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NOTES ON TWO CONJECTURES IN EXTENSION THEORY

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Abstract. It is noted that conjectures about the non-existence of universal compacta and compactifications of the given extension dimension for non finitely dominated complexes are not valid for all CW complexes of the form $L \vee S^2$, where L is of finite type and has a finite fundamental group, but is not finitely dominated.

1. Introduction

The following conjecture has been stated in [7, Conjecture 2.8] and later restated in [6, Conjecture 1.23]:

Conjecture A. *Suppose K is a countable CW complex. If the class of compacta $\{X\colon K \in AE(X)\}$ has a universal space, then K is homotopy dominated by a finite CW complex.*

Closely related to this is the following hypothesis [6, Conjecture 1.20]:

Conjecture B. *Let K be a countable CW complex. Any separable metrizable space X with $K \in AE(X)$ admits a metrizable compactification $\widetilde{X}$ with $K \in AE(\widetilde{X})$ if and only if K is homotopy dominated by a finite CW complex.*

Below we show that both conjectures fail (see [8, Lemma 1.11] where Conjecture A has been formally disproved) for a large class of complexes. This becomes clear by observing that the assertions of these conjectures are valid for arbitrary CW complexes of the form $K = L \vee S^2$, where L is a CW complex of finite type which is not finitely dominated and which has a finite fundamental group (consider, for instance, $L = \mathbb{R}P^\infty$). Note that K also is not finitely dominated and has a finite fundamental group.

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2. Disproving Conjectures A and B

For spaces X and L the notation $L \in \mathrm{AE}(X)$ means that every continuous map $f\colon A \to L$, defined on a closed subspace A of X, admits a continuous

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extension $\widetilde{f}\colon X \to L$. A. N. Dranishnikov introduced [5] the following partial order for CW complexes. We say that $L \leq K$ if for each space X the condition $L \in \mathrm{AE}(X)$ implies the condition $K \in \mathrm{AE}(X)$. Equivalence classes of complexes with respect to this relation are called *extension types*. The above defined relation $\leq$ creates a partial order in the class of extension types of complexes. This partial order is still denoted by $\leq$ and the extension type with representative K is denoted by $[K]$ ([1], where the reader can find a general overview of extension theory, discusses further properties of extension types in terms of this partial order).

For the reader's convenience we state here two statements which are needed below. Recall that a complex has a finite type if it contains finite number of cells in every dimension. βX denotes the Stone-Čech compactification of a space X.

Proposition 1 [2, Proposition 2.3]. *Let L be a CW complex of finite type with a finite fundamental group. Then the following conditions are equivalent for any normal space X and any integer $n \geq 2$:*

(a) $L \vee S^n \in \mathrm{AE}(X)$.
(b) $L \vee S^n \in \mathrm{AE}(\beta X)$.

Proposition 2 [2, Theorem 2.1]. *Let L be a countable CW complex. Then the following conditions are equivalent for any normal space X:*

(a) $L \in \mathrm{AE}(\beta X)$ *whenever* $L \in \mathrm{AE}(X)$.
(b) *There exists an L-invertible map $f_L\colon X_L \to Q$ of a metrizable compactum X_L, with $L \in \mathrm{AE}(X_L)$, onto the Hilbert cube Q.*

2.1. **Conjecture A.** In order to prove our claim related to Conjecture A, take a complex $K = L \vee S^2$ of the type indicated in the Introduction and note that, by Proposition 1, $K \in \mathrm{AE}(\beta Y)$ for any normal space Y with $K \in \mathrm{AE}(Y)$. Then, by Proposition 2, there exists a K-invertible map $f_K\colon Y_K \to Q$ of a metrizable compactum Y_K, with $K \in \mathrm{AE}(Y_K)$, onto the Hilbert cube Q. This implies that any embedding $Z \hookrightarrow Q$ of a metrizable compactum with $Z \in \mathrm{AE}(X)$ can be lifted to an embedding $Z \hookrightarrow X_K$. Consequently Y_K is an universal metrizable compactum for the class indicated in Conjecture A.

2.2. **Conjecture B.** Next consider a separable metrizable space X such that $K \in \mathrm{AE}(X)$. By [9], we may assume that X is completely metrizable. By Proposition 1, $K \in \mathrm{AE}(\beta X)$. By [3, Theorem 4.4], βX can be represented as the limit of a Polish spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of Polish spaces X_α such that $K \in \mathrm{AE}(X_\alpha)$, $\alpha \in A$. Since βX is compact, we may assume that all X_α's are metrizable compact spaces. We will show that there exists an index $\widetilde{\beta} \in A$ such that the restriction $p_{\widetilde{\beta}}|X\colon X \to p_{\widetilde{\beta}}(X)$ of the limit projection $p_{\widetilde{\beta}}\colon \beta X \to X_{\widetilde{\beta}}$ of the spectrum $\mathcal{S}$ is a homeomorphism. This would complete the proof since the metrizable compactum $\widetilde{X} = X_{\widetilde{\beta}}$ would then serve as the required compactification of the space X.

