EXTENSION DIMENSION AND C-SPACES

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Abstract. Some generalizations of the classical Hurewicz formula are obtained for extension dimension and C-spaces. A characterization of the class of metrizable spaces which are absolute neighborhood extensors for all metrizable C-spaces is also given.

1. Introduction

The dimension lowering Hurewicz theorem states that if $f\colon X \to Y$ is a closed map, then $\dim X \leq \dim f + \dim Y$, where $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$ (it was first proved by Hurewicz [18] for metric compacta and later extended [24] for paracompact spaces; see also [20], [21]). In the present paper we prove a version of Hurewicz's theorem for extension dimension e-dim (precise definition of this concept is given in Section 2). The "dimesional scale" corresponding to the extension dimension is much finer than the usual integer-valued one. Roughly speaking extension dimension of a space is (determined by) a complex. For instance, the inequality $\dim X \leq n$ is equivalent to e-dim$X \leq \mathbb{S}^n$ and the inequality $\dim_G X \leq n$ is equivalent to e-dim$X \leq K(G,n)$ ($K(G,n)$ denotes the corresponding Eilenberg-MacLain complex). Extension dimension allows us to detect new properties of spaces generated by the new scale. Moreover a variety of known facts can now be viewed from a more general point of view.

One of the first such generalizations of the classical Hurewicz inequality was obtained in [13]: If $f\colon X \to Y$ is a light map (i.e. $\dim f = 0$) between compact spaces, then e-dim$X \leq$ e-dim$Y$. This observation, combined with a result of Pasynkov [22], yields another generalization of the Hurewicz formula: If $\dim f \leq n$ and X, Y are finite-dimensional metric compacta, then e-dim$X \leq$ e-dim$(Y \times \mathbb{I}^n)$. The most general extension of the Hurewicz formula was obtained recently in [12]: If e-dim$(Y \times f^{-1}(y)) \leq K$ for every $y \in Y$, then e-dim$X \leq K$ provided X and Y are finite-dimensional metric compacta with Y being dimensionally full-valued and K being a countable CW-complex.

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It is clear now that the Hurewicz formula can be generalized in several possible directions. One of them is to replace the inequality $\dim f \leq n$ by e-dim$f^{-1}(y) \leq K$ for every $y \in Y$ and leave Y to be finite-dimensional, say $\dim Y = m$. Note that the inequality $\dim X \leq n + m$ from the Hurewicz formula is equivalent to e-dim$X \leq \mathbb{S}^{n+m}$ and $\mathbb{S}^{n+m} = \Sigma^m \mathbb{S}^n$, where $\Sigma^m \mathbb{S}^n$ denotes the m-iterated suspension of $\mathbb{S}^n$. Consequently $\dim X \leq n+m$ if and only if e-dim$X \leq \Sigma^m \mathbb{S}^n$. These observations "justify" our first results (for simplicity, they are not given in their most general forms; complete versions are recoreded below as Theorem 2.4 and Corollary 2.7) as "extensional analogues" of the Hurewicz formula.

Theorem. *Let $f\colon X \to Y$ be a closed surjection of metrizable spaces and* $\dim Y \leq m$. *If K is a CW-complex such that* e-dim$(\mathbb{I}^m \times f^{-1}(y)) \leq K$ *for any $y \in Y$, then* e-dim$X \leq K$.

Corollary. *Let $f\colon X \to Y$ and the spaces X, Y be as in the above Theorem. Then* e-dim$X \leq \Sigma^m K$ *provided* e-dim$f^{-1}(y) \leq K$ *for any $y \in Y$.*

