CHARACTERIZING THE TOPOLOGY OF PSEUDO-BOUNDARIES OF EUCLIDEAN SPACES

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Abstract. We give a topological characterization of the n-dimensional pseudo-boundary of the $(2n+1)$-dimensional Euclidean space.

1. Introduction

In [11] Geoghegan and Summerhill constructed the n-dimensional universal pseudo-boundary σ_n^k of the k-dimensional Euclidean space $\mathbb{R}^k$, $0 \leq n \leq k$, $k \geq 1$, as an $\mathcal{M}_n^k$-absorber of $\mathbb{R}^k$, where $\mathcal{M}_n^k$ denotes the collection of tame at most n-dimensional compacta in $\mathbb{R}^k$. In these notes we consider the space σ_n^{2n+1}. It has been remarked by several authors that from a certain point of view the space σ_n^{2n+1} can be considered as the n-dimensional counterpart of the pseudo-boundaries σ and Σ of the Hilbert cube Q. Topological characterizations of the latter spaces have been obtained by Mogilski [12], [4]. As for the problem of topological characterization of σ_n^{2n+1} (see, for instance, [14, Problem # 1017], [10, Problem # 607], [8, Conjecture 4.10], [15, Question 3], [5, Conjecture 5.6.9]) we mention here the following two related results. First of all we note that according to [8] $\sigma_n^{2n+1} \approx \sigma_n^k$ for each $k \geq 2n+1$. Secondly $\sigma_n^{2n+1} \approx \Sigma^n$ (see [7, Theorem 7.4], [5, Theorem 5.6.10]), where Σ^n denotes the pseudo-boundary of the universal n-dimensional Menger compactum μ^n [3] constructed in [6].

Below (Corollary 2.8) we give a topological characterization of the space σ_n^{2n+1}.

2. Topological characterization of finite-dimensional absorbing sets

2.1. **Preliminaries.** All spaces in these notes are assumed to be separable and metrizable. Maps are assumed to be continuous.

Let $n \in \omega$. A subset A of a space X is said to be locally connected in dimension n relative to X (briefly LC^n rel. X) if for each $k \leq n+1$, each $x \in X$ and each neighbourhood U of x in X there exists a neighbourhood V of x in X such that every map $f\colon \partial I^k \to V \cap A$ has an extension $F\colon I^k \to U \cap A$. A

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space X is said to be locally connected in dimension n (briefly LC^n) if X is LC^n rel. X. Recall that class of $LC^{n-1} \cap C^{n-1}$-spaces coincides with the class $AE(n)$ of absolute extensors in dimension n. Discussion of basic properties of LC^n-spaces can be found in [5].

Recall that for an open cover $\mathcal{U} \in \operatorname{cov}(Y)$ of a space Y two maps $f, g\colon X \to Y$ are said to be $\mathcal{U}$-close if for each point $x \in X$ there exists an element $U \in \mathcal{U}$ such that $f(x), g(x) \in U$. A space X satisfies the discrete n-cells property if the set

$$\{f \in C(I^n \times \mathbb{N}, X)\colon \{f(I^n \times \{k\})\colon k \in \mathbb{N}\} \text{ is discrete } \}$$

is dense in the space $C(I^n \times \mathbb{N}, X)$ equipped with the limitation topology. The latter topology on the space $C(X, Y)$ of all continuous maps of X into Y has a neighbourhood base at a point $f \in C(X, Y)$ consisting of the sets $\{g \in C(X, Y)\colon g \text{ is } \mathcal{U}\text{-close to } f\}$, $\mathcal{U} \in \operatorname{cov}(Y)$.

Proof of our main result is based on the following two statements obtained recently[1] in [1].

