ON COMMUTATIVE AND NON-COMMUTATIVE C^*-ALGEBRAS WITH THE APPROXIMATE n-TH ROOT PROPERTY

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Abstract. We say that a C^*-algebra X has the approximate n-th root property ($n \geq 2$) if for every $a \in X$ with $\|a\| \leq 1$ and every $\varepsilon > 0$ there exits $b \in X$ such that $\|b\| \leq 1$ and $\|a - b^n\| < \varepsilon$. Some properties of commutative and non-commutative C^*-algebras having the approximate n-th root property are investigated. In particular, it is shown that there exists a non-commutative (resp., commutative) separable unital C^*-algebra X such that any other (commutative) separable unital C^*-algebra is a quotient of X. Also we illustrate a commutative C^*-algebra, each element of which has a square root such that its maximal ideal space has infinitely generated first Čech cohomology.

1. Introduction

All topological spaces in this paper are assumed to be (at least) completely regular. A compact Hausdorff space is called a *compatum* for simplicity. By C^*-algebra and homomorphisms between C^*-algebras, we mean unital C^*-algebras and unital $*$-homomorphisms. For a space X and an integer $n \geq 2$, we consider the following conditions ($\|\cdot\|$ denotes the supremum norm):

- $(*)_n$ For each bounded continuous function $f\colon X \to \mathbb{C}$ and each $\varepsilon > 0$, there exists a continuous function $g\colon X \to \mathbb{C}$ such that $\|f - g^n\| < \varepsilon$.
- $(**)_n$ For each bounded continuous function $f\colon X \to \mathbb{C}$ and each $\varepsilon > 0$, there exist bounded continuous functions $g_1, .., g_n\colon X \to \mathbb{C}$ such that $f = \prod_{i=1}^{i=n} g_i$ and $\|g_i - g_j\| < \varepsilon$ for each i, j.

We say that the space $C^*(X)$ of all bounded complex-valued functions on X has the approximate n-th root property if X satisfies condition $(*)_n$. The results in this paper were inspired by the following theorem established by K. Kawamura and T. Miura [10]:

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Theorem 1.1. *Let X be a compactum with $\dim X \leq 1$ and n a positive integer. Then the following conditions are equivalent:*

(1) *$C(X)$ has the approximate n-th root property.*
(2) *X satisfies condition $(**)_n$.*
(3) *the first Čech cohomology $\check{H}^1(X;\mathbb{Z})$ is n-divisible, that is, each element of $\check{H}^1(X;\mathbb{Z})$ is divided by n.*

Let $\mathcal{A}(n)$ denote the class of all completely regular spaces satisfying condition $(*)_n$ and $\mathcal{A}_1(n)$ is the subclass of $\mathcal{A}(n)$ consisting of spaces X with $\dim X \leq 1$. Let also $\mathcal{H}(n)$ denote the class of all *compacta* X with $\check{H}^1(X;\mathbb{Z})$ being n-divisible.

In Section 2 we investigate some properties of the classes $\mathcal{A}(n)$, $\mathcal{A}_1(n)$ and $\mathcal{H}(n)$. In particular, the following theorem is established:

Theorem 1.2. *Let n be a positive integer and let $\mathcal{K}$ denote one of the classes $\mathcal{A}(n)$, $\mathcal{A}_1(n)$ or $\mathcal{H}(n)$. Then, for every cardinal $\tau \geq \omega$, there exists a compactum $X_\tau \in \mathcal{K}$ of weight $\leq \tau$ and a $\mathcal{K}$-invertible map $f_{\mathcal{K}}\colon X_\tau \to \mathbb{I}^\tau$.*

Here, a map $h\colon X \to Y$ is said to be *invertible* for the class $\mathcal{K}$ (or simply, $\mathcal{K}$-*invertible*) if for every map $g\colon Z \to Y$ with $Z \in \mathcal{K}$ there exists a map $\overline{g}\colon Z \to X$ such that $g = h \circ \overline{g}$.

Theorem 1.2 implies the next corollary.

Corollary 1.3. *Let n be a positive integer and let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$, $\mathcal{A}_1(n)$ or $\mathcal{H}(n)$. Then, for every $\tau \geq \omega$, there exists a compactum $X \in \mathcal{K}$ of weight τ which contains every space from $\mathcal{K}$ of weight $\leq \tau$.*

It is easily seen that the modification of condition $(*)_n$, obtained by requiring both f and g to be of norm ≤ 1, is equivalent to $(*)_n$. This observation leads us to consider the following classes of general (non-commutative) C^*-algebras. We say that a C^*-algebra X satisfies *the approximation n-th root property* if for every $a \in X$ with $\|a\| \leq 1$ and every $\varepsilon > 0$ there exists $b \in X$ such that $\|b\| \leq 1$ and $\|a - b^n\| < \varepsilon$. The class of all C^*-algebras with the approximate n-th root property is denoted by $\mathcal{AP}(n)$. Let $\mathcal{AP}_1(n)$ be the subclass of $\mathcal{AP}(n)$ consisting of C^*-algebras of bounded rank ≤ 1 (recall that bounded rank of C^*-algebras is a non-commutative analogue of the covering dimension dim, see [5]). We also consider the class $\mathcal{HP}(n)$ of C^*-algebras X with the following property: for every *invertible* element $a \in X$ with $\|a\| \leq 1$ and every $\varepsilon > 0$ there exists $b \in X$ such that $\|b\| \leq 1$ and $\|a - b^n\| < \varepsilon$.

In the sequel, $\mathcal{AP}(n)_s$ denotes the class of all separable C^*-algebras from $\mathcal{AP}(n)$. The notations $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$ have the same meaning.

Recall now the concept of $\Re$-invertibility introduced in [4], where $\Re$ is a given class of C^*-algebras. A homomorphism $p\colon X \to Y$ is said to be $\Re$-*invertible* if, for any homomorphism $g\colon X \to Z$ with $Z \in \Re$, there exists a homomorphism $\overline{g}\colon Y \to Z$ such that $g = \overline{g} \circ p$. We also introduce the notion of a *universal* C^*-algebra for a given class $\Re$ as a C^*-algebra $Y \in \Re$ such that any other C^*-algebra from $\Re$ is a quotient of Y.

Section 3 is devoted to the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. The results of this section can be considered as non-commutative counterparts of the results from Section 2. For example, Theorem 1.4 below is a non-commutative version of Theorem 1.2.

Theorem 1.4. *Let n be a positive integer and let $\mathcal{K}$ be one of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. Then there exists a $\mathcal{K}$-invertible unital $*$-homomorphism $p\colon C^*(\mathbb{F}_\infty) \to Z_{\mathcal{K}}$ of $C^*(F_\infty)$ to a separable unital C^*- algebra $Z_{\mathcal{K}} \in \mathcal{K}$, where $C^*(\mathbb{F}_\infty)$ is the group C^*-algebra of the free group on countable number of generators.*

It is well-known that every separable C^*-algebra is a surjective image of $C^*(\mathbb{F}_\infty)$. Therefore, if $\Re$ is a class of separable C^*-algebras and $p\colon C^*(F_\infty) \to Y_\Re$ is a $\Re$-invertible homomorphism with $Y_\Re \in \Re$, then $Y_\Re$ is universal for the class $\Re$. Hence, Theorem 1.4 implies that each of the classes $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$ has a universal element.

