ESSAYS ON λ-QUANTILE DEPENDENT CONVEX RISK MEASURES

by

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A dissertation submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Applied Mathematics

Charlotte

2011

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ABSTRACT

LIHONG XIA. Essays on λ-quantile dependent convex risk measures. (Under the direction of DR. MINGXIN XU.)

We define a class of convex measures of risk whose values depend on the random variables only up to the λ-quantiles for some given constant $\lambda \in (0, 1)$. For this class of convex risk measures, the assumption of Fatou property can be strengthened, and the robust representation theorem via convex duality method is provided. These results are specialized to the class of λ-quantile law invariant risk measures. We define the λ-quantile uniform preference (λ-quantile second order stochastic dominance) of two probability distribution measures and the λ-quantile dependent concave distortion and study their properties. The robust representation theorem of the λ-quantile dependent Weighted Value-at-Risk is proven via two different approaches: the λ-quantile uniform preference approach and the approach of maximizing the Choquet integral over the core of a λ-quantile dependent concave distortion. We demonstrate the two approaches in a classical example of Conditional Value-at-Risk and a new example of uniform λ-quantile dependent Weighted Value-at-Risk.

ACKNOWLEDGMENTS

My deepest appreciation goes to my advisor Dr. Mingxin Xu for numerous conversations that clarified my thinking on the study of this topic and for her wise advice at critical times. Without her direction and guidance I would never have successfully pursued the Ph.D degree.

I am also grateful to Dr. Volker Wihstutz, Dr. You-Lan Zhu, and Dr. Keener Hughen for serving as valuable members in my doctoral committee.

Finally, I am particularly grateful to my husband Wenhong Su for being always there for me with his unbounded love, tolerance, spiritual and material support.

This work is dedicated to my parents Xiaoguo Xia and Renjuan Cai.

INTRODUCTION

How to measure the riskiness of financial a position is an important yet complex topic for financial institutes such as banks and insurance companies as well as regulators. The financial crisis started in 2008 has shown us how critical risk measures are. Measuring risk, theoretically, involves how to define a proper measure to quantify the riskiness of a financial position and to study the properties of the measure; while practically, involves how to estimate and predict the selected measure using historic data or other methods such as Monte Carlo simulation. My research focuses on the first point. More precisely, we define the class of "λ-quantile dependent" convex risk measures and study its properties.

According to Artzner, Delbaen, Eber and Heath (1999), the nature of a risk measure lies in the capital requirements that can be added to a financial position to make it acceptable from the point of view of an agent or a regulator. The paper of Artzner et al. (1999) is the mathematical foundation of studying the measure of risk, in which "a unified framework for the definition, analysis, construction and implementation" of the measure of risk was proposed and axioms of the class of the "coherent measures of risk" and its "robust representation" were given. In their framework, Ω denotes the possible outcomes of market scenarios, which is assumed to be finite. A random variable $X : \Omega \to \mathbb{R}$ indicates the "final net worth" of a position for each element of Ω, and the collection of all these random variables is denoted by $\mathcal{X}$. Note that $\mathcal{X}$ is the collection of all bounded random variables due to the finiteness of Ω. A risk measure ρ is then a mapping from $\mathcal{X}$ to the real line $\mathbb{R}$, $\rho : \mathcal{X} \to \mathbb{R}$. From the point of view of an agent or a regulator, a financial position is either "acceptable" or "unacceptable" with respect to its final net worth. The collection of all acceptable financial positions is called the "acceptance set" and denoted by $\mathcal{A}$. Obviously, $\mathcal{A} \subset \mathcal{X}$. Given a measure of risk ρ, its acceptance set $\mathcal{A}_\rho$ is $\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \leq 0\}$, the collection of all financial positions whose risk is nonpositive under the measure ρ. On the other hand, since a risk measure is the minimum capital requirement that can to added to a financial position to make it acceptable, start from an acceptance set $\mathcal{A}$, a measure of risk ρ can be recovered as $\rho(X) = \inf\{m \in \mathbb{R} \,|\, m + X \in \mathcal{A}\}$, for $X \in \mathcal{X}$. As a special class of measure of risk, Artzner et al. defined the coherent measure of risk by postulating axioms it must satisfy. These axioms are monotonicity, translation invariance, subadditivity and positive homogeneity. Artzner et al. also showed that these axioms have correspondence to the axioms of the acceptance set, and for a coherent measure of risk ρ in the current setting (i.e., finite Ω), it is true that $\rho = \rho_{\mathcal{A}_\rho}$. In addition, Artzer et al. (1999) provided a representation, also known as the robust representation, of the coherent measure of risk ρ: $\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbf{Q}}[-X]$, where $\mathcal{Q}$ is a family of probability measures on Ω. Note that Ω is assumed to be finite.

Delbaen (2002) extended the coherent measure of risk to the general space $L^0 := L^0(\Omega, \mathcal{F}, \mathbf{P})$, the space of equivalent classes of all measurable functions on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with a general set of Ω. Since Ω can contain infinite number of elements, the random variables in L^0 are not anymore bounded. Thus, the coherent measure of risk ρ defined on the space L^0 could take infinite value. Delbaen pointed out that the probability measure $\mathbf{P}$ added to the space $(\Omega, \mathcal{F})$ is necessary to consider the probability space L^0, however, it is not really important which particular $\mathbf{P}$ is added, since the robust representation of the coherent measure of risk indicates that only the set of probability measures that are equivalent to $\mathbf{P}$ matters.

A more general class of measures of risk is the convex measure of risk. The law invariant measure of risk is a subclass of the convex measure of risk. In summary, there are three classes of risk measures that are precisely studied so far: the coherent measure of risk, the convex measures of risk, and the law invariant measure of risk.

The coherent measure of risk: As mentioned, this class of measures of risk was originally defined by Artzner et al. (1999) for finite market scenarios and was extended by Delbaen (2002) to the general state space. A coherent measure of risk satisfies axioms of translation invariance, monotonicity, subadditivity and positive homogeneity. Artzner et al. (1999) proposed the robust representation using the expectations of the random variables. Delbaen (2002) showed that for a general state space Ω, the robust representation of a coherent measure of risk exists under some condition. He proposed the equivalent conditions including the Fatou property which was first time defined for a measure of risk.

The convex measure of risk: It is a generalization of the coherent measure of risk independently made by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002). For a convex measure of risk, subadditivity and positive homogeneity are replaced by the convexity. For different probability spaces, the robust representation as well as equivalent conditions for the existence of the robust representation were proposed by Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002), Föllmer and Schied (2004), Biagini and Frittelli (2009), Kaina and Rüschendorf (2009).

The law invariant measure of risk: Kusuoka (2001) first studied those coherent measures of risk whose values depend on the random variables only through their probability distributions, and call them the "law invariant coherent risk measures". He further showed that the law invariance coherent risk measure ρ defined on the space $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, the equivalent classes of essentially bounded random variables, can be represented by the Weighted Value-at-Risk, if ρ has the Fatou property. The definition of the "law invariance" can be extended to the convex measure of risk, which was done by Föllmer and Schied (2004) and Frittelli and Rosazza Gianin (2005). Later, Jouini, Schachermayer and Touzi (2006) proved that all law invariant convex measures of risk on L^∞ already have the Fatou property.

In addition, as a particular subclass of the convex measure of risk, the Weighted Value-at-Risk is of great interest to researchers.

The Weighted Value-at-Risk (WVaR): This is a subclass of the convex measure of risk, it is coherent as well as law invariant. The WVaR includes the well-known Conditional Value-at-Risk (CVaR). It first appeared in Kusuoka (2001) as the one who represents the law invariant coherent measure of risk. Though Kusuoka did not give a particular name to it, he showed that on the space L^∞, the WVaR is a law invariant and comonotonic coherent risk measure and it has the Fatou property. Kusuoka also proposed a representation using the Choquet integral. Acerbi (2002) named this class of risk measures the "spectral measure of risk". Föllmer and Schied (2004) called it the "concave distortion" and provided the robust representation using the second order stochastic dominance of a concave core. Cherny (2006) named this class of risk measures the "Weighted Value-at-Risk" and extended it into the space L^0.

An important point on the study of the convex measure of risk (including the coherent measure of risk) is that under what conditions the robust representation exists. If Ω is finite, Artzner et al. (1999) showed that a measure of risk ρ is coherent if and only if there exists a family $\mathcal{Q}$ of probability measures on Ω such that

$$\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbf{Q}}[-X], \qquad \text{for } X \in \mathcal{X}. \tag{1}$$

The representation (1) links the coherent measure of risk to the expectations of the financial positions, and the supremum in (1) shows the robustness of ρ in the sense that the more probability measures are included in the representation set $\mathcal{Q}$, the more conservative is the risk measure. The representation (1) is called the "robust representation" of a coherent measure of risk. When a general Ω (i.e., Ω may contain infinitely many elements) is considered, more conditions are needed for a coherent measure of risk to be representable. Delbaen (2002) proposed these conditions. For the coherent measure of risk $\rho : L^\infty \to \mathbb{R}$, he defined the "Fatou property": for any sequence of random variables $(X_n) \subset L^\infty$ such that (X_n) is uniformly bounded by some constant C, ρ has the Fatou property if $\rho(X) \le \liminf_{n\to\infty} \rho(X_n)$ whenever $X_n \to X$ $\mathbf{P}$-a.s. for some $X \in L^\infty$. Delbaen showed that the Fatou property is sufficient as well as necessary for the acceptance set $\mathcal{A}_\rho$ of ρ to be weak* closed. Then, the robust representation (1) exists due to the bipolar theorem. Moreover, Delbaen showed that the Fatou property is in fact equivalent to that ρ is continuous from above. Since the representation (1) is continuous from above, ρ has the Fatou property is equivalent to that ρ has the representation (1).

For the robust representation of a convex measure of risk $\rho : L^\infty \to \mathbb{R}$, both Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) achieved the following representation:

$$\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})), \tag{2}$$

with ρ^* a penalty function and $\mathcal{Q}$ the representation set which can be identified. However, their approaches to show (2) are different: Föllmer and Schied used the approach similar to Delbaen (2002), while Frittelli and Rosazza Gianin (2002) used the Fenchel-Legendre duality.

To extend the robust representation (2) to the convex measure of risk defined on the functional space $L^p := L^p(\Omega, \mathcal{F}, \mathbf{P})$, $1 \le p < \infty$, the functional space such that for any $X \in L^p$, $\int_{\mathbb{R}} |X|^p d\mathbf{P} < \infty$, Biagini and Frittelli (2009) modified Delbaen's Fatou property: for any sequence $(X_n) \subset L^p$ such that (X_n) is dominated by some random variable $Y \in L^p$, a convex measure of risk $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ has the Fatou property if $\rho(X) \le \liminf \rho(X_n)$ whenever $X_n \to X$ $\mathbf{P}$-a.s. for some $X \in L^p$. The difference of the two Fatou properties is the boundedness of (X_n) required. In Delbaen's definition, (X_n) is bounded by some constant uniformly, while in Biagini and Frittelli's definition, (X_n) must be dominated by some random variable. Under the modified Fatou property, Biagini and Frittelli provided the robust representation of a convex and monotone functional defined on a locally convex Frechet lattice. Later on, Kaina and Rüschendorf (2009) specified Biagini and Frittelli's results onto the convex measure of risk $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ with $1 \leq p \leq \infty$.

When measuring risk, we are usually more concerned with downside risk than the upside profit of the portfolio. In fact, many convex risk measures we consider such as the Conditional Value-at-Risk (CVaR), the Weighted Value-at-Risk (WVaR), or non-convex risk measure such as Value-at-Risk (VaR), depend only on the lower quantiles of the financial positions up to some fixed significant level $\lambda \in (0,1)$. We call this class of convex risk measures the "λ-quantile dependent" convex risk measure and study it from the following points:

The definition: The λ-quantile dependent convex risk measure is a convex measure of risk $\rho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \leq p \leq \infty$, whose value depends on the random variables only up to a given level $\lambda \in (0,1)$. In other words, if the random variables are undistinguishable up to their λ-quantiles, their values of the risk measure must be same. Therefore, we first need to formally indicate the λ-quantile undistinguishable random variables, and then define the λ-quantile dependent convex risk measure.

The λ-quantile Fatou property: Since a λ-quantile dependent convex risk measure belongs to the class of the convex measures of risk, it is representable under the Fatou property defined by Biagini and Frittelli (2009). However, since the risk measure depends on the random variable only up to the level λ, it is more natural to require the upper λ-quantiles of the sequence (X_n) to be uniformly bounded above by some constant. Therefore, we define the λ-quantile Fatou property as the following: For a sequence of random variables $(X_n) \subset L^p$ such that their upper λ-quantiles are uniformly bounded by some real number, a convex measure of risk $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ has the λ-quantile Fatou property if $\rho(X) \leq \liminf \rho(X_n)$ whenever $X_n \to X$ $\mathbf{P}$-a.s. for some $X \in L^p$. The boundedness condition we adopt in defining the λ-quantile Fatou property yields the lower semicontinuity for the λ-quantile dependent risk measure and its robust representation (2). When the risk measure is restricted to be λ-quantile dependent, the corresponding λ-quantile Fatou property turns out to be stronger in the sense that the boundedness on the quantile function can be more readily satisfied than the boundedness on the entire random variable, so the continuity property works for a larger class of sequences of random variables.

The λ-quantile law invariant risk measure: This is a subclass of the λ-quantile dependent convex risk measure. A λ-quantile law invariant risk measure is a convex measure of risk that depends only on the law of the random variables up to the given significance level λ. We propose a representation theorem for this class of risk measures.

The λ-quantile dependent WVaR: As an important subclass of the λ-quantile dependent convex risk measure, we define the λ-quantile dependent Weighted Value-at-Risk (λ-quantile dependent WVaR), denoted as $\rho_{\mu,\lambda}$. We first define the λ-quantile dependent Weighted Value-at-Risk for some fixed $\lambda \in (0,1)$ as

$$\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} CVaR_\gamma(X)\mu(d\gamma), \qquad X \in L^p, \quad 1 \leq p \leq \infty,$$

with μ a probability measure on $[0,\lambda]$ and $\mu(\{0\}) = 0$. $\rho_{\mu,\lambda}$ is coherent, law invariant and λ-quantile dependent. We prove the representation theorem for the λ-quantile dependent WVaR by assuming that the probability space Ω is atomless. Similar to Carlier and Dana (2003), two approaches to the proof are adopted. The first one is to use the uniform preference of two probability distribution measures, also known as the second order stochastic dominance, which we extend the definition to the λ-quantile dependent case and study its properties for representation. The second approach hinges upon the relationship between comonotonic law invariant risk measures and Choquet integrals discovered by Schmeidler (1986). Showing the representation for the risk measure is reduced to finding the core of the Choquet integral. In the λ-quantile dependent WVaR case, we establish that

$$\rho_{\mu,\lambda}(X) = \begin{cases} \int_0^{q_X(\lambda)}(\Psi(\mathbf{P}(X<x))-1)dx + \int_{-\infty}^0 \Psi(\mathbf{P}(X<x))dx, & \text{if } q_X(\lambda)>0, \\ \int_{-\infty}^{q_X(\lambda)} \Psi(\mathbf{P}(X<x))dx, & \text{if } q_X(\lambda)\leq 0. \end{cases}$$

where Ψ is a λ-quantile dependent concave distortion. We verify that the above Choquet integral is the maximum of the expectation of $-X$ over the probability measures in the core of $\Psi \circ \mathbf{P}$. As an example, we give the robust representation of the Conditional Value-at-Risk $CVaR_\lambda$ using these two approaches and check that the representation sets achieved from these different methods are indeed the same, and they also coincide with the well-known result obtained from Neyman-Pearson Lemma by Föllmer and Schied (2004). We also show the robust representation in a new example which we call the uniform λ-quantile dependent Weighted Value-at-Risk.

This thesis is organized in the following way:

In Chapter 1, we review the definition of the convex measure of risk and the theorem of the robust representation. As a subclass of the convex measure of risk, we also review the law invariant measure of risk and its representation, as well as the Weighed Value-at-Risk and the corresponding representation.

In Chapter 2, we define the class of λ-quantile dependent convex risk measures and the λ-quantile Fatou property. We show the robust representation theorem using the λ-quantile Fatou property.

In Chapter 3, we define the class of λ-quantile law invariant risk measure and propose a representation similar to the law invariant measure of risk.

In Chapter 4, we first define the class of λ-quantile dependent WVaR. We then define the λ-quantile uniform preference of two probability distribution measures and study its properties. The robust representation of the λ-quantile dependent WVaR is proposed via the λ-quantile uniform preference and the core of the concave distortion respectively. As an example, we give the robust representation of the Conditional Value-at-Risk CVaR_λ using the above two approaches and check that this robust representation coincides to the well known one. Finally, we give the robust representation of the uniform λ-quantile dependent WVaR.

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LIST OF NOTATIONS

$(\Omega, \mathcal{F}, \mathbf{P})$: a probability space

$\mathcal{X}$: the collection of the random variables on Ω

$\mathcal{A}_\rho$: the acceptance set of the risk measure ρ

$L^0 := L^0(\Omega, \mathcal{F}, \mathbf{P})$: the space of equivalent classes of all measurable functions on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$

$L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbf{P})$: the space of equivalent classes of all essentially bounded functions on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$

$L^p := L^p(\Omega, \mathcal{F}, \mathbf{P})$, $1 \leq p < \infty$: the space of equivalent classes of measurable functions on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that for all $X \in L^p$, $\int |X|^p d\mathbf{P} < \infty$

$ba := ba(\Omega, \mathcal{F}, \mathbf{P})$: the space of all finitely additive measures μ which are absolutely continuous to $\mathbf{P}$ and whose total variation is finite.

$X \stackrel{\lambda}{=} Y$ $\mathbf{P}$-a.s.: the random variables X and Y are $\mathbf{P}$-a.s. equal up to their λ-quantiles

$\mathcal{Q}_p$: the set of probability measures on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{Q} \ll \mathbf{P}$ and $\frac{d\mathbf{Q}}{d\mathbf{P}} \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p \leq \infty$

$q_X(\lambda)$: the λ-quantile of the random variable X

$q_X^+(\lambda)$: the upper λ-quantile of the random variable X

$q_X^-(\lambda)$: the lower λ-quantile of the random variable X

$\sigma(L^\infty, L^1)$: the weak* topology on L^∞

$X \sim Y$: the random variables X and Y have the same probability distributions

$X \stackrel{\lambda}{\sim} Y$: the random variables X and Y have the same probability distributions up to their λ-quantiles

$\mu \underset{uni}{\succcurlyeq} \nu$: the probability measure μ is uniformly preferred (second order stochastic dominant) over the probability measure ν

$\mu \underset{uni(\lambda)}{\succcurlyeq} \nu$: the probability measure μ is λ-quantile uniformly preferred over the probability measure ν

CHAPTER 1: REVIEW OF CONVEX MEASURE OF RISK AND ITS ROBUST REPRESENTATION

Artzner et al. (1999) first defined the coherent measure of risk ρ for a finite set of market scenarios both through adding axioms on ρ and through adding axioms on the acceptance set of ρ. A more generalized class of risk measures is the convex measure of risk. In this chapter, we review the axiomatic definition of the convex measure of risk and the coherent measure of risk, the relation between the axioms on ρ and the axioms on the acceptance set $\mathcal{A}_\rho$, and the robust representation of the convex measure of risk and the coherent measure of risk.

1.1 The convex measure of risk and its acceptance set

Let Ω be a fixed set of scenarios, finite or infinite. The net worth of a financial position at the end of a trading period is modeled by a random variable $X : \Omega \to \mathbb{R} \cup \{\infty\}$. The collection of the net worth of all financial positions is denoted by $\mathcal{X}$.

Definition 1.1. (measure of risk) *Let $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a mapping.*

1. *ρ is a monetary measure of risk, if it satisfies the following axioms:*
   - *Monotonicity: For any $X, Y \in \mathcal{X}$ such that $X \leq Y$, $\rho(X) \geq \rho(Y)$.*
   - *Cash invariance: For any $X \in \mathcal{X}$ and any $m \in \mathbb{R}$, $\rho(X + m) = \rho(X) - m$.*
2. *ρ is a convex measure of risk, if it is a monetary measure of risk and satisfies:*
   - *Convexity: For any $X, Y \in \mathcal{X}$ and any $\lambda \in [0, 1]$,*
     $$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y).$$
3. *ρ is a coherent measure of risk, if it is a monetary measure of risk and satisfies:*
   - *Positive Homogeneity: For $\alpha \geq 0$, $\rho(\alpha X) = \alpha\rho(X)$, for any $X \in \mathcal{X}$.*

- *Subadditivity: For any* X, $Y \in \mathcal{X}$, $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

Remark 1.2. *We have the following remarks on Definition 1.1:*

1. *The definition of the convex measure of risk was proposed independently by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002), where* ρ *was assumed to be real-valued. However, when a random variable* $X \in \mathcal{X}$ *is not bounded, we can not eliminate the possibility that* ρ *takes infinite value. For example, for* $\mathcal{X} := L^0(\Omega, \mathcal{F}, \mathbf{P})$ *with* $(\Omega, \mathcal{F}, \mathbf{P})$ *an atomless probability space, Delbaen (2002) showed that there was no finite-valued coherent measure of risk since the functional space* L^0 *is not locally convex. Thus, we require* ρ *is a mapping from* $\mathcal{X}$ *to the extended real line* $\mathbb{R} \cup \{\infty\}$.

2. *Artzner et al. (1999) interpreted the axiom of subadditivity of the coherent measure of risk as "a merger does not create extra risk", which means that the risk of the aggregate position is bounded by the sum of the individual risk limits.*

3. *For the convex measure of risk, the axiom of convexity can be explained in a similar way: "diversification should not increase risks". More precisely, the risk of a diversified position* $\lambda X + (1-\lambda)Y$ *is not larger than the weighted sum of the positions* X *and* Y*. This interpretation can be found in Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002).*

Remark 1.3. *We make some remarks on the set* $\mathcal{X}$*. In the first paper on the coherent measure of risk by Artzner et al. (1999), the set of market scenarios* Ω *was assumed to be finite. Thus, all elements in* $\mathcal{X}$ *are naturally uniformly bounded. Delbaen (2002) considered a general set* Ω *which contains finite or infinite number of scenarios. He argued that it was necessary to consider a fixed probability space* $(\Omega, \mathcal{F}, \mathbf{P})$*. He then chose* $\mathcal{X} := L^0 := L^0(\Omega, \mathcal{F}, \mathbf{P})$*, the space of all equivalence classes of measurable functions on* $(\Omega, \mathcal{F}, \mathbf{P})$*, and considered the coherent measure of risk on* L^0*. Delbaen*

(2002) pointed out that there was actually no finite-valued coherent measure of risk ρ *on* L^0 *since the space* L^0 *is not locally convex. Therefore, he defined the coherent measure of risk* ρ *as a mapping from* L^0 *to* $\mathbb{R} \cup \{\infty\}$ *which satisfied axioms of the coherent measure of risk proposed by Artzner at al. (1999). If we choose locally convex set such as* $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbf{P})$*, the space of equivalence classes of essentially bounded random variables, or* $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbf{P})$*,* $1 \leq p < \infty$*, the space of equivalence classes of integrable random variables, it is much more convenient to define and study the coherent measure of risk. When defining the convex measure of risk, Föllmer and Schied (2002) chose* $\mathcal{X} = L^\infty$ *and considered* $\rho : \mathcal{X} \to \mathbb{R}$ *as a real-valued mapping. Frettelli and Rosazza Gianin (2002) assumed the set* $\mathcal{X}$ *to be an ordered locally convex topological vector space, where* $L^p := L^p(\Omega, \mathcal{F}, \mathbf{P})$*,* $1 \leq p \leq \infty$*, are included. Kaina and Rüschendorf (2009) studied the convex measure of risk defined on* L^p *space with* $1 \leq p \leq \infty$*, i.e.,* $\rho : L^p(\Omega, \mathcal{F}, \mathbf{P}) \to \mathbb{R} \cup \{\infty\}$*.*

Definition 1.4. (acceptance set) *The acceptance set* $\mathcal{A}_\rho$ *of a monetary measure of risk* $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ *is defined as*

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \leq 0\}. \tag{1.1}$$

Definition 1.4 means that a financial position is acceptable if no additional capital is required to make its risk be nonpositive. Artzner et al. (1999) demonstrated that there was certain correspondence between the axioms on the acceptance set and the axioms on the coherent measure of risk. Similar correspondence exists for the convex measure of risk, as Föllmer and Schied (2004) showed. Before we give the details of the correspondence, we first recall that a monetary measure of risk $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is proper, if $\rho(X) < \infty$ for some $X \in \mathcal{X}$. For a proper monetary measure of risk ρ with acceptance set $\mathcal{A}_\rho$, we have the following properties:

1. $\mathcal{A}_\rho$ is non-empty.

2. $\mathcal{A}_\rho$ is monotone: if $X \in \mathcal{A}_\rho$ and $Y \in \mathcal{X}$ such that $Y \geq X$, then $Y \in \mathcal{A}_\rho$.

3. ρ is convex if and only if $\mathcal{A}_\rho$ is convex.

4. ρ is coherent if and only if $\mathcal{A}_\rho$ is a convex cone. Recall that a set S is a cone if $s \in S$ implies $\alpha s \in S$ for every $\alpha \geq 0$.

