On the Existence of $S-$graphs

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Abstract

We answer in the affirmative a question posed by Al-Addasi and Al-Ezeh in 2008 on the existence of symmetric diametrical bipartite graphs of diameter 4. Bipartite symmetric diametrical graphs are called $S-$graphs by some authors and diametrical graphs have also been studied by other authors using different terminology, such as self-centered unique eccentric point graphs. We include a brief survey of some of this literature and advertise that the existence question was also answered by Berman and Kotzig in a 1980 paper, along with a study of different isomorphism classes of these graphs using a $(1, -1)-$matrix representation which includes the well-known Hadamard matrices. Our presentation focuses on a neighborhood characterization of $S-$graphs and we conclude our survey with a beautiful version of this characterization known to Janakiraman.

Keywords: cartesian product, geodesic, $S-$graphs, symmetric diametrical, neighborhood characterization

1 Introduction

We study finite connected bipartite graphs $G = G(V, E)$ with no multiple edges, no loops, and whose vertex set V can be partitioned into two partite sets U and W with the convention that $|U| \leq |W|$. The *cartesian product* of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$, where (u_1, v_1) is adjacent to (u_2, v_2) whenever $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. The *distance* $d(v_1, v_2)$ between two vertices v_1 and v_2 is the number of edges in a shortest path, or *geodesic*, connecting v_1 and v_2. The *diameter* of a graph is $d = max\{d(v_1, v_2) \mid v_1, v_2 \in V\}$. If $d(v, v') = d$ then v and v' are called *antipodes.* If every vertex v has a unique antipode v', the graph G is said to be *diametrical.* If

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$$d(v,z)+d(z,v')=d(v,v')=d \ for\ every\ antipodal\ pair\ (v,v') \\ of\ the\ diametrical\ graph\ G\ and\ for\ any\ z\in V, \tag{1}$$

G is said to be *symmetric* in the sense of [?].

A bipartite symmetric diametrical graph is called an $S-$graph. For example, the $n-$cube is an $S-$graph of diameter n. The inspiration of this work is a question posed by Al-Addasi and Al-Ezeh in [?]. For bipartite diametrical graphs of diameter 4, the orders of the partite sets U and W must be even since the antipodal pairs must be in the same partite set. Al-Addasi and Al-Ezeh showed that if $|U|=2m$, then $2m\leq |W|\leq 2^m$. These authors denote by $G(2m,2t)$ any $S-$graph of diameter 4 with $|U|=2m$ and $|W|=2t$ where $m\leq t$. They give constructions for $S-$graphs $G(2m,2m)$ and $G(2m,2^m)$ in the two extreme cases of the order of the largest partite set and they ask if $S-$graphs $G(2m,2t)$ exist, in general, for $m<t<2^{m-1}$.

The answer to the preceding question is yes, and can be found in the existing literature, namely the work of Berman and Kotzig on centrally symmetric graphs in [?]. Berman and Kotzig define centrally symmetric graphs (with at least one edge) by the property

$$for\ every\ vertex\ v\ there\ exists\ exactly\ one\ vertex\ v'\ which \\ is\ more\ remote\ from\ v\ than\ every\ vertex\ adjacent\ to\ v' \tag{2}$$

Berman and Kotzig call v' the opposite of v and show in [?] that the opposite v' of v must be the unique antipode of v, i.e. $d(v,v')=d$. They also show that property (2), which focuses on vertices adjacent to v', extends to all vertices of the graph and implies the symmetric condition (1). They then name bipartite centrally symmetric graphs $S-$graphs and construct a $(1,-1)-$matrix representation for these graphs, which shows that Hadamard graphs are examples of $G(2m,2m)$ for appropriate m. The existence of the proper $(1,-1)-$matrices, in the terminology of Berman and Kotzig, implies the answer to the existence question of $S-$graphs.

We offer [?], [?] and [?] as good gateways to the literature on this topic, and hope to advertise and add to this literature by providing a constructive argument and, therefore, a more explicit answer to the existence question. This construction has the further benefit of showing that the unique graph $G(2m,2^m)$ is universal in the set of all $S-$graphs with smaller partite set of order $2m$.

We begin with the neighborhood characterization of $S-$graphs, provide an example showing the importance of breaking condition (2) into the two pieces, diametrical and symmetric, give the inductive construction for $S-$graphs showing the universality of $G(2m,2^m)$, and close with some comments on $S-$graphs of higher diameters and potential further work.

