Z-SET UNKNOTTING IN LARGE CUBES

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Abstract. We introduce a notion of a Z_τ-set and prove various versions of Z_τ-set unknotting theorems in the Tychonov cube of weight τ. These results are applied to the study of Σ-products. In particular, we obtain a topological characterization of the space $\Sigma(\tau,\tau^+)$ consisting of those points of the cube I^{τ^+} whose at most τ-coordinates differ from zero.

1. Introduction

Concept of the Z-set and associated with it results are important in various areas of geometric topology (see [4], [1], [2], [12], [14]). However, in large cubes Z-sets do not posses nice properties. Consider, for example, the following three results.

Mapping Replacement for Z-sets. *If $f\colon Y \to I^\omega$ is a map of a metrizable compactum and its restriction $f|X\colon X \to I^\omega$ is a Z-embedding, then for any open cover $\mathcal{U}$ of I^ω there exists a Z-embedding $g\colon Y \to I^\omega$ such that $g|X = f|X$ and g is $\mathcal{U}$-close to f.*

Z-set Unknotting. *Every homeomorphism $h\colon Z_1 \to Z_2$ between Z-sets of the Hilbert cube I^ω can be extended to an autohomeomorphism $H\colon I^\omega \to I^\omega$.*

Local Z-set Unknotting. *For any open cover $\mathcal{U}$ of the Hilbert cube I^ω there exists an open cover $\mathcal{V}$ of I^ω such that any homeomorphism $h\colon Z_1 \to Z_2$ between Z-sets of I^ω, which is $\mathcal{V}$-close to the inclusion $Z_1 \hookrightarrow I^\omega$, admits an extension $H \in \mathrm{Auth}(I^\omega)$, which is $\mathcal{U}$-close to id_{I^ω}.*

Neither of these results is valid in the Tychonov cube I^τ, $\tau > \omega$. Indeed, consider a closed, but not functionally closed, subspace X of the Tychonov cube I^{ω_1} such that $X \approx I^{\omega_1}$. Now consider a Z-embedding $f\colon X \to I^{\omega_1}$ such that $f(X)$ is functionally closed in I^{ω_1}. It is clear that there is no embedding (let alone a Z-embedding) $g\colon I^{\omega_1} \to I^{\omega_1}$ such that $g|X = f$. Similarly, if $h\colon Z_1 \to Z_2$ is a homeomorphism between Z-sets of the cube I^{ω_1} both of which are copies of I^{ω_1}, but only one of which is functionally closed in I^{ω_1}, then there is no autohomeomorphism $H\colon I^{\omega_1} \to I^{\omega_1}$ extending h.

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In this note we introduce (Section 3) the notion of a Z_τ-set, $\tau > \omega$, and develop a theory consistent with the classical case. In particular, we prove the following counterparts of the results stated above (Theorems 4.4, 4.3 and 4.5 respectively).

Mapping Replacement for Z_τ-sets. *Let $\tau \geq \omega$ and F be a closed subspace of a compactum X with $w(X) \leq \tau$. Let also $f\colon X \to I^\tau$ be a map such that the restriction $f|F$ is a Z_τ-embedding. Then for any collection $\{\mathcal{U}_i \colon i \in I\} \subseteq \operatorname{cov}(I^\tau)$, with $|I| < \tau$, there exists a Z_τ-embedding $g\colon X \to I^\tau$ which is $\{\mathcal{U}_i \colon i \in I\}$-close to f and coincides with f on F.*

Z_τ-set Unknotting. *Every homeomorphism $h\colon Z_1 \to Z_2$ between Z_τ-sets of the Tychonov cube I^τ can be extended to an autohomeomorphism $H\colon I^\tau \to I^\tau$.*

Local Z_τ-set Unknotting. *Let $\tau > \omega$. For any collection $\{\mathcal{U}_i \colon i \in I\}$, $|I| < \tau$, of open covers of the Tychonov cube I^τ there exists a collection $\{\mathcal{V}_j \colon j \in J\}$, $|J| < \tau$, of open covers of I^τ such that any homeomorphism $h\colon Z_1 \to Z_2$ between Z_τ-sets of I^τ, which is $\{\mathcal{V}_j \colon j \in J\}$-close to the inclusion $Z_1 \hookrightarrow I^\tau$, admits an extension $H \in \operatorname{Auth}(I^\tau)$, which is also $\{\mathcal{V}_j \colon j \in J\}$-close to id_{I^τ}.*

While results in the countable and uncountable cases are quite similar, there are interesting differences indicating that in certain sense, perhaps contrary to expectations, the non-metrizable theory is simpler. One of the illustrations of this phenomenom can be seen by comparing local unknotting theorems in the countable and uncountable cases.

Definition of a Z_τ-set is based on the special type of topology on the set of maps which is discussed in Section 3. The corresponding space $C_\tau(Y, X)$ for an uncountable τ has a base consisting of open and closed sets which makes the space much simpler than its counterpart for the case $\tau = \omega$.

Based on the concept of a Z_τ-set it is possible to define absorbing sets in I^τ (as a non-metrizable counterpart of the theory developed in [3]). One of the approaches leads us to the study of Σ-products. Recall that for any cardinal numbers $\kappa > \tau \geq \omega$ the space $\Sigma(\tau, \kappa)$ is defined as follows

$$\Sigma(\tau, \kappa) = \{\{x_t \colon t \in T\} \in I^T \colon |\{t \in T \colon x_t \neq 0\}| \leq \tau\},$$

where $|T| = \kappa$. In section 5 we study regular τ-skeletoids which resemble properties of $\Sigma(\tau, \tau^+)$ and prove that any two regular τ-skeletoids in the cube I^{τ^+} are homeomorphic. Moreover, it is shown (Theorem 5.13) that $\mathrm{id}_{I^{\tau^+}}$ can be approximated (in $\operatorname{Auth}_{\tau^+}(I^{\tau^+}, I^{\tau^+})$) by homeomorphisms $H\colon I^{\tau^+} \to I^{\tau^+}$ such that $H(M) = N$, where M and N are given regular τ-skeletoids.

In Section 6 first we give a characterization of spaces embeddable into $\Sigma(\tau, \tau^+)$ as closed subpsaces (Theorem 6.1). It turns out that these are precisely regular τ-skeletoids (as a byproduct we obtain characterizations of τ^+-Valdivia and τ^+-Corson compact spaces of weight $\leq \tau^+$). Next we determine (Proposition 6.6) which closed subspaces of $\Sigma(\tau, \tau^+)$ are its retracts. Finally in Theorem 6.7 we prove that a retract of $\Sigma(\tau, \tau^+)$ is homeomorphic to $\Sigma(\tau, \tau^+)$ if and only if it contains no G_τ-points.

2. Preliminaries

2.1. **Directed sets.** Let A be a partially ordered *directed set* (i.e. for every two elements $a, b \in A$ there exists an element $c \in A$ such that $c \geq a$ and $c \geq b$). We say that a subset $A_1 \subseteq A$ of A *majorates* another subset $A_2 \subseteq A$ of A if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subseteq A$, denotes the lower upper bound of B (if such an element exists in A). Let $\tau \geq \omega$. A subset B of A is said to be τ-*closed* in A if for each chain $C \subseteq B$, with $|C| \leq \tau$, we have $\sup C \in B$ whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-*complete* if for each chain B of elements of A, with $|B| \leq \tau$, there exists an element $\sup B$ in A. A standard example of an τ-complete set is the set $\exp_\tau A$ of all subsets of cardinality $\leq \tau$ of any set A.

Proposition 2.1. *Let $\{A_i \colon i \in I\}$, $|I| \leq \tau$, be a collection of τ-closed and cofinal subsets of an τ-complete set A. Then the intersection $\cap\{A_i \colon i \in I\}$ is also cofinal and τ-closed in A.*

Corollary 2.2. *For each subset B, with $|B| \leq \tau$, of a τ-complete set A there exists an element $c \in A$ such that $c \geq b$ for each $b \in B$.*

The following proposition [6], known as the scheme of the spectral search, is a useful tool of studying inverse spectra.

Proposition 2.3. *Let A be an τ-complete set, $L \subseteq A^2$, and suppose the following three conditions are satisfied:*

- **Existence:** *For each $a \in A$ there exists $b \in A$ such that $(a, b) \in L$.*
- **Majorantness:** *If $(a, b) \in L$ and $c \geq b$, then $(a, c) \in L$.*
- **ω-closedness:** *Let $\{a_i \colon i \in I\}$, $|I| \leq \tau$, be a chain in A with $a = \sup\{a_k \colon i \in I\}$. If $(a_i, c) \in L$ for some $c \in A$ and each $i \in I$, then $(a, c) \in L$.*

*Then the set of all L-*reflexive *elements of A (an element $a \in A$ is said to be L-reflexive if $(a, a) \in L$) is cofinal and τ-closed in A .*

In section 5 we will also consider strongly τ-complete sets. These, by definition, are those τ-complete sets which together with any two of its elements $a, b \in A$ contain their infimum $\inf(a, b)$.

2.2. **Inverse Spectra.** All limit projections of inverse spectra considered below are surjective and all spaces are compact. Let $\tau \geq \omega$. An inverse spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of compact spaces is a τ-spectrum if:

(i) $w(X_\alpha) \leq \tau$, $\alpha \in A$;
(ii) The indexing set A is τ-complete;
(iii) $\mathcal{S}_X$ is τ-continuous, i.e. for each chain $\{\alpha_i \colon i \in I\} \subseteq A$ with $|I| \leq \tau$ and $\alpha = \sup\{\alpha_i \colon i \in I\}$, the diagonal product $\triangle\{p_{\alpha_i}^{\alpha} \colon i \in I\} \colon X_\alpha \to \lim\{X_{\alpha_i}, p_{\alpha_i}^{\alpha_j}, I\}$ is a homeomorphism.

One of the main results concerning τ-spectra is the following result of Ščepin (known as the Ščepin's Spectral Theorem).

Theorem 2.4. *Let $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ be two τ-spectra. Then for every map $f \colon \lim \mathcal{S}_X \to \lim \mathcal{S}_Y$ there exist a cofinal and τ-closed subset $B \subseteq A$ and maps $f_\alpha \colon X_\alpha \to Y_\alpha$, $\alpha \in B$, such that $f = \lim\{f_\alpha \colon \alpha \in B\}$. If f is a homeomorphism, then we may assume that each f_α, $\alpha \in B$, is also a homeomorphism.*

Note that τ-spectra are τ-factorizing, i.e. for every map $f \colon \lim \mathcal{S}_X \to Y$, where Y is a compact space with $w(Y) \leq \tau$, there exist $\alpha \in A$ and $f_\alpha \colon X_\alpha \to Y$ such that $f = f_\alpha p_\alpha$ ([6, Corollary 1.3.2]).

Proposition 2.5. *Let $\tau \geq \omega$, $|T| > \tau$, $T_0 \subseteq T$, $|T_0| < |T|$ and $g \colon \prod\{X_t \colon t \in T\} \to \prod\{X_t \colon t \in T\}$ be a map of the product of metrizable compact spaces such that $\pi_{T_0} g = \pi_{T_0}$. Then the set*

$$\mathcal{M}_{(g,T_0)} = \left\{ R \subseteq \exp_\tau(T \setminus T_0) \colon \exists g_{T_0 \cup R} \colon \prod\{X_t \colon t \in T_0 \cup R\} \to \prod\{X_t \colon t \in T_0 \cup R\} \right.$$
$$\left. \text{with } \pi_{T_0 \cup R} g = g_{T_0 \cup R} \pi_{T_0 \cup R} \right\}$$

is cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$.

Proof. By Theorem 2.4, the set

$$\mathcal{M}_g = \left\{ R \in \exp_\tau T \colon \exists g_R \colon \prod\{X_t \colon t \in R\} \to \prod\{X_t \colon t \in R\} \text{ with } \pi_R g = g_R \pi_R \right\}$$

is cofinal and τ-closed in $\exp_\tau T$.

Let $S \in \exp_\tau(T \setminus T_0)$ and choose $\tilde{R} \in \mathcal{M}_g$ such that $S \subseteq \tilde{R}$. The corresponding $g_{\tilde{R}}$ does not change the X_t-coordinate for $t \in \tilde{R} \cap T_0$ (since $\pi_{T_0} g = \pi_{T_0}$). Consequently the diagonal product

$$g_{T_0\cup\tilde{R}} = \pi^{T_0\cup\tilde{R}}_{T_0} \triangle \pi^{\tilde{R}}_{\tilde{R}\setminus T_0} g_{\tilde{R}} \pi^{T_0\cup\tilde{R}}_{\tilde{R}} \colon \prod\{X_t \colon t \in T_0 \cup \tilde{R}\} \to \prod\{X_t \colon t \in T_0\} \times \prod\{X_t \colon t \in \tilde{R} \setminus T_0\}$$

is well defined. Set $R = \tilde{R} \setminus T_0$. Obviously, $S \subseteq R$ and $R \in \mathcal{M}_{(g,T_0)}$ which proves that $\mathcal{M}_{(g,T_0)}$ is cofinal in $\exp_\tau(T \setminus T_0)$. The τ-completeness of $\mathcal{M}_{(g,T_0)}$ in $\exp_\tau(T \setminus T_0)$ is obvious. □

Corollary 2.6. *Let $\tau \geq \omega$, $|T| > \tau$, $T_0 \subseteq T$, $|T_0| < |T|$ and $f \colon X \to Y$ be a map between closed subspaces of the product $\prod\{X_t \colon t \in T\}$. If $\pi_{T_0} f = \pi_{T_0}|X$. Then the set*

$$\mathcal{M}_{(f,T_0)} = \Big\{ R \subseteq \exp_\tau(T \setminus T_0) \colon \exists f_{T_0\cup R} \colon \pi_{T_0\cup R}(X) \to \pi_{T_0\cup R}(Y) \text{ with } \pi_{T_0\cup R} f = f_{T_0\cup R}\pi_{T_0\cup R}|X \Big\}$$

is cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$.

Proof. Let $g \colon \prod\{X_t \colon t \in T\} \to \prod\{X_t \colon t \in T\}$ be a map such that $g|X = f$ and $\pi_{T_0} g = \pi_{T_0}$. By Proposition 2.5, the set

$$\mathcal{M}_{(g,T_0)} = \Big\{ R \subseteq \exp_\tau(T\setminus T_0) \colon \exists g_{T_0\cup R} \colon \prod\{X_t \colon t \in T_0\cup R\} \to \prod\{X_t \colon t \in T_0\cup R\} \text{ with } \pi_{T_0\cup R} g = g_{T_0\cup R}\pi_{T_0\cup R} \Big\}$$

is cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$.

For each $R \in \mathcal{M}_{(g,T_0)}$ let $f_{T_0\cup R} = g_{T_0\cup R}|g_{T_0\cup R}(X)$. □

Proposition 2.7. *If, in Proposition 2.5, the map g is a homeomorphism, then the set*

$$\mathcal{H}_{(g,T_0)} = \{ R \in \mathcal{M}_{(g,T_0)} \colon g_{T_0\cup R} \text{ is a homeomorphism} \}$$

is cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$.