The second part of this paper deals with C-spaces [1] (see also [16]), predominantly with the class $\mathcal{C}$ of all metrizable C-spaces. It is well known that $\mathcal{C}$ contains (strongly) countable-dimensional metrizable spaces, i.e. metrizable spaces which are countable union of (closed) finite-dimensional subsets, but there exists a metric C-compactum which is not countable-dimensional [23]. Hurewicz type theorem is known [17] to be true for paracompact C-spaces (i.e. if $f\colon X \to Y$ is a closed surjection between paracompact spaces and if Y and all fibers $f^{-1}(y)$, $y \in Y$, are C-spaces, then X also is a C-space). Extensional properties of X in such a situation are discussed in Theorem 3.2. In particular, we conclude (Corollary 3.3) that absolute extensors for the class $\mathcal{C}$, denoted by $AE(\mathcal{C})$, are precisely aspherical absolute neighbourhood extensors for the same class ($ANE(\mathcal{C})$). Moreover in Theorem 3.6 we present description of $ANE(\mathcal{C})$-spaces and provide an answer to a corresponding question of F. Ancel [3, Question 5.13(c)]. Another implication of Theorem 3.6 is that any subclass of $\mathcal{C}$ which contains strongly countable-dimensional spaces, has the same absolute (neighborhood) extensors as the class $\mathcal{C}$. In particular, if $\mathcal{M}$ is such a proper subclass of $\mathcal{C}$, then $\mathcal{M}$ can not be distinguished by existence of a metric space K such that e-dim$X \leq K$ if and only if $X \in \mathcal{M}$ (J. Dijkstra [8] arrived to the same observation for the classes $\mathcal{M}_\alpha$ of all metrizable spaces with transfinite inductive dimension $\leq \alpha$, where α is an infinite ordinal).

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2. Generalized Hurewicz's theorems for extension dimension

All spaces considered in this paper are at least completely regular and all single-valued maps are continuous.

A space K is called an absolute (neighborhood) extensor[1] of X (notation: $K \in A(N)E(X)$) if every map $f\colon A \to K$, where defined on a closed subspace A of X, admits an extention over the whole X (respectively, over a neighborhood of A in X). Next let us introduce a relation $\leq$ for CW-complexes. Following [10] (see also [11], [4]), we say that $L \leq K$ if for each space X the condition $L \in AE(X)$ implies the condition $K \in AE(X)$. Equivalence classes of CW-complexes with respect to this relation are called extension types. The above defined relation creates a partial order in the collection of extension types of complexes. This partial order is still denoted by $\leq$ and the extension type $[K]$ of a complex K for simplicity is still denoted by K. Note that under these definitions the collection of extension types of all complexes has both maximal ans minimal elements. The minimal element is the extension type of the 0-dimensional spehere $\mathbb{S}^0$ (i.e. the two-point discrete space) and the maximal element is obviously the extension type of the one-point space (or equivalently, of any contractible CW-complex). Finally the extension dimension of a space X is the minimum of extension types of complexes K satisfying the relation $K \in AE(X)$: e-dim$X = \min\{[K]\colon K \in AE(X)\}$. For simplicity below we write e-dim$X \leq K$ instead of e-dim$X \leq [K]$.

The cone of a space X (notation: $Cone(X)$) is the quotient set $X \times [0,1]/(X \times \{1\})$ with the following topology: U is open in $Cone(X)$ iff $U \cap (X \times [0,1))$ is open in $X \times [0,1)$ with the product topology and, if the vertex v belongs to U, then $X \times (t,1) \subset U$ for some $0 < t < 1$. We need the following result of Dydak [14]: If K is a space with at least two points, then $K \in ANE(X)$ if and only if e-dim$X \leq Cone(K)$.

Lemma 2.1. *Let $H \subset X$ be a zero-set in X and* e-dim$X \leq K$. *Then every map $f\colon H \to Cone(K)\backslash\{v\}$ extends to a map from X into $Cone(K)\backslash\{v\}$.*

Proof. Let $\pi_1\colon Cone(K)\backslash\{v\} \to K$ and $\pi_2\colon Cone(K) \to [0,1]$ be the natural projections. Then $f = (f_1, f_2)$ with $f_i = \pi_i \circ f$, $i = 1,2$. Since e-dim$X \leq K$ implies e-dim$X \leq Cone(K)$ (by the Dydak result mentioned above), there exists a map $g\colon X \to Cone(K)$ extending f. Then $p = \pi_2 \circ g$ extends f_2. Fix a function $q\colon X \to [0,1]$ such that $H = q^{-1}(0)$ and define $s\colon X \to [0,1)$ by $s(x) = (1 - q(x))p(x)$. Since e-dim$X \leq K$, f_1 can be extended to a map $h\colon X \to K$. Then $\overline{f} = (h,s)\colon X \to Cone(K)\backslash\{v\}$ is the required extension of f. □

Everywhere below $C(X,M)$ denotes the space of all continuous maps from X into M equipped with the compact-open topology. A set-valued map $\phi\colon X \to 2^Y$ is called strongly lower semi-continuous (br., strongly lsc) if for any $x \in X$

[1] In these notes we follow the standard definition of the concept of absolute (neighbourhood) extensor. It should be noted however that in certain situations this definition is not satisfactory and requires a modification. Such an approach is developed in [4], [5], [6].

and a compact set $P \subset \phi(x)$ there exists a neighborhood U of x such that $P \subset \phi(z)$ for every $z \in U$. Here, 2^Y stands for the family of all nonempty subsets of Y. We also write X is C^n to denote that every continuous image of a k-sphere in X, $k \le n$, is contractible in X.

