Domination and Dominion of Some Graphs

by

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ABSTRACT

Given a simple undirected graph $G = (V, E)$, a γ-set also known as dominating set is a subset $S \subseteq V$ such that for any vertex $v \in V$, either $v \in S$ or a neighbor u of v is in S. Given G, the size of its γ-set, denoted by γ, is its domination number and the dominion of G, denoted by ζ, counts the number of its γ-sets. The former is a well-studied concept while the latter is new. Each parameter accesses the reliability and the vulnerability of a network system when exposed to attacks. In this thesis, we introduce basic notions and topology of several graphs. For each such graph, we found and proved the aforementioned parameters while presenting several activities with solutions.

DEDICATION

ACKNOWLEDGEMENT

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Chapter 1 Introduction

In this chapter, we introduce the reader to graph theory, its history and basic notions.

1.1 History

Graph theory began in 1736 when Leonard Euler published a paper that contained the solution to the 7 bridges of Konigsberg (see Figure 1.1 left) problem [6]. Is it possible to take a walk around town crossing each bridge exactly once and wind up at your starting point? A graph (vertices and links) is used to model or represent the Konigsberg problem (see Figure 1.1 right). The answer to this problem is “no”. To help provide a solution to this problem, Euler used a drawing or a model that we call *graphs*, that reduces the problem down to its important elements, thus avoiding unnecessary details. We begin by introducing the basic information about graphs.

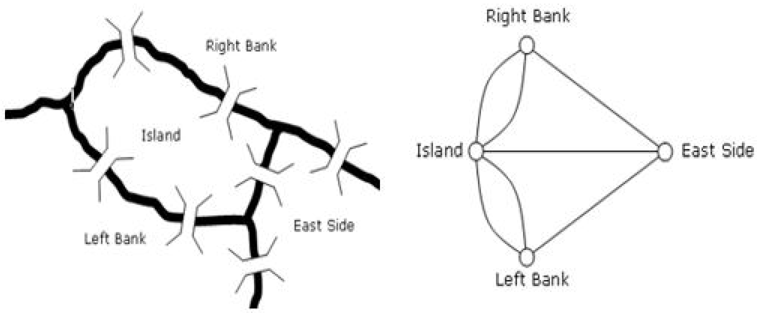

Figure 1.1: Konigsberg city and its corresponding graph model

James Sylvester of the 19^{th} century was said to be the first to use the word “graph” in the context of graph theory, who was one of many mathematicians that were intrigued with studying types of diagrams representing molecule [6]. Furthermore, in the mid-19^{th} century, Francis Guthrie presented the puzzle problem four-color problem led to the studies of graphs for theoretical and applied interests. The four-color problem seeks whether the countries on every map can be colored by using just four colors in a way that countries sharing an edge have different colors; however, this question can also be presented if the vertices of a planar graph can always be colored by using just four colors[7]. Although this problem would not be solved until more than a century later, it led to further studies within the broad field of graph theory. In 1862, De jaenish experimented to discover the minimum number of queens that can be placed on the chess board in such a way every square is either occupied by a queen or being attacked by at least one queen, which deals with domination. Through the work of solving practical problems in this time period, it made possible to obtain solutions important to graph theory such as Gustav Kirchhoff's complete set of equation for currents and voltages in electric circuits is summed up by representing his equations by a graph with skeleton tress and with the aid of these representations helped obtain linearly independent circuit systems. Arthur Cayley arrived at the situation of listing and describing trees with certain properties by starting from calculating the number of isomers of saturated hydrocarbons linking graphic theory and other sciences. At the start of the 20^{th} century, problems including graphs began to appear in physics, chemistry, electrical engineering, biology, economics, sociology and many other fields of study. Chemistry was a prominent field that would use lettered vertices to denote individual atoms while the lines denoted the chemical bonds with the degree of the vertices denoting the valence. The studies of connectivity properties, graph symmetry, and planarity are a few of a number of tools that helped direct the study of graph theory, which began to appear more frequently in the 1920s and 1930s, and soon extended throughout the 1940s and 1950s through the development of cybernetics and calculation techniques. As the range of problems that graph theory dealt with increased, so did the interest of graph theory; in addition, electronics computers became more useful with handling practical problems containing complex equations, leading to more discoveries using graphic theory. Methods were established to solve external problems, such as the construction of the maximum flow across a network, which could clearly be solved through graphs (tree and planar graphs) rather than arbitrary graphs. Problems in Graph theory could be less structured and free flowing (combinatorial) while others were more (geometric) structured. Problems such as graph circuits and graph imbedding were geometric in nature [6]. Other problems concerned with the modes of classification of graph such as classification by the properties of partitions of them. The results of these problems involving the existence of these graphs with certain properties can be shown by representing numbers by the degrees of the vertices of a given graph: "a collection of integers $0 < d_1 < \ldots < d_n$, the sum of which is even", can be understood by the degrees of the vertices of a graph not containing any loops and multiple edges. Problems questioning the enumeration of graphs with prescribed properties can be represented by problems of non-isomorphic graphs that contain the same number of vertices and/or edges.

Graphic Theory can be used to deal with problems pertaining to the connectivity of graphs and to study the structure of graphs based off of the connectivity of graphs; Analysis of the reliability of electronic circuits or communication networks raise the problem of solving the amount of non-intersecting edges that connect vertices within a graph [7]. The result of this problem yielded that the least number of vertices separating two non-adjacent vertices are equal to the greatest number of non-intersecting simple edges that connect this pair of vertices. Algorithms were developed to establish the degree of connectivity for of graphs. Other studies of graphs consisted of finding the number of edge progressions that include all the vertices or all the edges the graph; through multiple observations the resulting criteria is that a connected graph a cycle containing all the edges and passing through each edge once and only once exists if and only if the degrees of all except two vertices of the graph are even. If the set of vertices of a graph is traversed, only a number of sufficient conditions for the existence of a cycle passing through each vertex once is available. In all, results and methods of graph theory have been used extensively to aid in solving transportation problems, find optimal solutions for planning and control of project developments, establishing the best routes for supply of goods, and modeling complex technological processes in the creation of wide varieties of discrete situations.

The term graph in this branch of mathematics does not concern data charts such as line graphs or bars graphs yet involves a set of points (vertex) that are joined by lines that can be called edges[6]. A graph containing at most one edge between any two points without any loops is called a simple graph; if stated otherwise, the term graph is to be assumed to be a simple graph. When two points are connected by two or more edges, the graph is described as a multigraph. A complete graph is graph where every point contained in that graph is connected by and edge to every other point. In some cases, direction can be assigned to each edge to produce a graph called a directed graph or digraph. Other important basic concepts of graph theory are a point's degree and the types of path [6]. Each vertex has a number associated with it called its degree, which is the number of edges that are connected to it; a loop contributes 2 to the degree of the vertex. The number of vertices in a complete graph classifies its nature, therefore complete graphs are commonly denoted by K_n, where n refers to the number of vertices, and all vertices of K_n have a degree of $n - 1$. With this information, Euler's theorem pertaining to Konigsberg bridge Problem could be translated in modern terms as: if there is a path along edges of a *multigraph*, that travels along each edge only once, then there exists at most two vertices of odd degree; additionally, if the path begins and ends at the same vertex, then no vertices will have an odd degree. A path is described as the route of the edges of the graph; a path can follow one edge between two points or follow multiple edges through multiple points. When a path connects any two vertices in a graph, the graph is connected; furthermore, when a path begins at a point and ends at that same point without crossing any edge more than once is called a *circuit.* In 1750 Euler discovered the polyhedral formula $V - E + F = 2$, where the equation relates to the number of vertices (V), edges (E), and Face (F) of a polyhedron; the vertices and edges of this solid forms a graph leading to how graphs can be formulated on other surfaces.