5. ρ can be recovered from $\mathcal{A}_\rho$:

$$\rho(X) = \begin{cases} \infty, & \text{if } m + X \notin \mathcal{A}_\rho, \quad \forall m \in \mathbb{R}, \\ \inf\{m \in \mathbb{R} : \, m + X \in \mathcal{A}_\rho\}, & \text{otherwise.} \end{cases}$$

Conversely, start from a non-empty set $\mathcal{A} \subset \mathcal{X}$ such that $\mathcal{A}$ is convex and monotone, and

$$\inf\{m \in \mathbb{R} : \, m + X \in \mathcal{A}\} > -\infty, \qquad \text{for all } X \in \mathcal{X},$$

we can define a convex measure of risk as

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} : \, m + X \in \mathcal{A}\}, \qquad \text{for } X \in \mathcal{X}. \tag{1.2}$$

In addition, we have:

1. If $\mathcal{A}$ is a cone, then $\rho_{\mathcal{A}}$ is a coherent measure of risk.

2. $\mathcal{A} \subset \mathcal{A}_{\rho_{\mathcal{A}}}$.

3. $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$ if for any $X \in \mathcal{A}$ and any $Y \in \mathcal{X}$, the set $\{\lambda \in [0,1] : \, \lambda X + (1-\lambda) Y \in \mathcal{A}\}$ is closed in $[0,1]$.

As examples of risk measures, we look at the Value-at-Risk, the Conditional Value-at-Risk, and the Weighted Value-at-Risk. All these risk measures are quantile-dependent. The Value-at-Risk at a fixed level $\lambda \in (0,1)$, denoted by VaR_λ, is the negative λ-quantile of the random variables. The Value-at-Risk is widely used in practice, but it is not a convex measure of risk, which means that it may penalize diversification. The Conditional Value-at-Risk at some fixed level λ, denoted by

$CVaR_\lambda$, and also known as the Average Value-at-Risk or the expected shortfall, averages the Value-at-Risk up to the level λ equally weighted. Unlike the Value-at-Risk, the $CVaR_\lambda$ is a coherent measure of risk. The Weighted Value-at-Risk $WVaR$ averages the Conditional Value-at-Risk over the interval $(0,1)$ with the weights given by a probability measure μ on $(0,1)$. We list these examples in Example 1.5.

Let us recall some quantile-related definitions. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a fixed number $\lambda \in (0,1)$, a λ-quantile of a random variable X is any real number q such that

$$\mathbf{P}(X \leq q) \geq \lambda \qquad \text{and} \qquad \mathbf{P}(X < q) \leq \lambda. \tag{1.3}$$

We use $q_X(\lambda)$ to denote a λ-quantile of X. Note that $q_X(\lambda)$ may be not unique. The lower- and upper λ-quantile of the random variable X are denoted by $q_X^-(\lambda)$ and $q_X^+(\lambda)$ respectively, and they are defined by

$$\begin{aligned} q_X^-(\lambda) &:= \sup\{x : \mathbf{P}(X < x) < \lambda\} = \inf\{x : \mathbf{P}(X \leq x) \geq \lambda\}, \\ q_X^+(\lambda) &:= \inf\{x : \mathbf{P}(X \leq x) > \lambda\} = \sup\{x : \mathbf{P}(X < x) \leq \lambda\}. \end{aligned} \tag{1.4}$$

Note that if the random variable X is (essentially) bounded, the λ-quantiles as well as the upper- and lower λ-quantiles are well defined (real-valued) for all $\lambda \in [0,1]$. In particular, $q_X(0)$ can be taken as the lower (essential) bound of X and $q_X(1)$ can be taken as the upper (essential) bound of X. If X is not bounded, we may use the notation $q_X(0) = q_X^-(0) := -\infty$ and $q_X(1) = q_X^+(1) := \infty$.

Example 1.5. *(VaR, $CVaR$, and $WVaR$)*

In the following examples, we suppose $\mathcal{X} = L^p := L^p(\Omega, \mathcal{F}, \mathbf{P})$ with $1 \leq p \leq \infty$. Moreover, we assume $\lambda \in (0,1)$ to be a fixed number.

- *The Value-at-Risk $VaR_\lambda(X)$ of a financial position X has the following definition:*

$$VaR_\lambda(X) := -q_X^+(\lambda) = q_{-X}^-(1-\lambda). \tag{1.5}$$

$VaR_\lambda(X)$ controls the probability of a loss, but does not control the size of a loss

if it occurs. $VaR_\lambda(X)$ *is a monetary measure of risk, however, it is not convex. In particular, if* $\mathcal{X} = L^\infty$, *we define*

$$VaR_0(X) := -ess\,inf X = \inf\{m \in \mathbb{R} : \mathbf{P}(X + m < 0) = 0\},$$
$$VaR_1(X) := -esssup X = \inf\{m \in \mathbb{R} : \mathbf{P}(X - m > 0) = 0\}.$$

- *The Conditional Value-at-Risk at level* λ *is defined as*

$$CVaR_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda VaR_\gamma(X)d\gamma = -\frac{1}{\lambda}\int_0^\lambda q_X^+(\gamma)d\gamma. \tag{1.6}$$

  $CVaR_\lambda(X)$ *is a coherent measure of risk. Note that if* $\mathcal{X} = L^\infty$, $CVaR_\lambda$ *is finite for all* $X \in L^\infty$ *with* $CVaR_0(X) := -ess\,inf X$. *For an unbounded* $X \in L^p$, $1 \le p < \infty$, $CVaR_0(X)$ *may be* ∞. *However,* $CVaR_1(X) = -\int_0^1 q_X(t)dt = \mathbb{E}[-X] < \infty$.

- *Let* μ *be a probability measure on* $[0, 1]$. *The Weighted Value-at-Risk of a financial position* X, *denoted by* $\rho_\mu(X)$, *has the definition*

$$\rho_\mu(X) := \int_{[0,1]} CVaR_\gamma(X)\mu(d\gamma).$$

  ρ_μ *is a coherent measure of risk. Note that* ρ_μ *can be infinted valued. We will discuss more details on the Weighted Value-at-Risk in section 1.4.*

1.2 The robust representation of the convex measure of risk

In this section, $(\Omega, \mathcal{F}, \mathbf{P})$ is a fixed probability space. For $1 \le p \le \infty$, the L^p spaces are Banach spaces whose norms are defined by

$$\|X\|_p := \begin{cases} \left(\int |X|^p d\mathbf{P}\right)^{1/p}, & \text{for } 1 \le p < \infty, \\ \text{esssup}(X) := \inf\{x : \mathbf{P}(|X| > x) = 0\}, & \text{for } p = \infty. \end{cases}$$

The L^p spaces are locally convex spaces. For $1 \le p < \infty$, the dual space of $(L^p, \|\cdot\|_p)$ is the space $(L^q, \|\cdot\|_q)$ with $q \in \mathbb{R} \cup \{\infty\}$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, where we let $q = \infty$

for $p = 1$. If $p = \infty$, the dual space of $(L^\infty, \|\cdot\|_\infty)$ is the space $ba := ba(\Omega, \mathcal{F}, \mathbf{P})$, the space of all finitely additive measures μ which are absolutely continuous to $\mathbf{P}$ and whose total variation is finite. The space ba contains not only probability measures, but also the finitely additive measures.

The weak* topology on L^∞, denoted as $\sigma(L^\infty, L^1)$, is the coarsest topology on L^∞ to make every linear functional $\ell : L^\infty \to L^1$ be continuous. Endowed with the weak* topology, the dual space of $(L^\infty, \sigma(L^\infty, L^1))$ is L^1. We refer the book of Dunford and Schwartz (1964) for more details on the L^p spaces and their dual spaces.

For $1 \le p \le \infty$, we define the following set of probability measures:

$$\mathcal{Q}_p := \left\{ \mathbf{Q} \text{ probability measure on } (\Omega, \mathcal{F}, \mathbf{P}) \, : \, \mathbf{Q} \ll \mathbf{P} \text{ and } \frac{d\mathbf{Q}}{d\mathbf{P}} \in L^q \right\}, \tag{1.7}$$

where L^q is the dual space of L^p for $1 \le p < \infty$, and for $p = \infty$, we use L^q for convenience to denote the space L^1, the dual space of $(L^\infty, \sigma(L^\infty, L^1))$.

For the discussion of the robust representation of the convex measure of risk, we first recall the definition of a lower semicontinuous function.

Definition 1.6. *A function $f : E \to [-\infty, \infty]$ on a topological space E is lower semicontinuous if the set $\{x \in E : f(x) \le \alpha\}$ is closed for all $\alpha \in \mathbb{R}$.*

We refer the book of Aliprantis and Border (2006) for more related topics on the lower semicontinuity.

The following Definition 1.7 and Theorem 1.8 are quoted from Föllmer and Schied (2004).

Definition 1.7. (Fenchel-Legendre transform) *Let E be a topological space and E' be its dual space. The Fechel-Legendre transform of a function $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$ is the function $f^* : E' \to \mathbb{R} \cup \{\infty\}$ defined by*

$$f^*(\ell) := \sup_{x \in E} (\ell(x) - f(x)). \tag{1.8}$$

For the following theorem, we recall that a function $f : E \to \mathbb{R} \cup \{\infty\}$ is proper, if there is some $x \in E$ such that $f(x) < \infty$.

Theorem 1.8. *Let f be a proper convex function on a locally convex topological space E. If f is lower semicontinuous with respect to the weak topology $\sigma(E, E')$, then $f = f^{**}$, where f^{**} is the Fenchel-Legendre transform of f^*.*

Remark 1.9. *In Theorem 1.8, the topological space E is required to be locally convex. According to the definition, a topological space E is locally convex if it has a base of convex sets. By this definition, the functional spaces L^p, $1 \leq p \leq \infty$, are locally convex, but if the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless, the space L^0 is not locally convex.*

For a convex measure of risk ρ defined on the space L^∞, Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) showed the following theorem on its robust representation:

Theorem 1.10. *Suppose $\rho : L^\infty \to \mathbb{R}$ is a convex measure of risk. Then the following statements are equivalent:*

1. *ρ is lower semicontinuous with respect to the weak* topology $\sigma(L^\infty, L^1)$.*

2. *ρ admits the following robust representation:*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_1} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})), \qquad \text{for all } X \in L^\infty, \tag{1.9}$$

   where

$$\rho^*(\mathbf{Q}) := \sup_{X \in L^\infty} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X) \tag{1.10}$$

   is the Fenchel-Legendre transform of ρ.

3. *ρ is continuous from above: If $X_n \searrow X$ $\mathbf{P}$-a.s., then $\rho(X_n) \nearrow \rho(X)$.*

4. *ρ has the Fatou property: For any bounded sequence (X_n) which converges $\mathbf{P}$-a.s. to some X,*

$$\rho(X) \leq \liminf_{n \to \infty} \rho(X_n).$$

For the convex measure of risk defined on L^p, $1 \le p < \infty$, Kaina and Rüschendorf showed the following theorem on its representation by using the extended Namioka-Klee theorem proven by Biagina and Frettelli (2009):

Theorem 1.11. *Suppose $\rho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p < \infty$, is a proper convex measure of risk. Then the following statements are equivalent:*

1. *ρ is lower semicontinuous with respect to the weak topology $\sigma(L^p, L^q)$.*

2. *ρ has the following robust representation:*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})), \qquad \text{for all } X \in L^p, \tag{1.11}$$

   where

$$\rho^*(\mathbf{Q}) := \sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)) \tag{1.12}$$

   is the Fenchel-Legendre transform of ρ.

3. *ρ is continuous from above: If $X_n \searrow X$ $\mathbf{P}$-a.s., then $\rho(X_n) \nearrow \rho(X)$.*

4. *ρ has the Fatou property: For any sequence (X_n) such that for some $Y \in L^p$, $|X_n| \le Y$ $\mathbf{P}$-a.s., if X_n converges to X $\mathbf{P}$-a.s. for some $X \in L^p$, then*

$$\rho(X) \le \liminf_{n \to \infty} \rho(X_n).$$

Remark 1.12.

1. *The main difference of Theorem 1.10 and Theorem 1.11 lies in the definition of the Fatou property. As already mentioned in the section of Introduction, random variables in L^∞ are essentially bounded, so sequences $(X_n) \subset L^\infty$ uniformly bounded by a constant are considered to define the Fatou property, as what Artzner et al.(1999) did. However, random variables in L^p are most likely not essential bounded when $p \neq \infty$, to define the Fatou property, Biagini and Frittelli (2009) require the sequence $(X_n) \subset L^p$ to be dominated by some random*

variable. This dominance allows us to use the Dominated Convergence Theorem when prove the Theorem.

2. *A similar version to Theorem 1.10 for the coherent measure of risk was shown by Artzner et al. (1999). Since the coherent measure of risk is a subclass of the convex measure of risk, Theorem 1.10 can be applied to the coherent measure of risk. In particular, Föllmer and Schied (2002) showed that if a coherent measure of risk* $\rho : L^\infty \to \mathbb{R}$ *can be represented by (1.9), then for each* $\mathbf{Q} \in \mathcal{Q}_1$*, either* $\rho^*(\mathbf{Q}) = 0$ *or* $\rho^*(\mathbf{Q}) = \infty$*. Therefore, if we define*

$$\mathcal{Q}_{\max} := \{\mathbf{Q} \in \mathcal{Q}_1 : \rho^*(\mathbf{Q}) = 0\}, \tag{1.13}$$

then the coherent measure of risk ρ *can be represented as*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_{\max}} \mathbb{E}_{\mathbf{Q}}[-X], \qquad \textit{for } X \in L^\infty. \tag{1.14}$$

This representation coincides to the one proposed by Artzner et al. (1999). Moreover, as shown by Kaina and Rüschendorf (2009), these results remain true if the coherent measure of risk is defined on the L^p *space, in which case,* $\mathcal{Q}_{\max}$ *is defined as*

$$\mathcal{Q}_{\max} := \{\mathbf{Q} \in \mathcal{Q}_p : \rho^*(\mathbf{Q}) = 0\}. \tag{1.15}$$

3. *The equivalence of statement 1 and statement 2 in Theorem 1.10 and Theorem 1.11 is a direct consequence of Theorem 1.8, where* ρ^* *is the Fenchel-Legendre transform of* ρ*. This approach of the proof was first stated by Frittelli and Rosazza Gianin (2002). When Föllmer and Schied (2002) proved Theorem 1.10, they used the Hahn-Banach theorem. Föllmer and Schied called the function* ρ^* *the "penalty function" and showed an alternative form of* ρ^**, namely*

$$\rho^*(\mathbf{Q}) = \alpha_{\min}(\mathbf{Q}) := \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X], \qquad \textit{for } \mathbf{Q} \in \mathcal{Q}_1.$$

Föllmer and Schied (2002) demonstrated that if $\alpha(\mathbf{Q})$ is a penalty function, then it must be true that $\alpha(\mathbf{Q}) \geq \alpha_{\min}(\mathbf{Q})$ for all $\mathbf{Q} \in \mathcal{Q}_1$. This means that $\alpha_{\min}(\mathbf{Q})$ is the minimal penalty function of ρ.

1.3 The law invariant risk measure and its robust representation

Throughout this section, we consider the real-valued monetary measure of risk $\rho : L^\infty \to \mathbb{R}$ defined on the space L^∞. We further assume that the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless in the sense of the following definition:

Definition 1.13. (atomless probability space) *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. An atom of the probability measure $\mathbf{P}$ is some set $A \in \mathcal{F}$ such that $\mathbf{P}(A) > 0$ and for any $B \in \mathcal{F}$ and $B \subset A$, either $\mathbf{P}(B) = 0$ or $\mathbf{P}(B) = \mathbf{P}(A)$. A probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless if it contains no atoms.*

The study of the law invariant risk measure was mainly contributed by Kusuoka (2001), where he defined the law invariant risk measure and proposed the robust representation for the class of the law invariant coherent measure of risk. The following definition is due to Kusuoka (2001).

Definition 1.14. (law invariant risk measure) *A monetary measure of risk $\rho : L^\infty \to \mathbb{R}$ is law invariant, if $\rho(X) = \rho(Y)$ whenever X and Y have the same probability distribution under $\mathbf{P}$.*

Let $\rho : L^\infty \to \mathbb{R}$ be a convex measure of risk which is law invariant. If ρ has the Fatou property, then Theorem 1.10 ensures that ρ admits the robust representation (1.9). In addition, the law invariance property insures the following representation:

Theorem 1.15. *Let $\rho : L^\infty \to \mathbb{R}$ be a convex measure of risk that has the Fatou property formulated in Theorem 1.10. Then ρ is law invariant if and only if it can be represented as*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}1} \left(\int_0^1 q_X(t) q_{-\varphi_{\mathbf{Q}}}(t) dt - \rho^*(\mathbf{Q}) \right), \tag{1.16}$$

where $\varphi_{\mathbf{Q}} := \frac{d\mathbf{Q}}{d\mathbf{P}}$, *and*

$$\begin{aligned} \rho^*(\mathbf{Q}) &= \sup_{X \in L_1} \left(\int_0^1 q_X(t) q_{-\varphi_{\mathbf{Q}}}(t) dt - \rho(X) \right) \\ &= \sup_{X \in \mathcal{A}_\rho} \int_0^1 q_X(t) q_{-\varphi_{\mathbf{Q}}}(t) dt. \end{aligned} \tag{1.17}$$

Here we recall the set

$$\mathcal{Q}_1 = \{\mathbf{Q} \text{ probability measure on } (\Omega, \mathcal{F}, \mathbf{P}) : \mathbf{Q} \ll \mathbf{P}, \frac{d\mathbf{Q}}{d\mathbf{P}} \in L^1\},$$

and $\mathcal{A}_\rho = \{X \in L^\infty : \rho(X) \leq 0\}$ *is the acceptance set of* ρ.

Theorem 1.15 can be found as Theorem 4.54 of Föllmer and Schied (2004), it generalizes Lemma 10 of Kusuoka (2001) for the coherent measure of risk defined on L^∞. (1.16) and (1.17) reflects the "law invariance" of ρ, namely, ρ depends on the random variable X and the Radon-Nikodým derivatives $\frac{d\mathbf{Q}}{d\mathbf{P}}$ only through their laws.

Remark 1.16. *In Theorem 1.15, the Fatou property is a sufficient and necessary condition which leads* ρ *to the representation (1.16). For a law invariant convex measure of risk* $\rho : L^\infty \to \mathbb{R}$ *defined on* L^∞, *Jouini et al. (2006) showed that* ρ *has automatically the Fatou property. Therefore, in Theorem 1.15 we can eliminate the condition that* ρ *has the Fatou property, and the conclusions remain true.*

For a law invariant coherent risk measure defined on L^∞, Kusuoka (2001) proposed another representation through the Weighted Value-at-Risk ρ_μ introduced in section 1.1:

$$\rho_\mu(X) = \int_{(0,1]} CVaR_\gamma(X)\mu(\gamma).$$

with μ a probability measure on $(0, 1]$. Föllmer and Schied (2004) generalized this representation for the convex measure of risk $\rho : L^\infty \to \mathbb{R}$ that is law invariant. The following theorem is quoted from Theorem 4.57 of Föllmer and Schid (2004).

Theorem 1.17. *A convex measure of risk* $\rho : L^\infty \to \mathbb{R}$ *is law invariant if and only* *if ρ has the following representation:*

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left(\int_{(0,1]} CVaR_\gamma(X)\mu(d\gamma) - \beta_{\min}(\mu) \right),$$

where $\mathcal{M}_1((0,1])$ indicates the set of all probability measures on $(0,1]$, and

$$\beta_{\min} = \sup_{X \in \mathcal{A}_\rho} \int_{(0,1]} CVaR_\gamma(X)\mu(d\gamma).$$

In particular, ρ is law invariant and coherent if and only if there is some set of probability measures on $(0,1]$, denoted by $\mathcal{M}_0((0,1])$, such that

$$\rho(X) = \sup_{\mu \in \mathcal{M}_0((0,1])} \int_{(0,1]} CVaR_\gamma(X)\mu(d\gamma).$$

The proof of the theorem can be found in Kusuoka (2001) for the coherent case and in Föllmer and Schied (2004) for the convex case. Note that we do not need to assume ρ has the Fatou property since it is implied by the law invariance of ρ. For the Weighted Value-at-Risk ρ_μ, we will take a closer look in the next section.

1.4 The Weighted Value-at-Risk and its representation

In this section, we assume that the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless. The Weighted Value-at-Risk $\rho_\mu : L^\infty \to \mathbb{R}$ is defined on the functional space L^∞ and has the form of

$$\rho_\mu(X) := \int_{[0,1]} CVaR_\gamma(X)\mu(d\gamma), \qquad \text{for } X \in L^\infty, \tag{1.18}$$

where μ is a probability measure on $[0,1]$ and $CVaR_\gamma$ is defined by (1.6): $CVaR_\gamma(X) = -\frac{1}{\gamma}\int_0^\gamma q_X(t)dt$, for $\gamma \in [0,1]$ and $X \in L^\infty$. In particular, $CVaR_0(X) := -\liminf X$ and $CVaR_1(X) := -\limsup X$. The Weighted Value-at-Risk ρ_μ is a coherent measure of risk.

Substituting $CVaR_\gamma(X)$ into (1.18) and applying the Fubini's Theorem, we yield

$$\begin{aligned}\rho_\mu(X) &= \mu(\{0\})CVaR_0(X) + \int_{(0,1]} -\frac{1}{\gamma}\int_0^\gamma q_X(t)dt\mu(d\gamma)\\ &= \mu(\{0\})CVaR_0(X) - \int_{(0,\gamma]} q_X(t)\int_{(t,1]}\frac{1}{\gamma}\mu(d\gamma)dt.\end{aligned}$$

Define

$$\phi(t) := \int_{(t,1]}\frac{1}{\gamma}\mu(d\gamma), \qquad \text{for } 0 < t < 1, \tag{1.19}$$

then we obtain an alternative form for ρ_μ:

$$\rho_\mu(X) = \mu(\{0\})CVaR_0(X) - \int_{(0,1]} q_X(t)\phi(t)dt. \tag{1.20}$$

As pointed out by Föllmer and Schied (2004), equation (1.19) defines a one-to-one correspondence between the probability measures μ on $(0,1]$ and the increasing concave functions $\Psi : [0,1] \to [0,1]$, and the function Ψ satisfies $\Psi'(t+) = \phi(t)$, $\Psi(0) = 0$, $\Psi(0+) = \mu(0)$, $\Psi(1) = 1$.

Equation (1.20) is a slightly modified version of Kusuoka (2001). In addition, Föllmer and Schied (2004) showed the following equivalent form of ρ_μ:

$$\rho_\mu(X) = \int_{-\infty}^0 (\Psi(\mathbf{P}(X > x)) - 1)dx + \int_0^\infty \Psi(\mathbf{P}(X > x))dx, \qquad \text{for } X \in L^\infty. \tag{1.21}$$

The right hand side of (1.21) is called the Choquet integral. More precisely, we have the following definition:

Definition 1.18. (Choquet integral) *Let $c : \mathcal{F} \to [0,1]$ be any set function which is normalized and monotone. The Choquet integral of a bounded measurable function X on $(\Omega, \mathcal{F})$ with respect to c is defined as*

$$\int X\, dc := \int_{-\infty}^0 (c(X > x) - 1)dx + \int_0^\infty c(X > x)dx. \tag{1.22}$$

The Choquet integral $\int_\Omega X dc$ was originally defined by Choquet (1954) for a bounded, non-negative and $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$ with respect to a not necessarily additive set function $c : \mathcal{F} \to \mathbb{R}$. Schmeidler (1986) extended Choquet's definition by eliminating the non-negativeness of X. We call (1.21) the Choquet integral by following Carlier and Dana (2003), though the elements in L^∞ are only $\mathbf{P}$-almost surely bounded. In addition, the function $\Psi \circ \mathbf{P}$ appeared in (1.21) is called the concave distortion.

The robust representation of ρ_μ is given by Corollary 4.74 of Föllmer and Schied (2004), by using the uniform preference of two probability measures (also known as the second order stochastic dominance).

Definition 1.19. (uniform preference) *Let μ, ν be two probability measures. μ is uniformly preferred over ν, written as $\mu \underset{uni}{\succcurlyeq} \nu$, if for all utility functions u, it is true that*

$$\int u d\mu \geq \int u d\nu.$$

Note that a utility function u is a function $u : \mathbb{R} \to \mathbb{R}$ which is strictly concave and strictly increasing.