Theorem (Topological characterization of the Nöbeling space [1])**.** *Let $n \geq 0$. Then the following conditions are equivalent for any space X:*

1. *X is homeomorphic to the n-dimensional universal Nöbeling space N_n^{2n+1}.*
2. *X is a separable completely metrizable space satisfying the following properties:*
   (a) $\dim X = n$.
   (b) $X \in AE(n)$.
   (c) *X has the discrete n-cells property.*

Theorem (Z-set unknotting [1])**.** *Let $n \geq 0$. Then for each open cover $\mathcal{U} \in \operatorname{cov}(N_n^{2n+1})$ there exists an open cover $\mathcal{V} \in \operatorname{cov}(N_n^{2n+1})$ such that the following property is satisfied:*

- *Every homeomorphism $h\colon Z_1 \to Z_2$ between Z-sets of N_n^{2n+1} which is $\mathcal{V}$-close to the inclusion $Z_1 \hookrightarrow N_n^{2n+1}$ can be extended to a homeomorphism $H\colon N_n^{2n+1} \to N_n^{2n+1}$ which is $\mathcal{U}$-close to the identity $\mathrm{id}_{N_n^{2n+1}}$.*

2.2. **Uniqueness of finite-dimensional absorbing sets.** In this section we prove that any two "absorbing sets" for a class of finite-dimensional spaces are homeomorphic.

Let $\mathcal{K}$ be a class of spaces that is topological, finitely additive and hereditary with respect to closed subspaces. A space X is *strongly $\mathcal{K}$-universal* if, for every map $f\colon C \to X$ from a space $C \in \mathcal{K}$ into X, for every closed subspace $D \subseteq C$ such that $f/D\colon D \to X$ is a Z-embedding and for every open cover $\mathcal{U} \in \operatorname{cov}(X)$,

[1]The first named author recalls with satisfaction series of very interesting lectures given by S. Ageev during his stay at the University of Saskatchewan in May-June of 1999.

there exists a Z-embedding $g\colon C \to X$ such that $g/D = f/D$ and g is $\mathcal{U}$-close to f.

The class consisting of countable unions of members of $\mathcal{K}$ is denoted by $\mathcal{K}_\sigma$.

Let $n \in \omega$. An n-dimensional separable metrizable space X is a $\mathcal{K}$-*absorbing set* if:

(a) $X \in AE(n)$.
(b) X is a countable union of strong Z-sets.
(c) $X \in \mathcal{K}_\sigma$.
(d) X is strongly $\mathcal{K}$-universal.

Several examples of $\mathcal{K}$-absorbing sets (for various classes $\mathcal{K}$) can be found in [10].

First we show that spaces we are interested in can be nicely embedded into the Nöbeling space of the same dimension.

Proposition 2.1. *Let $n \geq 0$ and X be a separable metrizable $LC^{n-1}\&C^{n-1}$-space satisfying the discrete n-cells property. Then X can be embedded into a copy M of the universal n-dimensional Nöbeling space N_n^{2n+1} so that the set $\{f \in C(I^n, M)\colon f(I^n) \subseteq X\}$ is dense in the space $C(I^n, M)$. In particular, the following properties are satisfied:*

(a) *Every F_σ-subset F of M such that $F \cap X = \emptyset$ is a Z_σ-set in M.*
(b) *Every G_δ-subspace of M, containing X, is homeomorphic to N_n^{2n+1}.*
(c) *If A and B are G_δ-subsets of M such that $X \subseteq A \subseteq B$, then $B - A$ is a Z_σ-subset in B.*
(d) *If A and B are G_δ-subsets of M such that $X \subseteq A \subseteq B$, then the inclusion $A \hookrightarrow B$ is a near-homeomorphism.*

Proof. Let $\widetilde{X}$ be an n-dimensional metrizable compactification of X. By [2, Theorem 2], there exists a G_δ-set $M \subseteq \widetilde{X}$, containing X, so that

(1) X is LC^{n-1} rel. M;
(2) $M \in \mathrm{LC}^{n-1}$;
(3) For every at most n-dimensional Polish space Y the set of all closed embeddings is dense in $C(Y, M)$.

Let us show that $M \in C^{n-1}$. Indeed, let $f\colon \partial I^k \to M$ be a map defined on the boundary ∂I^k of the k-dimensional disk I^k, $k \leq n$. According to [5, Proposition 4.1.7], there exists an open cover $\mathcal{V} \in \mathrm{cov}(M)$ such that the following condition is satisfied:

$(*)_{n-1}$ If a $\mathcal{V}$-close to f map $g\colon \partial I^k \to M$, $k \leq n$, has an extension $G\colon I^k \to M$, then f also has an extension $F\colon I^k \to M$.

