On the Choosability of Some Graphs

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Abstract

Suppose $ch(G)$ and $\chi(G)$ denote, respectively, the choice number and the chromatic number of a graph $G = (V, E)$. If $ch(G) = \chi(G)$ then G is said to be chromatic-choosable. Recently, Reed et al. proved a conjecture by Ohba that states that G is chromatic-choosable whenever $|V(G)| \leq 2\chi(G) + 1$. Here, we present other classes of chromatic-choosable graphs that do not satisfy the hypothesis of the proven conjecture. Moreover, we give the upper bounds for the choice numbers of the Mycielski graphs and the cartesian product of any two graphs, in terms of a vertex-neighborhood condition.

Keywords: List coloring, chromatic-choosable, cartesian product.

1 Basic notions

Throughout this paper, $G = (V, E)$ denotes a loopless connected *graph* where $V = V(G)$ and $E = E(G)$ denote, respectively, the set of *vertices* and the set of *edges* of G. An edge $e \in E$ with endpoints $u, v \in V$ is denoted by uv; u and v are *adjacent* in G and e is said to be *incident* with u and v. We denote by $N_G(u) = \{x \in V \mid ux \in E\}$ the *(open) neighbor* set in G of $u \in V$. When it is unnecessary to distinguish the graph G, the subscript G on N will be omitted. $\Delta(G)$, K_n and C_n denote, respectively, the maximum degree of G, a complete graph and a cycle on n vertices.

2 Preliminaries

A *list assignment* to the graph $G = (V, E)$ is a function L which assigns a finite set (list) $L(v)$ to each vertex $v \in V$. A *proper L-coloring* of G is a function $\phi : V \to \cup_{v \in V} L(v)$ satisfying, for every u, $v \in V$, (i) $\phi(v) \in L(v)$ and (ii) $uv \in E \to \phi(v) \neq \phi(u)$.

The *choice number* or *list-chromatic number of* G, denoted by $ch(G)$, is the smallest integer k such that there is always a proper L-coloring of G if L satisfies $|L(v)| \geq k$ for every $v \in V$. We define G to be *k-choosable* if it admits a proper L-coloring whenever $|L(v)| \geq k$ for all $v \in V$; then $ch(G)$ is the smallest integer k such that G is k-choosable. The following is a well-known result in the estimation of choice number.

Theorem A. (Erdős, Rubin and Taylor [3]) *If G is a connected graph that is neither a complete graph nor an odd cycle, then $ch(G) \leq \Delta(G)$.*

Corollary A. *For any graph G, $ch(G) \leq \Delta(G) + 1$.*

Proof. Clearly $ch(G)$ is the maximum of the choice numbers of the components of G. If H is a complete graph or an odd cycle then $ch(G) = \Delta(G)+1$. The conclusion now follows from Theorem A. □

Corollary A also follows by a "greedy coloring" argument.

Since the chromatic number $\chi(G)$ is similarly defined with the restriction that the list assignment is to be constant, it is clear that for all G, $\chi(G) \leq ch(G)$. There are many graphs whose choice number exceeds (sometimes greatly) their chromatic number. Figure 1 depicts the smallest graph G whose choice number exceeds its chromatic number.

It is easy to see that G is not properly L-colorable, so $ch(G) > 2 = \chi(G)$. Since G is connected, and neither a complete graph nor an odd cycle, it follows from Theorem A that $ch(G) \leq \Delta(G) = 3$. Thus, $ch(G) = 3$.

Any graph G for which the extremal equality $\chi(G) = ch(G)$ holds is said to be *chromatic-choosable.* It is not hard to see that cycles, cliques and trees are all chromatic-choosable.

The topic of list colorings was introduced by Vizing [9] and independently by Erdős, Rubin and Taylor [3]. Since the beginning, several results have sought to address specifically which classes of graphs are chromatic-choosable; Ohba's conjecture [8] is a well-known problem which has recently been settled by Reed et al.[7]. We state their result (or Ohba's conjecture) without proof, in the next theorem.

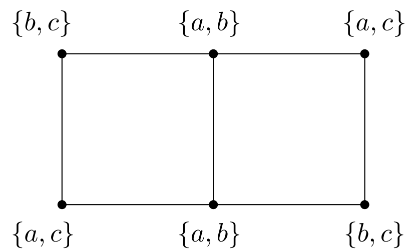

Figure 1: $G = \theta(1, 3, 3)$ with a list assignment L.