Föllmer and Schied (2004) showed that ρ_μ has the following robust representation:

$$\rho_\mu(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_\mu} \mathbb{E}_{\mathbf{Q}}[-X], \tag{1.23}$$

where

$$\mathcal{Q}_\mu := \left\{ \mathbf{Q} \in \mathcal{Q}_p : \mathbf{P} \circ \left(\frac{d\mathbf{Q}}{d\mathbf{P}} \right)^{-1} \underset{uni}{\succcurlyeq} \mathcal{L} \circ (\phi)^{-1} \right\}, \tag{1.24}$$

and $\mathcal{L}$ denotes the Lebesgue measure. Note that the supremum in (1.23) can be attained if and only if $\mu(\{0\}) = 0$, and in this case, an "optimal" measure $\mathbf{Q}_X$ has the density $\frac{d\mathbf{Q}_X}{d\mathbf{P}} =: f(X)$ given by

$$f(x) = \begin{cases} \Psi'(F_X(x)), & \text{if x is a continuous point of } F_X, \\ \frac{1}{F_X(x) - F_X(x-)} \int_{F_X(x-)}^{F_X(x)} \Psi'(t)dt, & \text{otherwise.} \end{cases}$$

CHAPTER 2: λ-QUANTILE DEPENDENT CONVEX RISK MEASURE AND ITS ROBUST REPRESENTATION

In Chapter 1, as examples, we looked at risk measures including the Value-at-Risk and the Conditional Value-at-Risk. Both risk measures depend on the random variables only up to some pre-determined level. This level, also called the significance level and denoted by λ, is some real number between 0 and 1. When the value of λ is fixed, the value of the financial positions beyond λ are irrelevant to the value of VaR_λ and $CVaR_\lambda$. We call these kind of risk measures the λ-quantile dependent risk measures. In this chapter, we give the mathematical definition of the λ-quantile dependent convex risk measure and propose its robust representation.

2.1 The λ-quantile dependent convex risk measure

The idea behind the λ-quantile dependent convex risk measure is that the value of the convex risk measure only depends on the tail behavior of the random variables. More precisely, if two random variable are $\mathbf{P}$-a.s. equal up to some fixed level λ, then the value of the convex risk measures must be same. We first define the λ-quantile $\mathbf{P}$-a.s. equality of random variables.

Definition 2.1. (λ-quantile $\mathbf{P}$-a.s. equality of two random variables) *Fix $\lambda \in (0,1)$. Two random variables X, Y on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ are $\mathbf{P}$-a.s. equal up to their λ-quantiles if there exists some set $A \in \mathcal{F}$ and some λ-quantiles $q_X(\lambda)$, $q_Y(\lambda)$ of X and Y respectively, such that the following conditions are satisfied:*

$$\begin{aligned} &\{X < q_X(\lambda)\} \cup \{Y < q_Y(\lambda)\} \subset A \quad a.e., \\ &A \subset \{X \le q_X(\lambda)\} \cap \{Y \le q_Y(\lambda)\} \quad a.e., \\ &\mathbf{P}(A) \ge \lambda, \quad and \quad X1_A = Y1_A \quad \mathbf{P}-\ a.s. \end{aligned} \tag{2.1}$$

If X and Y are $\mathbf{P}$-a.s. equal up to their λ-quantiles, we denote it as $X \overset{\lambda}{=} Y$ $\mathbf{P}$-a.s.

As mentioned earlier, for fixed $\lambda \in (0,1)$, if two random variables are indistinguishable up to their λ-quantiles, they must have the same value of the risk measure if the convex risk measure is λ-quantile dependent. We give the formal definition as the following:

Definition 2.2. (λ-quantile dependent convex risk measure) *Fix $\lambda \in (0,1)$. A convex measure of risk $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ is λ-quantile dependent if for any $X, Y \in L^p$ the following is true:*

$$X \overset{\lambda}{=} Y \quad \mathbf{P}- \ \ a.s. \quad implies \quad \rho(X) = \rho(Y). \tag{2.2}$$

The definition postulates the value of the risk measure ρ depends on the random variables only up to a given significance level λ.

If $X \overset{\lambda}{=} Y$ $\mathbf{P}$-a.s., then the set A which satisfies (2.1) exists. However, the choices of A, $q_X(\lambda)$, and $q_Y(\lambda)$ in Definition 2.1 are not unique. The following Lemma 2.4 and Lemma 2.5 discuss the structure of these choices which lead to an equivalent definition to Definition 2.1, given in Definition 2.3. Further discussions of λ-quantile equal random variables for the cases of atomless probability space, and continuously distributed random variables are given by Proposition 2.7 and Proposition 2.8. Example 2.11 contains computational examples for discretely distributed random variables.

We define the following random variables which are useful in this section and the rest of the chapters:

$$\begin{aligned} X_q &:= X 1_{\{X \le q_X(\lambda)\}} + q_X(\lambda) 1_{\{X > q_X(\lambda)\}}, \\ Y_q &:= Y 1_{\{Y \le q_Y(\lambda)\}} + q_Y(\lambda) 1_{\{Y > q_Y(\lambda)\}}, \end{aligned} \tag{2.3}$$

where $q_X(\lambda)$ is a λ-quantile of random variable X and $q_Y(\lambda)$ is a λ-quantile of random variable Y. We also define random variables

$$\begin{aligned} X_{q_\lambda} &:= X1_{\{X\le q_\lambda\}} + q_\lambda 1_{\{X>q_\lambda\}}, \\ Y_{q_\lambda} &:= Y1_{\{Y\le q_\lambda\}} + q_\lambda 1_{\{Y>q_\lambda\}}, \end{aligned} \tag{2.4}$$

provided that $q_\lambda \in \mathbb{R}$ is some common λ-quantile of X and Y.

Definition 2.3. *Fix $\lambda \in (0,1)$. Two random variables X, Y on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ are $\mathbf{P}$-a.s. equal up to their λ-quantiles if there is some $q_\lambda \in \mathbb{R}$ such that q_λ is a λ-quantile of X and Y, and*

$$\mathbf{P}(\{X \le q_\lambda\} \cap \{Y \le q_\lambda\}) \ge \lambda, \quad \textit{and} \quad X_{q_\lambda} = Y_{q_\lambda}\,\mathbf{P} - a.s. \tag{2.5}$$

To show the equivalence between Definition 2.1 and Definition 2.3, we use the following two lemmas:

Lemma 2.4. *Fix $\lambda \in (0,1)$. Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $X \overset{\lambda}{=} Y$ $\mathbf{P}$-a.s. Denote $q_X := q_X(\lambda)$ and $q_Y := q_Y(\lambda)$. Then one of the following cases must be true for the sets described in (2.1):*

Case 1: $\{X < q_X\} = \{Y < q_Y\} \subsetneq A$ *a.e., and* $\mathbf{P}(A) \ge \lambda$.

Case 2: $\{X < q_X\} = \{Y < q_Y\} = A$ *a.e., and* $\mathbf{P}(A) = \lambda$.

Case 3: *Either* $\{X < q_X\} \subsetneq \{Y < q_Y\} = A$ *or* $\{Y < q_Y\} \subsetneq \{X < q_X\} = A$ *a.e., and* $\mathbf{P}(A) = \lambda$.

PROOF. We use Figure 2.1 to illustrate the partition of the probability space Ω. Since $X \overset{\lambda}{=} Y$, there exists set $A \in \mathcal{F}$ satisfying (2.1), where $X1_A = Y1_A$ $\mathbf{P}$-a.s. Thus,

$$X = Y\,\mathbf{P}{-}a.s \quad \text{on both sets } \{X < q_X\} \cup \{Y < q_Y\} \text{ and } A \backslash (\{X < q_X\} \cup \{Y < q_Y\}). \tag{2.6}$$

Notice that

$$X = q_X \text{ and } Y = q_Y\,\mathbf{P} - a.s. \quad \text{on the set } A \backslash (\{X < q_X\} \cup \{Y < q_Y\}). \tag{2.7}$$

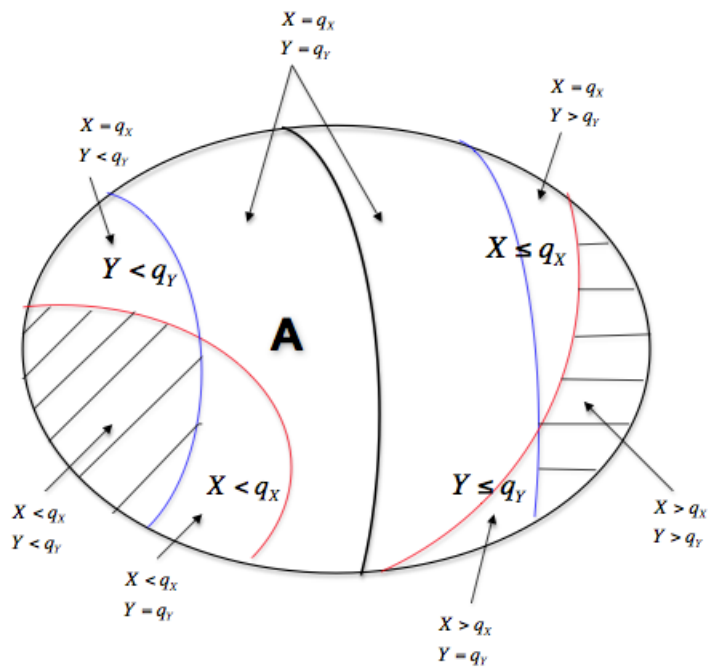

Figure 2.1: Partition of the probability space Ω

First, we assume

$$\mathbf{P}(A\backslash(\{X < q_X\} \cup \{Y < q_Y\})) > 0.$$

Then (2.6) and (2.7) imply $q_X = q_Y$. Consequently, from (2.6), we conclude

$$\mathbf{P}(\{X = q_X\} \cap \{Y < q_Y\}) = 0.$$

Similar argument yields

$$\mathbf{P}(\{X < q_X\} \cap \{Y = q_Y\}) = 0.$$

Hence, we obtain **Case 1** in the Lemma:

$$\{X < q_X\} = \{Y < q_Y\} \subsetneq A \quad a.e.$$

Second, assume

$$\mathbf{P}(A\backslash(\{X < q_X\} \cup \{Y < q_Y\})) = 0.$$

For the sets $\{X = q_X\} \cap \{Y < q_Y\}$ and $\{X < q_X\} \cap \{Y = q_Y\}$, we have the following possible cases:

- $\mathbf{P}(\{X = q_X\} \cap \{Y < q_Y\}) = \mathbf{P}(\{X < q_X\} \cap \{Y = q_Y\}) = 0$. In this case $A = \{X < q_X\} = \{Y < q_Y\}$ a.e., which is **Case 2** of the lemma. In addition, we note that $\lambda \leq \mathbf{P}(A) = \mathbf{P}(\{X < q_X\}) = \mathbf{P}(\{Y < q_Y\}) \leq \lambda$, which implies $\mathbf{P}(A) = \lambda$.

- One of the sets $\{X = q_X\} \cap \{Y < q_Y\}$ and $\{X < q_X\} \cap \{Y = q_Y\}$ has positive probability, and the other set has 0 probability. Without loss of generality, assume $\mathbf{P}(\{X = q_X\} \cap \{Y < q_Y\}) > 0$ and $\mathbf{P}(\{X < q_X\} \cap \{Y = q_Y\}) = 0$. Thus **Case 3** of the lemma is obtained. And again, we have $\lambda \leq \mathbf{P}(A) = \mathbf{P}(Y < q_Y) \leq \lambda$.

- Both sets $\{X = q_X\} \cap \{Y < q_Y\}$ and $\{X < q_X\} \cap \{Y = q_Y\}$ have positive probability. Then on the set $\{X = q_X\} \cap \{Y < q_Y\}$, $q_X = X = Y < q_Y$ **P**-a.s. On the other hand, on the set $\{X < q_X\} \cap \{Y = q_Y\}$, $q_Y = Y = X < q_X$. A contradiction occurs. $\diamond$

In Lemma 2.4, the quantiles q_X and q_Y are not necessarily same. The following lemma shows that q_X and q_Y can be chosen as a common number, and **Case 1** and **Case 3** can be combined.

Lemma 2.5. *Under the same assumptions as of Lemma 2.4, the λ-quantiles $q_X(\lambda)$, $q_Y(\lambda)$ of X and Y can be chosen as the same number $q_\lambda := q_X(\lambda) = q_Y(\lambda)$ and the sets $\{X < q_\lambda\}$ and $\{Y < q_\lambda\}$ described in (2.1) are almost everywhere equal, i.e., $\{X < q_\lambda\} = \{Y < q_\lambda\}$ a.e. In particular, they can be chosen as one of the following cases:*

Case 1': $\{X < q_\lambda\} = \{Y < q_\lambda\} \subsetneq A$ *a.e. and* $\mathbf{P}(A) \geq \lambda$.

Case 2': $\{X < q_\lambda\} = \{Y < q_\lambda\} = A$ *a.e. and* $\mathbf{P}(A) = \lambda$.

PROOF. We discuss each of the **Case 1** through **Case 3**. In particular, **Case 1** and **Case 3** can be combined into **Case 1'**, and **Case 2** can be rewritten into **Case 2'**.

Case 1: We already obtained $q_X = q_Y$ in the proof of this case in Lemma 2.4, where $q_X := q_X(\lambda)$ and $q_Y := q_Y(\lambda)$. Take $q_\lambda = q_X = q_Y$, then $\{X < q_\lambda\} = \{Y < q_\lambda\} \subsetneq A$ and $\mathbf{P}(A) \geq \lambda$.

Case 2: Without loss of generality, suppose $q_X < q_Y$. Then

$$A = \{Y < q_X\} \cup \{q_X \leq Y < q_Y\},$$

and these two subsets are disjoint. Since $X 1_A = Y 1_A$ $\mathbf{P}$-a.s., on the set $\{q_X \leq Y < q_Y\}$, we have $X = Y \geq q_X$ $\mathbf{P}$-a.s. Since $\{q_X \leq Y < q_Y\} \subset A = \{X < q_X\}$, we conclude that $\mathbf{P}(\{q_X \leq Y < q_Y\}) = 0$. Therefore,

$$A = \{Y < q_X\} \text{ a.e.} \quad \text{and} \quad \mathbf{P}(\{Y < q_X\}) = \mathbf{P}(A) = \lambda,$$

which implies that q_X is a λ-quantile of Y. Take $q_\lambda := q_X$, A can be chosen as $A = \{X < q_\lambda\} = \{Y < q_\lambda\}$ a.e., and $\mathbf{P}(A) = \lambda$.

Case 3: We discuss the case of $\{X < q_X\} \subsetneq \{Y < q_Y\} = A$ a.e., the proof for the other case $\{Y < q_Y\} \subsetneq \{X < q_X\} = A$ will be in a similar way.

As a first step, we show that q_X is a λ-quantile of Y. From (2.1), $A \subset \{X \leq q_X\}$ a.e., this implies

$$X = q_X \ \mathbf{P}- \text{ a.s. on the set } A \backslash \{X < q_X\}. \tag{2.8}$$

Together with $X 1_A = Y 1_A$ $\mathbf{P}$-a.s.,

$$X = Y = q_X \ \mathbf{P} - \text{a.s. on the set } A \backslash \{X < q_X\}. \tag{2.9}$$

(2.9) and the condition $A = \{Y < q_Y\}$ imply

$$Y = q_X < q_Y \ \mathbf{P} - \text{a.s. on } A \backslash \{X < q_X\}.$$

Thus, $\{Y < q_X\} \subset \{Y < q_Y\} = A$, which implies $\mathbf{P}(\{Y < q_X\}) \leq \lambda$. On the other hand,

$$Y = q_X \text{ on the set } A \backslash \{X < q_X\},$$

and

$$Y = X < q_X \text{ on the set } \{X < q_X\},$$

imply $A \subset \{Y \leq q_X\}$. Hence, $\mathbf{P}(Y \leq q_X) \geq \mathbf{P}(A) = \lambda$. Therefore, q_X is a λ-quantile of Y.

As a second step, we show that $\{X < q_X\} = \{Y < q_X\}$. This is the direct consequence of

$$\begin{aligned}
&\{Y < q_X\} \subset A \quad \text{and} \quad \{X < q_X\} \subset A, \\
&X = Y < q_X \, \mathbf{P} - a.s. \text{ on the set } \{X < q_X\}, \\
&X = Y = q_X \, \mathbf{P} - a.s. \text{ on the set } A \backslash \{X < q_X\}.
\end{aligned}$$

Take $q_\lambda := q_X$, we have $\{X < q_\lambda\} = \{Y < q_\lambda\} \subsetneq A$ a.e. and $\mathbf{P}(A) = \lambda$. ◇

We now show the equivalence of Definition 2.1 and Definition 2.3.

Proposition 2.6. *Definition 2.3 is equivalent to Definition 2.1.*

PROOF. " $\Leftarrow$ ": If $X \stackrel{\lambda}{=} Y$ **P**-a.s. in the sense of Definition 2.1, then due to Lemma 2.5, there is some $q_\lambda \in \mathbb{R}$ such that either **Case 1'** or **Case 2'** is true. In summary, $X = Y$ **P**-a.s. on the set $\{X < q_\lambda\} = \{Y < q_\lambda\}$. This implies $X_{q_\lambda} = Y_{q_\lambda}$ **P**-a.s.

" $\Rightarrow$ ": Suppose for X and Y there exists some q_λ satisfying the conditions expressed in Definition 2.3. We show the existence of set A that satisfies the conditions given by (2.1). First, $X_{q_\lambda} = Y_{q_\lambda}$ **P**-a.s. implies $\{X < q_\lambda\} = \{Y < q_\lambda\}$ a.e. If this is not the case, let

$$\begin{aligned}
B_X &:= \{\omega : \omega \in \{X < q_\lambda\} \text{ and } \omega \notin \{Y < q_\lambda\}\}, \\
B_Y &:= \{\omega : \omega \notin \{X < q_\lambda\} \text{ and } \omega \in \{Y < q_\lambda\}\},
\end{aligned}$$

then either B_X or B_Y has positive probability. Without loss of generality, suppose

$\mathbf{P}(B_X) > 0$. This implies that $Y(\omega) \geq q_\lambda$ while $X(\omega) < q_\lambda$ for all $\omega \in B_X$. This further implies for all $\omega \in B_X$, $X_{q_\lambda}(\omega) = X(\omega) < q_\lambda$ and $Y_{q_\lambda}(\omega) = q_\lambda$, which is a contradiction to the fact that $X_{q_\lambda} = Y_{q_\lambda}$ $\mathbf{P}$-a.s. Take $A := \{X \leq q_\lambda\} \cap \{Y \leq q_\lambda\}$, then A satisfies (2.1). Thus, $X \stackrel{\lambda}{=} Y$ $\mathbf{P}$-a.s. in the sense of Definition 2.1. $\diamond$

The following proposition discusses the atomless probability space case based on the results of Lemma 2.4.

Proposition 2.7. *Fix $\lambda \in (0, 1)$. Suppose X and Y are two random variables on $(\Omega, F, \mathbf{P})$ such that $X \stackrel{\lambda}{=} Y$ $\mathbf{P}$-a.s. If the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless, then the set A which satisfies (2.1) can be chosen such that $\mathbf{P}(A) = \lambda$.*

Proof. If **Case 2** or **Case 3** in Lemma 2.4 arise, the proof of the lemma shows that the set A satisfying (2.1) must satisfy $\mathbf{P}(A) = \lambda$. What remains is **Case 1** in Lemma 2.4. If $\{X < q_X\} = \{Y < q_Y\} \subsetneq A$ a.e. occurs, where $\mathbf{P}(A) \geq \lambda$ and $\mathbf{P}(\{X < q_X\}) = \mathbf{P}(\{Y < q_Y\}) \leq \lambda$, then we can always choose some subset B such that $\{X < q_X\} = \{Y < q_Y\} \subset B \subset A$ and $\mathbf{P}(B) = \lambda$ due to the atomlessness of the probability space. Moreover, the set B satisfies condition (2.1). $\diamond$

Next, we discuss the continuous distribution case.

Proposition 2.8. *Fix $\lambda \in (0, 1)$. Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. If both X and Y have continuous probability distribution, then*

$$X \stackrel{\lambda}{=} Y \quad \mathbf{P}-\ a.s. \quad \Leftrightarrow \quad X1_{\{X \leq q_X(\lambda)\}} = Y1_{\{Y \leq q_Y(\lambda)\}} \quad \mathbf{P}-\ a.s., \tag{2.10}$$

where $q_X(\lambda)$ and $q_Y(\lambda)$ are any λ-quantiles of X and Y respectively.

Proof. " $\Rightarrow$ ": Suppose $X \stackrel{\lambda}{=} Y$ $\mathbf{P}$-a.s. Then there is a set A which satisfies conditions given by (2.1). Therefore, we have

$$\{X < q_X(\lambda)\} \subset A \subset \{X \leq q_X(\lambda)\} \Rightarrow A = \{X < q_X(\lambda)\} = \{X \leq q_X(\lambda)\} \quad a.e.$$

Similarly, $A = \{Y < q_Y(\lambda)\} = \{Y \leq q_Y(\lambda)\}$ a.e. By (2.1), $X1_A = Y1_A$ $\mathbf{P}$-a.s. implies $X1_{\{X \leq q_X(\lambda)\}} = Y1_{\{Y \leq q_Y(\lambda)\}}$ $\mathbf{P}$-a.s.

" $\Leftarrow$ ": Suppose $X1_{\{X\leq q_X(\lambda)\}} = Y1_{\{Y\leq q_Y(\lambda)\}}$ **P**-a.s. If we can show

$$\{X < q_X(\lambda)\} = \{Y < q_Y(\lambda)\} = \{X \leq q_X(\lambda)\} = \{Y \leq q_Y(\lambda)\} \quad a.e., \tag{2.11}$$

then A can be chosen as any of the four sets and $X \overset{\lambda}{=} Y$ is proven. To be this end, we use contradiction. Suppose $\mathbf{P}(\{X \leq q_X(\lambda)\} \cap \{Y > q_Y(\lambda)\}) > 0$. Due to the condition of $X1_{\{X\leq q_X(\lambda)\}} = Y1_{\{Y\leq q_Y(\lambda)\}}$ **P**-a.s., we obtain $X = 0$ a.e. on the set $\{X \leq q_X(\lambda)\} \cap \{Y > q_Y(\lambda)\}$, which is a contradiction to the fact that X has continuous probability distribution. Thus, $\mathbf{P}(\{X \leq q_X(\lambda)\} \cap \{Y > q_Y(\lambda)\}) = 0$. Similarly, $\mathbf{P}(\{Y \leq q_Y(\lambda)\} \cap \{X > q_X(\lambda)\}) = 0$. Thus, we can conclude that $\{X \leq q_X(\lambda)\} = \{Y \leq q_Y(\lambda)\}$ a.e. Due to the continuity of the distributions of X and Y, we obtain (2.11). ◇

We give an example which we will frequently use in the sequel.

Example 2.9. *For $X \in L^p$, $1 \leq p \leq \infty$, we recall the definition of X_q as (2.3), i.e.,*

$$X_q := X1_{\{X\leq q_X(\lambda)\}} + q_X(\lambda)1_{\{X>q_X(\lambda)\}}, \tag{2.12}$$

*where $q_X(\lambda)$ is some λ-quantile of X. We check that $X \overset{\lambda}{=} X_q$ **P**-a.s. by the definitions.*

- *Choose $A = \{X \leq q_X(\lambda)\}$, then by Definition 2.1, $X \overset{\lambda}{=} X_q$ **P**-a.s.*
- *Let $Y := X_q = X1_{\{X\leq q_X(\lambda)\}} + q_X(\lambda)1_{\{X>q_X(\lambda)\}}$. Take $q_\lambda := q_X(\lambda)$, we then conclude $X \overset{\lambda}{=} X_q$ by Definition 2.3.*

*If ρ is λ-quantile dependent, then we conclude $\rho(X) = \rho(X_q)$ **P**-a.s. according to Definition 2.2.*

Remark 2.10. *In general, when we check whether two random variables are λ-quantile **P**-a.s. equal, we can use Definition 2.1 or Definition 2.3. If the random variables have continuous probability distribution, then we can also check whether the condition*

$$X1_{\{X\leq q_X(\lambda)\}} = Y1_{\{Y\leq q_Y(\lambda)\}} \quad \mathbf{P}-a.s. \tag{2.13}$$

is fulfilled due to Proposition 2.10. However, in the case the random variables do not have continuous distributions, (2.13) is not applicable. Here is a counter example: Fix $\lambda \in (0,1)$. *Let* X *be a random variable which has the standard normal distribution. Let* X_q *be defined by (2.12) and* $Y := X_q$. *Note that in this case* $q_X(\lambda)$ *and* $q_Y(\lambda)$ *are unique. We have* $q_Y(\lambda) = q_X(\lambda)$, *and* $\mathbf{P}(Y = q_Y(\lambda)) = 1 - \lambda$ *while* $\mathbf{P}(X = q_X(\lambda)) = 0$. *From Example 2.9,* $X \overset{\lambda}{=} Y$ $\mathbf{P}$*-a.s. However, we can not conclude that* $X 1_{\{X \le q_X(\lambda)\}} = Y 1_{\{Y \le q_Y(\lambda)\}}$ $\mathbf{P} - a.s.$

We continue the discussion on Definition 2.1 and Definition 2.3 with examples of discretely distributed random variables.

Example 2.11. *(*a trinomial example*) Let* $\Omega = \{\omega_1, \omega_2, \omega_3\}$, *and suppose* X *and* Y *are two random variables on* Ω.