Since X is LC^{n-1} rel. M, it follows by [13, Theorem 2.8] that $M - X$ is locally n-negligible in M. According to [13, Theorem 2.3] we can find a map

$g\colon \partial I^k \to X$ which is $\mathcal{V}$-close to f. Since $X \in C^{n-1}$, there exists an extension $G\colon I^k \to X$ of g. The above stated property $(*)_{n-1}$ of the cover $\mathcal{V}$ guarantees that f also has an extension $F\colon I^k \to M$. This shows that $M \in C^{n-1}$. Therefore M is an n-dimensional, separable, completely metrizable $LC^{n-1}\&C^{n-1}$-space satisfying property (3). Topological characterization of the Nöbeling space (see Section 1) implies that M is homeomorphic to N_n^{2n+1}. The fact that the set the set $\{f \in C(I^n, M)\colon f(I^n) \subseteq X\}$ is dense in the space $C(I^n, M)$ follows from [5, Theorem 2.8].

Let F be an F_σ-subset of M such that $F \cap X = \emptyset$. Since

$$\{f \in C(I^n, M)\colon f(I^n) \subseteq X\} \subseteq \{f \in C(I^n, M)\colon f(I^n) \cap F = \emptyset\},$$

it follows that the set $\{f \in C(I^n, M)\colon f(I^n) \cap F = \emptyset\}$ is dense in $C(I^n, M)$. Consequently, F is a Z_σ-subset of M. This proves property (a).

Next observe that since M is homeomorphic to N_n^{2n+1} it can be identified with the pseudo-interior ν^n of the universal n-dimensional Menger compactum (see [5, Theorem 5.5.5]). Let Y be a G_δ-subspace of M containing X. By (a) and [5, Proposition 5.7.7], the inclusion $Y \hookrightarrow M$ is a near-homeomorphism. In particular, Y is homeomorphic to N_n^{2n+1}. This proves (b). Properties (c) and (d) are proved
similarly. □

Remark 2.2. An n-dimensional $\mathcal{K}$-absorbing set is called *representable* in $\mathbb{R}^k$ [10] if there exists an embedding $i\colon M \to \mathbb{R}^k$ such that the set $\mathbb{R}^k - M$ is locally n-negigible in $\mathbb{R}^k$.

Every n-dimensional $\mathcal{K}$-absorbing set is representable in $\mathbb{R}^{2n+1}$.

Proof. It is showm in the proof of Proposition 2.1 that there exists an embedding of $\mathcal{K}$-absorbing set M into N_n^{2n+1} with locally n-negligible complement of the image. Now observe that the complement $\mathbb{R}^{2n+1} - N_n^{2n+1}$ as a σZ_n-set [5] in $\mathbb{R}^{2n+1}$ is locally n-negligible in $\mathbb{R}^{2n+1}$. This obviously implies that the complement $\mathbb{R}^{2n+1}$ is also n-negligible in $\mathbb{R}^{2n+1}$ as required. □

The above statement provides an affirmative solution of Problem 555 from [10].

Lemma 2.3. *Let X be an at most n-dimensional separable metrizable LC^{n-1}-space. If $X = \cup\{X_i\colon i \in \omega\}$, where each X_i is a strong Z-set in X, then each compact subset of X is a strong Z-set in X.*

Proof. Let $\widetilde{X}$ be an n-dimensional separable completely metrizable space containing X as a subspace in such a way that X is LC^{n-1} rel. $\widetilde{X}$ (see [9, Proposition 2.8]. As in the proof of Proposition 2.1, we conclude that

($*$) the set $\{f \in C(I^n, \widetilde{X})\colon f(I^n) \subseteq X\}$ is dense in $C(I^n, \widetilde{X})$.

Next we need the following observation.

Claim. *A compact subset K of X is a Z-set in X if and only if K is a Z-set in $\widetilde{X}$.*

Proof of Claim. First let K be a Z-set in X. Consider a map $f\colon I^n \to \widetilde{X}$ and open covers $\mathcal{U}, \mathcal{V} \in \operatorname{cov}(\widetilde{X})$ such that $\operatorname{St}(\mathcal{V})$ refines $\mathcal{U}$. By $(*)$, there exists a $\mathcal{V}$-close to f map $g \in C(I^n, \widetilde{X})$ such that $g(I^n) \subseteq X$. Since K is a Z-set in X, there exists a $\mathcal{V}$-close to g map $h\colon I^n \to X$ such that $h(I^n) \cap K = \emptyset$. Since h is $\mathcal{U}$-close to f, it follows that K is a Z-set in $\widetilde{X}$.