Proposition 2.2. *Suppose $f\colon X \to Y$ is a closed surjection such that X is a k-space, $K \in ANE(\mathbb{I}^m \times X)$ and* e-dim$(\mathbb{I}^m \times f^{-1}(y)) \le K$ *for any $y \in Y$. Let M be the cone of K with a vertex v and $h\colon A \to K$ a map with $A \subset X$ being a zero-set. Then the set-valued map $\phi\colon Y \to 2^{C(X,M)}$, $\phi(y) = \{g \in C(X,M) : g(f^{-1}(y)) \subset M\backslash\{v\}$ and $g(x) = h(x)$ for all $x \in A\}$ is strongly lsc and each $\phi(y)$ is C^{m-1}.*

Proof. Claim 1. *$\phi(y) \ne \emptyset$ for each $y \in Y$.*

Observe first that $K \in ANE(\mathbb{I}^m \times X)$ implies $M \in AE(\mathbb{I}^m \times X)$, in particular, $M \in AE(X)$. For fixed $y \in Y$ extend $h|(f^{-1}(y) \cap A)$ to a map $g_1\colon f^{-1}(y) \to K$ (such an extension exists because $f^{-1}(y)$ is a closed subset of $\mathbb{I}^m \times f^{-1}(y)$), so e-dim$f^{-1}(y) \le K$). Then g_1 and h define a map from $f^{-1}(y) \cup A$ into K which is extendable to a map $g\colon X \to M$. Obviously, $g(f^{-1}(y)) \subset K$ and $g|A = h|A$, so $g \in \phi(y)$.

Claim 2. *ϕ is strongly lsc.*

Let $y_0 \in Y$ and $P \subset \phi(y_0)$ be compact. We have to find a neighborhood V of y_0 in Y such that $P \subset \phi(y)$ for every $y \in V$. Let $P(x) = \{g(x) : g \in P\}$, $x \in X$. Since $P \subset C(X,M)$ is compact and X is a k-space, by the Ascoli theorem, each $P(x)$ is compact and P is evently continuous. This easily implies that the set $W = \{x \in X : P(x) \subset M\backslash\{v\}\}$ is open in X and, obviously, $f^{-1}(y_0) \subset W$. Because f is closed, there exists a neighborhood V of y_0 in Y with $f^{-1}(V) \subset W$. Then, according to the choice of W and the definition of ϕ, $P \subset \phi(y)$ for every $y \in V$.