Proof. By Proposition 2.5 applied both to g and g^{-1}, the sets $\mathcal{M}_{(g,T_0)}$ and $\mathcal{M}_{(g^{-1},T_0)}$ are cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$. By Proposition 2.1, $\mathcal{M}_{(g,T_0)} \cap \mathcal{M}_{(g^{-1},T_0)}$ is still cofinal and τ-closed. It only remains to note that for each R from this intersection the map $g_{T_0\cup R}$ is a homeomorphism. □

Corollary 2.8. *If, in Corollary 2.6, the map f is a homeomorphism, then the set*

$$\mathcal{H}_{(f,T_0)} = \{R \in \mathcal{M}_{(f,T_0)} \colon f_{T_0 \cup R} \text{ is a homeomorphism } \}$$

is cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$.

We need one more corollary of the Spectral Theorem

Proposition 2.9. *Let $\tau \geq \omega$ and X be a closed subset of the product $\prod\}X_t \colon t \in T\}$ of compact metrizable spaces. If $|T| > \tau$ and $w(X) \leq \tau$, then there exists a subset $S \subseteq T$ such that $|S| = \tau$ and the restriction $\pi_S|X \colon X \to \pi_S(X)$ is a homeomorphism.*

3. Z_τ-SETS

By $\text{cov}(X)$ we denote the collection of all countable functionally open covers of a space X. We introduce the following notation

$$B(f, \{\mathcal{U}_t \colon t \in T\}) = \{g \in C(X,Y) \colon g \text{ is } \mathcal{U}_t\text{-close to } f \text{ for each } t \in T\},$$

Let τ be an infinite cardinal. If X and Y are Tychonov spaces then $C_\tau(X,Y)$ denotes the space of all continuous maps $X \to Y$ with the topology defined as follows ([5], [6, p.273]): a set $G \subseteq C_\tau(X,Y)$ is open if for each $h \in G$ there exists a collection $\{\mathcal{U}_t \colon t \in T\} \subseteq \text{cov}(Y)$, with $|T| < \tau$, such that

$$h \in B(f, \{\mathcal{U}_t \colon t \in T\}) \subseteq G.$$

Obviously if $\tau = \omega$, then the above topology coincides with the limitation topology (see [15]).

The following observation directly follows from definition and we record it for future references.

Lemma 3.1. *Let $\tau \geq \omega$. If $\{G_i \colon i \in I\}$ is a collection of open subsets of $C_\tau(Y,X)$ and $|I| < \tau$, then $\cap\{G_i \colon i \in I\}$ is open.*

In a wide range of situations description of basic neighborhoods in $C_\tau(Y,X)$ is quite simple.

Lemma 3.2. *Let $\tau > \omega$ and X be a z-embedded subspace of a product $\prod\{X_t \colon t \in T\}$ of separable metrizable spaces. If $|T| = \tau$, then basic neighborhoods of a map $f \colon Y \to X$ in $C_\tau(Y,X)$ are of the form $B(f,S) = \{g \in C_\tau(Y,X) \colon \pi_S g = \pi_S f\}$, $S \subseteq T$, $|S| < \tau$, where $\pi_S \colon \prod\{X_t \colon t \in T\} \to \prod\{X_t \colon t \in S\}$ denotes the projection onto the corresponding subproduct.*

Proof. Since X is z-embedded $\prod\{X_t\colon t\in T\}$ every functionally open subset $U\subseteq X$ is of the form $U=\tilde{U}\cap X$ where $\tilde{U}$ is a functionally open subset in $\prod\{X_t\colon t\in T\}$. In turn, since each X_t is separable and metrizable, each such $\tilde{U}$ is of the form $\pi_{S_U}^{-1}(\tilde{V}_U)$, where S_U is a countable subset of T and $\tilde{V}_U$ is open in $\prod\{X_t\colon t\in S_U\}$. Consequently, for any $\mathcal{U}\in\operatorname{cov}(X)$ there exist a countable subset $S_{\mathcal{U}}$ of T and an open cover $\mathcal{V}\in\operatorname{cov}(\pi_{S_{\mathcal{U}}}(X)$ such that $\mathcal{U}=\pi_{S_{\mathcal{U}}}^{-1}(\mathcal{V})$. Further if $\{\mathcal{U}_i\colon i\in I\}\subseteq\operatorname{cov}(X)$ and $|I|<\tau$, then we can find $S\subseteq T$, with $|S|\leq\omega\cdot|I|<\tau$, such that $\mathcal{U}_i=\pi_S^{-1}(\mathcal{V}_i)$ for each $i\in I$ (here $\mathcal{V}_i\in\operatorname{cov}(\pi_S(X))$). Next let $\{\mathcal{W}_j\colon j\in J\}$ be a cofinal collection in $\operatorname{cov}(\pi_S(X))$ such that $|J|=w(\pi_S(X))<\tau$. It is easy to see that for any $f\in C_\tau(Y,X)$ we have

$$B(f,S)=\{g\in C_\tau(Y,X)\colon \pi_S g=\pi_S f\}=B(f,\{\pi_S^{-1}(\mathcal{W}_j)\colon j\in j\}).$$

□

Now we are ready to define Z_τ-sets.

Definition 3.1. Let $\tau\geq\omega$. A closed subset $A\subseteq X$ is a Z_τ-set in X if the set $\{f\in C_\tau(X,X)\colon f(X)\cap A=\emptyset\}$ is dense in the space $C_\tau(X,X)$.

Definition 3.2. Let $\tau\geq\omega$ and $\pi\colon X\to Y$ be a map. A closed subset $A\subseteq X$ is a fibered Z_τ-set in X if the set $\{f\in C_\tau^\pi(X,X)\colon f(X)\cap A=\emptyset\}$ is dense in the space $C_\tau^\pi(X,X)=\{f\in C_\tau(X,X)\colon \pi f=\pi\}$.

Lemma 3.3. *Let $\kappa\geq\omega$ and $\mathcal{S}=\{X_\alpha,p_\alpha^\beta,A\}$ be an inverse κ-spectrum consisting compact spaces X_α of weight $\leq\kappa$ and soft limit projections p_α, $\alpha\in A$. Suppose that F is a closed subset of $X=\lim\mathcal{S}$ containing no closed G_κ-subsets of X. Then for each $\alpha\in A$ there exists $\beta\in A$, with $\beta>\alpha$, such that there is a map $i_\alpha^\beta\colon X_\alpha\to X_\beta$ satisfying the following two properties:*

(1) $p_\alpha^\beta i_\alpha^\beta=\mathrm{id}_{X_\alpha}$,
(2) $i_\alpha^\beta(X_\alpha)\cap p_\beta(F)=\emptyset$

Proof. Let $\alpha_0=\alpha$ and $x_0\in X_{\alpha_0}$. By assumption, $p_{\alpha_0}^{-1}(x_0)\setminus F\neq\emptyset$. Take an index $\alpha_1\in A$ such that $\alpha_1>\alpha_0$ and $(p_{\alpha_0}^{\alpha_1})^{-1}(x_0)\setminus p_{\alpha_1}(F)\neq\emptyset$. Let $x_1\in(p_{\alpha_0}^{\alpha_1})^{-1}(x_0)\setminus p_{\alpha_1}(F)$. The softness of the projection $p_{\alpha_0}^{\alpha_1}\colon X_{\alpha_1}\to X_{\alpha_0}$ guarantees the existence of a map $i_0^1\colon X_{\alpha_0}\to X_{\alpha_1}$ such that $p_{\alpha_0}^{\alpha_1}i_0^1=\mathrm{id}_{X_{\alpha_0}}$ and $i_0^1(x_0)=x_1$. Let

$$V_1=\{x\in X_{\alpha_0}\colon i_0^1(x)\notin p_{\alpha_1}(F)\}.$$

Note that $x_0\in V_1$ and consequently V_1 is a non-empty open subset of X_{α_0}.

Let $\gamma<\kappa^+$. Suppose that for each λ, $1\leq\lambda<\gamma$, we have already constructed an index $\alpha_\lambda\in A$, an open subset $V_\lambda\subseteq X_{\alpha_0}$ and a section $i_0^\lambda\colon X_{\alpha_0}\to X_{\alpha_\lambda}$ of the projection $p_{\alpha_0}^{\alpha_\lambda}\colon X_{\alpha_\lambda}\to X_{\alpha_0}$, satisfying the following conditions:

(i) $\alpha_\lambda<\alpha_\mu$, whenever $\lambda<\mu<\gamma$,

(ii) $\alpha_\mu = \sup\{\alpha_\lambda \colon \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal,
(iii) $V_\lambda \subseteq V_\mu$, whenever $\lambda < \mu < \gamma$,
(iv) $V_\mu = \cup\{V_\lambda \colon \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal,
(v) $i_0^\mu = \triangle\{i_0^\lambda \colon \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal,
(vi) $i_0^\lambda = p_{\alpha_\lambda}^{\alpha_\mu} i_0^\mu$, whenever $\lambda < \mu < \gamma$,
(vii) $V_\lambda = \{x \in X_{\alpha_0} \colon i_0^\lambda(x) \notin p_{\alpha_\lambda}(F)\}$

We shall construct the index α_γ, the open subset $V_\gamma \subseteq X_{\alpha_0}$ and the section $i_0^\gamma \colon X_{\alpha_0} \to X_{\alpha_\gamma}$ of the projection $p_{\alpha_0}^{\alpha_\gamma} \colon X_{\alpha_\gamma} \to X_{\alpha_0}$.

Suppose that γ is a limit ordinal. By (i) and (ii), $\{\alpha_\mu \colon \mu < \gamma\}$ is a chain of length $\leq \kappa$ in A and we let (recall that the indexing set A of is a κ-complete set and therefore contains supremums of κ-chains of its elements)

$$\alpha_\gamma = \sup\{\alpha_\mu \colon \mu < \gamma\} \in A.$$

By the κ-continuity of the spectrum $\mathcal{S}$, compactum X_γ is naturally homeomorphic to the limit of the inverse spectrum $\{X_{\alpha_\mu}, p_{\alpha_\lambda}^{\alpha_\mu}, \lambda, \mu < \gamma\}$. Consequently, by (vi), the diagonal product

$$i_0^\gamma = \triangle\{i_0^\mu \colon \mu < \gamma\} \colon X_{\alpha_0} \to X_{\alpha_\gamma}$$

is well defined and satisfies corresponding conditions (v) and (vi). Let

$$V_\gamma = \{x \in X_{\alpha_0} \colon i_0^\gamma(x) \notin p_{\alpha_\gamma}(F)\}.$$

Note that $V_\gamma = \cup\{V_{\alpha_\mu} \colon \mu < \gamma\}$.

Next consider the case $\gamma = \mu + 1$. Suppose that $V_\mu \neq X_{\alpha_0}$ and let

$$x_\mu = i_0^\mu(z) \in i_0^\mu(X_{\alpha_0}) \subseteq X_{\alpha_\mu},$$

where $z \in X_{\alpha_0} \setminus V_\mu$. By assumption, $p_{\alpha_\mu}^{-1}(x_\mu) \setminus F \neq \emptyset$ (note that $w(X_{\alpha_\mu}) \leq \kappa$). Choose an index $\alpha_\gamma \in A$ so that $\alpha_\gamma > \alpha_\mu$ and $(p_{\alpha_\mu}^{\alpha_\gamma})^{-1}(x_\mu) \setminus p_{\alpha_\gamma}(F) \neq \emptyset$.

Softness of the projection $p_{\alpha_\mu}^{\alpha_\gamma} \colon X_{\alpha_\gamma} \to X_{\alpha_\mu}$ guarantees the existence of a map $i_\mu^\gamma \colon X_{\alpha_\mu} \to X_{\alpha_\gamma}$ such that $p_{\alpha_\mu}^{\alpha_\gamma} i_\mu^\gamma = \mathrm{id}_{X_{\alpha_\mu}}$ and $i_\mu(x_\mu) = z'$, where $z' \in (p_{\alpha_\mu}^{\alpha_\gamma})^{-1}(x_\mu) \setminus p_{\alpha_\gamma}(F)$. Let $i_0^\gamma = i_\mu^\gamma i_0^\mu$ and $V_\gamma = \{x \in X_{\alpha_0} \colon i_0^\gamma(x) \notin p_{\alpha_\gamma}(F)\}$. Note that $V_\mu \subseteq V_\gamma$ and $z \in V_\gamma \setminus V_\mu$. This completes construction of the needed objects in the case $\gamma = \mu + 1$.

Thus the construction can be carried out for each $\lambda < \kappa^+$ and we obtain an increasing collection $\{V_\lambda \colon \lambda < \kappa^+\}$ of length κ^+ of open subsets of the compactum X_{α_0}. Since $w(X_{\alpha_0}) \leq \kappa$, this collection must stabilize, which means that there in an index $\lambda_0 < \kappa^+$ such that $V_\lambda = V_{\lambda_0}$ for any $\lambda \geq \lambda_0$. By construction, this is only possible if $V_{\lambda_0} = X_{\alpha_0}$. Let $\beta = \alpha_{\lambda_0}$ and $i_\alpha^\beta = i_{\alpha_0}^{\lambda_0}$. Clearly $i_\alpha^\beta(X_\alpha) \cap p_\beta(F) = \emptyset$. □

Proposition 3.4. *Let $\tau > \omega$ and $|T| \geq \tau$. For a closed set $M \subseteq I^T$ the following conditions are equivalent:*

(1) *M is a Z_τ-set,*

(2) *If* $F \subseteq M$ *is a closed subset, then* $\psi(F, I^T) \geq \tau$.

Proof. (1) $\Longrightarrow$ (2). If there is a closed set $F \subseteq M$ with $\psi(F, I^T) = \kappa < \tau$, then we can find a subset $R \subseteq T$, with $|R| = \kappa$, such that $F = \pi_R^{-1}(\pi_R(F))$. Choose a collection $\{\mathcal{V}_i \colon i \in I\} \subseteq \mathrm{cov}(I^R)$ so that $|I| = \kappa$ and any two $\{\mathcal{V}_i \colon i \in I\}$-close maps (from any space) into I^R coincide. Let $\mathcal{U}_i = \pi_R^{-1}(\mathcal{V}_i)$, $i \in I$. We claim that image of no map $f \colon I^T \to I^T$, which is $\{\mathcal{U}_i \colon i \in I\}$-close to id_{I^T}, misses M. Indeed, for such a map we have $\pi_R f = \pi_R$. Consequently, $\pi_R(f(F)) = \pi_R(F)$ and $f(F) \subseteq \pi_R^{-1}(\pi_R(F)) = F \subseteq M$.