Conversely, let K be a Z-set in $\widetilde{X}$. Consider a map $f\colon I^n \to X$ and open covers $\mathcal{U}, \mathcal{V} \in \operatorname{cov}(X)$ so that $\operatorname{St}(\mathcal{V})$ refines $\mathcal{U}$. For each $V \in \mathcal{V}$ choose an open subset $\widetilde{V} \subseteq \widetilde{X}$ such that $V = \widetilde{V} \cap X$. It is easy to see that K is a Z-set in $Y = \cup\{\widetilde{V}\colon V \in \mathcal{V}\}$. Consequently there exists a $\widetilde{\mathcal{V}}$-close to f map $g\colon I^n \to Y$ such that $g(I^n) \cap K = \emptyset$, where $\widetilde{\mathcal{V}} = \{\widetilde{V}\colon V \in \mathcal{V}\} \in \operatorname{cov}(Y)$. Let G be an open subsets of Y such that $K \cap G = \emptyset$ and $g(I^n) \subseteq G$. By $(*)$, there exists a map $h\colon I^n \to X$ which is $\widetilde{\mathcal{V}} \wedge \{G, Y - g(I^n)\}$-close to g. Obviously, h is $\mathcal{U}$-close to f and $h(I^n) \cap K = \emptyset$. This shows that K is a Z-set in X and completes the proof of claim.

We continue the proof of Lemma 2.3. Let K be a compact subset of X. Clearly $K \cap X_i$ is a compact Z-set in X for each $i \in \omega$. By the above Claim, $K \cap X_i$ is a Z-set in $\widetilde{X}$. This means that the set $\{f \in C(I^n, \widetilde{X})\colon f(I^n) \cap (X_i \cap K) = \emptyset\}$ is open and dense in the space $C(I^n, \widetilde{X})$. Since $\widetilde{X}$ is completely metrizable, the space $C(I^n, \widetilde{X})$ has the Baire property (see, for instance, [5, Proposition 2.1.7]) and consequently, the set

$$
\begin{aligned}
&\{f \in C(I^n, \widetilde{X})\colon f(I^n) \cap K = \emptyset\} = \\
&\qquad \{f \in C(I^n, \widetilde{X})\colon f(I^n) \cap (\cup\{X_i \cap K\colon i \in \omega\}) = \emptyset\} = \\
&\qquad\qquad \bigcap \Big\{\{f \in C(I^n, \widetilde{X})\colon f(I^n) \cap (X_i \cap K) = \emptyset\}\colon i \in \omega\Big\}
\end{aligned}
$$

is also dense in the space $C(I^n, \widetilde{X})$. This simply means that K is a Z-set in $\widetilde{X}$. By the above Claim, we conclude that K is a Z-set in X as well. □

Proof of the following statement uses Lemma 2.3 and follows verbatim the proof of [10, Lemma 1.9].

Proposition 2.4. *Let X be an at most n-dimensional separable metrizable LC^{n-1}-space. If $X = \cup\{X_i\colon i \in \omega\}$, where each X_i is a strong Z-set in X, then X satisfies the discrete n-cells property.*

Now we are in position to prove the uniqueness result.

Theorem 2.5. *Let $n \geq 0$ and $\mathcal{K}$ be a class of spaces that is topological, finitely additive and hereditary with respect to closed subspaces. Then any two $\mathcal{K}(n)$-absorbing sets are homeomorphic.*

Proof. Let X and Y be $\mathcal{K}(n)$-absorbing sets. Proposition 2.4 guarantees that $X, Y \in n$-SDAP. Embed X and Y into a copy M of the universal n-dimensional Nöbeling space N_n^{2n+1} in such a way that properties (a)–(d) of Proposition 2.1 are satisfied. The rest of the proof follows the argument presented in the proof of [4, Theorem 3.1] (use the Z-set Unknotting Theorem for N_n^{2n+1} instead of the Z-set unknotting theorem for ℓ_2 at the appropriate place). □

2.3. **Characterization of σ_n^{2n+1}.** In order to obtain a topological characterization of a $\mathcal{K}(n)$-absorbing set Theorem 2.5 must be combined with the corresponding existence result. In other words, we need to know that there exists a $\mathcal{K}(n)$-absorbing set. For certain choices of $\mathcal{K}$ it is even possible to explicitly construct corresponding absorbing sets.

