COMPLEMENTED SUBSPACES OF PRODUCTS OF BANACH SPACES

ALEX CHIGOGIDZE

Abstract. We show that complemented subspaces of uncountable products of Banach spaces are products of complemented subspaces of countable subproducts.

1. Introduction

The following old unsolved problem (L. Nachbin [10]) of describing injective locally convex spaces is one of the general problems of the structure theory of locally convex spaces.

Problem 1. Is every injective locally convex space isomorphic to a product of Banach injective spaces?

In investigations related to this problem (see, for instance, [4], [2], [3], [5]) the following problem ([3, p. 71], [7, p. 147]) arose.

Problem 2. Is every complemented subspace of a product of a (countable) family of Banach spaces isomorphic to a product of Banach spaces?

G. Metafune and V. B. Moscatelli [6, p. 251] conjectured that this is false in general. Later this conjecture has been confirmed by M. Ostrovskii [11] who showed that not all complemented subspaces of countable products of Banach spaces are isomorphic to products of Banach spaces.

Our main result shows that for uncountable products situation is somewhat different.

Theorem. *A complemented subspace of an uncountable product of Banach spaces is a product of complemented subspaces of countable subproducts.*

The following immediate corollary of this result provides a partial solution to Problem 1.

Corollary. *Every injective locally convex space is isomorphic to a product of injective Fréchet spaces.*

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2. Results

The following statement expresses a key fact used in the proof of Theorem 2.2.

Proposition 2.1. *Let $r\colon \prod\{B_t\colon t \in T\} \to \prod\{B_t\colon t \in T\}$ be a continuous linear map of an uncountable product of Banach spaces into itself. Let also A be a countable subset of T. Then there exist a countable subset S of T and a continuous linear map $r_S\colon \prod\{B_t\colon t \in S\} \to \prod\{B_t\colon t \in S\}$ such that $A \subseteq S$ and $\pi_S \circ r = r_S \circ \pi_S$, where $\pi_S\colon \prod\{B_t\colon t \in T\} \to \prod\{B_t\colon t \in S\}$ denotes the projection onto the corresponding subproduct.*

Proof. Let $\exp_\omega T$ denote the set of all countable subsets of the indexing set T. Consider the following relation

$$\mathcal{L} = \{(S,R) \in (\exp_\omega T)^2 : S \subseteq R \text{ and there exists a continuous linear map}$$
$$r_S^R\colon \prod\{B_t\colon t \in R\} \to \prod\{B_t\colon t \in S\} \text{ such that } \pi_S \circ r = r_S^R \circ \pi_R\},$$

where

$$\pi_S\colon \prod\{B_t\colon t \in T\} \to \prod\{B_t\colon t \in S\}$$

and

$$\pi_S^R\colon \prod\{B_t\colon t \in R\} \to \prod\{B_t\colon t \in S\}$$

denote canonical projections onto the corresponding subproducts.

We need to verify the following three properties of the above defined relation.

Existence. If $S \in \exp_\omega T$, then there exists $R \in \exp_\omega T$ such that $(S,R) \in \mathcal{L}$.

Proof. Let $S = \{t_n\colon n \in \omega\}$. For each $n \in \omega$ consider the composition $\pi_{t_n} \circ r\colon \prod\{B_t\colon t \in T\} \to B_{t_n}$. Since B_{t_n} is a Banach space, it follows that every continuous linear map into B_{t_n}, defined on an infinite product of Banach space, can be factored through a finite subproduct (this is a well known fact; see, for instance, [8, Proposition 0.1.9]). Consequently there exist a finite subset R_n and a continuous linear map $r_{t_n}^{R_n}\colon \prod\{B_t\colon t \in R_n\} \to B_{t_n}$ such that such that $\pi_{t_n} \circ r = r_{t_n}^{R_n} \circ \pi_{R_n}$ for each $n \in \omega$. Without loss of generality we may assume that $t_n \in R_n$ for each $n \in \omega$ (otherwise consider the set $R_n \cup \{t_n\}$). Let $R = \cup\{R_n\colon n \in \omega\}$ and $r_{t_n}^R = r_{t_n}^{R_n} \circ \pi_{R_n}^R$, $n \in \omega$. Clearly

$$\pi_{t_n} \circ r = r_{t_n}^{R_n} \circ \pi_{R_n} = r_{t_n}^{R_n} \circ \pi_{R_n}^R \circ \pi_R = r_{tn}^R \circ \pi_R,\ n \in \omega.$$