Finally, graph theory and topology history are closely related and share common problems and techniques and the similarities between both topics led to a subsection named topological graph theory [6]. One problem in this area is called planar graphs, which are dotted graphs with edges on a plane no edges intersect.

1.2 Basic Definitions

A **simple graph** $G = (V, E)$ consists of $V = V(G)$, a nonempty set of objects called **vertices** (or nodes) and $E = E(G)$, a set of an unordered pair of distinct vertices called **edges**.

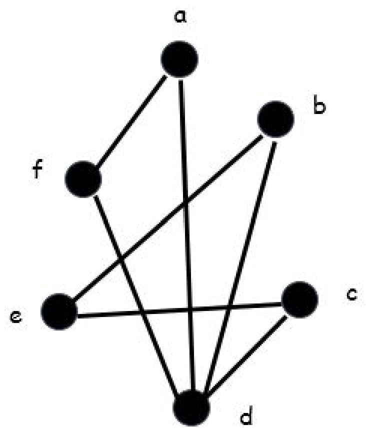

Figure 1.2: Example of a simple graph on 6 vertices

See Figure 1.2, for example. Vertices, say u and v that share an endpoint are said to be **adjacent**; u is also said to be a neighbor of v and vice-versa the edge denoted by uv is said to be incident to the vertices u and v. The **order** of the graph G is the size of its vertex set which we denote by $|V|$ and the size of the edge set, denoted by $|E|$, is called **size** of the graph G. The degree of vertex, v denoted by $deg(v)$, is the number of edges incident to v ; that is the size of its neighbor. A vertex of degree 0 is said to be **isolated** while a vertex of degree 1 is called a **leaf**. The **minimum degree** of G, denoted by $\delta(G)$, is its smallest vertex degree, and the **maximum degree** of G denoted by $\Delta(G)$ is the largest degree among its vertices. A vertex u is said to be connected to a vertex v, in a graph G, if there exists a sequence of edges (or path) from u to v in G. A graph G is **connected** if there is a path that connects every two of its vertices. There are other types of graphs such as multigraphs (when multiple edges are allowed between vertices), pseudographs (when a vertex is allowed to be connected to itself, as in a loop) and directed graphs (when each edge is given an orientation, using an arrow). However, our thesis will focus only on simple graphs, as previously defined.

In Chapter 2, we define and introduce some fundamental properties of nine graphs; one of them is new. In Chapter 3, we introduce the notions of dominion and domination with several examples, activities with solutions that can be introduced at a high school level. Further, in Chapter 4, we present several results on dominion and domination values of the nine graphs introduced in Chapter 2. We close this thesis in Chapter 5 with several useful directions and open problems.

Chapter 2 Topological Properties of Some Graphs

In this chapter, we present some basic graph properties, after we define them. We give seven such properties for seven common graphs on $n \geq 2$ vertices.

2.1 Definitions

Suppose G is a simple graph and $v \in V(G)$. The distance between two vertices $u, v \in V(G)$, often denoted by $d_G(u, v)$, is the length (number of edges) of their shortest path in G; this is also known as a **geodesic distance**. Given v, the **eccentricity** of v, written as $\epsilon(v)$ is the maximum of the distance to any vertex in the graph, i.e., $\epsilon(v) = \max_{u \in V}\{d_G(v, u)\}$. Further, the **diameter, d** of a graph is the maximum eccentricity of any vertex in the graph. In other words, the diameter is the longest distance between any two vertices in the graph. So, $d = \max_{v \in V} \epsilon(v)$. The **radius, r** of a graph is the minimum eccentricity among all vertices in the graph in which case $r = \min_{v \in V} \epsilon(v)$. These parameters are very useful in classifying acyclic (tree-like) graphs. Figure 2.1 shows two trees with the same radius but different diameters.

Likewise, for cyclic graphs, we define the following: the length of its shortest cycle is its **girth, g,** while the length of its longest cycle is its **circumference** which we denote by c. Note that, if a graph G is acyclic, then $g(G) = c(G) = \infty$ and if G is disconnected then $r(G) = \infty$.

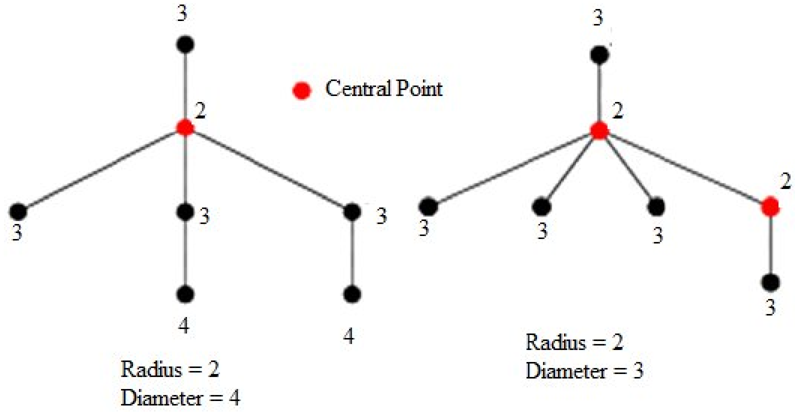

Figure 2.1: Diameters of two trees of radius 2

2.2 Topological Properties of Some Graphs

here, we discuss nine different graphs, which come from well-known classes of graphs. For each graph, we present their seven (7) topological properties, after their definition. We list these properties as they obviously stem from the definition. Hence, no proof or additional statements is necessary.

2.2.1 Complete graphs

A **complete graph** also known as **cliques** on n vertices, denoted by K_n is a graph where every pair of vertices are adjacent. Below is a family of complete graphs, K_2, K_3, K_4, and K_5 (from left to right).

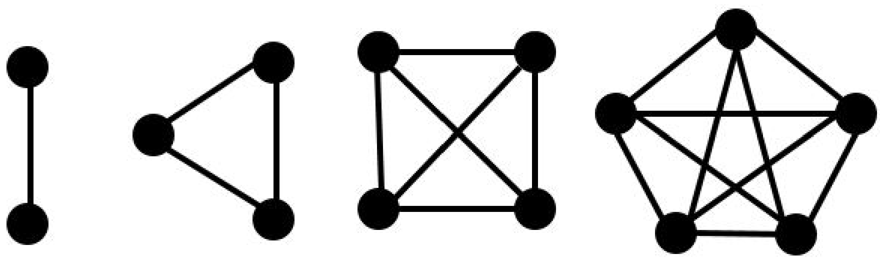

Figure 2.2: A family of four complete graphs

Topological Properties:

Given a complete graph on $n \geq 2$, we have

1. size (number of edges): $\binom{n}{2} = \dfrac{n(n-1)}{2}$, $n \geq 2$

2. δ (minimum degree): $n-1$

3. Δ (maximum degree): $n-1$

4. r (radius): 1

5. d (diameter): 1

6. g (girth): 3

7. c (circumference): n

2.2.2 Cycles

A **cycle** on n vertices, denoted by C_n is a graph with exactly one closed path. Here is a C_5, a cycle on 5 vertices.