Case I: $\mathbf{P}(\omega_1) = \lambda - \varepsilon, \mathbf{P}(\omega_2) = 2\epsilon, \mathbf{P}(\omega_3) = 1 - \lambda - \varepsilon$, *for some small* $\varepsilon > 0$. *Suppose*

$$X(\omega_1) = Y(\omega_1), \quad X(\omega_2) = Y(\omega_2), \quad X(\omega_3) \neq Y(\omega_3),$$
$$X(\omega_1) < X(\omega_2) < X(\omega_3), \quad \textit{and} \quad Y(\omega_1) < Y(\omega_2) < Y(\omega_3).$$

According to Definition 2.1 or Definiton 2.3, $X \overset{\lambda}{=} Y$ $\mathbf{P}$*-a.s. We check Lemma 2.4 and Lemma 2.5 for this case. Since*

$$q_X^-(\lambda) = q_X^+(\lambda) = X(\omega_2) = Y(\omega_2) = q_Y^+(\lambda) = q_Y^-(\lambda),$$

we have unique choices of q_X, q_Y *and* A *with* $A = \{\omega_1, \omega_2\}$ *and* $q_X = q_Y$. *Therefore, we have*

$$\{\omega_1\} = \{X < q_X\} = \{Y < q_Y\} \subsetneq A, \ \textit{and}\ \mathbf{P}(A) \geq \lambda,$$

*this is **Case 1** of Lemma 2.4, and also is **Case 1'** of 2.5.*

Case II: $\mathbf{P}(\omega_1) = \lambda, \mathbf{P}(\omega_2) = \mathbf{P}(\omega_3) = \frac{1-\lambda}{2}$. *Suppose*

$$X(\omega_1) = Y(\omega_1), \quad X(\omega_2) \neq Y(\omega_2),$$
$$X(\omega_1) < X(\omega_2) < X(\omega_3), \quad \text{and} \quad Y(\omega_1) < Y(\omega_2) < Y(\omega_3).$$

According to Definition 2.1 or Definition 2.3, $X \stackrel{\lambda}{=} Y$ $\mathbf{P}$*-a.s.*

We now check Lemma 2.4 and Lemma 2.5 for this case. Note that

$$q_X^-(\lambda) = q_Y^-(\lambda) = X(\omega_1) = Y(\omega_1),$$
$$q_X^+(\lambda) = X(\omega_2) \neq Y(\omega_2) = q_Y^+(\lambda).$$

If we choose $q_X = q_X^-(\lambda)$, $q_Y = q_Y^-(\lambda)$, *and* $A = \{\omega_1\}$, *then*

$$\emptyset = \{X < q_X\} = \{Y < q_Y\} \subsetneq A \text{ and } \mathbf{P}(A) = \lambda.$$

This is ***Case 1*** *of Lemma 2.4. Note that this is also* ***Case 1'*** *of Lemma 2.5.*

An alternative choice is to take $q_X \in (q_X^-(\lambda), q_X^+(\lambda)]$, $q_Y \in (q_Y^-(\lambda), q_Y^+(\lambda)]$, *and* $A = \{\omega_1\}$, *then*

$$\{\omega_1\} = \{X < q_X\} = \{Y < q_Y\} = A \text{ and } \mathbf{P}(A) = \lambda.$$

We see that this is ***Case 2*** *of Lemma 2.4. Note that* q_X *and* q_Y *do not have to be equal. If we maintain* $q_X = q_Y = q_\lambda$, *then we obtain* ***Case 2'*** *of Lemma 2.5.*

The choices of the quantiles and the set A in Definition 2.1 are not unique, therefore, they can fall into the different cases of Lemma 2.4 and Lemma 2.5.

Case III: $\mathbf{P}(\omega_1) = \lambda$, $\mathbf{P}(\omega_2) = \varepsilon$, *and* $\mathbf{P}(\omega_3) = 1 - \lambda - \varepsilon$, *for some small* $\varepsilon > 0$. *Suppose*

$$X(\omega_1) = Y(\omega_1), \quad X(\omega_1) = X(\omega_2) < X(\omega_3), \quad Y(\omega_1) < Y(\omega_2) < Y(\omega_3).$$

Then $X \stackrel{\lambda}{=} Y$ $\mathbf{P}$*-a.s. according to Definition 2.1 or Definition 2.3.*

To check Lemma 2.4 and Lemma 2.5 for this case, we note that

$$q_X^-(\lambda) = q_X^+(\lambda) = X(\omega_1), \quad q_Y^-(\lambda) = Y(\omega_1), \quad q_Y^+(\omega) = Y(\omega_2).$$

The choice of q_X *is unique. If we choose* $q_Y = q_Y^-(\lambda)$, *then*

$$\emptyset = \{X < q_X\} = \{Y < q_Y\} \subsetneq A \quad \textit{and } \mathbf{P}(A) = \lambda.$$

*This is **Case 1** of Lemma 2.4.*

Alternatively, we can choose $q_Y \in (q_Y^-(\lambda), q_Y^+(\lambda)]$, *and* $A = \{\omega_1\}$, *then*

$$\emptyset = \{X < q_X\} \subsetneq \{Y < q_Y\} = \{\omega_1\} = A, \quad \textit{and } \mathbf{P}(A) = \lambda.$$

*This is **Case 3** of Lemma 2.4.*

For Lemma 2.5, we have to choose $q_Y = q_X = q_\lambda$, *then*

$$\emptyset = \{X < q_\lambda\} = \{Y < q_\lambda\} \subsetneq A, \quad \textit{and } \mathbf{P}(A) = \lambda,$$

*which is **Case 1'** of Lemma 2.5.*

Case IV: $\mathbf{P}(\omega_1) = \lambda - \varepsilon$, $\mathbf{P}(\omega_2) = 2\varepsilon$, $\mathbf{P}(\omega_3) = 1 - \lambda - \varepsilon$, *for some small* $\varepsilon > 0$. *Suppose*

$$X(\omega_1) = Y(\omega_1), \quad X(\omega_1) = X(\omega_2) < X(\omega_3), \quad Y(\omega_1) < Y(\omega_2) < Y(\omega_3).$$

Then we can not find a set A *satisfying the conditions in Definition 2.1. Note that*

$$q_X^-(\lambda) = q_X^+(\lambda) = X(\omega_1), \quad q_Y^-(\lambda) = q_Y^+(\lambda) = Y(\omega_2),$$
$$\{X < q_X^+(\lambda)\} = \emptyset, \quad \{X \leq q_X^+(\lambda)\} = \{\omega_1, \omega_2\},$$
$$\{Y < q_X^+(\lambda)\} = \{\omega_1\}, \quad \{Y \leq q_Y^+(\lambda)\} = \{\omega_1, \omega_2\}.$$

If we chose $A = \{\omega_1\}$, *then* $\mathbf{P}(A) = \lambda - \varepsilon < \lambda$. *If we chose* $A = \{\omega_1, \omega_2\}$, *then on* A, $\mathbf{P}(X1_A \neq Y1_A) = \mathbf{P}(\omega_2) > 0$. *In this case, we can not find a proper set*

A such that (2.1) is satisfied. Therefore, we can not conclude $X \overset{\lambda}{=} Y$ $\mathbf{P}$-a.s.

2.2 The λ-quantile Fatou property

As discussed in Chapter 1, for a convex measure of risk on L^p, the Fatou property is an essential condition to make it representable. This is also true for the λ-quantile dependent convex measures of risk. However, in this case, we can use the λ-quantile Fatou property to substitute the Fatou property while maintaining the representability of the risk measure.

Definition 2.12. (λ-quantile Fatou property) *Fix $\lambda \in (0,1)$. A convex measure of risk $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ has the λ-quantile Fatou property if for any sequence $(X_n) \subset L^p$ such that $q^+_{X_n}(\lambda) \leq c_\lambda$ for some $c_\lambda \in \mathbb{R}$ and for all $n \in \mathbb{N}$, $X_n \to X$ $\mathbf{P}$-a.s. for some $X \in L^p$ implies $\rho(X) \leq \liminf \rho(X_n)$.*

Remark 2.13. *We have the following remarks on the evolution of the Fatou property developed over time for different spaces:*

1. *Let us recall the original Fatou property defined by Delbaen (2002) for a finite coherent measure of risk ρ on the space L^∞: for any sequence $(X_n) \subset L^\infty$ with $|X_n| \leq C$ for some constant C, $X_n \to X$ $\mathbf{P}-$ a.s. for some $X \in L^\infty$ implies $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$. We also recall the Fatou property of a convex measure of risk defined on the space L^p, $1 \leq p \leq \infty$, given by Biagini and Frettelli (2009): for any sequence $(X_n) \subset L^p$ such that for some $Y \in L^p$, $|X_n| \leq Y$ $\mathbf{P}$-a.s., $X_n \to X$ $\mathbf{P}$-a.s. for some $X \in L^p$ implies $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$. In the definition of the λ-quantile Fatou property given by Definition 2.2, the upper λ-quantiles of the sequence (X_n) is uniformly bounded above by some constant which depends on the level λ. This boundedness is the weakest compared to the boundedness in the Fatou property of Delbaen's and Biagini and Frettelli's in the sense that more sequences of random variables satisfy this condition, and therefore, the continuity condition turns out to be the strongest. In conclusion,*

we have the following implication: ρ has λ-quantile Fatou property $\Rightarrow\rho$ has the Fatou property of Biagini and Frittelli's $\Rightarrow\rho$ has the Fatou property of Delbaen's.

2. *The uniform boundedness of the upper quantiles $q^{+}_{X_n}$ in Definition 2.12 is easier to handle compared with finding a dominant random variable Y for the whole sequence (X_n). As in practice we are mostly concerned about the losses of the financial positions, a natural choice of $c_\lambda = 0$ is already included.*

2.3 The robust representation of the λ-quantile dependent convex risk measure

We defined the λ-quantile dependent convex risk measure in Definition 2.2 and the λ-quantile Fatou property in Definition 2.12, and mentioned that the λ-quantile Fatou property enables the λ-quantile dependent convex risk measure to be representable. In this section, we will develop a theorem on the robust representation of the λ-quantile dependent convex measure.

For the preparation of the proof, we first recall some theorems and Lemmas.

Theorem 2.14. *(S.Mazur) The closure and weak closure of a convex subset of a normed space are the same. In particular, a convex subset of a normed space is closed if and only if it is weakly closed.*

The following Lemma appeared as Exercise 2.84 of Megginson (1988). It states an result between the norm and weak topologies, this result is analogues to the Krein-Šmulian theorem on weakly* closed convex sets. For completeness, we give the proof here.

Lemma 2.15. *Let $\mathcal{C}$ be a convex subset of a normed space $(X, \|\cdot\|)$.*

1. *$\mathcal{C}$ is closed if and only if $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ is closed for all $t > 0$.*

2. *$\mathcal{C}$ is weakly closed if and only if $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ is weakly closed for all $t > 0$.*

Proof.

1. If $\mathcal{C}$ is closed, then it is obvious that $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ is closed. Suppose now for any $t > 0$, $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ is closed. Let (c_n) be a sequence in $\mathcal{C}$ converging to some c in norm. Then for $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for every $n \geq N$, $\|c_n\| \leq \|c\| + \epsilon$. Taking $t = \|c\| + \epsilon$, then $(c_n)_{n>N} \subset \mathcal{C} \cap \{x \in X : \|x\| \leq t\}$. Since the set $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ is closed, $c \in \mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ and therefore $c \in \mathcal{C}$.

2. Since for any $t > 0$ the closed ball $\{x \in X : \|x\| \leq t\}$ is convex, it is weakly closed by Theorem 2.14. Therefore, if $\mathcal{C}$ is weakly closed, so is $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$. Conversely, suppose for any $t > 0$, $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ is weakly closed, then since the intersection of two convex sets is still convex, again by Theorem 2.14, $\mathcal{C} \cap \{x \in X : \|x\| \leq t\}$ is strongly (norm) closed. Thus $\mathcal{C}$ is strongly closed. Since $\mathcal{C}$ is convex, it is weakly closed. $\diamond$

The following two Lemmata, Lemma 2.16 and Lemma 2.17, are quoted from Föllmer and Schied (2004), where a short proof was given to Lemma 2.16 and a more precise proof was proposed to Lemma 2.17.

Lemma 2.16. *Suppose that E is a locally convex space and that $\mathcal{C}$ is a convex subset of E. Then $\mathcal{C}$ is weakly closed if and only if $\mathcal{C}$ is closed in the original topology of E.*

Lemma 2.17. *A convex subset $\mathcal{C}$ of L^∞ is weak* closed if for every $r > 0$, the set*

$$\mathcal{C}_r := C \cap \{X \in L^\infty : \|X\|_\infty \leq r\}$$

is closed in L^1.

For the λ-quantile dependent convex risk measure ρ considered below, we make the following assumption:

Assumption 2.18. *Let $\lambda \in (0,1)$ be given. $\rho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \leq p \leq \infty$, is a proper λ-quantile dependent convex measure of risk.*

The following theorem states the robust representation of the λ-quantile dependent convex risk measure as well as the equivalent conditions. This theorem is comparable to Theorem 1.10 and Theorem 1.11 for the convex measure of risk.

Theorem 2.19. *Suppose Assumption 2.18 holds. The following statements are equivalent:*

1. *For* $1 \leq p < \infty$, ρ *is* $\sigma(L^p, L^q)$*-lower semicontinuous, where* $\sigma(L^p, L^q)$ *indicates the weak topology on* L^p*; For* $p = \infty$, ρ *is* $\sigma(L^\infty, L^1)$*-lower semicontinuous, where* $\sigma(L^\infty, L^1)$ *indicates the weak* topology on* L^∞*.*

2. *For all* $X \in L^p$, $\rho(X)$ *has the following representation:*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})), \tag{2.14}$$

where ρ^* *is the Fenchel-Legendre transformation of* ρ*:*

$$\rho^*(\mathbf{Q}) = \sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)), \tag{2.15}$$

and $\mathcal{Q}_p$ *is as defined by (1.7):*

$$\mathcal{Q}_p := \left\{ \mathbf{Q} \text{ probability measure on } (\Omega, \mathcal{F}, \mathbf{P}) \,:\, \mathbf{Q} \ll \mathbf{P} \text{ and } \frac{d\mathbf{Q}}{d\mathbf{P}} \in L^q \right\}.$$

3. *For all* $X \in L^p$, $\rho(X)$ *has the following representation:*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p} \left(\mathbb{E}_{\mathbf{Q}}[-X 1_{\{X \leq q_X(\lambda)\}}] - q_X(\lambda) \mathbf{Q}(X > q_X(\lambda)) - \rho^*(\mathbf{Q}) \right), \tag{2.16}$$

with

$$\rho^*(\mathbf{Q}) = \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X] = \sup_{X \in \mathcal{A}_\rho} \left(\mathbb{E}_{\mathbf{Q}}[-X 1_{\{X \leq q_X(\lambda)\}}] - q_X(\lambda) \mathbf{Q}(X > q_X(\lambda)) \right), \tag{2.17}$$

where $q_X(\lambda)$ *is a* λ*-quantile of* X *and* $\mathcal{A}_\rho := \{X \in L^p \,|\, \rho(X) \leq 0\}$ *is the acceptance set.*

4. ρ *is continuous from above: For any sequence* (X_n) *in* L^p, $X_n \searrow X$ $\mathbf{P}$*-a.s.*

implies $\rho(X_n) \nearrow \rho(X)$.

5. ρ *has the* λ*-quantile Fatou property.*

Proof. We adapt the proof of Theorem 4.31 of Föllmer and Schied (2004) and of Theorem 3.1 of Kaina and Rüschendorf (2009) to prove the equivalence of statements *1*, *2*, and *4*. First, we show that " *1⇒2⇒4⇒1* ".

" *1⇒2* ": This is true due Theorem 1.8. Note that for $1 \leq p < \infty$, the dual space of $(L^p, \|\cdot\|_p)$ is L^q with $\frac{1}{p} + \frac{1}{q} = 1$, and for $p = \infty$, the dual space of $(L^\infty, \sigma(L^\infty, L^1))$ is L^1. Therefore, due to Theorem 1.8, we have $\rho = \rho^{**}$, where ρ^{**} is the Fenchel-Legendre transform of ρ^*, the Fenchel-Legendre transform of ρ defined by Definition 1.8. We need to verify ρ^* and ρ^{**} of the form (2.15) and (2.14). First, consider the case of $1 \leq p < \infty$. Let ℓ be the linear functional from L^p to $\mathbb{R}$. By Definition 1.8,

$$\rho^*(\ell) = \sup_{X \in L^p} (\ell(X) - \rho(X)),$$

and

$$\rho^{**}(X) = \sup_{\ell \in L^q} (\ell(X) - \rho^*(\ell)).$$

The monotonicity and cash invariance of ρ (Definition 1.1) implies that $\ell(X) \leq 0$ for $X \geq 0$, and $\ell(1) = -1$ for all $\ell \in L^q$ such that $\rho^*(\ell) < \infty$. More precisely, for $X \in L^p$ and $X \geq 0$, $nX \geq X$. If $\rho^*(\ell) < \infty$, then

$$\ell(nX) - \rho(nX) \leq \rho^*(\ell) \quad \Rightarrow \quad n\ell(X) \leq \rho(nX) + \rho^*(\ell) \leq \rho(X) + \rho^*(\ell),$$

the last inequality is due to the monotonicity of ρ. Therefore,

$$\ell(X) \leq \frac{1}{n}(\rho(X) + \rho^*(\ell)) \to 0, \qquad \text{as } n \to \infty,$$

which implies $\ell(X) \leq 0$ for all $X \geq 0$. In particular, due to the cash invariance of ρ, for natural number n,

$$\begin{aligned}\ell(n) - \rho(n) \le \rho^*(\ell) \quad &\Leftrightarrow \quad n\ell(1) - \rho(0) + n \le \rho^*(\ell)\\ &\Leftrightarrow \quad \ell(1) \le \frac{\rho(0) + \rho^*(\ell) - n}{n} \to -1, \quad \text{as } n \to \infty,\end{aligned}$$

and

$$\begin{aligned}\ell(-n) - \rho(-n) \le \rho^*(\ell) \quad &\Leftrightarrow \quad -n\ell(1) - \rho(0) - n \le \rho^*(\ell)\\ &\Leftrightarrow \quad \ell(1) \ge -\frac{\rho(0) + \rho^*(\ell) + n}{n} \to -1, \quad \text{as } n \to \infty.\end{aligned}$$

These imply $\ell(1) = -1$. Thus, given $\ell \in L^q$ with $\rho^*(\ell) < \infty$, we can define a probability measure $\mathbf{Q}_\ell$ in the way that $\mathbf{Q}_\ell(A) := -\ell(1_A) = -\int_A \ell d\mathbf{P}$, for $A \in \mathcal{F}$. The Radon-Nikodým derivative of $\mathbf{Q}_\ell$ is given by $\frac{d\mathbf{Q}_\ell}{d\mathbf{P}} = -\ell$. Therefore, for $X \in L^p$, $\ell(X) = \mathbb{E}_{\mathbf{Q}_\ell}[-X]$, and $\rho^*(\ell) = \rho^*(\mathbf{Q}) = \sup_{X \in L^p}(\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X))$. If we define $\mathcal{Q}_p$ as of (1.7), then $\rho^{**}(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p}(\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q}))$. Thus, we obtain statement *1* for $1 \le p < \infty$. For the $p = \infty$, the argument is exactly same.

" *2*⇒*4* ": Let $(X_n) \subset L^p$ and $X_n \searrow X$ $\mathbf{P}$-a.s. for $X \in L^p$. We need to show $\rho(X_n) \nearrow \rho(X)$, where $\rho(X_n)$ and $\rho(X)$ are given by statement *2*. Due the Monotone Convergence Theorem,

$$\begin{aligned}\rho(X) &= \sup_{\mathbf{Q} \in \mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q}))\\ &\le \sup_{\mathbf{Q} \in \mathcal{Q}_p} (\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}}[-X_n] - \rho^*(\mathbf{Q}))\\ &\le \liminf_{n\to\infty} \sup_{\mathbf{Q} \in \mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X_n] - \rho^*(\mathbf{Q}))\\ &= \liminf_{n\to\infty} \rho(X_n).\end{aligned}$$

On the other hand, by the monotonicity of ρ, $\rho(X_n) \le \rho(X)$, for all n, implies that $\limsup_{n\to\infty} \rho(X_n) \le \rho(X)$. Thus, we obtain

$$\limsup_{n\to\infty} \rho(X_n) \le \rho(X) \le \liminf_{n\to\infty} \rho(X_n),$$

which implies $\rho(X_n) \searrow \rho(X)$.

" *4*⇒*1* ": Recall Definition 1.6, that ρ is lower semicontinuous is equivalent to that the set $\mathcal{C} := \{\rho \le c\}$ is weakly closed for $1 \le p < \infty$ or weak* closed for $p = \infty$, for all $c \in \mathbb{R}$. We first look at the case of $1 \le p < \infty$. Let

$$\mathcal{C}_r := \mathcal{C} \cap \{X \in L^p : \|X\|_p \le r\}$$

with $r > 0$. From Lemma 2.15, we need to show that $\mathcal{C}_r$ is weakly closed. Let (X_n) be a sequence in $\mathcal{C}_r$ such that $X_n \to X$ in L^p, then there is a subsequence (X_{n_k}) such that $X_{n_k} \to X$ $\mathbf{P}$-a.s. Define $Y_n := \sup_{n_j \ge n} X_{n_j}$, then $Y_n \searrow X$, and from *4*, $\rho(Y_n) \nearrow \rho(X)$. Thus,

$$\rho(X) = \lim_{n\to\infty} \rho(Y_n) \le \liminf_{n\to\infty} \rho(X_n) \le c,$$

which implies that $X \in \mathcal{C}$. Moreover, $X_n \to X$ in L^p implies that $\|X\|_p \le r$. Thus, we achieve $X \in \mathcal{C}_r$, which means the set $\mathcal{C}_r$ is norm (strongly) closed. Due to Lemma 2.16, $\mathcal{C}_r$ is weakly closed.

For the case of $p = \infty$, the proof is very similar to the case of $1 \le p < \infty$ except that in the last step, instead of using Lemma 2.16, we need the Lemma 2.17.

We now show the equivalence of *2*, *3*, and *5*.

" *2* ⇔*3* ": Example 2.9 showed that $X_q \overset{\lambda}{=} X$ with $X_q = X1_{\{X \le q_X(\lambda)\}} + q_X(\lambda)1_{\{X > q_X(\lambda)\}}$. If ρ has the representation (2.14), then

$$\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})) \le \sup_{\mathbf{Q}\in\mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X_q] - \rho^*(\mathbf{Q})) = \rho(X_q) = \rho(X),$$

where the inequality is due to the fact of $X \ge X_q$. Thus, we obtain (2.16).

Note that $\rho^*(\mathbf{Q})$ given by equation (2.15) in *2* is known as the penalty function of the representation (2.14). For (2.17), we have

$$\rho^*(\mathbf{Q}) = \sup_{X\in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)) \ge \sup_{X\in\mathcal{A}_\rho} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)) \ge \sup_{X\in\mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X].$$

And on the other hand, for any $X \in L^p$, $X + \rho(X) \in \mathcal{A}_\rho$, therefore,

$$\sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)) = \sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-(X + \rho(X)])) \leq \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X].$$

Therefore, $\rho^*(\mathbf{Q}) = \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X]$. We further have

$$\rho^*(\mathbf{Q}) = \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X] \leq \sup_{X \in \mathcal{A}_\rho} \left(\mathbb{E}_{\mathbf{Q}}[-X 1_{\{X \leq q_X(\lambda)\}}] - q_X(\lambda)\mathbf{Q}(X > q_X(\lambda))\right).$$

For each $X \in \mathcal{A}_\rho$, $\rho(X_q) = \rho(X) \leq 0$. Therefore,

$$\begin{aligned} &\sup_{X \in \mathcal{A}_\rho} \left(\mathbb{E}_{\mathbf{Q}}[-X 1_{\{X \leq q_X(\lambda)\}}] - q_X(\lambda)\mathbf{Q}(X > q_X(\lambda))\right) \\ &= \sup_{X_q \in \mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X_q] \leq \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbf{Q}}[-X] = \rho^*(\mathbf{Q}). \end{aligned}$$

We show the equivalence of statement *2* and statement *5*.

" *2*⇒*5* ": Let $(X_n) \in L^p$ be a sequence satisfying $q^+_{X_n}(\lambda) \leq c_\lambda$ for some $c_\lambda \in \mathbb{R}$ and for all $n \in \mathbb{N}$ and $X_n \to X$ $\mathbf{P}$-a.s. for some $X \in L^p$. The goal is to show that $\rho(X) \leq \liminf(X_n)$. Define $Y_n := X_n 1_{\{X_n \leq c_\lambda\}} + c_\lambda 1_{\{X_n > c_\lambda\}}$ and $Y := X 1_{\{X \leq c_\lambda\}} + c_\lambda 1_{\{X > c_\lambda\}}$. Then $Y_n \to Y$ $\mathbf{P}$-a.s. Since ρ is λ-quantile dependent, $\rho(Y_n) = \rho(X_n)$ and $\rho(Y) = \rho(X)$. Define $Z_n(\omega) := \sup_{k \geq n} Y_k(\omega)$ for all $\omega \in \Omega$, then $Z_n \searrow \limsup Y_n = Y$. Thus, by statement *3*, $\rho(Y) = \lim_{n \to \infty} \rho(Z_n)$. Since $Z_n(\omega) \geq Y_n(\omega)$ for all $\omega \in \Omega$, the monotonicity of ρ implies that $\rho(Z_n) \leq \rho(Y_n)$. Therefore, $\rho(Y) \leq \liminf \rho(Y_n)$. By the λ-quantile dependence of ρ, we obtain $\rho(X) \leq \liminf \rho(X_n)$.