Next consider the diagonal product

$$r_S^R = \triangle\{r_{t_n}^R \colon n \in \omega\} \colon \prod\{B_t \colon t \in R\} \to \prod\{B_{t_n} \colon n \in \omega\} = \prod\{B_t \colon t \in S\}$$

and note that r_S^R is a continuous linear map which satisfies the equality $\pi_S \circ r = r_S^R \circ \pi_R$. This shows that $(S, R) \in \mathcal{L}$.

Majorantness. If $(S, R) \in \mathcal{L}$, $P \in \exp_\omega T$ and $R \subseteq P$, then $(S, P) \in \mathcal{L}$.

Proof. This is trivial. Indeed let $r_S^R \colon \prod\{B_t \colon t \in R\} \to \prod\{B_t \colon t \in S\}$ be a continuous linear map such that $\pi_S \circ r = r_S^R \circ \pi_R$. Consider the map $r_S^P \colon \prod\{B_t \colon t \in P\} \to \prod\{B_t \colon t \in S\}$ defined as the composition $r_S^P = r_S^R \circ \pi_R^P$. Since $\pi_S \circ r = r_S^R \circ \pi_R = r_S^R \circ \pi_R^P \circ \pi_P = r_S^P \circ \pi_P$ it follows that $(S, P) \in \mathcal{L}$.

ω-closeness. Suppose that $(S_i, R) \in \mathcal{L}$ and $S_i \subseteq S_{i+1}$ for each $i \in \omega$. Then $(\cup\{S_i \colon i \in \omega\}, R) \in \mathcal{L}$.

Proof. Consider the following projective sequence

$$\prod\{B_t \colon t \in S_0\} \xleftarrow{\pi_{S_0}^{S_1}} \cdots \longleftarrow \prod\{B_t \colon t \in S_i\} \xleftarrow{\pi_{S_i}^{S_{i+1}}} \prod\{B_t \colon t \in S_{i+1}\} \longleftarrow \cdots$$

limit of which is isomorphic to the product $\prod\{B_t \colon t \in S\}$, where $S = \cup\{S_i \colon i \in \omega\}$.

Since $(S_i, R) \in \mathcal{L}$, there exists a continuous linear map $r_{S_i}^R \colon \prod\{B_t \colon t \in R\} \to \prod\{B_t \colon t \in S_i\}$ such that $\pi_{S_i} \circ r = r_{S_i}^R \circ \pi_R$, $i \in \omega$. Note that $\pi_{S_i}^{S_{i+1}} \circ r_{S_{i+1}}^R = r_{S_i}^R$ for each $i \in \omega$. Indeed let $x \in \prod\{B_t \colon t \in R\}$ and consider any point $y \in \prod\{B_t \colon t \in T\}$ such that $x = \pi_R(y)$. Since $(S_i, R), (S_{i+1}, R) \in \mathcal{L}$ we have

$$\pi_{S_i}^{S_{i+1}}\left(r_{S_{i+1}}^R(x)\right) = \pi_{S_i}^{S_{i+1}}\left(r_{S_{i+1}}^R(\pi_R(y))\right) = \pi_{S_i}^{S_{i+1}}\left(\pi_{S_{i+1}}(r(y))\right) = \pi_{S_i}(r(y)) = r_{S_i}^R(\pi_R(y)) = \pi_{S_i}^R(x).$$

In this situation the collection $\left\{\pi_{S_i}^R \colon \prod\{B_t \colon t \in R\} \to \prod\{B_t \colon t \in S_i\} \colon i \in \omega\right\}$ uniquely defines a continuous linear map $r_S^R \colon \prod\{B_t \colon t \in R\} \to \prod\{B_t \colon t \in S\}$ such that $\pi_{S_i}^S \circ r_S^R = r_{S_i}^R$ for each $i \in \omega$ (r_S^R is simply the diagonal product of $r_{S_i}^R$'s). It only remains to note that $\pi_S \circ r = r_S^R \circ \pi_R$ which completes the proof of the fact that $(S, R) \in \mathcal{L}$.

