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This dissertation focuses on the study of steady states of reaction diffusion problems that are motivated by applications. In particular, we focus on elliptic boundary value problems where the nonlinear reaction may appear in the interior or on the boundary of a domain in the Euclidean space.

First, we study linear elliptic problems with nonlinear reaction on the boundary. In this case, we establish the existence of maximal and minimal solutions for both monotone and non monotone cases. We then extend these results to the systems case. Next, we prove the existence, nonexistence, multiplicity and global bifurcation results of positive solutions of superlinear problems. To support our analytical results we numerically approximate solutions using finite difference methods including existence and stability analysis.

Second, we study problems that are nonlinear inside the domain and linear on the boundary in the context of a model arising in mathematical ecology. To begin with we perform computational simulations for the problem in the one dimensional setting. Then, motivated by the bifurcation diagrams that are obtained, we prove several analytical results such as existence, uniqueness and nonexistence.

SOLVABILITY OF NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

by

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Approved by

Committee Chair

To my parents.

APPROVAL PAGE

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Chapter 1: Introduction

The focus of this dissertation is to study solutions of problems of the form

$$\begin{cases} Lu = a(x,u) & \text{in} \quad \Omega\,; \\ Bu = b(x,u) & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where either $L = -\Delta + I$ or $L = -\Delta$ with $\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator, I is the identity operator and $B = \partial/\partial\eta := \eta(x) \cdot \nabla$ denotes the outer normal derivative on the boundary $\partial\Omega$. The functions a and b may also depend on a bifurcation parameter λ and Ω is a bounded smooth domain in the Euclidean space $\mathbb{R}^N$ $(N \geq 1)$. In particular, we focus on contributing to the study of two distinct sub classes of problems as follows:

(i) $a(x,u)$ is linear with $b(x,u)$ nonlinear,

(ii) $a(x,u)$ is nonlinear with $b(x,u)$ linear.

For the case (i), we begin by developing a theory of maximal and minimal solutions. Then we study existence, multiplicity, nonexistence, and local and global bifurcations of positive solutions with respect to a bifurcation parameter for a subcritical superlinear nonlinearity. In addition, we use the finite difference method to numerically approximate solutions for sublinear type nonlinearities including existence and stability analysis. Then we provide bifurcation diagrams for both sublinear and superlinear type nonlinearities which are generated by computational simulations in MATLAB to support our theoretical results.

For the case (ii), we begin with computational simulations in MATHEMATICA for a model arising from ecology, in the one dimensional setting, using the quadrature method and shooting method. Then motivated by the computational results, we study existence, uniqueness and non existence results using the sub and supersolutions method and Green's second identity.

1.1 Linear Elliptic PDEs with Nonlinear Boundary Conditions:

Throughout this section, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with $C^{2,\alpha}$ $(0 < \alpha < 1)$ boundary $\partial\Omega$. We consider, $L\,u = -\Delta u + u$ with $a(x,u) \equiv 0$, $Bu = \frac{\partial u}{\partial \eta}$ and $b(x,u)$ is at least a Carathéodory function which may depend on a bifurcation parameter λ. In particular, we consider the linear elliptic PDEs with nonlinear boundary conditions of the form:

$$\begin{cases} -\Delta u + u &= 0 \quad \text{in} \quad \Omega\,; \\ \frac{\partial u}{\partial \eta} &= \lambda f(x,u) \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.2}$$

where $\lambda > 0$ is a bifurcation parameter. Here $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $f(\cdot,u)$ is measurable for each u and $f(x,\cdot)$ is continuous for a.e. $x \in \partial\Omega$.

In general, reaction-diffusion problems are important for studying mathematical models in chemical reactions, ecology, population dynamics, combustion theory and so on. Extensive study has been done when the differential equation is linear on the boundary, which includes Dirichlet, Neumann and Robin boundary conditions. Motivation to study equations with nonlinear boundary conditions stems from the fact that chemical reactions, the biological bonding or species interactions may occur in a narrow layer or region near the boundary the reactions and may depend on the density itself. In such scenarios, linear boundary conditions (Dirichlet, Neumann, or

Robin) are often inadequate, see [10,12,19,34,39–41,45,60,66,71,72,75] and references therein for specific applications.

For example, in [66], authors studied a model of limb development in birds (primarily chick), mammals (primarily mouse) and amphibians to analyse the interactions between two different morphogen sources including the zone of polarising activity (ZPA), a specialized group of cells that lie in the posterior margin of the limb bud and the apical epidermal ridge (AER), detemined by the boundary between ectodermal cells. In order to represent the strength of AER, which depends on the ZPA factor, authors introduced nonlinear boundary conditions in their specific limb model.

Also, recently in [34], thermal effects of a blinking eye on the anterior corneal surface and the bio-heat transfer process during different blinking rates has been discussed in detail. The heat loss on the outer surface of the eye, which is exposed to the environment, caused by convection, radiation and evaporation was modeled using a nonlinear boundary condition. Therefore, the study of elliptic equations with nonlinear boundary conditions have attracted a lot of attention over the past decades, see for instance [4,5,8,9,19,33,35,38,50,54,55,57,65,69,73,74] and references therein. Results on the existence of positive solutions of problems with nonlinear boundary conditions can be found (without being exhaustive) using techniques such as, monotone methods and functional analysis in [3,38], concentration compactness method of Lions (see [30,49], bifurcation theory in [11,13,14,50,55,65], variational methods in [42,64,69], and topological degree in [21,27].

The main focus of this section is to study the existence, local bifurcation and global bifurcation of positive solutions of (1.2) when the nonlinearity on the boundary is superlinear but subcritical at infinity. In order to study multiplicity results for superlinear problems certain properties of sub and supersolutions of the related problem are needed. Therefore, we will first investigate maximal and minimal solutions of parameter free problems. Then proceed to study superlinear problems. We further extend maximal and minimal solutions results to a coupled system of equations.

In order to state the theorems in this section, we consider the following Steklov eigenvalue problem

$$\begin{cases} -\Delta\psi + \psi &= 0 \qquad \text{in} \quad \Omega\,; \\ \frac{\partial\psi}{\partial\eta} &= \mu\psi \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Let $\mu_1 > 0$ be the first Steklov eigenvalue and $0 < \varphi_1 \in H^1(\Omega)$ the corresponding positive eigenfunction associated with (1.3) such that $\|\varphi_1\|_{H^1(\Omega)} = 1$ (see [15, 54]).

1.1.1 Maximal and Minimal Solutions

We consider an elliptic equation with nonlinear boundary condition of the form

$$\begin{cases} -\Delta u + u &= 0 \qquad \text{in} \quad \Omega\,; \\ \frac{\partial u}{\partial\eta} &= f(x,u) \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We investigate the existence of maximal and minimal weak solutions (to be defined later, see Definition 2.1) between an ordered pair of sub and supersolution of (1.4) for both monotone and nonmonotone type nonlinearities.

Our first result deals with the monotone case.

Theorem 1.1. *Suppose there exists a pair of weak sub and supersolution* $\underline{u}$ *and* $\overline{u}$*, respectively, of* (1.4) *satisfying* $\underline{u} \le \overline{u}$ *in* $\overline{\Omega}$*. Assume that*

(H1) *there exists* $k \ge 0$ *such that the map* $s \mapsto f(x,s) + ks$ *is nondecreasing for all* $\underline{u}(x) \le s \le \overline{u}(x)$.

Then, there exist a minimal weak solution u_* *and a maximal weak solution* u^* *to* (1.4) *in the sense that, if* u *is any weak solution to* (1.4) *such that* $\underline{u} \le u \le \overline{u}$*, then* $u_* \le u \le u^*$.

Remark 1.1. If f is locally Lipschitz with respect to the second variable and the interval $[\underline{u}, \overline{u}]$ is bounded, then f satisfies the hypothesis **(H1)**. Indeed, let $s_1 > s_2$ and L be the Lipschitz constant of f. Then $f(x, s_1) - f(x, s_2) \geq L(s_2 - s_1)$ and, hence, $f(x, s_1) + Ls_1 \geq f(x, s_2) + Ls_2$.

Next, if f does not satisfy the monotonicity condition **(H1)**, we will establish the following existence result.

Theorem 1.2. *Suppose there exists a pair of weak sub and supersolutions $\underline{u}$ and $\overline{u}$, respectively, of* (1.4) *satisfying $\underline{u} \leq \overline{u}$ in $\overline{\Omega}$. Assume that*

(H2) *there exists a $K \in L^r(\partial\Omega)$, $r > \frac{2(N-1)}{N}$, such that $|f(x, s)| \leq K(x)$ a.e. $x \in \partial\Omega$, for all s satisfying $\underline{u}(x) \leq s \leq \overline{u}(x)$.*

Then (1.4) *has at least one weak solution u such that $\underline{u} \leq u \leq \overline{u}$.*

Our next result guarantees the existence of a maximal and a minimal weak solution without assuming the hypothesis **(H1)** on the nonlinearity f.

Theorem 1.3. *Assume the hypotheses of Theorem 1.2 hold. Then, there exists a minimal weak solution u_* and a maximal weak solution u^* to* (1.4) *in the sense that, if u is any weak solution to* (1.4) *such that $\underline{u} \leq u \leq \overline{u}$, then $u_* \leq u \leq u^*$.*

In [3, Thm. 1], authors studied elliptic problem with nonlinearities appearing both in the domain as well on the boundary. They proved existence of classical maximal and minimal solution in between an ordered pair of sub and supersolution, analogous to Theorem 1.1. In [38], authors extended the result in [3, Thm. 1] to include more general nonlinearity in Ω and considered strong solutions, that is, solutions in $W^{2,p}$. They also proved a result that guarantees the existence of three distinct solutions. In [36], Theorem 1.2 is established by assuming

$$\int_{\partial\Omega} \sup_{\underline{u}(x) \leq s \leq \overline{u}(x)} |f(x, s)|^q < \infty,$$

where $q = 2$. We improve their result by allowing $q = \frac{2(N-1)}{N} < 2$ in the definition of weak solutions in Theorem 1.2, see Definition 2.1. For results similar to Theorem 1.1-1.3, for the nonlinear elliptic problem with the Dirichlet boundary condition, see [25] for the Laplacian case and [24] for the p-Laplacian case.

Next, we give a simple application of our existence result, Theorem 1.1, to problems involving sublinear nonlinearities.

Theorem 1.4. *Consider*

$$\begin{cases} -\Delta u + u = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial u}{\partial \eta} = \lambda f(u) & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.5}$$

where $\lambda > 0$ parameter and $f : [0, \infty) \to [0, \infty)$ is a differentiable function satisfying $f(0) = 0$ with $f'(0) > 0$, and $\lim_{s\to\infty} \frac{f(s)}{s} = 0$. Then, (1.5) has a positive weak solution for $\lambda > \frac{\mu_1}{f'(0)}$.

Next, we extend Theorem 1.1, Theorem 1.2 and Theorem 1.3 to a coupled system of equations. In particular, we consider the following system of elliptic equations with nonlinear boundary conditions:

$$\begin{cases} -\Delta u_i + u_i = 0 & \text{in} \quad \Omega; \\ \frac{\partial u_i}{\partial \eta} = f_i(x, u_1, u_2) & \text{on} \quad \partial\Omega, \ i = 1, 2, \end{cases} \tag{1.6}$$

where for each $i = 1, 2$, $f_i : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function satisfying the following quasimonotonicity condition:

(Q) f_i is quasimonotone nondecreasing, that is, $f_1(x, u_1, u_2)$ is nondecreasing in u_2 for all fixed $x \in \partial\Omega, u_1 \in \mathbb{R}$, and $f_2(x, u_1, u_2)$ is nondecreasing in u_1 for all fixed $x \in \partial\Omega, u_2 \in \mathbb{R}$.

Remark 1.2. By $(x_1, y_1) \leq (x_2, y_2)$ in $\mathbb{R}^2$ we will mean $x_i \leq y_i$ for $i = 1, 2$.

First, we prove the analogoue of Theorem 1.1 for the monotone case.

Theorem 1.5. *Assume* **(Q)** *holds and that there exist a weak subsolution* $(\underline{u}_1, \underline{u}_2)$ *and a weak supersolution* $(\overline{u}_1, \overline{u}_2)$ *of* (1.6) *such that* $(\underline{u}_1, \underline{u}_2) \leq (\overline{u}_1, \overline{u}_2)$ *a.e. on* $\overline{\Omega}$*, and*

(C1) *there exist* $k_i \geq 0$ *such that the map* $s \mapsto f_i(x, s_1, s_2) + k_i s_i$ *is nondecreasing for all* $\underline{u}_i(x) \leq s_i \leq \overline{u}_i(x)$ *and for all* $x \in \partial\Omega$*,*$i = 1, 2$.

Then, there exist a minimal weak solution $(u_{1,*}, u_{2,*})$ *and a maximal weak solution* (u_1^*, u_2^*) *to* (1.6)*, that is, if* (u_1, u_2) *is any weak solution to* (1.6) *such that* $(\underline{u}_1, \underline{u}_2) \leq (u_1, u_2) \leq (\overline{u}_1, \overline{u}_2)$*, then* $(u_{1,*}, u_{2,*}) \leq (u_1, u_2) \leq (u_1^*, u_2^*)$.

Note that condition **(C1)** together with the quasimonotonicity condition **(Q)** will allow us to use the monotone iteration method for the proof.

In the next theorem, we remove the monotonicity condition **(C1)** and obtain the following existence result analogous to Theorem 1.2 and Theorem 1.3 combined.

Theorem 1.6. *Assume* **(Q)** *holds and that there exists a pair of ordered weak subsolution* $(\underline{u}_1, \underline{u}_2)$ *and supersolution* $(\overline{u}_1, \overline{u}_2)$ *of* (1.6)*, and that the following conditions holds:*

(C2) *there exist* $K_1, K_2 \in L^r(\partial\Omega), r > \frac{2(N-1)}{N}$*, such that* $|f_i(x, s_1, s_2)| \leq K_i(x)$ *a.e.* $x \in \partial\Omega$*, whenever* $\underline{u}_i(x) \leq s_i \leq \overline{u}_i(x), i = 1, 2$.

Then there exists a weak solution (u_1, u_2) *of* (1.6) *such that* $(\underline{u}_1, \underline{u}_2) \leq (u_1, u_2) \leq (\overline{u}_1, \overline{u}_2)$*. Moreover, there exist a minimal weak solution* $(u_{1,*}, u_{2,*})$ *and a maximal weak solution* (u_1^*, u_2^*) *to* (1.6)*; that is, for any weak solution* (u_1, u_2) *to* (1.6) *with* $(\underline{u}_1, \underline{u}_2) \leq (u_1, u_2) \leq (\overline{u}_1, \overline{u}_2)$*, we have* $(u_{1,*}, u_{2,*}) \leq (u_1, u_2) \leq (u_1^*, u_2^*)$.

In [56], authors studied the existence of a maximal solution in between ordered pairs of sub and supersolutions for a coupled system of elliptic problem with a Dirichlet boundary condition and nonlinearities inside the domain. In [24], authors extended the results in [56] to the p-Laplacian case. Finally, we note that the book [60] contains results regarding maximal and minimal solutions for elliptic problems with nonlinear boundary conditions for both the scalar case as well as coupled system of equations using monotone iteration methods for classical solutions.

To conclude our study of sub and supersolution results we discuss the following example as a simple application of Theorem 1.5:

Theorem 1.7. *Consider*

$$\begin{cases} -\Delta u_1 + u_1 = 0 & \text{in} \quad \Omega\,; \\ -\Delta u_2 + u_2 = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial u_1}{\partial \eta} = \lambda f_1(u_2) & \text{on} \quad \partial\Omega\,; \\ \frac{\partial u_2}{\partial \eta} = \lambda f_2(u_1) & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.7}$$

where $\lambda > 0$ is a parameter and $f_i : [0, \infty) \to [0, \infty)$ are nondecreasing differentiable functions satisfying $f_i(0) = 0$ with $f_i'(0) > 0$ and $\lim_{s\to\infty} \frac{f_i(s)}{s} = 0$. Then (1.7) *has a positive weak solution for $\lambda > \frac{\mu_1}{\sqrt{f_1'(0) f_2'(0)}}$.*

We provide the proofs of Theorem 1.1- Theorem 1.6 in Chapter 3. In particular, Theorem 1.1 is proved in Section 3.2 using monotone iteration method, Theorem 1.2 and Theorem 1.3 are proved in Section 3.3 using the surjectivity of a coercive operator and Zorn's lemma, respectively. Theorem 1.4 is proved in Section 3.4 using Theorem 1.1. Theorem 1.5 is proved in Section 3.5 using monotone iteration method, and Theorem 1.6 is proved in Section 3.6 using Theorem 1.2, Zorn's lemma and Kato's inequality. Theorem 1.7 is proved in Section 3.9 using Theorem 1.5. We also prove Kato's inequality up to the boundary for single equation in Section 3.1 and for systems in Section 3.7 and Section 3.8.

1.1.2 Superlinear Subcritical Problem

Here we consider the following nonlinear boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial u}{\partial \eta} = \lambda f(u) & \text{on} \quad \partial\Omega, \end{cases} \tag{1.8}$$

where $\lambda > 0$ is a bifurcation parameter and the nonlinearity on the boundary $f : [0, \infty) \to [0, \infty)$ is a locally Lipschtiz continuous function.

Regarding the nonlinear eigenproblem of the type (1.8), we refer to [29,50,64,68,69] where there are existence results with a parameter λ on the boundary but with a pure sublinear power. When the nonlinearity f is a pure power and superlinear, we mention for instance [29, 37]. See also [74], where $\Delta u = 0$ in Ω and the nonlinearity on the boundary is a pure power perturbation of a linear term involving the bifurcation parameter. They proved existence as well as nonexistence results.

Our first result aims to prove an existence result and a local bifurcation result assuming only that the nonlinearity is superlinear and subcritical.

Theorem 1.8. (Local bifurcation result) *Assume that f is superlinear and subcritical at infinity i.e., there exists a constant $b > 0$ such that*

$$(\mathrm{H})_\infty \qquad \lim_{s \to +\infty} \frac{f(s)}{s^p} = b \quad \textit{with} \quad \begin{cases} 1 < p < \frac{N}{N-2} & \textit{if } N \geq 3, \\ p > 1 & \textit{if } N = 2\,. \end{cases} \tag{1.9}$$

Then, there exists $\widehat{\lambda} > 0$ such that for all $\lambda \in (0, \widehat{\lambda}]$, (1.8) has a positive weak solution u such that $\|u\|_{C(\overline{\Omega})} \to \infty$ as $\lambda \to 0^+$. Moreover, there exists a connected component $\mathscr{C}^+ \subset \Sigma$, of positive weak solutions of (1.8), bifurcating from infinity at $\lambda = 0$, such *that* λ *takes all values in* $(0, \widehat{\lambda}]$ *along* $\mathscr{C}^+$*, where*

$$\Sigma := \left\{(\lambda, u) \in [0, +\infty) \times C(\overline{\Omega}) : (\lambda, u) \textit{ is a weak solution of } (1.8)\right\}$$

is the solution set.

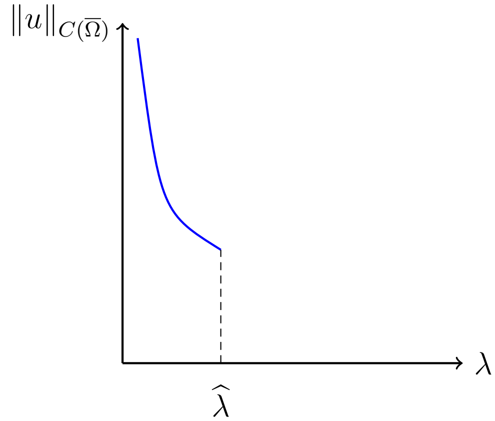


Figure 1.1. Bifurcation from infinity at $\lambda_\infty = 0$

We remark that Theorem 1.8 is independent of the behavior of the nonlinearity f away from infinity (see Figure 1.1). A simple example satisfying the hypotheses of Theorem 1.8 is $f(s) = s + bs^2$ for $s \geq 0$ with $b > 0$.

Next we state a result that says that if there is a bifurcation of solutions from the trivial solutions, then $\frac{\mu_1}{f'(0)}$ is the bifurcation point.

Proposition 1.9. *Assume that the nonlinearity* $f \in C^1([0, \infty))$ *satisfies the hypothesis* $(\mathrm{H})_0$*. Let* $\{\lambda_n\}$ *be a convergent sequence of real numbers and* u_n *be the corresponding sequence of positive weak solutions of equation* (1.8) *satisfying* $||u_n||_{C(\overline{\Omega})} \to 0$ *as* $n \to \infty$*. Then, necessarily* $\lambda_n \to \dfrac{\mu_1}{f'(0)}$*, and* $\{u_n\}$ *satisfies, up to a subsequence,*

$$\frac{u_n}{||u_n||_{C(\overline{\Omega})}} \to \varphi_1 \quad \textit{in } C^\beta(\overline{\Omega})$$

for some $\beta \in (0, 1)$*.*

Next, in order to discuss global bifurcation and multiplicity results, we impose additional conditions on the nonlinearity f near the origin. First, we discuss conditions on f that guarantee bifurcation from the trivial solution, that is, $f \in C^1([0,\infty))$ satisfies the following:

$$(\mathrm{H})_0 \begin{cases} f(0) = 0, \quad f'(0) > 0, \\ \text{and there exists a constant } \nu > 1 \text{ such that} \\ f(s) = f'(0)s + \mathcal{R}(s) \text{ for } s \geq 0 \text{ with } \mathcal{R}(s) = O(s^\nu) \text{ as } s \to 0. \end{cases}$$

We show that if the condition $(\mathrm{H})_0$ holds, then the bifurcation of positive solutions from the trivial solution must occur at the bifurcation point $\frac{\mu_1}{f'(0)}$. Second, to discuss the bifurcation direction of weak solutions near the bifurcation point $\frac{\mu_1}{f'(0)}$, the following quantities play a crucial role. For $\nu > 1$, as defined in $(\mathrm{H})_0$, set

$$\underline{\mathcal{R}}_0 := \liminf_{s\to 0^+} \frac{\mathcal{R}(s)}{s^\nu} \text{ and } \overline{\mathcal{R}}_0 := \limsup_{s\to 0^+} \frac{\mathcal{R}(s)}{s^\nu}.$$

We will characterize the subcritical (bifurcation to the left) or supercritical (bifurcation to the right) nature of weak solutions near the bifurcation point.

Theorem 1.10. (Direction of bifurcation) *Assume that the nonlinearity $f \in C^1([0,\infty))$ satisfies the hypothesis $(\mathrm{H})_0$. Then, the following holds.*

(i) **(Subcritical bifurcation)**. *If $\underline{\mathcal{R}}_0 > 0$, then the bifurcation of positive weak solutions from the trivial solution at $\lambda = \frac{\mu_1}{f'(0)}$ is subcritical, i.e. $\lambda < \frac{\mu_1}{f'(0)}$ for every positive solution (λ, u) of (1.8) with $(\lambda, \|u\|_{C(\overline{\Omega})})$ in a neighborhood of $(\frac{\mu_1}{f'(0)}, 0)$.*

(ii) **(Supercritical bifurcation)**. *If $\overline{\mathcal{R}}_0 < 0$, then the bifurcation of positive weak solutions from the trivial solution at $\lambda = \frac{\mu_1}{f'(0)}$ is supercritical, i.e. $\lambda > \frac{\mu_1}{f'(0)}$*

for every positive solution (λ, u) *of* (1.8) *with* $(\lambda, \|u\|_{C(\overline{\Omega})})$ *in a neighborhood of* $(\frac{\mu_1}{f'(0)}, 0)$.

Finally we state the global bifurcation and multiplicity result in the following theorem which is being illustrated in Figure 1.2.

Theorem 1.11. (Global bifurcation and multiplicity result) *Let* $f \in C^1([0,\infty))$ *be such that hypotheses* $(\mathrm{H})_0$ *and* $(\mathrm{H})_\infty$ *are satisfied. Suppose that there exists* $K > 0$ *such that*

$$f(s) \geq Ks \quad \text{for } s \geq 0\,.$$

Then, there exists a connected component $\mathscr{C}^+$ *of positive weak solutions of* (1.8) *emanating from the trivial solution at the bifurcation point* $\left(\frac{\mu_1}{f'(0)}, 0\right) \in \Sigma$ *possessing a unique bifurcation point from infinity at* $\lambda = 0$*. More precisely, if* $(\lambda, u_\lambda) \in \mathscr{C}^+$*, then the following holds:*

$$\begin{cases} \|u_\lambda\|_{C(\overline{\Omega})} \to 0 & \text{as} \quad \lambda \to \frac{\mu_1}{f'(0)}, \\ \|u_\lambda\|_{C(\overline{\Omega})} \to \infty & \text{as} \quad \lambda \to 0^+ \text{ and} \\ \text{if } (\lambda_\infty, \infty) \text{ is a bifurcation point from infinity, then } \lambda_\infty = 0. \end{cases}$$

Moreover, the problem (1.8) *has a positive weak solution for any* $\lambda \in \left(0, \frac{\mu_1}{f'(0)}\right)$ *and no positive weak solutions for* $\lambda > \frac{\mu_1}{K}$.

Furthermore, if $\overline{\mathcal{R}}_0 < 0$*, then the bifurcation from the trivial solution at* $\left(\frac{\mu_1}{f'(0)}, 0\right)$ *is supercritical. In addition, there exists* $\bar{\lambda} > \frac{\mu_1}{f'(0)}$ *such that problem* (1.8) *has at least two positive weak solutions for any* $\lambda \in \left(\frac{\mu_1}{f'(0)}, \bar{\lambda}\right)$*, and at least one positive weak solution for* $\lambda = \frac{\mu_1}{f'(0)}$*, and for* $\lambda = \bar{\lambda}$.

To the best of our knowledge, the only existence result for the superlinear, subcritical problem with nonlinear boundary condition for problems of the form (1.8) is

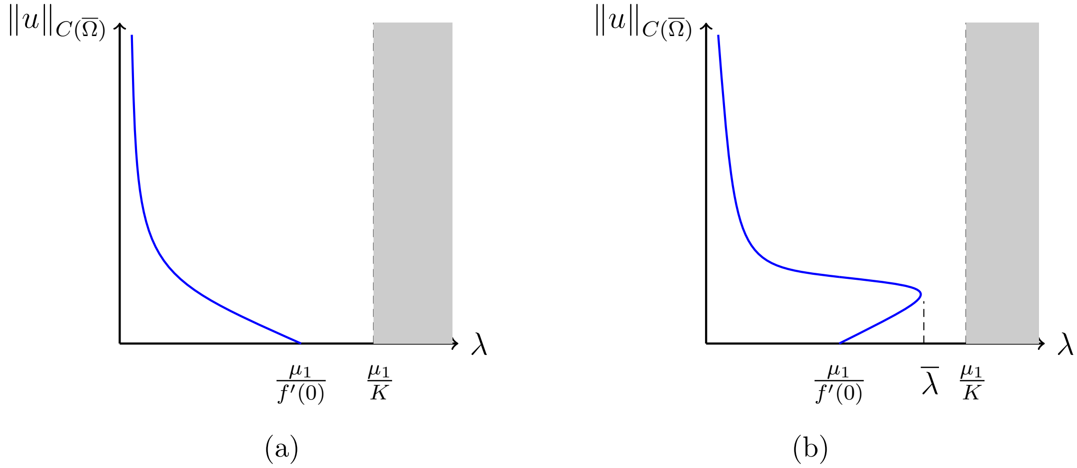


Figure 1.2. Possible global bifurcation diagrams in Theorem 1.11: (a) subcritical bifurcation; (b) supercritical bifurcation

proved in [27] when $\lambda = 1$. Therefore, our local bifurcation result complements the existence result in [27].

We proved regularity and positivity of solutions for nonlinear problems in Section 4.1. We provide the proofs of Theorem 1.8- Theorem 1.11 in Chapter 4. In particular, Theorem 1.8 is proved in Section 4.2 using rescaling method combined with degree theory. Proposition 1.9 and Theorem 1.10 are proved in Section 4.3. We use Rabinowitz global bifurcation theorem and Crandall-Rabinowitz theorem to prove Theorem 1.10. Theorem 1.11 is proved in Section 4.4 using degree theory, Corollary 3.4 and Theorem 1.2.

1.2 Finite Difference Approximations

In this section, we introduce the computational techniques we used to approximate solutions of (1.2) and generate bifurcation diagrams. The numerical study validates our theoretical analysis. In most scenarios, closed form solutions of PDEs do not exist. Therefore, in order to visualize solutions when they exist, we utilize numerical methods (see [58, 59, 61–63]). In this dissertation we employ the finite difference (FD) method to numerically study (1.2). There are two main goals for our computational work:

(i) To extend the techniques for analyzing continuous problems to the discrete setting by developing a rigorous theory for approximating the (unknown) PDE solutions. We have successfully extended results for the sublinear coupled system and are currently working on the superlinear case.

(ii) To generate solutions and bifurcation diagrams to visualize solutions and properties of the unknown PDE solutions for both sublinear and superlinear problems.

In Chapter 5, we formulate and provide a detailed analysis of a FD method for approximating solutions of a sublinear coupled system. In particular, we prove existence and stability results and formulate a methodology for generating bifurcation diagrams. The methods are tested in the one dimensional setting. The work for sublinear problems complements the results and techniques presented in Chapter 3 for the continuous problem. We also generate bifurcation diagrams for the superlinear problem (1.8) using the method of continuation in the one dimensional case to validate our PDE results described in Section 1.1.

1.2.1 Sublinear Problem

In this section, we describe the results for our FD method for approximating the solutions of

$$\begin{cases} -\Delta u_i + u_i = 0 & \text{in } \Omega; \\ \frac{\partial u_i}{\partial \eta} = \lambda f_i(u_1, u_2); & \text{on } \partial\Omega;\ i = 1, 2, \end{cases} \tag{1.10}$$

where $f_i : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfies the following:

(i) f_i's are quasimonotone nondecreasing, i.e. $f_1(s_1, s_2)$ is nondecreasing in s_2 and $f_2(s_1, s_2)$ is nondecreasing in s_1.

(ii) $f_1(s_1, s_2)$ is locally Lipschitz continuous in s_1 and $f_2(s_1, s_2)$ is locally Lipschitz continuous in s_2.

(iii) $f_i(0, 0) \geq 0$.

(iv) f_i's are sublinear, i.e. $\lim_{||(s_1,s_2)||_1 \to \infty} \frac{f_i(s_1,s_2)}{||(s_1,s_2)||_1} = 0$, where $||(s_1, s_2)||_1 = |s_1| + |s_2|$.

Remark 1.3. Notice that we can generalize (1.10) to the case $f_i : \partial\Omega \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ for $f_i(\cdot, u_1, u_2)$ bounded over $\partial\Omega$ for all $(u_1, u_2) \in [0, \infty) \times [0, \infty)$ due to the fact that the FD method is defined pointwise. Therefore, Problem (1.10) is a similar form as Problem (1.6) except in Problem (1.10) we introduced the bifurcation parameter λ in order to generate bifurcation diagrams.

We prove the existence of nonnegative solutions for the discrete problem generated by the FD method (see Chapter 2) in between an ordered pair of discrete sub and supersolutions which turn out to be uniformly bounded independent of the discretization parameter h. This result and the corresponding sub and supersolution technique is a discrete analogue of Theorem 1.5. In fact we find exact sub and supersolutions for the discrete problem. We formulate a monotone iteration to find the maximal nonnegative solution bounded above by the supersolution. Several bifurcation diagrams generated using MATLAB are provided.

1.2.2 Superlinear Problem

Next we computationally study Problem (1.8) in the one dimensional settings, where $f : [0, \infty) \to [0, \infty)$ is a locally Lipschitz continuous function that satisfies $(\mathrm{H})_\infty$, that is, f is superlinear at infinity. We implement the FD method in MATLAB for $\Omega = (a, b)$ when $N = 1$ with various choices for f satisfying the hypotheses of Theorem 1.8, Theorem 1.10 and Theorem 1.11 mentioned above to create approximate bifurcation diagrams and graphs for various positive solutions.

1.3 Mathematical Ecology

In this section, we study $L\,u = -\Delta u$ with $a(x,u) = \lambda f(u)$; $Bu = -\frac{\partial u}{\partial \eta}$, where η is the outward normal derivative on the boundary and $b(x,u)$ is a linear function in u depending on the parameters of the ecological model described below. The study of such problems is motivated from ecological scenarios where two species compete inside the landscape for resources but may compete or cooperate on the boundary of the landscape. In particular, we study a reaction diffusion model arising in ecology where a species A (with population density v), for survival, needs to compete with a species B (with population density u) in the habitat. Often times in ecology, in a heterogeneous patch, exponential growth of a population gets affected by intra-reaction (in other words competing among themselves for resources) as well as by inter-reaction (in other words competing with another species for resources). This particular model that we study is asymmetric in the sense that species A's population growth is affected by both intra- and inter-reaction whereas species B's growth is affected by only intra-reaction.

Further, A's movement across the boundary of the habitat depends on the density of species B on the boundary. Species A may feel threatened and leave the patch if species B's density is high on the boundary because of the crowding affect (competition on the boundary) or A may feel comfortable being inside the patch assuming there are enough resources to survive as it observes a higher density of B (cooperation on the boundary). This type of interaction on the boundary is also known as density dependent dispersal. The model with constant density dependent dispersal on the boundary has been studied recently in [1]. In this section we extend the study for density dependent dispersal on the boundary affected by the interaction of the species B, namely for a case of competitive interaction and a case of cooperative interaction.

To begin with, we describe the time dependent model and rescale the domain. Let $\Omega = (0,1)$ or a bounded domain in $\mathbb{R}^N$; $N = 2,3$ with smooth boundary $\partial\Omega$ and

$|\Omega| = 1$ where $|\cdot|$ denotes the measure of a set. Let $\Omega_0 = \{lx | x \in \Omega\}$, where l is a positive parameter representing the patch size of Ω_0. We will assume the diffusion rate in Ω_0 is D and that Ω_0 is surrounded by a matrix Ω_M, where the matrix diffusion rate is D_0 and the death rate is S_0, with D, D_0 and $S_0 > 0$.

Let $\alpha(u, v, \epsilon)$ be the probability of the population A staying in Ω_0 when it reaches the boundary, where $\epsilon > 0$ is the strength of the interaction (competition or cooperation) at the boundary between A and B. In our case, we consider the case when $\alpha(u, v, \epsilon) = \alpha(u, \epsilon)$. Hence the resulting model is (assuming unit carrying capacity)(see [23],[52], [53]):

$$\begin{cases} v_t = D\Delta v + rv(1 - v - bu) & \text{in } \Omega_0, \text{ for } t > 0; \\ D\alpha(u,\epsilon)\frac{\partial v}{\partial \eta} + \frac{\sqrt{S_0 D_0}}{k}(1 - \alpha(u,\epsilon))v = 0 & \text{on } \partial\Omega_0, \text{ for } t > 0; \\ v(0,x) = v_0(x) & \text{in } \Omega_0 \end{cases} \tag{1.11}$$

with the corresponding steady state equation:

$$\begin{cases} -\Delta v = \frac{l^2}{D} rv(1 - v - bu) & \text{in } \Omega; \\ \frac{D}{l}\alpha(u,\epsilon)\frac{\partial v}{\partial \eta} + S^*(1 - \alpha(u,\epsilon))v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $r > 0$ is the patch intrinsic growth rate, $S^* = \frac{\sqrt{S_0 D_0}}{k}$ for k a positive parameter that encapsulates the hypothesis regarding the patch-matrix interface (see [23]), and $b > 0$ is the competition rate. Now substituting $\lambda = \frac{l^2 r}{D}$ (hence λ is proportional to the measure of the patch size), $\gamma_2 = \frac{S^*}{\sqrt{rD}}$, and $h(s,\epsilon) = \frac{1-\alpha(s,\epsilon)}{\alpha(s,\epsilon)}$, we obtain the following model:

$$\begin{cases} -\Delta v = \lambda rv[1 - v - bu] & \text{in } \Omega; \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_2 h(s,\epsilon)v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

In particular, we study (1.12) for species A for the following cases:

$$h(s,\epsilon) = \begin{cases} 1 + \epsilon s \text{ (competitive interaction)}; s \geq 0, \ \epsilon > 0, \text{or} \\ \frac{1}{1+\epsilon s} \text{ (cooperative interaction)}; s \geq 0, \ \epsilon > 0. \end{cases}$$

We also assume that B satisfies logistic growth in the habitat (with no or negligible interference by A) and $\alpha(u,\epsilon)$, in (1.11) is a constant, which results its steady state dynamics to be described by an equation (assuming the patch intrinsic growth rate is 1):

$$\begin{cases} -\Delta u = \lambda u[1-u] & \text{in } \Omega; \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where $\gamma_1 > 0$ is a constant depending on the respective diffusion and death rates.

Remark 1.4. Observe that $v, u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

In Section 6.1, we will provide computational results for $N = 1$ for different values of b and ϵ using quadrature method and shooting method. We note that for $N = 1$, we can rewrite (1.12) and (1.13) as follows:

$$\begin{cases} -v'' = \lambda r v[1-v-bu]; \ (0,1), \\ -v'(0) + \gamma_2\sqrt{\lambda}h(u(0),\epsilon)v(0) = 0, \\ v'(1) + \gamma_2\sqrt{\lambda}h(u(1),\epsilon)v(1) = 0 \end{cases} \tag{1.14}$$

and

$$\begin{cases} -u'' = \lambda u(1-u); \quad (0,1), \\ -u'(0) + \gamma_1\sqrt{\lambda}u(0) = 0, \\ u'(1) + \gamma_1\sqrt{\lambda}u(1) = 0, \end{cases} \tag{1.15}$$

respectively. We observe several interesting results for both the competitive case and the cooperative case on the boundary.

In order to describe the bifurcation diagrams, we first recall from [32] that for the model:

$$\begin{cases} -\Delta z = \lambda r z(1-z) & \text{in } \Omega; \\ \frac{\partial z}{\partial \eta} + \sqrt{\lambda}kz = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

where λ, r, k are positive parameters, the bifurcation diagram for positive solutions is exact as shown in Figure 1.3:

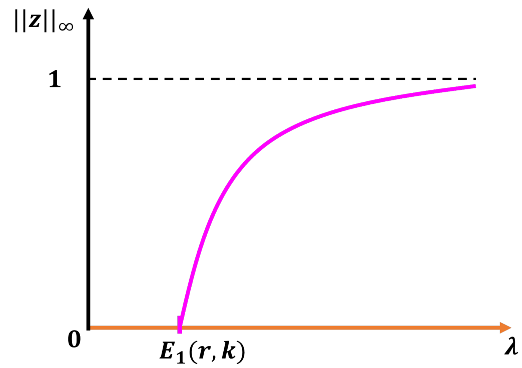


Figure 1.3. Exact bifurcation diagram for positive solutions of model (1.16)

Here, $E_1(r,k)$ is the principal eigenvalue of:

$$\begin{cases} -\Delta z = Erz & \text{in } \Omega; \\ \frac{\partial z}{\partial \eta} + k\sqrt{E}z = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

Hence, the problem (1.13) has a unique positive solution u for $\lambda > E_1(1,\gamma_1)$ and no positive solution for $\lambda \leq E_1(1,\gamma_1)$. Further, if the competition strength is negligible or none (i.e. $b = 0, \epsilon = 0$), observe that species A satisfies (1.16) with $k = \gamma_2$ and hence has a unique positive solution v for $\lambda > E_1(r,\gamma_2)$ and no solutions for $\lambda \leq E_1(r,\gamma_2)$. We also note here that it was established in [32] that $E_1(r,k)$ is increasing in k and decreasing in r.

Remark 1.5. In all the bifurcation diagrams below, the red curve represents bifurcation curves for the solution v when not affected by u ($b = 0$ and $\epsilon = 0$ case) and the blue curve represents bifurcation curves for the solution u. Green curves represent the bifurcation curves corresponding to the solution v when there is competition inside the domain ($b > 0$) but no interaction on the boundary ($\epsilon = 0$). The black curves represent the bifurcation curves corresponding to the solution v when there is competition inside the domain in addition to the interaction on the boundary.

We now present a few of our results (from Section 6.1) to illustrate interesting patchsize dynamics when the interaction level at the boundary varies. For instance, in Figure 1.4, we illustrate a result in the competitive case, i.e. $h(s, \epsilon) = 1 + \epsilon s$. For

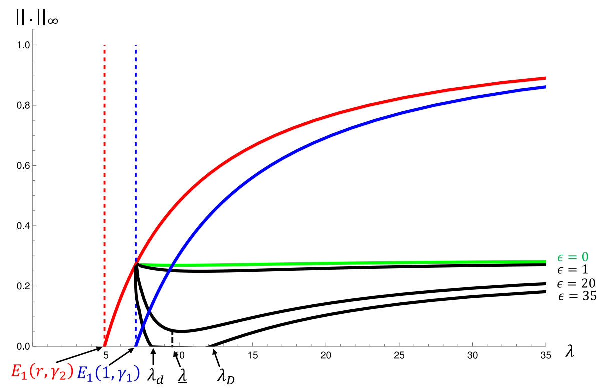


Figure 1.4. Typical bifurcation diagram of (1.14) for $h(s, \epsilon) = 1 + \epsilon s$ when $b = 0.728074(= b^*)$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $\epsilon > 0$.

$\epsilon > 0$, in the presence of species B, we observe there is a range of patchsize in which the population of species A increases as the patch size decreases. We also notice nonexistence of population of A for a certain range of patch sizes when the strength of competition ϵ is large enough.