Let us recall that for each space X and for each ordinal $\alpha < \omega_1$, we can define two classes of subspaces of X – the *additive Borelian class* α, $\mathcal{A}_\alpha(X)$, and the *multiplicative Borelian class* α, $\mathcal{M}_\alpha(X)$, – as follows: $\mathcal{A}_0(X)$ is the collection of all open subsets of X and $\mathcal{M}_0(X)$ is the collection of all closed subsets of X. Assuming that for each ordinal $\beta < \alpha$, where $\alpha < \omega_1$, the classes $\mathcal{A}_\beta$ and $\mathcal{M}_\beta$ have already been constructed, we proceed as follows: the class $\mathcal{A}_\alpha$ consists of countable unions of elements of $\cup\{\mathcal{M}_\beta \colon \beta < \alpha\}$ and the class $\mathcal{M}_\alpha$ consists of countable intersections of elements of $\cup\{\mathcal{A}_\beta \colon \beta < \alpha\}$.

Further, let $\alpha < \omega_1$ and X be a separable metrizable space. We say that X belongs to the *absolute additive Borelian class* $\mathcal{A}_\alpha$, if for any embedding $i \colon X \to Y$ into any separable metrizable space Y, we have $i(X) \in \mathcal{A}_\alpha(Y)$. Similarly, X belongs to the *absolute multiplicative Borelian class* $\mathcal{M}_\alpha$ if for any embedding $i \colon X \to Y$ into any separable metrizable space Y, we have $i(X) \in \mathcal{M}_\alpha(Y)$. It is well-known that: (a) $X \in \mathcal{A}_\alpha$, $\alpha \geq 2$, if and only if $X \in \mathcal{A}_\alpha(l_2)$ and (b) $X \in \mathcal{M}_\alpha$, $\alpha \geq 1$, if and only if $X \in \mathcal{M}_\alpha(l_2)$.

Obviously, $\mathcal{A}_0 = \emptyset$ and $\mathcal{M}_0$ coincides with the class of all metrizable compacta. Further, $\mathcal{A}_1 = \{\sigma\text{-compact spaces}\}$, $\mathcal{M}_1 = \{\text{Polish spaces}\}$, etc.

The existence problem for these classes of spaces is solved in the following statement [5, Theorem 5.7.21], [15, Theorem 2.5].

Theorem 2.6. *Let $n \in \omega$ and $1 \leq \alpha < \omega_1$. Then there exist an $\mathcal{A}_\alpha(n)$-absorbing set $\Lambda_\alpha(n)$ and $\mathcal{M}_\alpha(n)$-absorbing set $\Omega_\alpha(n)$.*

Theorems 2.5 and 2.6 imply the following characterization result.

Theorem 2.7. *Let X be an n-dimensional, $n \geq 0$, separable metrizable $AE(n)$-space and $1 \leq \alpha < \omega_1$. Then X is homeomorphic to $\Omega_\alpha(n)$ (respectively, $\Lambda_\alpha(n)$) if and only if the following two conditions are satisfied:*

(i) *$X = \cup\{X_i \colon i \in \omega\}$, where each $X_i \in \mathcal{M}_\alpha$ (respectively, $X_i \in \mathcal{A}_i$) and X_i is a strong Z-set in X,*

(ii) *X is strongly $\mathcal{M}_\alpha(n)$-universal (respectively, $\mathcal{A}_\alpha(n)$-universal.*

In particular ($\alpha = 1$), we obtain a topological characterization of σ_n^{2n+1}.

Corollary 2.8. *Let X be an n-dimensional, $n \geq 0$, σ-compact metrizable $AE(n)$-space. Then X is homeomorphic to σ_n^{2n+1} if and only if the following conditions are satisfied:*

(i) *X has the discrete n-cells property,*
(ii) *X is strongly $\mathcal{A}_1(n)$-universal.*

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