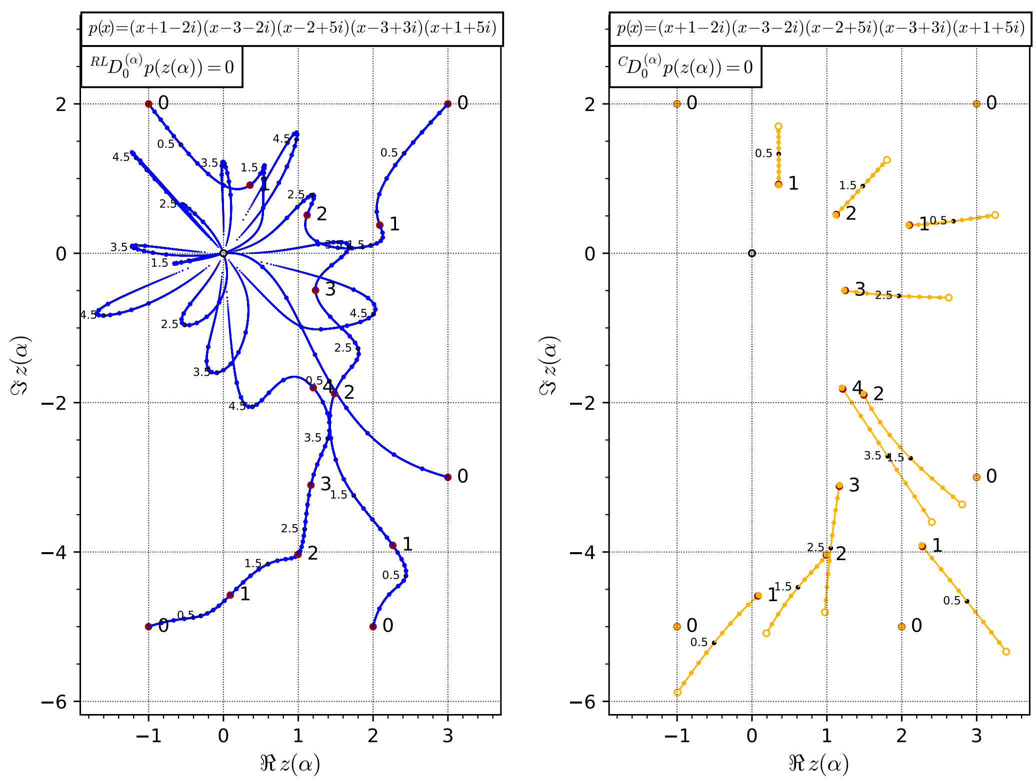

Figure 6.7. Comparing paths $z(\alpha)$ of zeros of the Riemann-Liouville (left) and Caputo (right) fractional derivatives of the quintic $p(x) = (x+1-2i)(x-3-2i)(x-2+5i)(x-3+3i)(x+1+5i)$, for $0 \leq \alpha \leq 5$.

Lemma 6.16. *Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{R} \setminus \mathbb{N}$. Let $d_j^\alpha = \left(\prod_{k=j+1}^{n}(k-\alpha)\right) \cdot \frac{j!}{n!}$ where $0 \le j \le n-1$. Then*

1. $|d_j^\alpha| \le \frac{n-\alpha}{n}$, *for* $0 < \alpha < n$.
2. $|d_j^\alpha| \le \frac{|n-\alpha|_n}{n}$, *where* $|n-\alpha|_n = \prod_{k=0}^{n-1} |n-\alpha-k|$, *for* $\alpha < 0$ *and* $\alpha > n$.
3. $|d_j^\alpha| \ge \frac{|n-\alpha|}{n}$, *for* $\alpha < 0$ *and* $\alpha > 2n$.
4. $|d_j^\alpha| \ge \frac{\alpha-n}{n}\binom{n-1}{(n-1)/2}^{-1}$, *for* $n < \alpha \le 2n$.