(2) $\Longrightarrow$ (1). Let $\{\mathcal{U}_i \colon i \in I\} \subseteq \mathrm{cov}(I^T)$ be a collection of functionally open covers of I^T, where $|I| = \kappa < \tau$. We wish to find a map $f \colon I^T \to I^T$ image of which avoids M and which is $\{\mathcal{U}_i \colon i \in I\}$-close to id_{I^T}. Choose a subset $R \subseteq T$, with $|R| = \kappa$, so that for each $i \in I$ we have $\mathcal{U}_i = \pi_R^{-1}(\mathcal{V}_i)$, where $\mathcal{V}_i \in \mathrm{cov}(I^R)$. By Lemma 3.3, there exist $S \subseteq T$, with $R \subseteq S$ and $|S| = \kappa$, and a section $i_R^S \colon I^R \hookrightarrow I^S$ of the projection $\pi_R^S \colon I^S \to I^R$ such that $i_R^S(I^R) \cap \pi_S(M) = \emptyset$. Consider also any section $i_S \colon I^S \hookrightarrow I^T$ of the projection $\pi_S \colon I^T \to I^S$. Then $f = i_S i_R^S \pi_S$ satisfies the needed conditions. □

Proposition 3.5. *Let* $\tau > \omega$ *and* $|T| = \tau$. *For a closed set* $M \subseteq I^T$ *the following conditions are equivalent:*

(1) M *is a* Z_τ*-set in* I^T,

(2) *For each* $T_0 \subseteq T$, *with* $|T_0| < \tau$, *the set*

$$\mathcal{Z}_{(M,T_0)} = \Big\{ S \subseteq \exp_\omega(T \setminus T_0) \colon \pi_{T_0 \cup S}(M) \textit{ is a fibered } Z\textit{-set in } I^{T_0 \cup S} \textit{ with respect to } \pi_{T_0}^{T_0 \cup S} \Big\}$$

is cofinal and ω*-closed in* $\exp_\omega(T \setminus T_0)$.

Proof. (1) $\Longrightarrow$ (2). Let $R \in \exp_\omega(T \setminus T_0)$ and $\{\mathcal{U}_i \colon i \in I\} \subseteq \mathrm{cov}(I^{T_0})$, $|I| = \max\{|T_0|, \omega\}$, be a collection of functionally open covers of I^{T_0} which is cofinal in $\mathrm{cov}(I^{T_0})$ (i.e. any two $\{\mathcal{U}_i \colon i \in I\}$-close maps (from any space) into I^{T_0} coincide).

Let $S_0 = R$ and denote by $\{\mathcal{V}_n \colon n \in \omega\}$ a cofinal collection of open covers in $\mathrm{cov}(I^{S_0 \setminus T_0})$ (again note that any two $\{\mathcal{V}_n \colon n \in \omega\}$-close maps from any space into $I^{S_0 \setminus T_0}$ coincide). Consider the collection $\{\pi_{T_0}^{-1}(\mathcal{U}_i) \colon i \in I\} \cup \{\pi_{S_0 \setminus T_0}^{-1}(\mathcal{V}_n) \colon n \in \omega\} \subseteq \mathrm{cov}(I^T)$. Since M is a Z_τ-set there exists a map $f_1 \colon I^T \to I^T$ which is $\{\pi_{T_0}^{-1}(\mathcal{U}_i) \colon i \in I\} \cup \{\pi_{S_0 \setminus T_0}^{-1}(\mathcal{V}_n) \colon n \in \omega\}$-close to id_{I^T} and such that $f_1(I^T) \cap M = \emptyset$. Clearly $\pi_{T_0} f_1 = \pi_{T_0}$ (choice of $\{\mathcal{U}_i \colon i \in I\}$). By Proposition 2.7, there exist a countable subset $S_1 \subseteq T \setminus T_0$, with $S_0 \subseteq S_1$, and a map $g_1 \colon I^{T_0 \cup S_1} \to I^{T_0 \cup S_1}$ such that $g_1(I^{T_0 \cup S_1}) \cap \pi_{T_0 \cup S_1}(M) = \emptyset$ and $\pi_{T_0 \cup S_1} f_1 = g_1 \pi_{T_0 \cup S_1}$. Note that

$\pi^{T_0\cup S_1}_{T_0} g_1 = \pi^{T_0\cup S_1}_{T_0}$. Choice of $\{\mathcal{V}_n\colon n\in\omega\}$ guarantees that $\pi^{T_0\cup S_1}_{S_0\setminus T_0} g_1 = \pi^{T_0\cup S_1}_{S_0\setminus T_0}$. Representing $I^{T_0\cup S_0}$ as the product $I^{T_0}\times I^{S_0\setminus T_0}$ we see that

$$\pi^{T_0\cup S_1}_{T_0\cup S_0} g_1 = \pi_{T_0} g_1 \triangle \pi_{S_0\setminus T_0} g_1 = \pi^{T_0\cup S_1}_{T_0\cup S_0}\triangle \pi_{S_0\setminus T_0} = \pi^{T_0\cup S_1}_{T_0\cup S_0}.$$

In other words, g_1 acts fiberwise with respect to the projection $\pi^{T_0\cup S_1}_{T_0\cup S_0}$.

Continuing this process we construct an increasing sequence $\{S_n\colon n\in\omega\}$ of countable subsets of $T\setminus T_0$ and maps $g_n\colon I^{T_0\cup S_n}\to I^{T_0\cup S_n}$ so that $\pi^{T_0\cup S_{n+1}}_{T_0\cup S_n} g_{n+1} = \pi^{T_0\cup S_{n+1}}_{T_0\cup S_n}$ and $g_n(I^{T_0\cup S_n})\cap \pi_{T_0\cup S_n}(M)=\emptyset$ for each $n\geq 1$. Let $S=\cup\{S_n\colon n\in\omega\}$. We leave to the reader verification of the fact that the set $\pi_{T_0\cup S}(M)$ is a fibered Z-set in the cube $I^{T_0\cup S}$ with respect to the projection $\pi^{T_0\cup S}_{T_0}$. This proves the cofinality of the set $\mathcal{Z}_{(M,T_0)}$. The ω-completeness of this set is obvious.

(2) $\Longrightarrow$ (1). Let $\{\mathcal{U}_i\colon i\in I\}\subseteq \mathrm{cov}(I^T)$ be a collection of functionally open covers of I^T, where $|I|=\kappa<\tau$. We wish to find a map $f\colon I^T\to I^T$ image of which avoids M and which is $\{\mathcal{U}_i\colon i\in I\}$-close to id_{I^T}. Choose a subset $T_0\subseteq T$, with $|T_0|=\kappa<\tau$, so that for each $i\in I$ we have $\mathcal{U}_i=\pi^{-1}_{T_0}(\mathcal{V}_i)$, where $\mathcal{V}_i\in \mathrm{cov}(I^{T_0})$. By (2), there exist a countable set $R\subseteq T\setminus T_0$ and a map $g\colon I^{T_0\cup R}\to I^{T_0\cup R}$ such that $\pi^{T_0\cup R}_{T_0} g=\pi^{T_0\cup R}_{T_0}$ and $g(I^{T_0\cup R})\cap \pi_{T_0\cup R}(M)=\emptyset$. Let $i_{T_0\cup R}\colon I^{T_0\cup R}\hookrightarrow I^T$ be a section of the projection $\pi_{T_0\cup R}\colon I^T\to I^{T_0\cup R}$ and $f=i_{T_0\cup R}\, g\, \pi_{T_0\cup R}$. Clearly,

$$\pi_{T_0} f = \pi^{T_0\cup R}_{T_0}\pi_{T_0\cup R} i_{T_0\cup R} g \pi_{T_0\cup R} = \pi^{T_0\cup R}_{T_0} g \pi_{T_0\cup R} = \pi^{T_0\cup R}_{T_0}\pi_{T_0\cup R} = \pi_{T_0}.$$

This shows that f and id_{I^T} are $\{\mathcal{U}_i\colon i\in I\}$-close. Since

$$\pi_{T_0\cup R}(f(I^T))\cap \pi_{T_0\cup R}(M) = \pi_{T_0\cup R}(i_{T_0\cup R}(g(\pi_{T_0\cup R}(I^T)))\cap \pi_{T_0\cup R}(M) = g(I^{T_0\cup R})\cap \pi_{T_0\cup R}(M)=\emptyset,$$

we conclude that $f(I^T)\cap M=\emptyset$. □

Proposition 3.6. *Let $\tau\geq\omega$ and X be a compact space of weight $w(X)\leq\tau$. Then the set of Z_τ-embeddings is dense in $C_\tau(X,I^\tau)$.*

Proof. For $\tau=\omega$ the statement is well known and we assume that $\tau>\omega$.

First we show that X admits a Z_τ-embedding $f\colon X\to I^T$.

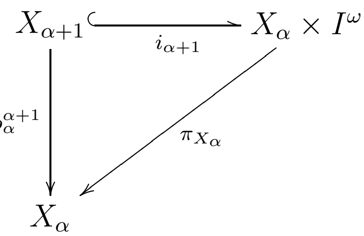
Let $|T|=\tau$ and represent I^T as the limit of the well-ordered continuous spectrum $\mathcal{S}=\{I^{T_\alpha}, \pi^{T_{\alpha+1}}_{T_\alpha}, \tau\}$ so that

(i) $|T_0|=\omega$,
(ii) $T_\alpha\subseteq T_{\alpha+1}$ and $|T_{\alpha+1}\setminus T_\alpha|=\omega$ for each $\alpha<\tau$,
(iii) $T_\beta=\cup\{T_\alpha\colon \alpha<\beta\}$ for each limit ordinal $\beta<\tau$.

Similarly, let $\mathcal{S}_X=\{X_\alpha, p^{\alpha+1}_\alpha, \tau\}$ be a well-ordered continuous spectrum such that

(iv) $\lim \mathcal{S}_X = X$,

(v) $p_\alpha^{\alpha+1}$ has a metrizable kernel, i.e. $X_{\alpha+1}$ can be identified with the subspace of the product $X_\alpha \times I^\omega$ so that $p_\alpha^{\alpha+1} = \pi_{X_\alpha}|X$; here is the diagram



(vi) X_0 is a metrizable compactum.

We will construct a morphism $\mathcal{F} = \{f_\alpha \colon \alpha < \tau\} \colon \mathcal{S}_X \to \mathcal{S}$ the limit map $f = \lim \mathcal{F}$ of which will be the required Z_τ-embedding.

Let $f_0 \colon X_0 \to I^{T_0}$ denote any embedding. Suppose that the embeddings $f_\alpha \colon X_\alpha \to I^{T_\alpha}$ have already been constructed for each $\alpha < \lambda$, where $\lambda < \tau$, so that the following conditions are satisfied:

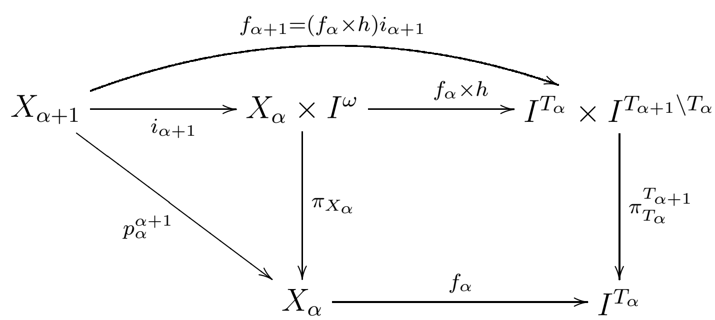
(iv) $f_\alpha p_\alpha^{\alpha+1} = \pi_{T_\alpha}^{T_{\alpha+1}} f_{\alpha+1}$ for each α with $\alpha + 1 < \lambda$,

(v) $f_{\alpha+1}(X_\alpha)$ is a fibered Z-set in $I^{T_{\alpha+1}}$ with respect to the projection $\pi_{T_\alpha}^{T_{\alpha+1}}$ for each α with $\alpha + 1 < \lambda$,

(vi) $f_\beta = \lim\{f_\alpha \colon \alpha < \beta\}$ for each limit ordinal $\beta < \lambda$.

If λ is a limit ordinal let $f_\lambda = \lim\{f_\alpha \colon \alpha < \lambda\}$.

Next consider the case when $\lambda = \alpha+1$. Pick a Z-embedding $h \colon I^\omega \to I^{T_{\alpha+1}\setminus T_\alpha}$. Then, as the diagram explains,



$f_{\alpha+1} = (f_\alpha \times h)i_{\alpha+1}$ is the needed fibered Z-embedding of $X_{\alpha+1}$ into $I^{T_{\alpha+1}}$.

This completes the construction. Now consider the embedding $f = \lim\{f_\alpha \colon \alpha < \tau\} \colon X \to I^T$. Proposition 3.5 guarantees that $f(X)$ is a Z_τ-set in I^T.

For the general case consider a map $g \colon X \to I^T$ and a collection $\{\mathcal{U}_i \colon i \in I\} \subseteq \mathrm{cov}(I^T)$ such that $|I| < \tau$. We wish to construct a Z_τ-embedding $f \colon X \to I^T$ which is $\{\mathcal{U}_i \colon i \in I\}$-close to g. Find a subset $T_0 \subseteq T$ such that $|T_0| < \tau$ and $\mathcal{U}_i = \pi_{T_0}^{-1}(\mathcal{V}_i)$ for each $i \in I$, where $\mathcal{V}_i \in \mathrm{cov}(I^{T_0})$. By the first part of this proof, there exists a Z_τ-embedding $h\colon X \to I^{T\setminus T_0}$. Let $f = \pi_{T_0} g \triangle h$. It is easy to see that f is a Z_τ-embedding. Since $\pi_{T_0} f = \pi_{T_0} g$ it follows that f and g are $\{\mathcal{U}_i \colon i \in I\}$-close. □

A stronger version of this statement will be proved in Section 4 (Theorem 4.4).

4. Z_τ-set Unknotting

In this section we prove several versions of Z_τ-set unknotting theorem. As was noted in the introduction Z_ω-set unknotting theorem is not valid in I^τ for $\tau > \omega$. Extension of homeomorphisms between Z-sets (i.e. Z_ω-sets) of I^τ was studied in [13].

Lemma 4.1. *Let $M, N \subseteq I^T \times I^\omega$ be fibered Z-sets with respect to the projection $\pi_1 \colon I^T \times I^\omega \to I^T$. Then every homeomorphism $h\colon M \to N$ such that $\pi_1 h = \pi_1 | M$ admits an extension $H \colon I^T \times I^\omega \to I^T \times I^\omega$ such that $\pi_1 H = \pi_1$.*

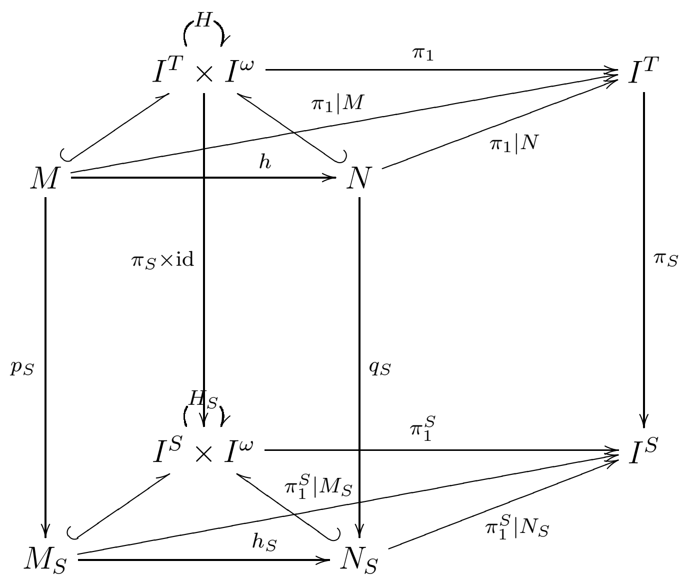
Proof. If $|R| \leq \omega$, then lemma is known [16]. Let $|R| > \omega$. Consider ω-spectrum $\mathcal{S} = \{I^S, \pi^R_S, \exp_\omega T\}$, consisting of countable subproducts of I^T and corresponding projections $\pi^R_S \colon I^R \to I^S$. Then the product $I^T \times I^\omega$ is the limit of the spectrum $\mathcal{S} \times \mathrm{id} = \{I^S \times I^\omega, \pi^R_S \times \mathrm{id}_{I^\omega}, \exp_\omega T\}$. Note that for each $S \in \exp_\omega T$ the following commutative diagram

$$\begin{array}{ccc} I^T \times I^\omega & \xrightarrow{\pi_1} & I^T \\ \downarrow{\scriptstyle \pi_S \times \mathrm{id}} & & \downarrow{\scriptstyle \pi_S} \\ I^S \times I^\omega & \xrightarrow{\pi_1^S} & I^S \end{array}$$

is a pullback square.