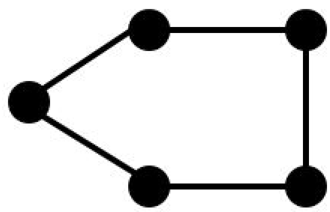

Figure 2.3: A Cycle on 5 vertices

Topological Properties:

For $n \geq 3$, we have

1. size (number of edges): n

2. δ (minimum degree): $n-1$

3. Δ (maximum degree): $n-1$

4. r (radius): $\lfloor \frac{n}{2} \rfloor$

5. d (diameter): $\lfloor \frac{n}{2} \rfloor$

6. g (girth): n

7. c (circumference): n

2.2.3 Trees

A **tree** also known as an **acyclic** graph on n vertices, denoted by T_n is a graph with no cycle. Figure 2.4 is an example of a tree on 6 vertices.

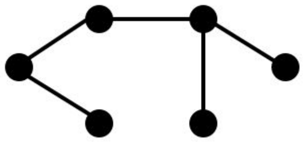

Figure 2.4: A tree on 6 vertices

Because trees are made of finitely many non-isomorphic members, we consider only one of of its members: the stars. See Figure 2.5 as an example of a Star on 8 vertices. Thus, a star graph on n vertices is simply a central vertex that is connected to $n-1$ leaves.

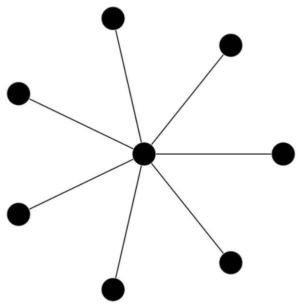

Figure 2.5: A Star on 8 vertices

Topological Properties:

Given a star graph on $n \geq 2$ vertices, we have

1. size (number of edges): $n-1$
2. δ (minimum degree): 1
3. Δ (maximum degree): $n-1$
4. r (radius): 2
5. d (diameter): 2
6. g (girth): ∞
7. c (circumference): ∞

2.2.4 2-trees

As a generalization of a tree, a *k-tree* is a graph which arises from a k-clique by 0 or more iterations of adding n new vertices, each joined to a k-clique in the old graph; This process generates finitely many non-isomorphic k-trees. When $k \geq 2$, are shown to be useful in constructing reliable network in [2]. When $k = 2$, we consider a particular 2-tree is also known as a *Fan*. Figure 2.6 is an example of a Fan on 5 vertices.

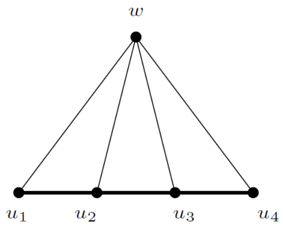

Figure 2.6: A Fan graph

Topological Properties:

Given a Fan on $n \geq 3$ vertices, we have

1. size (number of edges): $2(n-1) - 1 = 2n - 3$
2. δ (minimum degree): 2
3. Δ (maximum degree): $n - 1$
4. r (radius): 2
5. d (diameter): $n - 2$
6. g (girth): 3
7. c (circumference): n

2.2.5 Wheel

A Wheel on n vertices, denoted by W_n, is cycle on $n - 1$ joined to a central vertex, say w, and every vertex of the cycle is connected to w. The vertex w is sometimes referred to as *hub*. Figure 2.7 is an example of family of three Wheels.

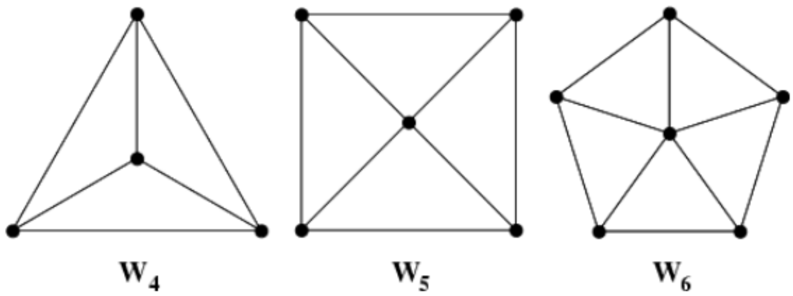

Figure 2.7: A family of three Wheels: W_4, W_5, W_6

Topological Properties:

Given a wheel on $n \geq 4$ vertices, we have

1. size (number of edges): $2(n-1)$
2. δ (minimum degree): 3
3. Δ (maximum degree): $n-1$
4. r (radius): 2
5. d (diameter): $\lfloor \frac{n}{2} \rfloor$
6. g (girth): 3
7. c (circumference): $n-1$

2.2.6 Barbell

The n-barbell graph is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge. Figure 2.8 is an example of 5-barbell on 10 vertices.

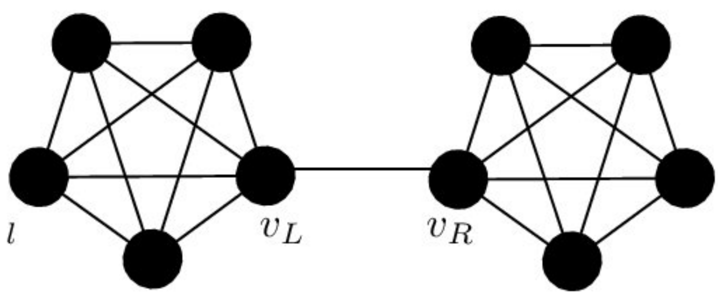

Figure 2.8: a 5-barbell

Topological Properties:

For $n \geq 2$, we have

1. size (number of edges): $2\binom{n}{2} + 1 = n(n-1) + 1$
2. δ (minimum degree): 1
3. Δ (maximum degree): $n - 1$
4. r (radius): 3
5. d (diameter): 3
6. g (girth): 3
7. c (circumference): n

2.2.7 Generalized Barbell

The (m, n)-barbell graph is a generalization of an n-barbell by connecting two complete graphs K_n and K_m by a bridge, for $m, g \geq 2$.

Topological Properties:

For $2 \leq n \leq m$, we have

1. size (number of edges): $\binom{n}{2} + \binom{m}{2}$
2. δ (minimum degree): 1
3. Δ (maximum degree): $n - 1$
4. r (radius): 3
5. d (diameter): 3
6. g (girth): 3
7. c (circumference): m

2.2.8 Complete bipartite graphs

A simple graph $G = (V, E)$ is called **bipartite** if its vertex set be divided into two disjoint groups, with edges connecting vertices from one group to the other; no edge connects vertices within the same group. We note that when each vertex from one group is connected to each vertex from the group, the resulting bipartite graph is said to **complete**; we write $K(m, n)$ where m, n, are the sizes of the two groups. Below is *complete bipartite* graph $K_{3,2}$ on $3 + 2 = 5$ vertices. We also note that $K(m, 1)$ is isomorphic a Star graph as discussed.