Proof. 1. For $0 \le \alpha < j+1$, we have

$$|d_j^\alpha| = \frac{n-\alpha}{n} \prod_{k=j+1}^{n-1} \frac{k-\alpha}{k} \cdot \frac{j!}{j!} < \frac{n-\alpha}{n}.$$

When $j+1 < \alpha$ set $h := \lfloor \alpha \rfloor$. We get

$$|d_j^\alpha| = \frac{n-\alpha}{n} \cdot \frac{(n-1-\alpha)\cdots(h+1-\alpha)\cdot(\alpha-j-1)\cdots(\alpha-h)\cdot j!}{(n-1)!} \le \frac{n-\alpha}{n}.$$

2. For $\alpha < 0$ and $\alpha > n$, we have

$$\begin{aligned} |d_j^\alpha| &= \frac{|n-\alpha|\cdots|j+1-\alpha|\cdot j!}{n!} \\ &= \frac{|\alpha-n|\cdots|\alpha-j-1|\cdot j!}{n!} \\ &\le \frac{|\alpha-n|\cdots|\alpha-1|}{n!} = \frac{|n-\alpha|_n}{n!}. \end{aligned}$$

3. For $\alpha < 0$ and $\alpha > 2n$, we have

$$|d_j^\alpha| \ge \frac{|n-\alpha|}{n} \cdot \frac{|n-1|\cdots|j+1|\cdot j!}{(n-1)!} = \frac{|n-\alpha|}{n}.$$

4. For $2n > \alpha > n$, we have

$$\begin{aligned} |d_j^\alpha| &= \frac{(\alpha-n)\cdots(\alpha-j-1)\cdot j!}{n!} \\ &\ge \frac{\alpha-n}{n} \cdot \frac{1\cdots(n-j-1)\cdot j!}{(n-1)!} \\ &\ge \frac{\alpha-n}{n}\binom{n-1}{(n-1)/2}^{-1}. \end{aligned}$$

$\square$

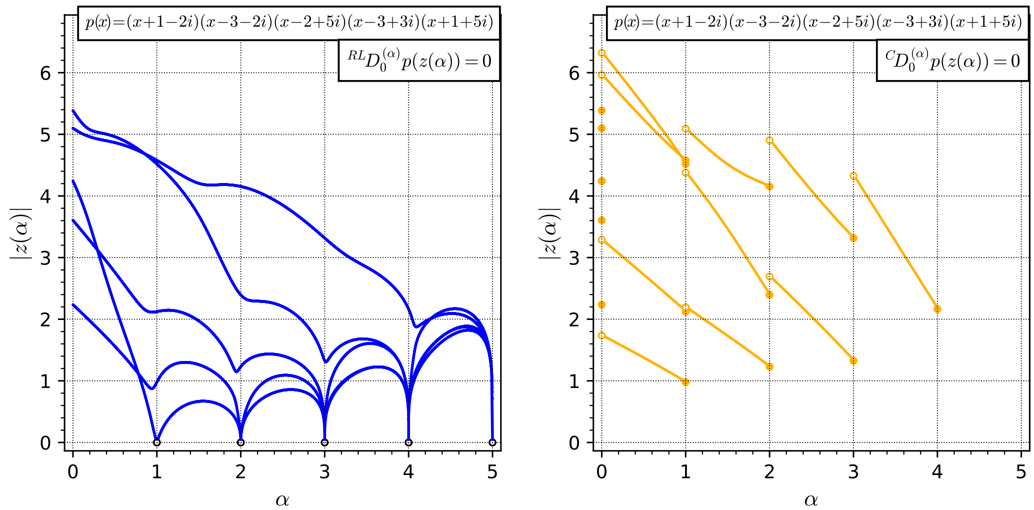

Figure 6.8. Comparing the absolute value of $z(\alpha)$ of zeros of the Riemann-Liouville (left) and Caputo (right) fractional derivatives of the quintic $p(x) = (x+1-2i)(x-3-2i)(x-2+5i)(x-3+3i)(x+1+5i)$, for $0 \le \alpha \le 5$.