Consider the following ω-spectra $\mathcal{S}_M = \{M_S, p^R_S, \exp_\omega T\}$ and $\mathcal{S}_N = \{N_S, q^R_S, \exp_\omega T\}$, where $M_S = (\pi_S \times \mathrm{id})(M)$, $N_S = (\pi_S \times \mathrm{id})(N)$, $p^R_S = (\pi^R_S \times \mathrm{id})|M_R$ and $q^R_S = (\pi^R_S \times \mathrm{id})|N_R$, $S \subseteq R$, $S, R \in \exp_\omega T$. It follows from Corollary 2.6 that the set of countable subsets $S \subseteq T$ for which there exists a homeomorphism $h_S \colon M_S \to N_S$ such that $(\pi_S \times \mathrm{id})h = h_S(\pi_S \times \mathrm{id})$ is cofinal and ω-closed in $\exp_\omega T$. It is clear that for any such $S \in \exp_\omega T$ we have $\pi_1^S h_S = \pi_1^S | M_S$.

Here is the corresponding diagram:



Next we show that the set

$$\{S \in \exp_\omega T\colon\ M_S \text{ and } N_S \text{ are fibered } Z\text{-sets in } I^S \times I^\omega\}$$

is cofinal and ω-closed in $\exp_\omega T$.

Consider the following relation

$$L_M = \{(S,R) \in (\exp_\omega T)^2\colon\ \forall\, \mathcal{U} \in \operatorname{cov}(I^S \times I^\omega)\ \exists f_{\mathcal{U}}\colon I^R \times I^\omega \to I^R \times I^\omega \text{ such that } f_{\mathcal{U}}(I^R \times I^\omega) \cap \pi_R(M) = \emptyset, \pi_1^R f_{\mathcal{U}} = \pi_1^R \text{ and } (\pi_S^R \times \mathrm{id}) f_{\mathcal{U}} \text{ is } \mathcal{U}\text{-close to } \pi_S^R \times \mathrm{id}\}$$

Existence. Let $\mathcal{U}_n \in \operatorname{cov}(I^S \times I^\omega)$ and $\operatorname{mesh}(\mathcal{U}_n) < 2^{-n}$. Since M is a fibered Z-set in $I^T \times I^\omega$ there exists a map $f_{\mathcal{U}_n}\colon I^T \times I^\omega \to I^T \times I^\omega$ such that $f_{\mathcal{U}_n}(I^T \times I^\omega) \cap M = \emptyset$, $\pi_1 f_{\mathcal{U}_n} = \pi_1$ and $f_{\mathcal{U}_n}$ is $(\pi_S \times \mathrm{id})^{-1}(\mathcal{U}_n)$-close to $\mathrm{id}_{I^T \times I^\omega}$. By Corollary 2.6, the set

$$\{R \in \exp_\omega T\colon \exists \text{ a map } f^R_{\mathcal{U}_n}\colon I^R \times I^\omega \to I^R \times I^\omega \text{ with } f^R_{\mathcal{U}_n}(\pi_R \times \mathrm{id}) = (\pi_R \times \mathrm{id}) f_{\mathcal{U}_n}\}$$

is cofinal and ω-closed in $\exp_\omega T$. Choose a countable subset $R_n \subseteq T$ in the above indicated set such that $S \subseteq R_n$. In addition, we may assume that $f^{R_n}_{\mathcal{U}_n}(I^T \times I^\omega) \cap M_{R_n} = \emptyset$. Note that by construction, $\pi_1^{R_n} f^{R_n}_{\mathcal{U}_n} = \pi_1^{R_n}$ and $f^{R_n}_{\mathcal{U}_n}$ is $(\pi_S^{R_n} \times \mathrm{id})^{-1}(\mathcal{U}_n)$-close to $\mathrm{id}_{I^{R_n} \times I^\omega}$. Select the sets R_n, $n \in \omega$, so that $R_n \subseteq R_{n+1}$. Straightforward verification shows that the set $R = \cup\{R_n\colon n \in \omega\}$ has all the required properties. This shows that $(S,R) \in L_M$.

We leave to the reader verification of the majorantness and ω-closedness conditions from Proposition 2.3. Thus, by this proposition, the set of L-reflexive elements is cofinal and ω-closed in $\exp_\omega T$. Let S be such a element. Then the homeomorphism h_S, as a homeomorphism between fibered Z-sets, has an extension $H_S\colon I^S\times I^\omega \to I^S\times I^\omega$ such that $\pi_1^S H_S = \pi_1^S$ (see [16]). The fact that the above indicated diagram is a pullback square uniquely defines a homeomorphism $H\colon I^T\times I^\omega \to I^T\times I^\omega$ such that $\pi_1 H = \pi_1$ and $(\pi_S\times \mathrm{id})H = H_S(\pi_S\times\mathrm{id})$. It only remains to note that $H|M = h$. □

Lemma 4.2. *Let M be a fibered Z-set in $I^S\times I^\omega$ with respect to the projection $\pi_1\colon I^S\times I^\omega \to I^S$. Suppose that $H\colon I^S\to I^S$ is a homeomorphism. If $\tilde{H}\colon I^S\times I^\omega \to I^S\times I^\omega$ is a homeomorphism such that $\pi_1\tilde{H} = H\pi_1$, then $\tilde{H}(M)$ is also a fibered Z-set $I^S\times I^\omega$.*

Proof. Let $\mathcal{U}\in \mathrm{cov}(I^S\times I^\omega)$. Since M is a fibered Z-set there exists a map $f\colon I^S\times I^\omega \to I^S\times I^\omega$ such that $f(I^S\times I^\omega)\cap M = \emptyset$, $\pi_1 f = \pi_1$ and f is $\tilde{H}^{-1}(\mathcal{U})$-close to $\mathrm{id}_{I^S\times I^\omega}$. We set $g = \tilde{H}f\tilde{H}^{-1}$. Let $x\in I^S\times I^\omega$ and consider the point $\tilde{H}^{-1}(x)$. Since f and id are $\tilde{H}^{-1}(\mathcal{U})$-close, there exists an element $U\in\mathcal{U}$ such that $f(\tilde{H}^{-1}(x)), \tilde{H}^{-1}(x) \in \tilde{H}^{-1}(U)$. Then $g(x) = \tilde{H}(f(\tilde{H}^{-1}(x))), x = \tilde{H}(\tilde{H}^{-1}(x)) \in \tilde{H}(\tilde{H}^{-1}(U)) = U$ which shows that g is $\mathcal{U}$-close to $\mathrm{id}_{I^S\times I^\omega}$. Note also that $\pi_1 g = \pi_1\tilde{H}f\tilde{H}^{-1} = H\pi_1 f\tilde{H}^{-1} = H\pi_1\tilde{H}^{-1} = \pi_1\tilde{H}\tilde{H}^{-1} = \pi_1$. Finally since $I^S\times I^\omega = \tilde{H}^{-1}(I^S\times I^\omega)$, we have $f(\tilde{H}^{-1}(I^S\times I^\omega))\cap M = f(I^S\times I^\omega)\cap M = \emptyset$. It only remains to note that

$$g(I^S\times I^\omega)\cap\tilde{H}(M) = \tilde{H}(f(\tilde{H}^{-1}(I^S\times I^\omega)))\cap\tilde{H}(M) = \tilde{H}(f(H^{-1}(I^S\times I^\omega))\cap M) = \emptyset.$$

□

Theorem 4.3. *Let $|T| = \tau > \omega$. For each collection $\{\mathcal{U}_\gamma\colon \gamma\in\Gamma\}) \subseteq \mathrm{cov}(I^T)$, with $|\Gamma| < |T|$, there exists collection $\{\mathcal{V}_\gamma\colon \gamma\in\Gamma'\}\subseteq \mathrm{cov}(I^T)$, with $|\Gamma'| < |T|$ such that any homeomorphism $h\colon M\to N$ of Z_τ-sets of I^T, which is $\{\mathcal{V}_\gamma\colon \gamma\in\Gamma'\}$-close to Id_M, can be extended to a homeomorphism $H\colon I^T\to I^T$ which is $\{\mathcal{V}_\gamma\colon\gamma\in\Gamma'\}$-close to Id_{I^T}.*

Proof. Since $|\Gamma| < |T|$ and $|T| > \omega$, we can find a subset $S\subseteq T$ and a cover $\mathcal{U}_\gamma^S \in \mathrm{cov}(I^S)$, $\gamma\in\Gamma$, so that $\mathcal{U}_\gamma = \pi_S^{-1}(\mathcal{U}_\gamma^S)$ for each $\gamma\in\Gamma$. Choose a collection $\{\mathcal{V}_\gamma^S\colon\gamma\in\Gamma'\}\subseteq\mathrm{cov}(I^S)$ in such a way that any two $\{\mathcal{V}_\gamma^S\colon\gamma\in\Gamma'\}$-close maps (from any space into I^S) coincide. Set $\mathcal{V}_\gamma = \pi_S^{-1}(\mathcal{V}_\gamma^S)$, $\gamma\in\Gamma'$.

Let also $h\colon M\to N$ be a homeomorphism between Z_τ-sets of I^T which is $\{\mathcal{V}_\gamma\colon\gamma\in\Gamma'\}$-close to Id_M. It is clear from the choice of the collection $\{\mathcal{V}_\gamma^S\colon\gamma\in\Gamma'\}$ that $\pi_S h = \pi_S|M$.

Next for each $\alpha<\tau$ we construct a subset $T_\alpha\subseteq T$ and a homeomorphism $H_\alpha\colon I^{T_\alpha}\to I^{T_\alpha}$, $\alpha<\tau$, satisfying the following conditions:

(i) $T_\alpha\subseteq T_\beta$ whenever $\alpha<\beta$;

(ii) $T_\beta = \cup\{T_\alpha \colon \alpha < \beta\}$, whenever β is a limit ordinal;
(iii) $|T_{\alpha+1} \setminus T_\alpha| \le \omega$ for each $\alpha < \tau$;
(iv) $\pi^{T_\beta}_{T_\alpha} H_\beta = H_\alpha \pi^{T_\beta}_{T_\alpha}$, whenever $\alpha < \beta$;
(v) $H_\beta = \lim\{H_\alpha \colon \alpha < \beta\}$, whenever β is a limit ordinal;
(vi $H_\alpha | \pi_{T_\alpha}(M) = h_\alpha$, where $h_\alpha \colon \pi_{T_\alpha}(M) \to \pi_{T_\alpha}(N)$ denotes the homeomorphism such that $\pi_{T_\alpha} h =_\alpha h\pi_{T_\alpha}|M$, $\alpha < \tau$.

Suppose that these objects, satisfying above conditions for corresponding indices, have already been constructed for each $\alpha < \beta$. We need to construct T_β and H_β.

If β is a limit ordinal we set $T_\beta = \cup\{T_\alpha \colon \alpha < \beta\}$ and $H_\beta = \lim\{H_\alpha \colon \alpha < \beta\}$.

Consider the case $\beta = \alpha + 1$. Choose, based on Corollary 2.6 and Proposition 3.5, a countable subset $R \subseteq T \setminus T_\alpha$ satisfying the following conditions:

(1) the sets $\pi_{T_\alpha \cup R}(M)$ and $\pi_{T_\alpha \cup R}(N)$ are fibered Z-sets with respect to the projection $\pi^{T_\alpha \cup R}_{T_\alpha} \colon I^{T_\alpha \cup R} \to I^{T_\alpha}$
(2) there is a homeomorphism $h_{(\alpha,R)} \colon \pi_{T_\alpha \cup R}(M) \to \pi_{T_\alpha \cup R}(N)$ such that $\pi_{T_\alpha \cup R} h = h_{\alpha,R)} \pi_{T_\alpha \cup R}|M$.
(3) R contains any element of $T \setminus T_\alpha$ specified in advance.

Let $T_\beta = T_\alpha \cup R$ and $h_\beta = h_{(\alpha,R)}$. Note that

(4) $\pi^{T_\beta}_{T_\alpha} h_\beta = h_\alpha \pi^{T_\beta}_{T_\alpha} | \pi_{T_\beta}(M) = H_\alpha \pi^{T_\beta}_{T_\alpha} | \pi_{T_\beta}(M)$.

Consider the homeomorphism

$$\tilde{H}_\beta = H_\alpha \times \mathrm{id}_{I^{T_\beta \setminus T_\alpha}} \colon I^{T_\alpha} \times I^{T_\beta \setminus T_\alpha} \to I^{T_\alpha} \times I^{T_\beta \setminus T_\alpha}$$

and let $\tilde{h}_\beta = \tilde{H}^{-1}_\beta h_\beta \colon \pi_{T_\beta}(M) \to \tilde{H}^{-1}_\beta(h_\beta(\pi_{T_\beta}(M)))$. Clearly $\pi^{T_\beta}_{T_\alpha} \tilde{h}_\beta = \pi^{T_\beta}_{T_\alpha} | \pi_{T_\beta}(M)$. By Lemmas 4.2 and 4.1, there exists a homeomorphism $F \colon I^{T_\beta} \to I^{T_\beta}$ such that $\pi^{T_\beta}_{T_\alpha} F = \pi^{T_\beta}_{T_\alpha}$ and $F|\pi_{T_\beta}(M) = \tilde{h}_\beta$. Finally, let $H_\beta = \tilde{H}_\beta F$. Note that $H_\beta | \pi_{T_\beta}(M) = h_\beta$ and $\pi^{T_\beta}_{T_\alpha} H_\beta = H_\alpha \pi^{T_\beta}_{T_\alpha}$.