According to [1, Proposition 1.1.29] the set of $\mathcal{L}$-reflexive elements of $\exp_\omega T$ is cofinal in $\exp_\omega T$. An element $S \in \exp_\omega T$ is $\mathcal{L}$-reflexive if $(S, S) \in \mathcal{L}$. In our situation this means that the given countable subset A of T is contained in a larger countable subset S for which there exists a continuous linear map $r_S =$

$r_S^S\colon \prod\{B_t\colon t\in S\}\to\prod\{B_t\colon t\in S\}$ satisfying the equality $\pi_S\circ r = r_S\circ\pi_S$. Proof is completed. $\square$

Theorem 2.2. *A complemented subspace of a product of uncountable family of Banach spaces is isomorphic to a product of Fréchet spaces. More formally, if X is a complemented subspace of the product $\prod\{B_t\colon t\in T\}$ of Banach spaces B_t, $t\in T$, then X is isomorphic to the product $\prod\{F_j\colon j\in J\}$, where F_j is a complemented subspace of the product $\prod\{B_t\colon t\in T_j\}$ with $|T_j|=\omega$ for each $j\in J$.*

Proof. Let us first of all set up a notation. For a subset $S\subseteq T$, where T is an indexing set with $|T|=\tau>\omega$, let

$$B_S=\prod\{B_t\colon t\in S\}\ \text{ and }\ B=\prod\{B_t\colon t\in T\}.$$

Let also for $S\subseteq R\subseteq T$

$$\pi_S\colon B=\prod\{B_t\colon t\in T\}\to B_S=\prod\{B_t\colon t\in S\}$$

and

$$\pi_S^R\colon B_R=\prod\{B_t\colon t\in R\}\to B_S=\prod\{B_t\colon t\in S\}$$

denote canonical projections onto the corresponding subproducts.

Let X be a complemented subspace of the product $B=\prod\{B_t\colon t\in T\}$. Choose a continuous homomorphism $r\colon B\to X$ such that $r(x)=x$ for each $x\in X$. Let us agree that a subset $S\subseteq T$ is called r-admissible if $\pi_S\left(r(z)\right)=\pi_S(z)$ for each point $z\in\pi_S^{-1}\left(\pi_S(X)\right)$.

Claim 1. *The union of an arbitrary family of r-admissible sets is r-admissible.*

Let $\{S_j\colon j\in J\}$ be a collection of r-admissible sets and $S=\bigcup\{S_j\colon j\in J\}$. Let $z\in\pi_S^{-1}\left(\pi_S(X)\right)$. Clearly $z\in\pi_{S_j}^{-1}\left(\pi_{S_j}(X)\right)$ for each $j\in J$ and consequently $\pi_{S_j}\left(r(z)\right)=\pi_{S_j}(z)$ for each $j\in J$. Assuming that there is a point $z_0\in\pi_S^{-1}\left(\pi_S(X)\right)$ such that $\pi_S(r(z_0))\neq\pi_S(z_0)$ we conclude that there exists an index $s\in S$ such that $\pi_{\{s\}}^S\left(\pi_S(r(z_0))\right)\neq\pi_{\{s\}}^S\left(\pi_S(z_0)\right)$. Since $S=\bigcup\{S_j\colon j\in J\}$ it follows that there exists an index $j\in J$ such that $s\in S_j$. Then we have $\pi_{S_j}^S\left(\pi_S(r(z_0))\right)\neq\pi_{S_j}^S\left(\pi_S(z_0)\right)$. But this is impossible

$$\pi_{S_j}^S\left(\pi_S\left(r(z_0)\right)\right)=\pi_{S_j}\left(r(z)\right)=\pi_{S_j}(z)=\pi_{S_j}^S\left(\pi_S(z_0)\right).$$

This contradiction proves the claim.

Claim 2. *If $S \subseteq T$ is r-admissible, then $\pi_S(X)$ is a complemented subspace of $B_S = \prod\{B_t \colon t \in S\}$.*

Indeed, let $i_S \colon B_S \to B$ be the canonical section of π_S (this means that $i_S = \mathrm{id}_{B_S} \triangle \mathbf{0} \colon B_S \to B_S \times B_{T-S} = B$). Consider a continuous linear map $r_S = \pi_S \circ r \circ i_S \colon B_S \to \pi_S(X)$. Obviously, $i_S(y) \in \pi_S^{-1}(\pi_S(X))$ for any point $y \in \pi_S(X)$. Since S is r-admissible the latter implies that

$$y = \pi_S(i_S(y)) = \pi_S(r(i_S(y))) = r_S(y).$$

This shows that $\pi_S(X)$ is a complemented subspace of B_S.