We show that $M\left({}^{RL}D_0^{(\alpha)}p\right)$ is bounded from above for $0 < \alpha < \deg p$ where the bound linearly decreases with α. Furthermore, in Theorem 6.18, we see that for $\alpha < 0$ and $\alpha > \deg p$ the Mahler measure $M\left({}^{RL}D_0^{(\alpha)}p\right)$ increases at least linearlily with α.

In our proofs we use the following notation. We set

$$ {}^{RL}D_{0*}^{(\alpha)}f(x) = \frac{\Gamma(n+1-\alpha)}{n!}x^{\alpha} \cdot {}^{RL}D_0^{(\alpha)}f(x). \tag{6.10} $$

Because ${}^{RL}D_{0*}^{(\alpha)}f(x)$ has the same roots as ${}^{RL}D_0^{(\alpha)}$ except for the additional root 0. we have

$$ M\left({}^{RL}D_{0*}^{(\alpha)}f\right) = M\left({}^{RL}D_a^{(\alpha)}f\right). $$

Theorem 6.17. *Let $f(x) \in \mathbb{C}[x]$ be monic and let $0 < \alpha < n$. Then*

1. $M({}^{RL}D_0^{(\alpha)}f) \le \frac{n-\alpha}{n}\|f\|_1 + 1$
2. $M({}^{RL}D_0^{(\alpha)}f) \le \sqrt{n+1}\max\left\{\frac{n-\alpha}{n}\|f\|_\infty, 1\right\}$
3. $M({}^{RL}D_0^{(\alpha)}f) \le \frac{n-\alpha}{n}\|f\|_2 + 1$

Proof. Write $f(x) = \sum_{j=1}^{n} c_j x^j$. With the notation from Lemma 6.3 we have

$$ {}^{RL}D_{0*}^{(\alpha)}(x) = x^n + \sum_{j=0}^{n-1}\left(\prod_{k=j+1}^{n}(k-\alpha)\right)\cdot\frac{j!}{n!}c_j x^j = x^n + \sum_{j=0}^{n-1} d_j^{\alpha}\, c_j\, x^j. $$

This yields the following three bounds for the Mahler measure of ${}^{RL}D_a^{(\alpha)} f$.

1. With Lemma 6.16 (1), we get

$$\begin{aligned}\|{}^{RL}D_{0*}^{(\alpha)}\|_1 &= |d_0^{(\alpha)}c_0| + |d_1^{(\alpha)}c_1| + \cdots + |d_{n-1}^{(\alpha)}c_{n-1}| + 1 \\ &\leq \frac{n-\alpha}{n}\left(|c_0| + |c_1| + \cdots + |c_{n-1}|\right) + 1 \\ &\leq \frac{n-\alpha}{n}\|f\|_1 + 1.\end{aligned}$$

   With Mahler's (6.8), we get

$$M\left({}^{RL}D_a^{(\alpha)} f\right) = M\left({}^{RL}D_{0*}^{(\alpha)}\right) \leq \|{}^{RL}D_{0*}^{(\alpha)}\|_1 \leq \frac{n-\alpha}{n}\|f\|_1 + 1.$$

2. By Lemma 6.16 (1), we have

$$\begin{aligned}\|{}^{RL}D_{0*}^{(\alpha)}\|_\infty &= \max\{|d_0^{(\alpha)}c_0|, |d_1^{(\alpha)}c_1|, \ldots, |d_{n-1}^{(\alpha)}c_{n-1}|, 1\} \\ &\leq \max\left\{\frac{n-\alpha}{n}\max\{|c_0|, |c_1|, \ldots, |c_{n-1}|\}, 1\right\} \\ &\leq \frac{n-\alpha}{n}\|f\|_\infty + 1.\end{aligned}$$