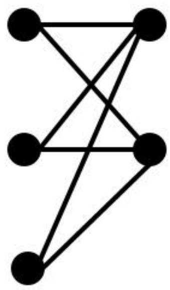

Figure 2.9: A complete bipartite graph with parts sizes 3 and 2

Topological Properties:

Given a complete bipartite graph $K(m, n)$ of order (number of vertices) $m + n$, with $1 \leq n \leq m$, we have

1. size (number of edges): mn
2. δ (minimum degree): n
3. Δ (maximum degree): m
4. r (radius): 1
5. d (diameter): 2
6. g (girth):

$$g = \begin{cases} \infty & n = 1, m \geq 2 \\ 4 & 2 \leq n \leq m \end{cases}$$

7. c (circumference):

$$c = \begin{cases} \infty & n = 1, m \geq 2 \\ n & 2 \leq n \leq m \end{cases}$$

2.2.9 Complete Multipartite graphs

A complete k-partite, $G = K(m_1, m_2, \ldots, m_k)$, is an extension of a complete bipartite with $k \geq 2$ disjoint parts, each of sizes $m_1, m_2, \ldots, m_k$. So, when $k = 2$, G is complete 2-partite also known as a complete bipartite graph and when $k = 3$, G is complete 3-partite also known as a complete tripartite. Figure 2.10 is a complete tripartite $K(5, 3, 2)$.

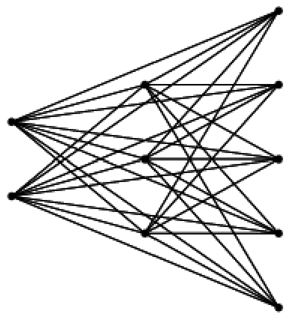

Figure 2.10: A complete 3-partite graph

Topological Properties:

For each $m_i \geq 1$, with $2 \leq i \leq k$, we have

1. size (number of edges): $\prod_{i=1}^{k} m_i$

2. δ (minimum degree): $\inf_i \{m_i\}$

3. Δ (maximum degree): $\sup_i \{m_i\}$

4. r (radius): 1

5. d (diameter): 1

6. g (girth):

$$g = \begin{cases} \infty & n = 1, m \geq 2 \\ 4 & 2 \leq n \leq m \end{cases}$$

7. c (circumference): varies

We think the last property (circumference) of complete multipartite graphs can simplified down to 3-4 cases, but it will require some proof which we are prepared to give in this thesis. For this reason, we leave it to reader or future researchers to explore it.

Chapter 3 Introduction to Domination and Dominion

In this chapter, we introduce the reader to the notion of domination and dominion with some examples. The latter is a newly developed concept.

3.1 Definitions

3.1.1 γ-set

Let A and B be two discrete sets. We define the mapping $f : A \to B$ with $f(x) := x \sim y$, whenever $x \in A$ and $y \in B$ are adjacent. The set A is said to **dominates** the set $B \iff \forall\, y \in B, \exists\, x \in A$ s.t. $f(x) = xy$. In which case, f is a surjection. If the set A dominates the set B and $|A| \leq |B|$, then $|A|$ is called a **domination number**.

Suppose $G = (V, E)$ is a simple graph and let $A \subseteq V(G)$. $|A|$ is a domination number of G if:

1. A is a dominating set of G
2. $|A| \leq |B|$ for every dominating set $B \subseteq V(G)$

It is customary to denote the domination number of A, with $\gamma = |A|$. Clearly, a graph G can have multiple dominating sets. For simplicity, we refer to such sets, **γ-sets**. Figure 3.1 is an example of a graph (Peterson) where $\gamma = 3$. Observe that the vertex 3 dominates the vertices in the set $\{0, 5, 7\}$. The vertex 9 dominates the vertices in the set $\{2, 4, 7\}$ and vertex 6 dominates the vertices in the set $\{8, 1, 7\}$. Thus, all vertices in $G \setminus \{3, 6, 9\}$ are covered.

Since a γ-set may not be unique for a given graph, this leads to the following natural question: How many γ-sets does a graph has?–To answer this question, we introduce the notion of *dominion*.

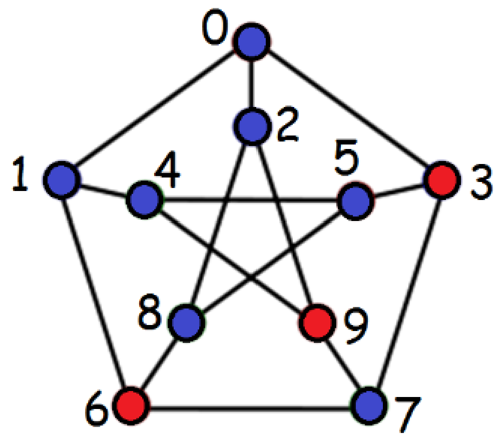

Figure 3.1: Peterson Graph with a dominating set $\{3, 6, 9\}$

3.1.2 Dominion

The dominion (number) of a graph G, denoted by ζ is the number of its γ-sets. In other words, $\zeta := |\{A : A \subseteq V(G), |A| = \gamma\}|$. For instance, for a path G on $n = 2$ vertices, $\zeta(G) = 2$ since each endpoint or leave forms a γ-set. We provide additional examples for paths on n vertices, with $2 \leq n \leq 12$ in Example 3.1.4. Further, in order to track the values of the two previously defined parameters, given a sequence of families of graphs, we introduce the next concepts.

3.1.3 Special sequences

Given $G = G_n$, a graph on $n \geq 1$ vertices, the sequence of domination numbers, $\{\gamma(G_1), \gamma(G_2), \gamma(G_3), \ldots, \gamma(G_n)\}$ is called a **γ-spectrum** of G. Likewise, the sequence of dominion numbers, $\{\zeta(G_1), \zeta(G_2), \zeta(G_3), \ldots, \zeta(G_n)\}$ is called a **ζ-spectrum** of G. For example, given a family of paths on n vertices, with $2 \leq n \leq 12$ as shown in Example 3.1.4, the γ-spectrum is

$$\{1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4\}$$

and the ζ-spectrum is

$$\{2, 1, 4, 3, 1, 8, 4, 1, 13, 5, 1\}.$$

It is clear from the values for each spectrum that, there is a general formula for the γ-spectrum sequence which is $\lfloor \frac{n}{3} \rfloor$, for $n \geq 2$. However, the a general formula for the ζ-spectrum sequence is unclear and we do not know it.

3.1.4 Examples: Paths on n-vertices with $2 \leq n \leq 12$.

A **path of length** n, denoted by P_n, is a graph that has exactly 2 leaves and every other vertex is of degree 2. Below is an example of a P_3.

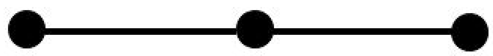

Figure 3.2: A Path on 3 vertices

Throughout this example, we assume $G = P_n$ is a path on n vertices and $x_i - x_{i+1}$ indicates that the vertices x_i and x_{i+1} of G are adjacent, for some $1 \leq i \leq 11$. We note that it is much harder to determine the ζ-spectrum values for a general path of any length n.