" *5*⇒*2* ": Suppose ρ has the λ-quantile Fatou property. We first show that ρ is $\sigma(L^p, L^q)$-lower semicontinuous for $1 \leq p < \infty$. This is equivalent to show that the convex subset $\mathcal{C} := \{\rho \leq c\} \subset L^p$ is weakly closed for any fixed constant c. By Lemma 2.15 in the Appendix, an analogous result to the Krein-Šmulian Lemma, this is true if and only if $\mathcal{C}_r := \mathcal{C} \cap \{X \in L^p : \|X\|_p \leq r\}$ is weakly closed for all $r > 0$. Since the space $(L^p, \|\cdot\|_p)$, $1 \leq p < \infty$, is locally convex, that $\mathcal{C}_r$ is weakly closed is equivalent to that $\mathcal{C}_r$ is strongly closed. In the following, we will show that $\mathcal{C}_r$ is strongly closed in L^p with respect to the norm topology. Let (X_n) be a sequence in

$\mathcal{C}_r$ converging to X in L^p-norm. Then there is a subsequence (X_{n_k}) converging to X $\mathbf{P}$-a.s. If we can show that $(q^+_{X_{n_k}}(\lambda))$ is uniformly bounded above, then statement *4* implies $\rho(X) \leq \liminf \rho(X_{n_k}) \leq c$. Therefore, $X \in \mathcal{C}_r$, i.e., $\mathcal{C}_r$ is strongly closed.
To complete the proof, it remains to show that $(q^+_{X_{n_k}}(\lambda))$ is uniformly bounded from above. If this is not true, then for any $m \in \mathbb{N}$, there exists a $Y_m \in (X_{n_k})$ such that $q^+_{Y_m}(\lambda) > m$. Thus

$$\begin{aligned}\|Y_m\|_p^p &= \int_{\{Y_m < q^+_{Y_m}(\lambda)\}} |Y_m|^p d\mathbf{P} + \int_{\{Y_m \geq q^+_{Y_m}(\lambda)\}} |Y_m|^p d\mathbf{P} \\ &\geq m^p \mathbf{P}(Y_m \geq q^+_{Y_m}(\lambda)) \geq m^p(1-\lambda) \to \infty, \qquad \text{as } m \to \infty.\end{aligned}$$

This is a contradiction to the fact that $Y_m \in \mathcal{C}_r$. For the case $p = \infty$, apply Lemma 2.17 instead of Lemma 2.15, the remaining part of the proof is similar to the case $1 \leq p < \infty$. ◇

Under certain continuity conditions or when the convex risk measure is finitely valued, the supremum in representation (2.14) can be attained, see Biagini and Frettelli (2009) and Kaina and Rüschendorf (2009). In this case, we can further narrow the representation set in (2.16) so that the probability measures concentrate on relevant sets.

Corollary 2.20. *Suppose $\rho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \leq p \leq \infty$, satisfies Assumption 2.18. Let $X \in L^p$ such that $\rho(X) < \infty$ and $\rho(X)$ can be represented by*

$$\rho(X) = \max_{\mathbf{Q} \in \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})), \tag{2.18}$$

where ρ^* is defined in equation (2.15) and $\mathcal{Q} \subset \mathcal{Q}_p$. Then there is a corresponding set $\mathcal{Q}_p^{\lambda,X} := \{\mathbf{Q} \in \mathcal{Q}_p : \mathbf{Q}(X > q_X(\lambda)) = 0\}$ such that

$$\rho(X) = \max_{\mathbf{Q} \in \mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})) = \max_{\mathbf{Q} \in \mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X 1_{\{X \leq q_X(\lambda)\}}] - \rho^*(\mathbf{Q})). \tag{2.19}$$

Proof. For each $X \in L^p$, there exists a $\mathbf{Q}_X \in \mathcal{Q}$ such that

$$\rho(X) = \mathbb{E}_{\mathbf{Q}_X}[-X] - \rho^*(\mathbf{Q}_X).$$

Since $X \overset{\lambda}{=} X_q$, by the λ-quantile dependence of ρ,

$$\rho(X_q) = \rho(X) = \mathbb{E}_{\mathbf{Q}_X}[-X] - \rho^*(\mathbf{Q}_X) \le \mathbb{E}_{\mathbf{Q}_X}[-X_q] - \rho^*(\mathbf{Q}_X) \le \rho(X_q).$$

This implies

$$\rho(X_q) = \mathbb{E}_{\mathbf{Q}_X}[-X] - \rho^*(\mathbf{Q}_X) = \mathbb{E}_{\mathbf{Q}_X}[-X_q] - \rho^*(\mathbf{Q}_X).$$

Thus, $\mathbb{E}_{\mathbf{Q}_X}[(X - q_X(\lambda))1_{\{X > q_X(\lambda)\}}] = 0$, which implies $\mathbf{Q}_X(X > q_X(\lambda)) = 0$. Since $\mathbf{Q}_X \in \mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q}$,

$$\begin{aligned}\rho(X) &= \mathbb{E}_{\mathbf{Q}_X}[-X] - \rho^*(\mathbf{Q}_X) \\ &\le \max_{\mathbf{Q} \in \mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})) \\ &= \max_{\mathbf{Q} \in \mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X 1_{\{X \le q_X(\lambda)\}}] - \rho^*(\mathbf{Q})).\end{aligned}$$

On the other hand, $\mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q} \subset \mathcal{Q}$, representation (2.18) implies

$$\rho(X) \ge \max_{\mathbf{Q} \in \mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})) = \max_{\mathbf{Q} \in \mathcal{Q}_p^{\lambda,X} \bigcap \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X 1_{\{X \le q_X(\lambda)\}}] - \rho^*(\mathbf{Q})).$$

$\diamond$

CHAPTER 3: λ-QUANTILE LAW INVARIANT CONVEX RISK MEASURE

The Conditional Value-at-Risk $CVaR_\lambda$ for a given significance level λ is a λ-quantile dependent convex risk measure. In fact, it is λ-quantile law invariant in the sense that if two financial positions X and Y have the same probability distributions up to the level λ, then they have the same Conditional Value-at-Risk. In this chapter, we give the formal definition of the λ-quantile law invariant convex risk measure and study the robust representation.

3.1 The λ-quantile law invariant convex risk measure

In Chapter 2, we gave two equivalent definitions of that two random variables are $\mathbf{P}$-a.s. equal up to their λ-quantiles. Definition 2.1 used the set A satisfying conditions (2.1), while Definition 2.3 depended on the equivalence of the random variables X_{q_λ} and Y_{q_λ} defined by (2.4):

$$X_{q_\lambda} := X1_{\{X \le q_\lambda\}} + q_\lambda 1_{\{X > q_\lambda\}},$$

$$Y_{q_\lambda} := Y1_{\{Y \le q_\lambda\}} + q_\lambda 1_{\{Y > q_\lambda\}}.$$

In this section, we define that two random variables have the same probability law up to their λ-quantiles using the similar approach as in Definition 2.3, and then define the λ-quantile law invariant convex risk measure.

Definition 3.1. (two random variables have the same law up to λ-quantile) *Fix* $\lambda \in (0, 1)$. *Let* X, Y *be two random variables on the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$. *If for some* $q_\lambda \in \mathbb{R}$ *such that* q_λ *is a* λ*-quantile of both* X *and* Y, *and the random variables* X_{q_λ} *and* Y_{q_λ} *have the same probability distributions, then we say that* X *and* Y *have the same law up to their* λ*-quantiles. We denote it as* $X \overset{\lambda}{\sim} Y$.

The following lemma shows that the λ-quantile $\mathbf{P}$-a.s. equality of two random variables implies that these random variables have the same probability distributions up to their λ-quantile. In **Case I** of Example 3.5 we will show the reverse implication does not hold.

Lemma 3.2. *Let X, Y be two random variables on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then*

$$X \overset{\lambda}{=} Y \quad \mathbf{P}-a.s. \quad \Rightarrow \quad X \overset{\lambda}{\sim} Y. \tag{3.1}$$

PROOF. Suppose $X \overset{\lambda}{=} Y$ $\mathbf{P}$-a.s. Let $q_X(\lambda)$ and $q_Y(\lambda)$ be λ-quantiles of X and Y respectively such that a set A satisfying (2.1) exists. By Lemma 2.5, there exists some number q_λ such that A can be chosen as $\{X < q_\lambda\} = \{Y < q_\lambda\} \subset A$ a.e. and $\mathbf{P}(A) \geq \lambda$. To prove $X \overset{\lambda}{\sim} Y$, it is sufficient to show that X_{q_λ} and Y_{q_λ} have the same probability distribution. We have

$$\begin{aligned} X_{q_\lambda} &= X 1_{\{X<q_\lambda\}} + q_\lambda 1_{A \setminus \{X<q_\lambda\}} + q_\lambda 1_{\Omega \setminus A} \\ &= Y 1_{\{Y<q_\lambda\}} + q_\lambda 1_{A \setminus \{Y<q_\lambda\}} + q_\lambda 1_{\{\Omega \setminus A\}} \\ &= Y_{q_\lambda} \qquad \mathbf{P}-a.s. \end{aligned}$$

Thus, $X \overset{\lambda}{\sim} Y$. $\diamond$

A convex risk measure is λ-quantile law invariant, if its value is only relevant to the tail distribution of the random variables up to a fixed level λ. In other words, if two random variables have the same probability distribution up to their λ-quantiles, then they must have the same value of the convex risk measure.

Definition 3.3. (λ-quantile law invariant convex risk measure) *Fix $\lambda \in (0,1)$. A convex measure of risk $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ is λ-quantile law invariant if for any $X, Y \in L^p$,*

$$X \overset{\lambda}{\sim} Y \quad \textit{implies} \quad \rho(X) = \rho(Y).$$

Namely, the value of the risk measure ρ depends on the distribution of the random variables only up to a given significance level λ.

Remark 3.4. *Under Definition 3.1 or alternatively, using Example 2.9 and Lemma 3.2, we have* $X \overset{\lambda}{\sim} X_q$*. Thus, if a convex measure of risk* ρ *is* λ*-quantile law invariant, then* $\rho(X) = \rho(X_q)$*.*

Definition 3.1 incorporates the flexibility of the choice of the quantiles for the two random variables X and Y. We illustrate this concept in a simple quadnomial tree.

Example 3.5. *Let* $\lambda \in (0, 1)$ *be fixed. Suppose the probability space is* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ *and the probability measure* $\mathbf{P}$ *is defined by* $\mathbf{P}(\omega_1) = \lambda - \varepsilon$, $\mathbf{P}(\omega_2) = \varepsilon$, $\mathbf{P}(\omega_3) = \varepsilon$, *and* $\mathbf{P}(\omega_4) = 1 - \lambda - \varepsilon$*, for some small* $\varepsilon > 0$*.*

Case I: *Suppose*

$$X(\omega_1) = -1, \quad X(\omega_2) = 0, \quad X(\omega_3) = 1, \quad X(\omega_4) = 2,$$

$$Y(\omega_1) = -1, \quad Y(\omega_2) = 1, \quad Y(\omega_3) = 0, \quad Y(\omega_4) = 3.$$

Then random variables X *and* Y *have the same probability distribution up to their* λ*-quantiles, but they are not* λ*-quantile* $\mathbf{P}$*-a.s. equal. Note that* $q_X^-(\lambda) = q_Y^-(\lambda) = 0$ *and* $q_X^+(\lambda) = q_Y^+(\lambda) = 1$*. We have infinitely many choices for* $q_X(\lambda)$ *and* $q_Y(\lambda)$*:*

1. $q_X(\lambda) = q_Y(\lambda) = q_X^-(\lambda) = q_Y^-(\lambda)$,
2. $q_X(\lambda) = q_Y(\lambda) = q_X^+(\lambda) = q_Y^+(\lambda)$,
3. $q_X^-(\lambda) < q_X(\lambda) = q_Y(\lambda) < q_X^+(\lambda)$,

under which $X \overset{\lambda}{\sim} Y$*.*

Case II: *Suppose*

$$X(\omega_1) = -1, \quad X(\omega_2) = 0, \quad X(\omega_3) = 1, \quad X(\omega_4) = 2,$$

$$Y(\omega_1) = -1, \quad Y(\omega_2) = 2, \quad Y(\omega_3) = 0, \quad Y(\omega_4) = 3.$$

Then the random variables X *and* Y *still have the same probability distribution up to their* λ*-quantiles. Note that* $q_X^-(\lambda) = q_Y^-(\lambda) = 0$ *and* $q_X^+(\lambda) = 1 < q_Y^+(\lambda) =$

2. *We now have the following choices for* $q_X(\lambda)$ *and* $q_Y(\lambda)$*:*

   1. $q_X(\lambda) = q_Y(\lambda) = q_X^-(\lambda) = q_Y^-(\lambda)$,

   2. $q_X^-(\lambda) < q_X(\lambda) = q_Y(\lambda) < q_X^+(\lambda)$.

Case III: *Suppose*

$$X(\omega_1) = -1, \quad X(\omega_2) = -1, \quad X(\omega_3) = -1, \quad X(\omega_4) = 2,$$
$$Y(\omega_1) = -1, \quad Y(\omega_2) = -1, \quad Y(\omega_3) = 1, \quad Y(\omega_4) = 3.$$

The random variables X *and* Y *still have the same probability distribution up to their* λ*-quantiles. Note that we have* $q_X^-(\lambda) = q_X^+(\lambda) = -1$ *and* $q_Y^-(\lambda) = -1 < q_Y^+(\lambda) = 1$*. We now have the only choice* $q_X(\lambda) = q_Y(\lambda) = q_X^-(\lambda) = q_Y^-(\lambda)$*.*

3.2 The robust representation of the λ-quantile law invariant convex risk measure

In this section and the rest of the dissertation, $X \sim Y$ denotes that X and Y have the same probability distribution.

The robust representation of the λ-quantile law invariant convex risk measure is given by Theorem 3.9. As a preparation of the proof, we will need the following Lemma 3.6, Lemma 3.7, and Lemma 3.8.

Lemma 3.6. *Let* X *be a random variable with a continuous cumulative distribution function* F_X *and quantile function* q_X*. Define* $U := F_X(X)$*. Then* U *is uniformly distributed on* $(0,1)$*, and* $X = q_X(U)$ $\mathbf{P}$*-almost surely.*

Lemma 3.6 is quoted from Lemma A.21 of Föllmer and Schied (2004), where a proof of the lemma is provided.

The next lemma and its proof can be found in Lemma A.24 of Föllmer and Schied (2004). This Lemma provides a version of the "Hardy-Littlewood inequalities". The original version of the Hardy-Littlewood inequalities can be found in Hardy, Littlewood, and Pólya (1952).

Lemma 3.7. *Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ with quantile functions q_X and q_Y. Then*

$$\int_0^1 q_X(1-s)q_Y(s)ds \le \mathbb{E}[XY] \le \int_0^1 q_X(s)q_Y(s)ds, \tag{3.2}$$

provided that all integrals are well defined. If $X = f(Y)$ and the lower (upper) bound is finite, then the lower (upper) bound is attained if and only if f can be chosen as a decreasing (increasing) function.

The following lemma generalizes Lemma 4.55 of Föllmer and Schied (2004) from L^∞ space to L^p space.

Lemma 3.8. *Suppose the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless. For random variables $X \in L^p$ and $Y \in L^q$, where $p, q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X}Y] = \int_0^1 q_X(t)q_Y(t)dt.$$

Proof. The idea of the proof is very similar to that of Lemma 4.55 of Föllmer and Schied (2004). First, the Hardy-Littlewood inequalities (3.2) ensures that

$$\mathbb{E}[\tilde{X}Y] \le \int_0^1 q_{\tilde{X}}(t)q_Y(t)dt = \int_0^1 q_X(t)q_Y(t)dt, \qquad \text{for all } \tilde{X} \sim X.$$

Thus, $\sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X}Y] \le \int_0^1 q_X(t)q_Y(t)dt$.

In general, to show $\sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X}Y] \ge \int_0^1 q_X(t)q_Y(t)dt$, we first assume Y has continuous distribution. Define $U := F_Y(Y)$, where $F_Y(\cdot)$ is the cumulative distribution function of random variable Y, then by Lemma 3.6, $Y = q_Y(U)$ $\mathbf{P}$-a.s. Define $\tilde{X} := q_X(U)$, then $\tilde{X} \sim X$. Therefore, for such defined $\tilde{X}$,

$$\mathbb{E}[\tilde{X}Y] = \mathbb{E}[q_X(U)q_Y(U)] = \int_0^1 q_X(t)q_Y(t)dt.$$

So indeed we find some $\tilde{X}$ that has the same law as X and attains $\int_0^1 q_X(t)q_Y(t)dt$.

In the case that Y does not have continuous probability distribution, we define for

$n \geq 1$ that

$$Y_n := Y + \frac{1}{n} Z 1_{\{Y \geq q_Y(a)\}} - \frac{1}{n} Z 1_{\{Y < q_Y(a)\}},$$

where $Z \in L^q$ is a nonnegative random variable having continuous probability distribution (such Z exists due to the atomlessness of the probability space), and the real number $a \in [0, 1]$ is chosen such that $q_X(t) \leq 0$ for all $t < a$ and $q_X(t) \geq 0$ for $t > a$. Then Y_n has continuous probability distribution. For the quantile function $q_{Y_n}(t)$, we have $q_{Y_n}(t) \leq q_Y(t)$, for $t < a$, and $q_{Y_n}(t) \geq q_Y(t)$, for $t > a$. We have

$$\begin{aligned}
\int_0^1 q_X(t) q_{Y_n}(t) dt &= \int_0^a q_X(t) q_{Y_n}(t) dt + \int_a^1 q_X(t) q_{Y_n}(t) dt \\
&\geq \int_0^a q_X(t) q_Y(t) dt + \int_a^1 q_X(t) q_Y(t) dt \\
&= \int_0^1 q_X(t) q_Y(t) dt.
\end{aligned}$$

Thus, by applying the Lemma for a continuously distributed random variable,

$$\int_0^1 q_X(t) q_Y(t) dt \leq \int_0^1 q_X(t) dt q_{Y_n}(t) dt = \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y_n].$$

If we can show that $\sup_{\tilde{X} \sim X} \mathbb{E}[X Y_n] \to \sup_{\tilde{X} \sim X} \mathbb{E}[XY] \quad$ as $n \to \infty$, then we can conclude $\int_0^1 q_X(t) q_Y(t) dt \leq \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y]$ and finish the proof. For any $\varepsilon > 0$, there exist $\tilde{X}_\varepsilon \sim X$ such that $\mathbb{E}[\tilde{X}_\varepsilon Y_n] \geq \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y_n] - \varepsilon$. Then for arbitrary ε,

$$\begin{aligned}
\sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y_n] - \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y] &\leq \mathbb{E}[\tilde{X}_\varepsilon Y_n] + \varepsilon - \mathbb{E}[\tilde{X}_\varepsilon Y] \\
&= \mathbb{E}[\tilde{X}_\varepsilon (Y_n - Y)] + \varepsilon \\
&\leq \|\tilde{X}_\varepsilon\|_p \, \|Y_n - Y\|_q + \varepsilon \\
&= \|\tilde{X}_\varepsilon\|_p \, \|\frac{1}{n} Z 1_{\{Y \geq q_X(a)\}} - \frac{1}{n} Z 1_{\{Y < q_X(a)\}}\|_q + \varepsilon \\
&\leq \frac{2}{n} \|\tilde{X}_\varepsilon\|_p \, \|Z\|_q + \varepsilon \to \varepsilon, \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, we obtain $\limsup_{n \to \infty} \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y_n] \leq \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y]$. Similar argument leads to the opposite inequality $\liminf_{n \to \infty} \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y_n] \geq \sup_{\tilde{X} \sim X} \mathbb{E}[\tilde{X} Y]$, hence, the equality holds. $\diamond$

We state a representation of a λ-quantile law invariant convex risk measure in the following theorem. It is a λ-quantile version of Theorem 1.15 for the law invariant convex risk measure.

Theorem 3.9. *Suppose the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless. Let $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ be a proper convex measure of risk (i.e., satisfy Assumption 2.18) that is λ-quantile law invariant. ρ has the λ-quantile Fatou property if and only if ρ has the following representation:*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p} \left(\int_0^\lambda q_X(t) q_{-\varphi}(t) dt + q_X(\lambda) \int_\lambda^1 q_{-\varphi}(t) dt - \rho^*(\mathbf{Q}) \right), \tag{3.3}$$

where $\varphi := \frac{d\mathbf{Q}}{d\mathbf{P}}$ for $\mathbf{Q} \in \mathcal{Q}_p$, and $\rho^*(\mathbf{Q})$ depends on $\mathbf{Q}$ only through its Radom-Nikodým derivative φ:

$$\rho^*(\mathbf{Q}) = \sup_{X \in L^p} \left(\int_0^\lambda q_X(t) q_{-\varphi}(t) dt + q_X(\lambda) \int_\lambda^1 q_{-\varphi}(t) dt - \rho(X) \right) \tag{3.4}$$

Proof. First, we show equation (3.4). For $X \in L^p$, $X_q \overset{\lambda}{\sim} X$, which implies

$$\rho(X_q) = \rho(X).$$

Together with the fact that $X \geq X_q$, we have the following:

$$\begin{aligned}
\rho^*(\mathbf{Q}) &= \sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)) \\
&\leq \sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X_q] - \rho(X)) \\
&= \sup_{X_q \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X_q] - \rho(X_q)) \\
&\leq \sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)) \\
&= \rho^*(\mathbf{Q})
\end{aligned}$$

Thus, all inequalities are equalities and $\rho^*(\mathbf{Q}) = \sup_{X \in L^p}(\mathbb{E}_{\mathbf{Q}}[-X_q] - \rho(X_q))$. By

Theorem 2.19 and Lemma 3.8, we further have

$$\begin{aligned}
\rho^*(\mathbf{Q}) &= \sup_{X\in L^p} (\mathbb{E}_{\mathbf{Q}}[-X_q] - \rho(X_q)) \\
&= \sup_{X\in L^p} (\sup_{\tilde{X}\sim X_q} (\mathbb{E}_{\mathbf{Q}}[-\tilde{X}] - \rho(\tilde{X}))) \\
&= \sup_{X\in L^p} (\sup_{\tilde{X}\sim X_q} \mathbb{E}_{\mathbf{Q}}[-\tilde{X}] - \rho(X)) \\
&= \sup_{X\in L^p} \left(\int_0^1 q_{X_q}(t) q_{-\varphi}(t) dt - \rho(X) \right) \\
&= \sup_{X\in L^p} \left(\int_0^\lambda q_X(t) q_{-\varphi}(t) dt + q_X(\lambda) \int_\lambda^1 q_{-\varphi}(t) dt - \rho(X) \right).
\end{aligned}$$

The last equality is true, since $q_{X_q}(t) = q_X(t)$ for $0 < t \le \lambda$ and $q_{X_q}(t) = q_X(\lambda)$ for $\lambda < t < 1$.

To show (3.3), we note that due to (3.4), $\rho^*(\tilde{\mathbf{Q}}) = \rho^*(\mathbf{Q})$ if $\varphi_{\tilde{\mathbf{Q}}} \sim \varphi_{\mathbf{Q}}$, where $\varphi_{\tilde{\mathbf{Q}}} := \frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}}$ and $\varphi_{\mathbf{Q}} := \frac{d\mathbf{Q}}{d\mathbf{P}}$. Again, by Theorem 2.19 and Lemma 3.8,

$$\begin{aligned}
\rho(X) = \rho(X_q) &= \sup_{\mathbf{Q}\in\mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X_q] - \rho^*(\mathbf{Q})) \\
&= \sup_{\mathbf{Q}\in\mathcal{Q}_p} (\sup_{\varphi_{\tilde{\mathbf{Q}}}\sim\varphi_{\mathbf{Q}}} (\mathbb{E}[X_q(-\varphi_{\tilde{Q}})] - \rho^*(\tilde{\mathbf{Q}}))) \\
&= \sup_{\mathbf{Q}\in\mathcal{Q}_p} (\sup_{\varphi_{\tilde{\mathbf{Q}}}\sim\varphi_{\mathbf{Q}}} \mathbb{E}[X_q(-\varphi_{\tilde{Q}})] - \rho^*(\mathbf{Q})) \\
&= \sup_{\mathbf{Q}\in\mathcal{Q}_p} \left(\int_0^\lambda q_X(t) q_{-\varphi_{\mathbf{Q}}}(t) dt + q_X(\lambda) \int_\lambda^1 q_{-\varphi_{\mathbf{Q}}}(t) dt - \rho^*(\mathbf{Q}) \right).
\end{aligned}$$

Hence, if we denote $\varphi_{\mathbf{Q}}$ by φ for simplicity, then

$$\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}_p} \left(\int_0^\lambda q_X(t) q_{-\varphi}(t) dt + q_X(\lambda) \int_\lambda^1 q_{-\varphi}(t) dt - \rho^*(\mathbf{Q}) \right).$$

◇

Remark 3.10. *In the special case when $p = \infty$, Jouini, Schachermayer and Touzi (2006) has shown that Fatou property is automatically satisfied, thus representation (2.14) is guaranteed without any additional continuity condition. Consequently, representation (3.3) will follow. For $1 \le p < \infty$, Kaina and Rüschendorf (2009) showed* *that ρ possesses the Fatou property if ρ is a finite convex risk measure.*

A λ-quantile law invariant convex risk measure is also law invariant. Föllmer and Schied (2004) gave a representation of the law invariant convex risk measure

$$\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}_1} \left(\int_0^1 q_X(t) q_{-\varphi}(t) dt - \rho^*(\mathbf{Q}) \right), \qquad \text{for } X \in L^\infty,$$

with

$$\rho^*(\mathbf{Q}) = \sup_{X\in L^\infty} \left(\int_0^1 q_X(t) q_{-\varphi}(t) dt - \rho(X) \right).$$

Due to Lemma 3.8, this representation can be extended to all X in the space L^p. The following lemma summarizes this result.