Claim 3. *Each $\phi(y)$ is C^{m-1}.*

For a fixed $y \in Y$ take an arbitrary map $u\colon \mathbb{S}^{n-1} \to \phi(y)$, where $n \le m$. We are going to show that u can be extended continuously to a map from $\mathbb{I}^n$ into $\phi(y)$ (we identify $\mathbb{S}^{n-1}$ with the boundary of $\mathbb{I}^n$). Since $\mathbb{S}^{n-1} \times X$ is a k-space (as a product of a compact space and a k-space), the map $u_1\colon \mathbb{S}^{n-1} \times X \to M$, $u_1(z,x) = u(z)(x)$, is continuous (see [15]). Because $u_1(z,x) = h(x)$ for every $(z,x) \in \mathbb{S}^{n-1} \times A$, we can extend $u_1|(\mathbb{S}^{n-1} \times A)$ to a map $u_2\colon \mathbb{I}^n \times A \to K$, $u_2(z,x) = h(x)$. Then, we have a closed subset $H = (\mathbb{S}^{n-1} \times f^{-1}(y)) \cup (\mathbb{I}^n \times (f^{-1}(y) \cap A))$ of $\mathbb{I}^n \times f^{-1}(y)$ and a map $u_3\colon H \to M\backslash\{v\}$ defined by $u_3|(\mathbb{S}^{n-1} \times f^{-1}(y)) = u_1|(\mathbb{S}^{n-1} \times f^{-1}(y))$ and $u_3|(\mathbb{I}^n \times (f^{-1}(y) \cap A)) = u_2|(\mathbb{I}^n \times (f^{-1}(y) \cap A))$. Since $\mathbb{S}^{n-1}$ and $f^{-1}(y) \cap A$ are zero-sets in $\mathbb{I}^n$ and $f^{-1}(y)$, respectively, both $\mathbb{S}^{n-1} \times f^{-1}(y)$ and $\mathbb{I}^n \times (f^{-1}(y) \cap A)$ are zero-sets in $\mathbb{I}^n \times f^{-1}(y)$, so is H. Note that e-dim$(\mathbb{I}^n \times f^{-1}(y)) \le K$ because $\mathbb{I}^n \times f^{-1}(y)$ is closed in $\mathbb{I}^m \times f^{-1}(y)$. Therefore, by Lemma 2.1, u_3 extends to a map $u_4\colon \mathbb{I}^n \times f^{-1}(y) \to M\backslash\{v\}$. Now, let F be the union of the sets $F_1 = \mathbb{I}^n \times f^{-1}(y)$, $F_2 = \mathbb{I}^n \times A$ and $F_3 = \mathbb{S}^{n-1} \times X$. We define the map $p\colon F \to M$ by $p|F_1 = u_4$, $p|F_2 = u_2$ and $p|F_3 = u_1$. Obviously, F is closed in $\mathbb{I}^n \times X$. Since $M \in AE(\mathbb{I}^n \times X)$, there exists an extension $q\colon \mathbb{I}^n \times X \to M$ of p. To finish the proof of Claim 3, observe that q generates the map $\overline{u}\colon \mathbb{I}^n \to C(X,M)$, $\overline{u}(z)(x) = q(z,x)$. Moreover, $q(z,x) = h(x)$ for any $(z,x) \in \mathbb{I}^n \times A$ and $q(\mathbb{I}^n \times f^{-1}(y)) \subset M\backslash\{v\}$. So, $\overline{u}$ is a map from $\mathbb{I}^n$ to $\phi(y)$ which extends u. □

Now we need the following result of E. Michael [25, Remark 2].

Proposition 2.3. *Let X be paracompact with $\dim X \leq m$ and Y an arbitrary space. Then every strongly lsc mapping $\varphi\colon X \to 2^Y$ has a continuous selection provided $\varphi(x)$ is C^{m-1} for each $x \in X$.*

Theorem 2.4. *Let $f\colon X \to Y$ be a closed surjection with X a k-space and Y paracompact of dimension $\dim Y \leq m$. If K is any space such that $K \in ANE(\mathbb{I}^m \times X)$ and $\text{e-dim}(\mathbb{I}^m \times f^{-1}(y)) \leq K$ for any $y \in Y$, then $\text{e-dim}X \leq K$.*

Proof. Suppose $A \subset X$ is closed and $h\colon A \to K$ is a map. We are going to find a continuous extension $\overline{h}\colon X \to K$ of h. Let M be the cone of K with a vertex v. Since $M \in AE(X)$, there exists a map $q\colon X \to M$ extending h. Then $q^{-1}(K)$ is a zero-set in X (because K is such a set in M) containing A. Therefore, we can assume that A is a zero-set in X. Next, define the set-valued map $\phi\colon Y \to 2^{C(X,M)}$, $\phi(y) = \{g \in C(X,M) : g(f^{-1}(y)) \subset M\backslash\{v\}$ and $g(x) = h(x)$ for all $x \in A\}$ (a similar idea was earlier used by V.Gutev and V. Valov). By Proposition 2.2, $\phi\colon Y \to C(X,M)$ is a strongly lsc map with each $\phi(y)$ being a C^{m-1}-set. Since $\dim Y \leq m$, we can apply Proposition 2.3 to obtain a continuous selection $t\colon Y \to C(X,M)$ for ϕ. Then $g\colon X \to M$, defined by $g(x) = t(f(x))(x)$, is continuous on every compact subset of X and because X is a k-space, g is continuous. Since $t(f(x)) \in \phi(f(x))$, we have $g(x) = h(x)$ for all $x \in A$ and $g(x) \in M\backslash\{v\}$, $x \in X$. Finally, if $\pi_1\colon M\backslash\{v\} \to K$ denotes the natural retraction, then $\overline{h} = \pi \circ g\colon X \to K$ is the required continuous extension of h. □

A k-space X is called a *cw*-space [11] if every contractible CW-complex is an $AE(X)$. In particular, if X is a *cw*-space and K any CW-complex, then $Cone(K) \in AE(X)$. Any metrizable space, more generally, every space admitting a perfect map onto a first countable paracompact space, is *cw* [14].

