NONMETRIZABLE ANR'S ADMITTING A GROUP STRUCTURE ARE MANIFOLDS

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Abstract. It is shown that a nonmetrizable ANR-space of weight $\tau > \omega$, admitting a group structure, is (topologically) an $\mathbb{R}^\tau$-manifold.

It is well known [4, Corollary 1] that if a separable complete ANR carries a topological group structure then either this is a Lie group or the ANR is an l_2-manifold. We show that in the non-metrizable case the situation, is certain sense, is simpler. Namely we prove the following result.

Theorem 0.1. *Let $\tau > \omega$ and G be an ANR-space of weight τ admitting a group structure. Then G is an $\mathbb{R}^\tau$-manifold.*

We refer the reader to [1] for comprehensive discussion of the general (non-metrizable) theory of absolute retracts, soft maps and spectral techniques. Here all we need to emphasize is that ANR-spaces are defined as retracts of functionally open subspaces of powers of the real line and that a map $p\colon X \to Y$ is soft iff there is a retraction $r\colon Y \times P \to X$, where P is an AR-space, such that $pr = \pi_Y$. Two main properties of soft maps used below are: (a) soft maps are retractions (and hence admit sections); (b) inverse images of singletons are AR's.

We begin with the following two statements needed in the proof of our result.

Lemma 0.2. *Let $p\colon X \to Y$ be a continuous surjective homomorphism of topological groups. If there exists a continuous map $q\colon Y \to X$ such that $pq = \mathrm{id}_Y$, then p is a trivial fibration with fiber $\ker(p)$. More precisely there exists a homeomorphism $h\colon X \to Y \times \ker(p)$ such that $\pi_Y h = p$.*

Proof. The required homeomorphism h and its inverse $t\colon Y \times \ker(p) \to X$ are defined as follows:

$$h(x) = \left(p(x), x \cdot q(p(x))^{-1}\right), x \in X; \quad t(y,a) = a \cdot q(y), (y,a) \in Y \times \ker(p).$$

First, note that h is well defined. Indeed, for $x \in X$ we have

$$p\left(x \cdot q(p(x))^{-1}\right) = p(x) \cdot p\left(q(p(x))^{-1}\right) = p(x) \cdot (p(q(p(x)))^{-1} = p(x) \cdot (p(x))^{-1} = e_Y,$$

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which shows that $h(x) \in Y \times \ker(p)$.

For each $(y,a) \in Y \times \ker(p)$ we have

$$h(a \cdot q(y)) = (p(a \cdot q(y)), (a \cdot q(y)) \cdot (q(p(a \cdot q(y)))^{-1}) =$$
$$(p(a) \cdot p(q(y)), a \cdot q(y) \cdot (q(p(a) \cdot p(q(y)))^{-1}) = \left(e_Y \cdot y, a \cdot q(y) \cdot (q(e_Y \cdot y))^{-1}\right) =$$
$$(y,a).$$

Also for each $x \in X$ we have

$$t(h(x)) = t\left(p(x), x \cdot (q(p(x))^{-1}\right) = x \cdot (q(p(x))^{-1} \cdot q(p(x)) = x \cdot e_X = x.$$

In other words $ht = \mathrm{id}_{Y \times \ker(p)}$ and $th = \mathrm{id}_X$. Thus h is a homeomorphism. □

Lemma 0.3. *Let $\tau \geq \omega$. Every ANR-group of weight $\tau \geq \omega$ is topologically and algebraically isomorphic to a closed and C-embedded subgroup of the product $\prod\{G_t \colon t \in T\}$, where $|T| = \tau$ and each G_t, $t \in T$, is a Polish AR-group homeomorphic to $\mathbb{R}^\omega$.*

Proof. This follows from [1, Propositions 6.1.4, 6.5] and [3, Proposition 4.1]. □

Now we prove our main result.

Proof of Theorem 0.1. First we construct a well-ordered inverse spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^{\alpha+1}, \tau\}$ satisfying the following properties:

(1) G is topologically and algebraically isomorphic to $\lim \mathcal{S}_G$.
(2) for each $\alpha < \tau$, G_α is a ANR-group and $p_\alpha^{\alpha+1} \colon G_{\alpha+1} \to G_\alpha$ is a soft homomorphism such that $\ker\,(p_\alpha^{\alpha+1})$ is a non-compact Polish AR-group.
(3) If $\beta < \tau$ is a limit ordinal, then the diagonal product

$$\triangle\{p_\alpha^\beta \colon \alpha < \beta\} \colon G_\beta \to \lim\{G_\alpha, p_\alpha^{\alpha+1}, \alpha < \beta\}$$

is a topological and algebraic isomorphism.
(4) G_0 is a Polish ANR-group.