2 Neighborhood characterization for symmetric diametrical graphs of any diameter

For any graph G, denote the rank k neighborhood of a vertex v by $N_k(v) = \{z \in V \mid d(v,z) = k\}$. A graph G is diametrical when $N_d(v)$ is a singleton and $N_k(v) = \emptyset$ for $k > d$ for all $v \in V$. It is clear that when $k = 0$ and $k = 1$, $\{v\} = N_0(v)$ and $N(v) = N_1(v)$ respectively for any vertex v. So we choose to use these notations interchangeably when it is convenient. We observe that graphs can be diametrical and not symmetric. For example, consider the graphs $G_1 = K_2 \square C_6$ and $G_2 = G_1 \setminus e$ with $e \notin E(C_6)$. It is easy to see that G_1 is symmetric diametrical but G_2 is only diametrical. Also, note that both G_1 and G_2 have diameter 4.

When a diametrical graph G is also symmetric, for an antipodal pair (v, v') we call v' the dual of v. The symmetry condition (1) can be written as

$$\begin{aligned} &N_k(v) = N_{d-k}(v') \; for \; all \; 0 \leq k \leq d \; and \\ &all \; pairs \; of \; antipodes(v, v') \end{aligned} \tag{3}$$

Note that condition (3) incorporates both the diametrical and symmetry conditions, since $N_d(v) = N_0(v') = \{v'\}$. It is important to point out that not every symmetric diametrical graph is bipartite. For instance, the graph $G \setminus uv$, where $G = K_2 \square W_4$ and $deg_G(u) = deg_G(v) = 5$ can easily be verified as symmetric diametrical of diameter 3 and yet is not bipartite. For the rest of this paper, we restrict our attention to bipartite graphs that are diametrical and symmetric, namely the $S-$graphs. Condition (3) can be revised to yield a useful characterization specifically for $S-$graphs of diameter 4.

3 Neighborhood characterization for $S-$graphs of diameter 4

Being an $S-$graph of diameter 4, antipodes v and v' must be in the same partite set and $N_2(v) = N_2(v')$ is the partite set containing v and v' minus v and v'. It is also an immediate consequence of being antipodes in a graph with diameter greater than 2 that

$$N(v) \cap N(v') = \emptyset \; for \; all \; v \tag{4}$$

Theorem 1. *A bipartite diametrical graph G of diameter 4 is an $S-$graph if and only if*

$$\begin{array}{l} N(v) \cup N(v') \textit{ is the partite set not containing } v \textit{ and } v' \\ \textit{for each antipodal pair } (v, v') \end{array} \tag{5}$$

Proof. If G is an $S-$graph, and since $N_1(v) \cup N_3(v)$ is the partite set not containing v and v', condition (3) gives condition (5). Conversely, if (5) holds, then $N_1(v')$ must be $N_3(v)$, since $N_1(v) \cup N_3(v) = N_1(v) \cup N_1(v')$ and $N_1(v) \cap N_1(v') = \emptyset = N_1(v) \cap N_3(v)$. □

Note that the graph G_2 in our previous example does not satisfy condition (5). As remarked in [**?**] and [**?**], it is easy to see that if G is diametrical and symmetric and $u \in N_1(w)$ then $u' \in N_1(w')$. Hence it is convenient to describe an $S-$graph of diameter 4, $G(2m, 2t)$, by describing the neighborhoods of half of the vertices w, no two of which are dual, in the partite set of order $2t$. The dual neighborhood of w' is the complement of the neighborhood of w in U and the description of each neighborhood need only specify the inclusion of one half of the vertices, no two of which are dual, of the smaller partite set. This observation immediately gives the bound $|W| \leq 2^m$ and motivates the use of $(1, -1)-$matrices, as seen in [**?**]: Relative to a labeling of the vertices of an $S-$graph G of diameter 4 as u_i, u_i', w_i and w_i', let $H = [a_{ij}]$ be a matrix with $1 \leq i \leq m$ and $1 \leq j \leq t$ where $a_{ij} = \pm 1$ is given by the rules:

1) $a_{ij} = 1$ if and only if u_i is adjacent to w_j and u_i' is adjacent to w_j', and

2) $a_{ij} = -1$ if and only if u_i is adjacent to w_j' and u_i' is adjacent to w_j.

A proper $(1, -1)-$matrix is one in which no two rows and no two columns are proportional. G is an $S-$graph of diameter 4 if and only if H is a proper $(1, -1)-$matrix.