Corollary 2.5. *Let $f\colon X \to Y$ be a closed surjection, where Y is paracompact with $\dim Y \leq m$ and $\mathbb{I}^m \times X$ is a cw-space. If K is a CW-complex such that $\text{e-dim}(\mathbb{I}^m \times f^{-1}(y)) \leq K$ for every $y \in Y$, then $\text{e-dim}X \leq K$.*

Proof. Since X is a k-space and $K \in ANE(\mathbb{I}^m \times X)$, we can apply Theorem 2.4. □

Lemma 2.6. *If* e-dim$X \leq K$*, where* $X \times \mathbb{I}$ *is a paracompact cw-space and* K *a CW-complex, then* e-dim$(X \times \mathbb{I}) \leq \Sigma K$*.*

Proof. This lemma was proved by Dranishnikov [9] for metric spaces X. His proof, coupled with [11, Propositions 1.17-1.18], works in our situation as well. □

Corollary 2.7. *Let* $X \times \mathbb{I}^m$ *be a paracompact cw-space,* K *be a CW-complex and* $f\colon X \to Y$ *be a closed surjection with* $\dim Y \leq m$*. If* e-dim$f^{-1}(y) \leq K$ *for every* $y \in Y$*, then* e-dim$X \leq \Sigma^m K$*.*

Proof. Observe first that Y is paracompact as a closed image of the paracompact X. By Lemma 2.6, e-dim$(\mathbb{I}^m \times f^{-1}(y)) \leq \Sigma^m K$ for any $y \in Y$. Then the proof follows from Corollary 2.5 with K replaced by $\Sigma^m K$. □

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3. C-spaces

Recall that X is a C-space [1] if for any sequence $\{\omega_n\}$ of open covers of X there exists a sequence $\{\gamma_n\}$ of open disjoint families in X such that each γ_n refines ω_n and $\bigcup\{\gamma_n : n \in \mathbb{N}\}$ covers X. Property C is a dimensional type property, and it admits a characterization similar to that one (see Proposition 2.3) of finite-dimensional spaces (everywhere below a space is said to be aspherical if it is C^n for all n).

Proposition 3.1. [20] *A paracompact* X *is a* C*-space if and only if every strongly lsc map* $\phi\colon X \to 2^Y$ *with aspherical images* $\phi(x)$*,* $x \in X$*, where* Y *is an arbitrary space, has a continuous selection.*

Theorem 3.2. *Let* $f\colon X \to Y$ *be a closed surjection with* X *a* k*-space and* Y *a paracompact* C*-space. If* K *is a space satisfying both conditions* $K \in ANE(\mathbb{I}^m \times X)$ *and* $K \in AE(\mathbb{I}^m \times f^{-1}(y))$ *for any* $m \in \mathbb{N}$ *and any* $y \in Y$*, then* $K \in AE(X)$*.*

Proof. We follow the proof of Theorem 2.4. Maintaining the same notations and applying now Proposition 3.1 (instead of Proposition 2.3), it suffices to show that if A is a zero-set in X, then the formula $\phi(y) = \{g \in C(X, M) : g(f^{-1}(y)) \subset M\backslash\{v\}$ and $g(x) = h(x)$ for all $x \in A\}$ defines a set-valued map $\phi\colon Y \to 2^{C(X,M)}$ which is strongly lsc and each $\phi(y)$ is aspherical. And this follows from Proposition 2.2. □

Theorem 3.2 is not of any interest when K is a CW-complex. Indeed, $K \in AE(\mathbb{I}^m \times f^{-1}(y))$ for all m implies that every homotopy group of K is trivial. So, K is contractible and therefore it is an absolute extensor for any cw-space. On the other hand, the Borsuk example of a contractible and locally contractible compact metric space which is not an AE for the class of all metrizable spaces shows that Theorem 3.2 has a meaning for general spaces K.

Let $\mathcal{C}$ denote the class of all metrizable C-spaces. We write $K \in A(N)E(\mathcal{C})$ if $K \in A(N)E(X)$ for any $X \in \mathcal{C}$; when the class of all metrizable spaces is considered, we simply write $K \in A(N)E$.