This completes the construction. By (3) we can ensure that $T = \cup\{T_\alpha \colon \alpha < \tau\}$. Then $H = \lim\{H_\alpha \colon \alpha < \tau\}$ is an extension of h such that $\pi_S H = \pi_S$ (i.e. H is $\{\mathcal{V}_\gamma \colon \gamma \in \Gamma'\}$-close to Id_{I^T}. □

The following statement strengthens Proposition 3.6

Theorem 4.4. *Let $\tau \ge \omega$. Let $f \colon X \to I^\tau$ be a map such that for a closed subset $F \subseteq X$ the restriction $f|F$ is a Z_τ-embedding. Then for any collection $\{\mathcal{U}_i \colon i \in I\} \subseteq \mathrm{cov}(I^\tau)$, with $|I| < \tau$, there exists a Z_τ-embedding $g \colon X \to I^\tau$ which is $\{\mathcal{U}_i \colon i \in I\}$-close to f and coincides with f on F.*

Proof. According to Theorem 4.3, choose a collection $\{\mathcal{V}_j \colon j \in J\} \subseteq \mathrm{cov}(I^\tau)$, with $|J| < \tau$, so that for any homeomorphism $h \colon M \to N$ between Z_τ-sets of I^τ which is $\{\mathcal{V}_j \colon j \in J\}$-close to the inclusion $M \hookrightarrow I^\tau$ there is a homeomorphism $H \colon I^\tau \to I^\tau$, extending h which is $\{\mathcal{V}_j \colon j \in J\}$-close to id_{I^τ}. By Proposition 3.6, there is a Z_τ-embedding $f_1\colon X \to I^\tau$ which is $\{\mathcal{V}_j\colon j \in J\}$-close to f. Note that $f_1(F)$ is a Z_τ-set in I^τ. Consider the homeomorphism $ff_1^{-1}|f_1(F)\colon f_1(F) \to f(F)$. Clearly, $ff_1^{-1}|f_1(F)$ is $\{\mathcal{V}_j\colon j \in J\}$-close to the inclusion $f_1(F) \hookrightarrow I^\tau$. Then, by the choice of the collection $\{\mathcal{V}_j\colon j \in J\}$, there exists a homeomorphism $H\colon I^\tau \to I^\tau$ such that $H|f_1(F) = ff_1^{-1}|f_1(F)$ and which is $\{\mathcal{V}_j\colon j \in J\}$-close to id_{I^τ}. Then the required Z_τ-embedding $g\colon X \to I^\tau$ can be defined as $g = Hf_1$. $\square$

Combining proofs of Theorem 4.3 and Lemma 4.1 we arrive to the following statement. Its proof is left to the reader.

Theorem 4.5. *Let $\omega \le |S| \le |T| = \tau$. Let $M, N \subseteq I^T \times I^S$ be fibered Z_τ-sets with respect to the projection $\pi\colon I^T \times I^S \to I^T$. Then every homeomorphism $h\colon M \to N$ such that $\pi h = \pi|M$ admits an extension $H\colon I^T \times I^S \to I^T \times I^S$ such that $\pi H = \pi$.*

Moreover, for any collection $\{\mathcal{U}_\gamma\colon \gamma \in \Gamma\}) \subseteq \mathrm{cov}(I^T \times I^S)$, with $|\Gamma| < |T|$, there exists a collection $\{\mathcal{V}_\gamma\colon \gamma \in \Gamma'\} \subseteq \mathrm{cov}(I^T \times I^S)$, with $|\Gamma'| < |T|$, such that any homeomorphisms $h\colon M \to N$, with $\pi h = \pi|M$, of fibered Z_τ-sets of $I^T \times I^S$, which is $\{\mathcal{V}_\gamma\colon \gamma \in \Gamma'\}$-close to Id_M, can be extended to a homeomorphism $H\colon I^T \times I^S \to I^T \times I^S$ which is $\{\mathcal{V}_\gamma\colon \gamma \in \Gamma'\}$-close to Id_{I^T} and such that $\pi H = \pi$.

5. τ-skeletoids and closed subspaces of Σ-products

In this section we introduce notions of τ-skeleton and τ-skeletoid. Our definition is a modification of the concept of τ-retrada [9, Definition 2.1].

5.1. General properties of τ-skeletoids.

Definition 5.1. Let $\tau \ge \omega$. A collection $\mathcal{L} = \{X_a, r_a, A\}$, consisting of compact subsets X_a of a space Y and retractions $r_a\colon Y \to X_a$, is a τ-skeleton in Y if the following conditions are satisfied:

(1) The indexing set A is strongly τ-complete;
(2) For each $a \in A$, X_a is a compact space of weight $\le \tau$ and there exists a retraction $r_a\colon Y \to X_a$;
(3) If $a \le b$, then $X_a \subseteq X_b$ and $r_a r_b = r_a$;
(4) For each $a, b \in A$, $X_{\inf(a,b)} = X_a \cap X_b$ and $r_a r_b = r_{\inf(a,b)} = r_b r_a$;
(5) If $\{a_i\colon i \in I\}$, $|I| \le \tau$, is a chain of elements of A with $a = \sup\{a_i\colon i \in I\}$, then $X_a = \mathrm{cl}(\cup\{X_{a_i}\colon i \in I\})$ and $r_a(y) = \lim\{r_{a_i}(y)\colon i \in I\}$ for any $y \in Y$.
(6) If F is a functionally closed subset in Y, then $F = r_a^{-1}(F_a)$, where $a \in A$ and F_a is a functionally closed subset in X_a.

Definition 5.2. Let $\tau \ge \omega$ and $\mathcal{L} = \{X_a, r_a, A\}$ be a τ-skeleton in Y. The subspace $X = \cup\{X_a\colon a \in A\}$ is called the τ-skeletoid in Y. If X is a τ-skeletoid in X, we simply say that X is a τ-skeletoid.

Proposition 5.1. *Let $\tau \geq \omega$ and X be a τ-skeletoid (in any space). Then $t(X) \leq \tau$.*

Proof. Let $M \subseteq X$ and $x \in \mathrm{cl}_X M$. We wish to find a subset $R \subseteq M$ such that $|R| \leq \tau$ and $x \in \mathrm{cl}_X R$. Let $\mathcal{L} = \{X_a, r_a, A\}$ be the τ-skeleton in Y such that $X = \cup\{X_a \colon a \in A\}$. Choose $a_0 \in A$ big enough to ensure that $x \in X_{a_0}$. Since $w(r_{a_0}(M)) \leq w(X_{a_0}) \leq \tau$, there exists a subset $S_0 \subseteq r_{a_0}(M)$ such that $|S_0| \leq \tau$ and $\mathrm{cl}_X S_0 = \mathrm{cl}_X(r_{a_0}(M))$. Then we have

$$x = r_{a_0}(x) \in r_{a_0}(\mathrm{cl}_X M) \subseteq \mathrm{cl}_X(r_{a_0}(M)) = \mathrm{cl}_X S_0.$$

Choose $R_1 \subseteq M$ such that $S_0 = r_{a_0}(R_1)$ and $|R_1| \leq \tau$. If $x \in \mathrm{cl}_X R_1$, then we are done. If not, we proceed as follows. Choose an index $a_1 \geq a_0$ so that $R_1 \subseteq X_{a_1}$. Note that $r_{a_0}|X_{a_1} \colon X_{a_1} \to X_{a_0}$, as map of compact spaces, is closed. Therefore,

$$x \in \mathrm{cl}_X S_0 = \mathrm{cl}_X(r_{a_0}(R_1)) = r_{a_0}(\mathrm{cl}_X R_1)$$

Suppose that for each $k \leq m$ we have already constructed an index $a_k \in A$ and a set R_k satisfying the following conditions:

(i) $a_k \leq a_{k+1}$;
(ii) $|R_k| \leq \tau$;
(iii) $R_k \subseteq M \cap X_{a_k}$;
(iv) $x \in r_{a_k}(\mathrm{cl}_X R_{k+1})$.

Let us construct an index a_{m+1} and a set R_{m+1}. Since $w(r_{a_m}(M)) \leq w(X_{a_m}) \leq \tau$ we can choose a set $S_m \subseteq r_{a_m}(M)$ such that $|S_m| \leq \tau$ and $\mathrm{cl}_X S_m = \mathrm{cl}_X(r_{a_m}(M))$. Then

$$x = r_{a_m}(x) \in r_{a_m}(\mathrm{cl}_X M) \subseteq \mathrm{cl}_X(r_{a_m}(M)) = \mathrm{cl}_X S_m.$$

Let $R_{m+1} \subseteq M$ such that $S_m = r_{a_m}(R_{m+1})$ and $|R_{m+1}| \leq \tau$. If $x \in \mathrm{cl}_X R_{m+1}$, then we are done. If not, we proceed as follows. Choose an index $a_{m+1} \geq a_m$ so that $R_{m+1} \subseteq X_{a_{m+1}}$. Note that $r_{a_m}|X_{a_{m+1}} \colon X_{a_{m+1}} \to X_{a_m}$, as map of compact spaces, is closed. Therefore,

$$x \in \mathrm{cl}_X S_m = \mathrm{cl}_X(r_{a_m}(R_{m+1})) = r_{a_m}(\mathrm{cl}_X R_{m+1})$$

This completes the inductive construction.

Now let $a = \sup\{a_k \colon k \in \omega\}$. Since $X_a \supseteq \cup\{X_{a_k} \colon k \in \omega\}$ we conclude that $R = \cup\{R_k \colon k \geq 1\} \subseteq M \cap X_a$. Clearly $|R| \leq \tau \cdot \omega = \tau$. We claim that $x \in \mathrm{cl}_X R$. Assuming the contrary, by condition 5 of Definition 5.1, we can find an index $n \geq 1$ such that $x = r_{a_n}(x) \notin r_{a_n}(\mathrm{cl}_X R)$. Consequently, $x \notin r_{a_n}(\mathrm{cl}_X R_{n+1})$, contradicting property (iv). Thus $x \in \mathrm{cl}_X R$ as required. □

Corollary 5.2. *Let $\tau \geq \omega$, X be a τ-skeletoid (in any space) and $\mathcal{L} = \{X_a, r_a, A\}$ be the corresponding τ-skeleton. If $F \subset X$ and $F \cap X_a$ is closed in X_a for each $a \in A$, then F is closed in X.*

Proof. Let $x \in \mathrm{cl}_X F$. Since $t(X) \leq \tau$ (Proposition 5.1), there exists a subset $R \subseteq F$ such that $|R| \leq \tau$ and $x \in \mathrm{cl}_X R$. Choose $a \in A$ so that $R \subseteq X_a$. Then $R \subseteq F \cap X_a$ and since the latter set is closed we have $x \in \mathrm{cl}_X R \subseteq F \cap X_a \subseteq F$. □

Corollary 5.3. *Let $\tau \geq \omega$ and X be a τ-skeletoid. If $\{M_\alpha \colon \alpha < \kappa\}$ is an increasing well ordered collection of closed subsets of X and $\kappa \geq \tau^+$, then $M = \cup\{M_\alpha \colon \alpha < \kappa\}$ is closed in X.*

Proof. Let $x \in \mathrm{cl}_X M$. Since $t(X) \leq \tau$ (Proposition 5.1), there exists a subset $R \subseteq M$ such that $x \in \mathrm{cl}_X R$ and $|R| \leq \tau$. Obviously $R \subseteq X_a$ for some $a \in A$, where $\mathcal{L} = \{X_a, r_a, A\}$ is the corresponding τ-skeleton. Thus $\mathrm{cl}_X R \subseteq X_a$ and $w(\mathrm{cl}_X R) \leq w(X_a) \leq \tau$. Next consider the increasing well ordered collection $\{M_\alpha \cap \mathrm{cl}_X R \colon \alpha < \kappa\}$ of closed subsets of $\mathrm{cl}_X R$. Since $w(\mathrm{cl}_X R) \leq \tau$, this collection must stabilize, i.e. there exists $\alpha < \tau^+$ such that $M_\alpha \cap \mathrm{cl}_X R = M_\beta \cap \mathrm{cl}_X R$ for each $\beta \geq \alpha$. Therefore, $M_\alpha \cap \mathrm{cl}_X R = M \cap \mathrm{cl}_X R$. Since $R \subseteq M \cap \mathrm{cl}_X R$ it follows that $x \in \mathrm{cl}_X R \subseteq M_\alpha \cap \mathrm{cl}_X R \subseteq M_\alpha \subseteq M$. □

Proposition 5.4. *Let $\tau \geq \omega$ and $\mathcal{L} = \{X_a, r_a, A\}$ be a τ-skeleton in Y. Suppose that a subset F of the τ-skeletoid X is such that $F \cap X_a$ is closed in X_a for each $a \in A$. Then the set*

$$A_F = \{a \in A \colon r_a(F) = F \cap X_a\}$$

is cofinal and τ-closed in A.

Proof. Let $a \in A$. In order to find $b \geq a$ such that $b \in A_F$ we inductively construct an increasing sequence of indexes. Let $a_0 = a$. Consider an open base $\mathcal{U}_0$ of cardinality $\leq \tau$ of a compactum X_{a_0} and let $\mathcal{V}_0 = \{U \in \mathcal{U}_0 \colon U \cap r_{a_0}(F) \neq \emptyset\}$. For each $U \in \mathcal{V}_0$ choose an index $a_U \geq a_0$ such that $r_{a_0}^{-1}(U) \cap F \cap X_{a_U} \neq \emptyset$. Since $\mathcal{V}_0| \leq \tau$, Corollary 2.2 guarantees that there exists an index $a_1 \in A$ such that $a_1 \geq a_U$ for each $U \in \mathcal{V}_0$. We proceed in the same manner. If an index $a_k \in A$ has already been constructed, then we consider an open base $\mathcal{U}_k$ of X_{a_k} of cardinality $\leq \tau$ and let $\mathcal{V}_k = \{U \in \mathcal{U}_k \colon U \cap r_{a_k}(F) \cap U \neq \emptyset\}$. For each $U \in \mathcal{V}_k$ choose an index $a_U \geq a_k$ such that $r_{a_k}^{-1}(U) \cap F \cap X_{a_U} \neq \emptyset$. As above, there is an index $a_{k+1} \in A$ such that $a_{k+1} \geq a_U$ for each $U \in \mathcal{V}_k$. Let $b = \sup\{a_k \colon k \in \omega\}$. We wish to show that $b \in A_F$. Assume the contrary. Then there exists a point $x \in r_b(F) \setminus (F \cap X_b)$. Since the intersection $F \cap X_b$ is closed in X_b we can find an open neighborhood G of x in X_b such that $G \cap (F \cap X_b) = \emptyset$. It follows from condition 5 of Definition 5.1 that there exist an index $k \in \omega$ and a set $U \in \mathcal{U}_k$ such that $r_{a_k}^{-1}(U) \cap X_b \subseteq G$. Clearly $r_{a_k}^{-1}(U) \cap F \neq \emptyset$, i.e. $U \in \mathcal{V}_k$. By construction, $r_{a_k}^{-1}(U) \cap F \cap X_{a_{k+1}} \neq \emptyset$. On the other hand we have

$$r_{a_k}^{-1}(U) \cap F \cap X_{a_{k+1}} \subseteq r_{a_k}^{-1}(U) \cap F \cap X_b \subseteq G \cap F \cap X_b = \emptyset.$$

This contradiction proves that $b \in A_F$. Thus A_F is cofinal in A. The τ-closedness of A_F is a straightforward consequence of condition 5. □

This statement has several useful corollaries.