Theorem B.(Noel, Reed and Wu [7]) *If $|V(G)| \leq 2\chi(G) + 1$ then G is chromatic-choosable.*

The graph in Figure 1 shows that this result is sharp when $\chi(G) = 2$. In [2], examples are given in which $|V(G)| = 2\chi(G) + 2$ and $ch(G) = \chi(G) + 1$, for all even $\chi(G) > 2$. We do not know if Ohba's Theorem is sharp for odd $\chi(G) \geq 3$.

It is obvious that the proposed bound in Theorem B is weak in characterizing chromatic-choosable graphs, especially graphs with low chromatic numbers. For instance, according to this theorem, any bipartite graph of order smaller than 6 is chromatic-choosable, while it is well-known that any even cycle (of any order) is chromatic-choosable.

It is important to point out that the problem of finding chromatic-choosable graphs contains the famous list coloring conjecture which is generally attributed to Vizing [9], namely: the line graph of any graph is chromatic-choosable. So far, this has been proved to be correct for the line graphs of bipartite graphs by Galvin, as stated in

Theorem C.(Galvin [4]) *The line graph of any bipartite multigraph is chromatic-choosable.*

3 Choice number of the Mycielski graphs

We construct the *Mycielski graph* $M(G)$ from a graph G whose vertices are $V(G) = \{v_1, v_2, \ldots, v_n\}$, by introducing the set
$U = \{u_1, u_2, \ldots, u_n\} \cup \{w\}$, disjoint from $V(G)$. The vertices of $M(G)$ are $V(G) \cup U$, and its edges are $E(G) \cup \bigcup_{i=1}^{n} \Big[\{u_i z \mid z \in N_G(v_i)\} \cup \{wu_i\}\Big]$. It is well known that $\chi(M(G)) = \chi(G) + 1$.

Lemma 3.1. *$ch(M(G)) \le \Delta(G) + 2$.*

Proof. Let $k = \Delta(G) + 2 \ge ch(G) + 1$, by Corollary A, and suppose $M(G)$ is supplied with lists L of length k. Since G is k-choosable, we can color each $v_i \in V(G)$ so that the coloring is proper. Remove from the $L(u_i)'s$ the colors from their neighbors in G. Because each u_i has at most $\Delta(G)$ neighbors in G, this leaves at least 2 colors in each $L(u_i)$, $i = 1, \ldots, n$. Now color w and then proceed to properly color the $u_i's$ from their lists, giving a proper L-coloring of $M(G)$. □

Corollary 3.1. *If G is a complete graph or an odd cycle, then $ch(M(G)) = \Delta(G) + 2$.*

Proof. Applying Lemma 3.1, $\Delta(G)+2 = \chi(G)+1 = \chi(M(G)) \le ch(M(G)) \le \Delta(G) + 2$. □

From the argument in the proof of Corollary 3.1 follows

Corollary 3.2. *The Mycielski graph $M(K_n)$ is chromatic-choosable for all $n \ge 1$.*

Corollary 3.3. *The Mycielski graph $M(C_{2r+1})$ is chromatic-choosable for all $r \ge 1$.*

$M(C_5)$ is the Grötzsch graph, which is 4-chromatic and triangle-free. It is not too hard to see that the Mycielski of a star graph, $K_{1,n}$, is chromatic-choosable and perhaps $M(G)$ is chromatic-choosable whenever G is chromatic-choosable. We leave this as an open question, but from the fact that $\chi(M(G)) = \chi(G) + 1$ it would suffice to show that $ch(M(G)) \le ch(G) + 1$ for any graph G.

4 Choice number of the cartesian product of some graphs

Here, we present some results on an upper bound on the choice numbers of the Cartesian product of two graphs.

The *Cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$, where (u_1, v_1) is adjacent to (u_2, v_2) whenever $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. For instance, $K_2\Box K_2 \cong C_4$.