Corollary 3.3. *A space $K \in ANE(\mathcal{C})$ is an $AE(\mathcal{C})$ if and only if K is aspherical.*

Proof. Any $AE(\mathcal{C})$ is aspherical (because the class $\mathcal{C}$ contains all finite-dimensional spaces) and an $ANE(\mathcal{C})$. Suppose $K \in ANE(\mathcal{C})$ is aspherical and $X \in \mathcal{C}$. We are going to apply Theorem 3.2 in the special case when $X = Y$ and f being the identity map. In this special case Proposition 2.2 is true if $\text{e-dim}(\mathbb{I}^m \times f^{-1}(y)) \leq K$ is replaced by $K \in C^{m-1}$. Indeed, Claim 1 becomes trivial; to prove Claim 2 we don't need to apply Lemma 2.1 because the set H is homeomorphic either to $\mathbb{I}^n$ if $y \in A$ or $\mathbb{S}^{n-1}$ otherwise, we need that any map from $\mathbb{S}^{n-1}$ into K is extendable to map from $\mathbb{I}^n$ into K, $n \leq m$. In order to apply Theorem 3.2, it remains only to check that $K \in ANE(X \times \mathbb{I}^m)$ for all m. And that is true because $X \times \mathbb{I}^m \in \mathcal{C}$ [17]. $\square$

Let discuss now some sufficient (and necessary) conditions for a metric space to be an $ANE(\mathcal{C})$. Let $\mathcal{P}$ be a topological property. We say that $X \subset E$ is a $UV(\mathcal{P})$ subset of E if each neighborhood U of X in E contains a neighborhood V of X in E such that any map $h\colon Z \to V$, where $Z \in \mathcal{P}$, extends to a map $\overline{h}\colon Cone(Z) \to U$. A closed surjection $f\colon X \to Y$ is called $UV(\mathcal{P})$ if each of its point inverses is a $UV(\mathcal{P})$ subset of X. Recall that if, in the above definition, V is contractible in U, then X is called UV^∞; a cell-like space is a compact metric space X such that X is a UV^∞ set in every ANE-space E in which it is embedded as a closed subset (see, for example, [3]). In the existing terminology, a UV^∞ (resp., cell-like) map is a perfect map with UV^∞ (cell-like) preimages. Obviously, every UV^∞ map is $UV(\mathcal{P})$ for any property $\mathcal{P}$.

Proposition 3.4. *Any one of the following two conditions is sufficient for a metrizable space Y to be an $ANE(\mathcal{C})$:*

(a) *Y is locally contractible, more generally, there exists a metrizable space X and a UV^∞ map from X onto Y.*

(b) *Y has a base of open aspherical sets.*

Proof. First condition was proved by Ancel [3, Theorem C.5.9], see also [1] for the case of local contractibility. Condition (b) can be obtained by using the arguments of Ageev and Repovš [2, proof of Theorem 1.3]. $\square$

Not every metrizable $ANE(\mathcal{C})$-space is locally contractible. J. van Mill provided an example of a cell-like image of the Hilbert cube such that no nonempty open subset is contractible in that space [19]. At the same time, by [3], this example is an $ANE(\mathcal{C})$. In view of mentioned above result of Ancel [3, Theorem C.5.9], it is interesting whether any metrizable $ANE(\mathcal{C})$ is a UV^∞ image of a metrizable space. In such a case, the class of metrizable $ANE(\mathcal{C})$ would be precisely the class of all UV^{∞} images of metrizable spaces. We can provide similar characterization $ANE(\mathcal{C})$ in terms of $UV(s.c.d.)$ maps, where s.c.d. denotes the property strong countable-dimensionality.

Proposition 3.5. *Let $f\colon M \to X$ be a surjective map between metrizable spaces. If for any $x \in X$ and its neighborhood $U(x)$ in X there exists another neighborhood $V(x)$ of x in X such that $\overline{V}(x) = f^{-1}(V(x))$ is contractible in $\overline{U}(x) = f^{-1}(U(x))$, then $X \in ANE(\mathcal{C})$.*

Proof. First step is to show that X is an approximate absolute neighborhood extensor for the class $\mathcal{C}$, i.e. if H is a metrizable C-space, $A \subset H$ closed and $h\colon A \to X$ a map, then for every open cover γ of X there is a neighborhood W_A of A in H and a map $\overline{h}\colon W_A \to X$ such that $\overline{h}|A$ is γ-close to h. We follow the construction from the proof of [2, Teorem 4.3, first part]. For every $x \in X$ and $n \geq 0$ fix points $z(x) \in f^{-1}(x)$ and neighborhoods $V_n(x) \subset U_n(x)$ of x in X such that:

(1) $\overline{V}_n(x)$ contracts in $\overline{U}_n(x)$ to $z(x)$ for all $n \geq 0$ and $x \in X$;
(2) the cover $\alpha_0 = \{U_0(x) : x \in X\}$ refines γ;
(3) the cover $\alpha_n = \{U_n(x) : x \in X\}$ star-refines $\beta_{n-1} = \{V_{n-1}(x) : x \in X\}$ for any $n \geq 1$, i.e. $\{St(U, \alpha_n) : U \in \alpha_n\}$ refines β_{n-1}.