- $n = 2$: $x_1 - x_2$

  γ sets: $\{x_1\}$, $\{x_2\}$.

  So, when $G = P_2$, we have $\gamma(G) = 1$ and $\zeta(G) = 2$

- $n = 3$: $x_1 - x_2 - x_3$

  γ sets: $\{x_2\}$.

  So, when $G = P_3$, we have $\gamma(G) = 1$ and $\zeta(G) = 1$

- $n = 4$: $x_1 - x_2 - x_3 - x_4$

  γ sets: $\{x_1, x_3\}$. $\{x_1, x_4\}$, $\{x_2, x_4\}$, $\{x_2, x_3\}$

  So, when $G = P_4$, we have $\gamma(G)= 2$ and $\zeta(G)= 4$

- $n = 5$: $x_1 - x_2 - x_3 - x_4 - x_5$

  γ sets: $\{x_1, x_4\}$, $\{x_2, x_4\}$, $\{x_2, x_5\}$

  So, when $G = P_5$, we have $\gamma(G)= 2$ and $\zeta(G)= 3$

- $n = 6$: $x_1 - x_2 - x_3 - x_4 - x_5 - x_6$

  γ sets: $\{x_2, x_5\}$

  So, when $G = P_6$, we have $\gamma(G)$= 2 and $\zeta(G)$= 1

- $n = 7$: $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7$

  γ sets: $\{x_1, x_3, x_6\}$. $\{x_1, x_4, x_6\}$, $\{x_1, x_4, x_7\}$, $\{x_2, x_3, x_6\}$, $\{x_2, x_4, x_6\}$, $\{x_2, x_4, x_7\}$, $\{x_2, x_5, x_6\}$, $\{x_2, x_5, x_7\}$

  So, when $G = P_7$, we have $\gamma(G)$= 3 and $\zeta(G)$= 8

- $n = 8$: $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8$

  γ sets: $\{x_1, x_4, x_7\}$, $\{x_2, x_4, x_7\}$, $\{x_2, x_5, x_7\}$, $\{x_2, x_5, x_8\}$

  So, when $G = P_8$, we have $\gamma(G)$= 3 and $\zeta(G)$= 4

- $n = 9$: $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9$

  γ sets: $\{x_2, x_5, x_8\}$

  So, when $G = P_9$, we have $\gamma(G)$= 3 and $\zeta(G)$= 1

- $n = 10$: $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10}$

  γ sets: $\{x_1, x_3, x_6, x_9\}$, $\{x_1, x_4, x_6, x_9\}$, $\{x_1, x_4, x_7, x_9\}$, $\{x_1, x_4, x_7, x_{10}\}$, $\{x_2, x_3, x_6, x_9\}$, $\{x_2, x_4, x_6, x_9\}$, $\{x_2, x_4, x_7, x_9\}$,$\{x_2, x_4, x_7, x_{10}\}$, $\{x_2, x_5, x_6, x_9\}$, $\{x_2, x_5, x_7, x_9\}$,$\{x_2, x_5, x_7, x_{10}\}$, $\{x_2, x_5, x_8, x_9\}$,$\{x_2, x_5, x_8, x_{10}\}$

  So, when $G = P_{10}$, we have $\gamma(G)$= 4 and $\zeta(G)$= 13

- $n = 11$: $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11}$

  γ sets: $\{x_1, x_4, x_7, x_{10}\}$, $\{x_2, x_4, x_7, x_{10}\}$, $\{x_2, x_5, x_7, x_{10}\}$, $\{x_2, x_5, x_8, x_{10}\}$, $\{x_2, x_5, x_8, x_{11}\}$,

  So, when $G = P_{11}$, we have $\gamma(G)$= 4 and $\zeta(G)$= 5

- $n = 12$: $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12}$

  γ sets: $\{x_2, x_5, x_8, x_{11}\}$

  So, when $G = P_{12}$, we have $\gamma(G)$= 4 and $\zeta(G)$= 1.

3.2 Brief History

The first reported instance of a related problem is thought to be the *queen's domination problem* which was originally called "coefficient of external stability" in a publication by Claude Berge in 1958. The *dominating set* was first used by Oystein Ore in 1962. Ever since, there has been an increased interest in this concept and other related problems with terms such as in "covering" and "location". In general, dominating set problems are concerned with testing whether $\gamma(G) \leq K$ for a given graph G and input K; it is a classical NP-complete decision problem in computational complexity theory. Therefore it is believed that there may be no efficient algorithm that finds a smallest dominating set for all graphs, although there are efficient approximation algorithms, as well as both efficient and exact algorithms for certain graph classes. It is also NP-Hard to determine $\gamma(G)$, given any graph G.

Further, estimates of the γ value for some graphs have also been studies.So, a lower bound is clearly based on the neighborhood of a vertex and the set of all vertices on a graph is by definition a dominating set. Hence, $n/(1+\Delta) \leq \gamma(G) \leq n$. This bound is substantially improved for any connected graph as $2 \leq \gamma(G) \leq n/2$.

3.3 Applications

Dominating sets are of practical interest in several areas. In wireless networking, dominating sets are used to find efficient routes within ad-block mobile networks. They have also been used in document summarization, and in designing secure systems for electrical grids. Dominating sets are useful in routing problems. For instance, how many internet routers do you need so that every computer in your business has internet access?

The notion of dominating set can and have been extended to other graph parameters as mentioned in Chapter 5. For instance, a **connected γ-set** is a γ-set whose elements induce a connected graph. Figure 3.3 is an example of tree where the elements (black) of the γ-set form a path on 3-vertices.

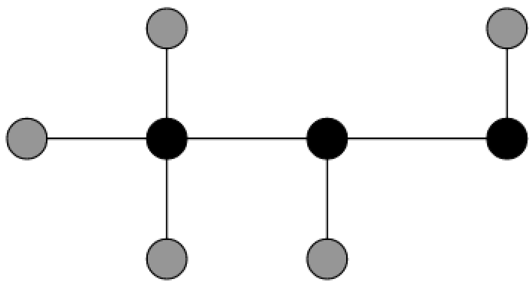

Figure 3.3: connected γ-set (in black) of a tree on 7 vertices

Thus, it is clear that only nodes (or computers) in a connected γ-set can relay messages, reduce communication cost and redundant traffic. They can maintain and keep routing information localized and save storage space. So, a message sent from a circled black node shown in Figure 3.4 can reach other centers, if it is suddenly under attack; this can be a scenario case of a back-up call by a police unit.