Lemma 3.11. *Let $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ be a proper convex measure of risk that is law invariant. Suppose ρ has the representation (1.11), i.e.,*

$$\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})),$$

then we have

$$\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}_p} \left(\int_0^1 q_X(t) q_{-\varphi}(t) dt - \rho^*(\mathbf{Q}) \right), \qquad \textit{for } X \in L^p, \tag{3.5}$$

where

$$\rho^*(\mathbf{Q}) = \sup_{X\in L^p} \left(\int_0^1 q_X(t) q_{-\varphi_{\mathbf{Q}}}(t) dt - \rho(X) \right). \tag{3.6}$$

Proof. The proof of the lemma is very similar to the proof of Theorem 4.54 of Föllmer and Schied (2004), where instead of Lemma 4.55 of Föllmer and Schied (2004), we need to use Lemma 3.8. For completeness, we sketch the proof here. Let $\varphi_{\mathbf{Q}}$ denote the Radon-Nikodým derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$. For any $\mathbf{Q} \in \mathcal{Q}_p$, by Lemma 3.8 we have

$$\begin{aligned}
\rho^*(\mathbf{Q}) &= \sup_{X\in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)) \\
&= \sup_{X\in L^p} \left(\sup_{\tilde{X}\sim X} (\mathbb{E}[-\tilde{X}\varphi_{\mathbf{Q}}] - \rho(\tilde{X})) \right) \\
&= \sup_{X\in L^p} \left(\int_0^1 q_X(t) q_{-\varphi_{\mathbf{Q}}}(t) dt - \rho(X) \right).
\end{aligned}$$

If we use $\mathbf{Q} \sim \tilde{\mathbf{Q}}$ to denote that the Radon-Nikodým derivatives $\frac{d\mathbf{Q}}{d\mathbf{P}}$ and $\frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}}$ have the same probability law, then we have

$$\begin{aligned}\rho(X) &= \sup_{\mathbf{Q}\in\mathcal{Q}_p} \left(\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})\right) \\ &= \sup_{\mathbf{Q}\in\mathcal{Q}_p} \left(\sup_{\tilde{\mathbf{Q}}\sim\mathbf{Q}} \left(\mathbb{E}_{\tilde{\mathbf{Q}}}[-X] - \rho^*(\tilde{Q})\right)\right) \\ &= \sup_{\mathbf{Q}\in\mathcal{Q}_p} \left(\int_0^1 q_X(t) q_{-\varphi_{\mathbf{Q}}}(t) dt - \rho^*(\mathbf{Q})) \right).\end{aligned}$$

The last equality is due to Lemma 3.8 and the fact that $\rho^*(\mathbf{Q})$ depends on $\mathbf{Q}$ only through the probability distribution of its Radon-Nikodým derivative. Using $-\varphi$ to substitute $-\varphi_{\mathbf{Q}}$, we obtain the results stated in the lemma. $\diamond$

The following proposition is the λ-quantile law invariant version of Corollary 2.20.

Corollary 3.12. *Suppose the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless. Let $\rho : L^p \to \mathbb{R}\cup\{\infty\}$ be a λ-quantile law invariant convex risk measure. Suppose for some $X \in L^p$, $\rho(X) < \infty$ and it has the representation of*

$$\rho(X) = \max_{\mathbf{Q}\in\mathcal{Q}} \left(\int_0^1 q_X(t) q_{-\varphi}(t) dt - \rho^*(\mathbf{Q}) \right),$$

where $\mathcal{Q} \subset \mathcal{Q}_p$ and $\varphi := \frac{d\mathbf{Q}}{d\mathbf{P}}$. Define

$$\lambda^+ := \inf\{s > \lambda : \, q_X(s) > q_X(\lambda)\}.$$

Then there exists a set

$$\mathcal{Q}_p^{\lambda^+,X} := \left\{ \mathbf{Q} \in \mathcal{Q}_p : \, \int_{\lambda^+}^1 q_{-\varphi}(t) dt = 0 \right\}$$

such that

$$\rho(X) = \max_{\mathbf{Q}\in\mathcal{Q}_p^{\lambda^+,X}\cap\mathcal{Q}} \left(\int_0^{\lambda^+} q_X(t) q_{-\varphi}(t) dt - \rho^*(\mathbf{Q}) \right).$$

Proof. For $X \in L^p$, let $\mathbf{Q}_X \in \mathcal{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ such that

$\rho(X) = \int_0^1 q_X(t) q_{-\varphi_X}(t)dt - \rho^*(\mathbf{Q}_X)$, where $\varphi_X := \frac{d\mathbf{Q}_X}{d\mathbf{P}}$. Then

$$\begin{aligned}
\rho(X) &= \int_0^\lambda q_X(t) q_{-\varphi_X}(t)dt + \int_\lambda^1 q_X(t) q_{-\varphi_X}(t)dt - \rho^*(\mathbf{Q}_X) \\
&\leq \int_0^\lambda q_X(t) q_{-\varphi_X}(t)dt + q_X(\lambda) \int_\lambda^1 q_{-\varphi_X}(t)dt - \rho^*(\mathbf{Q}_X) \\
&\leq \sup_{\mathbf{Q}\in\mathcal{Q}_p} \left(\int_0^\lambda q_X(t) q_{-\varphi}(t)dt + q_X(\lambda) \int_\lambda^1 q_{-\varphi}(t)dt - \rho^*(\mathbf{Q}) \right) \\
&= \rho(X_q),
\end{aligned}$$

where $X_q = X1_{\{X \leq q_X(\lambda)\}} + q_X(\lambda)1_{\{X > q_X(\lambda)\}}$. Note that the first inequality is due to the negativity of $q_{-\varphi_X}(t)$, and the last equality is due to the representation (3.5) of $\rho(X_q)$. Further, since $X_q \overset{\lambda}{\sim} X$, it is true that $\rho(X) = \rho(X_q)$. Therefore, all the inequalities are equalities, and we have

$$\int_\lambda^1 q_X(t) q_{-\varphi_X}(t)dt = q_X(\lambda) \int_\lambda^1 q_{-\varphi_X}(t)dt.$$

Thus, for any $s > \lambda$ such that such that $q_X(s) > q_X(\lambda)$, we have

$$\int_s^1 q_{-\varphi_X}(t)dt = 0. \tag{3.7}$$

If we define

$$\lambda^+ := \inf\{\, s > \lambda :\ q_X(s) > q_X(\lambda)\},$$

then $\lambda^+ \in [\lambda, 1]$. By Dominated Convergence Theorem and (3.7), we have

$$\int_{\lambda^+}^1 q_{-\varphi_X}(t)dt = \lim_{s \searrow \lambda^+} \int_s^1 q_{-\varphi_X}(t)dt = 0.$$

$\diamond$

CHAPTER 4: ROBUST REPRESENTATION OF λ-QUANTILE DEPENDENT WEIGHTED VALUE-AT-RISK $\rho_{\mu,\lambda}$

In this chapter, we study the λ-quantile dependent Weighted Value-at-Risk $\rho_{\mu,\lambda}$ defined on the space L^p, $1 \le p \le \infty$. Throughout this chapter, the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is assumed to be atomless.

4.1 The definition of the λ-quantile dependent Weighted Value-at-Risk $\rho_{\mu,\lambda}$

Recall that the Weighted Value-at-Risk is defined as $\rho_\mu(X) := \int_{[0,1]} CVaR_\gamma(X)\mu(d\gamma)$ with μ a probability measure on $[0,1]$. If μ is a probability measure on $[0,\lambda]$ for some fixed $\lambda \in (0,1]$, then the value of ρ_μ depends on the random variables only through their left tails. Formally, we have the following definition:

Definition 4.1. (the λ-quantile dependent WVaR) *The λ-quantile dependent Weighted Value-at-Risk is a mapping $\rho_{\mu,\lambda} : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p \le \infty$, defined as*

$$\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} CVaR_\gamma(X)\mu(d\gamma), \tag{4.1}$$

where $\lambda \in (0,1]$ is fixed, μ is a probability measure on $[0,\lambda]$ satisfying $\mu(\{0\}) = 0$, and $CVaR_\gamma(X) = -\frac{1}{\gamma}\int_0^\gamma q_X^+(t)dt$ is the Conditional Value-at-Risk at level γ.

Note that $\rho_{\mu,\lambda}$ is a coherent measure of risk. Applying the Fubini's theorem, we easily obtain an equivalent form of $\rho_{\mu,\lambda}$:

$$\rho_{\mu,\lambda}(X) = -\int_0^\lambda q_X(t)\phi(t)dt, \tag{4.2}$$

where

$$\phi(t) = \int_{(t,\lambda]} \frac{1}{s}\mu(ds), \qquad \text{for } t \in (0,\lambda]. \tag{4.3}$$

Hence, the measure of risk $\rho_{\mu,\lambda}$ is λ-quantile law invariant.

Remark 4.2.

- *Take λ as 1 and μ as a probability measure on $[0,1]$, then (4.1) defines the Weighted Value-at-Risk $\rho_\mu = \int_{[0,1]} CVaR_\gamma(X)\mu(d\gamma)$. For a given $\lambda \in (0,1)$, the Weighted Value-at-Risk ρ_μ is λ-quantile dependent if and only if $\mu([0,\lambda]) = 1$.*

- *Acerbi (2002) defined the Spectral Measure of Risk $M_{\tilde{\phi}}(X) = -\int_0^1 q_X(t)\tilde{\phi}(t)dt$, where $\tilde{\phi}(t) := \int_t^1 \mu(d\gamma)$ and μ is some measure on $[0,1]$ (not necessarily a probability measure). He showed that $M_{\tilde{\phi}}$ is a coherent measure of risk, if $\tilde{\phi}$ satisfies the admissibility conditions: $\tilde{\phi}$ is positive, decreasing and $\|\tilde{\phi}\| = \int_0^1 |\tilde{\phi}(p)|dp = 1$. Acerbi interpreted the function $\tilde{\phi}$ as the "risk spectrum". For given $\lambda \in (0,1)$, take $\tilde{\phi}(t) = \frac{1}{\lambda}1_{\{0 \le t \le \lambda\}}$, then $CVaR_\lambda(X) = M_{\tilde{\phi}}(X)$. In this case, the function $\tilde{\phi}(t)$ is the density of a uniform distribution on $[0,\lambda]$, $\tilde{\phi}$ assigns equal weights to every possible outcome under the threshold λ, so $CVaR$ represents the average of $\lambda 100\%$ worst losses of a financial position.*

- *The λ-quantile dependent coherent risk measure $\rho_{\mu,\lambda}$ defined by either (4.1) or (4.2) can be interpreted in a similar way. In (4.2), the function $\phi(t)$ assigns weights to the Value-at-Risk $-q_X(t)$ for $0 \le t \le \lambda$, a reason associated to the name of $\rho_{\mu,\lambda}$: the λ-quantile dependent Weighted Value-at-Risk. In Subsection 4.3.2, we will see a new example when the probability measure μ in (4.1) is uniformly distributed on $[0,\lambda]$. $\rho_{\mu,\lambda}$ averages the Conditional Value-at-Risk with equal weights up to the level λ and the "risk spectrum" in (4.2) turns out to be a natural logarithmic function.*

4.2 The relationship between the λ-quantile uniform preference and the core of λ-quantile dependent concave distortion

In this section, we extend the definitions of the uniform preference and the core of a concave distortion to the λ-quantile case. We confirm the relationship between the two discovered by Carlier and Dana (2003) holds in the λ-quantile case. This prepares for the robust representation of the λ-quantile dependent Weighted Value-at-Risk $\rho_{\mu,\lambda}$ in the next section.

4.2.1 λ-quantile uniform preference of two probability distribution measures

In this subsection, μ denotes a probability distribution measure on $(\Omega_1, \mathcal{F}_1)$ and ν denotes a probability distribution measure on $(\Omega_2, \mathcal{F}_2)$. We first define the "$\lambda$-quantile uniform preference" of two probability distribution measures. Recall that a quantile function of distribution measure ν is denoted by $q_\nu(\cdot)$ and satisfies

$$\nu((-\infty, q_\nu(t)]) \geq t \quad \text{and} \quad \nu((-\infty, q_\nu(t))) \leq t. \tag{4.4}$$

Similarly, $q_\mu(\cdot)$ is a quantile function of distribution measure μ. We do not restrict the definition to be on the upper quantile function.

We define

$$\mathcal{M} := \left\{ \mu \text{ probability measure on } \mathbb{R}\text{: } \int_{-\infty}^{q_\mu(\lambda)} x\mu(dx) > -\infty \right\}.$$

Note that

$$\int_{-\infty}^{q_\mu(\lambda)} x\mu(dx) > -\infty \Leftrightarrow \mathbb{E}_\mu[X 1_{\{X \leq q_X(\lambda)\}}] > -\infty \Leftrightarrow \int_0^\lambda q_\mu(s)ds > -\infty.$$

Since every concave function u is dominated by an affine function, for every $\mu \in \mathcal{M}$, $\int u d\mu \in [-\infty, \infty)$.

Definition 4.3. (λ-quantile uniform preference) *Fix* $\lambda \in (0,1)$. *Let* μ, ν *be in* $\mathcal{M}$. *The probability distribution measure* μ *is* λ*-quantile uniformly preferred over* ν, *denoted by* $\mu \underset{uni(\lambda)}{\succcurlyeq} \nu$, *if*

$$\int_0^t q_\mu(s)ds \geq \int_0^t q_\nu(s)ds, \quad \text{for all} \quad 0 < t \leq \lambda.$$

Remark 4.4. *For two random variables* X *and* Y *defined on two probability spaces*

$(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, we can similarly define the λ-quantile uniform preference of the random variables X and Y in the sense of the λ-quantile uniform preference of their respective probability distribution measures ν_X and ν_Y:

$$\begin{aligned} X \underset{uni(\lambda)}{\succcurlyeq} Y \quad &\Leftrightarrow \quad \nu_X \underset{uni(\lambda)}{\succcurlyeq} \nu_Y \\ &\Leftrightarrow \quad \int_0^t q_{\nu_X}(s)ds \geq \int_0^t q_{\nu_Y}(s)ds, \quad \textit{for all} \quad 0 < t \leq \lambda. \end{aligned}$$

The uniform preference of two probability distribution measures (Definition 1.19) is also known as the second order stochastic dominance. Definition 4.3 can be viewed as the λ-quantile dependent version of the second order stochastic dominance. The following theorem gives the equivalent conditions of the λ-quantile uniform preference of two probability distribution measures μ and ν.

Let us recall that a utility function $u : \mathbb{R} \to \mathbb{R}$ is a strictly concave, strictly increasing and continuous function. We define a ν-λ-quantile utility function $u_{\nu,\lambda} : \mathbb{R} \to \mathbb{R}$ as

$$u_{\nu,\lambda}(x) = u(x)1_{\{x \leq q_\nu(\lambda)\}} + u(q_\nu(\lambda))1_{\{x > q_\nu(\lambda)\}}, \tag{4.5}$$

with u a real-valued utility function on $\mathbb{R}$.

Theorem 4.5.

a. $\mu \underset{uni(\lambda)}{\succcurlyeq} \nu$ *if and only if for all decreasing functions $h : (0, \lambda] \to \mathbb{R}^+$, the following is true:*

$$\int_0^\lambda h(t)q_\mu(t)dt \geq \int_0^\lambda h(t)q_\nu(t)dt,$$

where q_μ and q_ν are quantile functions of μ and ν.

b. *The following equivalent conditions implies $\mu \underset{uni(\lambda)}{\succcurlyeq} \nu$:*

1. *For all ν-λ-quantile utility function $u_{\nu,\lambda} : \mathbb{R} \to \mathbb{R}$, the following is true:*

$$\int_{\mathbb{R}} u_{\nu,\lambda}(x)\mu(dx) \geq \int_{\mathbb{R}} u_{\nu,\lambda}(x)\nu(dx).$$

2. For all increasing, concave and continuous functions f on $\mathbb{R}$ such that $f(x) = f(q_\nu(\lambda))$ for all $x \geq q_\nu(\lambda)$,

$$\int_{\mathbb{R}} f(x)\mu(dx) \geq \int_{\mathbb{R}} f(x)\nu(dx).$$

3.

$$\int_{\mathbb{R}} (c-x)^+\mu(dx) \leq \int_{\mathbb{R}} (c-x)^+\nu(dx), \qquad \text{for all} \quad c \leq q_\nu(\lambda).$$

4. Let F_μ and F_ν denote the distribution functions of μ and ν, then

$$\int_{-\infty}^{c} F_\mu(x)dx \leq \int_{-\infty}^{c} F_\nu(x)dx, \qquad \text{for all} \quad c \leq q_\nu(\lambda).$$

c. $\mu \underset{uni(\lambda)}{\succcurlyeq} \nu$ *implies the following equivalent conditions:*

1. For all μ-λ-quantile utility function $u_{\mu,\lambda} : \mathbb{R} \to \mathbb{R}$, the following is true:

$$\int_{\mathbb{R}} u_{\mu,\lambda}(x)\mu(dx) \geq \int_{\mathbb{R}} u_{\mu,\lambda}(x)\nu(dx).$$

2. For all increasing, concave and continuous functions f on $\mathbb{R}$ such that $f(x) = f(q_\mu(\lambda))$ for all $x \geq q_\mu(\lambda)$,

$$\int_{\mathbb{R}} f(x)\mu(dx) \geq \int_{\mathbb{R}} f(x)\nu(dx).$$

3.

$$\int_{\mathbb{R}} (c-x)^+\mu(dx) \leq \int_{\mathbb{R}} (c-x)^+\nu(dx), \qquad \text{for all} \quad c \leq q_\mu(\lambda).$$

4. Let F_μ and F_ν denote the distribution functions of μ and ν, then

$$\int_{-\infty}^{c} F_\mu(x)dx \leq \int_{-\infty}^{c} F_\nu(x)dx, \qquad \text{for all} \quad c \leq q_\mu(\lambda).$$

PROOF.

"**a**": Take $h = 1_{(0,t)}$, $0 < t \leq \lambda$, then the "if" part is obviously true. For the proof of the "only if" part, since h is decreasing, without loss of generality, we may assume that h is left-continuous. Then a Radon measure η on $[0, \lambda]$ can be defined by $h(t) - h(\lambda) = \eta([t,\lambda])$. By Fubini's theorem,

$$\begin{aligned}\int_0^\lambda h(t)q_\mu(t)dt &= \int_0^\lambda h(\lambda)q_\mu(t)dt + \int_0^\lambda \int_{[t,\lambda]} \eta(ds)q_\mu(t)dt \\ &= \int_0^\lambda h(\lambda)q_\mu(t)dt + \int_{[0,\lambda]} \int_0^s q_\mu(t)dt\,\eta(ds) \\ &\geq \int_0^\lambda h(\lambda)q_\nu(t)dt + \int_{[0,\lambda]} \int_0^s q_\nu(t)dt\,\eta(ds) \\ &= \int_0^\lambda h(t)q_\nu(t)dt.\end{aligned}$$

"**b**": "1⇔2": That the statement 2 implies the statement 1 is obvious. To show "1⇒2", let u_0 be a utility function such that $\int_{\mathbb{R}} u_0 d\mu$ and $\int_{\mathbb{R}} u_0 d\nu$ are finite. Define $u(x) := u_0(x)1_{\{x\leq q_\nu(\lambda)\}} + u_0(q_\nu(\lambda))1_{\{x>q_\nu(\lambda)\}}$. Then u is a ν-λ-quantile utility function. For $\alpha \in (0,1)$ define

$$u_\alpha(x) = \alpha f(x) + (1-\alpha)u(x),$$

where f is an increasing concave and continuous function with $f(x) = f(q_\nu(\lambda))$ for $x \geq q_\nu(\lambda)$. Then, we have

$$u_\alpha(x) = \begin{cases} \alpha f(x) + (1-\alpha)u_0(x) & \text{for } x \leq q_\nu(\lambda), \\ u_\alpha(q_\nu(\lambda)) & \text{for } x > q_\nu(\lambda). \end{cases}$$

For $x \leq q_\nu(\lambda)$, $u_\alpha(x)$ is strictly increase, strictly concave and continuous. Thus, u_α is a ν-λ-quantile utility function. Statement 1 implies

$$\int_{\mathbb{R}} u_\alpha(x)\mu(dx) \geq \int_{\mathbb{R}} u_\alpha(x)\nu(dx).$$

Substituting u_α into this inequality and letting α goes to 1, yields

$$\int_{\mathbb{R}} f(x)\mu(dx) = \lim_{\alpha\to 1} \int_{\mathbb{R}} u_\alpha(x)\mu(dx) \geq \lim_{\alpha\to 1} \int_{\mathbb{R}} u_\alpha(x)\nu(dx) = \int_{\mathbb{R}} f(x)\nu(dx).$$

"2⇔3": Since the function $-(c-x)^+$ satisfies the conditions in statement 2, statement 2 implies statement 3. To show "3⇒2", let f be an increasing concave and continuous function on $\mathbb{R}$ satisfying $f(x) = f(q_\nu(\lambda))$ for all $x \geq q_\nu(\lambda)$. Define $h(x) := -(f(x) -$

$f(q_\nu(\lambda)))$, $x \in \mathbb{R}$, then h is decreasing, convex and continuous s.t. $h(x) = 0$ for $x \geq q_\nu(\lambda)$. Take $h'(x) = h'(x+)$, then h' is increasing and right-continous. For any real number a and b with $a < b$, define the Radon measure $\gamma((a,b]) = h'(b) - h'(a)$. Then

$$\begin{aligned} h(x) &= h(b) - \int_x^b h'(u)du \\ &= h(b) - \int_x^b h'(b)du + \int_x^b (h'(b) - h'(u))du \\ &= h(b) - \int_x^b h'(b)du + \int_x^b \int_{(u,b]} \gamma(dz)du \\ &= h(b) - h'(b)(b-x) + \int_{(-\infty,b]} (z-x)^+ \gamma(dz), \qquad x < b. \end{aligned}$$

For $b < q_\nu(\lambda)$ and $b \to q_\nu(\lambda)$, due to the assumption and the definition of h,

$$\begin{aligned} &\int_{(-\infty,b]} h(x)\mu(dx) \\ &= h(b)\mu((-\infty,b]) - h'(b)\int_{\mathbb{R}} (b-x)^+\mu(dx) + \int_{(-\infty,b]}\int_{\mathbb{R}} (z-x)^+\mu(dx)\gamma(dz) \\ &\to h(q_\nu(\lambda))\mu((-\infty,q_\nu(\lambda)]) - h'(q_\nu(\lambda))\int_{\mathbb{R}} (q_\nu(\lambda)-x)^+\mu(dx) \\ &\qquad + \int_{(-\infty,q_\nu(\lambda))}\int_{\mathbb{R}} (z-x)^+\mu(dx)\gamma(dz) \\ &\leq h(q_\nu(\lambda))\nu((-\infty,q_\nu(\lambda)]) - h'(q_\nu(\lambda))\int_{\mathbb{R}} (q_\nu(\lambda)-x)^+\nu(dx) \\ &\qquad + \int_{(-\infty,q_\nu(\lambda))}\int_{\mathbb{R}} (z-x)^+\nu(dx)\gamma(dz) \\ &= \int_{(-\infty,q_\nu(\lambda))} h(x)\nu(dx) \\ &= \int_{\mathbb{R}} h(x)\nu(dx). \end{aligned}$$

Note that $h(q_\nu(\lambda))\mu((-\infty,q_\nu(\lambda)]) = h(q_\nu(\lambda))\nu((-\infty,q_\nu(\lambda)]) = 0$ and the last equality is valid due to $h(x) = 0$ for all $x \geq q_\nu(\lambda)$. Since the inequality is true for all $b < q_\nu(\lambda)$, it is true in limit, and we have

$$\int_{(-\infty,q_\nu(\lambda))} h(x)\mu(dx) = \int_{\mathbb{R}} h(x)\mu(dx) \leq \int_{\mathbb{R}} h(x)\nu(dx).$$

Substituting $h(x) = -(f(x) - f(q_\nu(\lambda)))$ into the inequality yields

$$\int_{\mathbb{R}} f(x)\mu(dx) \geq \int_{\mathbb{R}} f(x)\nu(dx).$$

"3⇔4": From the Fubini's theorem, for all $c \in (-\infty, q_\nu(\lambda)]$, we have

$$\begin{aligned}\int_{-\infty}^{c} F_\mu(z)dz &= \int_{-\infty}^{c}\int_{(-\infty,z]} \mu(dx)dz \\ &= \int_{-\infty}^{c}\int_{x}^{c} dz\mu(dx) \\ &= \int_{(-\infty,c]} (c-x)\mu(dx) \\ &= \int_{\mathbb{R}} (c-x)^+\mu(dx),\end{aligned}$$

Similarly, $\int_{-\infty}^{c} F_\nu(z)dz = \int_{\mathbb{R}}(c-x)^+\nu(dx)$, for all $c \in (-\infty, q_\nu(\lambda)]$. This proves the equivalence.