Let us note that there exists a non-commutative separable C^*-algebra which belongs to any one of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. Indeed, let $X = M(m)$ be the algebra of all $m \times m$ complex matrixes, where $m \geq 2$ is a fixed integer. By [1], the bounded rank of X is 0. Moreover, using the canonical Jordan form representation, one can show that if $A \in X$ and $n \geq 2$, then A can be approximated by a matrix $B \in X$ with $C^n = B$ for some $C \in X$. Hence, X is a common element of $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. This implies that the universal elements of $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$ are also non-commutative.

Section 4 deals with *square root closed compacta*, compacta X such that, for every $f \in C(X)$, there is $g \in C(X)$ with $f = g^2$. It is known that if X is a first-countable connected compactum, then X is square-root closed if and only if X is locally connected, $\dim X \leq 1$ and $\check{H}^1(X;\mathbb{Z})$ is trivial, see [6], [8], [10] and [12]. A topological characterization of general square root closed compacta has not been known. Here we show that a square root closed compactum X with $\dim X \leq 2$, constructed based on the idea of B. Cole ([13], Chap.3, section 19) and M.I. Karahanjan [9] has infinitely generated first Čech cohomology $\check{H}^1(X;\mathbb{Z})$. This space X is the limit space of an inverse system $\left(X_\alpha, \pi^\beta_\alpha : \alpha < \omega_1\right)$ starting with the unit disk in the plane and such that each map $\pi^\beta_\alpha\colon X_\beta \to X_\alpha$ is invertible with respect to the class of square root closed compacta. A similar construction yields a one-dimensional such compactum. This illustrates that the topological characterization of (not necessarily first countable) square root closed compacta would be rather different than the one for first-countable compacta mentioned above. Also, the invertibility $\pi^\beta_\alpha\colon X_\beta \to X_\alpha$ allows us to obtain a universal element for the class of square root closed compacta with arbitrarily fixed weight.

2. Some properties of the classes $\mathcal{A}(n)$, $\mathcal{A}_1(n)$ and $\mathcal{H}(n)$

Lemma 2.1. *For a compactum X, the following conditions are equivalent:*

1. *For any $f\colon X \to S^1$ and any $\varepsilon > 0$ there exists $g\colon X \to S^1$ such that $\|f - g^n\| < \varepsilon$.*
2. *$\check{H}^1(X;\mathbb{Z})$ is n-divisible.*

Proof. When $\varepsilon = 0$ in (1), this equivalence was established by Kawamura-Miura in [10, Lemma 3.1]. Their arguments remain also valid in the present situation because any two sufficiently close functions from X into S^1 are homotopic. □

Lemma 2.2. *Let X be the limit space of an inverse system $\{X_\alpha, p^\beta_\alpha : \alpha, \beta \in A\}$ of compacta. Then, for every $f \in C(X)$ and every $\varepsilon > 0$, there exists $\alpha \in A$ and $g \in C(X_\alpha)$ such that $g \circ p_\alpha$ is ε-close to f, where $p_\alpha\colon X \to X_\alpha$ is the α-th limit projection. Moreover, $g \in C(X, S^1)$ provided $f \in C(X, S^1)$.*

Proof. We take a finite cover ω of $f(X)$ consisting of open and convex subsets of $\mathbb{C}$ each of diameter $< \varepsilon$. Since X is compact, we can find α and an open cover $\gamma = \{U_j : j = 1,..,m\}$ of X_α such that $p^{-1}_\alpha(\gamma)$ is a star-refinement of the cover $f^{-1}(\omega)$. Without loss of generality, we can assume that each U_j is functionally open in X_α, i.e., $U_j = h^{-1}_j((0, 1])$ for some function $h_j\colon X_\alpha \to [0, 1]$. For any j we fix a point $x_j \in p^{-1}_\alpha(U_j)$ and the required function $g\colon X_\alpha \to \mathbb{C}$ is defined by $g(y) = \displaystyle\sum_{j=1}^{j=m} h_j(y)f(x_j)$. When $f \in C(X, S^1)$ and ε is sufficiently small, $g(X_\alpha) \subset \mathbb{C}\backslash\{0\}$ and, by considering the composition of g and the usual retraction $r\colon \mathbb{C}\backslash\{0\} \to S^1$, we can assume $g \in C(X_\alpha, S^1)$. □

Corollary 2.3. *Let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$, $\mathcal{A}_1(n)$ and $\mathcal{H}(n)$. If X is the limit space of an inverse system $\{X_\alpha, p^\beta_\alpha : \alpha, \beta \in A\}$ of compacta with each $X_\alpha \in \mathcal{K}$, then $X \in \mathcal{K}$.*

Proof. This is a direct application of Lemma 2.2 for the class $\mathcal{A}(n)$. Since the limit space of any inverse system of at most one dimensional compacta is of dimension ≤ 1, the validity of our corollary for $\mathcal{A}(n)$ yields its validity for $\mathcal{A}_1(n)$. Finally, Lemma 2.1 and Lemma 2.2 settle the proof for the class $\mathcal{H}(n)$. □

We say that a class of spaces $\mathcal{K}$ is *factorizable* if, for every map $f\colon X \to Y$ of a *compactum* $X \in \mathcal{K}$, there exists a compactum $Z \in \mathcal{K}$ of weight $w(Z) \leq w(Y)$ and maps $\pi\colon X \to Z$ and $p\colon Z \to Y$ such that $f = p \circ \pi$.

Proposition 2.4. *Any one of the classes $\mathcal{A}(n)$, $\mathcal{A}_1(n)$ and $\mathcal{H}(n)$ is factorizable.*

Proof. We consider first the class $\mathcal{H}(n)$. Fix a map $f\colon X \to Y$ of a compactum $X \in \mathcal{H}(n)$ and assume $w(Y) \leq \tau$. Obviously, we can assume X is of weight $w(X) > \tau$ and Y is compact. By induction, we construct sequences of compacta X_k, dense subsets $M_k \subset C(X_k, S^1)$ of cardinality $\leq \tau$ and maps $\pi_k\colon X \to X_k$, $p_k^{k+1}\colon X_{k+1} \to X_k$, $k \geq 0$, satisfying the following conditions:

(0) $X_0 = Y$, $\pi_0 = f$,

(1) $p_k^{k+1} \circ \pi_{k+1} = \pi_k$, $w(X_k) \leq \tau$ and M_k separates points of X_k $(k \geq 0)$;

(2) For every $h \in M_k$ and every $\varepsilon > 0$, there exists $g \in M_{k+1}$ such that $\|h \circ p_k^{k+1} - g^n\| < \varepsilon$ $(k \geq 0)$.