Next we present some interesting behaviour of the population of A when species B is cooperating on the boundary, i.e. when $h(s, \epsilon) = \frac{1}{1+\epsilon s}$. For instance, in Figure 1.5, we see that cooperation on the boundary is causing the population of A to exist even

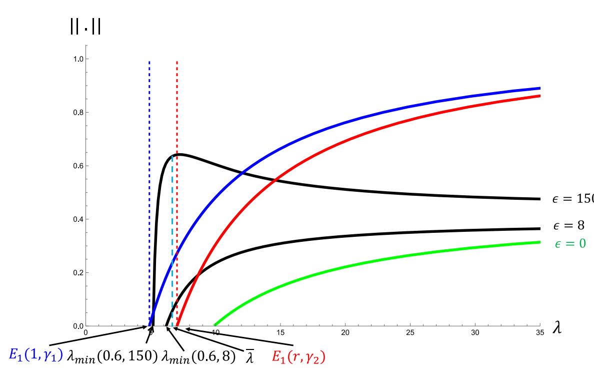


Figure 1.5. Typical bifurcation diagram of (1.14) for $h(s, \epsilon) = \frac{1}{1+\epsilon s}$ when $b = 0.6$, $r = 1, \gamma_1 = 2, \gamma_2 = 4$, and $\epsilon > 0$.

before the principal eigenvalue $E_1(r, \gamma_2)$. In Figure 1.6,

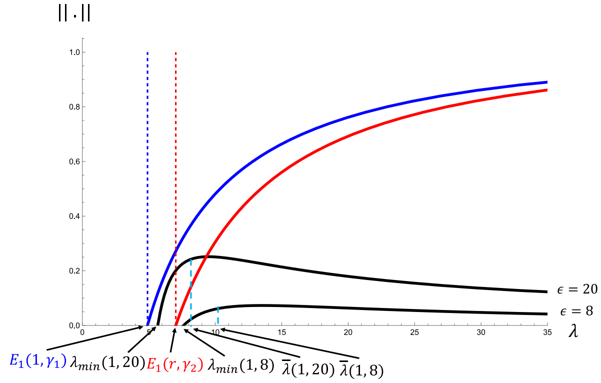


Figure 1.6. Typical bifurcation diagram of (1.14) for $h(s, \epsilon) = \frac{1}{1+\epsilon s}$ when $b = 1$, $r = 1, \gamma_1 = 2, \gamma_2 = 4$, and $\epsilon > 0$.

we notice for $b = 1$, that the population of species A starts to exists for large enough strength of cooperation ($\epsilon >> 1$), whereas from [1] we know that the solution does not exist for any patchsize when $\epsilon = 0$. In fact, we see here that a strong enough mutualism between two species on the boundary can create a hump shaped bifurcation curve (in other words hump shaped density area relationship).

Lastly, in Section 6.2, we state and prove several analytical results for $N \geq 1$ which are motivated by our computational results described in Section 6.1. We employ the method of sub and supersolutions and the Green's second identity to establish these results.

Chapter 2: Preliminaries

In this chapter we will recall several results and discuss various methods from functional analysis that are relevant to our study. First we discuss the functional framework of our PDEs which includes the underlying function spaces, definitions of weak solutions and weak sub and supersolutions, solution operators, regularity and positivity results. Next we discuss the mathematical tools such as topological degree theory and sub and supersolution method. At the end we briefly discuss the quadrature method, the shooting method, and the finite difference method which are used in our numerical approximations and computational simulations.

2.1 Functional Framework

2.1.1 Function Spaces

We start by defining some function spaces. Let D be a bounded, open subset of $\mathbb{R}^N (N \geq 2)$ with smooth boundary ∂D or $D = (0,1)$ if $N = 1$. We define the Lebesgue spaces and Sobolev spaces as follows:

- $L^p(D);\ 1 \leq p < \infty$ denotes the space of measurable functions u on D such that $\int_D |u|^p < \infty$. It is a Banach space with the norm $\|u\|_{L^p(D)} := \left(\int_D |u|^p\right)^{\frac{1}{p}}$.
- $L^\infty(D)$ denotes the space of measurable functions u with $|u(x)| \leq C$ a.e.in D. It is a Banach space with the norm $\|f\|_{L^\infty(D)} := inf\{C : |u(x)| \leq C$ for a.e. $x \in D\}$.

- $W^{1,p}(D)$; $1 \leq p \leq \infty$ denotes the space of measurable functions $u \in L^p(D)$ such that $\frac{\partial u}{\partial x_i} \in L^p(D)$ for $i = 1, 2, \cdots N\}$. It is a Banach space with the norm

$$||u||_{W^{1,p}(D)} = \left(\int_D |\nabla u|^p + \int_D |u|^p \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

  and

$$||u||_{W^{1,\infty}(D)} = max \left\{ ||\nabla u||_{L^\infty(D)}, ||u||_{L^\infty(D)} \right\}.$$

- If $p = 2$, then we denote $W^{1,2}(D)$ by $H^1(D)$. Then $H^1(D)$ is a Hilbert Space with the norm $\left(\int_D |\nabla u|^2 + \int_D |u|^2 \right)^{\frac{1}{2}}$ (see [16]).

Next we define the following Hölder spaces:

- The space $C(\overline{D})$ is the space of continuous functions $u : \overline{D} \to \mathbb{R}$. This is a Banach space with the norm

$$||u||_\infty = \sup_{x \in D} |u(x)|.$$

- The space $C^k(\overline{D})$ is the space of continuous functions $u : \overline{D} \to \mathbb{R}$ whose partial derivatives of order less than or equal to k are continuous on $\overline{D}$. $C^k(\overline{D})$ is a Banach space with the norm

$$||u||_{C^k(\overline{D})} = \sum_{|\beta| \leq k} \sup_D |\partial^\beta u|,$$

  where $\partial^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n}$ with $\partial_i^{\beta_i} = \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}$ for multi-index $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$.

- Suppose that $0 < \alpha \leq 1$. A function $u : D \to \mathbb{R}$ is uniformly Hölder continuous with the exponent α in D if the quantity

$$[u]_{\alpha,D} = \sup_{x,y \in D, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

is finite. We denote by $C^{0,\alpha}(\overline{D})$ the space of uniformly Hölder continuous functions with exponent α in D. $C^{0,\alpha}(\overline{D})$ is a Banach space with the norm

$$||u||_{C^{0,\alpha}(\overline{D})} = \sup_D |u| + [u]_{\alpha,D}.$$

- The space $C^{k,\alpha}(\overline{D})$ consists of functions with continuous partial derivatives in $\overline{D}$ of order less than or equal to k whose kth partial derivatives are Hölder continuous with exponent α in $\overline{D}$. The space $C^{k,\alpha}(\overline{D})$ is a Banach space with the norm

$$||u||_{C^{k,\alpha}(\overline{D})} = \sum_{|\beta|\leq k} \sup_D |\partial^\beta u| + \sum_{|\beta|=k} [\partial^\beta u]_{\alpha,D}.$$

2.1.2 Weak Solutions

In this section we provide the definitions of weak solutions along with weak sub and supersolutions of (1.4) and (1.6).

Definition 2.1. *We say that a function $u \in H^1(\Omega)$ is a weak solution to* (1.4) *whenever*

(i) $f(\cdot, u(\cdot)) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ *if* $N > 2$ *and* $f(\cdot, u(\cdot)) \in L^r(\partial\Omega)$ *for some* $r > 1$ *if* $N = 2$ *with*

(ii) $\displaystyle\int_\Omega (\nabla u \nabla \psi + u\psi) = \int_{\partial\Omega} f(x,u)\psi \qquad$ *for all* $\psi \in H^1(\Omega)$.

A weak sub and supersolution is defined by replacing "=" in (ii) above with "$\leq$" and "$\geq$", respectively, and letting $\psi \geq 0$.

Definition 2.2. *We say that a function $(u_1, u_2) \in H^1(\Omega) \times H^1(\Omega) = (H^1(\Omega))^2$ is a weak solution to* (1.6) *whenever:*

(i) *For every $i = 1, 2$, $f_i(., u_1(.), u_2(.)) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ if $N > 2$ and $f_i(., u_1(.), u_2(.)) \in L^r(\partial\Omega)$ for some $r > 1$ if $N = 2$ with*

(ii) $\int_{\Omega} (\nabla u_i \nabla \psi + u_i \psi) = \int_{\partial\Omega} f_i(x, u_1(x), u_2(x))\psi \qquad \textit{for all } \ \psi \in H^1(\Omega).$

A weak sub and supersolution is defined by replacing "$=$" in (ii) above with "$\leq$" and "$\geq$", respectively, and letting $\psi \geq 0$.

Next we define the weak solution of (1.8) as follows:

Definition 2.3. *By a weak solution of problem* (1.8)*, we mean a pair* $(\lambda, u) \in (0, \infty) \times H^1(\Omega)$ *such that*

$$\int_{\Omega} \nabla u \nabla \psi + \int_{\Omega} u\psi = \lambda \int_{\partial\Omega} f(u)\psi \qquad \textit{for all } \ \psi \in H^1(\Omega). \tag{2.1}$$

Remark 2.1. Let $\Gamma : H^1(\Omega) \to L^r(\partial\Omega)$ be the trace operator given by $\Gamma u = u|_{\partial\Omega}$. It is known that, see e.g. [2], [20, Thm 2.79], and [43, Chapter 6], Γ is continuous if

$$\begin{cases} 1 \leq r \leq \frac{2(N-1)}{N-2} \ \text{ if } N > 2, \\ r \geq 1 \text{ if } N = 2 \end{cases} \tag{2.2}$$

and Γ is compact if

$$\begin{cases} 1 \leq r < \frac{2(N-1)}{N-2} \ \text{ if } N > 2, \\ r \geq 1 \text{ if } N = 2. \end{cases} \tag{2.3}$$

Remark 2.2. We observe that the integrals on the right hand sides of (ii) of Definition 2.1 and Definition 2.2 are well defined by Remark 2.1. Also note that the right hand side of Definition 2.3 exists if, for example, f satisfies the subcritical condition 1.9.

2.1.3 Linear Problem and Regularity

In this subsection, we recall some regularity results of weak solutions of the following linear problem:

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega\,; \\ \frac{\partial v}{\partial \eta} = h & \text{on } \partial\Omega\,, \end{cases} \tag{2.4}$$

where $h \in L^q(\partial\Omega)$ for $q \geq 1$. It is known that for each $q \geq 1$, (2.4) has a unique solution v in $W^{1,m}(\Omega)$ and

$$\|v\|_{W^{1,m}(\Omega)} \leq C\|h\|_{L^q(\partial\Omega)}, \qquad \text{where} \quad 1 \leq m \leq Nq/(N-1)\,, \tag{2.5}$$

see, for instance, [55] for more details. In particular, if $q = \frac{2(N-1)}{N}$, then $u \in H^1(\Omega)$. We denote the solution operator corresponding to (2.4) by

$$T : L^q(\partial\Omega) \to W^{1,m}(\Omega) \text{ with } T\,h := v\,.$$

It is known that the trace operator

$$\Gamma : W^{1,m}(\Omega) \to L^r(\partial\Omega) \tag{2.6}$$

is a continuous linear operator for every r satisfying $\frac{N-1}{r} \geq \frac{N}{m} - 1$, and it is compact if $\frac{N-1}{r} > \frac{N}{m} - 1$, see [43, Ch. 6]. Now, we define the resolvent operator (also known as the *Neumann-to-Dirichlet operator*) $S := T \circ \Gamma$ as

$$S : L^q(\partial\Omega) \to L^r(\partial\Omega) \text{ given by } Sh := \Gamma\big(T\,h\big) = \Gamma v \tag{2.7}$$

for any $q \geq 1$ and for all r satisfying $\frac{N-1}{r} \geq \frac{N-m}{m}$ with $1 \leq m \leq Nq/(N-1)$, given schematically by

$$L^q(\partial\Omega) \xrightarrow{T} W^{1,m}(\Omega) \xrightarrow{\Gamma} L^r(\partial\Omega)\,.$$

Note that if $\frac{N-1}{r} > \frac{N-m}{m}$, then S is compact by the compactness of Γ.

The following Lemma states the regularity of the solution of the linear problem (2.4). In particular, if $q > N-1$, then $v \in C^{\alpha}(\overline{\Omega})$.

Lemma 2.1. *Let $N \geq 2$ and $h \in L^q(\partial\Omega)$ with $q \geq 1$. Then, the unique solution $v = Th$ of the linear problem* (2.4) *satisfies the following:*

(i) If $1 \leq q < N-1$, then $\Gamma v \in L^r(\partial\Omega)$ for all $1 \leq r \leq \frac{(N-1)q}{N-1-q}$ and the map $S : L^q(\partial\Omega) \to L^r(\partial\Omega)$ is continuous for $1 \leq r \leq \frac{(N-1)q}{N-1-q}$ and compact for $1 \leq r < \frac{(N-1)q}{N-1-q}$.

(ii) If $q = N-1$, then $\Gamma v \in L^r(\partial\Omega)$ for all $r \geq 1$ and the map $S : L^q(\partial\Omega) \to L^r(\partial\Omega)$ is continuous and compact for $1 \leq r < \infty$.

(iii) If $q > N-1$, then $v \in C^{\alpha}(\overline{\Omega})$ with $\|v\|_{C^{\alpha}(\overline{\Omega})} \leq C\|h\|_{L^q(\partial\Omega)}$ for some $\alpha \in (0,1)$. Moreover, $\Gamma v \in C^{\alpha}(\partial\Omega)$ and the map $S : L^q(\partial\Omega) \to C^{\alpha}(\partial\Omega)$ is continuous and compact.

(iv) If $h \in C^{\alpha}(\partial\Omega)$, then $v \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

Proof. See [13, Lemma 2.1] for proofs of items (i)-(iii).

(iv) Since $h \in C^{\alpha}(\partial\Omega)$, then by (iii) $v \in C^{\alpha}(\overline{\Omega})$. It follows from the first equation in (2.4) that $v \in C^{2,\alpha}(\Omega)$. Furthermore, using the second equation of (2.4), it follows that $v \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. □

Remark 2.3. ([25, 56]) A function $u \in H^1(\Omega)$ is called nonnegative (positive) on $E \subset \overline{\Omega}$ if there is a sequence $u_n \in W^{1,\infty} (\subset C(\overline{\Omega}))$ with $u_n(x) \geq 0$ ($u_n(x) > 0$) for $x \in E$ and $u_n \to u$ in $H^1(\Omega)$.

The following proposition shows that any nonnegative nontrivial solution of (2.4) is positive on $\overline{\Omega}$.

Proposition 2.1. *Let* $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *be a solution of* (2.4) *for* $h \geq 0$ *with* $h \not\equiv 0$. *Then* $v > 0$ *on* $\overline{\Omega}$.

Proof. Clearly $v > 0$ in Ω by the strong Maximum principal, see [3, p. 127]. Assume to the contrary that there exists an $x_0 \in \partial\Omega$ such that $v(x_0) = 0$. By Hopf's Lemma ([31, Lem. 3.4]) $\frac{\partial v}{\partial \eta}(x_0) < 0$, contradicting the boundary condition $\frac{\partial v}{\partial \eta}(x_0) = h(x_0) \geq 0$. As a conclusion, $v > 0$ for all $x \in \overline{\Omega}$. □

We conclude this subsection by the following proposition:

Proposition 2.2. *Let* $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *be a non-negative solution of* (2.4) *for* $h \equiv 0$. *Then* $v \equiv 0$ *on* $\overline{\Omega}$.

Proof. We have $\Delta v \geq 0$ which implies v cannot achieve an interior maximum M unless $v \equiv M$ in $\overline{\Omega}$ by the strong maximum principal (see [3]). If $v \equiv M$, then $M = 0$ by (2.4) and, hence, $v \equiv 0$ is trivial. If v is not a constant function, then the maximum should occur on $\partial\Omega$, say at $x_0 \in \partial\Omega$. Hence, by Hopf's lemma (see [31]), $\frac{\partial v}{\partial \eta}|_{x_0} > 0$ which contradicts the boundary condition of (2.4). □

2.2 Analytical Tools

2.2.1 Topological Degree Theory

In this section, we will discuss topological degree theory which has been employed to obtain the results for superlinear problems in Chapter 4. To begin with we will describe the topological degree and its properties in the finite dimensional setting, namely, Brouwer degree and eventually in infinite dimensional space, namely Leray-Shauder degree.

Brouwer degree

Let us assume D is a bounded open subset of $\mathbb{R}^N$ and $f : \overline{D} \to \mathbb{R}^N$ is a continuously differentiable function. A point $y \in \mathbb{R}^N$ is called a regular value of f if the determinant of the Jacobian $J_f(x) \neq 0$ for any $x \in D$ such that $f(x) = y$; in other words $J_f(x)$ is invertible. Therefore, by the Inverse Function Theorem the set $f^{-1}(y) = \{x \in D | f(x) = y\}$ is a closed subset of $\overline{D}$ and hence compact. Now, it follows that $f^{-1}(y)$ is finite since a compact set consisting of isolated points is finite. We also assume $y \notin f(\partial D)$. We define the degree of f at y as follows:

$$\deg(f, y, D) := \sum_{x \in f^{-1}(y)} \text{sign det } J_f(x),$$

where

$$\text{sign } p = \begin{cases} 1 & \text{if } p > 0, \\ -1 & \text{if } p < 0, \\ \text{undefined} & \text{if } p = 0. \end{cases}$$

Now, suppose y is not a regular value of f. Then we can define the degree of f at y as follows:

$$\deg(f, y, D) = \lim_{n \to \infty} \deg(f, y_n, D)$$

where $\{y_n\}$ is a sequence of regular values of f which converges to y. Here two problems needs to be dealt with; first, such a sequence exists. Indeed, Sard's Theorem which says set of regular values is dense in $\mathbb{R}^N$ and can be easily proved by showing the complement of the set of regular values has measure 0. Secondly, we need to show the limit exists and is independent of the choice of the sequence $\{y_n\}$; a detailed proof of this part can be found in [7]. Next, we will list all of the properties of the degree that we have defined above:

(i) **Solutions** If $\deg(f, y, D) \neq 0$, then there exists $x \in D$ such that $f(x) = y$.

(ii) **Excision** If D_i, for $i = 1, 2, \cdots, m$ are disjoint open subsets of D and if $f(x) \neq y$ whenever $x \in \overline{D} \setminus \cup_{i=1}^{m} D_i$, then

$$\deg(f, y, D) = \sum_{i=1}^{m} \deg(f, y, D_i).$$

(iii) **Normalization** If I is the identity operator and $y \in D$, then $\deg(I, y, D) = 1$. Also, if $y \notin \overline{D}$, then $\deg(I, y, D) = 0$.

(iv) **Homotopy Invariance** A homotopy between two continuous functions f and g is a continuous function of x, $H(t, x) : [0, 1] \times \overline{D} \to \mathbb{R}^N$, such that $H(0, x) = f(x)$, $H(1, x) = g(x)$ and $H(s, x) \to H(t, x)$ pointwise as $s \to t$. If $y \notin H(t, \partial D)$, then $\deg(H(t, \cdot), y, D)$ is independent of t when $0 \leq t \leq 1$; hence,

$$\deg(f, y, D) = \deg(g, y, D).$$

The homotopy invariance property of degree allows us to match our operator to a more simple operator for which calculating the degree is easier when compared to our operator. The excision property enables us to prove existence results.

Lerey-Schauder degree

Next, we extend the degree to infinite dimensions. We consider the compact perturbation of identity in the infinite dimensional space. Let X be a Banach space and D be an open bounded subset of X. We consider the map $I - T$, where I is the identity map and T is a compact map on $\overline{D}$. The map $T : X \to X$ is compact if T is continuous and $\overline{T(A)}$ is compact for every bounded subset $A \subset X$. Let $p \in X$ such that $p \notin T(\partial D)$. Since T is compact, there is a continuous map $\widehat{T} : \overline{D} \to X$ with finite dimensional range i.e. $\widehat{T}(D)$ is finite dimensional and $||T(x) - \widehat{T}(x)|| < dist(p, T(\partial D))$ for $x \in \overline{D}$. Let $\widehat{D} := D \cap span\{\widehat{T}(\overline{D}), p\}$. We define the degree of $I - T$ at the point

p when T is compact and $p \notin T(\partial D)$ as follows:

$$deg(I - T, D, p) := deg(I - \widehat{T}, \widehat{D}, p),$$

All of the properties of Brouwer degree hold for Leray-Schauder degree. See [7, 51] for details.

2.2.2 Method of Sub and Supersolutions

In this section, we recall an existence result via sub and supersolution theory which is relevant to our purposes. Let us consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega; \\ \frac{\partial u}{\partial \eta} + \mu(\lambda) g(u) u = 0 & \text{on } \partial\Omega \end{cases}, \tag{2.8}$$

where f, g are continuous functions, and $\mu \in C([0, \infty])$ is an increasing functions such that $\mu(0) \geq 0$. Next we define the the sub and supersolution for (2.8) as follows:

Definition 2.4. *$\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is called a subsolution of* (2.8)*, if*

$$\begin{cases} -\Delta \psi \leq \lambda f(\psi) & \text{in } \Omega, \\ \frac{\partial \psi}{\partial \eta} + \mu(\lambda) g(\psi) \psi \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 2.5. *$Z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is called a supersolution of* (2.8)*, if*

$$\begin{cases} -\Delta Z \geq \lambda f(Z) & \text{in } \Omega, \\ \frac{\partial Z}{\partial \eta} + \mu(\lambda) g(Z) Z \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.2. *([3, 70]) Let ψ and Z be a subsolution and a supersolution of* (2.8)*, respectively, such that $\psi \leq Z$. Then* (2.8) *has a solution $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $v \in [\psi, Z]$.*

2.3 Numerical Tools

2.3.1 Quadrature Method

Here we recall from [18, 28, 32] a quadrature method for two point boundary value problems with linear boundary conditions. Such a quadrature method was first introduced in [46] for Dirichlet boundary conditions. Consider the following two point boundary value problem:

$$\begin{cases} -u'' = \lambda f(u); \quad (0,1), \\ -u'(0) + \gamma_1\sqrt{\lambda}u(0) = 0, \\ u'(1) + \gamma_1\sqrt{\lambda}u(1) = 0, \end{cases} \tag{2.9}$$

where $f(u) = u(1-u)$. Let u be a positive solution of (2.9). Then it can be proved that u must be symmetric about $x = \frac{1}{2}$ (see [28] for more details). Let $u(\frac{1}{2}) = \rho$ and $u(0) = q$ (see Figure 2.1). Then $\rho \in (0,1)$ and by multiplying the differential equation in (2.9) by u' and integrating we get:

$$u'(x) = \sqrt{2\lambda(F(\rho) - F((u(x))}, \quad x \in \left[0, \frac{1}{2}\right], \tag{2.10}$$

where $F(s) = \int_0^s f(t)\, dt$. Further integration yields

$$\sqrt{2\lambda}x = \int_q^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}}, \qquad x \in \left[0, \frac{1}{2}\right).$$

Now, as $x \to \frac{1}{2}^-$, we get:

$$\sqrt{\lambda} = \sqrt{2}\int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}.$$

Note that this improper integral exists since $f(\rho) > 0$ and $F(\rho) > F(s)$ for all $s \in (q, \rho)$. Further, using the boundary condition at $x = 0$ and (2.10), we must have $\gamma_1\sqrt{\lambda}q = \sqrt{2\lambda(F(\rho) - F(q))}$ which implies

$$F(\rho) = \frac{2F(q) + \gamma_1^2 q^2}{2}. \tag{2.11}$$

It follows that given $\rho \in (0, 1)$, there exists a unique $q = q(\rho) \in (0, \rho)$ that satisfies (2.11), and hence we get

$$\sqrt{\lambda} = G(\rho) = \sqrt{2}\int_{q(\rho)}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{2.12}$$

Conversely, if λ, ρ satisfy (2.12), we can define $u : [0, \frac{1}{2}] \to [q, \rho]$ via

$$\int_{q(\rho)}^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x \tag{2.13}$$

and, further defining $u(x) = u(1 - x)$ for $x \in [\frac{1}{2}, 1)$, it can be shown that u satisfies (2.9). Hence, $S = \{(\lambda, \rho) | \rho \in (0, 1), G(\rho) = \sqrt{\lambda}\}$ describes the bifurcation diagram for positive solutions of (2.9). Also, for given λ, ρ and q satisfying (2.11)-(2.12), (2.13) can be used to numerically approximate u.

2.3.2 Shooting Method

Next, we discuss a numerical shooting method which will be employed to approximate the positive solution v of:

$$\begin{cases} -v'' = \lambda r v[1 - v - bu_\lambda]; (0, 1), \\ -v'(0) + \sqrt{\lambda}h(u_\lambda(0), \epsilon)\gamma_2 v(0) = 0, \\ v'(1) + \sqrt{\lambda}h(u_\lambda(1), \epsilon)\gamma_2 v(1) = 0, \end{cases} \tag{2.14}$$

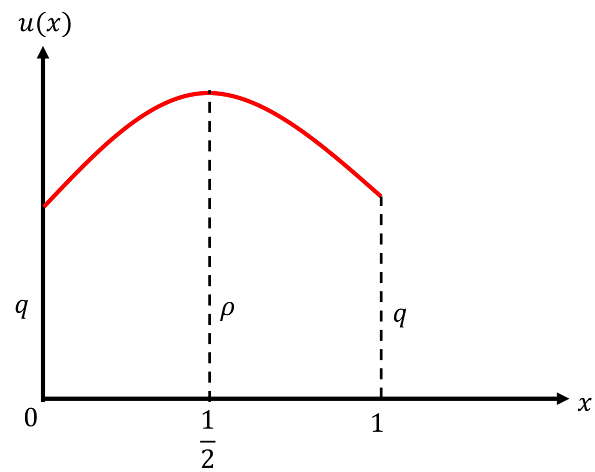


Figure 2.1. Shape of a positive solution to (2.9)

where $u = u_\lambda$ is the unique positive solution of (2.9) for $\lambda > E_1(1, \gamma_1)$ and is numerically approximated using the quadrature method described earlier. This method is illustrated in Figure 2.2.

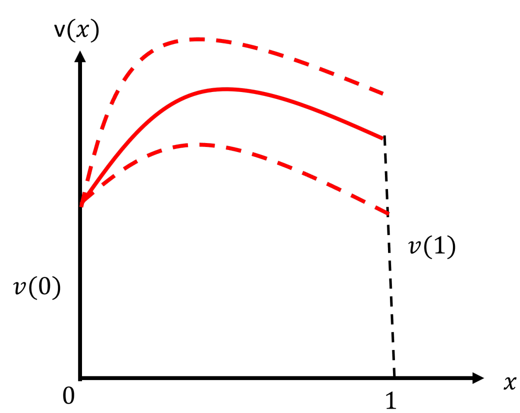


Figure 2.2. Shooting from $x = 0$ to $x = 1$

Let $v(0) = \delta$ and $v' = z$. Then we obtain the following system of ordinary differential equations:

$$\begin{cases} v' = z \text{ in } (0,1), \\ -z' = \lambda r v(1 - v - b_2 u_\lambda) \text{ in } (0,1), \\ z(1) = -\sqrt{\lambda}\gamma_2 h(u_\lambda, \epsilon) v(1), \\ v(0) = \delta, \\ z(0) = \sqrt{\lambda}\gamma_2 h(u_\lambda, \epsilon)\delta. \end{cases} \tag{2.15}$$

For a given value of $\delta > 0$, we use the ParametricNDSolve method in Wolfram Mathematica to approximate solutions of (2.15) using a Runge-Kutta numerical method to solve the differential equation. This process can be explained as shooting from $x = 0$ (where $v(0) = \delta$ and $z(0) = \sqrt{\lambda}\gamma_2 h(u_\lambda, \epsilon)\delta$) and checking at $x = 1$ to see if $z(1) = -\sqrt{\lambda} h(u_\lambda, \epsilon)\gamma_2 v(1)$.

2.3.3 Finite Difference Method

The finite difference method is an important numerical technique for solving partial differential equations. The main idea is to approximate all differential operators by discrete operators that are defined using difference quotients. To begin with, we first discretize the domain and at each of the grid points, we approximate the value of the solution by solving an algebraic system of equations resulting from replacing the differential operators with discrete difference operators.

Assume the domain Ω is an N-rectangle, where $N \geq 1$ is the dimension. In other words $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots (a_N, b_N)$. Let M_i be a positive integer and $h_i = \frac{b_i - a_i}{M_i - 1}$ for $i = 1, 2, \cdots N$. Define $h = (h_1, h_2, \cdots h_N) \in \mathbb{R}^N$, $M = \prod_{i=1}^{N}(M_i)$, and $\mathbb{N}_M^N = \{\alpha = (\alpha_1, \alpha_2, \cdots \alpha_N) \mid 1 \leq \alpha_i \leq M_i, i = 1, 2, \cdots N\}$. Next we partition Ω into $\prod_{i=1}^{N}(M_i - 1)$ sub-N rectangles with grid points $x_\alpha = (a_1 + (\alpha_1 - 1)h_1, a_2 + (\alpha_2 - 1)h_2, \cdots, a_N + (\alpha_N - 1)h_N)$ for each multi-index $\alpha \in \mathbb{N}_M^N$. We call $\mathscr{T}_h = \{x_\alpha\}_{\alpha \in \mathbb{N}_M^N}$ a grid for $\overline{\Omega}$.

Let $\{e_i\}_{i=1}^N$ denote the canonical basis vectors for $\mathbb{R}^N$. We define the discrete operators for approximating first order partial derivatives $\frac{\partial}{\partial x_i}u(x)$ by

$$\begin{cases} \delta^+_{x_i,h_i}u(x) &= \frac{u(x+h_ie_i)-u(x)}{h_i} \\ \delta^-_{x_i,h_i}u(x) &= \frac{u(x)-u(x-h_ie_i)}{h_i} \\ \delta_{x_i,h_i}u(x) &= \frac{1}{2}\delta^+_{x_i,h_i}u(x)+\frac{1}{2}\delta^-_{x_i,h_i}u(x) \quad = \frac{u(x+h_ie_i)-u(x-h_ie_i)}{2h_i} \end{cases} \tag{2.16}$$

for the function $u : \mathbb{R}^N \to \mathbb{R}$ and

$$\begin{cases} \delta^+_{x_i,h_i}u_h(x_\alpha) &= \frac{u_h(x_{\alpha+e_i})-u_h(x_\alpha)}{h_i} \\ \delta^-_{x_i,h_i}u_h(x_\alpha) &= \frac{u_h(x_\alpha)-u_h(x_{\alpha-e_i})}{h_i} \\ \delta_{x_i,h_i}u_h(x_\alpha) &= \frac{1}{2}\delta^+_{x_i,h_i}u_h(x_\alpha)+\frac{1}{2}\delta^-_{x_i,h_i}u_h(x_\alpha) \quad = \frac{u_h(x_{\alpha+e_i})-u_h(x_{\alpha-e_i})}{2h_i} \end{cases} \tag{2.17}$$

for all $x_\alpha \in \mathcal{T}_h \cap \Omega$ for the grid function $u_h : \mathcal{T}_h \to \mathbb{R}$. Note that the discrete operators $\delta^\pm_{x_i,h_i}$ are first-order accurate whereas δ_{x_i,h_i} is second-order accurate. We also define the corresponding discrete gradient operators

$$[\nabla_h^\pm]_i = \delta^\pm_{x_i,h_i}, \qquad [\nabla_h]_i = \delta_{x_i,h_i}.$$

Let $\widetilde{\partial\Omega} \subset \partial\Omega$ be such that $\widetilde{\partial\Omega} := \partial\Omega \setminus \{\text{the points where } \partial\Omega \text{ is not smooth}\}$. For $x \in \mathcal{T}_h \cap \widetilde{\partial\Omega}$, we define the discrete outer normal derivative using the discrete gradient operator ∇_h^* by

$$[\nabla_h^* u(x)]_i = \begin{cases} \delta^+_{x_i,h_i}u(x) & \text{if } \widehat{n}_i(x) < 0, \\ \delta^-_{x_i,h_i}u(x) & \text{if } \widehat{n}_i(x) > 0, \\ \delta_{x_i,h_i}u(x) & \text{if } \widehat{n}_i(x) = 0 \end{cases}$$

to ensure that $\nabla_h^* u(x) \cdot \widehat{n}$ does not require points outside of the domain $\overline{\Omega}$. Note that the discrete outward normal derivative approximation will only be first order accurate.

Next, we define the second order central difference operators for approximating second order partial derivatives $\frac{\partial^2}{\partial x_i^2}u(x)$ by

$$\delta^2_{x_i,h_i}u(x) = \delta^{\pm}_{x_i,h_i}(\delta^{\mp}_{x_i,h_i}(u(x))) \quad = \quad \frac{u(x+h_ie_i)-2u(x)+u(x-h_ie_i)}{h_i^2} \tag{2.18}$$

for the function $u : \mathbb{R}^N \to \mathbb{R}$ and

$$\delta^2_{x_i,h_i}u_h(x_\alpha) = \frac{u_h(x_{\alpha+e_i}) - 2u_h(x_\alpha) + u_h(x_{\alpha-e_i})}{h_i^2} \tag{2.19}$$

for the grid function $u_h : \mathcal{T}_h \cap \widetilde{\partial\Omega} \to \mathbb{R}$. Finally, we define the second order discrete Laplacian operator Δ_h by

$$\Delta_h = \sum_{i=1}^{N} \delta^2_{x_i,h_i}.$$

Chapter 3: Proofs of Theorems 1.1-1.6

In this Chapter, we start by collecting some results that we use in the sequel.

Surjectivity Result

Let X be a reflexive Banach space and $A : X \to X^*$, where X^* is the dual of X, in other words X^* is the set of all linear functionals of X. We say that the operator A is *coercive* if

$$\frac{\langle A(\psi), \psi \rangle}{\|\psi\|_X} \to \infty \text{ as } \|\psi\|_X \to \infty\,.$$

We say that A is *pseudo-monotone*, whenever

$$v_n \rightharpoonup v \qquad \text{in} \quad X \qquad \text{and} \quad \limsup_{n\to\infty} \langle A(v_n), v_n - v \rangle \le 0 \qquad \text{imply}$$

$$\liminf_{n\to\infty} \langle A(v_n), v_n - \psi \rangle \ge \langle A(v), v - \psi \rangle \qquad \text{for any} \quad \psi \in X. \tag{3.1}$$

We will utilize the following surjectivity result in the proof of Theorem 1.2.

Proposition 3.1. *[48, Thm. II. 2.8], [20, Thm. 2.99]) Let X be a reflexive Banach space. If $A : X \to X^*$ is a bounded, pseudomonotone and coercive operator, then for each $b \in X^*$, $Au = b$ has a solution.*

Zorn's Lemma

A partial order is a homogeneous relation $\le$ on a set P that is reflexive, antisymmetric and transitive. Hence if $a, b, c \in P$ then

(i) $a \leq a$,

(ii) $a \leq b$ and $b \leq a \implies a = b$,

(iii) $a \leq b$ and $b \leq c \implies a \leq c$.

Similarly, we can define partial order for $\geq$. We say that a subset Y of a partially ordered set $(P, \leq)$ is a chain if $x \leq y$ or $y \leq x$ for every $x, y \in P$. Then, to prove Theorem 1.3 and Theorem 1.6, we use the following version of Zorn's lemma (see [20]):

Proposition 3.2. *(Zorn's lemma) If in a partially ordered set $(P, \leq)$, every chain Y has an upper bound in P, then P possesses a maximal element.*
Moreover, if in a partially ordered set $(P, \geq)$, every chain P has an lower bound, then P possesses a minimal element.

3.1 Version of Kato's Inequality

In [17], authors establish Kato's inequality up to the boundary for a function $u \in W^{1,1}(\Omega)$. Here, we state and prove a a version of Kato's inequality up to the boundary, that is necessary in the proof of Theorem 1.3. This result can be rephrased as *the maximum of two weak subsolutions is also a weak subsolution*. In particular, *the maximum of two weak solutions is a weak subsolution.*

Theorem 3.3. *Let u_1 and u_2 be functions in $H^1(\Omega)$ such that there exist f_1 and f_2 in $L^r(\partial\Omega)$, for $r \geq \frac{2(N-1)}{N}$, satisfying*

$$\int_\Omega (\nabla u_i \nabla \psi + u_i \psi) \leq \int_{\partial\Omega} f_i \psi \qquad \textit{for all} \;\; 0 \leq \psi \in H^1(\Omega)\,, \tag{3.2}$$

for $i = 1, 2$*. Then,* $u := \max\{u_1, u_2\}$ *satisfies*

$$\int_{\Omega} (\nabla u \nabla \psi + u\psi) \le \int_{\partial\Omega} f\psi \qquad \textit{for all} \ \ 0 \le \psi \in H^1(\Omega)\,,$$

where $f(x) := \begin{cases} f_1(x) & \textit{if } u_1(x) > u_2(x) \\ f_2(x) & \textit{if } u_1(x) \le u_2(x), \end{cases}$ *a.e.* $x \in \partial\Omega$*.*

Proof. Define

$$\Omega_1 := \{x \in \Omega : u_1(x) > u_2(x)\} \text{ and } \Omega_2 := \Omega \setminus \Omega_1$$

and

$$\Gamma_1 := \{x \in \partial\Omega : u_1(x) > u_2(x)\} \text{ and } \Gamma_2 := \partial\Omega \setminus \Gamma_1\,.$$

Fix $0 \le \psi \in H^1(\Omega)$. Then,

$$\begin{aligned} I &= \int_{\Omega} \nabla u \nabla \psi + \int_{\Omega} u\psi \\ &= \underbrace{\int_{\Omega_1} (\nabla u_1 \nabla \psi + u_1\psi)}_{I_1} + \underbrace{\int_{\Omega_2} (\nabla u_2 \nabla \psi + u_2\psi)}_{I_2}\,. \end{aligned}$$

Consider a sequence $\xi_n \in C^1(\mathbb{R})$ such that

$$\xi_n(t) := \begin{cases} 1 & \text{if } t \ge 1/n \\ 0 & \text{if } t \le 0, \end{cases}$$

and $\xi_n' > 0$ on $(0, 1/n)$. Then, define the sequence of functions

$$r_n(x) := \xi_n((u_1 - u_2)(x)) \qquad \text{for } x \in \overline{\Omega}\,.$$

Observe that $r_n \in H^1(\Omega)$ and r_n converges pointwise to $\chi_{\Omega_1 \cup \Gamma_1}$, where the characteristic function is defined as $\chi_{\Omega_1 \cup \Gamma_1}(x) := \begin{cases} 1 & \text{if } x \in \Omega_1 \cup \Gamma_1 \\ 0 & \text{if otherwise}. \end{cases}$

Moreover, $||r_n||_{L^\infty(\Omega)\cap L^\infty(\partial\Omega)} \le 1$ and $\text{supp}(\nabla r_n) \subset \overline{D}_n$, where $D_n := \{x \in \Omega : 0 < u_1(x) - u_2(x) < \frac{1}{n}\}$. Then, using Lebesgue Dominated Convergence Theorem, we have that

$$I_1 = \lim_{n\to\infty} \left[\int_\Omega r_n \nabla u_1 \nabla \psi + \int_\Omega r_n u_1 \psi \right].$$

Since $r_n \in H^1(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$, it follows that $r_n\psi \in H^1(\Omega)$ for any test function $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$. Recalling that $\nabla r_n = 0$ on $\Omega \setminus D_n$, and that u_1 satisfies (3.2), we can write

$$\begin{aligned} \int_\Omega r_n \nabla u_1 \nabla \psi + r_n u_1 \psi &= \int_\Omega \nabla u_1 \nabla (r_n \psi) + u_1 (r_n \psi) - \int_{D_n} \psi \nabla u_1 \nabla r_n \\ &\le \int_{\partial\Omega} f_1 r_n \psi - \int_{D_n} \psi \nabla u_1 \nabla r_n\,. \end{aligned} \tag{3.3}$$

Taking the limit as $n \to \infty$ for the first term on the right-hand side of (3.3), using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_{\partial\Omega} f_1 r_n \psi = \int_{\Gamma_1} f_1 \psi\,.$$

Likewise, for I_2 we have

$$I_2 = \lim_{n\to\infty} \left[\int_\Omega (1 - r_n) \nabla u_2 \nabla \psi + \int_\Omega (1 - r_n) u_2 \psi \right],$$

and

$$\begin{aligned}
&\int_\Omega (1-r_n)\nabla u_2 \nabla\psi + \int_\Omega (1-r_n)u_2\psi \\
&\quad = \int_\Omega \nabla u_2 \nabla[(1-r_n)\psi] + u_2(1-r_n)\psi + \int_{D_n} \psi\nabla u_2 \nabla r_n \\
&\quad \le \int_{\partial\Omega} f_2(1-r_n)\psi + \int_{D_n} \psi \nabla u_2 \nabla r_n. \qquad (3.4)
\end{aligned}$$

Taking the limit as $n \to \infty$ for the first term on the right-hand side of (3.4) and using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_{\partial\Omega} f_2(1-r_n)\psi = \int_{\Gamma_2} f_2\psi\,.$$

Using the fact that $\nabla r_n = \xi_n'(u_1-u_2)\nabla(u_1-u_2)$, the sum of the second terms of the right-hand side of (3.3) and (3.4) yields

$$\begin{aligned}
-\int_{D_n} \psi\nabla u_1\nabla r_n + \int_{D_n}\psi\nabla u_2\nabla r_n &= -\int_{D_n} \psi\nabla(u_1-u_2)\nabla r_n \\
&= -\int_{D_n} \psi\xi_n'(u_1-u_2)|\nabla(u_1-u_2)|^2 \le 0, \qquad (3.5)
\end{aligned}$$

since $\psi \ge 0$ and $\xi_n' \ge 0$. Adding (3.3) and (3.4), taking the limit, and using (3.5), we get

$$I = I_1 + I_2 \le \int_{\Gamma_1} f_1\psi + \int_{\Gamma_2} f_2\psi = \int_{\partial\Omega} f\psi\,.$$

Thus, $u := \max\{u_1, u_2\}$ satisfies

$$\int_\Omega (\nabla u\nabla\psi + u\psi) \le \int_{\partial\Omega} f\psi \qquad \text{for all } \ 0 \le \psi \in H^1(\Omega)\,,$$

completing the proof of Theorem 3.3. $\square$

Likewise, we have the following result for the minimum of two supersolutions.

Corollary 3.4. *Let u_1 and u_2 be functions in $H^1(\Omega)$ such that there exist f_1 and f_2 in $L^r(\partial\Omega)$, for $r \geq \frac{2(N-1)}{N}$, satisfying*

$$\int_\Omega (\nabla u_i \nabla \psi + u_i \psi) \geq \int_{\partial\Omega} f_i \psi \qquad \textit{for all} \;\; 0 \leq \psi \in H^1(\Omega)\,,$$

for $i = 1, 2$. Then, $u := \min\{u_1, u_2\}$ satisfies

$$\int_\Omega (\nabla u \nabla \psi + u \psi) \geq \int_{\partial\Omega} f \psi, \qquad \textit{for all} \;\; 0 \leq \psi \in H^1(\Omega)\,,$$

where

$$f(x) := \begin{cases} f_1(x) & \textit{if } u_1(x) < u_2(x) \\ f_2(x) & \textit{if } u_1(x) \geq u_2(x), \end{cases} \qquad \textit{a.e. } x \in \partial\Omega\,.$$

Proof. Using the fact that $\min\{u_1, u_2\} = \max\{-u_1, -u_2\}$, the proof follows from Theorem 3.3. □

3.2 Proof of Theorem 1.1

We will construct a monotone operator, and show that the iterative scheme starting with a weak subsolution (supersolution) will converge to a minimal (maximal) weak solution.