Corollary 5.5. *Let $\tau \geq \omega$. Every τ-skeletoid (in any space) is a normal space.*

Proof. Let $\mathcal{L} = \{X_a, r_a, A\}$ be a τ-skeleton in a space Y and $X = \cup\{X_a \colon a \in A\}$ is the corresponding τ-skeletoid. If F_1 and F_2 are disjoint closed subsets of X, then, by Proposition 5.4, there exists an index $a \in A$ such that $r_a(F_k) = F_k \cap X_a$, $k = 1, 2$. Since X_a is normal, there are disjoint open subsets G_1, G_2 in X_a such that $r_a(F_k) \subseteq G_k$. Clearly $F_k \subseteq r_a^{-1}(G_k)$, $k = 1, 2$, and $r_a^{-1}(G_1) \cap r_a^{-1}(G_2) = \emptyset$. □

Lemma 5.6. *Let $\tau \geq \omega$ and X be τ-skeletoid in Y. Then X is G_τ-closed in Y, i.e. every non-empty set, representable as an intersection of at most τ open sets in Y, intersects X. In particular, X is dense in Y.*

Proof. Let $\mathcal{L} = \{X_a, r_a, A\}$ be the corresponding τ-skeleton. Consider a set $G = \cap\{G_i \colon i \in I\}$, where $|I| \leq \tau$ and G_i is open in Y. Let $y \in G$. By condition 6 of Definition 5.1, $y \in r_{a_i}^{-1}(F_i) \subseteq G_i$ for some functionally closed set $F_i \subseteq X_{a_i}$, $i \in I$. Choose an index $a \in A$ such that $a \geq a_i$ for each $i \in I$ and let $Z_i = r_{a_i}^{-1}(F_i) \cap X_a$. Note that $y \in \cap\{r_a^{-1}(Z_i) \colon i \in I\} \subseteq G$. Clearly $r_a(y) \in G \cap X_a$. □

Corollary 5.7. *Let $\tau \geq \omega$ and X be τ-skeletoid in Y. Then X is C-embedded in Y.*

Proof. X is C^*-embedded in Y because any two disjoint closed subsets in X are functionally separated in Y (see the proof of Corollary 5.5). Since X is G_δ-dense in Y it follows that X is C-embedded in Y. □

Lemma 5.8. *Let $\tau \geq \omega$ and X be a τ-skeletoid in Y. Let also $X \subseteq Z \subseteq Y$. Then for any map $f \colon Z \to F$, where $w(F) \leq \tau$, there exist $a \in A$ and a map $f_a \colon X_a \to F$ such that $f = f_a r_a | Z$.*

Proof. Let $\{U_i \colon i \in I\}$, where $|I| \leq \tau$, be a base in F consisting of functionally open subsets. According to condition 6 of Definition 5.1 there exists an index $a_i \in A$ and a functionally open subset U_{a_i} of X_{a_i} such that $f^{-1}(U_i) = r_{a_i}^{-1}(U_{a_i}) \cap Z$ for each $i \in I$. Choose $a \in A$ so that $a \geq a_i$ for each $i \in I$. By condition 3 of Definition 5.1, $f^{-1}(U_i) = r_a^{-1}(V_i) \cap Z$, where $V_i = r_{a_i}^{-1}(U_{a_i}) \cap X_a$. Note that if $x, y \in r_a^{-1}(x_a) \cap Z$ for a point $x_a \in X_a$, then $f(x) = f(y)$. Indeed, assuming the contrary, we can find an element U_i such that $x \in f^{-1}(U_i) \cap Z$ and $y \notin f^{-1}(U_i) \cap Z$. But $f^{-1}(U_i) = r_a^{-1}(V_i) \cap Z$ and therefore $x \in r_a^{-1}(V_i) \cap Z$ and

$y \notin r_a^{-1}(V_i) \cap Z$. Then $x_a = r_a(x) \in V_i$ and $x_a = r_a(y) \notin V_i$. This contradiction shows that the required map can be defined as $f_a(x_a) = f(r_a^{-1}(x_a) \cap Z)$, $x_a \in X_a$. Since, by definition, $f_a^{-1}(U_i) = V_i$, $i \in I$, it follows that f_a is continuous. □

Corollary 5.9. *Let $\tau \geq \omega$ and X be a τ-skeletoid in Y. If $X \subseteq Z \subseteq Y$, then Z is τ-pseudocompact, i.e. for any map $f \colon Z \to F$, where $w(F) \leq \tau$, $f(Z)$ is compact. In particular, Z is pseudocompact.*

Proposition 5.10. *Let $\tau \geq \omega$ and X be a τ-skeletoid in Y and $\mathcal{L} = \{X_a, r_a, A\}$ be the corresponding τ-skeleton. If $\tilde{r}_a \colon \beta Y \to X_a$ is an extension of r_a to the Stone-Čech compactification βY of Y, $a \in A$, then $\tilde{\mathcal{L}} = \{X_a, \tilde{r}_a, A\}$ is a τ-skeleton in βY.*

Proof. Let us verify conditions of Definition 5.1. Conditions 1 and 2 are obvious. Since $\tilde{r}_a \tilde{r}_b | Y = r_a r_b = r_a = \tilde{r}_a | Y$, it follows that $\tilde{r}_a \tilde{r}_b = \tilde{r}_a$ (condition 3). It is clear that if $a, b \in A$, then $\inf(\tilde{r}_a, \tilde{r}_b) = \tilde{r}_{\inf(a,b)}$ and condition 4 follows.

Let $\{a_i \colon i \in I\}$, $|I| \leq \tau$, be a chain of elements of A with $a = \sup\{a_i \colon i \in I\}$, and suppose that $y \in \beta Y$. Consider the point $x = \tilde{r}_a(y) \in X_a \subseteq X$. By condition 5 (for Y), $r_a(x) = \lim\{r_{a_i}(x) \colon i \in I\}$. Then condition 5 (for βY) follows from the following calculations:

$$\begin{aligned}\tilde{r}_a(y) = \tilde{r}_a(\tilde{r}_a(y)) = \tilde{r}_a(x) = \tilde{r}_a(\lim\{r_{a_i}(x) \colon i \in I\}) = \lim\{\tilde{r}_a(r_{a_i}(x)) \colon i \in I\} = \\ \lim\{r_a(r_{a_i}(x)) \colon i \in I\} = \lim\{r_{a_i}(x) \colon i \in I\} = \lim\{\tilde{r}_{a_i}(x) \colon i \in I\} = \\ \lim\{\tilde{r}_{a_i}(\tilde{r}_a(y)) \colon i \in I\} = \lim\{\tilde{r}_{a_i}(y) \colon i \in I\}.\end{aligned}$$

Since Y is pseudocompact (Corollary 5.9), a functionally closed subset F of βY is of the form $\mathrm{cl}_{\beta X} Z$, where Z is a functionally closed subset of Y. By condition 6 of Definition 5.1, $Z = r_a^{-1}(Z_a)$, where Z_a is a functionally closed subset of X_a. Consider a functionally closed set $\tilde{r}_a^{-1}(Z_a)$ of βY. Clearly $F \subseteq \tilde{r}_a^{-1}(Z_a)$ and $F \cap Y = \tilde{r}_a^{-1}(Z_a) \cap Y$. We claim that in this situation $F = \tilde{r}_a^{-1}(Z_a)$. Indeed, assuming the contrary, the complement $\tilde{r}_a^{-1}(Z_a) \setminus F$ would contain a non-empty G_δ-subset of βY. But this is impossible because Y is pseudocompact. Verification of condition 6 is completed. □

Proposition 5.11. *Let $\tau \geq \omega$ and $\mathcal{L} = \{X_a, r_a, A\}$ be a τ-skeleton in Y. Then*

(i) $y = \lim\{\tilde{r}_a(y) \colon a \in A\}$ *for each* $y \in \beta Y$*;*
(ii) $\mathcal{S}_{X,Y} = \{X_a, \tilde{r}_a | X_b, A\}$ *is a τ-spectrum and* $\beta Y = \lim \mathcal{S}_{X,Y}$.

Proof. Let $y \in \beta Y$ and G be any functionally open subset of βY containing y. By condition 6 (for βY, see Proposition 5.10), $G = \tilde{r}_a^{-1}(U)$, where $a \in A$ and U is functionally open in X_a. Note that if $b \geq a$, then we have

$$y \in G = \tilde{r}_a^{-1}(U) \Longrightarrow \tilde{r}_a(\tilde{r}_b(y)) = \tilde{r}_a(y) \in \tilde{r}_a(\tilde{r}_a^{-1}(U)) = U \Longrightarrow \tilde{r}_b(y) \in \tilde{r}_a^{-1}(U) = G.$$

This shows that $y = \lim\{\tilde{r}_a : a \in A\}$.

By condition 3 of Definition 5.1 (for βY), $\mathcal{S}_{X,Y}$ is indeed an inverse spectrum. Consider maps $\tilde{r}_a : \beta Y \to X_a$. Clearly $\tilde{r}_a = r_a \tilde{r}_b$ whenever $a \leq b$, $a, b \in A$. Thus the diagonal product $h = \triangle\{\tilde{r}_a : a \in A\}$ maps βY into $\lim \mathcal{S}_{X,Y}$. The fact that h is surjective follows form the compactness of βY. Condition (i) insures that h is injective. Similarly if $\{a_i : i \in I\}$, $|I| \leq \tau$, is a chain of elements in A with $a = \sup\{a_i : i \in I\}$, then the diagonal product $\triangle\{r_{a_i}|X_a : i \in I\}$ maps X_a into the limit of the well ordered inverse spectrum $\{X_{a_i}, r_{a_i}|X_{a_j}, I\}$. Condition 5 of Definition 5.1 and compactness of X_a insure that this map is also a homeomorphism. By condition 2, we conclude that $\mathcal{S}_X$ is a τ-spectrum. □

Proposition 5.12. *Let $\tau \geq \omega$ and $\mathcal{L} = \{X_a, r_a, A\}$ be a τ-skeleton in Y. If F is closed in $X = \cup\{X_a, r_a, A\}$, then $\mathcal{L}_F = \{F \cap X_a, r_a|F, A_F\}$ is a τ-skeleton in $\mathrm{cl}_Y F$.*

Proof. According to Proposition 5.4, the set $A_F = \{a \in A : r_a(F) = F \cap X_a\}$ is cofinal and τ-closed in A. Consequently, A_F is itself a τ-complete set. In order to see that A_F is strongly τ-complete consider any two elements $a, b \in A_F$. Then $c = \inf(a, b)$ exists in A. In order to show that $c \in A_F$ note that $r_c(F) = r_a(F) \cap r_b(F) = (F \cap X_a) \cap (F \cap X_b) = F \cap (X_a \cap X_b) = F \cap X_c$. Thus condition 1 of Definition 5.1 holds. Conditions 2–5 are obviously satisfied. Let Z be a functionally closed set in $\mathrm{cl}_Y F$. Choose a map $f : \mathrm{cl}_Y F \to [0,1]$ such that $Z = f^{-1}(0)$. By Corollaries 5.5 and 5.7, the map $f|F$ has an extension $g : Y \to [0,1]$. By Lemma 5.8, there is an index $a \in A$ and a function $f_a : X_a \to [0,1]$ such that $f = f_a r_a$. Since A_F is cofinal in A we may assume that $a \in A_F$. Let $g_a = f_a|(F \cap X_a)$. It is easy to see that $f = g_a r_a|\,\mathrm{cl}_Y F$. Then $Z = (r_a|\,\mathrm{cl}_Y F)^{-1}(Z_a)$, where $Z_a = g_a^{-1}(0)$. This completes verification of condition 6. □

5.2. **Regular τ-skeletoids.** Below we consider regular τ-skeletons, i.e. τ-skeletons of the form $\mathcal{L} = \{X_\alpha, r_\alpha, \tau^+\}$.

Theorem 5.13. *Let $\kappa = \tau^+ > \omega$. Suppose that M and N are two regular τ-skeletoids in the cube I^κ. Then there exists an autohomeomorphism $H : I^\kappa \to I^\kappa$ such that $H(M) = N$. Moreover, id_{I^κ} can be approximated (in $\mathrm{Auth}_\kappa(I^\kappa, I^\kappa)$) by autohomeomorphism $H : I^\kappa \to I^\kappa$ such that $H(M) = N$.*

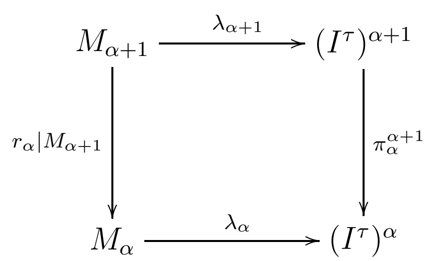
Proof. Represent the cube I^κ as the product $I^\kappa = (I^\tau)^\kappa$. By $\pi_\alpha : I^\kappa \to (I^\tau)^\alpha$ denote the projection onto the corresponding subproduct. Let also $i_\alpha : (I^\tau)^\alpha \to I^\kappa$ stand for the section of π_α which leaves first α coordinates unchanged and sends all other to $\mathbf{0}$. The sigma-product $\Sigma = \Sigma(\tau,\kappa) = \{\{x_\alpha\} \in I^\kappa \colon |\{\alpha \in \kappa \colon x_\alpha \neq \mathbf{0}\}| \leq \tau\}$ obviously coincides with the subspace $\cup\{i_\alpha((I^\tau)^\alpha) \colon \alpha < \kappa\}$.

It suffices to proof our theorem in the case when $N = \Sigma$. Suppose that $\mathcal{L} = \{M_\alpha, r_\alpha, \kappa\}$ is a regular τ-skeleton in I^κ corresponding to M and consider the spectrum $\mathcal{S}_M = \{M_\alpha, r_\alpha|M_{\alpha+1}, \kappa\}$ generated by $\mathcal{L}_M$ as in Proposition 5.11. Let also $\mathcal{S} = \{(I^\tau)^\alpha, \pi^{\alpha+1}_\alpha, \kappa\}$ be a standard spectrum consisting of subproducts of the cube I^κ and corresponding projections $\pi^{\alpha+1}_\alpha \colon (I^\tau)^{\alpha+1} \to (I^\tau)^\alpha$. Note that both $\mathcal{S}_M$ and $\mathcal{S}$ are τ-spectra with the same limit (i.e. I^κ).

Consider a neighborhood G of id_{I^κ} in $\mathrm{Auth}_\kappa(I^\kappa, I^\kappa)$. We may assume, without loss of generality, that $G = \{H \in \mathrm{Auth}_\kappa(I^\kappa, I^\kappa) \colon \pi_{\alpha_0}H = \pi_{\alpha_0}\}$ for some $\alpha_0 < \kappa$.

Theorem 2.4, applied to τ-spectra $\mathcal{S}_M$ and $\mathcal{S}$, guarantees that there exists a cofinal and τ-closed subset $\mathcal{T} \subseteq \kappa$ and homeomorphisms $\lambda_\alpha \colon M_\alpha \to (I^\tau)^\alpha$ such that $\lambda_\alpha r_\alpha = \pi_\alpha$ for each $\alpha \in \mathcal{T}$. Reindexing if necessary we may assume without loss of generality that $\mathcal{T} = \kappa$. We may also assume that $\alpha_0 = 0$.