The weight of the function space $C(Y, S^1)$ is $\leq \tau$, so $C(Y, S^1)$ contains a dense subset M_0 of cardinality $\leq \tau$, separating points of Y. Suppose the spaces X_i, the sets M_i and the maps π_i, p_{i-1}^i, $i \leq k$, have been constructed for some k. Since $X \in \mathcal{H}(n)$, for each $h \in M_k$ and each positive rational number $r \in Q^+$, there exists $g(h,r) \in C(X, S^1)$ with $\|h \circ \pi_k - g(h,r)^n\| < r$. Let $\pi_{k+1}\colon X \to X_k \times (S^1)^{M_k \times Q^+} \times (S^1)^{M_k}$ be the diagonal product of π_k and all maps $g(h,r)$ and $h \circ \pi_k$, where $h \in M_k$, $r \in Q^+$. Let $X_{k+1} = \pi_{k+1}(X)$ and $p_k^{k+1}\colon X_{k+1} \to X_k$ be the natural projection onto X_k. Since M_k separates points of X_k (condition (1)), π_{k+1} is an embedding and hence every $g(h,r)$ can be represented as $g_{k+1}(h,r) \circ \pi_{k+1}$ with $g_{k+1}(h,r) \in C(X_{k+1}, S^1)$. Because $w(X_{k+1}) \leq \tau$, $C(X_{k+1}, S^1)$ contains a dense subset M_{k+1} of cardinality $\leq \tau$ containing all $g_{k+1}(h,r)$, $h \in M_k$, $r \in Q^+$ and also separating points of X_{k+1}. Obviously, X_{k+1}, M_{k+1} and π_{k+1} satisfy conditions (1) and (2). Let Z be the limit of the inverse sequence $\{X_k, p_k^{k+1} : k = 1, 2..\}$, $p\colon Z \to Y$ the first limit projection and $\pi\colon X \to Z$ the limit of the maps π_k. Also let $p_k\colon Z \to X_k$ be the k-th limit projection. By Lemma 2.2, for every $h \in C(Z, S^1)$ and every $\varepsilon > 0$, there exists m and $g_m\colon X_m \to S^1$ such that $\|h - g_m \circ p_m\| < \varepsilon/3$. Now, take $h_m \in M_m$ with $\|g_m - h_m\| < \varepsilon/3$. According to our construction, $\|h_m \circ p_m^{m+1} - g^n\| < \varepsilon/3$ for some $g \in M_{m+1}$. Hence, $\|h - (g \circ p_{m+1})^n\| < \varepsilon$. Finally, by Lemma 2.1, we see $Z \in \mathcal{H}(n)$.

The same arguments remain valid when the class $\mathcal{H}(n)$ is replaced by $\mathcal{A}(n)$. The only difference is that we have to consider the function spaces $C(X_k)$ instead of $C(X_k, S^1)$. For the class $\mathcal{A}_1(n)$ we need the following modifications: all M_k, $k \geq 0$, are dense subsets of $C(X_k)$ of cardinality $|M_k| \leq \tau$ satisfying conditions (1) and (2), where the compactum X_k is of dimension ≤ 1 for each $k \geq 1$. It suffices to demonstrate the construction of X_1 and M_1. Using the above notations, take the diagonal product $q_1\colon X \to Y \times \mathbb{C}^{M_0 \times Q^+} \times \mathbb{C}^{M_0}$ of $\pi_0 = f$ and all maps $g(h,r)$ and $h \circ \pi_0$, where $h \in M_0$ and $r \in Q^+$. Let also $Z_1 = q_1(X)$ and $q_0\colon Z_1 \to Y$ be the natural projection. Then, $w(Z_1) \leq \tau$ and, by the Mardešič factorization theorem [11], there exists a compactum X_1 of weight $\leq \tau$ and $\dim X_1 \leq 1$, and maps $\pi_1\colon X \to X_1$ and $q_2\colon X_1 \to Z_1$ with $q_1 = q_2 \circ \pi_1$.

Obviously, every $g(h,r)$ can be represented as $g_1(h,r)\circ\pi_1$ with $g_1(h,r)\in C(X_1)$. We denote $p_0^1=q_0\circ q_2$ and choose a dense subset $M_1\subset C(X_1)$ such that $|M_1|\leq\tau$ and M_1 contains every $g_1(h,r)$ with $h\in M_0$ and $r\in Q^+$, and separates points of X_1. In this way we obtain the spaces X_k with $\dim X_k\leq 1$. The last inequalities imply that the limit space Z is also of dimension ≤ 1. Moreover, by Lemma 2.2, Z satisfies $(*)_n$, so $Z\in\mathcal{A}_1(n)$. □

Corollary 2.5. *Let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$. Then every space $X\in\mathcal{K}$ has a compactification $Z\in\mathcal{K}$ with $w(Z)=w(X)$.*

Proof. Obviously, $X\in\mathcal{K}$ implies $\beta X\in\mathcal{K}$. Let Y be an arbitrary compactification of X with $w(Y)=w(X)$ and let $f\colon\beta X\to Y$ the extension of the identity on X. Then, by Proposition 2.4, there exists a compactum $Z\in\mathcal{K}$ and maps $g\colon\beta X\to Z$ and $h\colon Z\to Y$ with $h\circ g=f$ and $w(Z)=w(X)$. It remains only to observe that Z is a compactification of X. □

Proposition 2.6. *Let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$, $\mathcal{A}_1(n)$ and $\mathcal{H}(n)$. Then every compactum $X\in\mathcal{K}$ can be represented as the limit space of an ω-spectrum $\{X_\alpha,p_\alpha^\beta:\alpha,\beta\in A\}$ of metrizable compacta with each $X_\alpha\in\mathcal{K}$.*

Proof. Because of similarity of the arguments, we consider only the class $\mathcal{A}(n)$. First, represent X as the limit space of an ω-spectrum $\{X_\alpha,p_\alpha^\beta:\alpha,\beta\in\Lambda\}$ and introduce the relation L on Λ^2 consisting of all $(\alpha,\beta)\in\Lambda^2$ such that $\alpha\leq\beta$ and for each $f\in C(X_\alpha)$ and $\varepsilon>0$ there is $g\in C(X_\beta)$ with $\|f\circ p_\alpha^\beta-g^n\|<\varepsilon$. The relation L has the following properties:
(i) for every $\alpha\in\Lambda$ there exists $\beta\in\Lambda$ with $(\alpha,\beta)\in L$:
(ii)if $(\alpha,\beta)\in L$ and $\beta\leq\gamma$, then $(\alpha,\gamma)\in L$;
(iii) if $\{\alpha_k\}$ is a chain in Λ with each $(\alpha_k,\beta)\in L$, then $(\alpha,\beta)\in L$, where $\alpha=\sup\{\alpha_k\}$.

Indeed, to show (i), we take a countable dense subset $M_\alpha\subset C(X_\alpha)$ and, as in Proposition 2.4, for every $h\in M_\alpha$ and $r\in Q^+$ choose $g(h,r)\in C(X)$ with $\|h\circ p_\alpha-g(h,r)^n\|<r$. Notice that, for each $f\in C(X)$, there is a $\gamma\in\Lambda$ and $\varphi\in C(X_\gamma)$ such that $f=\varphi\circ p_\gamma$. Applying this to $g(h,r)$, we can find $\beta\in\Lambda$, $\beta>\alpha$, such that for each $(h,r)\in M_\alpha\times Q^+$, we have $g(h,r)=g_\beta(h,r)\circ p_\beta$, where $g_\beta(h,r)\in C(X_\beta)$. Then $(\alpha,\beta)\in L$. Property (ii) follows directly and (iii) follows from Lemma 2.2 and the fact that X_α is the limit space of the inverse sequence generated by X_{α_k} and the projections $p_{\alpha_k}^{\alpha_{k+1}}\colon X_{\alpha_{k+1}}\to X_{\alpha_k}$, $k=1,..,$ because α is supremum of the chain $\{\alpha_k\}$.