By Lemma 0.3, we may assume that G is a closed and C-embedded subgroup of the product $X = \prod\{X_a \colon a \in A\}$, $|A| = \tau$, of Polish AR-groups X_a, $a \in A$. Since G is an ANR-space, there exist a functionally open subspace L of the product X and a retraction $r \colon L \to G$. Choose a countable subset $A_L \subseteq A$ and an open subset $L_G \subseteq \prod\{X_a \colon a \in A_L\}$ such that $L = L_G \times \prod\{X_a \colon a \in A \setminus A_L\}$.

Next let us denote by

$$\pi_B \colon \prod\{X_a \colon a \in A\} \to \prod\{X_a \colon a \in B\}$$

and

$$\pi^B_C\colon \prod\{X_a\colon a\in B\}\to\prod\{X_a\colon a\in C\}$$

the natural projections onto the corresponding subproducts ($C\subseteq B\subseteq A$). Let also G_B denote the subspace $\pi_B(G)$ of the product $X_B=\prod\{X_a\colon a\in B\}$ and $L_B=L_G\times\prod\{X_a\colon a\in B\setminus A_L\}$ for each $B\subseteq A$ with $A_L\subseteq B$.

We call a subset $B\subseteq A$ admissible if $A_L\subseteq B$ and the following equality

$$\pi_B(r(x))=\pi_B(x)$$

is true for each point $x\in\pi_B^{-1}(G_B)$. We need the following properties of admissible sets.

Claim 1. The union of arbitrary collection of admissible sets is admissible.

Indeed let $\{B_t\colon t\in T\}$ be a collection of admissible sets and $B=\cup\{B_t\colon t\in T\}$. Let $x\in\pi_B^{-1}(G_B)$. Clearly $x\in\pi_{B_t}^{-1}(G_{B_t})$ and we have

$$\pi_{B_t}(r(x))=\pi_{B_t}(x)\ \text{ for each }\ t\in T.$$

Since $B=\cup\{B_t\colon t\in T\}$ it follows that $\pi_B(x)=\pi_B(r(x))$.

Claim 2. If B is an admissible subset of A, then the restriction $\pi_B|G\colon G\to G_B$ is a soft map (in the sense of [1, Definition 6.1.12]*). If C and B are admissible subsets with $C\subseteq B$, then $\pi^B_C|G_B\colon G_B\to G_C$ is also soft.*

Since the restriction of the projection π_B onto $\pi_B^{-1}(G_B)$ is a trivial fibration (with the fiber $\prod\{X_a\colon a\in A\setminus B\}$) whose fiber is an AR-space (recall that each X_a is an absolute retract and consequently so are their products), it follows that $\pi_B|\pi_B^{-1}(G_B)$ is soft. The admissibility of B implies that $\pi_Br|\pi_B^{-1}(G_B)=\pi_B|\pi^{-1}(G_B)$. Since r is a retraction it follows that $\pi_B|G$ is also soft (as a retract of $\pi_B|\pi_B^{-1}(G_B)$). The second part of the claim follows from [1, Lemma 6.1.15].

Claim 3. Each countable subset of A is contained in a countable admissible subset of A.

The inverse spectrum $\mathcal{S}(G)=\{\mathrm{cl}_{L_B}(G_B),\pi^B_C|\,\mathrm{cl}_{L_B}(G_B),C,B\in\exp_\omega(A,A_L)\}$ is a factorizing (since G is C-embedded in X) ω-spectrum in the sense of [1, Section 1.3.2] and $\lim\mathcal{S}(G)=G$. Similarly $\mathcal{S}(L)=\{L_B,\pi^B_C|L_B,C,B\in\exp_\omega(A,A_L)\}$ is also a factorizing ω-spectrum and $\lim\mathcal{S}(L)=L$. Applying [1, Theorem 1.3.6] to the map $r\colon L\to G$ we conclude that the collection $\exp_\omega(A,A_L)$ contains a cofinal (and even ω-closed) subcollection $\mathcal{A}_r$ such that for each $B\in\mathcal{A}_r$ there exists a map $r_B\colon L_B\to\mathrm{cl}_{L_B}(G_B)$ such that $\pi_Br=r_B\pi_B|L$. Note that for every such B, G_B is closed in L_B and $r_B\colon L_B\to G_B$ is a retraction. If $x\in\pi_B^{-1}(G_B)$ and $B\in\mathcal{A}_r$, then

$$\pi_B(x)=r_B(\pi_B(x))=\pi_B(r(x))$$

which shows that every $B\in\mathcal{A}_r$ is admissible.