Note that the equality $\lambda_\alpha r_\alpha = \pi_\alpha$, $\alpha < \kappa$, implies that the following diagram



is commutative for each $\alpha < \kappa$. In particular, the map $r_\alpha|M_{\alpha+1}$ is a trivial fibration with fiber I^τ.

Next we are going to construct a morphism $\{H_\alpha \colon \alpha < \kappa\} \colon \mathcal{S}_M \to \mathcal{S}$ consisting of homeomorphisms $H_\alpha \colon M_\alpha \to (I^\tau)^\alpha$ satisfying the following properties:

(1) $H_\alpha r_\alpha|M_{\alpha+1} = \pi^{\alpha+1}_\alpha H_{\alpha+1}$, $\alpha < \kappa$;
(2) $H_\beta = \lim\{H_\alpha \colon \alpha < \beta\}$ for each limit ordinal $\beta < \kappa$;
(3) $H_{\alpha+1}|M_\alpha = i^{T_{\alpha+1}}_{T_\alpha} H_\alpha$ for each $\alpha < \kappa$;

Let $H_0 = \lambda_0$. Note that $H_0 r_0 = \pi_0$. This condition will insure that the resulting homeomorphism $H \colon I^\kappa \to I^\kappa$ is in the given neighborhood G of id_{I^κ}.

Suppose that above homeomorphisms have already been constructed for each $\alpha < \beta$, where $\beta < \kappa$. Let us construct H_β.

If β is a limit ordinal, then $H_\beta = \lim\{H_\alpha \colon \alpha < \beta\}$.

Let $\beta = \alpha + 1$. Note that $w(M_\alpha) < \kappa$ and consequently, according to Proposition 3.4, M_α is a Z_κ-set in I^κ. Using Proposition 3.5 we can find an index $\gamma > \alpha$ such that M_α is a fibered Z_τ-set in M_γ with respect to the retraction $r_\alpha|M_\gamma$. In order to keep notation simpler we may assume, without loss of generality, that $\gamma = \beta$.

Now consider a homeomorphism $F_{\alpha+1} = (H_\alpha \lambda_\alpha^{-1} \times \mathrm{id}_{I^\tau})\lambda_{\alpha+1} \colon M_{\alpha+1} \to (I^\tau)^{\alpha+1}$ – composition of the top arrows in the following diagram:

$$\begin{array}{ccccc}
M_{\alpha+1} & \xrightarrow{\lambda_{\alpha+1}} & (I^\tau)^{\alpha+1} & \xrightarrow{H_\alpha\lambda_\alpha^{-1}\times \mathrm{id}_{I^\tau}} & (I^\tau)^{\alpha+1} \\
\Big\downarrow{\scriptstyle r_\alpha|M_{\alpha+1}} & & \Big\downarrow{\scriptstyle \pi_\alpha^{\alpha+1}} & & \Big\downarrow{\scriptstyle \pi_\alpha^{\alpha+1}} \\
M_\alpha & \xrightarrow{\lambda_\alpha} & (I^\tau)^\alpha & \xrightarrow{H_\alpha\lambda_\alpha^{-1}} & (I^\tau)^\alpha
\end{array}$$

Then $F_{\alpha+1}(M_\alpha)$ is a fibered Z_τ-set in $(I^\tau)^{\alpha+1}$ with respect to the projection $\pi_\alpha^{\alpha+1}$. Consider a homeomorphism

$$H_\alpha F_{\alpha+1}^{-1}|F_{\alpha+1}(M_\alpha) \colon F_{\alpha+1}(M_\alpha) \to i_\alpha^{\alpha+1}((I^\tau)^\alpha)$$

which acts fiberwise. By Theorem 4.5, there exists a fiberwise auto-homeomorphism $G_{\alpha+1} \colon (I^\tau)^{\alpha+1} \to (I^\tau)^{\alpha+1}$ such that $G_{\alpha+1}|F_{\alpha+1}(M_\alpha) = H_\alpha F_{\alpha+1}^{-1}|F_{\alpha+1}(M_\alpha)$. Let $H_{\alpha+1} = G_{\alpha+1}F_{\alpha+1}$. Straightforward verifications shows that the homeomorphism $H_{\alpha+1}$ has the required properties.

This completes the inductive construction. Thus we can define $H \colon I^\kappa \to I^\kappa$ as $H = \lim\{H_\alpha \colon \alpha < \kappa\}$. Clearly, $H(M) = \Sigma$. Since $\pi_0 H = r_0 H_0 = \pi_0$ it follows that $H \in G$. □

6. Topological characterization of $\Sigma(\tau, \tau^+)$

As was pointed out in the Introduction in this Section we obtain a topological characterization of the sigma-product $\Sigma(\tau, \tau^+)$.

Theorem 6.1. *Let $\tau \geq \omega$. Then the following conditions are equivalent for any space X:*

(i) *X is a regular τ-skeletoid;*

(ii) *There exists an embedding $H \colon \beta X \to I^{\tau^+}$ such that $H(X) = H(\beta X) \cap \Sigma(\tau, \tau^+)$. In particular, X is homeomorphic to a closed subspace of $\Sigma(\tau, \tau^+)$.*

Proof. (ii) $\Longrightarrow$ (i). Note that $\Sigma(\tau, \tau^+)$ has the natural regular τ-skeleton and apply Proposition 5.12.

(i) $\Longrightarrow$ (ii). Let $\mathcal{L} = \{X_\alpha, r_\alpha, \tau^+\}$ be a regular τ-skeleton generating X and Y denote the Stone-Čech compactification of X. By Proposition 5.10, $\mathcal{L}$ forms a regular τ-skeleton in Y (we keep notation r_α for its extension $\tilde{r}_\alpha \colon Y \to X_\alpha$). By Proposition 5.11, $Y = \lim \mathcal{S}_X$, where $\mathcal{S}_X = \{X_\alpha, r_\alpha|X_{\alpha+1}, \tau^+\}$ is the associated with $\mathcal{L}$ well-ordered τ-spectrum. Let also $\mathcal{S} = \{(I^\tau)^\alpha, \pi_\alpha^{\alpha+1}, \tau^+\}$ be the standard well-ordered τ-spectrum consisting of subproducts of the cube $I^{\tau^+} = (I^\tau)^{\tau^+}$ and corresponding projections.

Required embedding $H\colon Y \to I^{\tau^+}$ will be constructed as the limit of a morphism $\{H_\alpha\colon X_\alpha \to (I^\tau)^\alpha, \alpha < \tau^+\}$, consisting of embeddings.

Let $H_1\colon X_1 \to (I^\tau)^1$ be any embedding. Suppose that embeddings H_α have already been constructed for each $\alpha < \beta$, where $\beta < \tau^+$, in such a way that the following conditions are satisfied:

(1) $H_\alpha r_\alpha | X_{\alpha+1} = \pi^{\alpha+1}_\alpha H_{\alpha+1}$ for each $\alpha < \beta$;
(2) $H_\gamma = \lim\{H_\alpha\colon \alpha < \gamma\}$ for each limit ordinal $\gamma < \beta$;
(3) $H_{\alpha+1}|X_\alpha = i^{\alpha+1}_\alpha H_\alpha$ for each $\alpha < \beta$;

If β is a limit ordinal, then $H_\beta = \lim\{H_\alpha\colon \alpha < \beta\}$. If $\beta = \alpha + 1$. By Proposition 3.6, there exists a Z_τ-embedding $j\colon X_{\alpha+1} \to I^\tau$. Then the diagonal product $F_{\alpha+1} = H_\alpha r_\alpha|X_{\alpha+1} \triangle j\colon X_{\alpha+1} \to (I^\tau)^\alpha \times I^\tau = (I^\tau)^{\alpha+1}$ is a fibered Z_τ-embedding such that $\pi^{\alpha+1}_\alpha F_{\alpha+1} = H_\alpha r_\alpha|X_{\alpha+1}$. Then the set $F_{\alpha+1}(X_\alpha)$ is also a fibered Z_τ-set. Consider a fibered homeomorphism

$$i^{\alpha+1}_\alpha H_\alpha F^{-1}_{\alpha+1}|F_{\alpha+1}(X_\alpha)\colon F_{\alpha+1}(X_\alpha) \to i^{\alpha+1}_\alpha(H_\alpha(X_\alpha))$$

By Theorem 4.5, there exists a fiberwise autohomeomorphism $G_{\alpha+1}\colon (I^\tau)^{\alpha+1} \to (I^\tau)^{\alpha+1}$ such that $G_{\alpha+1}|F_{\alpha+1}(X_\alpha) = i^{\alpha+1}_\alpha H_\alpha F^{-1}_{\alpha+1}|F_{\alpha+1}(X_\alpha)$. Let $H_{\alpha+1} = G_{\alpha+1}F_{\alpha+1}\colon X_{\alpha+1} \to (I^\tau)^{\alpha+1}$.

This completes inductive construction. Then the limit embedding $H = \lim\{H_\alpha\colon \alpha < \tau^+\}\colon Y \to I^{\tau^+}$ has the required property $H(Y) \cap \Sigma(\tau,\tau^+) = H(X)$. □

Recall [8], [10] that a compact space Y is called a κ-Valdivia compactum ($\kappa \geq \omega$) if there exist an embedding $H\colon Y \to I^T$ (for some T such that $H(Y) = \mathrm{cl}_{I^T}(H(Y) \cap \Sigma_\kappa(T)$, where $\Sigma_\kappa(T) = \{\{x_t\} \in I^T\colon |\{t \in T\colon x_t \neq 0\}| < \kappa\})$. Similarly, a compactum Y is a κ-Corson compactum if for some T there is an embedding $H\colon Y \to \Sigma_\kappa(T)$. Theorem 6.1 provides a characterization of κ-Valdivia and κ-Corson compact spaces of weight $\leq \kappa$ for regular cardinals (characterization of ω-Valdivia compact spaces has been obtained in [11]).

Corollary 6.2. *Let $\tau \geq \omega$. A compact space is a τ^+-Valdivia compactum if and only if it admits a regular τ-skeleton, i.e. is the Stone-Čech compactification of a regular τ-skeletoid.*

Corollary 6.3. *Let $\tau \geq \omega$. A compact space is a τ^+-Corson compactum if and only if it is a regular τ-skeletoid.*

6.1. **Retracts of Σ-products.** Since we now have a topological characterization of closed subspaces of $\Sigma(\tau,\tau^+)$, we are in a position to consider the next question: which closed subspaces of $\Sigma(\tau,\tau^+)$ are its retracts? First, however, we need to analyze properties of $\Sigma(\tau,\tau^+)$ itself. Namely those which are related to extension properties. It is clear that $\Sigma(\tau,\tau^+)$ is not an absolute extensor for the class of Tychonov spaces since any such space is realcompact [6, Proposition 6.1.7] and as we know (Corollary 5.9) $\Sigma(\tau,\tau^+)$ is not.

There are two key properties of $\Sigma(\tau,\kappa)$ making this space look like an absolute extensor.

($*$) Every element Σ_α, $\alpha<\tau^+$, of the natural regular τ-skeleton $\{\Sigma_\alpha, r_\alpha, \tau^+\}$ of $\Sigma(\tau,\tau^+)$ is a copy of the cube I^τ and hence is an absolute extensor.

($**$) Every retraction $r_\alpha|\Sigma_{\alpha+1}\colon \Sigma_{\alpha+1}\to\Sigma_\alpha$ is a soft map (it is even a trivial fibration with fiber I^τ).

We may require similar properties of natural τ-skeleton associated with a closed subspace (Proposition 5.12) of $\Sigma(\tau,\kappa)$. However we prefer to find equivalent properties in terms of the entire space (τ-skeletoid) rather than it terms of its individual elements (τ-skeleton).

Definition 6.1. Let $\tau\geq\omega$. A space X is an absolute extensor with respect to the class of compact spaces of weight $\leq\tau$ (notation $\mathrm{AE}(\tau)$) if for any compact space Y of weight $w(Y)\leq\tau$ and any map $f\colon F\to X$, defined on a closed subset F of Y, there exists a map $\tilde{f}\colon Y\to X$ such that $\tilde{f}|F=f$.

Lemma 6.4. *Let $\tau\geq\omega$ and X be a τ-skeletoid corresponding to a τ-skeleton $\{X_a, r_a, A\}$. Then the following conditions are equivalent:*

(i) $X\in\mathrm{AE}(\tau)$*;*

(ii) $X_a\in\mathrm{AE}$ *for each* $a\in A$*.*

Proof. (i) $\Longrightarrow$ (ii). Since $w(X_a)\leq\tau$ it suffices to show that X_a is an absolute extensor with respect to the class of compact spaces of weight $\leq\tau$. Let $f\colon F\to X_a$ be a map defined on a closed subset F of a compact space Y with $w(Y)\leq\tau$. Since $X\in\mathrm{AE}(\tau)$, there exists an extension $g\colon Y\to X$ of f. Then the composition $\tilde{f}=r_a g\colon Y\to X_a$ is a required extension of f.

(ii) $\Longrightarrow$ (i). Let $f\colon F\to X$ be a map defined on a closed subset F of a compact space Y with $w(Y)\leq\tau$. Choose a dense subset $\{y_i\colon i\in I\}$ in F so that $|I|\leq\tau$. Then we can find, for each $i\in I$, an index $a_i\in A$ such that $f(y_i)\in X_{a_i}$. Choose $a\in A$ so that $a\geq a_i$ for each $i\in I$. Then, as can be easily seen, $f(F)\subseteq X_a$. Since X_a is an absolute extensor, there exists an extension $\tilde{f}\colon Y\to X_a\subseteq X$ of f. □

Definition 6.2. Let $\tau\geq\omega$. A space X has an τ-estimated extension property if for each collection $\{\mathcal{U}_i\colon i\in I\}\subseteq\mathrm{cov}(X)$, with $|I|\leq\tau$, there exists a collection $\{\mathcal{V}_j\colon j\in J\}\subseteq\mathrm{cov}(X)$, with $|J|\leq\tau$, such that if maps $f,g\colon F\to X$, defined on a closed subset F of a compact space Y with $w(Y)\leq\tau$, are $\{\mathcal{V}_j\colon j\in J\}$-close and there exists an extension $\tilde{f}\colon Y\to X$ of f, then g also admits an extension $\tilde{g}\colon Y\to X$ which is $\{\mathcal{V}_j\colon j\in J\}$-close to $\tilde{f}$.

Lemma 6.5. *Let $\tau \geq \omega$ and X be a τ-skeletoid corresponding to a regular τ-skeleton $\{X_\alpha, r_\alpha, \tau^+\}$. If X has the τ-estimated extension property, then the set*

$$\mathcal{B} = \{\alpha < \tau^+ \colon r_\alpha | X_\beta \colon X_\beta \to X_\alpha \text{ is soft for each } \beta \text{ with } \beta \geq \alpha\}$$

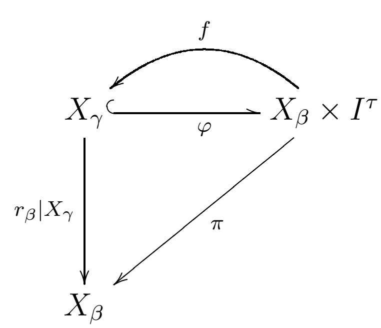
is cofinal and τ-closed in τ^+.