Let $J := \{u \in H^1(\Omega) : \underline{u} \leq u \leq \overline{u}\}$. Define the linear map $T : J \to H^1(\Omega)$ by $T(u) = v$, where v satisfies

$$\begin{cases} -\Delta v + v = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial v}{\partial \eta} + kv = f(x,u) + ku & \text{on} \quad \partial\Omega\,. \end{cases}$$

Step 1. T is well defined and maps J into itself.

For every $u \in J$, we have $\underline{u} \leq u \leq \overline{u}$. Then using **(H1)** and the fact that $\underline{u}$ and $\overline{u}$ are sub and supersolutions, we get

$$f(x,\underline{u}) + k\underline{u} \leq f(x,u) + ku \leq f(x,\overline{u}) + k\overline{u},$$

and

$$0 \leq |u| \leq \max\{|\underline{u}|, |\overline{u}|\} \leq |\underline{u}| + |\overline{u}|.$$

Taking into account the definitions of $\underline{u}$ and $\overline{u}$, we have that $f(.,\underline{u}(.)),\ f(.,\overline{u}(.))$ are in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Since $\underline{u},\ \overline{u} \in H^1(\Omega)$, then by the continuity of the trace operator (2.2) and the embedding of $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ into $L^{\frac{2(N-1)}{N}}(\partial\Omega)$, for every $u \in J$, we have

$$\begin{aligned} &\|f(x,u) + ku\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \\ &\qquad \leq \|f(x,\underline{u}) + k\underline{u}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} + \|f(x,\overline{u}) + k\overline{u}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \leq C\,. \end{aligned} \tag{3.6}$$

Therefore, $f(.,u(.)) + ku(.) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Then, by the uniqueness of linear problem discussed in Section 2.1.3 in Chapter 2 we get that $v = T(u) \in H^1(\Omega)$ is unique. Thus, the map T is well defined.

Further, if $u, w \in J$ with $u \leq w$, then by the weak maximum principal and the fact that f satisfies **(H1)**, $T(u) \leq T(w)$, that is, the map T is nondecreasing. Moreover, repeating the argument and using Definition 2.2(ii), it follows that

$$\underline{u} \leq T(\underline{u}) \leq T(\overline{u}) \leq \overline{u}. \tag{3.7}$$

Hence, T maps J to J.

Step 2. There exist weakly convergent monotone sequences in $H^1(\Omega)$.

Let's construct monotone sequences $\{u_n\}$ and $\{w_n\}$ successively from the (linear) iteration process

$$u_n = T(u_{n-1}) \text{ with } u_0 = \underline{u} \text{ and } w_n = T(w_{n-1}) \text{ with } w_0 = \overline{u}\,.$$

Using (3.7) and the monotonicity of T, we get

$$\underline{u} = u_0 \leq \cdots \leq u_n \leq \cdots \leq w_n \leq \cdots \leq w_0 = \overline{u}. \tag{3.8}$$

We show that $\{u_n\}$ is convergent. The proof for $\{w_n\}$ is analogous. We see that $u_n = T(u_{n-1})$ satisfies

$$\int_\Omega (\nabla u_n \nabla \psi + u_n \psi) + k \int_{\partial\Omega} u_n \psi = \int_{\partial\Omega} \big(f(x, u_{n-1}) + k u_{n-1}\big)\psi,$$

for all $\psi \in H^1(\Omega)$. Letting $u_n = T(u_{n-1})$ as a test function, we get

$$\int_\Omega \big(|\nabla u_n|^2 + u_n^2\big) + k \int_{\partial\Omega} u_n^2 = \int_{\partial\Omega} \big(f(x, u_{n-1}) + k u_{n-1}\big) u_n. \tag{3.9}$$

Since $u_{n-1}, u_n \in J$, using Hölder's inequality in (3.28), and the bound (3.6), we have

$$\begin{aligned}
\|u_n\|^2_{H^1(\Omega)} &\leq \|u_n\|^2_{H^1(\Omega)} + k\|u_n\|^2_{L^2(\partial\Omega)} \\
&\leq \|f(x, u_{n-1}) + k u_{n-1}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \|u_n\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} \\
&\leq C\left(\|\overline{u}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} + \|\underline{u}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)}\right).
\end{aligned}$$

Hence, there exists a uniform constant $C' > 0$, depending on $\Omega,\ f,\ k,\ \underline{u}$ and $\overline{u}$, such that

$$\|u_n\|_{H^1(\Omega)} \leq C'\,. \tag{3.10}$$

By the reflexivity of $H^1(\Omega)$, (3.10), there is a subsequence (relabeled) u_n which converges weakly to u_* in $H^1(\Omega)$. *Step 3. $f(x, u_n) + k u_n$ converges weakly to $f(x, u_*)+$*

ku_* *in* $L^{\frac{2(N-1)}{N}}(\partial\Omega)$.

Since the sequence u_n in Step 2 is nondecreasing and bounded (see (3.8)), it converges pointwise to u_*, that is,

$$u_*(x) = \lim_{n\to\infty} u_n(x) \in J. \tag{3.11}$$

Using the fact that f is continuous in the second variable u for a.e $x \in \partial\Omega$ and (3.11), we have that

$$f(x, u_*(x)) + ku_* = \lim_{n\to\infty} f(x, u_n(x)) + ku_n(x).$$

By (3.6), $f(x, u_n) + ku_n$ is bounded in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Then, Lebesgue Dominated Convergence Theorem yields

$$\|(f(x, u_n) + ku_n) - (f(x, u_*) + ku_*)\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \to 0 \qquad \text{as } n \to \infty\,.$$

Therefore, $f(x, u_n) + ku_n$ converges weakly to $f(x, u_*) + ku_*$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Thus, for all $\psi \in H^1(\Omega) \subset L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$, we have

$$\lim_{n\to\infty} \int_{\partial\Omega} (f(x, u_n) + ku_n)\psi = \int_{\partial\Omega} (f(x, u_*) + ku_*)\psi\,. \tag{3.12}$$

Step 4. u_* *is a weak solution to* (1.4).

First, since $u_* \in H^1(\Omega)$, the continuity of the trace (2.2) and $L^{\frac{2(N-1)}{N-2}}(\partial\Omega) \hookrightarrow L^{\frac{2(N-1)}{N}}(\partial\Omega)$, imply that $u_* \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Therefore, for some positive constant C'', we have

$$\begin{aligned}
\|f(x, u_*)\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} &= \|f(x, u_*) + ku_* - ku_*\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} \\
&\le \|f(x, u_*) + ku_*\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} + \|ku_*\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} \\
&\le C''\,.
\end{aligned}$$

Second, from the monotone iteration, we know that $u_n = T(u_{n-1})$ satisfies

$$\int_\Omega (\nabla u_n \nabla \psi + u_n \psi) + k \int_{\partial\Omega} u_n \psi = \int_{\partial\Omega} \big(f(x, u_{n-1}) + k u_{n-1}\big)\psi.$$

Observe that u_n converges weakly to u_* in $H^1(\Omega)$, strongly in $L^2(\partial\Omega)$ (see Step 2) and $f(x, u_n) + k u_n$ converges weakly to $f(x, u_*) + k u_*$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$ (see Step 3). Then taking the limit as $n \to \infty$ and using (3.12), we get for any $\psi \in H^1(\Omega)$

$$\begin{aligned}
\int_\Omega (\nabla u_* \nabla \psi + u_* \psi) + \int_{\partial\Omega} k u_* \psi &= \lim_{n\to\infty} \left(\int_\Omega (\nabla u_n \nabla \psi + u_n \psi) + \int_{\partial\Omega} k u_n \psi \right) \\
&= \lim_{n\to\infty} \left(\int_{\partial\Omega} (f(x, u_{n-1}) + k u_{n-1})\psi \right) \\
&= \int_{\partial\Omega} (f(x, u_*) + k u_*)\psi \,.
\end{aligned}$$

Hence,

$$\int_\Omega (\nabla u_* \nabla \psi + u_* \psi) = \int_{\partial\Omega} f(x, u_*)\psi \qquad \text{for all } \psi \in H^1(\Omega).$$

Moreover, we also have $f(x, u_*) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Thus u_* is a weak solution to (1.4).

Step 5. u_* *is the minimal weak solution in the interval* $[\underline{u}, \overline{u}]$.
Let v be a weak solution to (1.4) with $\underline{u} \le v \le \overline{u}$. Then v is a weak supersolution, and $\underline{u} \le v$. Repeating the above iteration procedure with $u_0 = \underline{u}$, we get $\underline{u} \le u_* \le v$. Thus u_* is a weak minimal solution.

Similarly, we can construct the maximal weak solution u^* from the sequence $\{w_n\}$ with $w_0 = \overline{u}$. This completes the proof of Theorem 1.1. □

3.3 Proofs of Theorem 1.2 and Theorem 1.3

We prove Theorem 1.2 by applying Proposition 3.1 to an appropriate operator related to our problem (1.4). Then, Theorem 1.3 is proved by using Zorn's lemma and Theorem 3.3.

3.3.1 Proof of Theorem 1.2

Let us consider a modified problem

$$\begin{cases} -\Delta u + u &= 0 \qquad \text{in} \quad \Omega\,; \\ \frac{\partial u}{\partial \eta} &= g(x,u) \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{3.13}$$

where

$$g(x,s) := \begin{cases} f(x,\underline{u}(x)), & s < \underline{u}(x), \\ f(x,s), & \underline{u}(x) \le s \le \overline{u}(x), \\ f(x,\overline{u}(x)), & s > \overline{u}(x) \end{cases} \tag{3.14}$$

is the truncated function. We observe that g is a Carathéodory function, since f is a Carathéodory function. We note that a weak solution u of (3.13) is a weak solution of (1.4) whenever $\underline{u} \le u \le \overline{u}$.

Our plan is to establish the existence of a weak solution u of (3.13), and verify that $\underline{u} \le u \le \overline{u}$. For the existence part, we use Proposition 3.1. For this, we define the map $B\colon H^1(\Omega) \to \big(H^1(\Omega)\big)^*$ given by

$$\langle B(v), \psi\rangle := \int_\Omega (\nabla v \nabla \psi + v\psi) - \int_{\partial\Omega} g(x,v)\psi, \tag{3.15}$$

for all $\psi \in H^1(\Omega)$.

First, we show that B is *well-defined and bounded.* First two terms of (3.15) are well defined since $v, \psi \in H^1(\Omega)$. By the Hölder's inequality combined with the continuity of trace operator (2.2) and hypothesis **(H2)**, we get

$$\int_{\{\underline{u} \le v \le \overline{u}\}} |f(x,v)\psi| \le \|K\|_{L^r(\partial\Omega)} \|\psi\|_{L^{r'}(\partial\Omega)}, \tag{3.16}$$

where $r' < \frac{2(N-1)}{N-2}$ is the conjugate of r. Then, the definition of g given in (3.14), Definition 2.1(i), and (3.16) yield

$$\begin{aligned} \left| \int_{\partial\Omega} g(x,v)\psi \right| &\le \int_{\{v<\underline{u}\}} |f(x,\underline{u})\psi| + \int_{\{\underline{u} \le v \le \overline{u}\}} |f(x,v)\psi| + \int_{\{v>\overline{u}\}} |f(x,\overline{u})\psi| \\ &\le C_2 \|\psi\|_{H^1(\Omega)}, \end{aligned} \tag{3.17}$$

where the last inequalities of (3.17) follow by (3.16) and (2.2), and the constant C_2 depends only on K and Ω.

Second, we show that B is *pseudo-monotone*, see definition (3.1). For this, we set $B = L - G$, where $L, G : H^1(\Omega) \to (H^1(\Omega))^*$ are defined by

$$\langle L(v), \psi \rangle := \int_\Omega (\nabla v \nabla \psi + v\psi) \ \text{ and } \langle G(v), \psi \rangle := \int_{\partial\Omega} g(x,v)\psi\,,$$

for all $\psi \in H^1(\Omega)$. Then we show that B is pseudomonotone in the following steps. Let $v_n \rightharpoonup v$ in $H^1(\Omega)$.

Step 1: $Lv_n \to Lv$ in $(H^1(\Omega))^*$.

Since $v_n \rightharpoonup v$, $\langle L(v_n) - L(v), \psi \rangle \to 0$ as $n \to \infty$ for all $\psi \in H^1(\Omega)$. Hence,

$$\|L(v_n) - L(v)\|_{(H^1(\Omega))^*} = \sup_{\|\psi\|_{H^1(\Omega)} \le 1} |\langle L(v_n) - L(v), \psi \rangle| \to 0 \text{ as } n \to \infty\,,$$

as desired.

Step 2: $G(v_n) \to G(v)$ in $(H^1(\Omega))^*$.

Suppose that $v_n \rightharpoonup v$ in $H^1(\Omega)$ but $G(v_n) \not\to G(v)$ in $(H^1(\Omega))^*$. Then there exists $\varepsilon_0 > 0$ and a subsequence $\{v_{n_j}\}$ such that

$$\|G(v_{n_j}) - G(v)\|_{(H^1(\Omega))^*} \geq \varepsilon_0. \tag{3.18}$$

Using the fact that $\{v_{n_j}\}$ is bounded in $H^1(\Omega)$ and the compactness of the trace operator (2.2), there exists a subsequence $\{v'_{n_j}\}$ such that $v'_{n_j} \to v$ in $L^{r'}(\partial\Omega)$, where $r' < \frac{2(N-1)}{N-2}$. By [16, Theorem 4.9], there exists a subsequence $\{v''_{n_j}\}$ such that

$$v''_{n_j}(x) \to v(x) \text{ a.e. } x \in \partial\Omega\,.$$

Since $g(x,.)$ is continuous for a.e. $x \in \partial\Omega$, then $g(x, v''_{n_j}(x)) \to g(x, v(x))$ a.e. $x \in \partial\Omega$ and $g(x, v''_{n_j}(x))$ is bounded in $L^r(\partial\Omega)$ by **(H2)**. Using the Lebesgue Dominated Convergence Theorem, we get

$$\|g(., v''_{n_j}(.)) - g(., v(.))\|_{L^r(\partial\Omega)} \to 0 \text{ as } n \to \infty\,.$$

By the Hölder's inequality, for all $\psi \in H^1(\Omega)$, we get

$$\langle G(v''_{n_j}) - G(v), \psi\rangle \to 0 \text{ as } j \to \infty\,.$$

Therefore, $\|G(v''_{n_j}) - G(v)\|_{(H^1(\Omega))^*} = \sup\limits_{\|\psi\|_{H^1(\Omega)} \leq 1} |\langle G(v''_{n_j}) - G(v), \psi\rangle| \to 0$ as $j \to \infty$. Hence, $G(v''_{n_j}) \to G(v)$ in $(H^1(\Omega))^*$ as $j \to \infty$, a contradiction to (3.18).

Step 3: B is pseudomonotone.

Let $v_n \rightharpoonup v$ in $H^1(\Omega)$. Using *Step 1-Step 2*, we get that

$$B(v_n) \to B(v) \qquad \text{in} \quad (H^1(\Omega))^*\,.$$

Therefore, $\langle B(v_n), \psi\rangle \to \langle B(v), \psi\rangle$ as $n \to \infty$ for all $\psi \in H^1(\Omega)$. Furthermore, by [16, Proposition 3.5 (iv)], $\langle B(v_n), v_n\rangle \to \langle B(v), v\rangle$ as $n \to \infty$. Hence,

$$\langle B(v_n), v_n - \psi\rangle \to \langle B(v), v - \psi\rangle \text{ as } n \to \infty\,,$$

establishing that B is pseudomonotone.

Finally, we show that B is *coercive*, i.e., $\langle B(\psi), \psi\rangle / \|\psi\|_{H^1(\Omega)} \to \infty$ as $\|\psi\|_{H^1(\Omega)} \to \infty$. For any $\psi \in H^1(\Omega)$, using (3.17) in the definition of the operator B, we have

$$\langle B(\psi), \psi\rangle \geq \|\psi\|^2_{H^1(\Omega)} - C_2\|\psi\|_{H^1(\Omega)} \geq \frac{1}{2}\|\psi\|^2_{H^1(\Omega)} - C_3.$$

Hence B is *coercive*. Thus B satisfies the hypotheses of Proposition 3.1 with $X = H^1(\Omega)$. Therefore, for $b = 0 \in (H^1(\Omega))^*$, there exists $u \in H^1(\Omega)$ such that

$$\langle B(u), \psi\rangle = 0 \quad \forall \psi \in H^1(\Omega)\,.$$

Moreover, $g(x, .)$ is bounded in $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ by **(H2)**, and therefore in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$ by continuous embedding of $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ into $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Hence u is a weak solution of (3.13). It remains to prove that u is a weak solution of our problem (1.4). For this, we will show that $\underline{u} \leq u \leq \overline{u}$ in $\overline{\Omega}$, so that $g = f$ in (3.13).

Clearly, $(u - \overline{u})_+ := \max\{0, u - \overline{u}\} \in H^1(\Omega)$ and $(\underline{u} - u)_+ := \max\{0, \underline{u} - u\} \in H^1(\Omega)$. Then, using the weak formulation of (3.13) with the test function $\psi := (u - \overline{u})_+ \geq 0$ in $H^1(\Omega)$, and the facts that $\overline{u}$ is a supersolution of (1.4) and $(u - \overline{u})_+ = 0$ in $\{u \leq \overline{u}\}$, we have

$$\int_\Omega (\nabla u \nabla (u-\overline{u})_+ + u(u-\overline{u})_+) = \int_{\partial\Omega} g(x,u)(u-\overline{u})_+ \tag{3.19}$$
$$= \int_{\{u>\overline{u}\}} f(x,\overline{u})(u-\overline{u})_+$$
$$= \int_{\partial\Omega} f(x,\overline{u})(u-\overline{u})_+$$
$$\leq \int_\Omega \nabla\overline{u}\,\nabla(u-\overline{u})_+ + \int_\Omega \overline{u}(u-\overline{u})_+.$$

Then, (3.19) yields

$$\begin{aligned} 0 &\leq \int_\Omega |\nabla(u-\overline{u})_+|^2 + \int_\Omega |(u-\overline{u})_+|^2 \\ &= \int_\Omega \nabla(u-\overline{u})\nabla(u-\overline{u})_+ + \int_\Omega (u-\overline{u})(u-\overline{u})_+ \\ &\leq 0, \end{aligned}$$

which implies that $\|(u-\overline{u})_+\|_{H^1(\Omega)} = 0$. That is, $u \leq \overline{u}$ a.e. in Ω. Using the the continuity of the trace operator (2.2), we get that $\|(u-\overline{u})_+\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} = 0$. Hence, $u \leq \overline{u}$ a.e. in $\overline{\Omega}$.

Analogously, taking the test function $\psi := (\underline{u}-u)_+ \geq 0$ and using the fact that $\underline{u}$ is a subsolution of (1.4), we obtain that

$$\begin{aligned} 0 &\leq \int_\Omega |\nabla(\underline{u}-u)_+|^2 + \int_\Omega |(\underline{u}-u)_+|^2 \\ &= \int_\Omega \nabla(\underline{u}-u)\nabla(\underline{u}-u)_+ + \int_\Omega (\underline{u}-u)(\underline{u}-u)_+ \leq 0, \end{aligned}$$

Therefore, $\underline{u} \leq u$ a.e. in $\overline{\Omega}$, and hence $\underline{u} \leq u \leq \overline{u}$ a.e. in $\overline{\Omega}$. Thus, u is a weak solution of (1.4), completing the proof of Theorem 1.2. $\square$

3.3.2 Proof of Theorem 1.3

We will use Zorn's Lemma and Proposition 3.2, to prove our result. Consider the set

$$\mathcal{A} := \{u \in H^1(\Omega)\colon \underline{u}(x) \le u(x) \le \overline{u}(x) \text{ a.e. in } \overline{\Omega}$$
$$\text{and } u \text{ is a weak solution of (1.4)}\},$$

and we note that $\mathcal{A}$ is nonempty by Theorem 1.2. Let $\{u_i\}_{i\in I} \subseteq \mathcal{A}$ be a family of *chain*. Since u_i is a weak solution of (1.4), taking u_i as the test function and using (3.16), we get

$$\|u_i\|_{H^1(\Omega)} = \int_\Omega \left(|\nabla u_i|^2 + u_i^2\right) = \int_{\partial\Omega} f(x, u_i)u_i \le C\,,$$

where C depends on $\underline{u}$, $\overline{u}$, K, $\partial\Omega$ but independent of $i \in I$. Notice that $(\mathcal{A}, \le)$ is a partial ordered set, hence $u_i \le u_j$ for all $i \le j \in I$. Since, $H^1(\Omega)$ is separable I is countable, in particular we can take $I = \mathbb{N}$. Also, $H^1(\Omega)$ is reflexive implies any bounded sequence sequence in $H^1(\Omega)$ has a weekly converegent subsequence (see Theorem 2.28 in [20]). Hence, $u_i \rightharpoonup u := \sup\limits_{i\in I} u_i$ in $H^1(\Omega)$ upto a subsequence (relabeled). Clearly, u is an upper bound of the chain $\{u_i\}_{i\in I}$. It suffices to show that $u \in \mathcal{A}$. Since $\{u_n\}$ is nondecreasing and $\underline{u} \le u_n \le \overline{u}$, we have that $u_n(x) \to u(x)$, and $u_n(x) \le u(x)$ for all n, and $\underline{u}(x) \le u(x) \le \overline{u}(x)$ pointwise a.e. in $\overline{\Omega}$. Furthermore, since f is Carathéodory, we have that

$$f(x, u_n(x)) \to f(x, u(x)) \text{ as } n \to \infty\,.$$

This, in conjunction with **(H2)**, and the Lebesgue Dominated Convergence Theorem yields $\|f(x, u_n) - f(x, u)\|_{L^r(\partial\Omega)} \to 0$ as $n \to \infty$. Therefore, using Hölder's inequality, we deduce that

$$\left|\int_{\partial\Omega} f(x,u_n)\psi - \int_{\partial\Omega} f(x,u)\psi\right| \leq \int_{\partial\Omega} |f(x,u_n) - f(x,u)|\,|\psi|$$
$$\leq \|f(x,u_n) - f(x,u)\|_{L^r(\partial\Omega)}\,\|\psi\|_{L^{r'}(\partial\Omega)} \to 0\,,$$

which yields

$$\lim_{n\to\infty}\int_{\partial\Omega} f(x,u_n)\psi = \int_{\partial\Omega} f(x,u)\psi \quad \text{for all } \psi \in H^1(\Omega).$$

Taking the limit as $n \to \infty$, we get for any $\psi \in H^1(\Omega)$

$$\int_{\Omega} (\nabla u \nabla \psi + u\psi) = \lim_{n\to\infty}\int_{\Omega} (\nabla u_n \nabla \psi + u_n\psi)$$
$$= \lim_{n\to\infty}\int_{\partial\Omega} f(x,u_n)\psi = \int_{\partial\Omega} f(x,u)\psi\,.$$

Hence, u is a weak solution of (1.4), thus concluding that $u \in \mathcal{A}$.

By Zorn's Lemma, there exists a maximal element $u^* \in \mathcal{A}$. It remains to show that u^* is maximal in the sense that if $\hat{u}$ is any other weak solution of (1.4) between $\underline{u}$ and $\overline{u}$, then $\widehat{u} \leq u^*$. So, let $\widehat{u}$ be a weak solution of (1.4) between $\underline{u}$ and $\overline{u}$, and u^* is the maximal element of $\mathcal{A}$. By Theorem 3.3, $u = \max\{\widehat{u}, u^*\}$ is a subsolution of (1.4). Then, by Theorem 1.2, there exists a weak solution u_0 of (1.4) satisfying

$$\underline{u} \leq u \leq u_0 \leq \overline{u}\,.$$

Thus, $u_0 \in \mathcal{A}$. On the other hand, $u^* \leq \max\{\widehat{u}, u^*\} = u \leq u_0$. But u^* is maximal element of $\mathcal{A}$, so necessarily $u^* = u_0$. Therefore, we readily see that $\widehat{u} \leq u \leq u_0 = u^*$, and hence

$$\underline{u} \leq \widehat{u} \leq u^* \leq \overline{u}\,,$$

as desired. The existence of a minimal element u_* of $\mathscr{A}$ is proved analogously. This completes the proof of Theorem 1.3. $\square$

3.4 Proof of Theorem 1.4

In this section, we apply our existence results, Theorem 1.1 and Theorem 1.2, to problems involving sublinear nonlinearities. In particular, in each case we construct an ordered pair of weak sub and supersolution. We apply Theorem 1.1 to establish Theorem 1.4 and Theorem 1.2 in Remark 3.1 below.

Let $\lambda > \frac{\mu_1}{f'(0)}$ be fixed. Using hypothesis (i), we verify that $\underline{u} := \epsilon\varphi_1$ is a subsolution of (1.5) for $\epsilon \approx 0$. Indeed, we observe that since $\lambda > \frac{\mu_1}{f'(0)}$ is fixed, $\xi(s) := \mu_1 s - \lambda f(s)$ satisfies $\xi(0) = 0$ and $\xi'(0) < 0$, then $\xi(s) < 0$ for $s \approx 0$. Therefore, for all $0 \leq \psi \in H^1(\Omega)$, the following holds for $\epsilon \approx 0$

$$\int_\Omega \nabla\underline{u}\nabla\psi + \int_\Omega \underline{u}\psi = \mu_1 \int_{\partial\Omega} (\epsilon\varphi_1)\psi \leq \lambda \int_{\partial\Omega} f(\epsilon\varphi_1)\psi = \lambda \int_{\partial\Omega} f(\underline{u})\psi \,.$$

Next, using hypothesis (ii), we show that there exists $M_\lambda > 0$ such that $\overline{u} := Me$ is a weak supersolution of (1.5) for all $M \geq M_\lambda$, where e is the unique positive solution of

$$\begin{cases} -\Delta e + e = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial e}{\partial \eta} = 1 & \text{on} \quad \partial\Omega\,. \end{cases}$$

We observe that while f is not assumed to be nondecreasing, $\overline{f}(t) := \max_{s\in[0,t]} f(s)$ is nondecreasing, and $f(t) \leq \overline{f}(t)$ for all $t \geq 0$. Moreover, due to hypothesis (ii), $\overline{f}$ satisfies the sublinear condition at infinity

$$\lim_{t\to+\infty} \frac{\overline{f}(t)}{t} = 0\,.$$

Therefore, there exists $M_\lambda > 0$ such that for all $M \geq M_\lambda$

$$\frac{\overline{f}(M\|e\|_{L^\infty(\partial\Omega)})}{M\|e\|_{L^\infty(\partial\Omega)}} \leq \frac{1}{\lambda\|e\|_{L^\infty(\partial\Omega)}} \text{ or equivalently } \lambda\overline{f}(M\|e\|_{L^\infty(\partial\Omega)}) \leq M\,.$$

Then $\overline{u} = Me \in H^1(\Omega)$ satisfies

$$\begin{aligned}\int_\Omega \nabla\overline{u}\nabla\psi + \int_\Omega \overline{u}\psi = M\int_{\partial\Omega}\psi \\ &\geq \lambda\int_{\partial\Omega}\overline{f}(M\|e\|_{L^\infty(\partial\Omega)})\psi \\ &\geq \lambda\int_{\partial\Omega}\overline{f}(Me)\psi \\ &\geq \lambda\int_{\partial\Omega}f(Me)\psi = \lambda\int_{\partial\Omega}f(\overline{u})\psi\end{aligned}$$

for all $0 \leq \phi \in H^1(\Omega)$. Therefore, $\overline{u}$ is a weak supersolution of (1.4) for each $\lambda > \frac{\mu_1}{f'(0)}$. Clearly $\overline{u} = Me \geq \epsilon(\lambda)\varphi_1 = \underline{u}$ a.e. in $\overline{\Omega}$. We remark that since f is locally Lipschitz and $[\underline{u}, \overline{u}]$ is bounded, f satisfies hypothesis **(H1)** of Theorem 1.1. Hence, there exists a positive weak solution u of (1.5) such that $\epsilon\varphi_1 \leq u \leq Me$ a.e. in $\overline{\Omega}$ for any $\lambda > \frac{\mu_1}{f'(0)}$. This completes the proof.

Remark 3.1. On the other hand, if f is continuous (not necessarily Lipschitz), satisfies hypothesis (ii) of Theorem 1.4 and $f(s) > 0$ for $s \geq 0$, the problem (1.5) has a positive weak solution for each $\lambda > 0$. Indeed, it is easy to see that $\underline{u} \equiv 0$ is a strict weak subsolution and for each $\lambda > 0$, there exists $M_\lambda > 0$ such that $\overline{u} = Me$ is a weak supersolution for all $M \geq M_\lambda$, as in the proof of Theorem 1.4. Then, the result follows by Theorem 1.2.

3.5 Proof of Theorem 1.5

Throughout this section, we use of the product Sobolev space $H^1(\Omega) \times H^1(\Omega)$, that we denote by $(H^1(\Omega))^2$, endowed with the norm $||(w_1, w_2)||_{(H^1(\Omega))^2} := ||w_1||_{H^1(\Omega)} + ||w_2||_{H^1(\Omega)}$. In order to keep the notation simple, we denote (u_1, u_2) by U and the inequality $U = (u_1, u_2) \leq V = (v_1, v_2)$ means $u_i(x) \leq v_i(x)$ for a.e. $x \in \overline{\Omega}$ in the sense of Kinderlehrer and Stampacchia (see [25, 56]), where $i = 1, 2$. In [25, 56] the function $u \in H^1(\Omega)$ satisfies $u \geq 0$ on $S \subset \overline{\Omega}$ in the sense of Kinderlehrer and Stampacchia, if there is a sequence $u_n \in W^{1,\infty}(\Omega)\,(\subset C(\overline{\Omega}))$ with $u_n(x) \geq 0$ for $x \in S$ and $u_n \to u$ in $H^1(\Omega)$. Now the system (1.6) can be rewritten in the vector form as follows:

$$\begin{cases} -\Delta U + U = 0 & \text{in } \Omega; \\ \frac{\partial U}{\partial \eta} = F(x, U) & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

where $F(x, U) = (f_1(x, u_1, u_2), f_2(x, u_1, u_2))$.

We will first construct a monotone operator and then use the iterative scheme to show the existence of a minimal (maximal) solution through the convergence of a sequence of weak subsolutions (supersolutions).

Step 1. *Construction of the monotone operator*

We define the bilinear map $T : J \to (H^1(\Omega))^2$ by $T(U) = V$, where $J := \{U = (u_1, u_2) \in (H^1(\Omega))^2 : \underline{U} \leq U \leq \overline{U}\}$ and $V = (v_1, v_2)$ is the unique weak solution of the decoupled system

$$\begin{cases} -\Delta v_i + v_i = 0 & \text{in } \Omega; \\ \frac{\partial v_i}{\partial \eta} + k v_i = f_i(x, u_1, u_2) + k u_i & \text{on } \partial\Omega;\ i = 1, 2 \end{cases} \tag{3.21}$$

where $k = \min\{k_1, k_2\}$.

Notice that the solution of (3.21) is guaranteed by the uniqueness of the solution of the linear problem in Section 2.1.3. Hence the map T *is well-defined.*

Now, let us prove that T is monotonically increasing and maps J into itself.

Take $U, V \in J$ with $U \leq V$. We will show $T(U) \leq T(V)$ where $T(U) = (w_1, w_2)$ and $T(V) = (z_1, z_2)$ satisfying the following:

$$\begin{cases} -\Delta w_i + w_i = 0 & \text{in } \Omega; \\ \frac{\partial w_i}{\partial \eta} + k w_i = f_i(x, u_1, u_2) + k u_i & \text{on } \partial\Omega \ i = 1, 2 \end{cases} \tag{3.22}$$

$$\begin{cases} -\Delta z_i + z_i = 0 & \text{in } \Omega; \\ \frac{\partial z_i}{\partial \eta} + k z_i = f_i(x, v_1, v_2) + k v_i & \text{on } \partial\Omega, \ i = 1, 2 \end{cases} \tag{3.23}$$

By (**C1**), $f_i(x, u_1, u_2) + k u_i \leq f_i(x, v_1, u_2) + k v_i$ and since each f_i satisfies the quasi-monotonicity condition (**Q**), it follows that $f_i(x, v_1, u_2) + k v_i \leq f_i(x, v_1, v_2) + k v_i$. Therefore, $f_i(x, u_1, u_2) + k u_i \leq f_i(x, v_1, v_2) + k v_i$. Observe that for each $i = 1, 2$, $w_i - z_i$ satisfies the following system:

$$\begin{cases} -\Delta(w_i - z_i) + (w_i - z_i) = 0, & \text{in } \Omega; \\ \frac{\partial(w_i - z_i)}{\partial \eta} + k(w_i - z_i) = f_i(x, u_1, u_2) + k u_i - (f_i(x, v_1, v_2) + k v_i) \leq 0 & \text{on } \partial\Omega. \end{cases} \tag{3.24}$$

Using the maximum principal, we get that $w_i \leq z_i$. Hence, $T(U) \leq T(V)$.

Let's show that $T(J) \subseteq J$. With the monotonicity of T, it is enough to show that $\underline{U} \leq T(\underline{U})$ and $T(\overline{U}) \leq \overline{U}$. Take $\underline{U} = (\underline{u_1}, \underline{u_2})$. Then $T(\underline{U}) = (\underline{w_1}, \underline{w_2})$ satisfies the system

$$\begin{cases} -\Delta \underline{w_i} + \underline{w_i} = 0 & \text{in } \Omega; \\ \frac{\partial \underline{w_i}}{\partial \eta} + k \underline{w_i} = f_i(x, \underline{u_1}, \underline{u_2}) + k \underline{u_i} & \text{on } \partial\Omega; \ i = 1, 2 \end{cases}$$

Using the fact that $\underline{U}$ is a subsolution to (3.20), we get that

$$\begin{cases} -\Delta(\underline{u_i} - \underline{w_i}) + (\underline{u_i} - \underline{w_i}) = 0 & \text{in } \Omega; \\ \frac{\partial(\underline{u_i} - \underline{w_i})}{\partial \eta} + k(\underline{u_i} - \underline{w_i}) \leq 0 & \text{on } \partial\Omega; \ i = 1, 2 \end{cases}$$

By the maximum principal, we have $\underline{u_i} - \underline{w_i} \leq 0$ in $\overline{\Omega}$. Hence, $\underline{u_i} \leq \underline{w_i}$ which implies $\underline{U} \leq T(\underline{U})$. Similarly, we can show $T(\overline{U}) \leq \overline{U}$. Hence,

$$\underline{U} \leq T(\underline{U}) \leq T(\overline{U}) \leq \overline{U} \tag{3.25}$$

Thus, T maps J into itself.

Step 2. *Construction of minimal and maximal weak solutions*

(i) We construct monotone sequences $\{U_n\} = \{(u_{1,n}, u_{2,n})\}$ and $\{W_n\} = \{(w_{1,n}, w_{2,n})\}$ using the linear iteration process as follows

$$U_n = T(U_{n-1}) \text{ with } U_0 = \underline{U}, \quad \text{and} \quad W_n = T(W_{n-1}) \text{ with } W_0 = \overline{U},$$

where U_n and W_n are weak solutions of the following systems respectively,

$$\begin{cases} -\Delta u_{i,n} + u_{i,n} &= 0 & \text{in } \Omega; \\ \frac{\partial u_{i,n}}{\partial \eta} + k u_{i,n} &= f_i(x, u_{1,n-1}, u_{2,n-1}) + k u_{i,n-1} & \text{on } \partial\Omega, \ i = 1, 2 \end{cases} \tag{3.26}$$

and

$$\begin{cases} -\Delta w_{i,n} + w_{i,n} &= 0 & \text{in } \Omega; \\ \frac{\partial w_{i,n}}{\partial \eta} + k w_{i,n} &= f_i(x, w_{1,n-1}, w_{2,n-1}) + k w_{i,n-1} & \text{on } \partial\Omega, \ i = 1, 2 \end{cases} \tag{3.27}$$

From the monotonicity of T and (3.25), it follows that

$$\underline{U} = U_0 \leq U_1 \leq U_2 \leq \cdots \leq W_n \leq W_{n-1} \leq \cdots \leq W_0 = \overline{U}.$$

(ii) We will show that the sequence U_n and W_n are weakly convergent. Since $U_n = T(U_{n-1})$ is a weak solution of (3.26), For each $i = 1, 2$, we have that

$$\int_\Omega (\nabla u_{i,n} \nabla \psi + u_{i,n} \psi) + k \int_{\partial\Omega} u_{i,n} \psi = \int_{\partial\Omega} \big(f_i(x, u_{1,n-1}, u_{1,n-1}) + k u_{i,n-1}\big) \psi,$$

for all $\psi \in H^1(\Omega)$. Taking $\psi = u_{i,n}$ for each $i = 1, 2$, we get

$$\int_\Omega \big(|\nabla u_{i,n}|^2 + u_{i,n}^2\big) + k\int_{\partial\Omega} u_{i,n}^2 = \int_{\partial\Omega} \big(f_i(x, u_{1,n-1}, u_{1,n-1}) + ku_{i,n-1}\big)u_{i,n}. \tag{3.28}$$

Observe that for every $U = (u_1, u_2) \in J$, we have

$$\begin{aligned}
&\|f_i(x, u_1, u_2) + ku_i\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \\
&\qquad \le \|f_i(x, \underline{u}_1, \underline{u}_2) + k\underline{u}_i\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} + \|f_i(x, \overline{u}_1, \overline{u}_2) + k\overline{u}_i\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \le \widetilde{C}\,.
\end{aligned} \tag{3.29}$$

Using Hölder's inequality in (3.28), and the bound (3.29), we get

$$\begin{aligned}
\|u_{i,n}\|^2_{H^1(\Omega)} &\le \|u_{i,n}\|^2_{H^1(\Omega)} + k\|u_{i,n}\|^2_{L^2(\partial\Omega)} \\
&\le \|f_i(x, U_{n-1}) + ku_{i,n-1}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \cdot \|u_{i,n}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} \\
&\le \widetilde{C}\left(\|\overline{U}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} + \|\underline{U}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)}\right) \\
&\le C',
\end{aligned}$$

where C' is a constant independent of n. Therefore,

$$\|U_n\|_{(H^1(\Omega))^2} = \|u_{1,n}\|_{H^1(\Omega)} + \|u_{2,n}\|_{H^1(\Omega)} \le C. \tag{3.30}$$

By the reflexivity of $(H^1(\Omega))^2$ (see [44, p.15]), there exists a subsequence (relabel) U_n which converges weakly to U_* in $(H^1(\Omega))^2$. We can show in a similar way that the the sequence W_n converges weakly to U^*.

(iii) We will show that for each i, $f_i(x, U_n) + ku_{i,n}$ converges weakly to $f_i(x, U_*) + ku_{i,*}$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. From (3.30), we have that the sequence U_n is monotone increasing and bounded. Therefore, U_n converges pointwise to U_*, that is, $U_*(x) = \lim_{n\to\infty} U_n(x)$.

Since f_i is Carathéodory (i.e. continuous with respect to the second and third variables), it follows that

$$f_i(x, U_*(x)) + ku_{i,*}(x) = \lim_{n\to\infty}[f_i(x, U_n(x)) + ku_{i,n}(x)].$$

Note that by (3.29) $f_i(x, U_n(x)) + ku_{i,n}(x)$ is bounded in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$ for $i = 1, 2$. By Lebesgue Dominated Convergence Theorem, $f_i(x, U_n(x)) + ku_{i,n}(x)$ converges strongly to $f_i(x, U_*(x)) + ku_{i,*}$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$.

Hence, $f_i(x, U_n(x)) + ku_{i,n}(x)$ converges weakly to $f_i(x, U_*(x)) + ku_{i,*}$, that is, for all $\psi \in H^1(\Omega)$

$$\left|\int_{\partial\Omega}[f_i(x, U_n) + ku_{i,n}]\psi - \int_{\partial\Omega}[f_i(x, U_*(x)) + ku_{i,*}]\psi\right| \to 0 \tag{3.31}$$

as $n \to \infty$.

(iv) We will show that $U_* = (u_{1,*}, u_{2,*})$ and $U^* = (u_1^*, u_2^*)$ are weak solutions of (1.6). From (ii) we have that $U_* \in (H^1(\Omega))^2$. Then by the continuity of the trace operator and the embedding $L^{\frac{2(N-1)}{N-2}}(\partial\Omega) \hookrightarrow L^{\frac{2(N-1)}{N}}(\partial\Omega)$, we get that $u_{i,*} \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ for $i = 1, 2$. Using (iii) and (3.29), we have

$$\begin{aligned}
\|f_i(x, U_*)\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} &= \|f_i(x, U_*) + ku_{i,*} - ku_{i,*}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} \\
&\le \|f_i(x, U_*) + ku_{i,*}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} + \|ku_{i,*}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} \\
&\le C'',
\end{aligned}$$

where C'' is a positive constant. Hence, $f_i(x, U_*) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$.

From (ii), we know that $U_n = T(U_{n-1})$ satisfies

$$\int_\Omega (\nabla u_{i,n} \nabla \psi + u_{i,n}\psi) + k\int_{\partial\Omega} u_{i,n}\psi = \int_{\partial\Omega} \big(f_i(x, U_{(n-1)} + ku_{i,(n-1)}\big)\psi. \qquad (3.32)$$

Since $u_{i,n}$ converges weakly to $u_{i,*}$ in $H^1(\Omega)$ and strongly in $L^2(\partial\Omega)$, and $f_i(x, U_n)+ku_{i,n}$ converges weakly to $f_i(x, U_*) + ku_{i,*}$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$, then taking the limit of (3.32) as $n \to \infty$ and using (3.31), we get for any $\psi \in H^1(\Omega)$

$$\begin{aligned}\int_\Omega (\nabla u_{i,*} \nabla \psi + u_{i,*}\psi) + \int_{\partial\Omega} ku_{i,*}\psi &= \lim_{n\to\infty} \left(\int_\Omega (\nabla u_{i,n} \nabla \psi + u_{i,n}\psi) + \int_{\partial\Omega} ku_n\psi \right) \\ &= \lim_{n\to\infty} \left(\int_{\partial\Omega} (f_i(x, U_{(n-1)}) + ku_{i,(n-1)})\psi \right) \\ &= \int_{\partial\Omega} (f_i(x, U_*) + ku_{i,*})\psi \,.\end{aligned}$$

Hence,

$$\int_\Omega (\nabla u_{i,*} \nabla \psi + u_{i,*}\psi) = \int_{\partial\Omega} f_i(x, U_*)\psi \qquad \text{for all } \ \psi \in H^1(\Omega).$$

Thus U_* is a weak solution to (1.6).