Theorem 4.1. *Suppose that $V(G) = V_1 \cup V_2$, V_1 and V_2 are disjoint and non-empty, $G_i = G[V_i]$, $i = 1, 2$, and no vertex in V_2 has, in G, more that r neighbors in V_1. Then $ch(G) \le \max\{ch(G_1), ch(G_2) + r\}$.*

Proof. Let $k = \max\{ch(G_1), ch(G_2) + r\}$ and suppose that L is a list assignment to G such that $|L(v)| \geq k$ for all $v \in V(G)$. Because $k \geq ch(G_1)$, there is a proper L-coloring ϕ of G_1. [More precisely, there is a proper L-restricted-to-V_1 coloring of G_1.] Define L' on G_2 by $L'(u) = L(u) \backslash \{\phi(v) \mid v \in N_G(u) \cap V_1\}$. By the hypothesis of the Theorem, $|L'(u)| \geq |L(u)| - r \geq k - r \geq ch(G_2)$, for all $u \in V_2$, and, therefore, there is a proper L'-coloring of G_2. Putting this coloring together with ϕ, we have a proper L-coloring of G. L was arbitrary, so $ch(G) \leq k$. $\square$

Corollary 4.1. *For $n \geq 2$, $ch(H \square P_n) \leq ch(H) + 1$.*

Proof. Let the vertices of P_n be $1, \ldots, n$; let $V_1 = V(H) \times \{1, \ldots, n-1\}$ and $V_2 = V(H) \times \{n\}$. Then V_1 and V_2 are disjoint and non-empty, $V(G) = V_1 \cup V_2$, and every vertex in V_2 is adjacent in $G = H \square P_n$ to exactly one vertex in V_1. Further, V_1 induces $G_1 \cong H \square P_{n-1}$ in G, and V_2 induces $G_2 \cong H$ in G. The conclusion follows from Theorem 4.1 by induction on n. $\square$

The graphs in the next two corollaries each contain the subgraph $H = \theta(1, 3, 3)$, giving a lower bound of their choice number. The upper bound follows from Corollary 4.1, giving each result.

Corollary 4.2. *For $m \geq 2$, $n \geq 3$, $ch(P_m \square P_n) = 3$.*

Corollary 4.3. *For $r \geq 2$, $n \geq 2$, $ch(C_{2r} \square P_n) = 3$.*

We note that Corollary 4.2 also follows from a result by Alon and Tarsi who showed that every bipartite planar graph is 3-choosable [1]. Also, it is worth noting that by Theorem B, $ch(C_m \square P_2) \leq 3$ for all $m \geq 3$. Therefore, $ch(C_m \square P_2) = 3$ when m is odd. Together with Corollary 4.3, we can conclude that $ch(C_m \square P_2) = 3$ for all $m \geq 3$.

Corollary 4.4. *Suppose that $u \in V(H)$ has degree $r > 0$ in H; then $ch(G \square H) \leq \max\{ch\Big(G \square (H - u)\Big), ch(G) + r\}$.*

Proof. $G \square H$ can be formed by connecting $G \square (H - u)$ to a disjoint copy of G in such a way that every vertex of the added G has r neighbors in $G \square (H - u)$. $\square$

Corollary 4.4 is just a special case of an obvious application of Theorem 4.1, in which $V(G \square H) = V(G) \times V(H)$ is partitioned into $V(G) \times U_1$ and $V(G) \times U_2$ for some partition of $V(H)$ into $U_1 \cup U_2$. In Corollary 4.4, $|U_2| = 1$. However, neither Corollary 4.4 nor its generalization seems to help in determining the choice numbers of the cylindrical grid $C_{2r+1} \square P_n$ or the toroidal grid $C_m \square C_n$, m, $n \geq 3$. Because these graphs contain $\theta(1, 3, 3)$ and have maximum degree 4, their choice numbers are in $\{3, 4\}$.

We close this section with a more general upper bound on the choice number of the cartesian product of any two graphs.

Theorem 4.2. *$ch(G\square H) \leq \max\{|V(G)|, |V(H)|\}$ with equality when $G = H = K_n$.*

Proof. If G and H are two graphs of order at most n, then $G\square H \subseteq K_n\square K_n$ and $ch(G\square H) \leq ch(K_n\square K_n) = ch(L(K_{n,n})) = n$, with the last equality following from Theorem C. $\square$

Corollary 4.5. *If $|V(G)| \leq n$ then $ch(G\square K_n)$ is chromatic-choosable.*

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