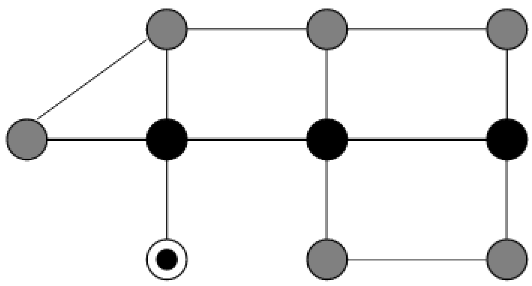

Figure 3.4: connected γ-set (in black) of a planar graph

Dominion counts all the γ-sets including the connected ones. There, the more γ sets a network has the more resilient that network is to attacks. Thus, the higher the ζ value, the less vulnerable is the network. Further, it is also clear that graphs for which $\zeta = 1$ have a *unique* dominion. In which case such graphs are more vulnerable to cyber attacks that target that γ-set. It would be nice to have an estimate that include the parameters ζ and γ. However, it is obvious that $1 \leq \zeta \leq n$, and for any graph on n vertices. Moreover, it is worth noting that any derived γ-set such as total dominating set is counted in the dominion (number). We defined these derived dominating sets later in the Chapter 5. Now, we close this chapter with some activities with solutions related to γ-sets and γ values, that can be introduced in a high school curriculum.

3.4 Activities

1. Find the γ values of the graphs in Figures 3.5 and 3.6.

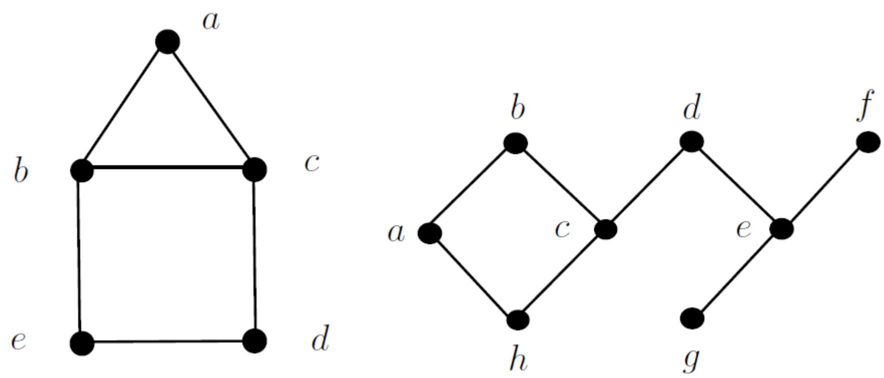

Figure 3.5: Graphs (A) and (B)

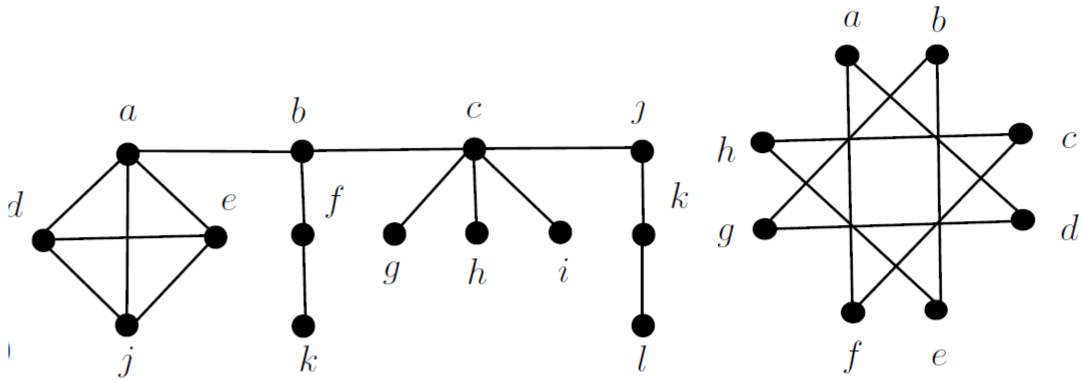

Figure 3.6: Graphs (C) and (D)

2. Suppose that a company contains eleven offices connected by hallways as indicated in Figure 3.7. The manager of the company wants to install top of the line photocopy machines so that each office has a copier within their office or is near an office that has one. Unfortunately, the company is new and funds are limited, thus only a minimum number of photocopiers can be installed.

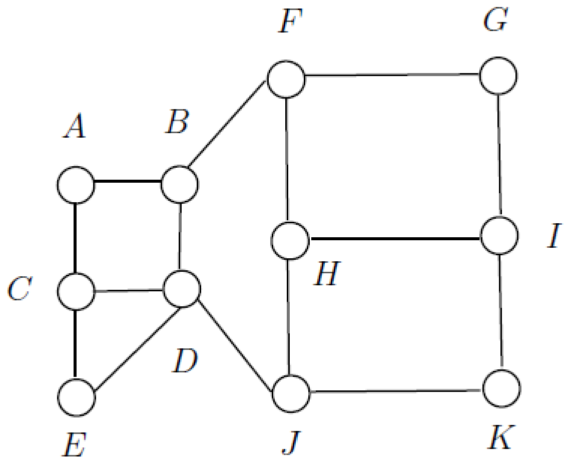

Figure 3.7: Office network

Tell the manager the minimum number of photocopier machines that need to be purchased and in which offices to place them.

3. Suppose that a contractor is building a new subdivision. The last decision that the contractor has to make is where to place the waste receptacles. Regret- tably, the contractor went over budget building the community center, so not every intersection can have a waste receptacle. The contractor would like for you to determine the number of receptacles that are needed so that, for each intersection, there is either a receptacle or there is one at an intersection one block away. The Figure 3.8 is a street grid of the new subdivision.

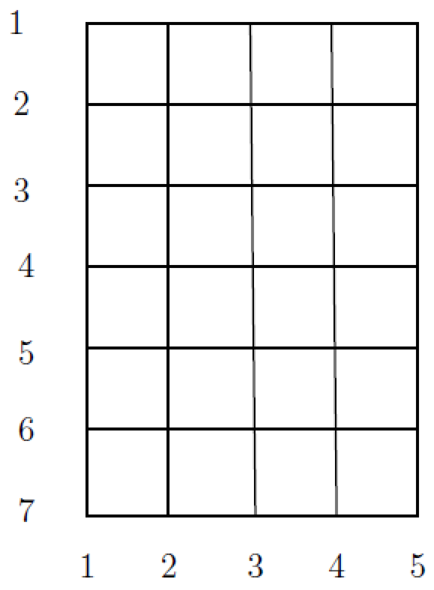

Figure 3.8: Street grid

4. Suppose that we have a collection of small villages in Alaska. We would like to locate radio stations in some of these villages so that messages can be broadcast to all of the villages in the region. Since each radio station has a limited broadcasting range, fifty miles, we need to use several stations to reach all the villages. The locations of the ten villages are given in Figure 3.9 with the distances between the villages in miles.

	A	B	C	D	E	F	G	H	I	J
A		40	45	40					65	
B	40		40		30	30				
C	45	40		40					50	
D	40		40				60			
E		30				30				70
F		30			30		45	45		
G				60		45		45		
H						45	45		50	50
I	65		50					50		50
J					70			50	50	

Figure 3.9: Radio Stations grid

What is the fewest number of stations that need to be constructed?