(v) We will show that, U_* is the minimal weak solution to (1.6) in the interval $[\underline{U}, \overline{U}]$.

Let V be a weak solution to (1.6) such that $\underline{U} \le V \le \overline{U}$. Then V is a weak supersolution of (1.6). Repeating the above iteration procedure with $U_0 = \underline{U}$, we get $\underline{U} \le U_* \le V$. Thus U_* is a weak minimal solution.

In a similar fashion, we can construct the maximal weak solution U^* from the sequence $\{W_n\}$ with $W_0 = \overline{U}$. This complete the proof of Theorem 1.5

3.6 Proof of Theorem 1.6

The proof of Theorem 1.6 is based on application of Zorn's lemma where we will construct a nonempty set of subsolutions and show that the set has a maximal element which will turn out to be a solution to (1.6). We will then use a version of Kato's Inequality up to the boundary for systems to prove that the maximal element of the set of subsolution is in fact a maximal solution to (1.6). Similarly, we can show the existence of a minimal solution to (1.6) by applying Zorn's lemma on a set of supersolutions.

3.6.1 Existence of Maximal Solution

This proof involves several steps described below:

Step 1. *Existence of a uniformly bounded subsolution of* (1.6).

Let $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2) \in (H(\Omega))^2$ be a subsolution to (1.6) such that $\underline{U} \leq \widetilde{U} \leq \overline{U}$. We will show the existence of a subsolution $W = (w_1, w_2)$ to (1.6) such that $\widetilde{U} \leq W \leq \overline{U}$ and $\|W\|_{(H^1(\Omega))^2} \leq C$, where C is a constant depending on $\overline{u}_i, \underline{u}_i, \Omega, K_i$.

Consider the following equations

$$\begin{cases} -\Delta u_1 + u_1 = 0 & \text{in } \Omega; \\ \frac{\partial u_1}{\partial \eta} = f_1(x, u_1, \widetilde{u}_2) & \text{on } \partial\Omega, \end{cases} \tag{3.33}$$

and

$$\begin{cases} -\Delta u_2 + u_2 = 0 & \text{in } \Omega; \\ \frac{\partial u_2}{\partial \eta} = f_2(x, \widetilde{u}_1, u_2) & \text{on } \partial\Omega. \end{cases} \tag{3.34}$$

Setting $f_1(x, s_1) := f_1(x, s_1, \widetilde{u}_2(x))$, it follows from assumption (**C2**) that $|f_1(x, s_1)| = |f_1(x, s_1, \widetilde{u}_2(x))| \leq K_1(x) \in L^r(\partial\Omega)$ whenever $\widetilde{u}_1 \leq s_1 \leq \overline{u}_1$ and $\widetilde{u}_2 \leq \overline{u}_2$, and $r > \frac{2(N-1)}{N}$. Then, applying Theorem 1.2 to (3.33), we get a solution $w_1 \in H^1(\Omega)$ such that $\widetilde{u}_1 \leq w_1 \leq \overline{u}_1$. Similarly, taking $f_2(x, s_2) = f_2(x, \widetilde{u}_1, s_2)$ and applying Theorem

1.2 to (3.34), we get a solution w_2 such that $\widetilde{u}_2 \leq w_2 \leq \overline{u}_2$. Since f_i is quasimonotone increasing, it follows that $f_1(x, w_1, \widetilde{u}_2) \leq f_1(x, w_1, w_2)$ and $f_2(x, \widetilde{u}_1, w_2) \leq f_2(x, w_1, w_2)$. Hence,

$$\int_\Omega (\nabla w_i \nabla \psi + w_i \psi) \leq \int_{\partial\Omega} f_i(x, w_1, w_2)\psi \qquad \text{for all} \;\; \psi \in H^1(\Omega), \; \text{for } i = 1, 2.$$

Using (**C2**) and the fact that $\underline{u}_i \leq \widetilde{u}_i \leq w_i \leq \overline{u}_i$, we have that $f_i(x, w_1, w_2) \in L^r(\partial\Omega)$ with $r > \frac{2(N-1)}{N}$. By the continuity of $L^r(\partial\Omega)$ into $L^{\frac{2(N-1)}{N}}(\partial\Omega)$, it follows that $f_i(x, w_1, w_2) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ for $i = 1, 2$. Hence, $W = (w_1, w_2)$ is a subsolution of (1.6).

Furthermore, $||W||_{(H_1(\overline{\Omega}))^2} \leq M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$, where $M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$ is a constant. Indeed,

$$\begin{aligned} ||w_1||_{H^1(\Omega)} = \int_\Omega |\nabla w_1|^2 + w_1^2 &= \int_{\partial\Omega} f_1(x, w_1, \widetilde{u}_2) w_1 \\ &\leq \int_{\partial\Omega} K_1(x) w_1 \leq \int_{\partial\Omega} K_1(x)\overline{u}_1 \\ &\leq ||K_1||_{L^r(\partial\Omega)}.||\overline{u}_1||_{L^{r'}(\partial\Omega)}, \quad \text{(Holder's Inequality)} \\ &\leq C_1, \qquad \text{((H2) and the continuity of trace operator)}, \end{aligned}$$

where r' is the conjugate of r and $r' < \frac{2(N-1)}{N-2}$. Similarly $||w_2||_{H^1(\Omega)} \leq C_2$. Hence, $||W||_{(H^1(\Omega))^2} \leq M$, where M is constant independent of W.

Step 2. *Zorn's lemma.*

Consider the set $\mathcal{A}$ consisting of $(w_1, w_2) \in (H^1(\Omega))^2$ such that there exists a subsolution $(\widetilde{u_1}, \widetilde{u_2})$ of (1.6) satisfying

$$\begin{cases} (\underline{u}_1, \underline{u}_2) \leq (\widetilde{u}_1, \widetilde{u}_2) \leq (w_1, w_2) \leq (\overline{u}_1, \overline{u}_2) \\ w_1, w_2 \text{ are solutions respectively of (3.33), (3.34) for the pair } \widetilde{u}_1, \widetilde{u}_2. \end{cases} \tag{3.35}$$

We will first check the hypothesis of Zorn's lemma and then derive the existence of a maximal element of $\mathscr{A}$. From **Step 1** we observe that $\mathscr{A} \neq \emptyset$. Let $Y = \{W_n = (w_{1,n}, w_{2,n})\}_{n\geq 1}$ be a chain in $(\mathscr{A}, \leq)$ where countability of the indexing set is guaranteed by the separability of the product space $(H^1(\Omega))^2$. Hence, $Y = \{W_n = (w_{1,n}, w_{2,n})\}_{n\geq 1}$ is basically an increasing sequence in $\mathscr{A}$. Now we will show that Y has an upper bound in $\mathscr{A}$. Since each element of Y, $W_n = (w_{1,n}, w_{2,n})$ belongs to $\mathscr{A}$, there exists a subsolution $(\widetilde{u}_{1,n}, \widetilde{u}_{2,n})$ of (1.6) such that $(\underline{u}_1, \underline{u}_2) \leq (\widetilde{u}_{1,n}, \widetilde{u}_{2,n}) \leq (w_{1,n}, w_{2,n}) \leq (\overline{u}_1, \overline{u}_2)$ and $w_{1,n}, w_{2,n}$ are solutions respectively of (3.33), (3.34) for the pair $\widetilde{u}_{1,n}, \widetilde{u}_{2,n}$. From **Step 1**, it follows that $||W_n||_{(H^1(\Omega))^2} \leq M$ where $M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$ is independent on n. By the reflexivity of $(H^1(\Omega))^2$, there is a subsequence (relabeled) W_n which converges weakly to $W_* = (w_{1,*}, w_{2,*})$. Since the sequence $\{w_{1,n}\}$ is monotonically increasing and bounded above, $\{w_{1,n}\}$ converges pointwise to $w_{1,*}$. Similarly, $\{w_{2,n}\}$ converges pointwise to $w_{2,*}$. Then, by the continuity of $f_i(x, \cdot, \cdot)$ it follows that

$$f_i(x, w_{1,n}(x), w_{2,n}(x)) \to f_i(x, w_{1,*}(x), w_{2,*}(x));\ i = 1, 2$$

as $n \to \infty$. Using the condition (**C2**), $|f_1(x, w_{1,n}, w_{2,n})| \leq K_1(x)$, where $K_1(x) \in L^r(\partial\Omega)$ with $r > \frac{2(N-1)}{N-2} > 1$. Therefore, by the Dominated Convergence Theorem, we have

$$||f_1(x, w_{1,n}(x), w_{2,n}(x)) - f_1(x, w_{1,*}(x), w_{2,*}(x))||_{L^r(\partial\Omega)} \longrightarrow 0 \text{ as } n \to \infty. \qquad (3.36)$$

Using the Holder's inequality and (3.36), we have that for any test function $\psi \in H^1(\Omega)$

$$\begin{aligned}
&\left| \int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi - \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x)))\psi \right| \\
&\leq \int_{\partial\Omega} |f_1(x, w_{1,n}(x), w_{2,n}(x)) - f_1(x, w_{1,*}(x), w_{2,*}(x))|.|\psi| \\
&\leq ||f_1(x, w_{1,n}(x), w_{2,n}(x)) - f_1(x, w_{1,*}(x), w_{2,*}(x))||_{L^r(\partial\Omega)}.||\psi||_{L^{r'}(\partial\Omega)} \\
&\longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi = \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x))\psi \quad \text{for all } \psi \in H^1(\Omega). \tag{3.37}$$

Utilizing (3.37) and the quasimonotonicity condition of f_1, we have

$$\begin{aligned}
\int_\Omega \nabla w_{1,*}\nabla\psi + w_{1,*}\psi &= \lim_{n\to\infty}\int_\Omega \nabla w_{1,n}\nabla\psi + w_{1,n}\psi \\
&= \lim_{n\to\infty}\int_{\partial\Omega} f_1(x, w_{1,n}(x), \widetilde{u}_{2,n}(x))\psi \\
&\le \lim_{n\to\infty}\int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi \\
&= \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x))\psi\,.
\end{aligned}$$

Similarly,

$$\int_\Omega \nabla w_{2,*}\nabla\psi + w_{2,*}\psi \le \int_{\partial\Omega} f_2(x, w_{1,*}(x), w_{2,*}(x))\psi\,.$$

Hence, $W_* = (w_{1,*}, w_{2,*})$ is a subsolution of (1.6).

We then apply **Step 1** taking $\widetilde{U} = W_* = (w_{1,*}, w_{2,*})$, there exists a subsolution $V = (v_1, v_2)$ of (1.6) such that $(\underline{u}_1, \underline{u}_2) \le (w_{1,*}, w_{2,*}) \le (v_1, v_2) \le (\overline{u}_1, \overline{u}_2)$, and v_1, v_2 are solutions respectively of (3.33), (3.34) for the pair $w_{1,*}, w_{2,*}$. This implies $V \in \mathscr{A}$ and is an upper bound of Y. By Proposion 3.2 (Zorn's Lemma), $\mathscr{A}$ has a maximal element $Z = (z_1, z_2) \in \mathscr{A}$. Notice that $Z = (z_1, z_2)$ is a subsolution of (1.6). Indeed, $Z = (z_1, z_2) \in \mathscr{A}$ implies that there exists a subsolution to (1.6), $(\widetilde{z}_1, \widetilde{z}_2)$, such that

$$\begin{cases} (\underline{u}_1, \underline{u}_2) \le (\widetilde{z}_1, \widetilde{z}_2) \le (z_1, z_2) \le (\overline{u}_1, \overline{u}_2) \\ z_1, z_2 \text{ are solutions respectively of (3.33), (3.34) for the pair } \widetilde{z}_1, \widetilde{z}_2. \end{cases} \tag{3.38}$$

By the quasimonotonicity of f_1 and f_2, it follows that

$$\int_\Omega \nabla z_1 \nabla \psi + z_1 \psi = \int_{\partial\Omega} f_1(x, z_1, \widetilde{z}_2)\psi \leq \int_{\partial\Omega} f_1(x, z_1, z_2),$$

and

$$\int_\Omega \nabla z_2 \nabla \psi + z_2 \psi = \int_{\partial\Omega} f_2(x, \widetilde{z}_1, z_2)\psi \leq \int_{\partial\Omega} f_2(x, z_1, z_2).$$

Therefore, $Z = (z_1, z_2)$ is a subsolution of (1.6).

Step 3: $Z = (z_1, z_2)$ *is a solution of* (1.6)

We now apply **Step 1**, by taking $\widetilde{U} = Z$, then there exists $Z^* = (z_1^*, z_2^*)$, a subsolution of (1.6) such that $z_i \leq z_i^* \leq \overline{u}_i$, $i = 1, 2$, and (z_1^*, z_2^*) is solution to (3.33) and (3.34) respectively, for the pair (z_1, z_2). From the definition of the set $\mathcal{A}$, we have that $Z^* = (z_1^*, z_2^*) \in \mathcal{A}$. Using the fact that Z is a maximal element of $\mathcal{A}$, we get that $Z^* \leq Z$. Hence, $Z = Z^*$, and so, for every $\psi \in H^1(\Omega)$

$$\int_\Omega \nabla z_1 \nabla \psi + z_1 \psi = \int_\Omega \nabla z_1^* \nabla \psi + z_1^* \psi = \int_{\partial\Omega} f_1(x, z_1^*, z_2)\psi = \int_{\partial\Omega} f_1(x, z_1, z_2)\psi,$$

and

$$\int_\Omega \nabla z_2 \nabla \psi + z_2 \psi = \int_\Omega \nabla z_2^* \nabla \psi + z_2^* \psi = \int_{\partial\Omega} f_2(x, z_1, z_2^*)\psi = \int_{\partial\Omega} f_2(x, z_1, z_2)\psi$$

Therefore, $Z = (z_1, z_2)$ is a solution of (1.6).

Step 4: $Z = (z_1, z_2)$ *is a maximal solution of* (1.6).

Let $U = (u_1, u_2)$ be any solution of (1.6) such that $\underline{U} \leq U \leq \overline{U}$ and hence a subsolution of (1.6). Clearly, $U = (u_1, u_2) \in \mathcal{A}$ as (u_1, u_2) is a solution of (3.33) and (3.34) for the pair (u_1, u_2). Let $v_1 = \max\{u_1, z_1\}$ and $v_2 = \max\{u_2, z_2\}$. By Section 3.7, $V = (v_1, v_2)$ is a subsolution of (1.6). Using **Step 1**, there exists $\widehat{Z} = (\widehat{z}_1, \widehat{z}_2) \in \mathcal{A}$ such that

$\underline{U} \leq V \leq \widehat{Z} \leq \overline{U}$. Since $V = (\max\{u_1, z_1\}, \max\{u_2, z_2\})$, then Kato's inequality implies $Z \leq V \leq \widehat{Z}$ (see Section 3.7 for details). As Z is a maximal element of $\mathscr{A}$, $V \leq Z$. Consequently, $Z = V$, and hence $U \leq Z$. Thus, $Z = (z_1, z_2)$ is the unique maximal solution of (1.6).

3.6.2 Existence of Minimal Solution

This proof consists of several steps similar to the proof in Subsection 3.6.1.

Step I. *Existence of a uniformly bounded supersolution of* (1.6).

Let $\widehat{U} = (\widehat{u}_1, \widehat{u}_2) \in (H(\Omega))^2$ be a supersolution to (1.6) such that $\underline{U} \leq \widehat{U} \leq \overline{U}$. We will show the existence of a supersolution $W = (w_1, w_2)$ to (1.6) such that $\underline{U} \leq W \leq \widehat{U}$ and $\|W\|_{(H^1(\Omega))^2} \leq C$, where C is a constant depending on $\overline{u}_i, \underline{u}_i, \Omega, K_i$.

Consider the following equations

$$\begin{cases} -\Delta u_1 + u_1 = 0 & \text{in } \Omega; \\ \frac{\partial u_1}{\partial \eta} = f_1(x, u_1, \widehat{u}_2) & \text{on } \partial\Omega, \end{cases} \tag{3.39}$$

and

$$\begin{cases} -\Delta u_2 + u_2 = 0 & \text{in } \Omega; \\ \frac{\partial u_2}{\partial \eta} = f_2(x, \widehat{u}_1, u_2) & \text{on } \partial\Omega. \end{cases} \tag{3.40}$$

Setting $f_1(x, s_1) := f_1(x, s_1, \widehat{u}_2(x))$, it follows from assumption (**C2**) that $|f_1(x, s_1)| = |f_1(x, s_1, \widehat{u_2}(x))| \leq K_1(x) \in L^r(\partial\Omega)$ whenever $\underline{u}_1 \leq s_1 \leq \widehat{u}_1 \leq \overline{u}_1$ and $\widehat{u}_2 \leq \overline{u}_2$, and $r > \frac{2(N-1)}{N}$. Then, applying Theorem 1.2 to (3.39), we get a weak solution $w_1 \in H^1(\Omega)$ such that $\underline{u}_1 \leq w_1 \leq \widehat{u}_1$. Similarly, taking $f_2(x, s_2) = f_2(x, \widehat{u}_1, s_2)$ and then applying Theorem 1.2 to (3.40), we get a weak solution w_2 such that $\underline{u_2} \leq w_2 \leq \widehat{u}_2$. Since f_i is quasimonotone increasing, it follows that $f_1(x, w_1, \widehat{u}_2) \geq f_1(x, w_1, w_2)$ and

$f_2(x, \widehat{u_1}, w_2) \geq f_2(x, w_1, w_2)$. Hence,

$$\int_\Omega (\nabla w_i \nabla \psi + w_i \psi) \geq \int_{\partial\Omega} f_i(x, w_1, w_2)\psi \qquad \text{for all } \ \psi \in H^1(\Omega), \text{ for } i = 1, 2.$$

Using (**C2**) and the fact that $\underline{u}_i \leq w_i \leq \widehat{u}_i \leq \overline{u}_i$, we have that $f_i(x, w_1, w_2) \in L^r(\partial\Omega)$ with $r > \frac{2(N-1)}{N}$. By the continuity of $L^r(\partial\Omega)$ into $L^{\frac{2(N-1)}{N}}(\partial\Omega)$, it follows that $f_i(x, w_1, w_2) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ for $i = 1, 2$. Hence, $W = (w_1, w_2)$ is a supersolution of (1.6).

Using similar arguments as in **Step 1** of the proof of existence of a maximal solution, we have that $||W||_{(H_1(\Omega))^2} \leq M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$, where $M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$ is a constant independent of W.

Step II. *Zorn's lemma.*

Consider the set $\mathcal{B}$ consisting of $(w_1, w_2) \in (H^1(\Omega))^2$ such that there exists a supersolution $(\widehat{u_1}, \widehat{u_2})$ of (1.6) satisfying

$$\begin{cases} (\underline{u}_1, \underline{u}_2) \leq (w_1, w_2) \leq (\widehat{u}_1, \widehat{u}_2) \leq (\overline{u}_1, \overline{u}_2) \\ w_1, w_2 \text{ are solutions respectively of (3.39), (3.40) for the pair } \widehat{u}_1, \widehat{u}_2. \end{cases} \tag{3.41}$$

We will first check the hypothesis of Zorn's lemma, and then derive the existence of a minimal element of $\mathcal{B}$. From **Step I** we observe that $\mathcal{B} \neq \emptyset$. Let $X = \{W_n = (w_{1,n}, w_{2,n})\}_{n\geq 1}$ be a chain in $(\mathcal{A}, \geq)$ where countability of the indexing set is guaranteed by the separability of the product space $(H^1(\Omega))^2$. Hence, $X = \{W_n = (w_{1,n}, w_{2,n})\}_{n\geq 1}$ is basically a decreasing sequence in $\mathcal{A}$. Now we will show that X has an lower bound in $\mathcal{B}$. Since each element of X, $W_n = (w_{1,n}, w_{2,n})$ is in $\mathcal{B}$, there exists a supersolution $(\widehat{u}_{1,n}, \widehat{u}_{2,n})$ of (1.6) such that $(\underline{u}_1, \underline{u}_2) \leq (w_{1,n}, w_{2,n}) \leq (\widehat{u}_{1,n}, \widehat{u}_{2,n}) \leq (\overline{u}_1, \overline{u}_2))$ and $w_{1,n}$ and $w_{2,n}$ are solutions of (3.39), (3.40) for the pair $(\widehat{u}_{1,n}, \widehat{u}_{2,n})$, respectively.

From **Step I**, it follows that $||W_n||_{(H^1(\Omega))^2} \leq M$ where $M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$ is independent on n. By the reflexivity of $(H^1(\Omega))^2$, there is a subsequence (relabeled) W_n which converges weakly to $W_* = (w_{1,*}, w_{2,*})$. Since the sequence $\{w_{1,n}\}$ is monotonically decreasing and bounded below by $\underline{u}$, then $\{w_{1,n}\}$ converges pointwise to $w_{1,*}$. Similarly, $\{w_{2,n}\}$ converges pointwise to $w_{2,*}$. Then, by the continuity of $f_i(x, \cdot, \cdot)$ it follows that

$$f_i(x, w_{1,n}(x), w_{2,n}(x)) \to f_i(x, w_{1,*}(x), w_{2,*}(x))$$

as $n \to \infty$, for $i = 1, 2$. Using the condition **(C2)**, $|f_1(x, w_{1,n}, w_{2,n})| \leq K_1(x)$, where $K_1(x) \in L^r(\partial\Omega)$ with $r > \frac{2(N-1)}{N-2} > 1$. As in **Step 2**, by employing the Dominated Convergence Theorem followed by the Holder's inequality we get

$$\lim_{n\to\infty} \int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi = \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x))\psi \quad \text{for all } \psi \in H^1(\Omega). \tag{3.42}$$

Utilizing (3.42) and the quasimonotonicity condition of f_1 we have,

$$\begin{aligned}
\int_\Omega \nabla w_{1,*} \nabla \psi + w_{1,*}\psi &= \lim_{n\to\infty} \int_\Omega \nabla w_{1,n} \nabla \psi + w_{1,n}\psi \\
&= \lim_{n\to\infty} \int_{\partial\Omega} f_1(x, w_{1,n}(x), \widehat{u}_{2,n}(x))\psi \\
&\geq \lim_{n\to\infty} \int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi \\
&= \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x))\psi\,.
\end{aligned}$$

Similarly,

$$\int_\Omega \nabla w_{2,*} \nabla \psi + w_{2,*}\psi \geq \int_{\partial\Omega} f_2(x, w_{1,*}(x), w_{2,*}(x))\psi\,.$$

Hence, $W_* = (w_{1,*}, w_{2,*})$ is a supersolution of (1.6).

We then apply Step I by taking $\widehat{U} = W_* = (w_{1,*}, w_{2,*})$, there exists a supersolution $V = (v_1, v_2)$ of (1.6) such that $(\underline{u}_1, \underline{u}_2) \leq (v_1, v_2) \leq (w_{1,*}, w_{2,*}) \leq (\overline{u}_1, \overline{u}_2)$, and v_1, v_2

are solutions respectively of (3.39), (3.40) for the pair $w_{1,*}, w_{2,*}$. This implies $V \in \mathcal{B}$ and is a lower bound of X. By Proposition 3.2 (Zorn's Lemma), $\mathcal{B}$ has a minimal element $Z = (z_1, z_2) \in \mathcal{B}$. Notice that $Z = (z_1, z_2)$ is a supersolution of (1.6). Indeed, $Z = (z_1, z_2) \in \mathcal{B}$ implies that there exists a supersolution to (1.6), $(\widehat{z}_1, \widehat{z}_2)$, such that

$$\begin{cases} (\underline{u}_1, \underline{u}_2) \leq (z_1, z_2) \leq (\widehat{z}_1, \widehat{z}_2) \leq (\overline{u}_1, \overline{u}_2) \\ z_1, z_2 \text{ are solutions respectively of } (3.39), (3.40) \text{ for the pair } \widehat{z}_1, \widehat{z}_2. \end{cases} \tag{3.43}$$

By the quasimonotonicity of f_1 and f_2, it follows that

$$\int_\Omega \nabla z_1 \nabla \psi + z_1 \psi = \int_{\partial\Omega} f_1(x, z_1, \widehat{z}_2)\psi \geq \int_{\partial\Omega} f_1(x, z_1, z_2),$$

and

$$\int_\Omega \nabla z_2 \nabla \psi + z_2 \psi = \int_{\partial\Omega} f_2(x, \widehat{z}_1, z_2)\psi \geq \int_{\partial\Omega} f_2(x, z_1, z_2).$$

Therefore, $Z = (z_1, z_2)$ is a supersolution of (1.6).

Step III: $Z = (z_1, z_2)$ *is a solution of* (1.6)

We now apply Step I, by taking $\widehat{U} = Z$, then there exists $Z^* = (z_1^*, z_2^*)$, a supersolution of (1.6) such that $\underline{u}_i \leq z_i^* \leq z_i \leq \overline{u}_i$, $i = 1, 2$, and (z_1^*, z_2^*) is solution to (3.39) and (3.40) respectively, for the pair (z_1, z_2). From the definition of the set $\mathcal{B}$, we have that $Z^* = (z_1^*, z_2^*) \in \mathcal{B}$. Using the fact that Z is a minimal element of $\mathcal{A}$, we get that $Z \leq Z^*$. Hence, $Z = Z^*$, and so, for every $\psi \in H^1(\Omega)$

$$\int_\Omega \nabla z_1 \nabla \psi + z_1 \psi = \int_\Omega \nabla z_1^* \nabla \psi + z_1^* \psi = \int_{\partial\Omega} f_1(x, z_1^*, z_2)\psi = \int_{\partial\Omega} f_1(x, z_1, z_2)\psi,$$

and

$$\int_\Omega \nabla z_2 \nabla \psi + z_2 \psi = \int_\Omega \nabla z_2^* \nabla \psi + z_2^* \psi = \int_{\partial\Omega} f_2(x, z_1, z_2^*)\psi = \int_{\partial\Omega} f_2(x, z_1, z_2)\psi$$

Therefore, $Z = (z_1, z_2)$ is a solution of (1.6).

Step IV: $Z = (z_1, z_2)$ *is a minimal solution of* (1.6).
Let $U = (u_1, u_2)$ be any solution of (1.6) such that $\underline{U} \leq U \leq \overline{U}$ and hence a supersolution of (1.6). Clearly, $U = (u_1, u_2) \in \mathcal{B}$ as (u_1, u_2) is a solution of (3.39) and (3.40) for the pair (u_1, u_2). Let $v_1 = \min\{u_1, z_1\}$ and $v_2 = \min\{u_2, z_2\}$. By applying kato's Inequality for system described in Section 3.8, $V = (v_1, v_2)$ is a supersolution of (1.6). Using Step I, there exists $\widetilde{Z} = (\widetilde{z}_1, \widetilde{z}_2) \in \mathcal{B}$ such that $\underline{U} \leq \widetilde{Z} \leq V \leq \overline{U}$ which implies $\widetilde{Z} \leq V = \min\{U, Z\} \leq Z$. Since Z is a minimal element of $\mathcal{B}$, $Z \leq V$. Consequently, $Z = V$ and hence $Z \leq U$. Thus, $Z = (z_1, z_2)$ is the unique minimal solution of (1.6). This completes the proof of the theorem.

Next we will prove two versions of Kato's Inequality which we used in **Step 4** of Section 3.6.1 and in **Step IV** of Section 3.6.2.

3.7 Version of Kato's Inequality for Systems Part I

Let $\mathcal{A}$ be a set consisting of $(w_1, w_2) \in (H^1(\Omega))^2$ such that there exists a subsolution $(\widetilde{u}_1, \widetilde{u}_2)$ of (1.6) satisfying

$$\begin{cases} (\underline{u}_1, \underline{u}_2) \leq (\widetilde{u}_1, \widetilde{u}_2) \leq (w_1, w_2) \leq (\overline{u}_1, \overline{u}_2) \\ w_1, w_2 \text{ are solutions respectively of } (3.33), (3.34) \text{ for the pair } \widetilde{u}_1, \widetilde{u}_2. \end{cases}$$

We will prove a version of Kato's inequality up to the boundary for systems; that is, the maximum of two subsolutions in $\mathcal{A}$ is a subsolution of (1.6), in a sense that $(\max\{\alpha_1, \beta_1\}, \max\{\alpha_2, \beta_2\})$ is a subsolution of (1.6), where $(\alpha_1, \alpha_2) \in \mathcal{A}$ and

$(\beta_1, \beta_2) \in \mathscr{A}$ are any two subsolutions of (1.6). Since (α_1, α_2) and (β_1, β_2) belong to $\mathscr{A}$, there exist $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ and $(\widetilde{\beta}_1, \widetilde{\beta}_2)$ such that $\underline{u}_i \le \widetilde{\alpha}_i \le \alpha_i \le \overline{u}_i$ $i = 1, 2$ and $\underline{u}_i \le \widetilde{\beta}_i \le \beta_i \le \overline{u}_i$ $i = 1, 2$, which satisfy the following:

$$\begin{cases} -\Delta\alpha_1 + \alpha_1 &= 0 \text{ in } \Omega \\ \frac{\partial \alpha_1}{\partial \eta} &= f_1(x, \alpha_1, \widetilde{\alpha}_2) \text{ on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta\alpha_2 + \alpha_2 &= 0 \text{ in } \Omega \\ \frac{\partial \alpha_2}{\partial \eta} &= f_2(x, \widetilde{\alpha}_1, \alpha_2) \text{ on } \partial\Omega \end{cases}$$

$$\begin{cases} -\Delta\beta_1 + \beta_1 &= 0 \text{ in } \Omega \\ \frac{\partial \beta_1}{\partial \eta} &= f_1(x, \beta_1, \widetilde{\beta}_2) \text{ on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta\beta_2 + \beta_2 &= 0 \text{ in } \Omega \\ \frac{\partial \beta_2}{\partial \eta} &= f_2(x, \widetilde{\beta}_1, \beta_2) \text{ on } \partial\Omega \end{cases}$$

Now, define $\gamma_1 := \max\{\alpha_1, \beta_1\}$ and $\gamma_2 := \max\{\alpha_2, \beta_2\}$. Observe that, the quasimonotinicity of f_i leads to the following inequalities

$$\begin{cases} f_1(x, \alpha_1, \widetilde{\alpha}_2) &\le f_1(x, \alpha_1, \gamma_2), \\ f_1(x, \beta_1, \widetilde{\beta}_2) &\le f_1(x, \beta_1, \gamma_2) \end{cases} \qquad \begin{cases} f_2(x, \widetilde{\alpha}_1, \alpha_2) &\le f_2(x, \gamma_1, \alpha_2) \\ f_2(x, \widetilde{\beta}_1, \beta_2) &\le f_2(x, \gamma_1, \beta_2) \end{cases}$$

Let $g_1(x) := \begin{cases} f_1(x, \alpha_1, \gamma_2) & \text{if } \alpha_1(x) > \beta_1(x) \\ f_1(x, \beta_1, \gamma_2) & \text{if } \alpha_1(x) \le \beta_1(x) \end{cases}$ a.e. $x \in \partial\Omega$,

and

$$g_2(x) := \begin{cases} f_2(x, \gamma_1, \alpha_2) & \text{if } \alpha_2(x) \le \beta_2(x) \\ f_1(x, \gamma_1, \beta_2) & \text{if } \alpha_2(x) > \beta_2(x) \end{cases} \text{ a.e. } x \in \partial\Omega.$$

Notice that α_1 and β_1 are subsolutions of the following:

$$\begin{cases} -\Delta v + v &= 0 \text{ in } \Omega \\ \frac{\partial v}{\partial \eta} &= g_1(x) \text{ on } \partial\Omega; \end{cases} \tag{3.44}$$

and α_2 and β_2 are subsolutions of the following:

$$\begin{cases} -\Delta v + v &= 0 \text{ in } \Omega \\ \frac{\partial v}{\partial \eta} &= g_2(x) \text{ on } \partial\Omega; \end{cases} \tag{3.45}$$

Then by applying Kato's inequality up to the boundary for single equation proved in Section 3.1, along with (3.44) and (3.45), we have that (γ_1, γ_2) is a subsolution of (1.6). This completes the proof.

Remark 3.2. Any solution of (1.6) also belongs to $\mathscr{A}$. Therefore, if (u_1, u_2) and (v_1, v_2) are two solutions of (1.6) then $(\max\{u_1, v_1\}, \max\{u_2, v_2\})$ is a subsolution of (1.6).

3.8 Version of Kato's Inequality for Systems Part II

Let $\mathscr{B}$ consisting of $(w_1, w_2) \in (H^1(\Omega))^2$ such that there exists a supersolution $(\widehat{u}_1, \widehat{u}_2)$ of (1.6) satisfying

$$\begin{cases} (\underline{u}_1, \underline{u}_2) \leq (w_1, w_2) \leq (\widehat{u}_1, \widehat{u}_2) \leq (\overline{u}_1, \overline{u}_2) \\ w_1, w_2 \text{ are solutions respectively of } (3.39), (3.40) \text{ for the pair } \widehat{u}_1, \widehat{u}_2. \end{cases} \tag{3.46}$$

We will prove a version of Kato's inequality up to the boundary; that is, the minimum of two supersolutions in $\mathscr{B}$ is a supersolution of (1.6), in a sense that $(\min\{\alpha_1, \beta_1\}, \min\{\alpha_2, \beta_2\})$ is a supersolution of (1.6), where $(\alpha_1, \alpha_2) \in \mathscr{B}$ and $(\beta_1, \beta_2) \in \mathscr{B}$ are any two supersolutions of (1.6). Since (α_1, α_2) and (β_1, β_2) belong to $\mathscr{B}$, there exist $(\widehat{\alpha}_1, \widehat{\alpha}_2)$ and $(\widehat{\beta}_1, \widehat{\beta}_2)$ such that $\underline{u}_i \leq \alpha_i \leq \widehat{\alpha}_i \leq \overline{u}_i$ $i = 1, 2$ and $\underline{u}_i \leq \beta_i \leq \widehat{\beta}_i \leq \overline{u}_i$ $i = 1, 2$, which satisfy the following:

$$\begin{cases} -\Delta\alpha_1 + \alpha_1 &= 0 \text{ in } \Omega \\ \frac{\partial\alpha_1}{\partial\eta} &= f_1(x, \alpha_1, \widehat{\alpha}_2) \text{ on } \partial\Omega \end{cases} \quad \begin{cases} -\Delta\alpha_2 + \alpha_2 &= 0 \text{ in } \Omega \\ \frac{\partial\alpha_2}{\partial\eta} &= f_2(x, \widehat{\alpha}_1, \alpha_2) \text{ on } \partial\Omega \end{cases}$$

$$\begin{cases} -\Delta\beta_1 + \beta_1 &= 0 \text{ in } \Omega \\ \frac{\partial\beta_1}{\partial\eta} &= f_1(x, \beta_1, \widehat{\beta}_2) \text{ on } \partial\Omega \end{cases} \quad \begin{cases} -\Delta\beta_2 + \beta_2 &= 0 \text{ in } \Omega \\ \frac{\partial\beta_2}{\partial\eta} &= f_2(x, \widehat{\beta}_1, \beta_2) \text{ on } \partial\Omega \end{cases}$$

Now, define $\gamma_1 := \min\{\alpha_1, \beta_1\}$ and $\gamma_2 := \min\{\alpha_2, \beta_2\}$. Observe that the quasimonotinicity of f_i leads to the following inequalities

$$\begin{cases} f_1(x, \alpha_1, \widehat{\alpha}_2) &\geq f_1(x, \alpha_1, \gamma_2), \\ f_1(x, \beta_1, \widehat{\beta}_2) &\geq f_1(x, \beta_1, \gamma_2) \end{cases} \quad \begin{cases} f_2(x, \widehat{\alpha}_1, \alpha_2) &\geq f_2(x, \gamma_1, \alpha_2) \\ f_2(x, \widehat{\beta}_1, \beta_2) &\geq f_2(x, \gamma_1, \beta_2) \end{cases}$$

Let $g_1(x) := \begin{cases} f_1(x, \alpha_1, \gamma_2) & \text{if } \alpha_1(x) > \beta_1(x) \\ f_1(x, \beta_1, \gamma_2) & \text{if } \alpha_1(x) \leq \beta_1(x), \end{cases}$ a.e. $x \in \partial\Omega$.

and

$$g_2(x) := \begin{cases} f_2(x, \gamma_1, \alpha_2) & \text{if } \alpha_2(x) \leq \beta_2(x) \\ f_1(x, \gamma_1, \beta_2) & \text{if } \alpha_2(x) > \beta_2(x). \end{cases} \text{ a.e. } x \in \partial\Omega.$$

Notice that α_1 and β_1 are supersolutions of the following:

$$\begin{cases} -\Delta v + v &= 0 \text{ in } \Omega \\ \frac{\partial v}{\partial\eta} &= g_1(x) \text{ on } \partial\Omega, \end{cases} \tag{3.47}$$

and α_2 and β_2 are supersolutions of the following:

$$\begin{cases} -\Delta v + v &= 0 \text{ in } \Omega \\ \frac{\partial v}{\partial \eta} &= g_2(x) \text{ on } \partial\Omega. \end{cases} \tag{3.48}$$

Then by applying the corollary of Kato's inequality up to the boundary for single equation (see Corollary 3.4), along with (3.47) and (3.48) we have that (γ_1, γ_2) is a supersolution of (1.6). This completes the proof.

Remark 3.3. Any solution of (1.6) is also belongs to $\mathscr{B}$. Hence, if (u_1, u_2) and (v_1, v_2) are two solutions of (1.6) then $(\min\{u_1, v_1\}, \min\{u_2, v_2\})$ is a supersolution of (1.6).

3.9 Proof of Theorem 1.7

We will apply Theorem 1.5 to establish the existence result to the problem involving the sublinear quasimonotone nonlinearities. We will construct an ordered pair of weak sub and supersolution to the system (1.7).

Step 1 *Construction of a subsolution*

We will show that $\underline{U} = (\underline{u}_1, \underline{u}_2) = (a\epsilon\varphi_1, b\epsilon\varphi_1)$ is a subsolution to (1.7), where $a = \sqrt{f_1'(0)}$ and $b = \sqrt{f_2'(0)}$. Indeed, let $\zeta(s) = \mu_1 \frac{a}{b} s - \lambda f_1(s)$. Observe that $\zeta(0) = 0$ as $f_1(0) = 0$. Taking $\lambda > \frac{\mu_1}{\sqrt{f_1'(0) f_2'(0)}}$, we have

$$\zeta'(0) = \mu_1 \frac{a}{b} - \lambda f_1'(0) < \lambda\sqrt{f_1'(0) f_2'(0)}\, \frac{a}{b} - \lambda f_1'(0) = \lambda f_1'(0) - \lambda f_1'(0) = 0.$$

Since $\zeta(0) = 0$, $\zeta'(0) < 0$, ζ is continuous, then there exists $\delta > 0$ such that $\zeta(s) < 0$ for every $0 < s < \delta$. Now, choosing ϵ small enough such that $0 < \epsilon b\varphi_1 < \delta$, we have that $\zeta(\epsilon b\varphi_1) < 0$. Therefore,

$$\int_\Omega \nabla \underline{u}_1 \nabla \psi + \underline{u}_1 \psi = \int_{\partial\Omega} \mu_1 \frac{a}{b} \epsilon b \varphi_1 \psi \le \lambda \int_{\partial\Omega} f_1(\epsilon b\varphi_1)\psi.$$

Similarly one can show,

$$\int_{\Omega} \nabla \underline{u}_2 \nabla \psi + \underline{u}_2 \psi \leq \lambda \int_{\partial\Omega} f_2(\epsilon a \varphi_1)\psi.$$

Hence, $\underline{U} = (a\epsilon\varphi_1, b\epsilon\varphi_1)$ is a subsolution to (1.7).

Step 2 *Construction of a supersolution*

Consider the problem

$$\begin{cases} -\Delta e + e &= 0 \quad \text{in} \quad \Omega\,; \\ \frac{\partial e}{\partial \eta} &= 1 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

We will show that, there exists M_λ such that $\overline{U} = (\overline{u}_1, \overline{u}_2) = (Me, Me)$ is a supersolution to (1.7) for $M > M_\lambda$.

Define, $\overline{f}_i(t) := \max_{s\in[0,t]} f_i(s)$. Clearly,

(i) $\overline{f}_i(t)$ is nondecreasing,

(ii) $f_i(t) \leq \overline{f}_i(t)$ for all $t \geq 0$,

(iii) Moreover, $\overline{f}_i$ satisfies the sublinear condition at infinity i.e. $\lim_{t\to+\infty} \frac{\overline{f}_i(t)}{t} = 0\,.$

Hence, there exists $M_\lambda > 0$ such that for all $M \geq M_\lambda$, $\dfrac{\overline{f}_i(M\|e\|_{L^\infty(\partial\Omega)})}{M\|e\|_{L^\infty(\partial\Omega)}} \leq \dfrac{1}{\lambda\|e\|_{L^\infty(\partial\Omega)}}$.

Consequently,

$$\lambda \overline{f}_i(M\|e\|_{L^\infty(\partial\Omega)}) \leq M\,. \tag{3.49}$$

It follows from (3.49) that, for any $\psi \in H^1(\Omega)$

$$\begin{aligned}\int_\Omega \nabla \overline{u}_i \nabla \psi + \int_\Omega \overline{u}_i \psi &= M \int_{\partial\Omega} \psi \\ &\geq \lambda \int_{\partial\Omega} \overline{f}_i(M\|e\|_{L^\infty(\partial\Omega)})\psi \\ &\geq \lambda \int_{\partial\Omega} \overline{f}_i(Me)\psi \\ &\geq \lambda \int_{\partial\Omega} f_i(Me)\psi = \lambda \int_{\partial\Omega} f_i(\overline{u}_i)\psi\end{aligned}$$

which implies that $\overline{U} = (Me, Me)$ is a supersolution of (1.6). Observe that, we can choose ϵ sufficiently small and M sufficiently large as necessary such that $\underline{U} \leq \overline{U}$. Finally, by Theorem 1.5 there exists a solution $U = (u_1, u_2)$ of the problem (1.7) such that $\underline{U} \leq U \leq \overline{U}$, where $u_1 > 0$ and $u_2 > 0$ since $\underline{u}_1 > 0$ and $\underline{u}_2 > 0$ in $\overline{\Omega}$.

Chapter 4: Proofs of Theorems 1.8-1.11

In this chapter, we begin with some definitions important for the proof of our results.