By [2, Proposition 1.1.29], the set $A=\{\alpha\in\Lambda:(\alpha,\alpha)\in L\}$ is cofinal and ω-closed in Λ. Obviously, $X_\alpha\in\mathcal{A}(n)$ for each $\alpha\in A$ and X is the limit of the inverse system $\{X_\alpha,p_\alpha^\beta:\alpha,\beta\in A\}$. □

Proof of Theorem 1.2. We consider the family of all maps $\{h_\alpha\colon Y_\alpha \to \mathbb{I}^\tau\}_{\alpha\in\Lambda}$ such that each Y_α is a closed subset of $\mathbb{I}^\tau$ with $Y_\alpha \in \mathcal{K}$. Let Y be the disjoint sum of all Y_α and the map $h\colon Y \to \mathbb{I}^\tau$ coincides with h_α on every Y_α. We extend h to a map $\overline{h}\colon \beta Y \to \mathbb{I}^\tau$. Since $\beta Y \in \mathcal{K}$, by Proposition 2.4, there exists a compactum X of weight $\leq \tau$ and maps $p\colon \beta Y \to X$ and $f\colon X \to \mathbb{I}^\tau$ such that $X \in \mathcal{K}$ and $f \circ p = \overline{h}$.

Let us show that f is $\mathcal{K}$-invertible. Take a space $Z \in \mathcal{K}$ and a map $g\colon Z \to \mathbb{I}^\tau$. Considering βZ and the extension $\overline{g}\colon \beta Z \to \mathbb{I}^\tau$ of g, we can assume that Z is compact. We also can assume that the weight of Z is $\leq \tau$ (otherwise we apply again Proposition 2.4 to find a compact space $T \in \mathcal{K}$ of weight $\leq \tau$ and maps $g_1\colon Z \to T$ and $g_2\colon T \to \mathbb{I}^\tau$ with $g_2 \circ g_1 = g$, and then consider the space T and the map g_2 instead, respectively, of Z and g). Therefore, without loss of generality, we can assume that Z is a closed subset of $\mathbb{I}^\tau$. According to the definition of Y and the map h, there is an index $\alpha \in \Lambda$ such that $Z = Y_\alpha$ and $g = h_\alpha$. The restriction $p|Z\colon Z \to X$ is a lifting of g, i.e. $f \circ (p|Z) = g$.

3. C^*-algebras with the approximate n-th root property

In this Section we investigate the behavior of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$ with respect to direct systems and then use the result to prove the existence of universal elements in the classes $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$.

When we refer to a unital C^*-subalgebra of a unital C^*-algebra we always assume that the inclusion is a unital $*$-homomorphism. The product in the category of (unital) C^*-algebras, i.e. the ℓ^∞-direct sum, is denoted by $\prod\{X_t\colon t \in T\}$. For a given set Y and a cardinal number τ, the symbol $\exp_\tau Y$ denotes the partially ordered (by inclusion) set of all subsets of Y of cardinality not exceeding τ.

Recall that a direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital C^*-algebras consists of a partially ordered directed indexing set A, unital C^*-algebras X_α, $\alpha \in A$, and unital $*$-homomorphisms $i_\alpha^\beta\colon X_\alpha \to X_\beta$, defined for each pair of indexes $\alpha, \beta \in A$ with $\alpha \leq \beta$, and satisfying the condition $i_\alpha^\gamma = i_\beta^\gamma \circ i_\alpha^\beta$ for each triple of indexes $\alpha, \beta, \gamma \in A$ with $\alpha \leq \beta \leq \gamma$. The (inductive) limit of the above direct system is a unital C^*-algebra which is denoted by $\varinjlim \mathcal{S}$. For each $\alpha \in A$ there exists a unital $*$-homomorphism $i_\alpha\colon X_\alpha \to \varinjlim \mathcal{S}$ which will be called the α-th limit homomorphism of $\mathcal{S}$.

If A' is a directed subset of the indexing set A, then the subsystem $\{X_\alpha, i_\alpha^\beta, A'\}$ of $\mathcal{S}$ is denoted $\mathcal{S}|A'$.

Let $\tau \geq \omega$ be a cardinal number. A direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital C^*-algebras X_α and unital $*$-homomorphisms $i_\alpha^\beta\colon X_\alpha \to X_\beta$ is called a *direct C_τ^*-system* [3] if the following conditions are satisfied:

(a) A is a τ-complete set, that is, for each chain C of elements of the directed set A with $|C| \leq \tau$, there exists an element $\sup C$ in A. See [2] for details.

(b) The density $d(X_\alpha)$ of X_α is at most τ, for each $\alpha \in A$.
(c) The α-th limit homomorphism $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ is an injective $*$-homomorphism for each $\alpha \in A$.
(d) If $B = \{\alpha_t \colon t \in T\}$ is a chain of elements of A with $|T| \leq \tau$ and $\alpha = \sup B$, then the limit homomorphism $\varinjlim\{i^{\alpha}_{\alpha_t} \colon t \in T\} \colon \varinjlim (\mathcal{S}|B) \to X_\alpha$ is an isomorphism.

Proposition 3.1 (Proposition 3.2, [3])**.** *Let τ be an infinite cardinal number. Every unital C^*-algebra X can be represented as the limit of a direct C^*_τ-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, A\}$ where the index set $A = \exp_\tau Y$ for some (any) dense subset Y of X with $|Y| = d(X)$.*

Lemma 3.2 (Lemma 3.3, [3])**.** *If $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, A\}$ is a direct C^*_τ-system, then $\varinjlim \mathcal{S}_X = \bigcup\{i_\alpha(X_\alpha) \colon \alpha \in A\}$.*

The next proposition is a non-commutative version of Corollary 2.3

Proposition 3.3. *Let $\mathcal{K}$ be one of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. If X is the limit of a direct system $\mathcal{S} = \{X_\alpha, i^\beta_\alpha, A\}$ consisting of unital C^*-algebras and unital $*$-inclusions with $X_\alpha \in \mathcal{K}$ for each α, then $X \in \mathcal{K}$.*

Proof. We consider first the case $\mathcal{K} = \mathcal{AP}(n)$. Let $a \in X$ with $\|a\| \leq 1$ and $\varepsilon > 0$. Since $\bigcup\{X_\alpha \colon \alpha \in A\}$ is dense in X (we identify each $i_\alpha(X_\alpha)$ with X_α), there exist α and $y \in X_\alpha$ with $\|a - y\| < \frac{\varepsilon}{4}$. Then, $\|y\| < \|a\| + \frac{\varepsilon}{4} \leq 1 + \frac{\varepsilon}{4}$, so $\|\frac{y}{1+\varepsilon/4}\| < 1$. Since $X_\alpha \in \mathcal{AP}(n)$, there is $b \in X_\alpha$ with $\|\frac{y}{1+\varepsilon/4} - b^n\| < \frac{\varepsilon}{2}$ and $\|b\| \leq 1$. Then $\|a - b^n\| \leq \|a - \frac{y}{1+\varepsilon/4}\| + \|\frac{y}{1+\varepsilon/4} - b^n\| < \varepsilon$. Hence, $X \in \mathcal{AP}(n)$. The above arguments work also for the class $\mathcal{HP}(n)$ because of the fact that the set of invertible elements of a C^*-algebra is open. Indeed, for an invertible element a of X, the above fact allows us to choose y in the above argument as an invertible element of X. Consequently, $\frac{y}{1+\varepsilon/4}$ is invertible in X_α and, since $X_\alpha \in \mathcal{HP}(n)$, there is $b \in X_\alpha$ with the required properties. Because the limit of any direct system consisting of C^*-algebras with bounded rank ≤ 1 has a bounded rank ≤ 1 [5, Proposition 4.1], the above proof remains valid for the class $\mathcal{AP}_1(n)$. $\square$

As in the commutative case (see Proposition 2.6), we can establish a decomposition theorem for the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$.