Observe that we have corresponding covers $\overline{\gamma} = f^{-1}(\gamma)$, $\overline{\alpha}_n = \{\overline{U}_n(x) : x \in X\}$ and $\overline{\beta}_n = \{\overline{V}_n(x) : x \in X\}$ of M such that $\overline{\alpha}_0$ refines $\overline{\gamma}$ and $\overline{\alpha}_n$ star-refines $\overline{\beta}_{n-1}$, $n \geq 1$. For every $n \geq 0$ and $x \in X$ we fix a contraction map $F^{x,n}\colon \overline{V}_n(x) \times [0,1] \to \overline{U}_n(x)$ with $F^{x,n}(z,1) = z(x)$. Since A is a C-space (as a closed subset of H), there is a sequence of disjoint open families $\{\mu_n : n = 1,2,..\}$ in H such that the restriction of each μ_n on A refines $h^{-1}(\beta_n)$ and $\mu = \bigcup\{\mu_n : n = 1,2..\}$ covers A. Further, let $\mathcal{K}$ be the nerve of μ and $\theta\colon W_A = \cup\{W : W \in \mu\} \to |\mathcal{K}|$ a barycentric map. We are going to define a map $g\colon |\mathcal{K}| \to M$ such that the family $\{g(\theta(y)) \cup f^{-1}(h(y)) : y \in A\}$ refines $\overline{\gamma}$. Then the map $\overline{h} = f \circ g \circ \theta$ will be the required γ-approximation of h. Any simplex $(W_0, W_1, .., W_k)$ from $\mathcal{K}$, where $W_i \in \mu_{n(i)}$, can be ordered such that $n(0) < n(1) < ..., n(k)$ (this is possible because $\cap\{W_i : i = 1,2,..,k\} \neq \emptyset$, so the numbers $n(i)$ are different). By (3), for any $W \in \mu_n$ there exists $x(W) \in X$ with $St(h(W \cap A), \alpha_n) \subset V_{n-1}(x(W))$. We define $g_0\colon |\mathcal{K}^0| \to M$ by $g_0(W) = z(x(W))$, $W \in \mu$. Using the contractions $F^{x,n}$, as in [2, proof of Theorem 4.3], we can define by induction maps $g_n\colon |\mathcal{K}^n| \to M$ such that the restriction of g_n on $|\mathcal{K}^i|$ is g_i, $i \leq n$, and for any simplex $\triangle^n = (W_0, W_1, .., W_n) \in |\mathcal{K}^n|$ we have

(4) $f^{-1}(h(W_0 \cap A)) \cup g_n(\triangle^n) \subset \overline{U}_{n_0-1}(x(W_0))$.

So, we obtain a map $g\colon |\mathcal{K}| \to M$ and, by (4), $\overline{h}|A$ and h are γ-close, where $\overline{h} = f \circ g \circ \theta$. Indeed, if $y \in A$ and $\theta(y) \in \triangle^n$ for some simplex $\triangle^n = (W_0, W_1, .., W_n)$, then $\overline{h}(y) \in f(g_n(\triangle^n))$ and $h(y) \in h(W_0 \cap A)$. According to (4), the last two inclusions imply that both $\overline{h}(y)$ and $h(y)$ belong to $U_{n_0-1}(x(W_0))$. So, $\overline{h}(y)$ and $h(y)$ are α_{n_0-1}-close and, since $n_0 - 1 \geq 0$, they are also γ-close. Therefore, X is an approximate absolute neighborhood extensor for the class $\mathcal{C}$.