Definition 4.1. *Let* $\lambda_0, \lambda_\infty \in [0, +\infty)$*. We say that* $(\lambda_0, 0)$ $\big($*respectively,* $(\lambda_\infty, \infty)\big)$ *is a bifurcation point from the trivial solution (respectively, from infinity) of* (1.8) *if there exists a sequence* $(\lambda_n, u_n) \in \Sigma$ *such that* $\lambda_n \to \lambda_0$ *and* $\|u_n\|_{C(\overline{\Omega})} \to 0$ *(respectively,* $\lambda_n \to \lambda_\infty$ *and* $\|u_n\|_{C(\overline{\Omega})} \to +\infty$*) as* $n \to +\infty$*.*

Definition 4.2. *We say that a connected component* $\mathscr{C}$ *bifurcates from the trivial solution at* $(\lambda_0, 0)$ *if* $\mathscr{C}$ *is a maximal closed connected subset of* $\mathscr{S} \cup (\lambda_0, 0)$ *containing* $(\lambda_0, 0)$ *where* $\mathscr{S}$ *denote the closure of the set of nontrivial weak solutions of* (1.8)*. A connected component bifurcating from infinity can be defined similarly.*

Next we recall a-priori bound result from [27] which we use for our purposes. Our main tool in the proof of Theorem 1.8 and Theorem 1.11 is degree theory, for which the following uniform a-priori bound is crucial. To state the result, consider

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega\,; \\ \frac{\partial u}{\partial \eta} = bu^p + \zeta(x, u) & \text{on } \partial\Omega\,, \end{cases} \tag{4.1}$$

where p is as in $(\mathrm{H})_\infty$, and for a.e. $x \in \overline{\Omega}$, all $\sigma \in \mathbb{R}$, and

$$\lim_{\sigma \to \infty} \frac{|\zeta(x,\sigma)|}{|\sigma|^p} = 0. \tag{4.2}$$

While we are not aware of any paper that establishes uniform a-priori estimate for (4.1), the result below follows by adapting the proof for systems case in [27, Thm. 3.7]. Their proof is written for $|\zeta(x,\sigma)| \leq c(1+|\sigma|^r)$ for some $0 < r < p$, but the same arguments can be used to prove the existence of a priori bound under condition (4.2).

Proposition 4.1. *There exists a constant $M > 0$ such that every positive solution $u \in C(\overline{\Omega})$ of (4.1) satisfies*

$$\|u\|_{C(\overline{\Omega})} \leq M\,.$$

4.1 Non-linear Problem and Regularity

Here, we state and prove regularity results for nonlinear problem, which is relevant for our purposes. In particular, we prove that any weak solution of (1.8) is in fact Hölder continuous, see Proposition 4.2 and Corollary 4.3.

Next, will show that any weak solution u of our nonlinear problem (1.8) lies in fact in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. To accomplish this, we will establish regularity results for problems with nonlinearities satisfying $(\mathrm{H})_\infty$. Hereafter, we will use the same symbol to denote both the function and the associated Nemytskii operator.

Proposition 4.2. *Let $N \geq 2$ and $h : [0,\infty) \to [0,\infty)$ be locally Lipschitz continuous satisfying condition $(\mathrm{H})_\infty$. Let v be a nontrivial weak solution of the following problem*

$$\begin{cases} -\Delta v + v &= 0 \qquad \text{in} \quad \Omega\,; \\ \frac{\partial v}{\partial \eta} &= h(v) \qquad \text{on} \quad \partial\Omega\,. \end{cases}$$

Then,

$$||v||_{C^\alpha(\overline{\Omega})} \leq C(1 + ||\Gamma v||_{L^r(\partial\Omega)})$$

for some positive $\alpha \in (0,1)$, *where* $r = \dfrac{2(N-1)}{N-2}$ *if* $N > 2$, *and* $r \geq 1$ *when* $N = 2$.

Proof. We assume $N > 2$, since the proof is trivial when $N = 2$. By definition of a weak solution and the trace operator, (2.6), $v \in H^1(\Omega)$ and its trace $\Gamma v \in L^r(\partial\Omega)$, where $1 \leq r \leq r_0 := \frac{2(N-1)}{N-2}$, respectively. It follows from the condition $(\mathrm{H})_\infty$ that

$$h\big(\Gamma v\big) \leq C(1 + |\Gamma v|^p), \tag{4.3}$$

and by the continuity of the Nemytskii operator

$$h\big(\Gamma v\big) \in L^{q_0}(\partial\Omega), \qquad \text{where} \qquad q_0 := \frac{r_0}{p} = \frac{2(N-1)}{p(N-2)}.$$

Now we proceed with the bootstrap argument. For $h\big(\Gamma v\big) \in L^{q_{i-1}}(\partial\Omega)$ and (2.5), we have

$$v \in W^{1,s_i}(\Omega), \text{ where } s_i := \frac{Nq_{i-1}}{N-1} \quad \text{ for } i = 1, 2, \dots .$$

By (2.6), we get

$$\Gamma v \in L^{r_i}(\partial\Omega), \text{ where } r_i := \frac{(N-1)q_{i-1}}{N-1-q_{i-1}} \quad \text{ for } i = 1, 2, \dots .$$

Then, using (4.3) and the continuity of the Nemytskii operator

$$h\big(\Gamma v\big) \in L^{q_i}(\partial\Omega), \text{ where } q_i := \frac{r_i}{p} \quad \text{ for } i = 1, 2, \dots .$$

If $q_i > N-1$ for some $i \in \mathbb{N}$, then $v \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ by Lemma 2.1 (iii).

If $q_i = N-1$ for some $i \in \mathbb{N}$, then by Lemma 2.1 (ii), $\Gamma v \in L^r(\partial\Omega)$ for all $r \geq 1$. By (4.3), $h(\Gamma v) \in L^m$ for all $m \geq 1$. Using the L^q-estimates for second-order linear elliptic equations, we get that v is actually in $W^{1,s}(\Omega)$ for any $s > 1$, in particular for $s > N$. By the continuity of the embedding $W^{1,s}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ for $s > N$, one has that $v \in C^{\alpha}(\overline{\Omega})$, see e.g [16, p. 285].

Now suppose $q_i < N-1$. Then,

$$\frac{1}{r_1} = \frac{1}{q_0} - \frac{1}{N-1} = \frac{p(N-2)-2}{2(N-1)} < \frac{N-2}{2(N-1)} = \frac{1}{r_0} \qquad \text{iff } p < \frac{N}{N-2}\,.$$

If $r_i > r_{i-1}$, then

$$\frac{1}{r_{i+1}} = \frac{1}{q_i} - \frac{1}{N-1} = \frac{p}{r_i} - \frac{1}{N-1} < \frac{p}{r_{i-1}} - \frac{1}{N-1} = \frac{1}{q_{i-1}} - \frac{1}{N-1} = \frac{1}{r_i}\,.$$

Hence, by induction $\{r_i\}$ is strictly increasing. Then, clearly $\{s_i\}$ and $\{q_i\}$ are strictly increasing as well.

If there is some $q_{i_0} \geq N-1$ the proof is finished. Suppose $q_i < N-1$ for all $i \in \mathbb{N}$. Since $\{q_i\}$ is strictly increasing and $1 \leq q_i < N-1$ for all $i \in \mathbb{N}$, $q_i \to q_\infty$ for some $1 \leq q_\infty \leq N-1$. If $q_\infty = N-1$, then fixing $\varepsilon > 0$, there exists an $i_0 \in \mathbb{N}$ such that $N-1 > q_{i_0} \geq N-1-\varepsilon$. However, $\{q_i\}$ is strictly increasing, hence $q_{i_0+1} > q_{i_0} \geq N-1-\varepsilon$, a contradiction. As a consequence, $q_\infty < N-1$.

Define $r_\infty := \lim_{i\to\infty} r_i = \lim_{i\to\infty} pq_i = pq_\infty\, 1$. Note that

$$\begin{aligned} q_{i+1} - q_i &= \frac{r_{i+1} - r_i}{p} = \frac{r_{i+1}\, r_i}{p}\left(\frac{1}{r_i} - \frac{1}{r_{i+1}}\right) \\ &= \frac{r_{i+1}\, r_i}{p}\left(\frac{1}{q_{i-1}} - \frac{1}{q_i}\right) = \frac{r_{i+1}\, r_i}{p}\,\frac{q_i - q_{i-1}}{q_{i-1}\, q_i} \end{aligned}$$

hence

$$\frac{q_{i+1} - q_i}{q_i - q_{i-1}} = \frac{r_{i+1}\, r_i}{p}\,\frac{1}{q_{i-1}\, q_i} = \frac{r_{i+1}\, r_i}{p}\,\frac{p^2}{r_{i-1}\, r_i} = p\,\frac{r_{i+1}}{r_{i-1}}\,.$$

Taking the limit as i goes to infinity and noting that $r_\infty > 1$, we have

$$\lim_{i\to\infty} \frac{q_{i+1} - q_i}{q_i - q_{i-1}} = p > 1\,.$$

This contradicts the boundedness of $\{q_i\}$. Therefore, there exists $i_0 \in \mathbb{N}$ such that $q_{i_0} \geq N-1$ and hence $v \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, as desired. Furthermore, the estimate in Lemma 2.1, (4.3) and (2.5) give

$$||v||_{C^\alpha(\overline{\Omega})} \leq \|h(\Gamma v)\|_{L^{q_0}(\partial\Omega)} \leq \|C(1+|\Gamma v|^p)\|_{L^{q_0}(\partial\Omega)} \leq C(1+||\Gamma v||_{L^r(\partial\Omega)}) \tag{4.4}$$

where $r = pq_0 = \dfrac{2(N-1)}{N-2}$ if $N > 2$. □

Corollary 4.3. Assume that the nonlinearity $f : [0,\infty) \to [0,\infty)$ is locally Lipschitz continuous and satisfies condition $(\mathrm{H})_\infty$. Fix any $\Lambda > 0$ and let u be a weak solution of the nonlinear problem (1.8) for some $0 < \lambda \leq \Lambda$. Then

$$||u||_{C^\alpha(\overline{\Omega})} \leq C(1+||\Gamma u||_{L^r(\partial\Omega)}),$$

for some $\alpha \in (0,1)$ and some $C = C(\Lambda) > 0$, where $r = \dfrac{2(N-1)}{N-2}$ for $N > 2$ and $r \geq 1$ for $N = 2$. Moreover, $u \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

Proof. Proposition 4.2 yields the proof for the first part.

Since $u \in C^\alpha(\overline{\Omega})$, f is locally Lipschtiz continuous, $f(u) \in C^\alpha(\partial\Omega)$. The conclusion follows from Lemma 2.1 (iv). □

Under additional assumption on the nonlinearity f, Corollary 4.3 can be rewritten in the following way.

Proposition 4.4. *Assume that the nonlinearity $f \in C^1([0,\infty))$ satisfies conditions $(\mathrm{H})_0$ and $(\mathrm{H})_\infty$. For any fixed $\Lambda > 0$, if u is a weak solution of the nonlinear problem*

(1.8) *for some* $0 \leq \lambda \leq \Lambda$*, then*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C||\Gamma u||_{L^r(\partial\Omega)},$$

for some $\alpha \in (0,1)$*, where* $C = C(\Lambda)$ *and* $r = \frac{2(N-1)}{N-2}$ *if* $N > 2$*, and* $r \geq 1$ *when* $N = 2$.

Proof. Note that under conditions $(\mathrm{H})_0$ and $(\mathrm{H})_\infty$, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$f(s) \leq (1+\varepsilon)f'(0)|s| + C_\varepsilon|s|^p.$$

In particular, there exists a constant $C > 0$ such that $f(s) \leq C(|s| + |s|^p)$. Hence, the conclusion follows from (4.4). □

Remark 4.1. Let $f : [0,\infty) \to [0,\infty)$ be a locally Lipschitz continuous satisfying condition $(\mathrm{H})_\infty$ and u be a weak solution of (1.8) for some $\lambda > 0$. Then, Corollary 4.3 implies that $u \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ and hence $u > 0$ on $\overline{\Omega}$ by Proposition 2.1.

Remark 4.2. Let $f : [0,\infty) \to [0,\infty)$ be a locally Lipschitz continuous function satisfying condition $(\mathrm{H})_\infty$. Then, for a given $u \in C(\overline{\Omega})$, $f(\Gamma u)$ maps $C(\overline{\Omega})$ into $L^q(\partial\Omega)$ with $q > 1$ by the continuity of the Nemytskii operator associated with f, see [4, Lemma 3.1]. Then, using (2.7), we have that

$$S \circ f \circ \Gamma : C(\overline{\Omega}) \longrightarrow L^r(\partial\Omega) \overset{\text{Cor.4.3}}{\longrightarrow} C^\alpha(\overline{\Omega}) \overset{c}{\hookrightarrow} C(\overline{\Omega}), \tag{4.5}$$

is compact, and $v = (S \circ f \circ \Gamma)u$ is the weak solution of

$$\begin{cases} -\Delta v + v &= 0 \qquad \text{in} \quad \Omega\,; \\ \frac{\partial v}{\partial \eta} &= \lambda f(u) \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

More precisely,

$$u \text{ is a weak solution of (1.8) for } \lambda > 0 \iff u = \lambda S\big(f(\Gamma u)\big)\,.$$

4.2 Proof of Theorem 1.8

Our proof is motivated by [6]. In particular, we re-scale (1.8) in such a way that the transformed problem approaches a limiting problem of "pure power type" as $\lambda \to 0^+$. Then, using $\lambda \geq 0$ as the homotopy parameter, we obtain a positive weak solution of the re-scaled problem, hence of (1.8) for $\lambda > 0$ small.

First, let us extend f to $\mathbb{R}$ by setting $f(s) = f(|s|)$ for $s \in \mathbb{R}$. Now consider the problem

$$\begin{cases} -\Delta u + u &= 0 & \text{in} \quad \Omega\,; \\ \frac{\partial u}{\partial \eta} &= \lambda f(|u|) & \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.6}$$

Note that for $\lambda > 0$, u is a solution of (4.6) if and only if $w = \lambda^{\frac{1}{p-1}} u$ satisfies

$$\begin{cases} -\Delta w + w &= 0 & \text{in} \quad \Omega\,; \\ \frac{\partial w}{\partial \eta} &= \lambda^{\frac{p}{p-1}} f\big(\lambda^{-\frac{1}{p-1}}|w|\big) & \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.7}$$

For $\lambda > 0$, define

$$\begin{aligned} \widetilde{f}(\lambda, s) &:= \lambda^{\frac{p}{p-1}} f\big(\lambda^{-\frac{1}{p-1}}|s|\big) \\ &= \lambda^{\frac{p}{p-1}} \Big[f\big(\lambda^{-\frac{1}{p-1}}|s|\big) - b\big(\lambda^{-\frac{1}{p-1}}|s|\big)^p \Big] + b|s|^p\,. \end{aligned}$$

We observe that

$$\lim_{\substack{\lambda \to 0^+ \\ s \to s_0}} \lambda^{\frac{p}{p-1}} f\big(\lambda^{-\frac{1}{p-1}}|s|\big) = b|s_0|^p\,,$$

due to superlinear condition at infinity $(\mathrm{H})_\infty$ for $s_0 \neq 0$, and by the continuity of f

at $s_0 = 0$. Therefore, we can define $\widetilde{f}$ at $\lambda = 0$ by setting $\widetilde{f}(0, s) := b|s|^p$. Therefore, since f is Lipschitz continuous, so is $\widetilde{f} : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ defined above.

Then the goal is to study the following re-scaled problem for $\lambda \geq 0$

$$\begin{cases} -\Delta w + w = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial w}{\partial \eta} = \widetilde{f}(\lambda, w) & \text{on} \quad \partial\Omega\,, \end{cases} \tag{4.8}$$

while keeping in mind that (4.8) reduces to the limiting problem for $\lambda = 0$

$$\begin{cases} -\Delta w + w = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial w}{\partial \eta} = b|w|^p & \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.9}$$

Our strategy to proceed with the proof of Theorem 1.8 is as follows: 1) we show that the limiting problem (4.9), corresponding to $\lambda = 0$, has a positive solution using the Leray-Schauder degree, 2) show that the re-scaled problem (4.8) has a positive solution using 1) and $\lambda \geq 0$ as the homotopy parameter, then 3) return to the original problem via the re-scaling.

To set up for the Leray-Schauder degree, we formulate the problem (4.8) in an abstract setting in terms of the compact and Nemytskii operators. For this, we define the compact map $\widetilde{\mathscr{F}} : [0, +\infty) \times C(\overline{\Omega}) \to C(\overline{\Omega})$ given by

$$\widetilde{\mathscr{F}}(\lambda, v) := S(\widetilde{f}(\lambda, \Gamma(v)))\,,$$

where $\widetilde{f}(\lambda, \cdot)$ denotes the Nemytskii operator corresponding to $\widetilde{f}(\lambda, \cdot)$, and S is as defined in Remark 4.2. It follows from Remark 4.2 that

$$(\lambda, w) \;\text{ is a weak solution of (4.8)} \iff \widetilde{\mathscr{F}}(\lambda, w) = w\,.$$

First we establish the following result regarding the limiting problem (4.9).

Lemma 4.1. *There exists $r > 0$ such that for all $\theta \in [0,1]$ and all $w \in C(\overline{\Omega})$ with $\|w\|_{C(\overline{\Omega})} = r$, $w \neq \theta\widetilde{\mathcal{F}}(0,w)$. Consequently deg$(I - \widetilde{\mathcal{F}}(0,\cdot), B_r(0), 0) = 1$.*

Proof. Suppose to the contrary that for each $r > 0$, there exists $\theta \in [0,1]$ such that the operator equation

$$w = \theta\widetilde{\mathcal{F}}(0,w)$$

has a solution $w \in C(\overline{\Omega})$ with $\|w\|_{C(\overline{\Omega})} = r$, that is, w is a solution of

$$\begin{cases} -\Delta w + w &= 0 \qquad \text{in} \quad \Omega\,; \\ \frac{\partial w}{\partial \eta} &= \theta\, b|w|^p \qquad \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.10}$$

Clearly $w \neq 0$ since $\|w\|_{C(\overline{\Omega})} = r > 0$. Hence $w > 0$ in $\overline{\Omega}$ by Proposition 2.1.

Now, let $0 < \varepsilon < \mu_1$ be fixed. Since $p > 1$, there exists $r^* > 0$ such that $bs^p < \varepsilon s$ for $0 < s \leq r^*$. Then there exists $\theta_{r^*} \in [0,1]$ and a solution $w_{r^*} > 0$ of (4.10) such that $\|w_{r^*}\|_{C(\overline{\Omega})} = r^*$, and w_{r^*} satisfies $bw_{r^*}^p < \varepsilon w_{r^*}$ whenever $\|w_{r^*}\|_{C(\overline{\Omega})} = r^*$. Using $\varphi_1 \geq 0$ as the test function and the fact that $\theta_{r^*} \in [0,1]$, we have

$$\begin{aligned} 0 &= \int_\Omega \nabla w_{r^*}\nabla\varphi_1 + \int_\Omega w_{r^*}\varphi_1 - \theta_{r^*} b \int_{\partial\Omega} w_{r^*}^p\varphi_1 \\ &\geq \int_\Omega \nabla w_{r^*}\nabla\varphi_1 + \int_\Omega w_{r^*}\varphi_1 - \varepsilon \int_{\partial\Omega} w_{r^*}\varphi_1 = (\mu_1 - \varepsilon)\int_{\partial\Omega} w_{r^*}\varphi_1\,, \end{aligned}$$

a contradiction since $\varepsilon < \mu_1$. Thus there exists $r > 0$ such that for all $\theta \in [0,1]$ and all $w \in C(\overline{\Omega})$ with $\|w\|_{C(\overline{\Omega})} = r$, $w \neq \theta\widetilde{\mathcal{F}}(0,w)$. Therefore, using $\theta \in [0,1]$ as a homotopy parameter, we get

$$\deg(I - \widetilde{\mathcal{F}}(0,\cdot), B_r(0), 0) = \deg(I - \theta\widetilde{\mathcal{F}}(0,\cdot), B_r(0), 0) = \deg(I, B_r(0), 0) = 1\,,$$

as desired. This completes the proof of Lemma 4.1. □

Lemma 4.2. *There exists $R > r > 0$ and $0 \leq z \in C(\overline{\Omega})$ such that $w \neq \widetilde{\mathcal{F}}(0,w) +$*

tz for all $t \geq 0$ and all $w \in C(\overline{\Omega})$ with $\|w\|_{C(\overline{\Omega})} = R$. Consequently, deg$(I - \widetilde{\mathcal{F}}(0,\cdot), B_R(0), 0) = 0$.

Proof. Let $0 \leq z \in C(\overline{\Omega})$ be the unique solution of

$$\begin{cases} -\Delta z + z &= 0 \quad \text{in} \quad \Omega\,; \\ \frac{\partial z}{\partial \eta} &= 1 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

Then, we observe that the operator equation

$$w = \widetilde{\mathcal{F}}(0, w) + t\, z$$

corresponds to the PDE

$$\begin{cases} -\Delta w + w &= 0 \quad \text{in} \quad \Omega\,; \\ \frac{\partial w}{\partial \eta} &= b|w|^p + t \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.11}$$

Step 1: We show that there exists $t_0 > 0$ such that (4.11) does not have a solution for $t \geq t_0$.

For this, let $\mu > \mu_1$ be fixed. Then there exists $t_0 > 0$ such that $bs^p + t > \mu s + t - t_0$ for $t \geq 0$. Suppose by contradiction that there exists $t_1 \geq t_0$ such that $w \geq 0$ is a solution of (4.11). Using $\varphi_1 \geq 0$ as the test function, we get

$$\begin{aligned} 0 &= \int_\Omega \nabla w \nabla \varphi_1 + \int_\Omega w\varphi_1 - \int_{\partial\Omega} [bw^p + t_1]\varphi_1 \\ &= \int_{\partial\Omega} [bw^p + t_1]\varphi_1 - \int_\Omega \nabla w \nabla \varphi_1 - \int_\Omega w\varphi_1 \\ &> \int_{\partial\Omega} [\mu w + (t_1 - t_0)]\varphi_1 - \int_\Omega \nabla w \nabla \varphi_1 - \int_\Omega w\varphi_1 \\ &\geq \mu \int_{\partial\Omega} w\varphi_1 - \int_\Omega \nabla w \nabla \varphi_1 - \int_\Omega w\varphi_1 = (\mu - \mu_1) \int_{\partial\Omega} w\varphi_1\,, \end{aligned}$$

which is a contradiction since $\mu > \mu_1$. This establishes Step 1, which implies that for all $a > 0$, $w \neq \widetilde{\mathcal{F}}(0, w) + t_0\, z$ for all $w \in C(\overline{\Omega})$ with $\|w\|_{C(\overline{\Omega})} = a$ for any $a > 0$. Hence, for any $a > 0$, we have

$$\deg(I - \widetilde{\mathcal{F}}(0, w) + t_0\, z, B_a, 0) = 0\,. \tag{4.12}$$

Step 2: We show there exists $R > r > 0$ such that for all $t \in [0, t_0]$,

$$\deg(I - \widetilde{\mathcal{F}}(0, w) + t\, z, B_R, 0) = 0\,.$$

Indeed, by Proposition 4.1 with $\xi(t) \equiv t \in [0, t_0]$, there exists $M > 0$ such that $\|w\|_{C(\overline{\Omega})} \leq M$. By taking $R > \max\{r, M\}$, we get $w \neq \widetilde{\mathcal{F}}(0, w) + t\, z$ for all $w \in C(\overline{\Omega})$ with $\|w\|_{C(\overline{\Omega})} = R$ and $t \in [0, t_0]$. Then, using (4.12), we get

$$\deg(I - \widetilde{\mathcal{F}}(0, w), B_R, 0) = \deg(I - \widetilde{\mathcal{F}}(0, w) + t_0\, z, B_R, 0) = 0\,,$$

as desired, establishing Step 2. This completes the proof of Lemma 4.2. □

Now we show that the limiting problem (4.9) has a positive solution.
Indeed, it follows from Lemma 4.1, Lemma 4.2 and the excision property of degree that

$$\deg(I - \widetilde{\mathcal{F}}(0, w), B_R \setminus \overline{B}_r, 0) = -1\,. \tag{4.13}$$

Therefore, there exists a solution of $w = \widetilde{\mathcal{F}}(0, w)$, or equivalently a weak solution of (4.9), say $w_0 \in B_R \setminus \overline{B}_r$. Using the fact that $\|w_0\|_{C(\overline{\Omega})} > r > 0$, it follows from Proposition 2.2 that $w_0 > 0$ in $\overline{\Omega}$.

Now we use $\lambda \geq 0$ as homotopy parameter to establish the following existence result for the re-scaled problem (4.8).

Lemma 4.3. *There exists $\widehat{\lambda} > 0$ such that*

(a) $\widetilde{\mathcal{F}}(\lambda, w) \neq w$ *for all* $\lambda \in [0, \widehat{\lambda}]$ *whenever* $\|w\|_{C(\overline{\Omega})} \in \{r, R\}$; *and*

(b) *deg*$(I - \widetilde{\mathscr{F}}(\lambda, \cdot), B_R \setminus \overline{B}_r, 0) = -1$ *for all* $\lambda \in [0, \widehat{\lambda}]$.

Proof. (a) Suppose not. Then there exist sequences $\lambda_n \geq 0$ with $\lambda_n \to 0$ and $w_n \in C(\overline{\Omega})$ such that $\widetilde{\mathscr{F}}(\lambda_n, w_n) = w_n$ and $\|w_n\|_{C(\overline{\Omega})} = r$ (or $\|w_n\|_{C(\overline{\Omega})} = R$). Since w_n is bounded and $\widetilde{\mathscr{F}}$ is compact, $(\lambda_n, w_n) \to (0, w)$ for some $w \in C(\overline{\Omega})$ with $\|w\|_{C(\overline{\Omega})} = r$ or $\|w\|_{C(\overline{\Omega})} = R$, a contradiction to Lemma 4.1 or Lemma 4.2, respectively. Hence there exists $\widehat{\lambda} > 0$ satisfying (a).

(b) Now using $\lambda \in [0, \widehat{\lambda}]$ as the homotopy parameter, it follows from part (a) that

$$\deg(I - \widetilde{\mathscr{F}}(\lambda, \cdot), B_R \setminus \overline{B}_r, 0) = \text{const.} \quad \forall \lambda \in [0, \widehat{\lambda}]\,.$$

In particular, it follows from (4.13) that for all $\lambda \in [0, \widehat{\lambda}]$

$$\deg(I - \widetilde{\mathscr{F}}(\lambda, \cdot), B_R \setminus \overline{B}_r, 0) = \deg(I - \widetilde{\mathscr{F}}(0, \cdot), B_R \setminus \overline{B}_r, 0) = -1\,.$$

This complete the proof of Lemma 4.3. □

Lemma 4.3 implies that the re-scaled problem (4.8) has a nontrivial solution $w_\lambda \in C(\widetilde{\Omega})$ for all $\lambda \in [0, \widehat{\lambda}]$ satisfying $r < \|w_\lambda\|_{C(\widetilde{\Omega})} < R$. Moreover, since f is nonnegative and satisfies $(\mathrm{H})_\infty$, so does $\widetilde{f}$ and hence $w_\lambda > 0$ in $\overline{\Omega}$ by Proposition 2.1.

Now we return to the original problem (1.8). Using the re-scaling

$$u = \lambda^{-\frac{1}{p-1}} w_\lambda\,,$$

we can conclude that (1.8) has a positive solution (λ, u) for $\lambda \in (0, \widehat{\lambda}]$. Also, since $\|w_\lambda\|_{C(\overline{\Omega})} > r > 0$, it follows that $\|u\|_{C(\overline{\Omega})} \to +\infty$ as $\lambda \to 0^+$.

We use the following Leray-Schauder continuation theorem to establish the last part of Theorem 1.8.

Proposition 4.5. *([26, Prop. 2.3]) Let X be a Banach space and U a bounded open sub-*

set of X. Let $\mathscr{T} : [a,b]\times\overline{U} \to X$ be a compact map and $\mathscr{S} := \{(t,x) \in [a,b] \times U : \mathscr{T}(t,x) = x\}$ is the set of all fixed points of $\mathscr{T}$. Assume that

- *$\mathscr{T}(t,x) \neq x$ for $(t,x) \in [a,b] \times \partial U$;*

- *$deg(I - \mathscr{T}(t,\cdot), 0) \neq 0$ for all $t \in [a,b]$.*

Then there exists a connected component $\mathscr{D}$ of $\mathscr{S}$ such that $\mathscr{D} \cap (\{a\} \times U)$ and $\mathscr{D} \cap (\{b\} \times U)$ are nonempty.

Now, by taking $[a,b] = [0,\widehat{\lambda}]$, $U = B_R \setminus \overline{B_r}$, $\mathscr{T} = \widetilde{\mathscr{F}}(\cdot,\cdot)$ in Proposition 4.5, it follows, using Lemma 4.3, that the re-scaled problem (4.8) has a connected component $\mathscr{D}$ of positive weak solutions along which λ takes all values in $[0,\widehat{\lambda}]$. This in turn, again using $u = \lambda^{-\frac{1}{p-1}} w_\lambda$, implies that there exists a connected component $\mathscr{C}^+ \subset \Sigma$ of positive weak solutions of (1.8) bifurcating from infinity at $\lambda_\infty = 0$. This completes the proof of Theorem 1.8.

4.3 Proof of Theorem 1.10

In this section, we will prove that there exists a connected set of positive weak solutions $\mathscr{C}^+$ of (1.8) bifurcating from the trivial solution at $\lambda = \frac{\mu_1}{f'(0)}$, and bifurcating from infinity at $\lambda = 0$. Furthermore, we discuss the direction of bifurcation of positive weak solutions at $(\frac{\mu_1}{f'(0)}, 0)$. We first show that the condition $(\mathrm{H})_0$ guarantees solutions bifurcating from the trivial solution. The proof is similar to the case of bifurcation from infinity, see for instance [13, Proposition 3.1]. We provide the proof below for completeness.

Proof of Proposition 1.9

Suppose that $\lambda_n \to \underline{\lambda}$ for some $\underline{\lambda} \in \mathbb{R}$ and set $v_n := \dfrac{u_n}{||u_n||_{C(\overline{\Omega})}}$. Observe that v_n is a weak solution of the problem

$$\begin{cases} -\Delta v_n + v_n = 0 & \text{in } \Omega\,; \\ \dfrac{\partial v_n}{\partial \eta} = \lambda_n f'(0) v_n + \lambda_n \dfrac{\mathcal{R}(u_n)}{||u_n||_{C(\overline{\Omega})}} & \text{on } \partial\Omega\,. \end{cases} \tag{4.14}$$

It follows from $(\mathrm{H})_0$ that $\frac{\mathcal{R}(u_n)}{||u_n||_{C(\overline{\Omega})}} \to 0$ in $C(\overline{\Omega})$ as $n \to \infty$. Therefore, the right-hand side of the second equation in (4.14) is bounded in $C(\overline{\Omega})$. Hence, by the elliptic regularity, $v_n \in W^{1,s}(\Omega)$ for any $s > 1$, in particular for $s > N$. Then, the Sobolev embedding theorem implies that $||v_n||_{C^\alpha(\overline{\Omega})}$ is bounded by a constant C that is independent of n. Then, the compact embedding of $C^\alpha(\overline{\Omega})$ into $C^\beta(\overline{\Omega})$ for $0 < \beta < \alpha$ yields, up to a subsequence, $v_n \to \Phi \geq 0$ in $C^\beta(\overline{\Omega})$. Since $||v_n||_{C(\overline{\Omega})} = 1$, we have that $||\Phi||_{C(\overline{\Omega})} = 1$. Hence, $\Phi \not\equiv 0$.

Using the weak formulation of equation (4.14), passing to the limit, and taking into account that $\lambda_n \to \underline{\lambda}$ for some $\underline{\lambda} \in \mathbb{R}$ and $v_n \to \Phi$, we obtain that Φ is a weak solution of the equation

$$\begin{cases} -\Delta \Phi + \Phi = 0 & \text{in } \Omega\,; \\ \dfrac{\partial \Phi}{\partial \eta} = \underline{\lambda} f'(0) \Phi & \text{on } \partial\Omega. \end{cases}$$

Then, it follows that $\underline{\lambda} f'(0) = \mu_1$, the first Steklov eigenvalue, and $\Phi = \varphi_1 > 0$ is its corresponding eigenfunction, ending the proof.

Now, we will show that $\left(\frac{\mu_1}{f'(0)}, 0\right)$ is a bifurcation point from the trivial solution of positive weak solutions of (1.8). That is, there exists a sequence $(\lambda_n, u_n) \in \Sigma$ such that $\lambda_n \to \frac{\mu_1}{f'(0)}$, $u_n > 0$ on $\overline{\Omega}$, and that $||u_n||_{C(\overline{\Omega})} \to 0$. In particular, we have the following result.

Theorem 4.6. *Assume that the nonlinearity* $f \in C^1([0,\infty))$ *satisfies hypothesis* $(\mathrm{H})_0$. *Then, there exists a connected component* $\mathscr{C}^+ \subset \Sigma$ *of positive weak solutions of* (1.8) *emanating from the trivial solution at* $\left(\frac{\mu_1}{f'(0)}, 0\right) \in \mathbb{R} \times C(\overline{\Omega})$. *Moreover,* $\mathscr{C}^+$ *is* *unbounded in* $\mathbb{R} \times C(\overline{\Omega})$.

Proof. The proof follows from the general results on bifurcation from the trivial solutions given in [67, Thm. 1.3]. More precisely, there exists a connected component $\mathscr{C}^+ \subset \Sigma$ of positive weak solutions of (1.8) emanating from the trivial solution at $\left(\frac{\mu_1}{f'(0)}, 0\right) \in \mathbb{R} \times C(\overline{\Omega})$ and, the branch $\mathscr{C}^+$ either meets another bifurcation point from the trivial solution, or it is unbounded in $\mathbb{R} \times C(\overline{\Omega})$. Since $f \geq 0$ satisfies $(\mathrm{H})_0$, it follows from Lemma 2.1 (iv) and Proposition 2.1 that the branch contains only positive solutions. From the Crandall-Rabinowitz Theorem, see [22], $\mathscr{C}^+$ can neither meet another bifurcation point from zero (that is, another point $\left(\frac{\mu'}{f'(0)}, 0\right)$ for another Steklov eigenvalue μ'), nor can meet $\left(\frac{\mu_1}{f'(0)}, 0\right)$ again, so the branch is unbounded in $\mathbb{R} \times C(\overline{\Omega})$. □

Next, we discuss sufficient conditions for the bifurcation from the trivial solution to be either subcritical (to the left) or supercritical (to the right). Following lemma is key in determining the direction of bifurcation from the trivial solution at $\left(\frac{\mu_1}{f'(0)}, 0\right)$.

Lemma 4.4. *Assume that the nonlinearity* $f \in C^1([0,\infty))$ *satisfies the hypothesis* $(\mathrm{H})_0$. *Consider a sequence of positive weak solutions* u_n *of* (1.8) *corresponding to the parameters* λ_n *such that* $\lambda_n \to \frac{\mu_1}{f'(0)}$ *and* $\|u_n\|_{C(\overline{\Omega})} \to 0$. *Then, we have*

$$\begin{aligned} \underline{\mathscr{R}}_0 \, \frac{\mu_1}{\big(f'(0)\big)^2} \frac{\int_{\partial\Omega} \varphi_1^{1+\nu}}{\int_{\partial\Omega} \varphi_1^2} &\leq \liminf_{n\to\infty} \frac{\frac{\mu_1}{f'(0)} - \lambda_n}{\|u_n\|_{C(\overline{\Omega})}^{\nu-1}} \\ &\leq \limsup_{n\to\infty} \frac{\frac{\mu_1}{f'(0)} - \lambda_n}{\|u_n\|_{C(\overline{\Omega})}^{\nu-1}} \leq \overline{\mathscr{R}}_0 \, \frac{\mu_1}{\big(f'(0)\big)^2} \frac{\int_{\partial\Omega} \varphi_1^{1+\nu}}{\int_{\partial\Omega} \varphi_1^2}\,, \end{aligned} \tag{4.15}$$

where $\underline{\mathscr{R}}_0$ *and* $\overline{\mathscr{R}}_0$ *are defined in* (1.1.2), *and* $\nu > 1$ *as defined in* $(\mathrm{H})_0$.

Proof. Using the weak formulation of (1.8) with φ_1 as the test function, we get

$$\int_\Omega \nabla u_n \nabla \varphi_1 + \int_\Omega u_n \varphi_1 = \lambda_n f'(0) \int_{\partial\Omega} u_n \varphi_1 + \lambda_n \int_{\partial\Omega} \mathscr{R}(u_n) \varphi_1\,,$$

which yields

$$(\mu_1 - \lambda_n f'(0)) \int_{\partial\Omega} u_n \varphi_1 = \lambda_n \int_{\partial\Omega} \mathscr{R}(u_n)\varphi_1 \,.$$

Consequently, we get

$$\frac{(\mu_1 - \lambda_n f'(0))}{||u_n||^{\nu-1}_{C(\overline{\Omega})}} \int_{\partial\Omega} \frac{u_n}{||u_n||_{C(\overline{\Omega})}} \varphi_1 = \lambda_n \int_{\partial\Omega} \frac{\mathscr{R}(u_n)}{||u_n||^{\nu}_{C(\overline{\Omega})}} \varphi_1. \tag{4.16}$$

From Fatou's Lemma,

$$\begin{aligned} \liminf_{n\to\infty} \int_{\partial\Omega} \frac{\mathscr{R}(u_n)}{u_n^\nu} \left(\frac{u_n}{\|u_n\|_{C(\overline{\Omega})}} \right)^\nu \varphi_1 \\ \geq \int_{\partial\Omega} \liminf_{n\to\infty} \left[\frac{\mathscr{R}(u_n)}{u_n^\nu} \left(\frac{u_n}{\|u_n\|_{C(\overline{\Omega})}} \right)^\nu \varphi_1 \right] \geq \underline{\mathscr{R}}_0 \int_{\partial\Omega} \varphi_1^{1+\nu} \,, \end{aligned} \tag{4.17}$$

where we have used the definition of $\underline{\mathscr{R}}_0$ (see (1.1.2)), that $\varphi_1 > 0$ on $\partial\Omega$ and the fact that $\frac{u_n}{\|u_n\|_{C(\overline{\Omega})}} \to \varphi_1$ uniformly on $\partial\Omega$ (see Proposition 1.9).

Passing to the limit in (4.16) and using (4.17), we obtain the first inequality of (4.15). The second inequality is trivial and the third is obtained likewise. □

Finally, we prove the Theorem 1.10 as follows:

Proof. Consider a sequence of positive weak solutions u_n of (1.8) corresponding to the parameters λ_n such that $\lambda_n \to \frac{\mu_1}{f'(0)}$ and $\|u_n\|_{C(\overline{\Omega})} \to 0$. Observe that, by (4.15), conditions $\underline{\mathscr{R}}_0 > 0$ and $\overline{\mathscr{R}}_0 < 0$ imply that $\frac{\mu_1}{f'(0)} > \lambda_n$ and $\frac{\mu_1}{f'(0)} < \lambda_n$, respectively, for sufficiently large n. This completes the proof. □

4.4 Proof of Theorem 1.11

This particular proof will be completed in several steps.

Step 1: By Theorem 4.6, there exists a connected component $\mathscr{C}^+$ of positive weak solutions of (1.8) bifurcating from the trivial solution at the bifurcation point $\left(\frac{\mu_1}{f'(0)}, 0\right)$ and that $\mathscr{C}^+$ is unbounded in $\mathbb{R} \times C(\overline{\Omega})$.

Step 2: At this step, we show that (1.8) has no positive weak solution for $\lambda > \frac{\mu_1}{K}$, where $K > 0$ is as given in the hypothesis 1.11.

Indeed, let u be a positive weak solution of (1.8) for some $\lambda > 0$. Then, using $\varphi_1 \geq 0$ as the test function, we get

$$\begin{aligned} 0 &= \int_\Omega \nabla u \nabla \varphi_1 + \int_\Omega u\varphi_1 - \lambda \int_{\partial\Omega} f(u)\varphi_1 \\ &= \lambda \int_{\partial\Omega} f(u)\varphi_1 - \int_\Omega \nabla u \nabla \varphi_1 - \int_\Omega u\varphi_1 \\ &\geq \lambda K \int_{\partial\Omega} u\varphi_1 - \int_\Omega \nabla u \nabla \varphi_1 - \int_\Omega u\varphi_1 = (\lambda K - \mu_1) \int_{\partial\Omega} u\varphi_1 \,. \end{aligned}$$

This yields $\lambda \leq \frac{\mu_1}{K}$. Hence there exists no positive weak solution u of (1.8) for $\lambda > \frac{\mu_1}{K}$, completing the proof of this step.

Step 3: Here, we show that $\mathscr{C}^+$ from Step 1 contains weak positive solutions that bifurcate from infinity at $\lambda = 0$, and establish (1.11).

By Step 1-Step 2, if $(\lambda, u) \in \mathscr{C}^+$ then $\|u\|_{C(\overline{\Omega})} \to 0$ as $\lambda \to \frac{\mu_1}{f'(0)}$, and $\mathscr{C}^+$ is bounded in the λ-direction. Hence, there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}^+$ such that $\lambda_n \in (0, K)$ and $\|u_n\|_{C(\overline{\Omega})} \to \infty$. By choosing a subsequence if necessary, there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}^+$ with the property that $\lambda_n \to \widetilde{\lambda}$ and $\|u_n\|_{C(\overline{\Omega})} \to \infty$. It suffices to show $\widetilde{\lambda} = 0$.

Assume to the contrary that $\widetilde{\lambda} > 0$. For $a_0 > 0$, let $[a_0, b_0]$ be any fixed compact interval with $\widetilde{\lambda} \in (a_0, b_0)$. By Proposition 4.1, for any $\lambda \in [a_0, b_0]$, there exists a uniform constant $M = M(a_0, b_0) > 0$ such that for every (λ, w) with $\lambda \in [a_0, b_0]$ and w a positive weak solution of the re-scaled problem $(4.7)_\lambda$, we have

$$\|w\|_{C(\overline{\Omega})} \leq M \,.$$

Here we recall from Section 4.2 that for any $\lambda > 0$, u is a positive weak solution of (1.8) if and only if $w = \lambda^{\frac{1}{p-1}} u$ is a weak solution of (4.7). Hence,

$$\|u\|_{C(\overline{\Omega})} \leq \lambda^{-\frac{1}{p-1}} M \leq a_0^{-\frac{1}{p-1}} M =: M' \qquad \text{for any} \quad \lambda \in [a_0, b_0], \tag{4.18}$$

which contradicts that $\|u_n\|_{C(\overline{\Omega})} \to \infty$ with $\lambda_n \to \widetilde{\lambda} > 0$. Hence $\widetilde{\lambda} = 0$. As a conclusion, necessarily, $\mathscr{C}^+$ contains a unique bifurcation point from infinity at $\lambda = 0$ and (1.11) holds. Then, (1.8) has a positive weak solution for any $\lambda \in \left(0, \frac{\mu_1}{f'(0)}\right)$. This completes Step 3. Now, set

$$\bar{\lambda} := \sup\{\lambda > 0 : (\lambda, u) \in \mathscr{C}^+\}.$$

Then, $\bar{\lambda} < \infty$ by Step 2.