Proposition 3.4. *Let $\mathcal{K}$ be one of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. The following conditions are equivalent for any unital C^*-algebra X:*

(1) *$X \in \mathcal{K}$.*

(2) *X can be represented as the direct limit of a direct C^*_ω-system $\{X_\alpha, i^\beta_\alpha, A\}$ satisfying the following properties:*
   (a) *The indexing set A is cofinal and ω-closed in the ω-complete set $\exp_\omega Y$ for some (any) dense subset Y of X such that $|Y| = d(X)$.*
   (b) *X_α is a (separable) C^*-subalgebra of X with $X_\alpha \in \mathcal{K}$, $\alpha \in A$.*

Proof. A similar statement holds for the class of all C^*-algebras of bounded rank $\leq n$ (see [5, Proposition 4.2]). So, it suffices to consider the classes $\mathcal{AP}(n)$ and $\mathcal{HP}(n)$. We suppose $\mathcal{K} = \mathcal{AP}(n)$. The implication (2) $\Longrightarrow$ (1) follows from Proposition 3.3.

In order to prove the implication (1) $\Longrightarrow$ (2) we first consider a direct C^*_ω-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, \Lambda\}$ with the properties indicated in Proposition 3.1 . Each X_α is identified with $i_\alpha(X_\alpha)$. We next introduce the following relation $L \subseteq A^2$: $(\alpha, \beta) \in \Lambda^2$ if and only if $\alpha \leq \beta$ and for each $x \in X_\alpha$ with $\|x\| \leq 1$ and each $\varepsilon > 0$ there exists $y \in X_\beta$ such that $\|y\| \leq 1$ and $\|x - y^n\| < \varepsilon$.

Let us show that L satisfies the following conditions:
(i) for every $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ with $(\alpha, \beta) \in L$:
(ii) If $(\alpha, \beta) \in L$ and $\beta \leq \gamma$, then $(\alpha, \gamma) \in L$;
(iii) if $\{\alpha_k\}$ is a chain in Λ with each $(\alpha_k, \beta) \in L$, then $(\alpha, \beta) \in L$, where $\alpha = \sup\{\alpha_k\}$.

To verify (i), we take $\alpha \in \Lambda$ and a countable set $M \subset X_\alpha$ which is dense in the unit ball $B_\alpha = \{x \in X_\alpha : \|x\| \leq 1\}$. Since $X \in \mathcal{AP}(n)$, for each $x \in M$ and each $r \in Q^+$, we may take (and fix) $y(x, r) \in X$ with $\|x - y(x, r)^n\| < r$ and $\|y(x, r)\| \leq 1$. By Lemma 3.2, every $y(x, r)$ belongs to some $X_{\alpha(x,r)}$. Since Λ is ω-complete, according to [2, Corollary 1.1.28], there exists $\beta \in \Lambda$ such that $\beta \geq \alpha$ and $\beta \geq \alpha(x, r)$ for each $x \in M$ and $r \in Q^+$. Then, X_β contains all $y(x, r)$ and $(\alpha, \beta) \in L$. Condition (ii) follows directly because $\beta \leq \gamma$ implies $X_\beta \subset X_\gamma$. Let us establish condition (iii). If α is the supremum of the countable chain $\{\alpha_k\}$, then X_α is the direct limit of the direct system generated by the C^*-subalgebras X_{α_k}, $k = 1, 2, ..$, and the corresponding inclusion homomorphisms. This fact and $(\alpha_k, \beta) \in L$ for all k yield $(\alpha, \beta) \in L$.

Since L satisfies the conditions (i)-(iii), we can apply [2, Proposition 1.1.29] to conclude that the set $A = \{\alpha \in \Lambda : (\alpha, \alpha) \in L\}$ is cofinal and ω-closed in Λ. Note that $(\alpha, \alpha) \in L$ precisely when $X_\alpha \in \mathcal{AP}(n)$. Therefore, we obtain a direct C^*_ω-system $\mathcal{S}'_X = \{X_\alpha, i^\beta_\alpha, A\}$ consisting of C^*-subalgebras $X_\alpha \in \mathcal{AP}(n)$ of X. Clearly $\varinjlim \mathcal{S}'_X = X$. This completes the proof for the class $\mathcal{AP}(n)$. The case $\mathcal{K} = \mathcal{AP}(n)$ is similar. $\square$

Proof of Theorem 1.4. Let $\mathcal{B} = \{f_t : C^*(\mathbb{F}_\infty) \to X_t : t \in T\}$ denote the set of all unital $*$-homomorphisms on $C^*(\mathbb{F}_\infty)$ such that $X_t \in \mathcal{K}$. We claim that the product $\prod\{X_t : t \in T\}$ belongs to $\mathcal{K}$. This is obviously true if $\mathcal{K}$ is either $\mathcal{AP}(n)$ or $\mathcal{HP}(n)$. Since the bounded rank of this product is ≤ 1 provided each X_t is of bounded rank ≤ 1 [5, Proposition 3.16], the claim holds for the class $\mathcal{AP}_1(n)$ as well. The $*$-homomorphisms f_t, $t \in T$, define the unital $*$-homomorphism $f\colon C^*(\mathbb{F}_\infty) \to \prod\{X_t\colon t \in T\}$ such that $\pi_t \circ f = f_t$ for each $t \in T$, where $\pi_t\colon \prod\{X_t\colon t \in T\} \to X_t$ denotes the canonical projection $*$-homomorphism onto X_t. By Proposition 3.4, $\prod\{X_t\colon t \in T\}$ can be represented as the limit of the C^*_ω-system $\mathcal{S} = \{C_\alpha, i^\beta_\alpha, A\}$ such that C_α is a separable unital C^*-algebra with $C_\alpha \in \mathcal{K}$ for each $\alpha \in A$. Suppressing the injective unital $*$-homomorphisms $i^\beta_\alpha\colon C_\alpha \to C_\beta$, we may assume, for notational simplicity, that C_α's are unital C^*-subalgebras of $\prod\{X_t\colon t \in T\}$. Let $\{a_k\colon k \in \omega\}$ be a countable dense subset of $C^*(\mathbb{F}_\infty)$. By Lemma 3.2, for each $k \in \omega$ there exists an index $\alpha_k \in A$ such that $f(a_k) \in C_{\alpha_k}$. Since A is ω-complete, there exists an index $\alpha_0 \in A$ such that $\alpha_0 \geq \alpha_k$ for each $k \in \omega$. Then $f(a_k) \in C_{\alpha_k} \subseteq C_{\alpha_0}$ for each $k \in \omega$. This observation coupled with the continuity of f guarantees that $f(C^*(\mathbb{F}_\infty)) = f(\mathrm{cl}\{a_k\colon k \in \omega\}) \subseteq \mathrm{cl}\{f(\{a_k\colon k \in \omega\})\} \subseteq \mathrm{cl}\, C_{\alpha_0} = C_{\alpha_0}$.