To complete the proof we state the following result which was actually proved in [2] but not explicitely formulated: If $\mathcal{M}$ is a class of metrizable spaces such that $Y \times [0,1) \in \mathcal{M}$ for every $Y \in \mathcal{M}$, then any approximate absolute neighborhood extensor for $\mathcal{M}$ is an $ANE(\mathcal{M})$. Since $\mathcal{C}$ is closed with respect to multiplication by $[0,1)$, we have $X \in ANE(\mathcal{C})$. $\square$

Theorem 3.6. *For a metrizable space X the following conditions are equivalent:*

(a) *X is an ANE for the class of metrizable (strongly) countable-dimensional spaces.*
(b) *X is a $UV(s.c.d.)$ image of a metrizable space.*
(c) *X is an $ANE(\mathcal{C})$.*

Proof. Since every metrizable (strongly) countable-dimensional space has property C, (c) implies (a). Standard arguments show that every metrizable X which is an ANE for the class of metrizable (strongly) countable-dimensional spaces has the following property $(*)$:

For every $x \in X$ and its neighborhood $U(x)$ in X there is a neighborhood $V(x) \subset U(x)$ such that any map from a closed subset of a (strongly) countable-dimensional metrizable space Z into $V(x)$ extends to a map from Z into $U(x)$.

Hence, (a) yields that the identity map of X is $UV(s.c.d)$. So, it remains to prove (b)$\Rightarrow$(c). Let $f\colon Y \to X$ be a $UV(s.c.d.)$ map with Y metrizable. We need the following result of M. Zarichnyi [26]: There exists an ω-soft map from a σ-compact strongly countable-dimensional metrizable space onto the Hilbert cube. Here, a map $g\colon M \to H$ is called ω-soft if for every strongly countable-dimensional metrizable space Z, its closed subset $B \subset Z$ and any two maps $\phi\colon Z \to H$, $\psi\colon B \to M$ such that $g \circ \psi = \phi|B$ there exists a map $\Phi\colon Z \to M$ extending ψ with $g \circ \Phi = \phi$. Using the Zarichnyi result, for every cardinal τ we can construct a strongly countable-dimensional metrizable space $M(\tau)$ of weight τ and an ω-soft map $g\colon M(\tau) \to l_2(\tau)$ (see [7] for a similar reduction), where $l_2(\tau)$ denotes the Hilbert space of weight τ. Embedding Y into $l_2(\tau)$ for some τ and considering the restriction g_Y of g onto $M_Y = g^{-1}(Y)$, we obtain a strongly countable-dimensional metrizable space M_Y and an ω-soft map $g_Y\colon M_Y \to Y$. Let $q = f \circ g_Y$. We are going to show that $q\colon M_Y \to X$ satisfies the hypotheses of Proposition 3.5. To this end, let $U(x)$ be a neighborhood of $x \in X$. Since f is $UV(s.c.d.)$, there exists a neighborhood $W(x) \subset f^{-1}(U(x))$ such that every map from a strongly countable-dimensional metrizable space Z into $W(x)$ extends to a map from $Cone(Z)$ into $f^{-1}(U(x))$. Then $f^{-1}(V(x)) \subset W(x)$ for some neighborhood $V(x)$ of x in X because f is closed. Now consider $\overline{V}(x) = q^{-1}(V(x))$

and $\overline{U}(x) = q^{-1}(U(x))$. Since $\overline{V}(x)$ is strongly countable-dimensional, there exists a map $\phi\colon Cone(\overline{V}(x)) \to f^{-1}(U(x))$ extending the restriction $g_Y|\overline{V}(x)$. Finally, using that g_Y ω-soft, we can lift ϕ to a map $\Phi\colon Cone(\overline{V}(x)) \to \overline{U}(x)$ such that $\Phi|\overline{V}(x)$ is the identity. Therefore, $\overline{V}(x)$ is contractible in $\overline{U}(x)$ and, by Proposition 3.5, $X \in ANE(\mathcal{C})$. □

The equivalence of conditions (a) and (c) from Theorem 3.6, yields the following observation: if $\mathcal{M}$ is a subclass of $\mathcal{C}$ containing all strongly countable-dimensional spaces, then $ANE(\mathcal{M})$ coincides with $ANE(\mathcal{C})$ in the realm of metrizable spaces. Consequently, since every $K \in AE(\mathcal{M})$ is aspherical, the above observation combined with Corollary 3.3 implies also that $\mathcal{M}$ and $\mathcal{C}$ have the same metrizable AE-spaces. Finally, we would like to point out that Theorem 3.6 provides an answer to the question [3, Question 5.13(c)] asking whether a metrizable space X is an ANE for the class of countable-dimensional spaces if X has the property $(*)$ mentioned in the proof of Theorem 3.6.

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