Step 4: Assuming $\overline{\mathscr{R}}_0 < 0$, we prove the existence of two positive weak solutions for each $\lambda \in \left(\frac{\mu_1}{f'(0)}, \bar{\lambda}\right)$.

It follows from Theorem 1.10 (ii), that the bifurcation is supercritical at the bifurcation point $\left(\frac{\mu_1}{f'(0)}, 0\right)$ from the trivial solution. Note that since $\overline{\mathscr{R}}_0 < 0$, $\bar{\lambda} > \frac{\mu_1}{f'(0)}$. Let $\lambda_0 \in \left(\frac{\mu_1}{f'(0)}, \overline{\lambda}\right)$ and u_0 be a positive weak solution corresponding to λ_0. Now, let $\lambda \in \left(\frac{\mu_1}{f'(0)}, \lambda_0\right)$ be fixed. We show that there exist two distinct positive weak solutions of (1.8) corresponding to λ using degree theory. For this, first we extend f to $\mathbb{R}$ by setting $f(t) = 0$ for $t < 0$.

First solution corresponding to λ:

First we note that, since f is Lipschitz continuous, there exists $c \in \mathbb{R}$ such that $\lambda f(s) + cs$ is nondecreasing on $[0, M']$, where $M' > M$, and $M > 0$ is given by Proposition 4.1. Now let $\theta \in [0, 1]$ and $\beta > \mu_1$. For a given $u \in C(\overline{\Omega})$, define the operator $T_\theta : C(\overline{\Omega}) \to C(\overline{\Omega})$ by $v = T_\theta(u) := (S \circ f_\theta \circ \Gamma)u$, where v is given by

$$\begin{cases} -\Delta v + v &= 0 \quad \text{in} \quad \Omega\,; \\ \frac{\partial v}{\partial \eta} + \theta c\, v &= \theta(\lambda f(u) + cu) + (1-\theta)(\beta u^+ + 1) \quad \text{on} \quad \partial\Omega\,, \end{cases}$$

and $f_\theta(u) := \theta(\lambda f(u) + cu) + (1-\theta)(\beta u^+ + 1)$. We note that T_θ is compact by Remark 4.2, and fixed point of the operator T_1 is a weak solution of (1.8).

We begin by establishing that $u_0 > \epsilon\varphi_1$ for sufficiently small $\epsilon > 0$. Clearly, $u_0 - \epsilon\varphi_1$ satisfies

$$-\Delta(u_0 - \epsilon\varphi_1) + (u_0 - \epsilon\varphi_1) = 0 \text{ in } \Omega\,.$$

Now, using the hypothesis (1.11), and the facts that $\lambda > \frac{\mu_1}{f'(0)}$, $\|u_0\|_{C(\overline{\Omega})} < M'$ and f is continuous, we get

$$\lambda f(u_0) - \epsilon\mu_1\varphi_1 \geq \frac{\mu_1}{f'(0)}\left(f(u_0) - \epsilon f'(0)\varphi_1\right) \geq 0$$

for $\epsilon > 0$ sufficiently small. Then

$$\frac{\partial(u_0 - \epsilon\varphi_1)}{\partial\eta} = \lambda f(u_0) - \epsilon\mu_1\varphi_1 \geq 0 \text{ on } \partial\Omega\,.$$

Therefore, by Proposition 2.1, $u_0 > \epsilon\varphi_1$ for $\epsilon > 0$ sufficiently small.

Now define

$$Y := \left\{v \in C(\overline{\Omega}) : \|v\|_{C(\overline{\Omega})} < M' \text{ and } v > \epsilon\varphi_1 \text{ on } \overline{\Omega}\right\}\,,$$

and

$$Z := \left\{v \in Y : v < u_0 \text{ on } \overline{\Omega}\right\}\,,$$

where $\epsilon > 0$ to be chosen sufficiently small later such that in particular $u_0 > \epsilon\varphi_1$ in $\overline{\Omega}$.

Claim I: $\deg(I - T_1, Y, 0) = 0$.

First, we justify that the degree $\deg(I - T_\theta, Y, 0)$ is well defined and independent of $\theta \in [0,1]$. That is, $u \neq T_\theta u$ for any u on the boundary of Y, ∂Y. We note that if $u \in \partial Y$, then either $\|u\|_{C(\overline{\Omega})} = M'$ or $u = \epsilon\varphi_1$. Now, if $\|u\|_{C(\overline{\Omega})} = M'$, then by Proposition 4.1, $u \neq T_\theta u$ for any $\theta \in [0,1]$. On the other hand, if $u = \epsilon\varphi_1$ is a solution of $u = T_\theta u$ for $\theta = 0$, then $\beta > \mu_1$ yields the contradiction

$$\beta\epsilon\varphi_1 + 1 = \frac{\partial \epsilon\varphi_1}{\partial \eta} = \mu_1\epsilon\varphi_1 < \beta\epsilon\varphi_1 \,.$$

Thus, $u \neq T_\theta u$ when $u = \epsilon\varphi_1$.

Now, by repeating arguments in Step 2 with $\lambda f(u)$ replaced by $\beta u^+ + 1$ and using $\beta > \mu_1$, we see that $u \neq T_0 u$ for any $u \in Y$. Then, using $\theta \in [0,1]$ as a homotopy parameter, we conclude that

$$\deg(I - T_1, Y, 0) = \deg(I - T_\theta, Y, 0) = \deg(I - T_0, Y, 0) = 0\,. \tag{4.19}$$

Claim II: $\deg(I - T_1, Z, 0) = 1$.

We fix $\psi_0 \in Z$ and show $\deg(I - (\theta T_1 + (1-\theta)T_{\psi_0}), Z, 0) = 1$ for $\theta \in [0,1]$, where T_{ψ_0} maps every element of Z to ψ_0. By $v = (\theta T_1 + (1-\theta)T_{\psi_0})u$, for $\theta \in [0,1]$, we mean

$$\begin{cases} -\Delta v + v &= 0 \quad \text{in} \quad \Omega\,; \\ \frac{\partial v}{\partial \eta} + \theta c\, v &= \theta(\lambda f(u) + cu) + (1-\theta)\psi_0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

Now we show that $\deg(I - (\theta T_1 + (1-\theta)T_{\psi_0}), Z, 0)$ is well defined and independent of $\theta \in [0,1]$. Indeed, note that if $u \in \overline{Z}$, that is, $u \leq u_0$, then by Proposition 2.1 $v = T_1 u \in Z$, since $-\Delta v + v = 0$ in Ω and

$$\frac{\partial v}{\partial \eta} + cv = \lambda f(u) + cu \leq \lambda f(u_0) + cu_0 < \lambda_0 f(u_0) + cu_0 \quad \text{on } \partial\Omega\,.$$

Also, $T_{\psi_0} u \in Z$ for $u \in Z$. Then $\theta T_1 u + (1-\theta)T_{\psi_0} u \in Z$ for all $\theta \in [0,1]$, since Z is convex. Hence, there is no solution of $I - (\theta T_1 + (1-\theta)T_{\psi_0})$ on the boundary of Z, and $\deg(I - (\theta T_1 + (1-\theta)T_{\psi_0}), Z, 0)$ is well defined for all $\theta \in [0,1]$. Therefore, since

$\psi_0 \in Z$, we have

$$\deg(I - T_1, Z, 0) = \deg(I - T_{\psi_0}, Z, 0) = \deg(I, Z, \psi_0) = 1\,, \tag{4.20}$$

completing Claim II.

Combining (4.19) and (4.20), one has $\deg(I - T_1, Y \setminus \overline{Z}, 0) = -1$ and hence there exists a positive weak solution $u_2 \in Y \setminus \overline{Z}$ of (1.8) corresponding to the fixed λ.

Second solution corresponding to λ:

We construct the second positive weak solution distinct from u_2 by the method of sub and supersolutions. Using the facts that $f(0) = 0$ and $f'(0) > 0$, we verify that $\underline{u} = \epsilon\varphi_1$ is a subsolution of (1.8) for $\epsilon \approx 0$. Indeed, we observe that since $\lambda > \frac{\mu_1}{f'(0)}$ is fixed, $\xi(s) := \mu_1 s - \lambda f(s)$ satisfies $\xi(0) = 0$ and $\xi'(0) < 0$, then $\xi(s) < 0$ for $s \approx 0$. Therefore, for all $0 \leq \psi \in H^1(\Omega)$, the following holds for $\epsilon \approx 0$

$$\int_\Omega \nabla\underline{u}\nabla\psi + \int_\Omega \underline{u}\psi = \mu_1 \int_{\partial\Omega} (\epsilon\varphi_1)\psi \leq \lambda \int_{\partial\Omega} f(\epsilon\varphi_1)\psi = \lambda \int_{\partial\Omega} f(\underline{u})\psi\,.$$

Note that $u_0 \in Y$ since $\epsilon\varphi_1 < u_0 < M < M'$ for sufficiently small $\epsilon > 0$. It follows from Corollary 3.4 in Section 3.1 of Chapter 3 that $\min(u_2, u_0)$ is a strict supersolution of (1.8). Since $u_0, u_2 \in Y$, $\underline{u} = \epsilon\varphi_1 < \min(u_2, u_0)$ on $\overline{\Omega}$. Hence, there exists a positive weak solution u_1 of (1.6) corresponding to the fixed λ satisfying $\epsilon\varphi_1 \leq u_1 < u_2$ on Ω by Proposition 2.1. This completes Step 4.

Step 5: At this step, we prove the existence of a solution for $\lambda = \bar{\lambda}$. For each $\lambda \in (\frac{\mu_1}{f'(0)}, \bar{\lambda})$, problem (1.8) admits a positive weak solution u_λ.

Using Proposition 4.1, (4.18) for $\lambda \in [\frac{\mu_1}{f'(0)}, \bar{\lambda}]$, and Proposition 4.4, there exists a uniform constant $C > 0$ such that $\|u_\lambda\|_{C^\alpha(\overline{\Omega})} \leq C$ for any $\lambda \in (\frac{\mu_1}{f'(0)}, \bar{\lambda})$. By compact embeddings, u_λ has a subsequence that converges to (say), $u_{\bar{\lambda}}$ in $C^\beta(\overline{\Omega})$ as $\lambda \to \bar{\lambda}$, where $\beta < \alpha$.

Moreover,

$$\|u_\lambda\|^2_{H^1(\Omega)} = \int_\Omega |\nabla u_\lambda|^2 + \int_\Omega |u_\lambda|^2 = \lambda \int_{\partial\Omega} f(u_\lambda) u_\lambda \leq C, \quad \forall \lambda \in \left(\frac{\mu_1}{f'(0)}, \bar{\lambda} \right).$$

By the reflexivity of $H^1(\Omega)$, u_λ has a subsequence that converges weakly to (say), $u_{\bar{\lambda}}$ in $H^1(\Omega)$ as $\lambda \to \bar{\lambda}$. On the other hand, since $u_\lambda \to u_{\bar{\lambda}} \in C^\beta(\overline{\Omega})$ and f is locally Lipschitz, then $f(u_\lambda) \to f(u_{\bar{\lambda}})$ in $C^\beta(\overline{\Omega})$ as $\lambda \to \bar{\lambda}$.

Then, by taking limits in the weak formulation of u_λ as $\lambda \to \bar{\lambda}$, we get

$$\int_\Omega \nabla u_{\bar{\lambda}} \nabla \psi + \int_\Omega u_{\bar{\lambda}} \psi = \bar{\lambda} \int_{\partial\Omega} f(u_{\bar{\lambda}}) \psi \,.$$

Hence $u_{\bar{\lambda}}$ is a positive weak solution of $(1.8)_{\bar{\lambda}}$.

Therefore, (1.8) has at least two positive weak solutions for $\lambda \in \left(\frac{\mu_1}{f'(0)}, \bar{\lambda}\right)$, and at least one positive weak solution for $\lambda = \bar{\lambda}$. Finally, since the connected set $\mathscr{C}^+$ bifurcates to the right at $\left(\frac{\mu_1}{f'(0)}, 0\right)$ and bifurcates from infinity at $\lambda = 0$, $\mathscr{C}^+$ must cross the hyperplane $\lambda = \frac{\mu_1}{f'(0)}$ at a point distinct from $u = 0$. Hence, the problem (1.8) has a positive weak solution for $\lambda = \frac{\mu_1}{f'(0)}$. This completes the proof of Theorem 1.11.

□

Chapter 5: Finite Difference Approximations

In this chapter we first discuss the discretization of (1.10) using the finite difference (FD) method and prove an existence result for the resulting discrete problem. Next, in Section 5.2, we apply the results to Problem (1.8) in order to generate bifurcation diagrams using MATLAB. For the detailed notations please check Chapter 2. The formulation uses the difference operators defined in Section 2.3.3. Due to our choice of approximating the outward normal derivative, the method will be first order accurate. In the future we aim to extend the methodology to a second order accurate formulation.

5.1 Sublinear Problem

In this section we use the following discrete problem to approximate solutions to (1.10), where the grid functions $u_{i,h}$ are an approximation for u_i over the grid $\mathcal{T}_h$ for $i = 1, 2$:

$$\begin{cases} -\Delta_h u_{1,h} + u_{1,h} &= 0 \quad \text{in } \mathcal{T}_h \cap \Omega, \\ -\Delta_h u_{2,h} + u_{2,h} &= 0 \quad \text{in } \mathcal{T}_h \cap \Omega, \\ \nabla_h^* u_{1,h} \cdot \widehat{n} - \lambda f_1(u_{1,h}, u_{2,h}) &= 0 \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}, \\ \nabla_h^* u_{2,h} \cdot \widehat{n} - \lambda f_2(u_{1,h}, u_{2,h}) &= 0 \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}. \end{cases} \tag{5.1}$$

Here, $\widetilde{\partial\Omega} \subset \partial\Omega$ such that $\widetilde{\partial\Omega} := \partial\Omega \setminus \{\text{the points where } \partial\Omega \text{ is not smooth}\}$.

Remark 5.1. Note that we are eliminating the set of points (a set with measure 0) where the outward normal derivative is not defined. Once U_h is defined over $\mathcal{T}_h \cap (\Omega \cup \widetilde{\partial\Omega})$, it can be extended to $\mathcal{T}_h \cap \overline{\Omega}$ in post-processing.

5.1.1 Existence and Stability

We use sub and supersolution theory in the discrete setting to prove existence and stability results for solutions to 5.1. To begin with, we will define discrete sub and supersolutions of 5.1. We say $\underline{U}_h = (\underline{u}_{1,h}, \underline{u}_{2,h})$ is a subsolution of (5.1) if it satisfies the following:

$$\begin{cases} -\Delta_h \underline{u}_{1,h} + \underline{u}_{1,h} & \leq & 0 & \text{in } \mathcal{T}_h \cap \Omega, \\ -\Delta_h \underline{u}_{2,h} + \underline{u}_{2,h} & \leq & 0 & \text{in } \mathcal{T}_h \cap \Omega, \\ \nabla_h^* \underline{u}_{1,h} \cdot \widehat{n} - \lambda f_1(\underline{u}_{1,h}, \underline{u}_{2,h}) & \leq & 0 & \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}, \\ \nabla_h^* \underline{u}_{2,h}) \cdot \widehat{n} - \lambda f_2(\underline{u}_{1,h}, \underline{u}_{2,h}) & \leq & 0 & \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}. \end{cases} \tag{5.2}$$

We can define the supersolution $\overline{U}_h = (\overline{u}_{1,h}, \overline{u}_{2,h})$ by reversing the inequalities in (5.2). In this dissertation we focus on proving the existence and stability of nonnegative solutions to (5.1). Convergence results follow using the techniques described in [47] where the authors studied a similar boundary value problem with positone and semipositone nonlinearities.

Step 1: *Constructing discrete sub and supersolutions of* (5.1)

It is clear that $\phi \equiv 0$ is a subsolution of (5.1) since $f_i(0) \geq 0$. To construct a supersolution of (5.1), let us define $m_i = \frac{a_i+b_i}{2}$ for $i = 1, 2, \cdots, N$ to be the midpoint of the domain Ω along the x_i direction, and, for some $c \gg 1$ specified later, define the quadratic function $\phi : \overline{\Omega} \to \mathbb{R}$ as follows:

$$\phi(x) = c \sum_{i=1}^{N} [(x_i - m_i)^2 + 4].$$

Clearly, $\phi(x) > 4cN$ on $\overline{\Omega}$. Observe that $\frac{\partial^2}{\partial x_i^2}\phi(x) = 2c$. Since the operators $\delta^2_{x_i,h_i}$ are exact for quadratic functions, we have

$$\begin{aligned} -\Delta_h \phi(x) + \phi(x) &= -\Delta\phi(x) + \phi(x) \\ &= -2cN + c\sum_{i=1}^{N}(x_i - m_i)^2 + 4cN \\ &= 2cN + c\sum_{i=1}^{N}(x_i - m_i)^2 > 0 \end{aligned} \tag{5.3}$$

for all $x \in \mathcal{T}_h \cap \Omega$. Choose $x \in \mathcal{T}_h \cap \widetilde{\partial\Omega}$, and define $H_i = m_i - a_i = \frac{b_i - a_i}{2}$. Suppose $x_i = a_i$. Then $\widehat{n} = -\widehat{e}_i$, and by the convexity of ϕ, we have

$$\begin{aligned} \nabla_h^* \phi(x) \cdot \widehat{n} &= -\delta^+_{x_i,h_i}\phi(x) \\ &\geq -\delta^+_{x_i,H_i}\phi(x) \\ &= -\frac{\phi(x + H_i e_i) - \phi(x)}{H_i} \\ &= \frac{-c(a_i + H_i - m_i)^2 + 4 + c(a_i - m_i)^2 - 4}{H_i} \\ &= c\frac{b_i - a_i}{2}. \end{aligned}$$

Similarly, if $x_i = b_i$, then, $\widehat{n} = \widehat{e}_i$, and by the convexity of $\phi(x)$ there holds

$$\begin{aligned} \nabla_{h_i}^* \phi(x) \cdot \widehat{n} &= \delta^-_{x_i,h_i}\phi(x) \\ &\geq \delta^-_{x_i,H_i}\phi(x) \\ &= \frac{\phi(x) - \phi(x - H_i e_i)}{H_i} \\ &= \frac{c(b_i - m_i)^2 + 4 - c(b_i - H_i - m_i)^2 - 4}{H_i} \\ &= c\frac{b_i - a_i}{2}. \end{aligned}$$

Observe that $||(\phi(x),\phi(x))||_1 \geq 8cN \to \infty$ as $c \to \infty$. Furthermore, $||(\phi(x),\phi(x))||_1 \leq cM$ where $M = (\sum_{i=1}^{N} 4H_i^2 + 4N)$. Hence, by the sublinearity of f_i, $0 \leq \frac{f_i(\phi(x),\phi(x))}{||(\phi(x),\phi(x))||_1} \leq \frac{f_i(cM,cM)}{8cN} = \frac{M}{8N}.\frac{f_i(cM,cM)}{cM} \to 0$ as $c \to \infty$, since $\frac{M}{8N}$ is a constant independent of c. Therefore, there exists $c \gg 1$ such that

$$\frac{\nabla_h^*\phi(x)\cdot\widehat{n}}{(||(\phi(x),\phi(x))||_1)} \geq \frac{c(b_i-a_i)}{2(||(\phi(x),\phi(x))||_1)} \geq \frac{b_i-a_i}{(\sum_{i=1}^{N} 4H_i^2+4N)} > \frac{\lambda f_j(\phi(x),\phi(x))}{||(\phi(x),\phi(x))||_1} \tag{5.4}$$

for $j = 1, 2$. Thus, $\nabla_h^*\phi(x) \geq \lambda f_j(\phi(x),\phi(x))$ for all $x \in \widetilde{\partial\Omega}$ and for $j = 1, 2$. Finally, combining (5.3) and (5.4), we conclude that $(\phi(x),\phi(x))$ is a supersolution of (5.1).

Step 2: *Forming a monotone iteration*

Let us assume the reaction terms f_1, f_2 are Lipschitz continuous and let $U_h^{(0)}$ be a discrete supersolution of (5.1). Consider the fixed point iteration

$$U_h^{(n+1)} = \mathcal{M}_K U_h^{(n)} \tag{5.5}$$

for all $n \geq 0$, where K is the maximum of the Lipschitz constants for f_1, f_2 in (1.10) and $\mathcal{M}_K$ is defined such that

$$\begin{aligned}
-\Delta_h u_{1,h}^{(n+1)} + u_{1,h}^{(n+1)} + \lambda K u_{1,h}^{(n+1)} &= \lambda K u_{1,h}^{(n)} && \text{in } \mathcal{T}_h \cap \Omega,\\
-\Delta_h u_{2,h}^{(n+1)} + u_{2,h}^{(n+1)} + \lambda K u_{2,h}^{(n+1)} &= \lambda K u_{2,h}^{(n)} && \text{in } \mathcal{T}_h \cap \Omega,\\
\nabla_h^* \cdot \widehat{n} u_{1,h}^{(n+1)} + \lambda K u_{1,h}^{(n+1)} &= \lambda f_1(u_{1,h}^{(n)}, u_{2,h}^{(n)}) + \lambda K u_{1,h}^{(n)} && \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega},\\
\nabla_h^* \cdot \widehat{n} u_{2,h}^{(n+1)} + \lambda K u_{2,h}^{(n+1)} &= \lambda f_2(u_{1,h}^{(n)}, u_{2,h}^{(n)}) + \lambda K u_{2,h}^{(n)} && \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}.
\end{aligned} \tag{5.6}$$

Before we proceed with the following theorem, let us write the above mapping on grid functions as an equivalent matrix transformation for vectors. Let $J_0 = |\mathcal{T}_h \cap (\Omega \cup (\widetilde{\partial\Omega}))|$ and $\mathbf{U} \in \mathbb{R}^{2J_0}$ denote the vectorization of the grid function U_h. Notationally, the I subscript corresponds to grid function values in $\mathcal{T}_h \cap \Omega$ and the B

subscript corresponds to grid function values in $\mathcal{T}_h \cap \widetilde{\partial\Omega}$. Then, (5.6) is equivalent to

$$M\mathbf{U}^{(n+1)} = \lambda \mathbf{F}(\mathbf{U}^{(n)}), \tag{5.7}$$

where $M = \begin{bmatrix} L_I & \underline{\underline{0}} & L_B & \underline{\underline{0}} \\ \underline{\underline{0}} & L_I & \underline{\underline{0}} & L_B \\ B_I & \underline{\underline{0}} & B_B & \underline{\underline{0}} \\ \underline{\underline{0}} & B_I & \underline{\underline{0}} & B_B \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} \mathbf{u_{1,I}} \\ \mathbf{u_{2,I}} \\ \mathbf{u_{1,B}} \\ \mathbf{u_{2,B}} \end{bmatrix}$, and $\mathbf{F} = \begin{bmatrix} \mathbf{F_{1,I}} \\ \mathbf{F_{2,I}} \\ \mathbf{F_{1,B}} \\ \mathbf{F_{2,B}} \end{bmatrix}$,

for L_I and L_B are matrices corresponding to $-\Delta_h + (1 + \lambda K)I$, B_I and B_B are matrices corresponding to $\nabla_h^* \cdot \widehat{n} + \lambda K I$, $\mathbf{F_{i,I}}$ corresponding to λK for $i = 1, 2$, and $\mathbf{F_{i,B}}$ corresponding to $\lambda f_i(\mathbf{u_1}, \mathbf{u_2}) + \lambda K$ for $i = 1, 2$. Clearly, M is diagonally dominant since it is positive on the diagonal, negative for all off-diagonal terms, and the row sum is positive. Hence, M is a Z-matrix, and by the Gershgorin Circle Theorem, M is non-singular since the real part of all of its eigenvalues are always positive. Therefore, it follows that M is a monotone matrix.

Remark 5.2. Note that the iteration (5.6) is well defined. Also notice that $f_i(s_1, s_2) + Ks_i$ is increasing in s_1 and s_2 for $i = 1, 2$.

Theorem 5.1. *Let $U_h = (u_{1,h}, u_{2,h})$ be a subsolution of* (5.1) *and $U_h^{(0)} = (u_{1,h}^{(0)}, u_{2,h}^{(0)})$ be a supersolution of* (5.1) *such that $U_h^{(0)} \geq U_h$. Then, $U_h^{(1)} = (u_{1,h}^{(1)}, u_{2,h}^{(1)}) = \mathcal{M}_K U_h^{(0)}$ is a supersolution of* (5.1) *with $U_h \leq U_h^{(1)} \leq U_h^{(0)}$.*

Proof. Observe that, by the definition of $\mathcal{M}_K$ described in (5.6), for $i = 1, 2$ in $\mathcal{T}_h \cap \Omega$, we have,

$$\begin{aligned} -\Delta_h u_{i,h}^{(1)} + u_{i,h}^{(1)} + \lambda K u_{i,h}^{(1)} &= \lambda K u_{i,h}^{(0)} \\ &\geq \lambda K u_{i,h} \\ &\geq -\Delta_h u_{i,h} + u_{i,h} + \lambda K u_{i,h}. \end{aligned} \tag{5.8}$$

Also, on $\mathcal{T}_h \cup \widetilde{\partial\Omega}$, we get

$$\begin{aligned}
\nabla_h^* \cdot \widehat{n} u_{1,h}^{(1)} + \lambda K u_{1,h}^{(1)} &= \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{1,h}^{(0)} \\
&\geq \lambda f_1(u_{1,h}^{(0)}, u_{2,h}) + \lambda K u_{1,h}^{(0)} \\
&\geq \lambda f_1(u_{1,h}, u_{2,h}) + \lambda K u_{1,h} \\
&\geq \nabla_h^* \cdot \widehat{n} u_{1,h} + \lambda K u_{1,h}
\end{aligned}$$

by the quasimonotonicity of f_i and Remark 5.2. Similarly,

$$\begin{aligned}
\nabla_h^* \cdot \widehat{n} u_{2,h}^{(1)} + \lambda K u_{2,h}^{(1)} &= \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{2,h}^{(0)} \\
&\geq \lambda f_1(u_{1,h}, u_{2,h}^{(0)}) + \lambda K u_{2,h}^{(0)} \\
&\geq \lambda f_1(u_{1,h}, u_{2,h}) + \lambda K u_{2,h} \\
&\geq \nabla_h^* \cdot \widehat{n} u_{2,h} + \lambda K u_{2,h}.
\end{aligned}$$

Hence $M\mathbf{U}^{(1)} \geq M\mathbf{U}$, and it follows that $\mathbf{U}^{(1)} \geq \mathbf{U}$. Thus $U_h^{(1)} \geq U_h$.

Next, for $i = 1, 2$,

$$\begin{aligned}
-\Delta_h u_{i,h}^{(1)} + u_{i,h}^{(1)} + \lambda K u_{i,h}^{(1)} &= \lambda K u_{i,h}^{(0)} \\
&= \lambda K u_{i,h}^{(0)} + 0 \\
&\leq -\Delta_h u_{i,h}^{(0)} + u_{i,h}^{(0)} + \lambda K u_{i,h}^{(0)} \text{ in } \mathcal{T}_h \cap \Omega
\end{aligned}$$

since $U_h^{(0)}$ is a nonnegative supersolution of (5.1). Furthermore,

$$\begin{aligned}
\nabla_h^* \cdot \widehat{n} u_{i,h}^{(1)} + \lambda K u_{i,h}^{(1)} &= \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{i,h}^{(0)} \\
&\leq \nabla_h^* \cdot \widehat{n} u_{i,h}^{(0)} + \lambda K u_{i,h}^{(0)} \text{ on } \mathcal{T}_h \cap \widetilde{\partial\Omega}
\end{aligned}$$

since $U_h^{(0)}$ is a nonnegative supersolution of (5.1). Hence, $M\mathbf{U}^{(1)} \leq M\mathbf{U}^{(0)}$ which implies $\mathbf{U}^{(1)} \leq \mathbf{U}^{(0)}$. Thus, $U_h^{(1)} \leq U_h^{(0)}$.

Since, $U_h^{(1)} \leq U^{(0)}$, (5.8) implies

$$-\Delta_h u_{i,h}^{(1)} + u_{i,h}^{(1)} = \lambda K u_{i,h}^{(0)} - \lambda K u_{i,h}^{(1)} \geq 0 \text{ in } \mathcal{T}_h \cap \Omega$$

for $i = 1, 2$.

Also, from the definition of $\mathcal{M}_K$ combined with the quasimonotonicity of f_i and Remark 5.2, we get

$$\begin{aligned}\nabla_h^* \cdot \widehat{n} u_{1,h}^{(1)} + \lambda K u_{1,h}^{(1)} =& \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{1,h}^{(0)} \\ &\geq f_1(u_{1,h}^{(1)}, u_{2,h}^{(0)}) + \lambda K u_{1,h}^{(1)} \\ &\geq f_1(u_{1,h}^{(1)}, u_{2,h}^{(1)}) + \lambda K u_{1,h}^{(1)} \text{ on } \mathcal{T}_h \cap \widetilde{\partial\Omega}.\end{aligned}$$

Thus

$$\nabla_h^* \cdot \widehat{n} u_{1,h}^{(1)} \geq f_1(u_{1,h}^{(1)}, u_{2,h}^{(1)}) \text{ on } \mathcal{T}_h \cap \widetilde{\partial\Omega}.$$

Similarly,

$$\begin{aligned}\nabla_h^* \cdot \widehat{n} u_{2,h}^{(1)} + \lambda K u_{2,h}^{(1)} &= \lambda f_2(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{2,h}^{(0)} \\ &\geq f_2(u_{1,h}^{(0)}, u_{2,h}^{(1)}) + \lambda K u_{2,h}^{(1)} \\ &\geq f_1(u_{1,h}^{(1)}, u_{2,h}^{(1)}) + \lambda K u_{2,h}^{(1)} \text{ on } \mathcal{T}_h \cap \widetilde{\partial\Omega}.\end{aligned}$$

Thus

$$\nabla_h^* \cdot \widehat{n} u_{2,h}^{(1)} \geq f_2(u_{1,h}^{(1)}, u_{2,h}^{(1)}) \text{ on } \mathcal{T}_h \cap \widetilde{\partial\Omega}.$$

Therefore, $U_h^{(1)}$ is a supersolution of (5.1). □

Step 3 *Existence of a fixed point*

Theorem 5.2. *Let U_h be a subsolution to* (5.1) *and $U_h^{(0)} \geq U_h$ be a supersolution of* (5.1)*. Then the sequence $U_h^{(n)}$ defined by* (5.6) *converges to a solution of* (5.1)*.*

Proof. Observe that, by Theorem 5.1, we have

$$0 \leq U_h \leq U_h^{(n+1)} \leq U_h^{(n)} \leq U_h^{(n-1)} \leq \cdots \leq U_h^{(1)} \leq U_h^{(0)}$$

for all $n \geq 1$. Thus, the sequence $\{U_h^{(n)}\}_{n=0}^{\infty}$ is convergent since it is monotone and bounded. Suppose there exists $V_h : [\widetilde{\mathcal{T}}_h]^2 \to \mathbb{R}^2$ such that $U_h^{(n)} \to V_h$ in $l^\infty([\widetilde{\mathcal{T}}_h]^2)$ for $\widetilde{\mathcal{T}}_h = \mathcal{T}_h \cap (\Omega \cup \widetilde{\partial\Omega})$. Clearly, V_h is a fixed point of (5.5). Thus V_h is a solution of (5.1) with

$$0 \leq U_h \leq V_h \leq U_h^{(n+1)} \leq U_h^{(n)} \leq U_h^{(n-1)} \leq \cdots \leq U_h^{(1)} \leq U_h^{(0)}.$$

□

Remark 5.3. Note that V_h is non-negative and uniformly bounded independent of h. Hence, V_h is the stable approximation of a solution to (1.10) independent of h.

5.1.2 Example 1

In this example, we consider the following nonlinearities

$$\begin{cases} f_1(u_1,u_2) &= 100\sqrt{u_2} - 2\sqrt{u_1}, \\ f_2(u_1,u_2) &= 4\sqrt{u_1} + \sqrt{u_2}. \end{cases} \tag{5.9}$$

Clearly the f_i's are Caratheódory, quasimonotone non decreasing and sublinear functions satisfying the assumption **C1** in Theorem 1.5. Moreover, $f_i(0,0) = 0$, $\frac{\partial f_1}{\partial u_2} = \infty$ and $\frac{\partial f_2}{\partial u_1} = \infty$. A solution can be found in Figure 5.1 for $\lambda = 0.5$ and $\lambda = 3$ and computed bifurcation diagrams can be found in Figure 5.2.

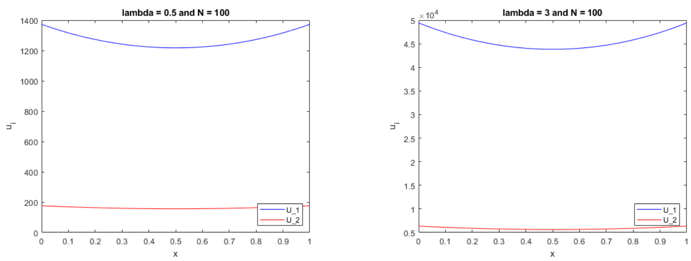
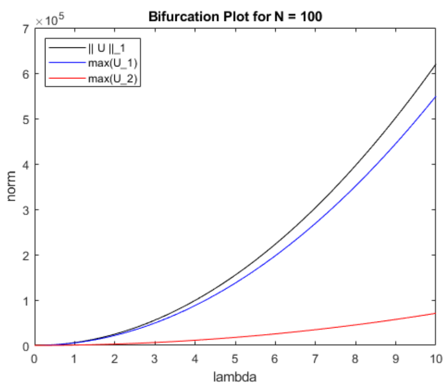
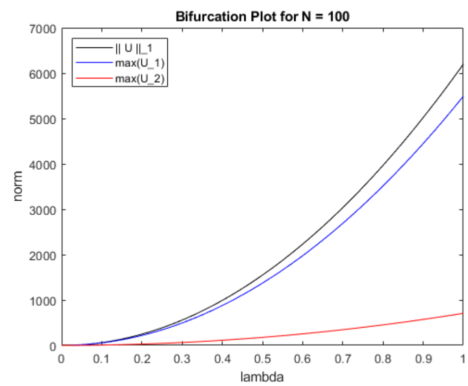


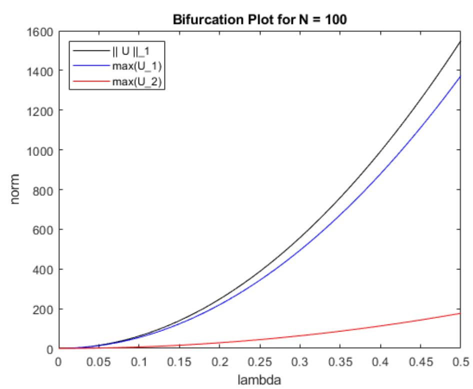
Figure 5.1. Graph of U_1 and U_2 for $\lambda = 0.5$ and $\lambda = 3$



(a) Bifurcation diagram when $\lambda \in (0, 10)$.



(b) Bifurcation diagram when $\lambda \in (0, 1)$.



(c) Bifurcation diagram when $\lambda \in (0, 0.5)$.

Figure 5.2. Bifurcation diagrams of (1.10) for different ranges of λ when $f_1(u_1, u_2) = 100\sqrt{u_2} - 2\sqrt{u_1}, f_2(u_1, u_2) = 4\sqrt{u_1} + \sqrt{u_2}$.

5.1.3 Example 2

In this example, we consider the following nonlinearities

$$\begin{cases} f_1(u_1, u_2) &= \arctan(u_2) \\ f_2(u_1, u_2) &= 3\arctan(u_1) \end{cases} \tag{5.10}$$

Clearly f_i's are Caratheodory, quasimonotone non decreasing and sublinear functions satisfying the assumption **C1** in Theorem 1.5. Moreover, $f_i(0,0) = 0$, $\frac{\partial f_1}{\partial u_2} > 0$ and $\frac{\partial f_2}{\partial u_1} > 0$. A solution can be found in Figure 5.3 for $\lambda = 0.5$ and $\lambda = 3$ and computed bifurcation diagrams can be found in Figure 5.4. We see nonexistence of positive solution for λ small.

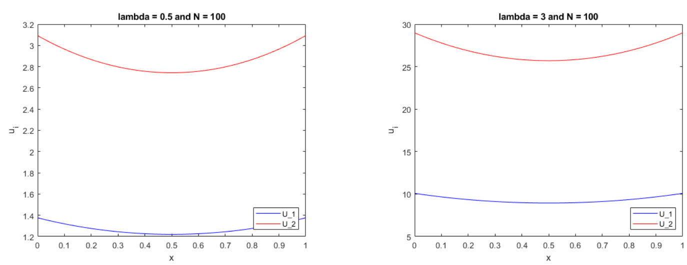
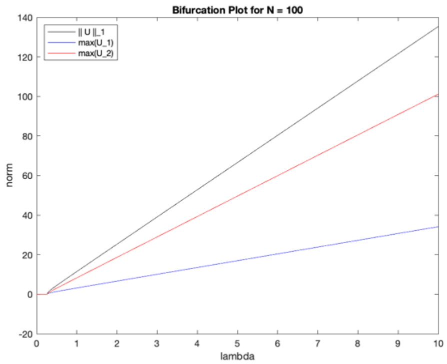
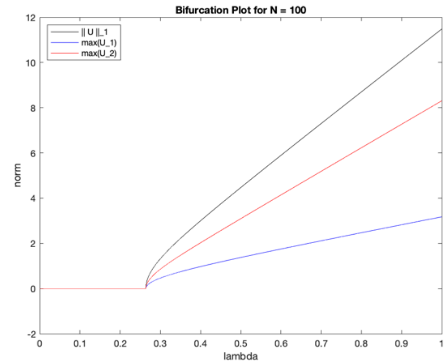


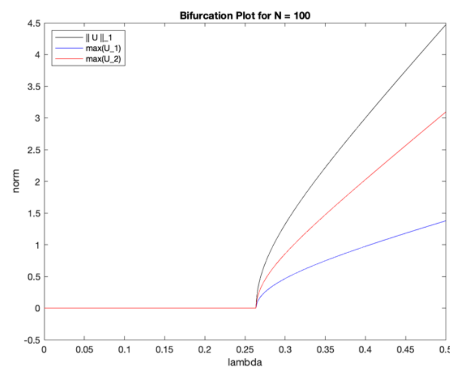
Figure 5.3. Graph of U_1 and U_2 for $\lambda = 0.5$ and $\lambda = 3$.



(a) Bifurcation diagram when $\lambda \in (0, 10)$.



(b) Bifurcation diagram when $\lambda \in (0, 1)$.



(c) Bifurcation diagram when $\lambda \in (0, 0.5)$.

Figure 5.4. Bifurcation diagrams of (1.10) for different ranges of λ when $f_1(u_1, u_2) = \arctan(u_2)$, $f_2(u_1, u_2) = 3\arctan(u_1)$.

5.1.4 Example 3

In this example, we consider the following nonlinearities

$$\begin{cases} f_1(u_1, u_2) &= e^{\frac{10u_1}{1+u_1^2}} - 1 + 5\sqrt[3]{u_2^2 + 1} \\ f_2(u_1, u_2) &= e^{\frac{u_2}{1+u_2^2}} - 1 \end{cases}$$

Clearly f_i's are Caratheodory, quasimonotone non decreasing and sublinear functions satisfying the assumption **C1** in Theorem 1.5. Moreover, $f_i(0, 0) = 0$, $\frac{\partial f_1}{\partial u_2} = \infty$ and

$\frac{\partial f_2}{\partial u_1} = 0$. A solution can be found in Figure 5.5 for $\lambda = 0.5$ and $\lambda = 3$ and computed bifurcation diagrams can be found in Figure 5.6. Also in Figure 5.6, notice that there exists a range of λ for which we see non coexistence of the positive solutions. Our method and monotone solver found the maximal solution based on our supersolution. Observe that, in Subfigure 5.6b and 5.6c the vertical lines are jump discontinuity. We used the method of continuation based on a good initial guess to find the other branches (see Figure 5.6).

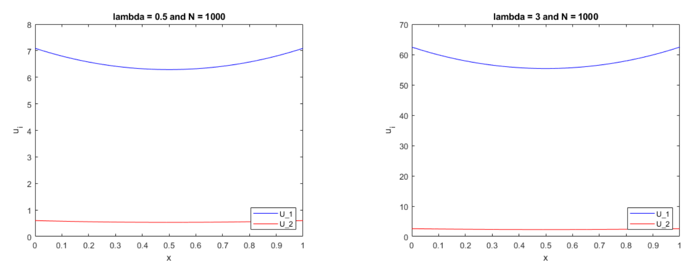
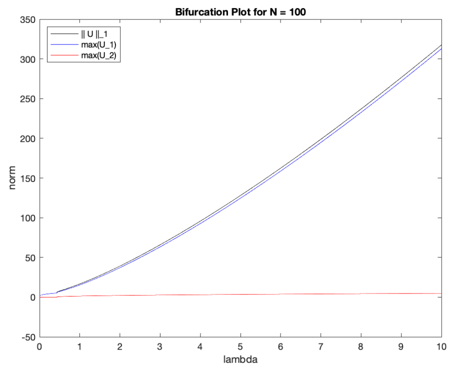
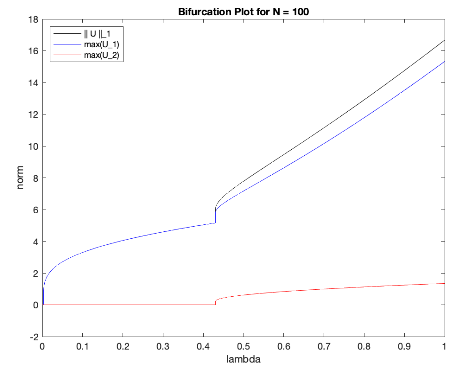


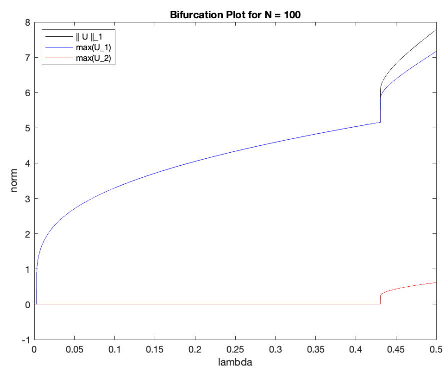
Figure 5.5. Graph of U_1 and U_2 for $\lambda = 0.5$ and $\lambda = 3$.