Let $Z_{\mathcal{K}} = C_{\alpha_0}$ and define the unital $*$-homomorphism $p\colon C^*(\mathbb{F}_\infty) \to Z_{\mathcal{K}}$ as f, regarded as a homomorphism of $C^*(\mathbb{F}_\infty)$ into $Z_{\mathcal{K}}$. Note that $f = i \circ p$, where $i\colon Z_{\mathcal{K}} = C_{\alpha_0} \hookrightarrow \prod\{X_t\colon t \in T\}$ stands for the inclusion.

By construction, we see $Z_{\mathcal{K}} \in \mathcal{K}$. Let us show that $p\colon C^*(\mathbb{F}_\infty) \to Z_{\mathcal{K}}$ is $\mathcal{K}$-invertible. For a given unital $*$-homomorphism $g\colon C^*(\mathbb{F}_\infty) \to X$, where X is a separable unital C^*-algebra with $X \in \mathcal{K}$, we need to establish the existence of a unital $*$-homomorphism $h\colon Z_{\mathcal{K}} \to X$ such that $g = h \circ p$. Indeed, by definition of the set $\mathcal{B}$, we conclude that $g = f_t\colon C^*(\mathbb{F}_\infty) \to X_t = X$ for some index $t \in T$. Observe that $g = f_t = \pi_t \circ f = \pi_t \circ i \circ p$. This allows us to define the required unital $*$-homomorphism $h\colon Z_{\mathcal{K}} \to X$ as the composition $h = \pi_t \circ i$. Hence, p is $\mathcal{K}$-invertible.

4. Example

In this section, we show that a construction due to B. Cole (cf.[13, Chap.3, section 19]) and M. Karahanjan [9, Thoerem 5] yields a square root closed compatum X such that $\check{H}^1(X;\mathbb{Z})$ is infinitely generated. In the sequel, we shall omit the coefficient group $\mathbb{Z}$. We will need the following theorem which is a consequence of [7, Theorem 3.2].

Theorem 4.1. *Let $f\colon X \to Y$ be an open surjective map between compacta. Then $f^*\colon \check{H}^1(Y) \to \check{H}^1(X)$ is a monomorphism.*

Now we outline the construction due to B. Cole. This is based on the exposition in [13, Chapter 3, §19, p.194-197]. Let X be a compactum and define

$$S_X = \{(x, (z_f)_{f \in C(X)})\colon f(x) = z_f^2 \quad \text{for each } f \in C(X)\} \subset X \times \mathbb{C}^{C(X)}$$

Note that S_X is a closed subset of $X \times \prod\{f(X) | f \in C(X)\}$ and hence is a compactum. Also, it is easy to see that S_X is a pull-back in the following diagram:

$$\begin{array}{ccc} S_X & \longrightarrow & \mathbb{C}^{C(X)} \\ \big\downarrow & & \big\downarrow S \\ X & \xrightarrow{F} & \mathbb{C}^{C(X)} \end{array}$$

where $F\colon X \to \mathbb{C}$ is defined by $F(x) = (f(x))_{f\in C(X)} (x \in X)$, and $S\colon \mathbb{C}^{C(X)} \to \mathbb{C}^{C(X)}$ is defined by $S((z_f)_{f\in C(X)}) = (z_f^2)_{f\in C(X)}$.

Let $\pi\colon S_X \to X$ be the map defined by $\pi[(x, (z_f)_{f\in C(X)})] = x$ for all $x \in X$. Then π is an open map with zero-dimensional fibers. The critical property of S_X and π is the following:

$(*)$ for any $f \in C(X)$ there exists $g \in C(X)$ such that $f \circ \pi = g^2$.

Indeed, define $g\colon S_X \to \mathbb{C}$ by $g[(x, (z_f)_{f\in C(X)})] = z_f$.

Note that $(*)$ implies:

$(**)$ π is invertible with respect to the class of square root closed compacta.

Starting with a compactum X_0, by transfinite induction we define an inverse spectrum $\{X_\alpha, \pi_\alpha^\beta\colon X_\beta \to X_\alpha : \alpha \leq \beta < \omega_1\}$ as follows. If $\beta = \alpha + 1$ then $X_\beta = S_{X_\alpha}$ and $\pi_\alpha = \pi\colon X_\beta = S_{X_\alpha} \to X_\alpha$ is the map defined above. If β is a limit ordinal, then $X_\beta = \varprojlim(X_\alpha, \pi_\alpha^\gamma\colon X_\gamma \to X_\alpha : \alpha \leq \gamma < \beta)$ and, for $\alpha < \beta$, let $\pi_\alpha^\beta = \varprojlim(\pi_\alpha^\gamma\colon X_\gamma \to X_\alpha : \gamma < \beta)$.

We let $X_\Omega = \varprojlim X_\alpha$. The α-th limit projection is denoted by $\pi_\alpha : X_\Omega \to X_\alpha$. As the length of the above spectrum is ω_1, the spectrum is factorizing in the sense that each $f \in C(X_\Omega)$ is represented as $f = f_\alpha \circ \pi_\alpha$ for some $\alpha < \omega_1$ and $f_\alpha \in C(X_\alpha)$. since its length is ω_1. This implies that $C(X_\Omega)$ is square root closed due to the property $(*)$.

In what follows, the unit disk in the complex plane $\{z \in \mathbb{C} : |z| \leq 1\}$ is denoted by Δ.

Theorem 4.2. *$C(\Delta_\Omega)$ is square-root closed,* $\dim\Delta_\Omega \leq 2$*, $\check{H}^1(\Delta_\Omega)$ is infinitely generated and 2-divisible.*

Notice that for each square root closed compactum X, $\check{H}^1(X)$ is 2-divisible. Hence, in view of the discussion above, we need only to show that $\check{H}_1(\Delta_\Omega)$ is infinitely generated. To show this, we need the following.

Theorem 4.3. *$\check{H}^1(S_\Delta)$ is infinitely generated.*

Note that Theorem 4.2 immediately follows from Theorems 4.1 and Theorem 4.3. The proof of Theorem 4.3 is divided into two parts.

Step 1. If $\check{H}^1(S_\Delta)$ is finitely generated then $\check{H}^1(S_\Delta) = 0$.

Step 2. $\check{H}^1(S_\Delta) \neq 0$.

Now we shall accomplish Steps 1 and 2.