Proof. We proceed by using the spectral search (Proposition 2.3) with respect to the following relation

$$L = \{(\alpha,\beta) \in (\tau^+)^2 \colon \alpha \leq \beta \text{ and } r_\beta | X_\gamma \colon X_\gamma \to X_\beta \text{ is soft for each } \gamma \text{ with } \gamma \geq \beta\}$$

Existence. Let $\alpha < \tau^+$ and consider a cofinal subset $\{\mathcal{U}_i \colon i \in I\}$, $|I| \leq \tau$, of $\text{cov}(X_\alpha)$. Then $\{r_\alpha^{-1}(\mathcal{U}_i) \colon i \in I\} \subseteq \text{cov}(X)$. Since X has the estimated τ-extension property, we can find a collection $\{\mathcal{V}_j \colon j \in J\} \subseteq \text{cov}(X)$ with the property specified in Definition 6.2. Based on condition 6 of Definition 5.1, we can find an index $\beta < \tau^+$, with $\alpha \leq \beta$, and covers $\mathcal{W}_j \in \text{cov}(X_\beta)$ such that $\mathcal{V}_j = r_\beta^{-1}(\mathcal{W}_j)$ for each $j \in J$. We claim that for each $\gamma < \tau^+$, with $\beta \leq \gamma$, the map $r_\beta | X_\gamma \colon X_\gamma \to X_\beta$ is soft.

Since $w(X_\gamma) \leq \tau$, there exists an embedding $\psi \colon X_\gamma \to I^\tau$. Clearly, the diagonal product $\varphi = (r_\beta | X_\gamma) \triangle \psi \colon X_\gamma \to X_\beta \times I^\tau$ is also an embedding. Let $F = \varphi(X_\gamma) \subseteq X_\beta \times I^\tau$. Now consider two maps $\varphi^{-1}, \pi|\varphi(X_\gamma) \colon \varphi(X_\gamma) \to X$, where $\pi \colon X_\beta \times I^\tau \to X_\beta$ denotes the projection onto the first coordinate. Note that $r_\beta\varphi^{-1}|\varphi(X_\gamma) = \pi|\varphi(X_\gamma) = r_\beta\pi|\varphi(X_\gamma)$. Consequently, the maps $\varphi^{-1}|\varphi(X_\gamma)$ and $\pi|\varphi(X_\gamma)$, as maps into X, are $\{r_\beta^{-1}(\mathcal{W}_j) \colon j \in J\}$-close. But $\pi|\varphi(X_\gamma)$ has an extension $\pi \colon X_\beta \times I^\tau \to X_\beta \subseteq X$. By the choice of the collection $\{V_j = r_\beta^{-1}(W_j) \colon j \in J\}$, the map $\varphi^{-1}|\varphi(X_\gamma)$ also has an extention $g \colon X_\beta \times I^\tau \to X$ which is $\{r_\beta^{-1}(\mathcal{W}_j) \colon j \in J\}$-close to π. Let $f = r_\gamma g \colon X_\beta \times I^\tau \to X_\gamma$. Obviously, $f|\varphi(X_\gamma) = \varphi^{-1}$. Because of the specified closeness of the maps g and π, we conclude that $r_\beta g = r_\beta \pi = \pi$. Also note that $r_\beta f = r_\beta r_\gamma g = r_\beta g$. Thus $r_\beta f = \pi$. In other words, $f \colon X_\beta \times I^\tau \to X_\gamma$ is a retraction making the diagram



commutative. But π, as a trivial bundle with fiber I^τ, is soft. Consequently, $r_\beta|X_\gamma\colon X_\gamma \to X_\beta$, as a retract of π, is also soft. This shows that $(\alpha,\beta) \in L$.

Majorantness and τ-closedness conditions are trivially satisfies. Consequently, according to Proposition 2.3, the set of L-reflexive elements is cofinal and τ-closed in τ^+T. In only remains to note that if α is L-reflexive (i.e. $(\alpha,\alpha) \in L$), then $r_\alpha|X_\beta\colon X_\beta \to X_\alpha$ is soft for each β with $\beta \geq \alpha$. □

Proposition 6.6. *Let $\tau \geq \omega$ and X be a closed subspace of $\Sigma(\tau,\tau^+)$. Then the following conditions are equivalent:*

(a) *X is a retract of $\Sigma(\tau,\tau^+)$;*
(b) *$X \in \mathrm{AE}(\tau)$ and X has the τ-estimated extension property.*

Proof. (a) $\Longrightarrow$ (b). Obviously $\Sigma(\tau,\tau^+) \in \mathrm{AE}(\tau)$ and it has the τ-estimated extension property. It only remains to note that both of these properties are preserved by retractions.

(b) $\Longrightarrow$ (a). Represent $\Sigma(\tau,\tau^+)$ as the increasing union $\cup\{\Sigma_\alpha = i_\alpha((I^\tau)^\alpha)\colon \alpha < \tau^+\}$, where $i_\alpha\colon (I^\tau)^\alpha \to (I^\tau)^{\tau^+}$ is the standard section of the projection $\pi_\alpha\colon (I^\tau)^{\tau^+} \to (I^\tau)^\alpha$. By Proposition 5.12 (see also Proposition 5.4) and Lemmas 6.4 and 6.5, we may assume (after reindexing if necessary) without loss of generality that

(1) $r_\alpha(X) = X \cap \Sigma_\alpha$, $\alpha < \tau^+$;
(2) $X_\alpha \in \mathrm{AE}$, $\alpha < \tau^+$;
(3) $r_\alpha|X_{\alpha+1}\colon X_{\alpha+1} \to X_\alpha$ is soft, $\alpha < \tau^+$.

Since $r_0(X) = X \cap \Sigma_0$ is a compact absolute retract, there exists a retraction $s_0\colon \Sigma_0 \to X \cap \Sigma_0$. Suppose that retractions $s_\alpha\colon \Sigma_\alpha \to X \cap \Sigma_\alpha$ have already been constructed for each $\alpha < \beta$, where $\beta < \tau^+$, in such a way that the following conditions are satisfied:

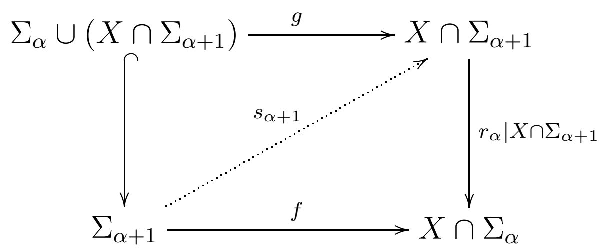
(1) $r_\alpha s_{\alpha+1} = s_\alpha r_\alpha|\Sigma_{\alpha+1}$, $\alpha < \beta$;
(2) $s_\gamma = \lim\{s_\alpha \alpha < \gamma\}$ for a limit ordinal $\gamma < \beta$;
(3) $s_{\alpha+1}|\Sigma_\alpha = s_\alpha$, $\alpha < \beta$.

Let us construct retraction $s_\beta\colon \Sigma_\beta \to X \cap \Sigma_\beta$. If β is a limit ordinal let $s_\beta = \lim\{s_\alpha\colon \alpha < \beta\}$. If $\beta = \alpha+1$, then consider the map $g\colon \Sigma_\alpha \cup (X \cap \Sigma_{\alpha+1}) \to X \cap \Sigma_{\alpha+1}$ which coincides with s_α on Σ_α and with $\mathrm{id}_{X\cap\Sigma_{\alpha+1}}$ on $X \cap \Sigma_{\alpha+1}$. Consider also the map $f\colon \Sigma_{\alpha+1} \to X \cap \Sigma_\alpha$, defined as the composition $f = s_\alpha r_\alpha|\Sigma_{\alpha+1}\colon \Sigma_{\alpha+1} \to X \cap \Sigma_\alpha$. Note that

$$(r_\alpha|(X \cap \Sigma_{\alpha+1}))g = f|(\Sigma_\alpha \cup (X \cap \Sigma_{\alpha+1})).$$

Indeed, if $x \in \Sigma_\alpha$, then $r_\alpha(x) = x$ and $g(x) = s_\alpha(x)$. Consequently $(r_\alpha|(X \cap \Sigma_{\alpha+1}))g(x) = r_\alpha(s_\alpha(x)) = s_\alpha(x) = s_\alpha(r_\alpha(x)) = f(x)$. If $x \in X \cap \Sigma_{\alpha+1}$, then $g(x) = x$ and $(r_\alpha|(X \cap \Sigma_{\alpha+1}))g(x) = r_\alpha(x)$. But in this case $r_\alpha(x) \in X \cap \Sigma_\alpha$ and consequently $s_\alpha(r_\alpha(x)) = r_\alpha(x)$. Thus $(r_\alpha|(X \cap \Sigma_{\alpha+1}))g(x) = s_\alpha(r_\alpha(x)) = f(x)$.

In other words we have the following commutative diagram consisting of solid arrows



Since $r_\alpha|(X \cap \Sigma_{\alpha+1})$ (the right vertical arrow) is a soft map, there exists a map $s_{\alpha+1} \colon \Sigma_{\alpha+1} \to X \cap \Sigma_{\alpha+1}$ (the dotted arrow) such that

$$(r_\alpha|(X \cap \Sigma_{\alpha+1}))s_{\alpha+1} = f = s_\alpha r_\alpha|\Sigma_{\alpha+1} \text{ and } s_{\alpha+1}|(\Sigma_\alpha \cup (X \cap \Sigma_{\alpha+1})) = s_\alpha.$$

It is clear that $s_{\alpha+1} \colon \Sigma_{\alpha+1} \to X \cap \Sigma_{\alpha+1}$ is a retraction satisfying conditions (1) and (3).

This completes inductive construction. Required retraction $s \colon \Sigma(\tau, \tau^+) \to X$ is now defined by letting $s(x) = s_\alpha(x)$ for $x \in \Sigma_\alpha$, $\alpha < \tau^+$. Continuity of s follows from Corollary 5.2. □

6.2. **Characterization of $\Sigma(\tau, \tau^+)$.**

Theorem 6.7. *Let $\tau > \omega$ and X be a regular τ-skeletoid. Then the following conditions are equivalent:*

(i) *X is homeomorphic to $\Sigma(\tau, \tau^+)$;*
(ii) *$X \in \mathrm{AE}(\tau)$, X has the τ-estimated extension property and does not contain G_τ-points.*

Proof. (i) $\Longrightarrow$ (ii). Clearly $\Sigma(\tau, \tau^+) \in \mathrm{AE}(\tau)$ and it has the τ-estimated extension property. Suppose that there is a G_τ-point $x \in \Sigma(\tau, \tau^+)$. Then $x = \cap\{Z_\alpha \colon \alpha < \tau\}$, where Z_α is a functionally closed subset of $\Sigma(\tau, \tau^+)$. Let $F_\alpha = \mathrm{cl}_{I^{\tau^+}} Z_\alpha$, $\alpha < \tau$. Since, by Corollary 5.9, $\Sigma(\tau, \tau^+)$ is pseudocompact, it follows that F_α is functionally closed subset of the cube I^{τ^+}. Note that $\{x\} = \cap\{F_\alpha \colon \alpha < \tau\}$. Indeed, if there were $y \in \cap\{F_\alpha \colon \alpha < \tau\}$ such that $y \neq x$, then $y \in I^{\tau^+} \setminus \Sigma(\tau, \tau^+)$ and consequently $I^{\tau^+} \setminus \Sigma(\tau, \tau^+)$ would contain a non-empty G_τ-set of I^{τ^+}. This contradicts conclusion of Lemma 5.6. Thus x is a G_τ-point in I^{τ^+}. Contradiction.

(ii) $\Longrightarrow$ (i). By Proposition 6.6, X can be embedded into $\Sigma(\tau, \tau^+)$ as its retract. Let $s \colon \Sigma(\tau, \tau^+) \to X$ denote the corresponding retraction. Since I^{τ^+} is the Stone-Čech compactification of $\Sigma(\tau, \tau^+)$ there exists extension $\tilde{s} \colon I^{\tau^+} \to \beta X$ of s. It is easy to verify that $\tilde{s}$ is also a retraction. Consequently βX is a compact absolute retract. Since X does not contain G_τ-points, it follows from Lemma

5.6 that βX also does not contain G_τ-points. Therefore βX is a copy of the cube I^{τ^+} (see [6, Theorem 7.2.9]). If $\{X_\alpha, r_\alpha, \tau^+\}$ is a regular τ-skeleton (in X) generating X, then, by Proposition 5.10, $\{X_\alpha, \tilde{r}_\alpha, \tau^+\}$ is a regular τ-skeleton in βX. It only remains to note that according to Theorem 5.13, $X \approx \Sigma(\tau, \tau^+)$. □

Corollary 6.8. *Let $\tau \geq \omega$. Then the following conditions are equivalent:*

(i) $X \times \Sigma(\tau, \tau^+)$ *is homeomorphic to* $\Sigma(\tau, \tau^+)$;
(ii) X *is a retract of* $\Sigma(\tau, \tau^+)$.

Proof. Implication (i) $\Longrightarrow$ (ii) is trivial. In order to prove implication (ii) $\Longrightarrow$ (i) note that $X \times \Sigma(\tau, \tau^+)$, as a retract of $\Sigma(\tau, \tau^+) \approx \Sigma(\tau, \tau^+) \times \Sigma(\tau, \tau^+)$, belongs to the class AE(τ), has the estimated τ-extension property and does not contain G_τ-points. Thus, by Theorem 6.7, $X \times \Sigma(\tau, \tau^+) \approx \Sigma(\tau, \tau^+)$. □

Corollary 6.9. *Let $\tau \geq \omega$. Then the following conditions are equivalent for a compact space X:*

(i) $X \times \Sigma(\tau, \tau^+)$ *is homeomorphic to* Σ,
(ii) X *is an absolute retract of weight* $\leq \tau$.

Proof. (i) $\Rightarrow$ (ii). Let $h\colon \Sigma(\tau, \tau^+) \to X \times \Sigma(\tau, \tau^+)$ be a homeomorphism and $\pi\colon X \times \Sigma(\tau, \tau^+) \to X$ be the projection. Clearly $r = \pi h\colon \Sigma(\tau, \tau^+) \to X$ is a retraction. Since I^{τ^+} is the Stone-Čech compactification of $\Sigma(\tau, \tau^+)$ (and since X is compact), r admits the extension $\tilde{r}\colon I^{\tau^+} \to X$. Therefore X, as a retract of I^{τ^+}, is an absolute retract. Note also that density of X does not exceed τ (as an image of I^{τ^+}). But then $w(X) \leq \tau$.

(ii) $\Rightarrow$ (i). Apply Corollary 6.8. □

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