3.5 Solutions

1. **Figure A.**

We can clearly notice that all the vertices form a dominating set, but we we want to find the least number. Notice that we could choose $\{b, d\}$ as a dominating set, since b dominates itself and the vertices adjacent to it, a,c, and e. Next we notice $\{b, d\}$ dominates itself and its neighbors c and e. Other minimum dominating sets are $\{a, e\}, \{a, d\}, \{c, e\}, \{b, e\}, \{b, c\}$, and $\{c, d\}$. So we know that we need at most two vertices to dominate the graph, $\gamma(G) = 2$. To see that $\gamma(G) = 2$, we must show that one vertex can not dominate the graph. To see this, note that no one vertex is adjacent to every vertex in the graph. Hence $\gamma(G) = 2$.

Figure B.

For the graph (B), the domination number is 3. A possible dominating set is $\{a, c, e\}$, so $\gamma(G) \leq 3$. We know that there is a C_4 subgraph is contained in the graph, so automatically you need at least two vertices to dominate it. No matter which two vertices of the cycle you choose, they cannot dominate the entire graph, so $\gamma(G) \geq 3$. Hence, $\gamma(G) = 3$.

Figure C.

For the graph (C), the domination number is 3. A possible dominating set is $\{a, b, c\}$, so so $\gamma(G) \leq 3$. We know that G is just a C_8, so at least three vertices are needed. Hence, $\gamma(G) = 3$.

Figure D.

For the graph (D), the domination number is 4. A possible dominating set is $\{a, f, c, k\}$, so $\gamma(G) \leq 4$. We know that G contains a subgraph of a K_4, two P_3's, and a $K(5, 1)$. We know that the domination number of any complete graph is 1. Likewise, it is easy to see that the domination number of any star is 1. However, the placement of the P_3's in G make it impossible to dominate the entire graph G with only one more additional vertex. Thus, $\gamma(G) \geq 4$. Hence, $\gamma(G) = 4$.

2.

We model the problem using a graph where the vertices of our graph represent the different offices, and, two vertices are adjacent if the office they represent are connected by a hallway. We need to find the smallest dominating set of the graph. The dominating number for the graph is 4 and a dominating set consists of $\{A, D, F, I\}$. So, now let us interpret our solution in terms of our real-world problem. We can see (Figure 3.10) that four photocopier machines need to be purchased and placed in office A, office D, office F, and office I.

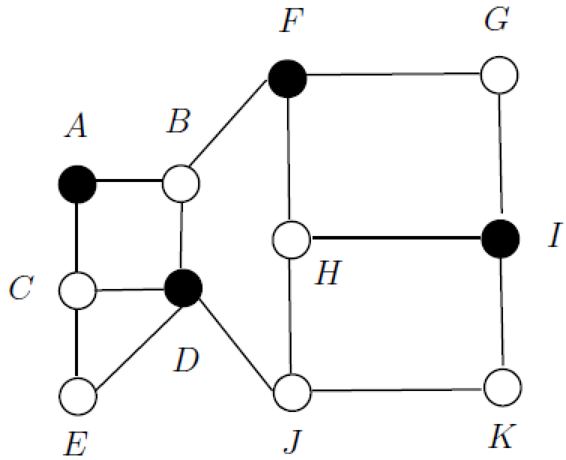

Figure 3.10: Office network $\gamma(G)$-set

3.

We use the original map, which is a grid (graph) and highlight the γ-set as shown in Figure 3.11. We need to find a dominating set that consists of the smallest number of intersections for waste receptacles. Our smallest dominating set consists of the vertices $\{(1,1),(1,5),(2,3),(3,1),(3,5),(4,3),(5,1),(5,5),(6,3),(7,1),(7,5)\}$. Now, let us interpret our solution in terms of our real-world problem. We can see (Figure 3.11) that eleven waste receptacles need to be placed at intersections $(1.1),(1.5),(2,3),(3,1),(3,5),(4,3),(5,1),(5,5),(6,3),(7,1)$, and $(7,5)$ so that the residents in the subdivision have a receptacle at their intersection or have a receptacle one block away.

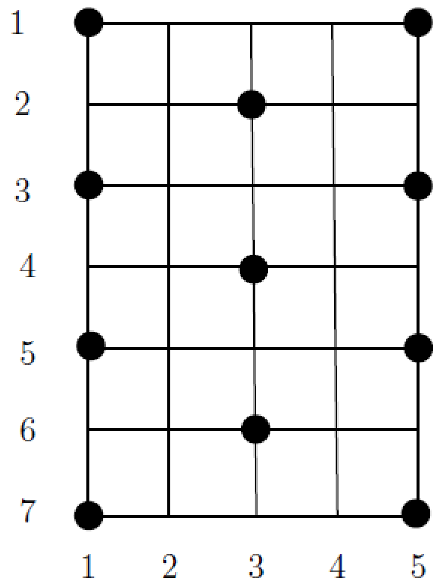

Figure 3.11: Street grid $\gamma(G)$-set

4.

Once again, we need to translate the given grid into a graph model. Since we know that that a radio station has a broadcast range of only fifty miles, we can disregard towns that are more than fifty miles apart. Therefore we can now have the vertices represent towns and connect two vertices by an edge whenever the towns they represent are 50 miles or less apart. This gives us the graph shown in Figure3.12. We want to find a set of the least number of stations which dominate all other vertices. A possible smallest dominating set consists of $\{B, D, H\}$. Now let us interpret our solution in terms of our real-world problem. We only need to construct three radio stations in the towns of B, D, and H, so that all of the other towns can be reached.

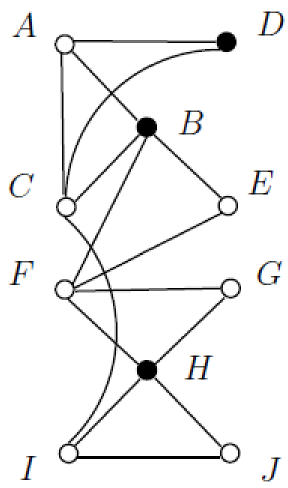

Figure 3.12: Radio grid $\gamma(G)$-set

Chapter 4 γ and ζ Spectral Values of Some Graphs

For each graph mentioned in Chapter 2, we determine sequences of γ and ζ values as defined in Chapter 3. These spectral values are sequentially derived from a sequence of families of graphs as we increase their order. Throughout, we assume G is a graph on $n \geq 2$ vertices, unless stated otherwise.

4.0.1 Complete Graphs

Theorem 4.0.1. *Given a complete graph K_n on $n \geq 2$ vertices, the following hold:*

1. *γ-spectrum:* $\{1, 1, \ldots, 1\}$
2. *γ-spectrum general form:* $\{1\}_{i=1}^{n}$
3. *ζ-spectrum:* $\{1, 2, \ldots, n\}$
4. *ζ-spectrum general form:* $\{i\}_{i=1}^{n}$

Proof. Assume $G = (V, E)$ is a complete graph on $n \geq 2$ vertices. Since every vertex $v \in V$ has exactly $n - 1$ neighbors, it is clear that any vertex $u \in V$ is dominated by v. Hence, the γ-spectral value. Further, because each vertex in V can by them self be a dominating set, the γ-spectral values follow. $\square$

4.0.2 Cycles

Theorem 4.0.2. *Given a cycle C_n on $n \geq 3$ vertices, the following hold:*

1. *γ-spectrum:* $\{1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, \ldots\}$
2. *γ-spectrum general form:* $\lfloor \frac{n}{3} \rfloor$*, $n \geq 2$.*
3. *ζ-spectrum: unknown*

4. *ζ-spectrum general form: unknown*

Proof. Assume $G = (V, E)$ is a cycle graph on $n \geq 3$ vertices. When $n = 3$, the result follows since $G \cong K_3$. Now, for all $n \geq 3$, consider $e \in E$. Delete it, in which case, $G \cong P_n$, a path on n vertices whose γ spectral values are known, following Example 3.1.4. Hence, any other γ set of G must have cardinality at least that of P_n. It is easy to see that their cardinality is at most that of P_n, giving the result. The ζ values are more intractable. $\square$

4.0.3 Star Trees

As discussed in Chapter 2, we focus only on the star graph on $n \geq 2$ vertices.