Proposition 4.4. *Let Y be a closed subspace of a compactum X such that there exists a retraction $r\colon X \to Y$. Let also $i\colon Y \hookrightarrow X$ be the inclusion. Then there exist an embedding $\overline{i}\colon S_Y \hookrightarrow S_X$ and a retraction $\overline{r}\colon S_X \to S_Y$ such that the following diagram is commutative.*

$$\begin{array}{ccccc}
S_Y & \xrightarrow{\overline{i}} & S_X & \xrightarrow{\overline{r}} & S_Y \\
\downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\
Y & \xrightarrow{i} & X & \xrightarrow{r} & Y
\end{array}$$

Proof. Define $\overline{i}$ by

$$\overline{i}[(y, (\eta_g)_{g \in C(Y)})] = (y, (\xi_f)_{f \in C(X)})$$

where $\xi_f = \eta_{f|Y}$ for all $f \in C(X)$. Define $\overline{r}$ by

$$\overline{r}[(x, (\xi_f)_{f \in C(X)})] = (r(x), (\eta_g)_{g \in C(X)})$$

where $\eta_g = \xi_{g \circ r}$ for all $g \in C(Y)$. □

Now we are ready to accomplish Step 1. Let $\Delta_m = \{z \in \mathbb{C}\colon |z| \leq \frac{1}{m}\} \subset \Delta$. Let $r_n\colon \Delta_n \to \Delta_{n+1}$ be the radial retraction and $i_n\colon \Delta_{n+1} \hookrightarrow \Delta_n$ be the inclusion. Consider the following sequence of commutative diagrams.

$$\begin{array}{ccccccccccc}
S_{\Delta_1} & \xleftarrow{\overline{i}_1} & S_{\Delta_2} & \xleftarrow{\overline{i}_2} & \cdots \longleftarrow & S_{\Delta_n} & \xleftarrow{\overline{i}_n} & S_{\Delta_{n+1}} & \longleftarrow \cdots \longleftarrow & \varprojlim S_{\Delta_n} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & & \downarrow{\scriptstyle \pi_n} & & & & \downarrow{\scriptstyle \varprojlim \pi_n = \pi_\infty} \\
\Delta_1 & \xleftarrow{i_1} & \Delta_2 & \xleftarrow{i_2} & \cdots \longleftarrow & \Delta_n & \xleftarrow{i_n} & \Delta_{n+1} & \longleftarrow \cdots \longleftarrow & \{0\}
\end{array}$$

It follows easily form the commutativity of the diagram that $\lim_{\longleftarrow} S_{\Delta_n}$ is homeomorphic to the inverse limit of the sequence

$$\pi_1^{-1}(0) \xleftarrow{\overline{i}_1|} \pi_2^{-1}(0) \longleftarrow \cdots \longleftarrow \pi_n^{-1}(0) \xleftarrow{\overline{i}_n|} \pi_{n+1}^{-1}(0) \longleftarrow \cdots$$

Since each fiber $\pi_n^{-1}(0)$ is 0-dimensional, we have $\dim \lim_{\longleftarrow} S_{\Delta_n} = 0$. This implies that $\check{H}^1(\lim\limits_{\longleftarrow} S_{\Delta_n}) = \lim\limits_{\longrightarrow} \check{H}^1(S_{\Delta_n}) = 0$, which is equivalent to the following observation.

Proposition 4.5. *For each $\alpha \in \check{H}^1(S_{\Delta_1}) = \check{H}^1(S_\Delta)$, there exists an n such that $(\bar{i}_1 \circ \cdots \circ \bar{i}_n)^*(\alpha) = 0$.*

Let A_n be the annulus defined by $A_n = \{z \in \mathbb{C} | \frac{1}{m+1} \le |z| \le \frac{1}{m}\}$, so that $\Delta_n = \{0\} \cup (\cup\{A_j | j \ge n\})$. Let $h\colon \Delta = \Delta_1 \to \Delta_2$ be the homeomorphism which maps A_j to A_{j+1} ($j \ge 1$) by "radial homeomorphisms" and such that $h(0) = 0$. Then the following diagram is commutative

$$\begin{array}{ccc} \Delta_n & \xrightarrow{h|} & \Delta_{n+1} \\ \uparrow{\scriptstyle i_n} & & \uparrow{\scriptstyle i_{n+1}} \\ \Delta_{n+1} & \xrightarrow{h|} & \Delta_{n+1} \end{array}$$

Define $h_n\colon S_{\Delta_n} \to S_{\Delta_{n+1}}$ by $h_n[(x, (u_f)_{f \in C(\Delta_n)})] = (h(x), (v_g)_{g \in C(\Delta_{n+1})})$, where $v_g = u_{g \circ h}$, $g \in C(\Delta_{n+1})$. Note that h_n is a homeomorphism.

Proposition 4.6. *The following diagram is commutative.*

$$\begin{array}{ccc} S_{\Delta_{n+1}} & \xrightarrow{\bar{i}_n} & S_{\Delta_n} \\ {\scriptstyle h_{n+1}}\downarrow & & \downarrow{\scriptstyle h_n} \\ S_{\Delta_{n+2}} & \xrightarrow{\bar{i}_{n+1}} & S_{\Delta_{n+1}} \end{array}$$

Proof. For each $(x_{n+1}, (z_f)_{f \in C(\Delta_{n+1})}) \in S_{\Delta_{n+1}}$ we have

$$\bar{i}_n[(x_{n+1}, (z_f)_{f \in C(\Delta_{n+1})})] = (x_{n+1}, (u_f)_{f \in C(\Delta_n)})$$

where $u_f = z_{f|\Delta_n} = z_{f \circ i_n}$, $f \in C(\Delta_n)$, and

$$h_n[(x_{n+1}, (u_f)_{f \in C(\Delta_n)})] = (h(x_{n+1}), (v_f)_{f \in C(\Delta_{n+1})})$$

where $v_f = u_{f \circ h} = z_{(f \circ h) \circ i_n} = z_{f \circ (h \circ i_n)}$. On the other hand,

$$h_{n+1}[(x_{n+1}, (z_f)_{f \in C(\Delta_{n+1})})] = (h(x_{n+1}), (u_g)_{g \in C(\Delta_{n+1})})$$

where $u_g = z_{g \circ h}$, $g \in C(\Delta_{n+2})$, and

$$\bar{i}_{n+1}[(h(x_{n+1}), (u_g)_{g \in C(\Delta_{n+1})})] = (h(x_{n+1}), (v_f)_{f \in C(\Delta_{n+1})})$$

where $v_f = u_{f \circ i_{n+1}} = z_{(f \circ i_{n+1}) \circ h} = z_{f \circ (i_{n+1} \circ h)}$. Since $h \circ i_n = i_{n+1} \circ h$, we conclude that the diagram is commutative. $\square$

The above lemma provides a commutative diagram in cohomologies:

$$(\dagger)\qquad \begin{array}{ccc} \check{H}^1(S_{\Delta_{n+1}}) & \xleftarrow{\;\overline{i}_n^*\;} & \check{H}^1(S_{\Delta_n}) \\ {\scriptstyle h_{n+1}^*}\Big\uparrow & & \Big\uparrow{\scriptstyle h_n^*} \\ \check{H}^1(S_{\Delta_{n+2}}) & \xleftarrow{\;\overline{i}_{n+1}^*\;} & \check{H}^1(S_{\Delta_{n+1}}) \end{array}$$

Let $\phi = h_1^* \circ i_1^* \colon \check{H}^1(S_\Delta) \to \check{H}^1(S_\Delta)$. Since $\overline{r}_1 \circ \overline{i}_1 = \mathrm{id}_{S_\Delta}$ we have $\overline{i}_1^* \circ \overline{r}_1^* = \mathrm{id}_{\check{H}^1(S_\Delta)}$ and hence ϕ is an epimorphism. We use diagram $(\dagger)$ to obtain the following diagram, in which all vertical arrows are isomorphisms.