Claim 3. *Let S and R be r-admissible subsets of T and $S \subseteq R \subseteq T$. Then there exists a topological isomorphism $h_S^R \colon X_R \to X_S \times \ker\left(\pi_S^R\right)$ which makes the diagram*

$$\begin{tikzcd}
\pi_R(X) \arrow[r, "h_S^R"] \arrow[d, "\pi_S^R|\pi_R(X)"'] & X_S \times \ker\left(\pi_S^R|\pi_R(X)\right) \arrow[dl, "pi_1"] \\
X_S &
\end{tikzcd}$$

commutative.

Obviously $\pi_R(X) \subseteq \pi_S(X) \times B_{R-S} \subseteq B_R = B_S \times B_{R-S}$. Consider the map $i_R = \mathrm{id}_{B_R} \triangle \mathbf{0} \colon B_R \to B_R \times B_{T-R} = B$. Also let $r_R = \pi_R \circ r \circ i_R \colon B_R \to \pi_R(X)$.

Observe that $\pi_S^R \circ r_R | (\pi_S(X) \times B_{R-S}) = \pi_S^R | (\pi_S(X) \times B_{R-S})$. Indeed, if $x \in \pi_S(X) \times B_{R-S}$, then $i_R(x) \in \pi_S^{-1}(\pi_S(X))$. Since S is r-admissible, we have $\pi_S(r(i_R(x))) = \pi_S(i_R(x))$. Consequently,

$$\begin{aligned}\pi_S^R(r_R(x)) = \pi_S^R(\pi_R(r(i_R(x)))) = \pi_S(r(i_R(x))) = \pi_S(i_R(x)) = \\ \pi_S^R(\pi_R(i_R(x))) = \pi_S^R(x).\end{aligned}$$

Next observe that $r_R(x) = x$ for any point $x \in \pi_R(X)$. Indeed, since R is r-admissible and since $i_R(x) \in \pi_R^{-1}(\pi_R(X))$ we have

$$r_R(x) = \pi_R(r(i_R(x))) = \pi_R(i_R(x)) = x.$$

In this situation we can define a map $h_S^R \colon \pi_R(X) \to X_S \times \ker\left(\pi_S^R|\pi_R(X)\right)$ by letting

$$h_S^R(x) = \left(\pi_S^R(x), x - r_R\left(\pi_S^R(x)\right)\right) \text{ for each } x \in \pi_R(X).$$

A straightforward verification shows that h_S^R is a continuous linear map which satisfies the required equality $\pi_1 \circ h_S^R = \pi_S^R|\pi_R(X)$. Also note that by letting

$$g_S^R(y, x) = r_R(y, 0) + x \text{ for each } (y, x) \in \pi_S(X) \times \ker\left(\pi_S^R|\pi_R(X)\right)$$

we define a continuous linear map $g^R_S\colon \pi_S(X)\times \ker\left(\pi^R_S|\pi_R(X)\right)\to \pi_R(X)$. It is easy to see that

$$g^R_S\circ h^R_S = \mathrm{id}_{\pi_R(X)} \quad\text{and}\quad h^R_S\circ g^R_S = \mathrm{id}_{\pi_S(X)\times\ker\left(\pi^R_S|\pi_R(X)\right)}\,.$$

This proves that h^R_S is a topological isomorphism and finishes the proof of Claim 3.

Claim 4. *Every countable subset of T is contained in a countable r-admissible subset of T.*

Let A be a countable subset of T. Our goal is to find a countable r-admissible subset S such that $A\subseteq S$. By Proposition 2.1, there exist a countable subset S of T and a continuous homomorphism $r_S\colon B_S\to B_S$ such that $A\subseteq S$ and $\pi_S\circ r = r_S\circ\pi_S$. Consider a point $y\in\pi_S(X)$. Also pick a point $x\in X$ such that $\pi_S(x)=y$. Then

$$y=\pi_S(x)=\pi_S\left(r(x)\right)=r_S\left(\pi_S(x)\right)=r_S(y).$$

This shows that $r_S|\pi_S(X)=\mathrm{id}_{\pi_S(X)}$ (this shows, in fact, that $\pi_S(X)$ is complemented in B_S).