"4⇒**a**": This is based on the duality relationship between the integral of the cumulative distribution function and the integral of the quantile function. First, recall Lemma A.22 from Föllmer and Schied (2004): For a random variable X with distribution function F_X and quantile function q_X such that $\mathbb{E}[|X|] < \infty$,

$$\sup_{c\in\mathbb{R}} \left(ct - \int_{-\infty}^{c} F_X(x)dx \right) = \int_0^t q_X(s)ds, \qquad \text{for} \quad t \in [0,1]. \tag{4.6}$$

Moreover, the supremum is attained by $c = q_X(t)$. If

$$\int_{-\infty}^{c} F_\mu(x)dx \leq \int_{-\infty}^{c} F_\nu(x)dx, \quad \forall c \in (-\infty, q_\nu(\lambda)],$$

then for each fixed number t we have

$$ct - \int_{-\infty}^{c} F_\mu(x)dx \geq ct - \int_{-\infty}^{c} F_\nu(x)dx,$$

and thus,

$$\sup_{c\in(-\infty,q_\nu(\lambda)]} \left(ct - \int_{-\infty}^{c} F_\mu(x)dx \right) \geq \sup_{c\in(-\infty,q_\nu(\lambda)]} \left(ct - \int_{-\infty}^{c} F_\nu(x)dx \right).$$

If $q_\mu(\lambda) \le q_\nu(\lambda)$, then by (4.6), for all $t \in [0, \lambda]$,

$$\begin{aligned}\int_0^t q_\mu(s)ds &= \sup_{c\in(-\infty,q_\mu(\lambda)]}\left(ct - \int_{-\infty}^c F_\mu(x)dx\right)\\ &= \sup_{c\in(-\infty,q_\nu(\lambda)]}\left(ct - \int_{-\infty}^c F_\mu(x)dx\right)\\ &\ge \sup_{c\in(-\infty,q_\nu(\lambda)]}\left(ct - \int_{-\infty}^c F_\nu(x)dx\right)\\ &= \int_0^t q_\nu(s)ds.\end{aligned}$$

Therefore, $\int_0^t q_\mu(s)ds \ge \int_0^t q_\nu(s)ds$ for all $t \in [0, \lambda]$.

If $q_\mu(\lambda) > q_\nu(\lambda)$, then the same conclusion can be obtained since

$$\begin{aligned}\int_0^t q_\mu(s)ds &= \sup_{c\in(-\infty,q_\mu(\lambda)]}\left(ct - \int_{-\infty}^c F_\mu(x)dx\right)\\ &\ge \sup_{c\in(-\infty,q_\nu(\lambda)]}\left(ct - \int_{-\infty}^c F_\mu(x)dx\right)\\ &\ge \sup_{c\in(-\infty,q_\nu(\lambda)]}\left(ct - \int_{-\infty}^c F_\nu(x)dx\right)\\ &= \int_0^t q_\nu(s)ds.\end{aligned}$$

"**c**": The equivalence "1 ⇔ 2 ⇔ 3 ⇔ 4" can be proved similarly as in **b**. To show "**a**⇒4", let us recall Theorem 1.8: Let f be a proper convex function on a locally convex space E. If f is lower semicontinuous with respect to the weak topology $\sigma(E, E')$, then $f = f^{**}$, where f^* denotes the Fenchel-Legendre transform of f.

The function $\psi(c) := \int_{-\infty}^c F_X(x)dx$ is obviously lower semicontinuous on $\mathbb{R}$. From the Lemma A.22 of Föllmer and Schied (2004) and Theorem 1.8,

$$\int_{-\infty}^c F_X(x)dx = \psi^{**}(c) = \sup_{t\in[0,1]}\left(ct - \int_0^t q_X(s)ds\right), \quad \text{for all } c \in \mathbb{R}$$

and the supremum is obtained when t is chosen such that $c = q_X(t)$. Thus, if $\int_0^t q_\mu(s)ds \ge \int_0^t q_\nu(s)ds$ for all $0 < t \le \lambda$, then the following is true for fixed value c:

$$\sup_{t\in[0,\lambda]}\left(ct - \int_0^t q_\mu(s)ds\right) \le \sup_{t\in[0,\lambda]}\left(ct - \int_0^t q_\nu(s)ds\right).$$

For all $c \in (-\infty, q_\mu(\lambda) \wedge q_\nu(\lambda)]$,

$$\int_{-\infty}^{c} F_\mu(x)dx = \sup_{t\in[0,\lambda]} \left(ct - \int_0^t q_\mu(s)ds \right),$$
$$\int_{-\infty}^{c} F_\nu(x)dx = \sup_{t\in[0,\lambda]} \left(ct - \int_0^t q_\nu(s)ds \right),$$

and we conclude

$$\int_{-\infty}^{c} F_\mu(x)dx \le \int_{-\infty}^{c} F_\nu(x)dx, \quad \forall c \in (-\infty, q_\mu(\lambda) \wedge q_\nu(\lambda)].$$

If $q_\mu(\lambda) \ge q_\nu(\lambda)$, then for all $c \in (q_\nu(\lambda), q_\mu(\lambda)]$,

$$\int_{-\infty}^{c} F_\mu(x)dx = \sup_{t\in[0,\lambda]} \left(ct - \int_0^t q_\mu(s)ds \right),$$
$$\int_{-\infty}^{c} F_\nu(x)dx \ge \sup_{t\in[0,\lambda]} \left(ct - \int_0^t q_\nu(s)ds \right),$$

and the result follows. $\diamond$

4.2.2 Core of a λ-quantile dependent concave distortion

In this subsection, we define the core of a λ-quantile dependent concave distortion and study its relation to the λ-quantile uniform preference.

Definition 4.6. (λ-quantile dependent concave distortion and its core) *Let $\lambda \in (0,1)$ be fixed and $(\Omega, \mathcal{F}, \mathbf{P})$ be an atomless probability space. $\Psi : [0,1] \to [0,1]$ is called a concave distortion function if it is increasing, concave, and it satisfies $\Psi(0) = 0$, $\Psi(x) = 1$ for all $x \in [\lambda, 1]$. In this case we call $\Psi \circ \mathbf{P} : \mathcal{F} \to [0,1]$ a λ-quantile dependent concave distortion of the probability measure $\mathbf{P}$. The core of the λ-quantile dependent concave distortion $\Psi \circ \mathbf{P}$ is naturally defined as:*

$$\text{core}(\Psi \circ \mathbf{P}) = \{\mathbf{Q} \textit{ finitely additive on } (\Omega, \mathcal{F}) \ : \mathbf{Q}(A) \le \Psi(\mathbf{P}(A)), \forall A \in \mathcal{F}\}. \quad (4.7)$$

According to Schmeidler (1972), the elements of core$(\Psi \circ \mathbf{P})$ are probability measures that are absolutely continuous to $\mathbf{P}$. Therefore,

$$\begin{aligned}&\mathrm{core}(\Psi \circ \mathbf{P})\\&= \{\mathbf{Q} \text{ probability measure on } (\Omega, \mathcal{F}) \ : \mathbf{Q} \ll \mathbf{P},\ \mathbf{Q}(A) \le \Psi(\mathbf{P}(A)),\ \forall A \in \mathcal{F}\}.\end{aligned} \tag{4.8}$$

The elements $\mathbf{Q}$ in core$(\Psi \circ \mathbf{P})$ can be identified by the Radon-Nikodým derivatives $h := \frac{d\mathbf{Q}}{d\mathbf{P}}$. We do not distinguish the notations $\mathbf{Q} \in \mathrm{core}(\Psi \circ \mathbf{P})$ and $h \in \mathrm{core}(\Psi \circ \mathbf{P})$. For a λ-quantile dependent concave distortion function Ψ, we denote $\phi(t) := \Psi'(t+)$ as its right-hand derivative. Then $\phi(t)$ is positive and monotone decreasing on $[0, \lambda]$ and $\phi(t) = 0$, $\forall t \in [\lambda, 1]$. Consequently, $-\phi$ can be viewed as the upper quantile function of some probability distribution function $\nu_{-\phi}$ such that

$$q^{+}_{\nu_{-\phi}}(t) = -\phi(t), \quad \forall t \in [0, 1]. \tag{4.9}$$

The next theorem describes the relation of core$(\Psi \circ \mathbf{P})$ and the λ-quantile uniform preference. It is the λ-quantile version of Theorem 1 in Carlier and Dana (2003).

Theorem 4.7. *Suppose $\Psi \circ \mathbf{P}$ is a λ-quantile dependent concave distortion of the probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$. Let $h : \Omega \to \mathbb{R}^+$ be a probability density function, ν_{-h} be its probability distribution function, and $q_{-h} := q_{\nu_{-h}}$ be a quantile function. Let $-\phi(t)$ be defined as (4.11). Then the following statements are equivalent.*

1. *$h \in \mathrm{core}(\Psi \circ \mathbf{P})$.*
2. *For all $x \in (0, \lambda]$,*

$$-\int_0^x q_{-h}(t)dt \le \Psi(x) = -\int_0^x -\phi(t)dt.$$

3. *$\nu_{-h} \underset{uni(\lambda)}{\succcurlyeq} \nu_{-\phi}$.*
4. *$-q_{-h} \in \mathrm{core}(\Psi \circ \mathcal{L})$, where $\mathcal{L}$ indicates the Lebesgue measure on $[0, 1]$.*

Proof. The equivalence between *2* and *3* is obvious due to equation (4.11) and Definition 4.3. We first show the equivalence between *1* and *2*. Recall from (4.8) that

h denotes the Radon-Nikodým derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$ for some $\mathbf{Q} \in \text{core}(\Psi \circ \mathbf{P})$.

"*1⇒2*": Suppose $h \in \text{core}(\Psi \circ \mathbf{P})$. From the two equivalent forms of Conditional Value-at-Risk by Acerbi and Tasche (2002):

$$CVaR_\lambda(X) = -\frac{1}{\lambda}\int_0^\lambda q_X(t)dt = -\frac{1}{\lambda}\mathbb{E}[X1_{\{X<q_X(\lambda)\}}] - q_X(\lambda)\tfrac{\lambda - P(X<q_X(\lambda))}{\lambda},$$

we know that,

$$\begin{aligned}
-\int_0^x q_{-h}(t)dt &= xCVaR_x(-h) \\
&= -\mathbb{E}_\mathbf{P}[-h1_{\{-h<q_{-h}(x)\}}] - q_{-h}(x)(x - \mathbf{P}(-h < q_{-h}(x))) \\
&= \mathbb{E}_\mathbf{P}[h1_{\{-h<q_{-h}(x)\}}] - q_{-h}(x)(x - \mathbf{P}(-h < q_{-h}(x))).
\end{aligned}$$

Since the probability space is assumed to be atomless, we may find a set $B \subset \{-h = q_{-h}(x)\}$ so that $\mathbf{P}(\{-h < q_{-h}(x)\} \cup B) = x$. Then for $x \in (0, \lambda]$,

$$\begin{aligned}
-\int_0^x q_{-h}(t)dt &= \mathbb{E}_\mathbf{P}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}1_{\{-h<q_{-h}(x)\}}\right] - q_{-h}(x)\mathbf{P}(B) \\
&= \mathbb{E}_\mathbf{Q}[1_{\{-h<q_{-h}(x)\}}] - \mathbb{E}_\mathbf{P}[-h1_B] \\
&= \mathbf{Q}(\{-h < q_{-h}(x)\} \cup B) \\
&\leq \Psi(\mathbf{P}(\{-h < q_{-h}(x)\} \cup B)) \\
&= \Psi(x).
\end{aligned}$$

"*2⇒1*": Let $\mathbf{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbf{Q} \ll \mathbf{P}$ and $h := \frac{d\mathbf{Q}}{d\mathbf{P}}$. For any $A \in \mathcal{F}$ such that $\mathbf{P}(A) \leq \lambda$, $q_{1_A}(t) = 0$ for $0 \leq t < 1 - \mathbf{P}(A)$ and $q_{I_A}(t) = 1$ for $1 - \mathbf{P}(A) \leq t \leq 1$. Due to the Hardy-Littlewood inequalities (3.2), we have

$$\mathbf{Q}(A) = \int 1_A d\mathbf{Q} = \int 1_A h d\mathbf{P} \leq \int_0^1 q_{1_A}(t)q_h(t)dt = \int_{1-\mathbf{P}(A)}^1 q_h(t)dt.$$

For the quantile function $q_h(t)$, it is true that $-q_h^+(t) = q_{-h}^-(1-t)$, for $t \in (0,1)$. Therefore, if $-\int_0^x q_{-h}(t)dt \leq \Psi(x) = -\int_0^x -\phi(t)dt$, then

$$\int_{1-\mathbf{P}(A)}^1 q_h(t)dt = -\int_{1-\mathbf{P}(A)}^1 q_{-h}(1-t)dt = -\int_0^{\mathbf{P}(A)} q_{-h}(t)dt \leq \int_0^{\mathbf{P}(A)} \phi(t)dt = \Psi(\mathbf{P}(A)).$$

When $\mathbf{P}(A) \geq \lambda$, $\Psi(\mathbf{P}(A)) = 1 \geq \mathbf{Q}(A)$. Thus $\mathbf{Q} \in \text{core}(\Psi \circ \mathbf{P})$, or equivalently, $h \in \text{core}(\Psi \circ \mathbf{P})$.

As a next step, we show the equivalence of *2* and *4*.

"*4*⇒*2*": Let $\mathcal{B}[0,1]$ be the Borel σ-algebra on $[0,1]$ and $\mathcal{L}$ be the Lebesgue measure on $([0,1], \mathcal{B}[0,1])$. Then

$$\text{core}(\Psi \circ \mathcal{L}) :=$$

$$\{\mathbf{Q} \text{ probability measure on } ([0,1], \mathcal{B}[0,1]) \; : \; \mathbf{Q} \ll \mathcal{L},\, \mathbf{Q}(A) \leq \Psi(\mathcal{L}(A)),\, \forall A \in \mathcal{B}[0,1]\}.$$

Suppose $\mathbf{Q} \in \text{core}(\Psi \circ \mathcal{L})$ such that $d\mathbf{Q} = -q_{-h} d\mathcal{L}$. For $x \in (0, \lambda]$,

$$\int_0^x -q_{-h}(t)dt = \int_{[0,x]} \frac{d\mathbf{Q}}{d\mathcal{L}} d\mathcal{L} = \mathbf{Q}([0,x]) \leq \Psi(\mathcal{L}[0,x]) = \Psi(x).$$

"*2*⇒*4*": Suppose $\mathbf{Q}$ is a probability measure such that $d\mathbf{Q} = -q_{-h} d\mathcal{L}$ and for all $x \in (0, \lambda]$ it holds that $-\int_0^x q_{-h}(t)dt \leq \Psi(x)$. For any $A \in \mathcal{B}[0,1]$,

$$\begin{aligned} \mathbf{Q}(A) = \int 1_A d\mathbf{Q} &= -\int 1_A q_{-h} d\mathcal{L} \\ &\leq -\int_0^1 q_{1_A}(t) q_{q_{-h}}(1-t)dt = -\int_{1-\mathcal{L}(A)}^1 q_{q_{-h}}(1-t)dt, \end{aligned}$$

due to the Hardy-Littlewood inequalities (3.2). It is not hard to use the definition of quantiles (1.3) to show that for any random variable X, $q_{q_X}(t) = q_X(t)$ Lebesgue-almost surely, $\forall t \in [0,1]$. Thus

$$\begin{aligned} \mathbf{Q}(A) &= -\int_{1-\mathcal{L}(A)}^1 q_{q_{-h}}(1-t)dt \\ &= -\int_{1-\mathcal{L}(A)}^1 q_{-h}(1-t)dt \\ &= -\int_0^{\mathcal{L}(A)} q_{-h}(t)dt \\ &\leq \Psi(\mathcal{L}(A)). \end{aligned}$$

We conclude that $\mathbf{Q} \in \text{core}(\Psi \circ \mathcal{L})$, i.e., $-q_{-h} \in \text{core}(\Psi \circ \mathcal{L})$. ◇

4.3 The robust representation of $\rho_{\mu,\lambda}$

Recall the setup in section 4.1, the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is assumed to be atomless, $\lambda \in (0,1)$ is fixed, and the probability measure μ on $[0,\lambda]$ satisfies $\mu(\{0\}) = 0$. The λ-quantile dependent Weighted Value-at-Risk is

$$\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} CVaR_\gamma(X)\mu(d\gamma) = -\int_0^\lambda q_X(t)\phi(t)dt,$$

with

$$\phi(t) = \int_{(t,\lambda]} \frac{1}{s}\mu(ds), \quad t \in (0,\lambda]. \tag{4.10}$$

The function $\phi(t)$ is positive and monotone decreasing on $(0,\lambda]$ with $\phi(\lambda) = 0$. Consequently, $-\phi$ can be viewed as the upper quantile function of some probability distribution function $\nu_{-\phi}$ such that

$$q^+_{\nu_{-\phi}}(t) = -\phi(t), \quad \forall t \in (0,\lambda]. \tag{4.11}$$

Thus, another equivalent form of $\rho_{\mu,\lambda}$ is obtained since $q^+_{\nu_{-\phi}}(t) = q_{\nu_{-\phi}}(t)$ Lebesgue-a.e.:

$$\rho_{\mu,\lambda}(X) = \int_0^\lambda q_X(t)q_{\nu_{-\phi}}(t)dt. \tag{4.12}$$

In this section, we give the robust representation of $\rho_{\mu,\lambda}$ via two representation sets. The first one is the set of all probability distribution measures that are λ-quantile uniformly preferred over $\nu_{-\phi}$, and the second one is given by the core of λ-quantile concave distortion $\Psi \circ \mathbf{P}$ defined by (4.7). Finally, we show that these two representation sets coincide.

4.3.1 The robust representation of $\rho_{\mu,\lambda}$ via the λ-quantile uniform preference

Let $\mathbb{R}^- := (-\infty, 0]$ and $\mathcal{B}(\mathbb{R}^-) := \mathcal{B}(-\infty, 0]$ be the Borel σ-algebra on $(-\infty, 0]$. We define

$$\Phi := \{\nu \text{ probability distribution measure on } (\mathbb{R}^-, \mathcal{B}(\mathbb{R}^-)) : \nu \underset{uni(\lambda)}{\succcurlyeq} \nu_{-\phi}\}.$$

Lemma 4.8. *For $X \in L^p$, $1 \le p \le \infty$, it is true that*

$$\rho_{\mu,\lambda}(X) = \max_{\nu \in \Phi} \left\{ \int_0^\lambda q_X(t) q_\nu(t) dt + q_X^+(\lambda) \int_\lambda^1 q_\nu(t) dt \right\}. \tag{4.13}$$

Proof. Let ν be in Φ. Define $C_X := q_X^+(\lambda)$. So $C_X - q_X(t) \ge 0$ is decreasing on $(0, \lambda]$. By Theorem 4.5,

$$\int_0^\lambda (C_X - q_X(t)) q_\nu(t) dt \ge \int_0^\lambda (C_X - q_X(t)) q_{\nu_{-\phi}}(t) dt. \tag{4.14}$$

Since $\mu([0,\lambda]) = 1$, we have $\int_0^\lambda q_{\nu_{-\phi}}(t)dt = -1$. Therefore, (4.14) becomes

$$C_X \int_0^\lambda q_\nu(t)dt - \int_0^\lambda q_X(t) q_\nu(t) dt \ge -C_X - \int_0^\lambda q_X(t) q_{\nu_{-\phi}}(t) dt = -C_X - \rho_{\mu,\lambda}(X).$$

And

$$\begin{aligned}
\rho_{\mu,\lambda}(X) &\ge \int_0^\lambda q_X(t) q_\nu(t) dt - C_X - C_X \int_0^\lambda q_\nu(t) dt \\
&= \int_0^\lambda q_X(t) q_\nu(t) dt + C_X \left(-1 - \int_0^\lambda q_\nu(t) dt \right) \\
&= \int_0^\lambda q_X(t) q_\nu(t) dt + q_X^+(\lambda) \int_\lambda^1 q_\nu(t) dt.
\end{aligned}$$

Since it is obvious that $\nu_{-\phi} \in \Phi$, by (4.12), we obtain (4.13). ◇

Let $\mathcal{Q}_\mu$ be the set of all probability measures such that the probability distribution measure $\nu_{-\varphi}$ of the negative value of the Radon-Nikodým derivative $\varphi := \frac{d\mathbf{Q}}{d\mathbf{P}}$ is in Φ, i.e.,

$$\mathcal{Q}_\mu := \left\{ \mathbf{Q} \, \text{probability measure on } (\Omega, \mathcal{F}) : \mathbf{Q} \ll \mathbf{P}, \, \varphi := \frac{d\mathbf{Q}}{d\mathbf{P}}, \text{ and } \nu_{-\varphi} \in \Phi \right\}.$$

The following Theorem gives the robust representation of $\rho_{\mu,\lambda}$, which is the λ-quantile variation of Corollary 4.74 in Föllmer and Schied (2004) based on uniform preference instead of concave distortion.

Theorem 4.9. *For all $X \in L^p$, $1 \le p \le \infty$, it is true that*

$$\rho_{\mu,\lambda}(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_\mu} \mathbb{E}_{\mathbf{Q}}[-X]. \tag{4.15}$$

If $\rho_{\mu,\lambda}(X) < \infty$, then the supremum can be attained by choosing $\mathbf{Q}_X$ such that its Radon-Nikodým derivative $\frac{d\mathbf{Q}_X}{d\mathbf{P}} = f(X)$ with f a decreasing function, $f(x) = 0$ for $F_X(x) > \lambda$, and

$$f(x) = \begin{cases} \phi(F_X(x)), & \text{if } x \text{ is a continuous point of } F_X \text{ and } F_X(x) \le \lambda, \\ \frac{\int_{F_X(x-)}^{F_X(x)} \phi(t)dt}{F_X(x) - F_X(x-)}, & \text{if } x \text{ is a discontinuous point of } F_X \text{ and } F_X(x) \le \lambda. \end{cases} \tag{4.16}$$

The set $\mathcal{Q}_\mu$ is the maximum set of probability measures that represents $\rho_{\mu,\lambda}$ in the sense that for all $\mathbf{Q} \in \mathcal{Q}_\mu$, $\rho^*(\mathbf{Q})$ defined in Theorem 2.19 is equal to 0, and for all $\mathbf{Q} \ll \mathbf{P}$ such that $\mathbf{Q} \notin \mathcal{Q}_\mu$, $\rho^*(\mathbf{Q}) = \infty$.

Proof. We show the theorem in four steps.

Step 1: We show that $\rho_{\mu,\lambda}(X) \ge \sup_{\mathbf{Q}\in\mathcal{Q}_\mu} \mathbb{E}_{\mathbf{Q}}[-X]$. For $\mathbf{Q} \in \mathcal{Q}_\mu$, let $\varphi := \frac{d\mathbf{Q}}{d\mathbf{P}}$ and $q_{-\varphi}$ be a quantile function. By Lemma 4.8 and the Hardy-Littlewood inequality (3.2),

$$\begin{aligned} \rho_{\mu,\lambda}(X) &\ge \int_0^\lambda q_X(t)q_{-\varphi}(t)dt + q_X^+(\lambda)\int_\lambda^1 q_{-\varphi}(t)dt \\ &\ge \int_0^\lambda q_X(t)q_{-\varphi}(t)dt + \int_\lambda^1 q_X(t)q_{-\varphi}(t)dt \\ &= \int_0^1 q_X(t)q_{-\varphi}(t)dt \\ &\ge \mathbb{E}[-X\varphi] \\ &= \mathbb{E}_{\mathbf{Q}}[-X]. \end{aligned}$$

Thus,

$$\rho_{\mu,\lambda}(X) \ge \sup_{\mathbf{Q}\in\mathcal{Q}_\mu} \mathbb{E}_{\mathbf{Q}}[-X], \qquad \forall X \in L^p. \tag{4.17}$$

Step 2: We show f defined by (4.16) is a probability density function and $\mathbf{Q}_X$ with density (4.16) is in the set $\mathcal{Q}_\mu$. Let U be a uniformly distributed random variable on $[0,1]$. Obviously, $f \ge 0$. To show $\mathbb{E}[f(X)] = 1$, we first use the definition of the conditional expectation to check

$$f(q_X(U)) = \mathbb{E}[\phi(U)|q_X(U)].$$

First, $f(q_X(U))$ is obviously $\sigma(q_X(U))$-measurable, and $\mathbb{E}[|f(q_X(u))|] = \mathbb{E}[|f(X)|] = 1 < \infty$, since X and $q_X(U)$ have the same distribution. We check that the partial averaging property is satisfied. Let $A \in \sigma(q_X(U))$, $A = A_c \cup A_d$, where

$$A_c := \{\omega \in \Omega : q_X(U(\omega)) \text{ is a continuous point of } F_X \text{ and } F_X(q_X(U(\omega))) \leq \lambda\},$$

and

$$\begin{aligned} A_d &:= \{\omega \in \Omega : q_X(U(\omega)) \text{ is a discrete point of } F_X \text{ and } F_X(q_X(U(\omega))) \leq \lambda\} \\ &= \cup_i \{\omega : q_X(U(\omega)) = x_i\}. \end{aligned}$$

Denote $A_{d_i} := \{\omega : \ q_X(U(\omega)) = x_i\}$, then $\mathbf{P}(A_{d_i}) = F_X(x_i) - F_X(x_i-)$, and

$$\begin{aligned} \int_{A_{d_i}} f(q_X(U))d\mathbf{P} &= \int_{A_{d_i}} \frac{1}{F_X(x_i) - F_X(x_i-)} \int_{F_X(x_i-)}^{F_X(x_i)} \phi(t)dtd\mathbf{P} \\ &= \mathbf{P}(A_{d_i}) \frac{1}{F_X(x_i) - F_X(x_i-)} \int_{F_X(x_i-)}^{F_X(x_i)} \phi(t)dt \\ &= \int_{F_X(x_i-)}^{F_X(x_i)} \phi(t)dt \\ &= \int_{A_{d_i}} \phi(U)d\mathbf{P}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_A f(q_X(U))d\mathbf{P} &= \int_{A_c} f(q_X(U))d\mathbf{P} + \int_{A_d} f(q_X(U))d\mathbf{P} \\ &= \int_{A_c} \phi(F_X(q_X(U)))d\mathbf{P} + \sum_i \int_{A_{d_i}} f(q_X(U))d\mathbf{P} \\ &= \int_{A_c} \phi(U)d\mathbf{P} + \sum_i \int_{A_{d_i}} \phi(U)d\mathbf{P} \\ &= \int_A \phi(U)d\mathbf{P}. \end{aligned}$$

Hence, we obtain $f(q_X(U)) = \mathbb{E}[\phi(U)|q_X(U)]$. By properties of conditional expectation,

$$\mathbb{E}[f(X)] = \mathbb{E}[f(q_X(U))] = \mathbb{E}[\mathbb{E}[\phi(U)|q_X(U)]] = \mathbb{E}[\phi] = 1.$$

Note that the last equality is valid due to the definition of ϕ. So the function f

defined by (4.16) is a probability density function.