$$\begin{array}{ccccccccc} \check{H}^1(S_\Delta) & \xrightarrow{\overline{i}_1^*} & \check{H}^1(S_{\Delta_2}) & \xrightarrow{\overline{i}_2^*} & \check{H}^1(S_{\Delta_3}) & \xrightarrow{\overline{i}_3^*} & \check{H}^1(S_{\Delta_4}) & \longrightarrow & \cdots \\ & \searrow^{\phi} & \Big\downarrow{\scriptstyle h_1^*} & & \Big\downarrow{\scriptstyle h_2^*} & & \Big\downarrow{\scriptstyle h_3^*} & & \\ & & \check{H}^1(S_\Delta) & \xrightarrow{\overline{i}_1^*} & \check{H}^1(S_{\Delta_2}) & \xrightarrow{\overline{i}_2^*} & \check{H}^1(S_{\Delta_3}) & \longrightarrow & \cdots \\ & & & \searrow^{\phi} & \Big\downarrow{\scriptstyle h_1^*} & & \Big\downarrow{\scriptstyle h_2^*} & & \\ & & & & \check{H}^1(S_\Delta) & \xrightarrow{\overline{i}_1^*} & \check{H}^1(S_{\Delta_2}) & \longrightarrow & \cdots \\ & & & & & \searrow^{\phi} & \Big\downarrow{\scriptstyle h_1^*} & & \\ & & & & & & \check{H}^1(S_\Delta) & \longrightarrow & \cdots \\ & & & & & & & \searrow^{\phi} & \\ & & & & & & & & \vdots \end{array}$$

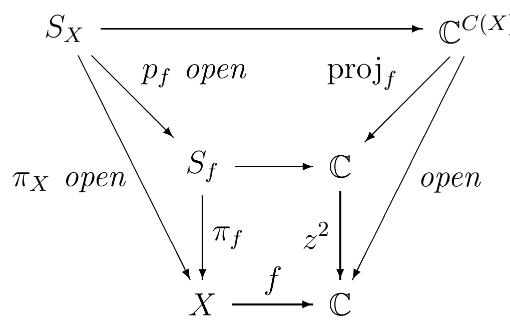
The above diagram together with Proposition 4.5 imply that, for each $\alpha \in \check{H}^1(S_\Delta)$, there exists n such that $\phi^n(\alpha) = 0$. If $\check{H}^1(S_\Delta)$ were finitely generated, we then would have $\check{H}^1(S_\Delta) = 0$ because of the following observation.

Proposition 4.7. *Let A be a finitely generated Abelian group. If there exists an epimorphism $f \colon A \to A$ such that for any $a \in A$ there exists n with $f^n(a) = 0$, then A is trivial.*

Proof. Note that $f \otimes 1_{\mathbb{Q}} \colon A \otimes \mathbb{Q} \to A \otimes \mathbb{Q}$ is an epimorphism of a vector space $A \otimes \mathbb{Q}$, which is finite-dimensional over $\mathbb{Q}$. Hence $f \otimes 1_{\mathbb{Q}}$ is an isomorphism with the property in the hypothesis. This implies rank$A = 0$ and therefore A is a finite Abelian group. Then f is an isomorphism and therefore $A = 0$. $\square$

Thus Step 1 is completed and we proceed to Step 2.

Proposition 4.8. *For a continuous function $f \in C(X)$, let* $S_f = \{(x, z) : f(x) = z^2 \text{ for each } x \in X\} \subset X \times \mathbb{C}$. *Let also $\pi_f \colon S_f \to X$ be the projection. Then the natural map $p_f \colon S_X \to S_f$, $(x, (z_g)_{g \in C(X)}) \mapsto (x, z_f)$ is open. Thus we have the following diagram.*



Proof. Consider $g_1, g_2, \ldots, g_n \in C(X)$ and open subset $U_X \subset X$, $V_f, V_{g_1}, \ldots, V_{g_n} \subset \mathbb{C}$. It suffices to show that

$$p_f[(U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1, \ldots, g_n, f} \mathbb{C}) \cap S_X]$$

is open in S_f. Take a point

$$(x, z_f, (z_{g_i})_{i=1}^n, (z_g)_{g \neq f, g_1, \ldots, g_n}) \in U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1, \ldots, g_n, f} \mathbb{C}$$

and choose $\epsilon > 0$ such that $B(z_f, \epsilon) = \{w \in \mathbb{C} \colon |w - z_f| < \epsilon\} \subset V_f$ and $B(z_{g_i}, \epsilon) \subset V_{g_i}$ for all $i = 1, 2, \ldots, n$. Let $a = f(x)$, $a_i = g_i(x)$, $i = 1, 2, \ldots, n$. There exists $\delta > 0$ such that if $|b - a| < \delta$ and $|b_i - a_i| < \delta$, $i = 1, \ldots, n$, then the equations

$$\begin{aligned} z^2 - b &= 0 \\ z_i^2 - b_i &= 0, i = 1, \ldots, n \end{aligned}$$

have solutions z_b and z_{b_i} respectively such that $|z_b - z_f| < \epsilon$, $|z_{b_i} - z_{g_i}| < \epsilon$. Choose a neighborhood N of x such that $|f(y) - f(x)| < \delta$ and $|g_i(y) - g_i(x)| < \delta$ for all $y \in N$ and $i = 1, \ldots, n$. We claim that

$$N \times B(z_f, \epsilon) \subset p_f[(U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1, \ldots, g_n, f} \mathbb{C}) \cap S_X]$$

Indeed, for each pint $(y,w) \in N \times B(z_f,\epsilon) \subset N \times V_f$ we have $|g_i(y) - g_i(x)| < \delta$, $i = 1,2,\dots,n$ by the choice of N. Then we can find $z_i \in B(z_{g_i},\epsilon)$ such that $z_i^2 = g_i(y)$. Now for arbitrary choice of z_g, where $g \neq f, g_1, g_2, \dots, g_n$ with $z_g^2 = g(x)$, we have

$$(y, w, (z_i)_{i=1}^n, (z_g)) \in U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1,\dots,g_n,f} \mathbb{C}$$

and $p_f[(y, w, (z_i)_{i=1}^n, (z_g))] = (y, w)$. This proves the claim and hence completes the proof of the proposition. □

By Proposition 4.8 and Theorem 4.1, the statement of the Step 2 follows from the next observation.

Proposition 4.9. *There exists a mapping $f\colon \Delta \to \mathbb{C}$ such that $\check{H}^1(S_f) \neq 0$.*

Proof. Let $f(x,y) = (-2|x| + \sqrt{1-y^2}, y)$ for all $(x,y) \in \Delta$. Then S_f is homeomorphic to cylinder $S^1 \times I$. □

This completes the proof of Theorem 4.2.

The above construction is carried out word by word for disks of arbitrary dimensions. In particular, applying the above to the one-dimensional disk $[-1,1]$, we have the following corollary which suggests that a topological characterization of general square root closed compacta could be rather different than the one for first-countable such compacta by [8] and [12].

Corollary 4.10. *There exists an one-dimensional square root closed compactum X with infinitely generated first Čech cohomology.*

For an infinite cardinal $\tau \geq \omega$, we consider $(\mathbb{I}^\tau)_\Omega$ and the limit projection $\pi_\Omega : (\mathbb{I}^\tau)_\Omega \to \mathbb{I}^\tau$. By the invertibility property (**) of $\pi : S_X \to X$ for arbitrary compactum X and the standard spectral argument, it follows easily that π_Ω is also invertible with respect to the class of square root closed compacta. Hence we have

Proposition 4.11. *The square root closed compactum $(\mathbb{I}^\tau)_\Omega$ contains every square root closed compactum of weight $\leq \tau$.*

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