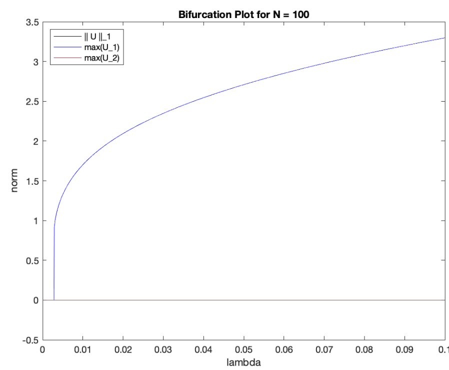
(a) Bifurcation diagram when $\lambda \in (0, 10)$.



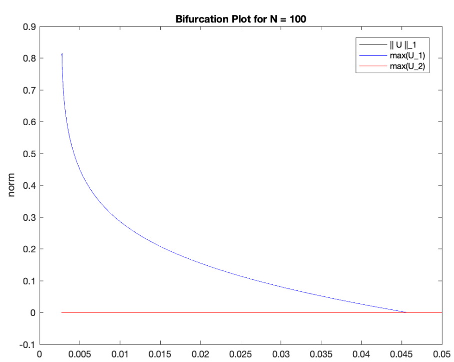
(b) Bifurcation diagram when $\lambda \in (0, 1)$.



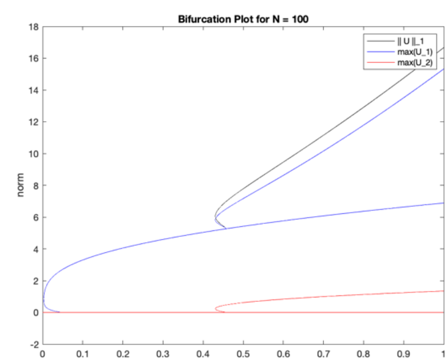
(c) Bifurcation diagram when $\lambda \in (0, 0.5)$.



(d) No solution for very small λ



(e) Second branch using continuation



(f) No co-existence for small λ

Figure 5.6. Bifurcation diagrams of (1.10) for different ranges of λ when $f_1(u_1, u_2) = e^{\frac{10u_1}{1+u_1^2}} - 1 + 5\sqrt[3]{u_2^2 + 1}$, $f_2(u_1, u_2) = e^{\frac{u_2}{1+u_2^2}} - 1$.

5.1.5 Example 4

In this example, we consider the following nonlinearities

$$\begin{cases} f_1(u_1,u_2) &=& e^{\frac{u_1}{1+u_1^2}} - 1 + \frac{1}{3}\sqrt[3]{u_2^2+1} \\ f_2(u_1,u_2) &=& \arctan(u_2) + e^{\frac{u_2}{1+u_2^2}} - 1 \end{cases}$$

Clearly f_i's are Caratheodory, quasimonotone non decreasing and sublinear functions satisfying the assumption **C1** in Theorem 1.5. Moreover, $f_i(0,0) = 0$, $\frac{\partial f_1}{\partial u_2} = \infty$ and $\frac{\partial f_2}{\partial u_1} > 0$. A solution can be found in Figure 5.7 for $\lambda = 0.5$ and $\lambda = 3$ and computed bifurcation diagrams can be found in Figure 5.8. In Figure 5.7, notice that for $\lambda = 0.5$, $u_2 > u_1$ whereas for $\lambda = 3$, $u_1 > u_2$.

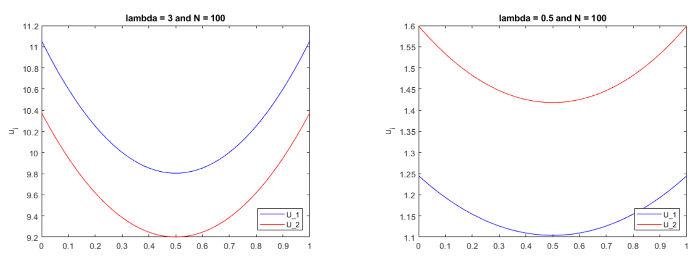
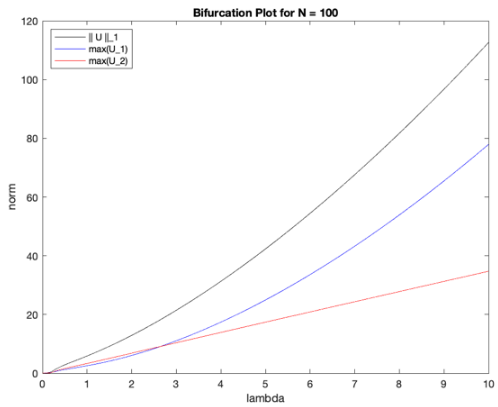
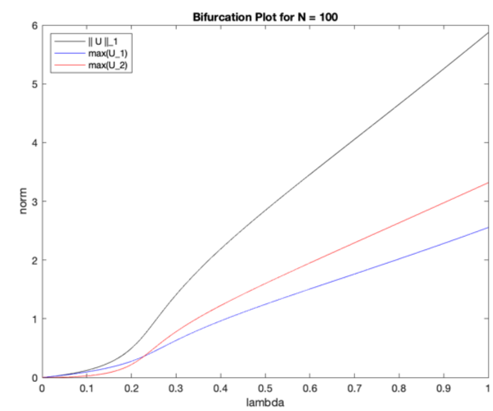


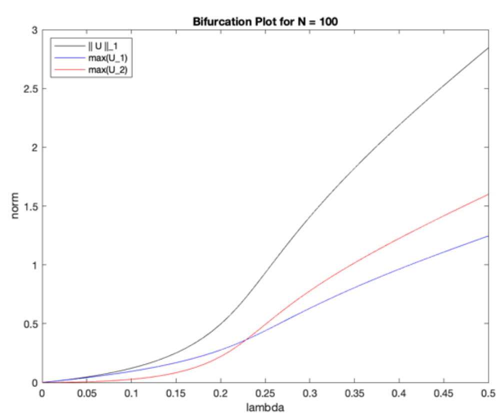
Figure 5.7. Graph of U_1 and U_2 for $\lambda = 0.5$ and $\lambda = 3$.



(a) Bifurcation diagram when $\lambda \in (0, 10)$.



(b) Bifurcation diagram when $\lambda \in (0, 1)$.



(c) Bifurcation diagram when $\lambda \in (0, 0.5)$.

Figure 5.8. Bifurcation diagrams of (1.10) for different ranges of λ when $f_1(u_1, u_2) = e^{\frac{u_1}{1+u_1^2}} - 1 + \frac{1}{3}\sqrt[3]{u_2^2 + 1}$, $f_2(u_1, u_2) = \arctan(u_2) + e^{\frac{u_2}{1+u_2^2}} - 1$.

5.2 Superlinear Problem Bifurcation Diagrams and the Shape of Solutions

In this section, we use the following discrete problem to approximate solutions to (1.2), where the grid function u_h is an approximation for u over the grid $\mathcal{T}_h$:

$$\begin{cases} -\Delta_h u_h + u_h &= 0 \quad \text{in } \mathcal{T}_h \cap \Omega. \\ \nabla_h^* u_h \cdot \widehat{n} - \lambda f(u_h) &= 0 \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}, \end{cases} \tag{5.11}$$

We use `fsolve` in MATLAB. From the theoretical results, there is a bifurcation of positive solution from infinity at $\lambda = 0$. We start with a large constant value for the initial guess when solving for a small λ value. We then use the method of continuation as we move λ along the positive real line. We also make sure the minimum of the solution u_h is positive so that we can guarantee `fsolve` only finds positive solutions. Once we find a solution we plot $\max\{|u_h|\}$ vs λ to approximate the $\|u\|_\infty$ vs λ bifurcation curve for (1.8).

Next, we provide the computed bifurcation diagrams for positive solutions of (1.8) in the one dimensional setting in order to validate and gain intuition for our theoretical results proposed in Section 1.1.2. In the following figures we see positive solutions bifurcating from infinity at $\lambda = 0$ whenever the nonlinearity f satisfies $(H)_\infty$, verifying Theorem 1.8. Also, positive solutions bifurcate from $\frac{\mu_1}{f'(0)}$ whenever f satisfies $(H)_0$. Whether the bifurcation curve near the bifurcation point $\frac{\mu_1}{f'(0)}$ is supercritical or subcritical depends on the choice of f, as proposed in Theorem 1.10. We provide bifurcation diagrams for several examples of nonlinearities f that satisfy the hypotheses of Theorem 1.8 and Theorem 1.11. We also graph various positive solutions and record $\|\cdot\|_\infty$.

Remark 5.4. Note that the bifurcation diagrams provided below are approximate.

Hence, the bifurcation point from the trivial solution is an approximation and may not match with the exact values which we obtained in our theoretical study.

5.2.1 Bifurcation of Positive Solutions for $f(s) = s^2$

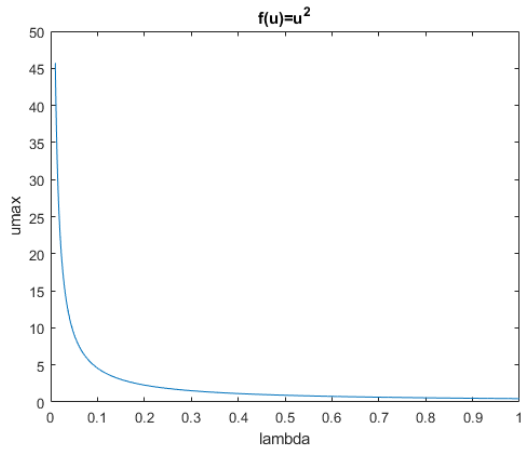


Figure 5.9. Bifurcation diagram for (1.8) with $f(s) = s^2$

It follows from a simple re-scaling argument that (1.8) has a positive solution for each $\lambda > 0$ consistent with the bifurcation curve in Figure 5.9.

5.2.2 Bifurcation of Positive solutions for $f(s) = s + s^2$

In this case $f'(0) = 1$. Hence the branch of positive solutions bifurcates from the trivial solution at the point $\mu_1 \approx 0.45$ and $\mathscr{R}_0 = 1 > 0$; hence, by Theorem 1.10 there is a positive solution for each $\lambda \in (0, \mu_1)$, i.e. the bifurcation from the trivial solution at $\lambda = 0.45$ is subcritical. The computational results are in Figure 5.10.

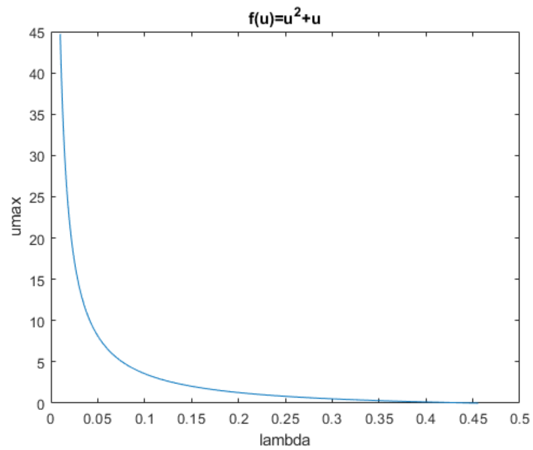


Figure 5.10. Bifurcation diagram for (1.8) with $f(s) = s^2 + s$

5.2.3 Bifurcation of Positive Solutions for $f(s) = 2s + s^2$

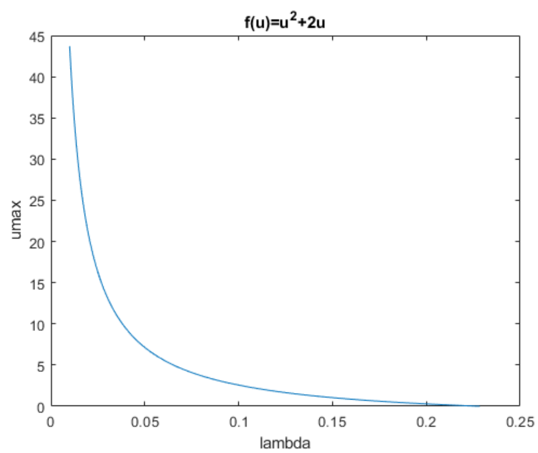


Figure 5.11. Bifurcation diagram for (1.8) with $f(s) = s^2 + 2s$

In this case $f'(0) = 2$. Hence the branch of positive solutions bifurcates from the trivial solution at the point $\frac{\mu_1}{2} \approx \frac{0.45}{2} = 0.225$ and $\mathcal{R}_0 = 1 > 0$. Hence, by Theorem 1.10, there is a positive solution for each $\lambda \in (0, \frac{\mu_1}{2})$, i.e., the bifurcation from the trivial solution at $\lambda = 0.225$ is subcritical (see Figure 5.11).

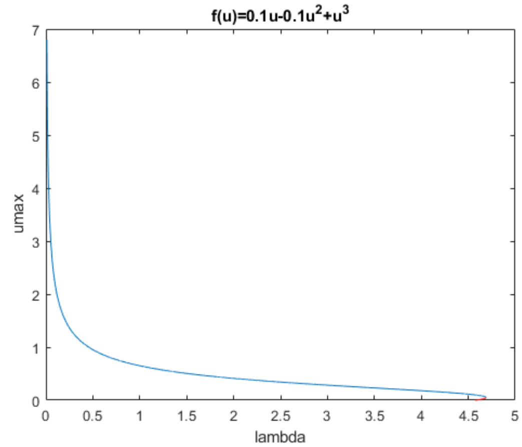


Figure 5.12. Bifurcation diagrams for (1.8) with $f(s) = 0.1s - 0.1s^2 + s^3$

5.2.4 Bifurcation of Positive Solutions for $f(s) = 0.1s - 0.1s^2 + s^3$

In this case $f'(0) = 0.1$. Hence, the branch of positive solutions bifurcates from the trivial solution at the point $\frac{\mu_1}{0.1} = \frac{0.45}{0.1} \approx 4.5$. Further, since $\mathscr{R}_0 = -0.1 < 0$, and the bifurcation in Figure 5.12 is supercritical, this confirms Theorem 1.10. We see at least one positive solution for each $\lambda \in (0, \overline{\lambda})$, where $\overline{\lambda} > \mu_1$, at least two solutions for $\lambda \in (\mu_1, \overline{\lambda})$ and at least one solution for $\lambda = \mu_1$ and for $\lambda = \overline{\lambda}$. This validates the conclusions of Theorem 1.11. Figure 5.12 shows the corresponding bifurcation diagram and Figure 5.13 gives a closer look near the bifurcation point and the turning point clearly showing two distinct solutions for $\lambda \in (\mu_1, \overline{\lambda})$.

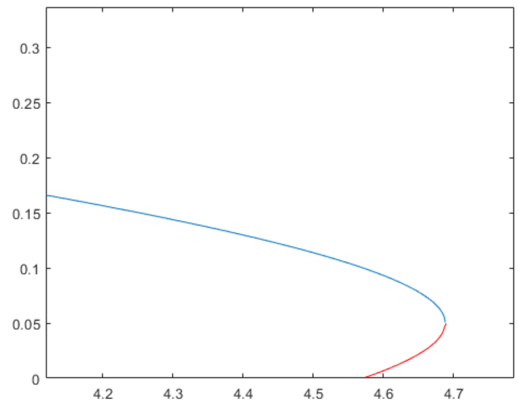


Figure 5.13. A closer look at Figure 5.12 near the bifurcation and turning point

5.2.5 Shape of The Solution u in Single Equation Case

We now analyze the shape of positive solutions of (1.8), i.e. when$u_h > 0$. Say $u_h(x_j)$ is a local maximum of u_h with $u_h(x_j) > 0$. Suppose $u_h(x_j)$ is in the interior of (a, b), i.e. $j \in \{2, 3, \cdots, N-1\}$. Now, since $u_h(x_j)$ is a maximum, we have $u_h(x_j) = \frac{1}{h^2}[u_{j-1} - 2u_j + u_{j+1}] \leq 0$. Thus $-\Delta_h u_h(x_j) + u_h(x_j) \geq u_h(x_j) > 0$. Hence, $x_j \in \mathcal{T}_h \cap \widetilde{\partial\Omega}$ which justifies the shape of the solution to be concave up. We plot solutions of (1.8) with $f(s) = s^2 + 2s$ and various λ values in Figure 5.14, Figures 5.15, and Figures 5.16. We see the graph is concave up with the maximum values along the boundary. We also see that the maximum of the solution decreases as λ increases.

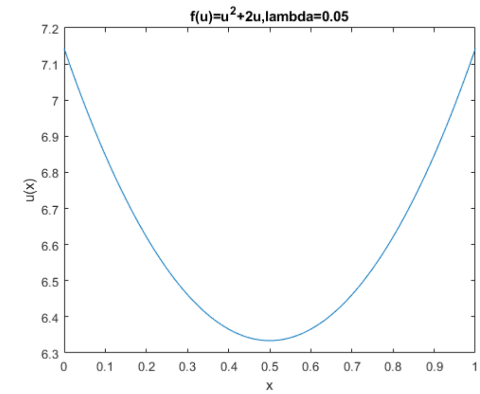


Figure 5.14. The solution of (1.8) with $||u_h||_\infty \approx 7.15$ for $\lambda = 0.05$ and $f(s) = s^2 + 2s$

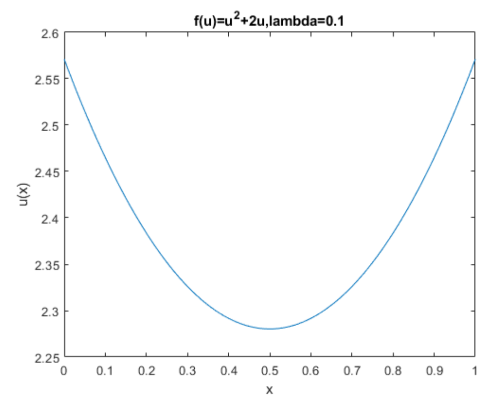


Figure 5.15. The solution of (1.8) with $||u||_\infty \approx 2.556$ for $\lambda = 0.1$ and $f(s) = s^2 + 2s$

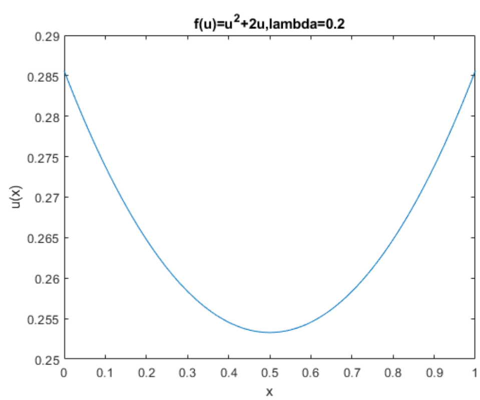


Figure 5.16. The solution of (1.8) with $||u||_\infty \approx 0.285$ for $\lambda = 0.2$ and $f(s) = s^2 + 2s$

Chapter 6: Mathematical Ecology

6.1 Computational Results when $N = 1$ and $\Omega = (0, 1)$

In this chapter we study a mathematical model arising in ecology. First we perform numerical simulations for the model in the one dimensional setting. Then motivated by the bifurcation diagrams obtained, we also prove several analytical results. In this section, we will study and provide complete bifurcation diagrams of positive solutions of (1.12) in the case when $N = 1$, namely when $\Omega = (0, 1)$, by using the combination of the quadrature and the shooting method described in Chapter 2. First we obtain **computational results 1.1-1.4**, for the case $h(s, \epsilon) = 1 + \epsilon s$ (competition on the boundary), when $E_1(r, \gamma_2) \leq E_1(1, \gamma_1)$ and $E_1(1, \gamma_1) < E_1(r, \gamma_2)$, respectively. Then we obtain **computational results 1.5-1.9**, for the case $h(s, \epsilon) = \frac{1}{1+\epsilon s}$ (cooperation on the boundary), when $E_1(1, \gamma_1) < E_1(r, \gamma_2)$ and $E_1(1, \gamma_1) < E_1(r, \gamma_2)$, respectively.

6.1.1 Case I: $h(s, \epsilon) = 1 + \epsilon s$

Case IA: $E_1(r, \gamma_2) \leq E_1(1, \gamma_1)$

Computational Result 1.1. (when $0 < b < 1$)

Part (a)

There exists $\epsilon_1(b) > 0$ such that for $\epsilon \in (0, \epsilon_1(b))$, (1.14) has a unique positive solution v_λ for $\lambda > E_1(r, \gamma_2)$, where $||v_\lambda||_\infty$ increases for $\lambda \in (E_1(r, \gamma_2), E_1(1, \gamma_1))$ and

$||v_\lambda||_\infty \to 1-b$ as $\lambda \to \infty$. Further, there exist $b^*(\gamma_1, \gamma_2, r) \in (0,1)$ (also see [1] for existence of this $b^*(\gamma_1, \gamma_2, r)$) and $\epsilon_2(b) \in (0, \epsilon_1(b))$ for $b \neq b^*(\gamma_1, \gamma_2, r)$ such that

(i) if $b < b^*(\gamma_1, \gamma_2, r)$ and $\epsilon < \epsilon_2(b)$, then $||v_\lambda||_\infty$ increases for $\lambda > E_1(1, \gamma_1)$

(ii) if $b > b^*(\gamma_1, \gamma_2, r)$ and $\epsilon < \epsilon_2(b)$, then $||v_\lambda||_\infty$ decreases for $\lambda > E_1(1, \gamma_1)$

(iii) if $b \neq b^*(\gamma_1, \gamma_2, r)$ and $\epsilon \in [\epsilon_2(b), \epsilon_1(b))$, then there exists $\underline{\lambda}(b, \epsilon) > E_1(1, \gamma_1)$ such that $||v_\lambda||_\infty$ decreases for $\lambda \in (E_1(1, \gamma_1), \underline{\lambda}(b, \epsilon))$ and increases for $\lambda \in (\underline{\lambda}(b, \epsilon), \infty)$

(iv) if $b = b^*(\gamma_1, \gamma_2, r)$ and $\epsilon \in (0, \epsilon_1(b))$, then there exists $\underline{\lambda}(b, \epsilon) > E_1(1, \gamma_1)$ such that $||v_\lambda||_\infty$ decreases for $\lambda \in (E_1(1, \gamma_1), \underline{\lambda}(b, \epsilon))$ and increases for $\lambda \in (\underline{\lambda}(b, \epsilon), \infty)$.

Part (b)

For $\epsilon \geq \epsilon_1(b)$ there exist $\lambda_d(b, \epsilon), \lambda_D(b, \epsilon) > E_1(1, \gamma_1)$ with $\lambda_d(b, \epsilon) \leq \lambda_D(b, \epsilon)$ such that (1.14) has a unique positive solution v_λ for $\lambda \in (E_1(r, \gamma_2), \lambda_d(b, \epsilon)) \cup (\lambda_D(b, \epsilon), \infty)$ and $||v_\lambda||_\infty \to 1-b$ as $\lambda \to \infty$. Further $||v_\lambda||_\infty$ increases in $(E_1(r, \gamma_2), E_1(1, \gamma_1)) \cup (\lambda_D(b, \epsilon), \infty)$ and decreases in $(E_1(1, \gamma_1), \lambda_d(b, \epsilon))$. Also, (1.14) has no positive solution for $\lambda \in [\lambda_d(b, \epsilon), \lambda_D(b, \epsilon)]$ (See Figure 6.1).

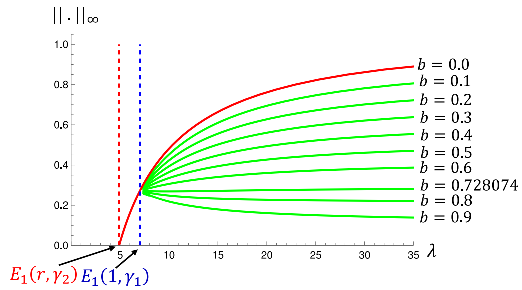
Furthermore,

(i) $\lambda_d(b, \epsilon)$ decreases in ϵ and $\lambda_d(b, \epsilon) \to E_1(1, \gamma_1)$ as $\epsilon \to \infty$

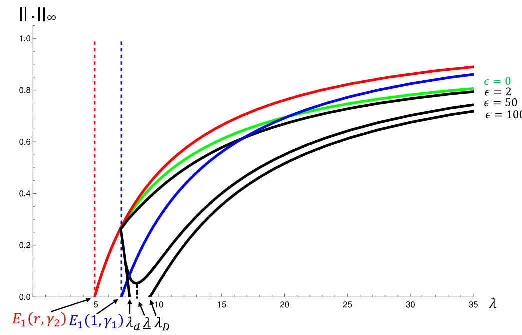
(ii) $\lambda_D(b, \epsilon)$ increases in ϵ and $\lambda_D(b, \epsilon) \to \tilde{\lambda}(b) = \frac{\pi^2}{r(1-b)}$ as $\epsilon \to \infty$ (See Figure 6.2).

Remark 6.1. Note that $\tilde{\lambda}(b) = \frac{\pi^2}{r(1-b)}$ is the principal eigenvalue of the problem

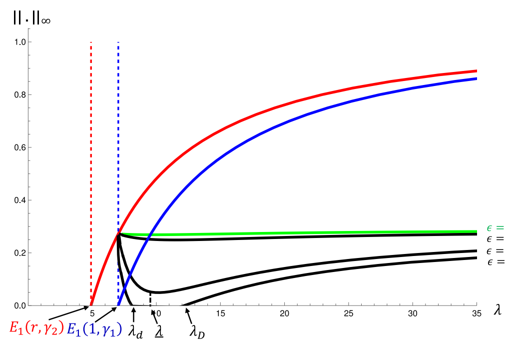
$$\begin{cases} -v'' &= \lambda r(1-b)v \quad \text{in } (0,1) \\ v(0) &= 0 \\ v(1) &= 0. \end{cases}$$



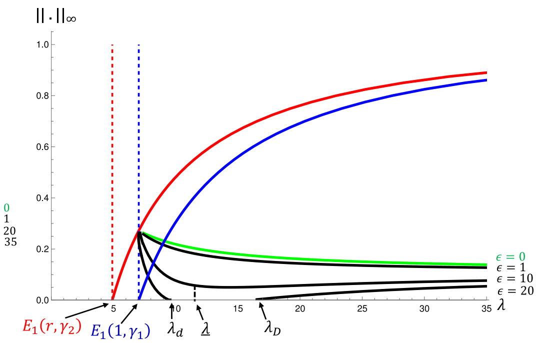
(a) $b \in [0, 1)$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $\epsilon = 0$, from [1].



(b) $b = 0.1$ $(< b^*)$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $\epsilon > 0$.



(c) $b = 0.728074$ $(= b^*)$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $\epsilon > 0$.



(d) $b = 0.9$ $(> b^*)$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$ and $\epsilon > 0$.

Figure 6.1. Typical bifurcation diagrams of (1.14) for different values of $b < 1$ and $\epsilon \geq 0$ when $h(s, \epsilon) = 1 + \epsilon s$

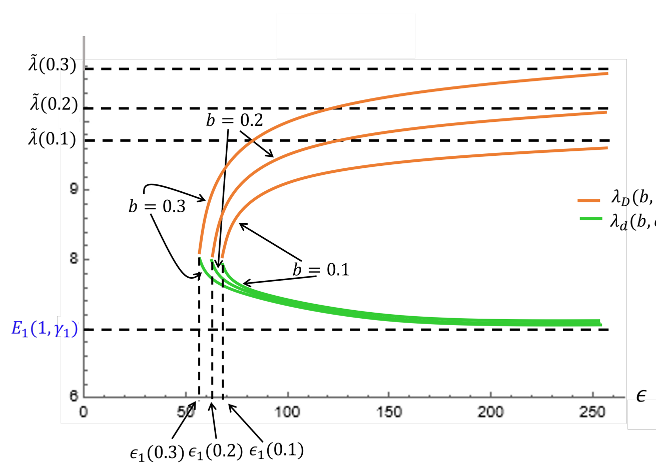


Figure 6.2. Evolution of λ_d and λ_D with respect to $\epsilon \gg 1$ and b when $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $h(s, \epsilon) = 1 + \epsilon s$.

Computational Result 1.2. (when $b \geq 1$)

There exists $\lambda_{max}(b,\epsilon) > E_1(1,\gamma_1)$ such that (1.14) has a unique positive solution v_λ for $\lambda \in (E_1(r,\gamma_2), \lambda_{max}(b,\epsilon))$ and has no positive solution for $\lambda \in [\lambda_{max}(b,\epsilon), \infty)$. Further $||v_\lambda||_\infty$ increases in $(E_1(r,\gamma_2), E_1(1,\gamma_1))$ and decreases in $(E_1(1,\gamma_1), \lambda_{max}(b,\epsilon))$ (see Figure 6.3).

Furthermore,

(i) $||v_\lambda||_\infty \to 0$ as $\lambda \to \lambda_{max}(b,\epsilon)$

(ii) $\lambda_{max}(b,\epsilon)$ decreases in ϵ and $\lambda_{max}(b,\epsilon) \to E_1(1,\gamma_1)$ as $\epsilon \to \infty$ (see Figure 6.4).

Remark 6.2. *If $b = 1$ then $\lambda_{max}(b,\epsilon) \to \infty$ as $\epsilon \to 0$.*

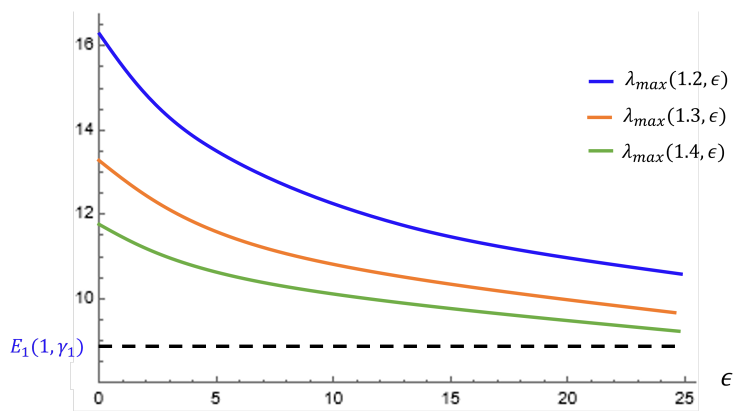


Figure 6.4. Evolution of λ_{max} with respect to ϵ and b when $h(s,\epsilon) = 1 + \epsilon s$, $r = 1, \gamma_1 = 4, \gamma_2 = 2$.

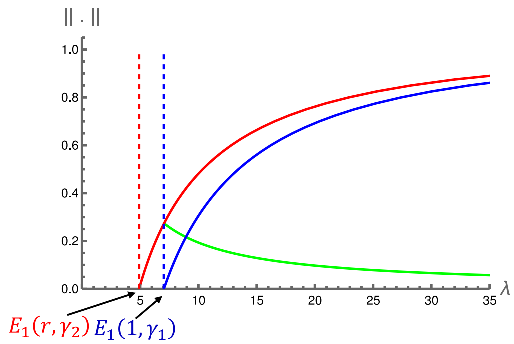
Case IB: $E_1(1,\gamma_1) < E_1(r,\gamma_2)$

Computational Result 1.3. (when $0 < b < 1$).

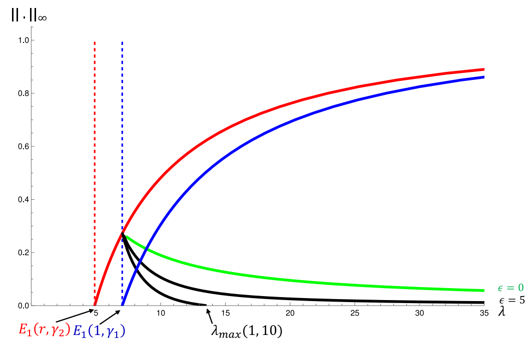
There exists $\lambda_{min}(b,\epsilon) > E_1(r,\gamma_2)$ such that (1.14) has a unique positive solution v_λ for $\lambda > \lambda_{min}(b,\epsilon)$ and no positive solution for $\lambda \leq \lambda_{min}(b,\epsilon)$ (see Figure 6.5).

Further,

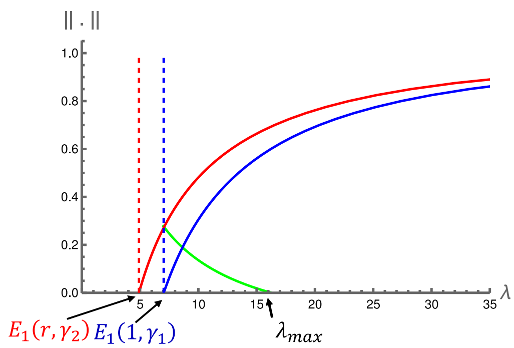
(i) $||v_\lambda||_\infty \to 1 - b$ as $\lambda \to \infty$



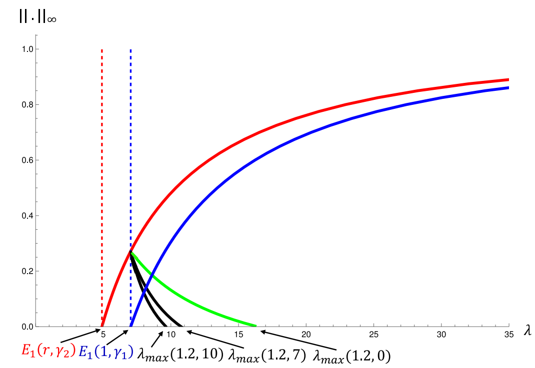
(a) $b = 1$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $\epsilon = 0$, from [1].



(b) $b = 1$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$ and $\epsilon > 0$.



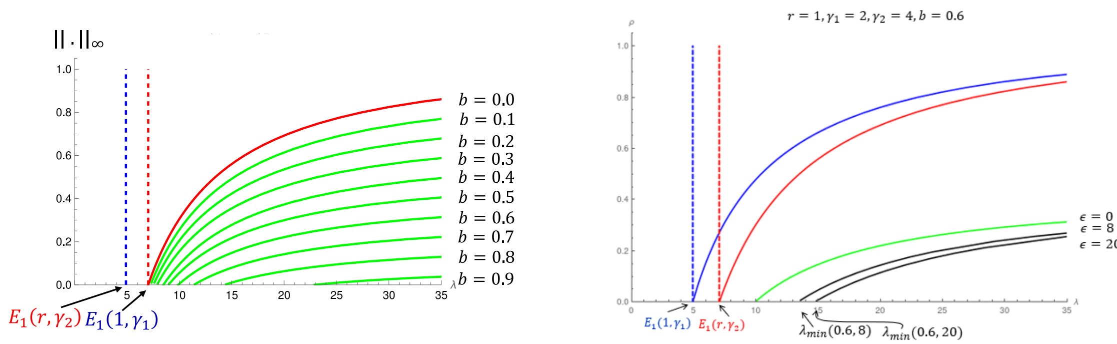
(c) $b = 1.2$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $\epsilon = 0$, from [1].



(d) $b = 1.2$, $\gamma_1 = 4, \gamma_2 = 2, r = 1$, and $\epsilon > 0$.

Figure 6.3. Typical bifurcation diagrams of (1.14) for different values of $b \geq 1$ and $\epsilon \geq 0$ when $h(s, \epsilon) = 1 + \epsilon s$

(ii) $\lambda_{min}(b, \epsilon)$ is increasing in ϵ and $\lambda_{min}(b, \epsilon) \to \frac{\pi^2}{1-b}$ as $\epsilon \to \infty$ (see Figure 6.6).



(a) $b \in [0, 1)$ and $\epsilon = 0$, from [1]

(b) $b = 0.6$ and $\epsilon > 0$

Figure 6.5. Typical bifurcation diagrams of (1.14) for different values of $b < 1$ and $\epsilon \geq 0$ when $h(s, \epsilon) = 1 + \epsilon s$

Computational Result 1.4. (when $b \geq 1$)

(1.14) has no positive solution for $\lambda > 0$.

6.1.2 Case II: $h(s, \epsilon) = \frac{1}{1+\epsilon s}$

Case IIA: $E_1(r, \gamma_2) \leq E_1(1, \gamma_1)$

Computational Result 1.5 (when $0 < b < 1$)

There exists a unique positive solution v_λ of (1.14) for $\lambda > E_1(r, \gamma_2)$, where $||v_\lambda||_\infty$ increases for $\lambda \in (E_1(r, \gamma_2), E_1(1, \gamma_1))$ and $||v_\lambda||_\infty \to 1 - b$ as $\lambda \to \infty$. Moreover, there exists $b^*(\gamma_1, \gamma_2, r) \in (0, 1)$ (also see [1] for existence of this $b^*(\gamma_1, \gamma_2, r)$) and $\epsilon_1(b) > 0$ for $b \neq b^*(\gamma_1, \gamma_2, r)$ such that

(i) if $b < b^*(\gamma_1, \gamma_2, r)$ and $\epsilon < \epsilon_1(b)$, then $||v_\lambda||_\infty$ increases for $\lambda \in (E_1(1, \gamma_1), \infty)$

(ii) if $b > b^*(\gamma_1, \gamma_2, r)$ and $\epsilon < \epsilon_1(b)$, then $||v_\lambda||_\infty$ decreases for $\lambda \in (E_1(1, \gamma_1), \infty)$

(iii) if $b \neq b^*(\gamma_1, \gamma_2, r)$ and $\epsilon \geq \epsilon_1(b)$, then there exists $\bar{\lambda}(b, \epsilon) > E_1(1, \gamma_1)$ such that $||v_\lambda||_\infty$ increases for $\lambda \in (E_1(1, \gamma_1), \bar{\lambda}(b, \epsilon))$ and decreases in $(\bar{\lambda}(b, \epsilon), \infty)$

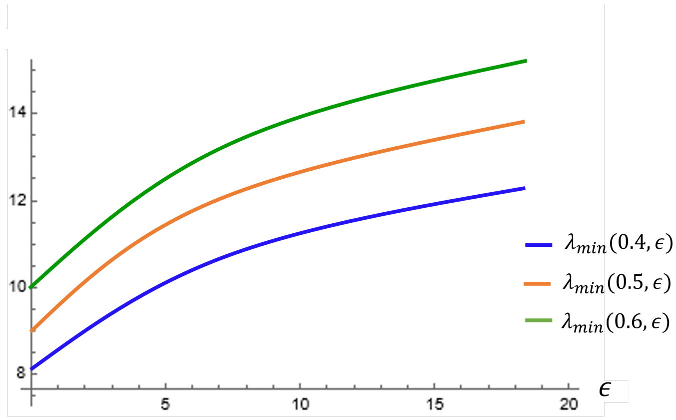
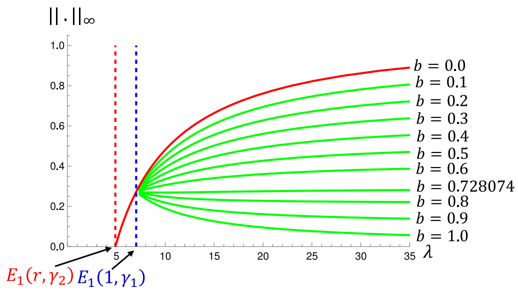
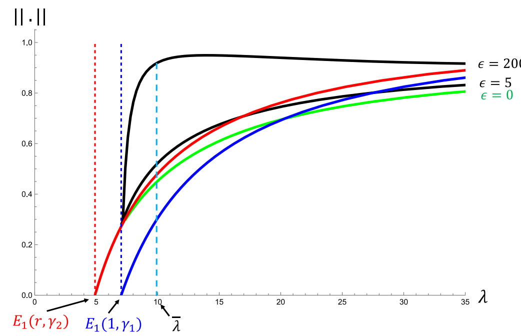


Figure 6.6. Evolution of λ_{min} with respect to ϵ and b when $h(s, \epsilon) = 1 + \epsilon s$, $r = 1, \gamma_1 = 2, \gamma_2 = 4$.

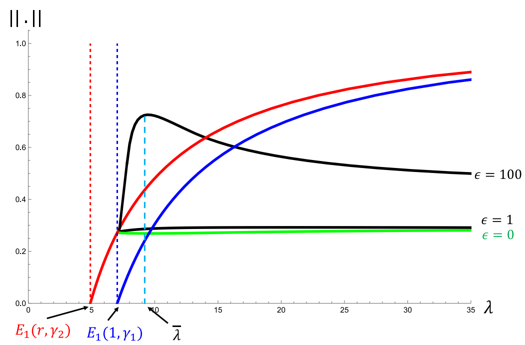
(iv) when $b = b^*(\gamma_1, \gamma_2, r)$ for any $\epsilon > 0$, there exists $\bar{\lambda}(b, \epsilon) > E_1(1, \gamma_1)$ such that $||v_\lambda||_\infty$ increases for $\lambda \in (E_1(1, \gamma_1), \bar{\lambda}(b, \epsilon))$ and decreases for $\lambda \in (\bar{\lambda}(b, \epsilon), \infty)$ (see Figure 6.7).