To show $-\frac{d\mathbf{Q}_X}{d\mathbf{P}}$ is uniformly preferred over $-\phi$, we use statement *1*, part **b** of Theorem 4.5. Let u_λ be a $\nu_{-\phi}$-λ-quantile utility function, then u_λ is concave. By applying the Jensen's Inequality, we yield

$$\begin{aligned}\mathbb{E}\left[u_\lambda\left(-\frac{d\mathbf{Q}_X}{d\mathbf{P}}\right)\right] &= \mathbb{E}[u_\lambda(-f(X))] = \mathbb{E}[u_\lambda(-f(q_X(U)))] = \mathbb{E}[u_\lambda(\mathbb{E}[-\phi|q_X(U)])] \\ &\geq \mathbb{E}[\mathbb{E}[u_\lambda(-\phi)|q_X(U)]] = \mathbb{E}[u_\lambda(-\phi)].\end{aligned}$$

Step 3: Show that $\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q}\in\mathcal{Q}_\mu} \mathbb{E}_\mathbf{Q}[-X]$. The proof of the Hardy-Littlewood inequality (Theorem A.24 of Föllmer and Schied (2004)) provides an optimal $\mathbf{Q}_X$ which has the probability density function $f(X)$ given by (4.16) such that

$$\rho_{\mu,\lambda}(X) = \int_0^\lambda q_X(t) q_{\mu_{-\phi}}(t)dt = \mathbb{E}_{\mathbf{Q}_X}[-X].$$

Together with Step 1 and Step 2, we obtain $\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q}\in\mathcal{Q}_\mu} \mathbb{E}_\mathbf{Q}[-X]$.

Step 4: Show that $\mathcal{Q}_\mu$ is the maximal set that represents $\rho_{\mu,\lambda}$. We denote the maximal set by $\mathcal{Q}_{\max}$. In Step 1 and Step 2, we have shown that $\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q}\in\mathcal{Q}_\mu} \mathbb{E}_\mathbf{Q}[-X]$, which means $\mathcal{Q}_\mu \subset \mathcal{Q}_{\max}$. Note that $\rho_{\mu,\lambda}$ is a λ-quantile law invariant risk measure, therefore, by Theorem 2.19 and Theorem 3.3, we obtain two forms of $\rho^*_{\mu,\lambda}(\mathbf{Q})$ for all $\mathbf{Q} \in \mathcal{Q}_p$ (where $\varphi = \frac{d\mathbf{Q}}{d\mathbf{P}}$):

$$\begin{aligned}\rho^*_{\mu,\lambda}(\mathbf{Q}) &= \sup_{X\in L^p} \left(\mathbb{E}_\mathbf{Q}[-X] - \rho_{\mu,\lambda}(X)\right) \\ &= \sup_{X\in L^p} \left(\int_0^\lambda q_X(t) q_{-\varphi}(t)dt + q_X^+(\lambda)\int_\lambda^1 q_{-\varphi}(t)dt - \rho_{\mu,\lambda}(X)\right).\end{aligned}$$

Consider a $\tilde{\mathbf{Q}}$ such that $\tilde{\mathbf{Q}} \ll \mathbf{P}$ but $\nu_{-\tilde{\varphi}}$ is not λ-quantile preferred over $\nu_{-\phi}$, where $\tilde{\varphi} = \frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}}$. Therefore, by Theorem 4.5, there is a $r \in (0, \lambda)$ such that

$$\int_0^r q_{\nu_{-\tilde{\varphi}}}(t)dt < \int_0^r q_{\nu_{-\phi}}(t)dt.$$

We show that for some $X \in L^p$, $\rho^*_{\mu,\lambda}(\tilde{\mathbf{Q}}) = \infty$. Let $X \in L^p$ be a random variable such that $\mathbf{P}(X=-N)=r$ and $\mathbf{P}(X=0)=1-r$. Then

$$\begin{aligned}
&\int_0^\lambda q_X(t)q_{-\tilde{\varphi}}(t)dt + q_X^+(\lambda)\int_\lambda^1 q_{-\phi}(t)dt - \rho_{\mu,\lambda}(X)\\
&= \int_0^r (-N)q_{-\tilde{\varphi}}(t)dt - \int_0^r (-N)q_{\nu_{-\phi}}(t)dt\\
&= N\left(\int_0^r q_{\nu_{-\phi}}(t)dt - \int_0^r q_{-\tilde{\varphi}}(t)dt\right) \to \infty, \quad \text{as } N\to\infty.
\end{aligned}$$

Hence, $\rho^*_{\mu,\lambda}(\tilde{\mathbf{Q}})=\infty$. ◇

4.3.2 The robust representation of $\rho_{\mu,\lambda}$ via the core of the λ-quantile concave distortion

For $X\in L^\infty(\Omega,\mathcal{F},\mathbf{P})$, Kusuoka (2001) showed that any Weighted Value-at-Risk ρ_μ can be written as a Choquet integral. Applying the result to our case of λ-quantile dependent Weighted Value-at-Risk $\rho_{\mu,\lambda}$, we have

$$\rho_{\mu,\lambda}(X) = \begin{cases} \int_0^{q_X(\lambda)}(\Psi(\mathbf{P}(X<x))-1)dx + \int_{-\infty}^0 \Psi(\mathbf{P}(X<x))dx, & \text{if } q_X(\lambda)>0,\\ \int_{-\infty}^{q_X(\lambda)} \Psi(\mathbf{P}(X<x))dx, & \text{if } q_X(\lambda)\le 0. \end{cases} \tag{4.18}$$

where the function $\Psi:[0,1]\to[0,1]$ is defined as

$$\Psi(x) = \int_0^x \phi(t)dt, \tag{4.19}$$

with ϕ given by (4.10). Obviously, Ψ is increasing and concave with right-hand side derivative $\Psi'(t+)=\phi(t)$ such that $\Psi(0)=0$, $\Psi(x)=1$ for $x\in[\lambda,1]$.

The Choquet integral (4.18) is well defined when X is $\mathbf{P}$-almost surely bounded. Otherwise, we extend the Choquet integral to be ∞ when the integral on the right hand side of (4.18) is infinite. Under this definition, we show that equation (4.18) can be extended to all random variables that are in the space $L^p(\Omega,\mathcal{F},\mathbf{P})$ with $1\le p\le\infty$.

Lemma 4.10. *Suppose $1\le p\le\infty$ and $\lambda\in(0,1)$ is fixed. Let $\rho_{\mu,\lambda}: L^p\to\mathbb{R}\cup\{\infty\}$ be a λ-quantile dependent Weighted Value-at-Risk defined by (4.1). Then for all $X\in L^p$,* *the extended Choquet integral (4.18) holds true.*

PROOF. For $X \in L^p$, recall X_q is defined as $X_q := X1_{\{X<q_X(\lambda)\}} + q_X(\lambda)1_{\{X \geq q_X(\lambda)\}}$. Let $X_{q,n} := X_q \vee -n$ for some natural number n. When n is sufficiently large, $X_{q,n} \in L^\infty$. Thus, by Theorem 23 of Kusuoka (2001) or Theroem 4.64 of Föllmer and Schied (2004),

$$\rho_{\mu,\lambda}(-X_{q,n}) = \int_{-\infty}^{0} (\Psi(\mathbf{P}(X_{q,n} > x)) - 1)dx + \int_{0}^{\infty} \Psi(\mathbf{P}(X_{q,n} > x))dx = \int_{\Omega} X_{q,n} d(\Psi \circ \mathbf{P}) \tag{4.20}$$

Substituting $-X_{q,n}$ by $X_{q,n}$, yield

$$\begin{aligned}\rho_{\mu,\lambda}(X_{q,n}) &= \int_{\Omega} -X_{q,n} d(\Psi \circ \mathbf{P}) \\ &= \begin{cases} \int_0^{q_X(\lambda)} (\Psi(\mathbf{P}(X < x)) - 1)dx + \int_{-n}^{0} \Psi(\mathbf{P}(X < x))dx, & \text{if } q_X(\lambda) > 0, \\ \int_{-n}^{q_X(\lambda)} \Psi(\mathbf{P}(X < x))dx, & \text{if } q_X(\lambda) \leq 0. \end{cases}\end{aligned}$$

Since $\rho_{\mu,\lambda}$ is continuous from above, which implies $\rho_{\mu,\lambda}(X_{q,n}) \nearrow \rho_{\mu,\lambda}(X_q)$, equivalently,

$$\begin{aligned}\rho_{\mu,\lambda}(X_q) &= \lim_{n\to\infty} \int_{\Omega} -X_{q,n} d(\Psi \circ \mathbf{P}), \\ &= \begin{cases} \int_0^{q_X(\lambda)} (\Psi(\mathbf{P}(X < x)) - 1)dx + \int_{-\infty}^{0} \Psi(\mathbf{P}(X < x))dx, & \text{if } q_X(\lambda) > 0, \\ \int_{-\infty}^{q_X(\lambda)} \Psi(\mathbf{P}(X < x))dx, & \text{if } q_X(\lambda) \leq 0. \end{cases}\end{aligned}$$

Note that it is possible for the limit to be ∞. Since $\rho_{\mu,\lambda}(X) = \rho_{\mu,\lambda}(X_q)$, we obtain equation (4.18). ◇

By Definition 4.6, the composite function $\Psi \circ \mathbf{P}$ appeared in (4.18) is the λ-quantile dependent concave distortion of the probability measure $\mathbf{P}$. Observe that it is a normalized monotone set function on $\mathcal{F}$ which is a submodular satisfying the following definition (Denneberg (1994)):

Definition 4.11. (submodular) *A set function $\mu : \mathcal{F} \to [0, \infty]$ is a submodular if for any $A, B \in \mathcal{F}$ such that $A \cup B$, $A \cap B \in F$ implies $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$.*

The composite function $\Psi \circ \mathbf{P}$ satisfies the following:

$$\Psi(\mathbf{P}(\emptyset)) = 0, \quad \Psi(\mathbf{P}(\Omega)) = 1,$$

$$\Psi(\mathbf{P}(A)) \leq \Psi(\mathbf{P}(B)), \qquad \text{for any } A, B \in \mathcal{F} \text{ such that } A \subset B,$$

$$\Psi(\mathbf{P}(A \cup B)) + \Psi(\mathbf{P}(A \cap B)) \leq \Psi(\mathbf{P}(A)) + \Psi(\mathbf{P}(B)), \qquad \text{for all } A, B \in \mathcal{F}.$$

For the representation of equation (4.18), let us recall Proposition 10.3 of Denneberg (1994).

Proposition 4.12. *Let μ be a monotone set function on an algebra $\mathcal{A}$, where $\mathcal{A}$ is a subset of the family of subsets of Ω and define*

$$core(\mu) := \{\nu : \nu \text{ additive on } \mathcal{A}, \, \nu(\Omega) = \mu(\Omega), \, \nu(A) \leq \mu(A), \, \forall A \in \mathcal{A}\}.$$

μ is submodular if and only if $core(\mu) \neq \emptyset$ and for all X such that X is $\mathcal{A}$-measurable and $\int |X| d\mu < \infty$,

$$\int X d\mu = \sup_{\nu \in core(\mu)} \int X d\nu.$$

Under this condition μ is the upper envelop of $core(\mu)$, i.e., $\mu = \sup_{\nu \in core(\mu)} \nu$.

Rewrite equation (4.18) as

$$\rho_{\mu,\lambda}(X) = -\int X d(\Psi \circ \mathbf{P}),$$

and recall from (4.8)

$$\text{core}(\Psi \circ \mathbf{P}) = \{\mathbf{Q} \text{ probability measure on } (\Omega, \mathcal{F}) : \mathbf{Q} \ll \mathbf{P}, \, \mathbf{Q}(A) \leq \Psi(\mathbf{P}(A)), \, \forall A \in \mathcal{F}\}.$$

We also recall Proposition 5.2(iii) and Proposition 9.3 of Denneberg (1994):

Proposition 4.13. *Let $X : \Omega \to \mathbb{R}$ be an upper S-measurable function and μ, ν monotone set functions on $S \subset 2^{\Omega}$. If $\mu(\Omega) = \nu(\Omega)$ or $X \geq 0$, then $\mu \leq \nu$ implies $\int X d\mu \leq \int X d\nu$.*

Proposition 4.14. *Let 2^{Ω} be the collection of all subsets of Ω and μ be a monotone* *set function on 2^Ω. If μ is submodular, then*

$$\int |X| d\mu < \infty \Leftrightarrow \int X d\mu < \infty \text{ and } \int -X d\mu < \infty.$$

Apply Proposition 4.12 and Proposition 4.14, and note that $\mathbf{P} \in \text{core}(\Psi \circ \mathbf{P})$, we have the following representation for the λ-quantile dependent Weighted Value-at-Risk:

Theorem 4.15. *For all $X \in L^p(\Omega, \mathcal{F}, \mathbf{P})$, $1 \leq p \leq \infty$, it is true that*

$$\rho_{\mu,\lambda}(X) = \sup_{\mathbf{Q} \in core(\Psi \circ \mathbf{P})} \int_\Omega -X d\mathbf{Q} = \sup_{\mathbf{Q} \in core(\Psi \circ \mathbf{P})} \mathbb{E}_{\mathbf{Q}}[-X]. \tag{4.21}$$

The supremum can be attained if $\rho_{\mu,\lambda}(X) < \infty$.

Proof. From Proposition 4.13, we have

$$\rho_{\mu,\lambda}(X) \geq \sup_{\mathbf{Q} \in \text{core}(\Psi \circ \mathbf{P})} \mathbb{E}_{\mathbf{Q}}[-X], \qquad \text{for } X \in L^p.$$

For the reverse inequality, we first assume $\rho_{\mu,\lambda}(X) = \int -X d(\Psi \circ \mathbf{P}) < \infty$. If X is bounded above, then $\int X d(\Psi \circ \mathbf{P}) = \rho_{\mu,\lambda}(-X) < \infty$. Then, Proposition 4.14 implies $\int |X| d(\Psi \circ \mathbf{P}) < \infty$. Since $(\Psi \circ \mathbf{P})$ is submodular, there exists some $\mathbf{Q}_X \in \text{core}(\Psi \circ \mathbf{P})$ such that $\rho_{\mu,\lambda}(X) = \mathbb{E}_{\mathbf{Q}_X}[-X]$. Thus, $\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q} \in \text{core}(\Psi \circ \mathbf{P})} \mathbb{E}_{\mathbf{Q}}[-X]$.
If X is not bounded above, we can find some natural number n such that

$$X_n := X \wedge n \qquad \text{and} \qquad q^+_{X_n}(\lambda) = q^+_X(\lambda). \tag{4.22}$$

Then $\{X > q^+_X(\lambda)\} = \{X_n > q^+_{X_n}(\lambda)\}$. In addition, by the definition of λ-quantile dependence (Definition 2.2), $\rho(X_n) = \rho(X)$. Since X_n is bounded above, there is some $\mathbf{Q}_n \in \text{core}(\Psi \circ \mathbf{P})$ such that $\rho_{\mu,\lambda}(X_n) = \mathbb{E}_{\mathbf{Q}_n}[-X_n]$. The proof of Lemma 2.9 shows that $\mathbf{Q}_n(X_n > q^+_{X_n}(\lambda)) = 0$. Due to the equality of the sets $\{X > q^+_X(\lambda)\}$ and $\{X_n > q^+_{X_n}(\lambda)\}$, $\mathbf{Q}_n(X > q^+_X(\lambda)) = 0$. Therefore,

$$\rho_{\mu,\lambda}(X) = \rho_{\mu,\lambda}(X_n) = \mathbb{E}_{\mathbf{Q}_n}[-X_n] = \mathbb{E}_{\mathbf{Q}_n}[-X].$$

Thus, we obtain $\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q}\in\text{core}(\Psi\circ\mathbf{P})} \mathbb{E}_{\mathbf{Q}}[-X]$.

Now, assume $\rho_{\mu,\lambda}(X) = \infty$. We need to show $\sup_{\mathbf{Q}\in\text{core}(\Psi\circ\mathbf{P})} \mathbb{E}_{\mathbf{Q}}[-X] = \infty$. We can assume without loss of generality that X is bounded above. (If not, use X_n as (4.22) and Lemma 2.20). Define

$$X_m := X \vee (-m).$$

Then $X_m \searrow X$. Since $\rho_{\mu,\lambda}$ is continuous from above, $\rho_{\mu,\lambda}(X_m) \nearrow \rho_{\mu,\lambda}(X) = \infty$. For sufficiently large m, X_m is bounded, which implies $\int |X_m| d(\Psi\circ\mathbf{P}) < \infty$. Applying Proposition 4.12, there exists a $\mathbf{Q}_m \in \text{core}(\Psi\circ\mathbf{P})$ such that

$$\rho_{\mu,\lambda}(X_m) = \mathbb{E}_{\mathbf{Q}_m}[-X_m] \le \mathbb{E}_{\mathbf{Q}_m}[-X] \to \infty, \quad \text{as } n \to \infty.$$

This shows that

$$\sup_{\mathbf{Q}\in\text{core}(\Psi\circ\mathbf{P})} \mathbb{E}_{\mathbf{Q}}[-X] = \infty.$$

◇

Theorem 4.7 implies that the two approaches in representing $\rho_{\mu,\lambda}$ by Theorem 4.9 and Theorem 4.15 are equivalent, and the representation sets $\mathcal{Q}_\mu$ and core$(\Psi\circ\mathbf{P})$ coincide.

4.4 Two examples

In this section, we discuss two examples of the λ-quantile dependent Weighted Value-at-Risk. The Conditional Value-at-Risk is a well known convex risk measure, we give its robust representation using the approach we derived in subsection 4.3.2 and check that the representation set coincides with the classic one. The uniform λ-quantile dependent Weighted Value-at-Risk is a new convex risk measure, for which the robust representation will be given using the approach discussed in subsection 4.3.2.

4.4.1 The Conditional Value-at-Risk

Let $\lambda \in (0,1)$ be fixed. Take $\mu(\{\lambda\}) = 1$. Then

$$\rho_{\mu,\lambda}(X) = CVaR_\lambda(X) = -\frac{1}{\lambda}\int_0^\lambda q_X(t)dt.$$

Its robust representation is well known (see Föllmer and Schied (2004) for the L^∞ case, Kaina and Rüschendorf (2009) for the L^1 case, and Cherny (2006) for the L^0 case) as $CVaR_\lambda(X) = \sup_{\mathbf{Q}\in\mathcal{Q}_\lambda} \mathbb{E}_{\mathbf{Q}}[-X]$, where the maximal representation set $\mathcal{Q}_\lambda$ is given by

$$\mathcal{Q}_\lambda := \left\{\mathbf{Q}\,\text{probability measure on } (\Omega,\mathcal{F}) \ : \mathbf{Q} \ll \mathbf{P}, \quad \frac{d\mathbf{Q}}{d\mathbf{P}} \le \frac{1}{\lambda} \quad \mathbf{P} - a.s.\right\}.$$

On the other hand, we can calculate from (4.10), (4.11) and (4.19) the functions

$$\phi(t) = \frac{1}{\lambda}1_{[0,\lambda)}, \quad \Psi(t) = \frac{t}{\lambda}1_{[0,\lambda)} + 1_{(\lambda,1]},$$
$$\nu_{-\phi}(\{-\tfrac{1}{\lambda}\}) = \lambda, \quad \nu_{-\phi}(\{0\}) = 1-\lambda.$$

Theorem 4.9 and Theorem 4.15 provides alternative representations

$$CVaR_\lambda(X) = \max_{\mathbf{Q}\in\mathcal{Q}_\mu} \mathbb{E}_Q[-X] = \max_{\mathbf{Q}\in\text{core}(\Psi\circ\mathbf{P})} \mathbb{E}_Q[-X],$$

where the representations sets

$$\mathcal{Q}_\mu = \left\{\mathbf{Q}\,\text{probability measure on } (\Omega,\mathcal{F}) : \mathbf{Q} \ll \mathbf{P}, \quad \nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_{-\phi}\right\}$$

and

$$\text{core}(\Psi\circ\mathbf{P}) = \{\mathbf{Q} \text{ probability measure on } (\Omega,\mathcal{F}) : \mathbf{Q} \ll \mathbf{P}, \mathbf{Q}(A) \le \Psi(\mathbf{P}(A)), \forall A \in \mathcal{F}\}$$

are coincide by Theorem 4.7. In the CVaR case, it is also straight-forward to check that for all $A \in \mathcal{F}$,

$$\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_{-\phi} \Leftrightarrow \frac{d\mathbf{Q}}{d\mathbf{P}} \le \frac{1}{\lambda}\mathbf{P} - a.s. \Leftrightarrow \mathbf{Q}(A) \le \frac{\mathbf{P}(A)}{\lambda} \wedge 1 \Leftrightarrow \mathbf{Q}(A) \le \Psi(\mathbf{P}(A)).$$

Therefore, the representation theorems derived in Section 4 match the classical result in the CVaR case as sets $\mathcal{Q}_\lambda = \mathcal{Q}_\mu = \text{core}(\Psi \circ \mathbf{P})$.

4.4.2 The uniform λ-quantile dependent Weighted Value-at-Risk

In the remark before Section 4.1, we mentioned a particular choice of probability measure μ with uniform distribution on $[0, \lambda]$, i.e., $\mu(ds) = \frac{1}{\lambda} ds$, $\forall s \in [0, \lambda]$. Then $\rho_{\mu,\lambda}(X) = \frac{1}{\lambda} \int_{[0,\lambda]} CVaR_\gamma(X) d\gamma$ is the average of $CVaR$ over the interval $[0, \lambda]$. The function ϕ, $\nu_{-\phi}$ and Ψ can be calculated from (4.10), (4.11) and (4.19) as

$$\phi(t) = \frac{1}{\lambda}(\ln \lambda - \ln t) 1_{[0,\lambda)}, \quad \Psi(t) = \frac{t}{\lambda}(\ln \lambda + 1 - \ln t) \wedge 1,$$
$$\nu_{-\phi}(dt) = \lambda e^{\lambda t + \ln \lambda}, \ \forall t \in [-\infty, 0), \quad \nu_{-\phi}(\{0\}) = 1 - \lambda.$$

Consequently the robust representations

$$CVaR_\lambda(X) = \max_{\mathbf{Q} \in \mathcal{Q}_\mu} \mathbb{E}_Q[-X] = \max_{\mathbf{Q} \in \text{core}(\Psi \circ \mathbf{P})} \mathbb{E}_Q[-X],$$

are characterized by sets

$$\mathcal{Q}_\mu = \left\{ \mathbf{Q} \, \text{probability measure on } (\Omega, \mathcal{F}) : \mathbf{Q} \ll \mathbf{P}, \quad \nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_{-\phi} \right\},$$

and

$$\text{core}(\Psi \circ \mathbf{P}) = \{ \mathbf{Q} \text{ probability measure on } (\Omega, \mathcal{F}) : \mathbf{Q} \ll \mathbf{P}, \, \mathbf{Q}(A) \leq \Psi(\mathbf{P}(A)), \, \forall A \in \mathcal{F} \}.$$

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