   With Mahler's (6.7), we get

$$M\left({}^{RL}D_a^{(\alpha)} f\right) = M\left({}^{RL}D_{a*}^{(\alpha)} f\right) \leq \sqrt{n+1}\|{}^{RL}D_{0*}^{(\alpha)}\|_\infty \leq \sqrt{n+1}\max\left\{\frac{n-\alpha}{n}\|f\|_\infty, 1\right\}.$$

3. By Lemma 6.16 (1), we have

$$\begin{aligned}\|{}^{RL}D_{0*}^{(\alpha)}\|_2 &= \sqrt{(d_0^{(\alpha)}c_0)^2 + \cdots + (d_{n-1}^{(\alpha)}c_{n-1})^2 + 1} \\ &\leq \frac{n-\alpha}{n}\sqrt{c_0^2 + \cdots + c_{n-1}^2} + 1 \\ &\leq \frac{n-\alpha}{n}\|f\|_2 + 1.\end{aligned}$$

   With Landau's (6.9), we get

$$M({}^{RL}D_a^{(\alpha)} f) = M\left({}^{RL}D_{0*}^{(\alpha)}\right) \leq \|{}^{RL}D_{0*}^{(\alpha)}\|_2 \leq \frac{n-\alpha}{n}||f||_2 + 1.$$

□

Now we are able to prove:

Theorem 6.18. *Let $f(x) \in \mathbb{C}[x]$ be monic of degree n. Then*

1. $M({}^{RL}D_0^{(\alpha)} f) \geq 2^{-n} \frac{n-\alpha}{n}(\|f\|_1 - 1) + 1$, *for $\alpha < 0$ and $\alpha > 2n$*

2. $M({}^{RL}D_0^{(\alpha)} f) \leq \frac{|n-\alpha|_n}{n!}\|f\|_1 + 1$, *for $\alpha < 0$ and $\alpha > n$*

3. $M({}^{RL}D_0^{(\alpha)} f) \geq 2^{-n} \binom{n-1}{\lceil (n-1)/2 \rceil}^{-1} \frac{\alpha-n}{n}\|f\|_1$, *for $n < \alpha \leq 2n$*

Proof. 1. By Lemma 6.16 (3), we have

$$\begin{aligned} \|{}^{RL}D_{0*}^{(\alpha)} f\|_1 &= |d_0^{(\alpha)} c_0| + |d_1^{(\alpha)} c_1| + \cdots + |d_{n-1}^{(\alpha)} c_{n-1}| + 1 \\ &\geq \frac{|n-\alpha|}{n}(|c_0| + |c_1| + \cdots + |c_{n-1}| + 1 - 1) + 1 \\ &= \frac{|n-\alpha|}{n}(\|f\|_1 - 1) + 1. \end{aligned}$$

With Mahler's (6.8), we get

$$M({}^{RL}D_0^{(\alpha)} f) = M\left({}^{RL}D_{0*}^{(\alpha)}\right) \geq 2^{-n}\|{}^{RL}D_{0*}^{(\alpha)}\|_1 \geq 2^{-n}\frac{n-\alpha}{n}(\|f\|_1 - 1) + 1.$$

2. By Lemma 6.16 (2), we have

$$\begin{aligned} \|{}^{RL}D_{0*}^{(\alpha)} f\|_1 &= \sqrt{(d_0^{(\alpha)} c_0)^2 + \cdots + (d_{n-1}^{(\alpha)} c_{n-1})^2 + 1} \\ &\leq \frac{|n-\alpha|_n}{n!}\sqrt{c_0^2 + \cdots + c_{n-1}^2} + 1 \\ &\leq \frac{|n-\alpha|_n}{n!}\|f\|_2 + 1. \end{aligned}$$