Since $|A|=\tau$ and $|A_L|=\omega$ we can write $A\setminus A_L=\{a_\alpha\colon\alpha<\tau\}$. By Claim 3, each $a_\alpha\in A$ is contained in a countable admissible subset $B_\alpha\subseteq A$.

Let $A_\alpha = \cup\{B_\beta \colon \beta \leq \alpha\}$. We use these sets to define a transfinite inverse spectrum $\mathcal{S} = \{G_\alpha, p_\alpha^{\alpha+1}, \tau\}$ as follows.

Let A_0 be a countable admissible subset of A containing $A_L \cup \{a_0\}$ and $G_0 = G_{A_0}$.

Assume that the admissible subsets $A_\beta \subseteq A$ have already been constructed for each $\beta < \alpha$, $\alpha < \tau$, so that $a_\beta \in A_\beta$ for $\beta < \alpha$, $A_\gamma \subseteq A_\beta$ for $\gamma < \beta$ and the spaces $G_\beta = G_{A_\beta}$ and the homomorphisms $p_\gamma^\beta = \pi_{A_\gamma}^{A_\beta}|G_\beta \colon G_\beta \to G_\gamma$ satisfy the required properties (2) and (3).

If α is a limit ordinal, then let $A_\alpha = \cup\{A_\beta \colon \beta < \alpha\}$ and $G_\alpha = G_{A_\alpha}$.

If $\alpha = \beta + 1$ then proceed as follows. Consider the fiber $\ker(\pi_\beta|G)$ of the homomorphism $\pi_\beta|G \colon G \to G_\beta$. This fiber is an AR-space (as an inverse image of a point under a soft map; see Claim 2). Since it is also a topological group we conclude that $\ker(\pi_\beta|G)$ is non-compact (much more general statement is true – every compact AE(4)-group is trivial [2]) – otherwise it must be a singleton in which case the homomorphism $\pi_\beta \colon G \to G_\beta$ must be a homeomorphism. But this is impossible since $w(G) > w(G_\beta)$. Consequently there exists a countable subset $C \subseteq A$ such that $\pi_C(\ker(\pi_\beta|G))$ is also non-compact. By claim 3, there exists a countable admissible subset B such that $C \cup \{a_\alpha\} \subset B$. We now let $A_\alpha = A_\beta \cup B$, $G_\alpha = G_{A_\alpha}$ and $p_\beta^{\beta+1} = \pi_{A_\beta}^{A_\alpha}|G_{A_\alpha}$. Note that $\ker\left(p_\beta^{\beta+1}\right)$ is a non-compact Polish AR-group.

This completes the inductive step and the construction of the spectrum $\mathcal{S}_G$. Conditions (2) and (3) are satisfied by construction. Condition (1) can be insured by choosing the sets A_α so that $A = \cup\{A_\alpha \colon \alpha < \tau\}$ which in turn is possible since the collection of all countable admissible sets in cofinal in $\exp_\omega(A, A_L)$ (see the inductive step above).

The straightforward transfinite induction coupled with Lemma 0.2 shows that G is homeomorphic to the product $G_0 \times \prod\{\ker(p_\alpha^{\alpha+1}) : \alpha < \tau\}$. Since $p_\alpha^{\alpha+1}$ is a non-compact Polish AR-space for each $\alpha < \tau$, it follows (by [5, Theorem 5.1]) that all countable (but infinite) subproducts of the product $\prod\{\ker(p_\alpha^{\alpha+1}) : \alpha < \tau\}$ are homeomorphic to $\mathbb{R}^\omega$. Thus, $\prod\{\ker(p_\alpha^{\alpha+1}) : \alpha < \tau\} \approx \mathbb{R}^\tau$ and $G \approx G_0 \times \mathbb{R}^\tau$. Since G_0 is a Polish ANR-space, it follows from [1, Corollary 7.3.4] that G is an $\mathbb{R}^\tau$-manifold.

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