4 Construction of $S-$graphs of diameter 4 in all possible orders

Let $G_t = G(2m, 2t)$ be an $S-$graph for some integer $t \in \{m, \ldots, 2^{m-1} - 1\}$. For convenience, enumerate the vertices in pairs $U = \{u_1, u_1', \ldots, u_m, u_m'\}$ and $W = \{w_1, w_1', \ldots, w_t, w_t'\}$ so that v' is the unique dual vertex of v. The neighborhood of each w_i is an $m-$tuple set of U of the form $\{u_{i_1}, \ldots, u_{i_m}\}$ such that each u_{i_j} is either u_j or its antipode u_j' and the neighborhood of w_i' is the complement $N(w_i') = U - N(w_i) = \{u_{i_1}', \ldots, u_{i_m}'\}$. Construct a new graph G_{t+1} by adding two vertices w_{t+1} and w'_{t+1} to W and add edges connecting these two new vertices to U by defining their neighborhoods to be any pair of complementary $m-$tuples, of the form previously described, that do not already appear as neighborhoods of the $w_i \in G_t$. Since $t < 2^{m-1}$ there must exist such a pair of $m-$tuples.

Theorem 2. *G_{t+1} is an $S-$graph $G(2m, 2t+2)$ of diameter 4.*

Proof. The construction maintains the diameter, makes w_{t+1} and w'_{t+1} antipodes, and ensures that condition (5) is satisfied for each of the old pairs of dual vertices and for the new pair (w_{t+1}, w'_{t+1}). □

Corollary 2.1. *$S-$graphs $G(2m, 2t)$ exist for all $2 \le m \le t \le 2^{m-1}$.*

Proof. [**?**] gives a construction for an $S-$graph $G(2m, 2m)$ when $m \ge 2$ and the rest follows inductively using the previously described construction. □

Corollary 2.2. *Every graph $G(2m, 2t)$ is isomorphic to a subgraph of any representative of $G(2m, 2^m)$.*

Proof. It is clear by the neighborhood characterization as $m-$tuples, from the perspective of W, that there is a unique isomorphism class of $S-$graphs when $|W| = 2^m$. The iterated construction builds from any $S-$graph $G(2m, 2t)$ a representative of $G(2m, 2^m)$ as a supergraph. Restriction of an isomorphism between two representatives of $G(2m, 2^m)$ to the initial $G(2m, 2t)$ produces an isomorphism between $G(2m, 2t)$ and its image. □

5 $S-$graphs of diameter $d \ge 5$

As remarked in [**?**], the cartesian product of any two $S-$graphs is again an $S-$graph, and its diameter is the sum of the two diameters. Hence the existence of $S-$graphs of diameter 1 (K_2), diameter 2 ($K_2 \Box K_2$), diameter 3 (any complete bipartite graph $K_{m,m}$ minus a $1-$factor, with $m \ge 3$) and diameter 4 imply the existence of many $S-$graphs of any diameter greater than 4.

A graph which cannot be represented as the cartesian product of two smaller graphs is said to be *primitive.* However, not all $S-$graphs are cartesian products, and it remains an area of research to describe these primitive $S-$graphs. Moreover, the convenient $(1, -1)-$matrix representation of [**?**] only applies to $S-$graphs of diameter 4. Is there an analogous representation for $S-$graphs of higher diameters?

For $S-$graphs of diameter 4, $|N_k(v_1)| = |N_k(v_2)|$ when v_1 and v_2 are in the same partite set. Considering cartesian products of K_2 and $G(2m, 2t)$

when $m \neq t$, it can be seen for $S-$graphs of diameter greater than 4 that the degrees of vertices in the same partite set can be different. This observation implies that any representation of $S-$graphs of higher diameters must be much more flexible than the $(1,-1)-$matrix representation for $S-$graphs of diameter 4.

Though the previous observation indicates that the complexity of $S-$graphs increases as the diameter increases, Janakiraman in [**?**] shows that there is some uniformity of the orders of ranked neighborhoods with the interesting characterization given by:

Theorem 3. *([**?**]) A diametrical graph G of diameter d is symmetric if and only if $|N_k(v)| = |N_{d-k}(v)|$ for all $v \in V(G)$ and all $0 \leq k \leq d$.*

Perhaps more can be said about the structure of the primitive $S-$graphs, leading to a useful representation that could be extended to all $S-$graphs upon factorization into cartesian products.

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