With Landau's (6.9), we get

$$M\left({}^{RL}D_a^{(\alpha)} f\right) = M\left({}^{RL}D_{a*}^{(\alpha)} f\right) \leq \|{}^{RL}D_{a*}^{(\alpha)} f\|_1 \leq \frac{|n-\alpha|_n}{n!}\|f\|_2 + 1.$$

3. By Lemma 6.16 (4)

$$\begin{aligned} \|{}^{RL}D_{0*}^{(\alpha)}\|_1 &= |d_0^{(\alpha)} c_0| + |d_1^{(\alpha)} c_1| + \cdots + |d_{n-1}^{(\alpha)} c_{n-1}| + 1 \\ &\geq \frac{\alpha-n}{n}\binom{n-1}{\lceil (n-1)/2 \rceil}^{-1}(|c_0| + |c_1| + \cdots + |c_{n-1}| + 1) \\ &= \frac{\alpha-n}{n}\binom{n-1}{\lceil (n-1)/2 \rceil}^{-1}\|f\|_1 \end{aligned}$$

With Mahler's (6.8), we get

$$M({}^{RL}D_0^{(\alpha)}f) = M\left({}^{RL}D_{0*}^{(\alpha)}\right) \geq 2^{-n}\|{}^{RL}D_{0*}^{(\alpha)}\|_1 \geq 2^{-n}\frac{\alpha - n}{n}\binom{n-1}{(n-1)/2}^{-1}\|f\|_1.$$

□

In Figure 6.6 we present the paths of zeros and the Mahler measures of the fractional derivatives of a degree 3 polynomial along with the bounds from Theorems 6.18 and 6.17. Furthermore Figures 6.3, 6.4, and 6.6 show the growth of $M({}^{RL}D_0^{(\alpha)}p)$ for $\alpha < 0$ and $\alpha > 0$.

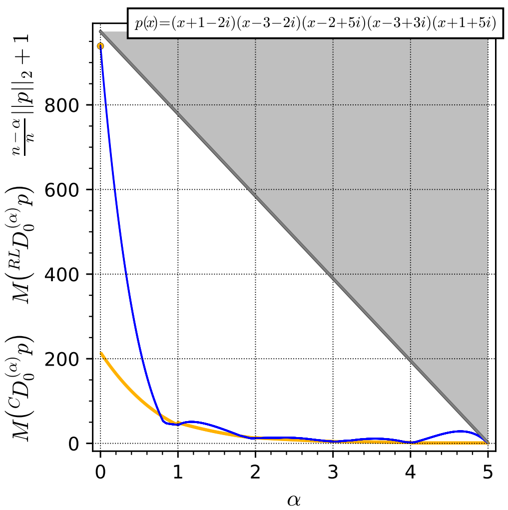

Figure 6.9. Mahler measure of the Riemann-Liouville and Caputo fractional derivatives of the quintic $p(x) = (x + 1 - 2i)(x - 3 - 2i)(x - 2 + 5i)(x - 3 + 3i)(x + 1 + 5i)$ for $0 \leq \alpha \leq 5$ along with the bound from Theorem 6.17 and Corollary 6.19.

6.5.1 Caputo Fractional Derivatives

The Riemann-Liouville and Caputo fractional derivatives differ, but the zeros of their derivatives obey some common general trends. Figures 6.7 and 6.8 compare the paths of zeros of the Riemann-Liouville and Caputo derivatives of a degree 5 polynomial.