Theorem 4.0.3. *Given a Star S_n on $n \geq 2$ vertices, the following hold:*

1. *γ-spectrum:* $\{1, 1, \ldots, 1\}$

2. *γ-spectrum general form:* $\{1\}_{i=1}^{n}$

3. *ζ-spectrum:*$\{1, 1, \ldots, 1\}$

4. *ζ-spectrum general form:* $\{i\}_{i=1}^{n}$

Proof. Assume $G = (V, E)$ is a Star on $n \geq 2$ vertices. When $n = 3$, the middle vertex is the only element of a dominating set. For all $n \geq 3$, the vertex of maximal degree, which is unique, is the only element of a dominating set. Hence the result. $\square$

4.0.4 Fans

Theorem 4.0.4. *Given a Fan F_n on $n \geq 3$ vertices, the following hold:*

1. *γ-spectrum:* $\{1, 1, \ldots\}$

2. *γ-spectrum general form:* $\{1\}_{i=1}^{n}$

3. *ζ-spectrum:* $\{3, 2, 1, \ldots, 1\}$

4. *ζ-spectrum general form:* $\{3, 2\} \cup \{1 : n \geq 5\}$

Proof. Assume $G = (V, E)$ is a Fan on $n \geq 3$ vertices. When $n = 3$, any of the vertices of F_3 forms a dominating set, since each has a degree 2. Hence, $\zeta = 3$. For $n = 4$, exactly two vertices (they have degrees 3). Hence, $\zeta = 2$. For all $n \geq 5$, only the vertex of degree $n - 1$ dominates the remaining vertices of F_n. Hence, $\zeta = 1$, giving the result. □

4.0.5 Wheel

Theorem 4.0.5. *Given a Wheel W_n on $n \geq 4$ vertices, the following hold:*

1. *γ-spectrum:* $\{1, 1, \ldots, 1\}$
2. *γ-spectrum general form:* $\{1\}_{i=1}^{n}$
3. *ζ-spectrum:*$\{1, 1, \ldots, 1\}$
4. *ζ-spectrum general form:* $\{i\}_{i=1}^{n}$

Proof. Assume $G = (V, E)$ is a Wheel on $n \geq 4$ vertices. When $n = 3$, the hub, is the vertex of maximal degree, which is unique. It is the only element of a dominating set. Hence the result. □

4.0.6 Barbell

Theorem 4.0.6. *Given an n-barbell $B(n)$ on $n \geq 3$ vertices, the following hold:*

1. *γ-spectrum:* $\{2, 2, 2, \ldots\}$
2. *γ-spectrum general form:* $\{2\}_{i=1}^{n}$
3. *ζ-spectrum:* $\{4, 9, 16, \ldots\}$
4. *ζ-spectrum general form:* $\{i^2\}_{i=2}^{n}$

Proof. Assume $G = (V, E)$ is an n-barbell on $n \geq 3$ vertices. Let u and v be the endpoints of the bridge connecting the two cliques, K_n's. It is easy to see that u covers one of the clique while v covers the other. Further, since any member of a K_n covers its neighbors that form a clique, and with it, we can form a γ set when paired with any other member from the second clique. Hence, the γ-spectrum. □

4.0.7 Generalized Barbell

Theorem 4.0.7. *Given an (m, n)-barbell $B(m, n)$ with $2 \leq n \leq m$ vertices, the following hold:*

1. *γ-spectrum:* $\{2, 2, 2, \ldots\}$
2. *γ-spectrum general form:* $\{2\}_{i=1}^{n}$
3. *ζ-spectrum: not applicable*
4. *ζ-spectrum general form:* $m \times n$

Proof. Assume $G = (V, E)$ is an an (m, n)-barbell $B(m, n)$ with $2 \leq n \leq m$ vertices. Let u and v be the endpoints of the bridge connecting the two cliques, K_m and K_n. It is easy to see that u covers one of the clique while v covers the other. Further, any vertex x of a K_j, with $j \in \{m, n\}$ covers its neighbors as they form a clique. So, with each such vertex x we can form a γ set of size $m \times n$ when paired with the other member from the second clique. Hence, the γ-spectrum. □

We list the results for the next two graphs without proof, as they are beyond the scope of our research to prove.

4.0.8 Complete bipartite graphs

Theorem 4.0.8. *Given $K(m, n)$, a complete bipartite graph with $2 \leq n \leq m$, the following hold:*

1. *γ-spectrum:* $\{2, 2, 2, \ldots, 2\}$
2. *γ-spectrum general form:* $\{2\}_{i=1}^{n}$
3. *ζ-spectrum: not applicable*
4. *ζ-spectrum general form: $mn + 1$, with $2 \leq n \leq m$.*

We note that when $n = 1$ and $m \geq 2$, $\zeta = 1$ and when $n = 1$ and $m = 1$, we have $\zeta = 2$.

4.0.9 Complete multipartite graphs

Theorem 4.0.9. *Given $K(m_1, m_2, \ldots, m_k)$, a complete k-partite graph with $m_j \geq 2$ and $k \geq 2$, the following hold:*

1. *γ-spectrum: $\{2, 2, 2, \ldots, 2\}$ with $2 \leq n \leq m$*

2. *γ-spectrum general form: $\{2\}_{i=1}^{n}$.*

3. *ζ-spectrum: not applicable*

4. *ζ-spectrum general form: $\displaystyle\sum_{i \neq j}^{\binom{k}{2}} m_i m_j$*

$$+ \begin{cases} \alpha & if\ some\ m_j = 2, j = 1, \ldots, \alpha \\ 0 & otherwise \end{cases}$$

Chapter 5 Conclusion and Future Research

The notion of γ-sets and dominion can further be extended to other graphs. Moreover, there are other types of related parameters that can be applied to our research. We list them here, briefly.

1. **Connected Dominating Set:** it requires that a graph induced by a γ-set must be connected.

2. **Total Dominating Set:** it requires no isolated vertices on graph induced by a γ-set.

3. **Independent Dominating Set:** it requires all vertices in a γ-set to be isolated vertices.

4. **Dominating Clique:** it requires all vertices in a γ-set to form a clique.

5. **Red-Blue Dominating Set:** it requires that the vertices in a γ-set can be partitioned into two sets, Red and Blue, and the vertices in Red dominate those in Blue.

Dominion includes the counting of each of the aforementioned variations of dominating sets as they are γ sets. Future research can also look into those special dominions.

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