Since X is completely metrizable there exist functionally open subsets G_k in βX such that $X = \cap_{k=1}^{\infty} G_k$. Since the spectrum $\mathcal{S}$ is factorizing σ-spectrum (see [4, Sections 1.3.1 and 1.3.2]) we can conclude that for each k there exists an index $\alpha_k \in A$ such that $G_k = p_{\alpha_k}^{-1}\left(p_{\alpha_k}(G_k)\right)$. According to [4, Corollary 1.1.28] there exists an index $\widetilde{\alpha} \in A$ such that $\widetilde{\alpha} \geq \alpha_k$ for each k. Then $X = p_{\widetilde{\alpha}}^{-1}\left(p_{\widetilde{\alpha}}(X)\right)$. Moreover, it is easy to see that $X = p_{\alpha}^{-1}\left(p_{\alpha}(X)\right)$ for each $\alpha \geq \widetilde{\alpha}$. It then follows that for each such α the restriction $p_{\alpha}|X \colon X \to p_{\alpha}(X)$ is a perfect surjection.

Finally we show that there exists an index $\widetilde{\beta} \geq \widetilde{\alpha}$ such that every its fiber $\left(p_{\widetilde{\beta}}|X\right)^{-1}(y)$, $y \in p_{\widetilde{\beta}}(X)$, consists of precisely one point. Choose a countable open base $\{U\}_{k=1}^{\infty}$ of X (such a base exists since X is a separable metrizable space). Let V_k be a functionally open subset of the Stone-Čech compactification βX such that $V_k \cap X = U_k$. Since the spectrum $\mathcal{S}$ is factorizing, there exists an index $\beta_k \geq \widetilde{\alpha}$ such that $V_k = p_{\beta_k}^{-1}\left(p_{\beta_k}(V_k)\right)$. By [4, Corollary 1.1.28], there exists an index $\widetilde{\beta} \in A$ such that $\widetilde{\beta} \geq \beta_k$ for each k. Let us show that this index has the required property. Assume the contrary, i.e. suppose that the exists a point $y \in p_{\widetilde{\beta}}(X)$ such that the fiber $\left(p_{\widetilde{\beta}}|X\right)^{-1}(y) = p_{\widetilde{\beta}}^{-1}(y) \cap X$ contains at least two distinct points. Denote them by x_1 and x_2 respectively. Since $\{U_k\}_{k=1}^{\infty}$ is an open base of X we can find an index n such that $x_1 \in U_n$ and $x_2 \notin U_k$. Note that

$$U_k = V_k \cap X = p_{\beta_k}^{-1}\left(p_{\beta_k}(U_k)\right) \cap X = p_{\widetilde{\beta}}^{-1}\left(\left(p_{\beta_k}^{\widetilde{\beta}}\right)^{-1}\left(p_{\beta_k}^{\widetilde{\beta}}(p_{\widetilde{\beta}}(U_k))\right)\right) \cap X = p_{\widetilde{\beta}}^{-1}\left(p_{\widetilde{\beta}}(U_k)\right) \cap X.$$

Then, by the choice of the set U_k we obtain two contradictory conclusions:

$$y = p_{\widetilde{\beta}}(x_1) \in p_{\widetilde{\beta}}\left(p_{\widetilde{\beta}}^{-1}\left(p_{\widetilde{\beta}}(U_k)\right) \cap X\right) = p_{\widetilde{\beta}}(U_k) \cap p_{\widetilde{\beta}}(X)$$

and

$$y = p_{\widetilde{\beta}}(x_2) \notin p_{\widetilde{\beta}}\left(p_{\widetilde{\beta}}^{-1}\left(p_{\widetilde{\beta}}(U_k)\right) \cap X\right) = p_{\widetilde{\beta}}(U_k) \cap p_{\widetilde{\beta}}(X).$$

This shows, as required, that the restriction $p_{\widetilde{\beta}}|X \colon X \to p_{\widetilde{\beta}}$ is a homeomorphism.

2.3. **Concluding remarks.** CW complexes we used to disprove Conjectures A and B are of the form $K = L \vee S^n$, $n \geq 2$, where L has a finite type (and a finite fundamental group). Of course, such complexes need not be finitely dominated. However, $[K] = [L \vee S^n] \leq [S^n]$ and consequently, according to [1, Example 2.4(iv)], $[K] = [K^{(n)} \vee S^n] = [(L \vee S^n)^{(n)} \vee S^n] = [L^{(n)} \vee S^n \vee S^n] = [L^{(n)} \vee S^n]$.

This shows that the extension type $[K]$ contains a finite complex (note that $L^{(n)}$ is finite). This leads to a more plausible versions of the above conjectures. Namely, is it true that under the same assumptions it follows that the extension type $[K]$ of the complex contains a finitely dominated complex?

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