The upper bounds from Theorem 6.17 easily transfer to the Caputo fractional derivative. Because the coefficients of the Caputo fractional derivatives of a polynomial $p \in \mathbb{C}[x]$ are either the same as those of the Riemann-Liouville fractional derivative or zero, see equation (2.6), we have

$$\|{}^{C}D_{0*}^{(\alpha)}\|_k \leq \|{}^{RL}D_{0*}^{(\alpha)}\|_k,$$

for $k \in \{1, 2, \infty\}$. This yields the bounds:

Corollary 6.19. *Let $p \in \mathbb{C}[x]$ be monic of degree n and let $0 < \alpha < n$. Then*

1. $M({}^{C}D_0^{(\alpha)}p) \leq \frac{n-\alpha}{n}\|p\|_1 + 1$,
2. $M({}^{C}D_0^{(\alpha)}p) \leq \sqrt{n+1}\max\left\{\frac{n-\alpha}{n}\|p\|_\infty, 1\right\}$,
3. $M({}^{C}D_0^{(\alpha)}p) \leq \frac{n-\alpha}{n}\|p\|_2 + 1$.

6.6 Open Questions

The bounds we have established in Section 5 and Section 6 were sufficient for our purposes, but they are far from best possible. Figure 6.9 illustrates the decline of $M({}^{RL}D_0^{(\alpha)}p)$ and $M({}^{C}D_0^{(\alpha)}f)$, when α approaches n, as described by Theorem 6.17 and Corollary 6.19 and Theorem 6.14. In a future work, it would be interesting to consider the true growth of the paths of zeros, for $\alpha < 0$ and $\alpha > n$. Also, the maximal extent of loops of paths, after traversing the origin a, is something that could be worth looking at. Moreover, as Figure 6.6 clearly shows, the paths exhibit very distinct linear asymptotes in both directions $\alpha \to \infty$ and $\alpha \to -\infty$. For polynomials of higher degrees, their exact directions are not yet known.

In addition to this, as we have noted earlier, the useful Gauss-Lucas property does not hold universally for the fractional derivatives of polynomials. However, some of the dynamical properties we have observed could be investigated with insights related to those that play a key role in the integral case. In particular, Gauss himself suggested a very intriguing physical interpretation of the nontrivial critical points of a polynomial (the critical points which are not zeros) as the equilibrium points in certain force fields, generated by particles placed at the zeros of the polynomial, with masses equal to the multiplicity of the zeros and repelling with a force inversely proportional to the distance. This amazing physical application of a purely theoretical polynomial concept is exceedingly intriguing and should be investigated further. It could go a long way in explaining the profound intricacies of the paths of zeros, and their seemingly chaotic local behavior.

Chapter 7: Conclusion

The common tread running through the thesis is the chain of zeros created by the fractional derivatives. In the Dirichlet series case the chains form continuous curves that continue as one takes higher derivatives. In the polynomial case the zeros tend to a center a discussed in Theorem 6.14. Both form zero-free regions that can be investigated and produce several important consequences. There is a large number of unsolved questions one could consider. The following list gives a handful of them.

(i) Within the realm of polynomials, one pressing inquiry emerges: is there an optimal value for a in Lemma 6.3 that leads to the shortest path of zeros? Could this optimal a coincide with the zero of the $(n-1)$th derivative? Moreover, does the asymptotic behavior exhibit variance across different a values?

(ii) For degrees higher than three, is there an universal ordering of distances of zeros from the origin a? Could the angles between asymptotes be explicitly evaluated in terms of the coefficients?

(iii) A direct correlation between polynomials and Dirichlet series becomes apparent when the Riemann-Liouville fractional derivative is applied. Might this fractional derivative offer deeper insights into Dirichlet series? Are the asymptotic behaviors of Dirichlet series akin to those of polynomials? Do polynomials exhibit uniform zero spacing as α approaches infinity, mirroring Dirichlet series?

(iv) The left half-plane of Dirichlet series adds another layer of intrigue. Here the curves have different shapes that we don't understand. The number of zeros on these paths still remain unknown. The distributions of higher derivative zeros along the negative real axis also needs to be investigated further.

(v) In Figure 4.7, the existence of double zeros is shown. This extremely rare phenomenon still needs to be explored further.

These unresolved inquiries, among others, could lay the groundwork for future endeavors and potentially inspire forthcoming scholars and researchers.

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