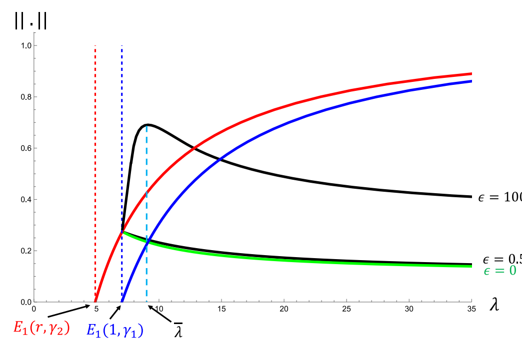
(a) $b \in [0,1]$ and $\epsilon = 0$, from [1]



(b) $b = 0.1(< b^*)$, $r = 1, \gamma_1 = 4, \gamma_2 = 2$ and $\epsilon > 0$



(c) $b = 0.728074(\approx b^*)$ $r = 1, \gamma_1 = 4, \gamma_2 = 2$ and $\epsilon > 0$



(d) $b = 0.9(> b^*)$, $r = 1, \gamma_1 = 4, \gamma_2 = 2$, and $\epsilon > 0$

Figure 6.7. Typical bifurcation diagram of (1.14) for different values of $b \leq 1$ and $\epsilon \geq 0$ when $h(s,\epsilon) = \frac{1}{1+\epsilon s}$

Computational Result 1.6. (when $b \geq 1$)

Part (a)

When $b = 1$ for any $\epsilon > 0$, (1.14) has a unique positive solution v_λ for all $\lambda > E_1(r,\gamma_2)$ where $||v_\lambda||_\infty \to 0$ as $\lambda \to \infty$. Further there exists $\bar{\lambda}_2(b,\epsilon)$ such that $||v||_\infty$ increases in $(E_1(r,\gamma_2), \bar{\lambda}_2(b,\epsilon)$ and decreases in $(\bar{\lambda}_2(b,\epsilon), \infty)$

Part (b)

When $b > 1$ there exists $\lambda_{max}(b,\epsilon) > E_1(1,\gamma_1)$ such that (1.14) has a unique positive solution v_λ for $\lambda \in (E_1(r,\gamma_2), \lambda_{max}(b,\epsilon))$ and no positive solution for $\lambda \geq \lambda_{max}(b,\epsilon)$. Further, there exists $\epsilon_1(b) > 0$ such that

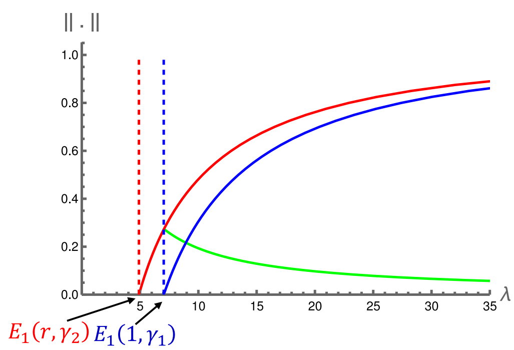
(i) if $\epsilon \leq \epsilon_1(b)$, then $||v_\lambda||_\infty$ increases for $\lambda \in (E_1(r, \gamma_2), E_1(1, \gamma_1))$ and decreases for $\lambda \in (E_1(1, \gamma_1), \lambda_{max}(b, \epsilon))$

(ii) if $\epsilon > \epsilon_1(b)$, then there exists $\bar{\lambda}(b, \epsilon) > E_1(1, \gamma_1)$ such that $||v_\lambda||_\infty$ increases for $\lambda \in (E_1(r, \gamma_2), \bar{\lambda}(b, \epsilon))$ and decreases for $\lambda \in (\bar{\lambda}(b, \epsilon), \lambda_{max}(b, \epsilon))$ (see Figure 6.8)

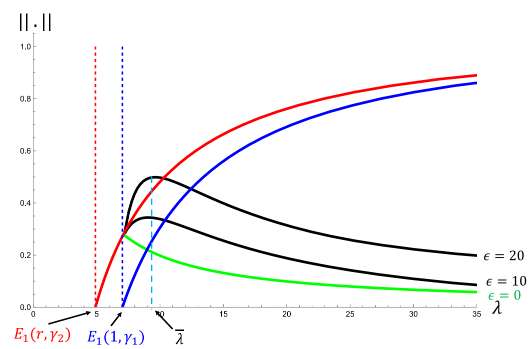
Furthermore,

(i) $\lambda_{max}(b, \epsilon)$ is increasing in ϵ

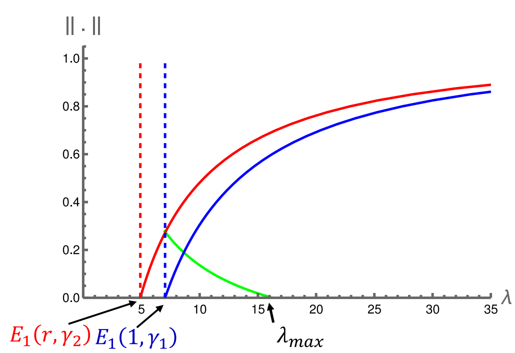
(ii) $\lambda_{max}(b, \epsilon) \to E_1(1, \gamma_1)$ as $\epsilon \to 0$ and $\lambda_{max}(b, \epsilon) \to \infty$ as $\epsilon \to \infty$ (see Figure 6.9).



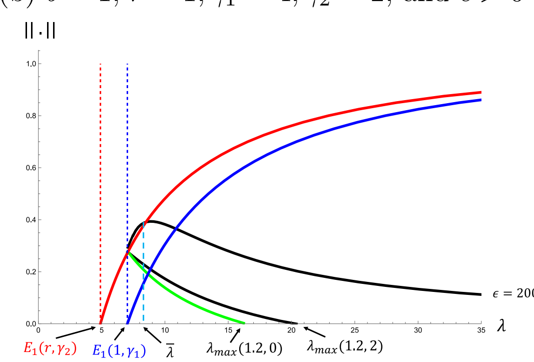
(a) $b = 1$ and $\epsilon = 0$, from [1]



(b) $b = 1$, $r = 1, \gamma_1 = 4, \gamma_2 = 2$, and $\epsilon > 0$



(c) $b = 1.2$ and $\epsilon = 0$, from [1]



(d) $b = 1.2$, $r = 1, \gamma_1 = 4, \gamma_2 = 2$, and $\epsilon > 0$

Figure 6.8. Typical bifurcation diagrams of (1.14) for different values of $b \geq 1$ and $\epsilon \geq 0$ when $h(s, \epsilon) = \frac{1}{1+\epsilon s}$

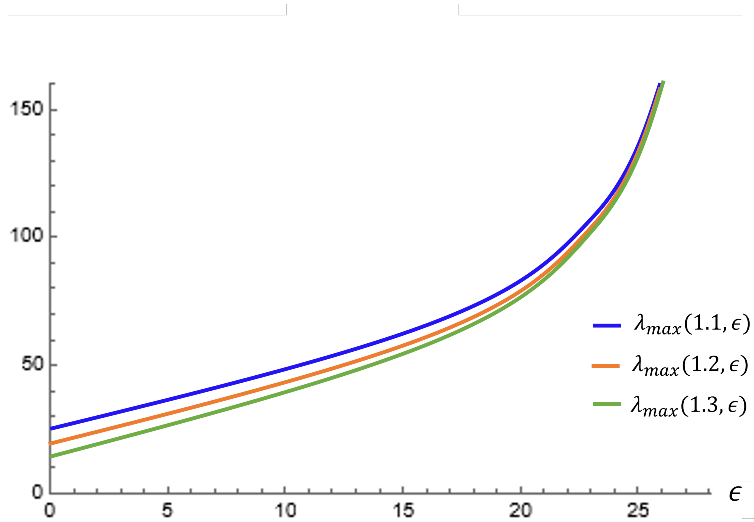


Figure 6.9. Evolution of λ_{max} with respect to ϵ and b when $h(s, \epsilon) = \frac{1}{1+\epsilon s}$, $r = 1, \gamma_1 = 4, \gamma_2 = 2$.

Case IIB: $E_1(1, \gamma_1) < E_1(r, \gamma_2)$

Computational Result 1.7. (when $0 < b < 1$)

There exists $\lambda_{min}(b, \epsilon) > E_1(1, \gamma_1)$ such that (1.14) has a unique positive solution v_λ for $\lambda > \lambda_{min}(b, \epsilon)$ and no positive solution for $\lambda \leq \lambda_{min}(b, \epsilon)$. Further, there exits $\epsilon_1(b) > 0$ such that

(i) if $\epsilon < \epsilon_1(b)$, then $||v_\lambda||_\infty$ increases for $\lambda > \lambda_{min}(b, \epsilon)$

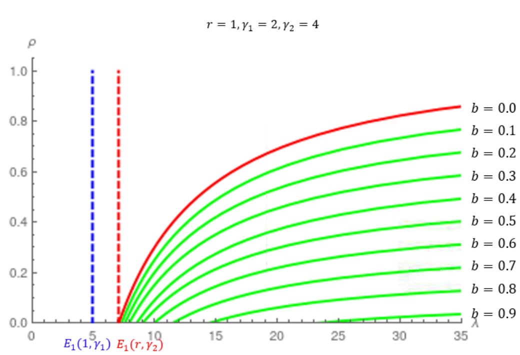
(ii) if $\epsilon \geq \epsilon_1(b)$, then there exists $\bar{\lambda}(b, \epsilon) > \lambda_{min}(b, \epsilon)$ such that $||v_\lambda||_\infty$ increases for $\lambda \in (\lambda_{min}(b, \epsilon), \bar{\lambda}(b, \epsilon))$ and decreases for $\lambda > \bar{\lambda}(b, \epsilon)$ (see Figure 6.10).

Furthermore,

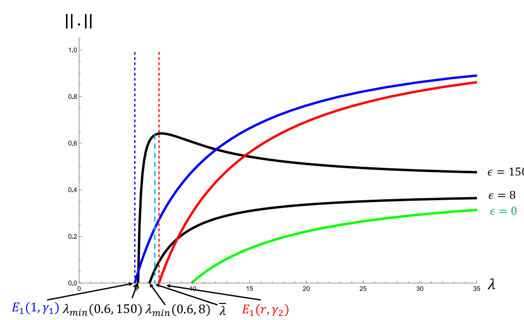
(i) $||v_\lambda||_\infty \to 1 - b$ as $\lambda \to \infty$

(ii) $\lambda_{min}(b, \epsilon)$ is decreasing in ϵ and $\lambda_{min}(b, \epsilon) \to E_1(1, \gamma_1)$ as $\epsilon \to \infty$ (see Figure 6.11).

Remark 6.3. In [1], when $\epsilon = 0$, the authors proved that there exists $\delta(b) > 0$ such that $\lambda_{min}(b, 0) > E_1(r, \gamma_2) + \delta(b)$ for $b \in (0, 1)$. But here, when there is cooperation on the boundary, $\lambda_{min}(b, \epsilon)$ (minimum patchsize) is a decreasing function of ϵ and $\lambda_{min}(b, \epsilon) \in (E_1(1, \gamma_1), E_1(r, \gamma_2))$ for $\epsilon \gg 1$. Further, $\lambda_{min}(b, \epsilon) \to E_1(1, \gamma_1)$ as $\epsilon \to \infty$. This shows that for large enough cooperation the solution starts existing for $\lambda < E_1(r, \gamma_2)$ (minimum patchsize of (1.15)).



(a) $b \in (0, 1)$ and $\epsilon = 0$, from [1]



(b) $b = 0.6$, $r = 1, \gamma_1 = 2, \gamma_2 = 4$, and $\epsilon > 0$

Figure 6.10. Typical bifurcation diagrams of (1.14) for different values of $b < 1$ and $\epsilon \geq 0$ when $h(s, \epsilon) = \frac{1}{1+\epsilon s}$

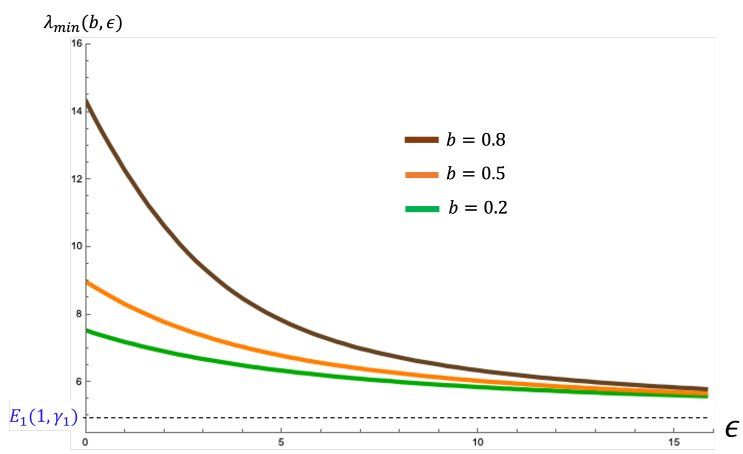


Figure 6.11. Evolution of λ_{min} with respect to b and ϵ when $h(s, \epsilon) = \frac{1}{1+\epsilon s}$, $r = 1, \gamma_1 = 2, \gamma_2 = 4$.

Computational Result 1.8. (When $b = 1$)

There exist $\lambda_{min}(b, \epsilon) > E_1(1, \gamma_1)$ such that (1.14) has a unique positive solution v_λ for $\lambda > \lambda_{min}(b, \epsilon)$ and no positive solution for $\lambda \leq \lambda_{min}(b, \epsilon)$. Further, there exists $\bar{\lambda}(b, \epsilon) > \lambda_{min}(b, \epsilon)$ such that $||v_\lambda||_\infty$ increases for $\lambda \in (\lambda_{min}(b, \epsilon), \bar{\lambda}(b, \epsilon))$ and decreases for $\lambda \in ((\bar{\lambda}(b, \epsilon), \infty)$ (see Figure 6.12).

Furthermore,

(i) $||v_\lambda||_\infty \to 0$ as $\lambda \to \infty$

(ii) $\lambda_{min}(b, \epsilon)$ decreases in ϵ and $\lambda_{min}(b, \epsilon) \to E_1(1, \gamma_1)$ as $\epsilon \to \infty$ (see Figure 6.13).

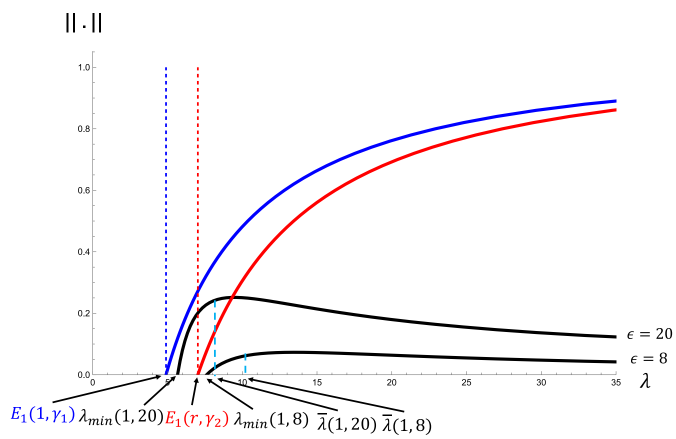


Figure 6.12. Typical bifurcation diagrams for different $\epsilon > 0$ when $h(s, \epsilon) = \frac{1}{1+\epsilon s}$; $b = 1$, $r = 1$, $\gamma_1 = 2$, $\gamma_2 = 4$

Computational Result 1.9. (when $b > 1$)

There exists $\epsilon_1(b) > 0$ such that (1.14) has no positive solution for $\epsilon \leq \epsilon_1(b)$. For $\epsilon > \epsilon_1(b)$, there exist $\lambda_{min}(b, \epsilon), \lambda_{max}(b, \epsilon) > E_1(1, \gamma_1)$ with $\lambda_{min}(b, \epsilon) < \lambda_{max}(b, \epsilon)$ such that (1.14) has a unique positive solution v_λ for $\lambda \in (\lambda_{min}(b, \epsilon), \lambda_{max}(b, \epsilon))$ and no positive solution for $\lambda \in (0, \lambda_{min}(b, \epsilon)) \cup (\lambda_{max}(b, \epsilon), \infty)$. Further, there exists $\bar{\lambda} \in (\lambda_{min}(b, \epsilon), \lambda_{max}(b, \epsilon))$ such that $||v_\lambda||_\infty$ increases for $\lambda \in (\lambda_{min}(b, \epsilon), \bar{\lambda}(b, \epsilon))$ and decreases for $\lambda \in (\bar{\lambda}(b, \epsilon), \lambda_{max}(b, \epsilon))$ (see Figure 6.14).

Furthermore,

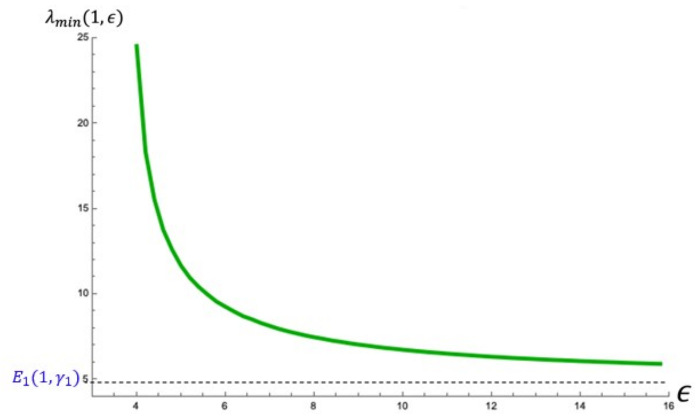


Figure 6.13. Evolution of $\lambda_{min}(b, \epsilon)$ with respect to ϵ

(i) $\lambda_{max}(b, \epsilon)$ increases in ϵ and $\lambda_{max}(b, \epsilon) \to \infty$ as $\epsilon \to \infty$

(ii) $\lambda_{min}(b, \epsilon)$ decreases in ϵ and $\lambda_{min}(b, \epsilon) \to E_1(1, \gamma_1)$ as $\epsilon \to \infty$ (see Figure 6.15).

Remark 6.4. In [1], the authors prove that (1.14) has no positive solution for any λ when $\epsilon = 0$ and $b \geq 1$. But here, when there is cooperation on the boundary, there exists $\lambda_{min}(b, \epsilon) > 0$ (minimum patchsize) such that (1.14) has a unique positive solution for $\lambda > \lambda_{min}(b, \epsilon) > 0$. Further, $\lambda_{min}(b, \epsilon)$ is a decreasing function of ϵ, $\lambda_{min}(b, \epsilon) \in (E_1(1, \gamma_1), E_1(r, \gamma_2))$ for $\epsilon \gg 1$ and $\lambda_{min}(b, \epsilon) \to E_1(1, \gamma_1)$ as $\epsilon \to \infty$. This shows that for large enough cooperation the solution starts existing for $\lambda < E_1(r, \gamma_2)$ (minimum patchsize of (1.15)).

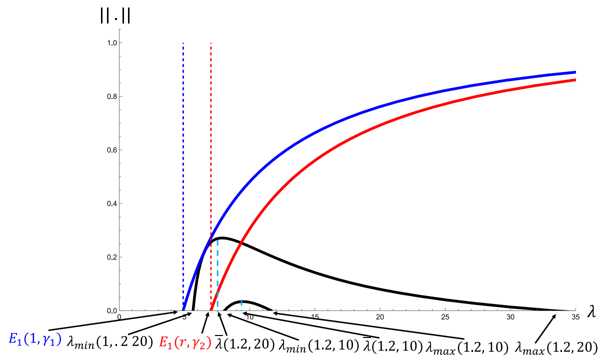


Figure 6.14. Typical bifurcation diagrams for different $\epsilon > 0$ values when $h(s,\epsilon) = \frac{1}{1+\epsilon s}$; $b = 1.2$, $r = 1, \gamma_1 = 2, \gamma_2 = 4$

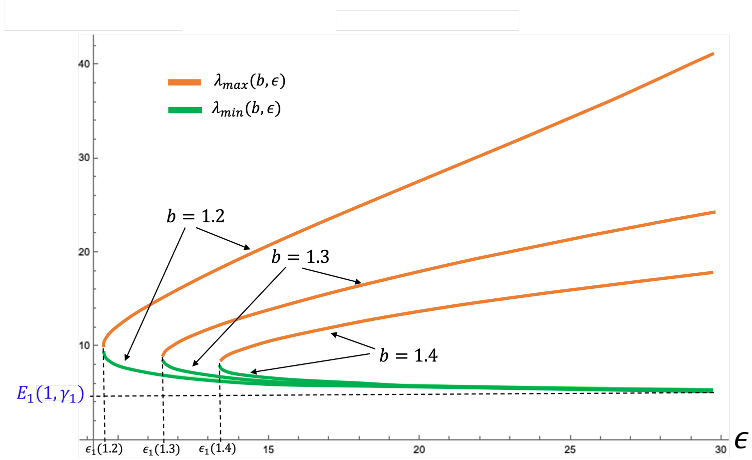


Figure 6.15. Evolution of $\lambda_{min}(b,\epsilon)$ and $\lambda_{max}(b,\epsilon)$ with respect to ϵ and b when $h(s,\epsilon) = \frac{1}{1+\epsilon s}$, $r = 1, \gamma_1 = 2, \gamma_2 = 4$.

6.2 Analytical results when Ω is a bounded domain in $\mathbb{R}^N : N \geq 1$

In this section, we prove analytically for $N \geq 1$ some of the results observed earlier (computationally for $N = 1$). First we note that v is a nonegative solutions of (1.12) then $v > 0$ in Ω (see Remark 6.5) below. Next we note that when $\epsilon > 0$ and $b < 1$, (1.12) has at most one positive solution (see Theorem 6.1 below).

Remark 6.5. If v is a nonnegative nontrivial solution of (1.12), then $v > 0$ in Ω. Indeed, by maximum principal we see that the positive solution $u = u_\lambda$ of (1.13) satisfies $0 < u_\lambda < 1; \overline{\Omega}$ for $\lambda > E_1(1, \gamma_1)$. Thus there exists some $\sigma_\lambda > 0$ such that $1 - bu_\lambda \geq \sigma_\lambda > 0;\ \overline{\Omega}$. Suppose v is a nonnegative nontrivial solution of (1.12). Assume that there exists $x_0 \in \Omega$ such that $v(x_0) = 0$, then there exists $D \subset \Omega$ such that $x_0 \in D$, $v < \sigma_\lambda \leq 1 - bu_\lambda;\ D$ and $v \not\equiv 0; D$. This with (1.12) give $-\Delta v \geq 0$ in D. Thus v cannot achieve an interior minimum in D unless it is a constant. But $v(x_0) = 0$, hence $v \equiv 0; D$ which is a contradiction.

Theorem 6.1. *For any $\epsilon > 0$ and $b < 1$, (1.12) has at most one positive solution.*

Proof. Let v_1 and v_2 be two distinct positive solutions of (1.12). Since $z \equiv 1;\ \overline{\Omega}$ is a global supersolution of (1.12), (1.12) has a maximal solution. Therefore, without loss of generality we can assume $v_1 \leq v_2;\ \overline{\Omega}$. Now we consider

$$\begin{aligned} J &= \int_\Omega [(\Delta v_1)v_2 - (\Delta v_2)v_1]dx \\ &= \int_\Omega [(-\lambda r v_1(1 - v_1 - bu_\lambda)v_2) + (\lambda r v_2(1 - v_2 - bu_\lambda)v_1)]dx \\ &= \int_\Omega [\lambda r v_1 v_2(1 - v_2 - bu_\lambda - 1 + v_1 + bu_\lambda)]dx \\ &= \int_\Omega [\lambda r v_1 v_2(v_1 - v_2)]dx < 0 \end{aligned}$$

since v_1 and v_2 are distinct. However, by Green's second identity

$$\begin{aligned} J &= \int_\Omega [(\Delta v_1)v_2 - (\Delta v_2)v_1]\, dx \\ &= \int_{\partial\Omega} \left[\frac{\partial v_1}{\partial \eta} v_2 - \frac{\partial v_2}{\partial \eta} v_1\right] dx \\ &= \int_{\partial\Omega} \left[-\sqrt{\lambda}\gamma_2 h(u_\lambda, \epsilon) v_1 v_2 + \sqrt{\lambda}\gamma_2 h(u_\lambda, \epsilon) v_2 v_1\right] dx = 0. \end{aligned}$$

Hence we have a contradiction and this implies $v_1 \equiv v_2;\ \overline{\Omega}$. This completes the proof. □

6.2.1 Competition on the boundary

In this subsection we suppose $h(s,\epsilon) = 1+\epsilon s$ (competition interaction on the boundary) and establish the following results.

Theorem 6.2. *For a fixed $\epsilon > 0$ and $b < 1$, (1.12) has a positive solution for $\lambda > E_1(r(1-b), \gamma_2 h(1,\epsilon))$.*

Proof. Here, we begin with constructing a subsolution of (1.12) as it is clear that $z \equiv 1;\ \bar{\Omega}$ is a global strict supersolution for (1.12). To create a subsolution of (1.12), we consider the following boundary value problem

$$\begin{cases} -\Delta\omega = \lambda r\omega(1-\omega-b) & \text{in } \Omega; \\ \frac{\partial\omega}{\partial\eta} + \sqrt{\lambda}\gamma_2 h(1,\epsilon)\omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Note that (6.1) has a unique positive solution $w < 1\,;\overline{\Omega}$, for $\lambda > E_1(r(1-b), \gamma_2 h(1,\epsilon))$(see [32]).
We have

$$\begin{aligned} -\Delta\omega - \lambda r\omega(1-\omega-bu_\lambda) &= \lambda r\omega(1-\omega-b) - \lambda r\omega(1-\omega-bu_\lambda) \\ &= \lambda r\omega b(u_\lambda - 1) < 0;\ \Omega. \end{aligned} \tag{6.2}$$

and

$$\frac{\partial\omega}{\partial\eta} + \sqrt{\lambda}\gamma_2 h(u_\lambda,\epsilon)\omega = \sqrt{\lambda}\gamma_2\omega\epsilon(u_\lambda - 1) < 0;\ \partial\Omega \tag{6.3}$$

Thus ω is a subsolution of (1.12). Hence by Lemma 2.2, (1.12) has a positive solution for all $\lambda > E_1(r(1-b), \gamma_2 h(1,\epsilon))$. This completes the proof. □

Remark 6.6. Let $b < 1$, r, b, γ_2, ϵ are chosen such that $r(1-b) \geq 1$, and $\gamma_2 h(1,\epsilon) \leq \gamma_1$. Then $E_1(r(1-b), \gamma_2 h(1,\epsilon)) \in (E_1(r,\gamma_2), E_1(1,\gamma_1))$. Hence (1.12) has a positive solution for all $\lambda > E_1(r,\gamma_2)$.

Theorem 6.3. *If $b > 1$ then (1.12) has no positive solution for $\lambda \gg 1$.*

Proof. Let v be a positive solution of (1.12) for $\lambda \gg 1$ and W be the unique positive solution of:

$$\begin{cases} -\Delta W = \lambda r W(1-W) & \text{in } \Omega; \\ \frac{\partial V}{\partial \eta} + \sqrt{\lambda}\gamma_2 h(u_\lambda,\epsilon) W = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.4}$$

Existence of W follows from Lemma 2.2. Indeed, $V \equiv 1$ is a supersolution to (6.4) and the solution of the problem

$$\begin{cases} -\Delta V = \lambda r V(1-V) & \text{in } \Omega; \\ \frac{\partial V}{\partial \eta} + \sqrt{\lambda}\gamma_2 h(1,\epsilon) V = 0 & \text{on } \partial\Omega \end{cases} \tag{6.5}$$

is a subsolution to (6.4). By Green's second identity we get,

$$\begin{aligned} J &= \int_\Omega [-(\Delta v)W + (\Delta W)v]\, dx \\ &= \int_{\partial\Omega} \left[-\frac{\partial v}{\partial \eta} W + \frac{\partial W}{\partial \eta} v \right] ds = 0 \end{aligned}$$

However,

$$\begin{aligned}J &= \int_\Omega [(-\Delta v)W + (\Delta W)v]dx \\ &= \int_\Omega [(\lambda r v(1 - v - bu_\lambda)W - \lambda r W(1 - W)v]dx \\ &= \int_\Omega \lambda r v W(1 - v - bu_\lambda - 1 + W)dx \\ &= \int_\Omega \lambda r v W(W - v - bu_\lambda)dx \\ &< \lambda r \int_\Omega vW(W - bu_\lambda)dx\end{aligned}$$

Note that $W - bu_\lambda \to 1 - b$ on all closed subsets of Ω as $\lambda \to \infty$ (see [32]). Since $b > 1$ we have $1 - b < 0$ and can choose $\lambda \gg 1$ such that $\int_\Omega vW(W - bu_\lambda)dx < 0$, which is a contradiction. □

Case I: $E_1(r, \gamma_2) \leq E_1(1, \gamma_1)$

Theorem 6.4. *For a fixed $\epsilon > 0$ and $0 < b < 1$ there exists $\delta(\epsilon, b) > 0$ such that* (1.12) *has a positive solution for $\lambda \in (E_1(r, \gamma_2), E_1(1, \gamma_1) + \delta)$.*

Proof. Problem (1.12) has a positive solution v for $\lambda \in (E_1(r, \gamma_2), E_1(1, \gamma_1)]$. Since v is continuous and strictly positive at $\lambda = E_1(1, \gamma_1)$ in $\overline{\Omega}$, by continuity there exist some $\delta(\epsilon, b) > 0$ such that $v > 0$ for $\lambda \in (E_1(1, \gamma_1), E_1(1, \gamma_1) + \delta)$. Hence, (1.12) has a positive solution for $\lambda \in (E_1(r, \gamma_2), E_1(1, \gamma_1) + \delta(\epsilon, b))$. □

Remark 6.7. Observe that, in Theorem 6.4, $\delta(\epsilon, b) \to 0$ as $\epsilon \to \infty$.

Case II: $E_1(1, \gamma_1) < E_1(r, \gamma_2)$

Theorem 6.5. *Let $\epsilon > 0$ and $0 < b < 1$ be fixed. Then there exists $\delta(b, \epsilon, r, \gamma_2) > 0$ such that* (1.12) *has no positive solution for $\lambda < E_1(r, \gamma_2) + \delta$.*

Proof. First, we consider the eigenvalue problem:

$$\begin{cases} -\Delta z = \mu z & \text{in } \Omega; \\ \frac{\partial z}{\partial \eta} + \beta(x) z = 0 & \text{on } \partial\Omega; \end{cases} \tag{6.6}$$

where β is a nonnegative continuous function on $\partial\Omega$. Let $\mu_1 = \mu(\beta)$ denote the principal eigencurve of (6.6). Next, for a fixed λ, let $\sigma_1(\lambda)$ be the principal eigenvalue of the problem

$$\begin{cases} -\Delta\psi - \lambda r\psi = \sigma_1\psi & \text{in } \Omega; \\ \frac{\partial \psi}{\partial \eta} + \sqrt{\lambda}\gamma_2\psi = 0 & \text{on } \partial\Omega; \end{cases} \tag{6.7}$$

and let $\sigma_2(\epsilon, \lambda)$ be the principal eigenvalue and $\phi > 0; \overline{\Omega}$ be the corresponding eigenfunction of the problem

$$\begin{cases} -\Delta\phi_2 - \lambda r\phi_2 = \sigma_2\phi & \text{in } \Omega; \\ \frac{\partial \phi_2}{\partial \eta} + \sqrt{\lambda}\gamma_2(1+\epsilon u_\lambda)\phi_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.8}$$

Comparing (6.6) with (6.7) and (6.8) we obtain

$$\mu_1(\sqrt{\lambda}\gamma_2) = \lambda r + \sigma_1$$

and

$$\mu_1(\sqrt{\lambda}\gamma_2(1+\epsilon u_\lambda)) = \lambda r + \sigma_2.$$

In [32], authors showed that $\mu_1(\sqrt{\lambda}\gamma_2)$ is increasing and concave in λ, $\mu_1(\sqrt{\lambda}\gamma_2) \to 0$ as $\lambda \to 0$ and $\mu_1(\sqrt{\lambda}\gamma_2)$ is bounded above by E_1^D where E_1^D is the principal eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta z = Ez & \text{in } \Omega; \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, we see that $\mu_1(\sqrt{\lambda}\gamma_2(1+\epsilon u_\lambda)) = \mu_1(\sqrt{\lambda}\gamma_2)$ for $\lambda \le E_1(1,\gamma_1)$ since $u_\lambda \equiv 0$; $\overline{\Omega}$ for $\lambda \le E_1(1,\gamma_1)$ (see [32]). Also we see that $\mu_1(\sqrt{\lambda}\gamma_2(1+\epsilon u_\lambda)) > \mu_1(\sqrt{\lambda}\gamma_2)$ for $\lambda > E_1(1,\gamma_1)$ since $\mu_1(\beta)$ is increasing in β (see [18]) and $u_\lambda > 0$; $\overline{\Omega}$ for $\lambda > E_1(1,\gamma_1)$. Moreover $\mu_1(\sqrt{\lambda}\gamma_2(1+\epsilon u_\lambda)) > \mu_1(\sqrt{\lambda}\gamma_2) = r\lambda$ at $\lambda = E_1(r,\gamma_2)$ and this implies that there exists $\delta > 0$ such that $\mu_1(\sqrt{\lambda}\gamma_2(1+\epsilon u_\lambda)) > r\lambda$ for $\lambda < E_1(r,\gamma_2)+\delta$. Hence we have $\sigma_2 > 0$ for $\lambda < E_1(r,\gamma_2)+\delta$. The following eigencurve diagram (Figure (6.16)) illustrates this argument.

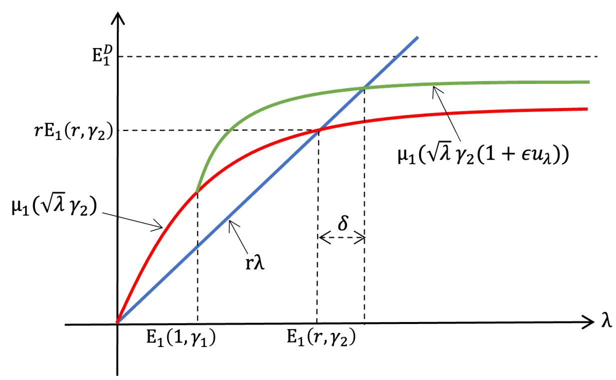


Figure 6.16. Existence of δ.

By Green's second identity we get,

$$
\begin{aligned}
J &= \int_\Omega [(\Delta\phi)v - (\Delta v)\phi]\,dx \\
&= \int_{\partial\Omega} \left[\frac{\partial\phi}{\partial\eta}v - \frac{\partial v}{\partial\eta}\phi\right] ds \\
&= \int_{\partial\Omega} [-\gamma_2\sqrt{\lambda}(1+\epsilon u_\lambda)\phi v + \gamma_2\sqrt{\lambda}h(u_\lambda,\epsilon)v\phi]ds \\
&= \int_{\partial\Omega} \gamma_2\sqrt{\lambda}v\phi[1+\epsilon u_\lambda - 1 - \epsilon u_\lambda]ds = 0
\end{aligned}
$$

However, for $\lambda < E_1(r, \gamma_2) + \delta$

$$\begin{aligned} J &= \int_\Omega [(\Delta\phi)v - (\Delta v)\phi]dx \\ &= \int_\Omega [(-\lambda r\phi - \sigma_2\phi)v + \lambda rv(1 - v - bu_\lambda)\phi]dx \\ &= \int_\Omega \phi v(-\lambda r - \sigma_2 + \lambda r - \lambda rv - \lambda rbu_\lambda)dx \\ &= \lambda r \int_\Omega \phi v \left(\frac{-\sigma_2}{\lambda r} - v - bu_\lambda\right) dx < 0 \end{aligned}$$

which is a contradiction. Thus (1.12) has no positive solutions for $\lambda < E_1(r, \gamma_2)) + \delta$.

□

6.2.2 Cooperation on the boundary

In this subsection we assume $h(s, \epsilon) = \frac{1}{1+\epsilon s}$ (cooperation interaction on the boundary) and establish the following results.

Remark 6.8. When $0 < b < 1$ and $E_1(r, \gamma_2) < E_1(1, \gamma_1)$ then (1.12) has a positive solution for all $\lambda > E_1(r, \gamma_2)$. This holds even if $\epsilon = 0$ i.e. no cooperation on the boundary (see [1]). Hence we only consider the case $E_1(1, \gamma_1) < E_1(r, \gamma_2)$.

Theorem 6.6. *Let $E_1(1, \gamma_1) < E_1(r, \gamma_2)$, $0 < b < 1$, $\lambda_0 \in (E_1(1, \gamma_1), E_1(r(1-b), \gamma_2))$ and $A(\lambda_0) = \min\{u_\lambda(x) : \lambda \in [\lambda_0, E_1(r(1-b), \gamma_2)],\ x \in \overline{\Omega}\}$. Then there exists $\epsilon_0(r, \gamma_2, \lambda_0) > 0$ such that $E_1\left(r(1-b), \frac{\gamma_2}{1+\epsilon_0 A(\lambda_0)}\right) = \lambda_0$ and a positive solution of (1.12) for $\lambda \in \left(E_1(r(1-b), \frac{\gamma_2}{1+\epsilon_0 A(\lambda_0)}), E_1(r(1-b), \gamma_2)\right]$ whenever $\epsilon \geq \epsilon_0$. Moreover, there exists a positive solution of (1.12) for $\lambda > E_1(r(1-b), \gamma_2)$ for any $\epsilon > 0$ and $0 < b < 1$.*

Proof. For fixed $0 < b < 1$ and $\lambda_0 \in (E_1(1, \gamma_1), E_1(r(1-b), \gamma_2))$, it is easy to see the existence of $\epsilon_0 > 0$ such that $E_1(r(1-b), \frac{\gamma_2}{1+\epsilon_0 A(\lambda_0)}) = \lambda_0$ since $E_1(r(1-b), \frac{\gamma_2}{1+\epsilon A(\lambda_0)})$ is continuous and decreases in ϵ (see [32]).

We will employ the sub-supersolution theorem to prove the existence of a positive solution of (1.12). Clearly, $z \equiv 1$ is a global strict supersolution of (1.12) for all $\lambda > 0$. Now for fixed $\epsilon \in (0, \epsilon_0)$, we take ω to be the unique positive solution to the problem:

$$\begin{cases} -\Delta\omega = \lambda r\omega(1-b-\omega) & \text{in } \Omega; \\ \frac{\partial\omega}{\partial\eta} + \sqrt{\lambda}\{\frac{\gamma_2}{1+\epsilon A(\lambda_0)}\}\omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.9}$$

Note that (6.9) has a unique positive solution for $\lambda > E_1(r(1-b), \frac{\gamma_2}{1+\epsilon A(\lambda_0)})$ (see [32]). Next, observe that in Ω, we have

$$\begin{aligned} -\Delta\omega - \lambda r\omega(1-\omega-bu_\lambda) &= \lambda r\omega(1-\omega-b) - \lambda r\omega(1-\omega-bu_\lambda) \\ &= \lambda r\omega b(u_\lambda - 1) < 0 \end{aligned} \tag{6.10}$$

and on $\partial\Omega$

$$\begin{aligned} \frac{\partial\omega}{\partial\eta} + \sqrt{\lambda}\gamma_2 h(u_\lambda,\epsilon)\omega &= -\sqrt{\lambda}\{\frac{\gamma_2}{1+\epsilon A(\lambda_0)}\}\omega + \sqrt{\lambda}\gamma_2\{\frac{1}{1+\epsilon u_\lambda}\}\omega \\ &= \sqrt{\lambda}\gamma_2\omega\{\frac{-1-\epsilon u_\lambda+1+\epsilon A(\lambda_0)}{(1+\epsilon u_\lambda)(1+\epsilon A(\lambda_0))}\} \\ &= \frac{\sqrt{\lambda}\gamma_2\omega\epsilon[A(\lambda_0)-u_\lambda]}{(1+\epsilon A(\lambda_0))(1+\epsilon)} \\ &\leq 0 \end{aligned} \tag{6.11}$$

for all $\lambda \in (E_1(r(1-b), \frac{\gamma_2}{1+\epsilon A(\lambda_0)}), E_1(r(1-b), \gamma_2)]$ and $\epsilon \geq \epsilon_0$. Thus ω is a subsolution of (1.12) for $\lambda \in (E_1(r(1-b), \frac{\gamma_2}{1+\epsilon A(\lambda_0)}), E_1(r(1-b), \gamma_2)]$. Hence (1.12) has a solution $v \in (\omega, 1)$ for $\lambda \in (E_1(r(1-b), \frac{\gamma_2}{1+\epsilon A(\lambda_0)}), E_1(r(1-b), \gamma_2)]$.

Next, for any $0 < b < 1$ and $\epsilon > 0$ fixed, let $\widehat{w}$ be the unique positive solution of

$$\begin{cases} -\Delta\widehat{w} = \lambda r\widehat{w}(1-b-\widehat{w}) & \text{in } \Omega; \\ \frac{\partial\widehat{w}}{\partial\eta} + \sqrt{\lambda}\gamma_2\widehat{w} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.12}$$

Note that (6.12) has a positive solution for $\lambda > E_1(r(1-b), \gamma_2)$. We have

$$\begin{aligned} -\Delta\widehat{w} - \lambda r\widehat{w}(1 - w - bu_\lambda) &= \lambda r\widehat{w}(1 - \widehat{w} - b) - \lambda r\widehat{w}(1 - \widehat{w} - bu_\lambda) \\ &= \lambda r\widehat{w}b(u_\lambda - 1) < 0 \text{ in } \Omega \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} \frac{\partial\widehat{w}}{\partial\eta} + \sqrt{\lambda}\gamma_2 h(u_\lambda, \epsilon)\widehat{w} &= -\sqrt{\lambda}\gamma_2\widehat{w} + \sqrt{\lambda}\gamma_2\{\frac{1}{1+\epsilon u_\lambda}\}\widehat{w} \\ &= \sqrt{\lambda}\gamma_2\widehat{w}\{\frac{-1-\epsilon u_\lambda + 1}{(1+\epsilon u_\lambda)}\} \leq 0 \text{ on } \partial\Omega. \end{aligned} \tag{6.14}$$

Therefore, $\widehat{w}$ is a subsolution of (1.12) for $\lambda > E_1(r(1-b), \gamma_2)$. Hence there exists a solution $v \in (\widehat{w}, 1)$ of (1.12) for $\lambda > E_1(r(1-b), \gamma_2)$ for any $\epsilon > 0$. This completes the proof. □

Theorem 6.7. *If $b > 1$ then (1.12) has no positive solution for $\lambda \gg 1$.*

Proof. This theorem can be proved with an argument similar to the proof of Theorem (6.3). □

Chapter 7: Conclusions and Future Directions

7.1 Conclusions

In this dissertation we studied two distinct sub classes of nonlinear elliptic boundary boundary value problem. First we studied problems that are linear inside the domain and nonlinear on the boundary. We established the existence of a maximal and a minimal weak solution. When the nonlinearity on the boundary is monotone, we used monotone iteration method. For the non monotone case we used the surjectivity of a coercive operator and combined with Zorn's lemma to prove our result. We extend these results to the coupled system of equations case as well. Next, when the nonlinearity on the boundary is superlinear and subcritical, we proved existence, multiplicity, nonexistence, local and global bifurcation results using degree theory, bifurcation theory combined with the sub and supersolution method. To validate our analytical results, we numerically approximated solutions and generated bifurcation diagrams using finite difference method.

Next we studied problems that are nonlinear inside the domain and linear on the boundary, in the context of a model arising in mathematical ecology. To begin with we performed computational simulations of the problem in one dimensional setting, then motivated by the bifurcation diagrams, that are obtained, we proved several analytical results such as existence, uniqueness and non-existence results. We employed method of sub and supersolutions to prove the existence of classical solutions and employed Green's identity to prove non-existence and uniqueness results.

7.2 Future Direction

In recent future we aim to study the following:

(i) Coupled systems case for (1.8),

(ii) p-Laplacian case for (1.4),

(iii) Numerical approximations for (1.8) including convergence, stability and existence results,

(iv) Second order accurate numerical simulations in Chapter 5,

(v) Nonlinear boundary condition for the ecology (1.12).

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