In order to show that S is r-admissible let us consider a point $z\in\pi_S^{-1}\left(\pi_S(X)\right)$. By the observation made above, $r_S\left(\pi_S(z)\right)=\pi_S(z)$. Finally

$$\pi_S(z)=r_S\left(\pi_S(z)\right)=\pi_S\left(r(x)\right)$$

which implies that S is r-admissible.

We now use the above listed properties of r-admissible subsets and proceed as follows. By Claim 4, each element $t_\alpha\in T$ is contained in a countable r-admissible subset $S_\alpha\subseteq T$. According to Claim 1, the set $T_\alpha=\bigcup\{S_\beta\colon \beta\leq\alpha\}$ is r-admissible for each $\alpha<\tau$. Consider the projective system

$$\mathcal{S}_X=\{X_\alpha, p^{\alpha+1}_\alpha,\tau\},$$

where

$$X_\alpha=\pi_{T_\alpha}(X)\ \text{ and }\ p^{\alpha+1}_\alpha=\pi^{T_{\alpha+1}}_{T_\alpha}|\pi_{T_{\alpha+1}}(X)\colon X_{\alpha+1}\to X_\alpha\ \text{ for each }\ \alpha<\tau.$$

Since $T=\bigcup\{T_\alpha\colon\alpha<\tau\}$, it follows that $X=\operatorname{proj}\lim\mathcal{S}$. Obvious transfinite induction based on Claim 3 shows that

$$X=\operatorname{proj}\lim\mathcal{S}=X_0\times\prod\{\ker\left(p^{\alpha+1}_\alpha\right):\alpha<\tau\}.$$

Since, by the construction, S_α is a countable r-admissible subset of T, it follows from Claim 2 that X_0 and $\ker\left(p^{\alpha+1}_\alpha\right)$, $\alpha<\tau$, being complemented subspaces of countable products of Banach spaces, are Fréchet spaces. This finishes the proof of Theorem 2.2. $\square$

Recall that an object X of the category $\mathcal{L}SC$ of locally convex spaces and their continuous linear maps is *injective* if any continuous linear map $f\colon A \to X$, defined on a linear subspace of a space B, admits a continuous linear extension $g\colon B \to X$ (i.e. $g|A = f$).

The following statement is related to Problem 1 stated in the Introduction.

Corollary 2.3. *The following conditions are equivalent for a locally convex topological vector space X:*

(1) *X is an injective object of the category $\mathcal{L}CS$.*
(2) *X is isomorphic to the product $\prod\{F_t\colon t \in T\}$, where each F_t, $t \in T$, is a complemented subspace of a product $\prod\{\ell_\infty(J_{t_n})\colon n \in \omega\}$.*

Proof. (2) $\Longrightarrow$ (1). By [4, Lemma 0] and [9, p.105], $\ell_\infty(J)$ is an injective object of the category $\mathcal{L}CS$ for any set J. Obviously (see, for instance, [4, Lemma 1.9]) product of an arbitrary collection of injective objects of the category $\mathcal{L}CS$ is also an injective object of this category. Consequently, the Fréchet space F_t, $t \in T$, as a complemented subspace of $\prod\{\ell_\infty(J_{t_n})\colon n \in \omega\}$, is injective. Finally, the space X, as a product of injectives, is an injective object of the category $\mathcal{L}CS$.

(1) $\Longrightarrow$ (2). The space X can be identified with a closed linear subspace of the product $\prod\{B_t\colon t \in T\}$ of Banach spaces B_t, $T \in T$. Each of the spaces B_t can in turn be identified with a closed linear subspace of the space $\ell_\infty(J_t)$ for some set J_t, $t \in T$. Condition (1) implies in this situation that X is a complemented subspace of the product $\prod\{\ell_\infty(J_t)\colon t \in T\}$. The required conclusion now follows from Theorem 2.2. $\square$

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Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada

E-mail address: chigogid@math.usask.ca