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Formulas for the computation of the Tutte polynomial of graphs with parallel classes

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Abstract

We give some reduction formulas for computing the Tutte polynomial of any graph with parallel classes. Several examples are given to illustrate our results.

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1. Introduction

Many polynomials have been extensively researched in relations to certain properties of graphs; graph polynomials play some important roles as they encode various information about graphs. Tutte and chromatic polynomials are two of the most studied graph polynomials. In addition, the Tutte polynomial is applicable in many fields, such as, colorings, flows, network relaibility, knot theory, statistical physics, etc.

This paper was motivated by the fact that the chromatic polynomial of a graph is an evaluation of its Tutte polynomial. See [2, 4]. Further, it is easier to find the chromatic polynomial of a given graph than to compute its Tutte polynomial which is often intractable. In fact, multigraphs of same parallel classes share the same chromatic polynomials even though their Tutte polynomials are different. This adds to the level of complexity for finding explicit formulas of Tutte polynomial of graphs in general.

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For this reason, many research have been focused on finding more efficient algorithms to reduce the computational steps; see for instance [1, 3], for some reduction formulas for graphs and matroids. Our results are inspired particularly by the work done in [3].

In this paper, we first present a short algorithm describing a sequence of deletion of some edges of a simple graph to get a minor. This process gives rise to three types of minors. Later, we classify multigraphs according to the type of minors obtained from their simplification. Then, for each multigraph of these classes, we give a reduction formula for their Tutte polynomial in terms of the Tutte polynomial of their simplification and the minor of their simplification. Finally, we provide examples to illustrate the reduction formulas obtained for each class of multigraphs.

2. Preliminaries

In this section we review some basic definitions and methods for computing the Tutte polynomial relevant to this paper.

$G = (V, E)$ denotes an undirected *(multi)graph* where $V = V(G)$ and $E = E(G)$ denote, respectively, the set of *vertices* and the multiset of unordered pair of vertices called *edges* or elements of G. An edge $e \in E(G)$ with ends $u, v \in V$ is denoted by $\{u, v\}$ and when $\{u, v\}$ occurs more than once in E, it is said to be *parallel*. An edge in a connected graph is an *isthmus* if its removal leaves a disconnected graph. The special edge $\{u, u\}$ is called a *loop*. A graph that admits no parallel edges or loops is said to be *simple*. The *simplification* or underlying graph of G is obtained by removing any loop and repeated edge of E. A graph G is said to be *isomorphic* to a graph H if G can be obtained by relabeling the vertices of H; and we write $G \cong H$. Given G, the *deletion* of an edge $e \in E$ is denoted by $G\backslash e = (V, E\backslash e)$. The *contraction* of e, denoted by G/e, results in identifying the endpoints of e after its deletion. A *minor* G' of G is a graph obtained from G through a sequence of edge deletions/contractions. A collection of multigraphs $\mathcal{G} = G_n, G_{n-1}, \cdots, G_1$ forms a *parallel class* if their simplifications are isomorphic.

There are several methods for computing the Tutte polynomial of a graph, see [4, 5]. The most widely used technique involves the deletion/contraction operation and it is given by the following:

T1. $T(I; x, y) = x$ and $T(L; x, y) = y$ where I is an isthmus and L is a loop.

T2. If e is an edge of the graph G and e is neither a loop nor an isthmus, then

$$T(G; x, y) = T(G\backslash e; x, y) + T(G/e; x, y).$$

T3. If e is a loop or an isthmus of the graph G, then

$$T(G; x, y) = T(G(e); x, y)T(G/e; x, y).$$

T4. $T(G; x, y) = 1$ if G is an edgeless or null graph.

3. Types of graphs with parallel classes

In this section, we present an algorithm called DEL that is used to classify any graph. From a simple graph G, we obtain a minor through a series of edge deletion; G is classified according to the type of minor we obtain from DEL.

Suppose G is a simple graph with some of its edges labeled, $e_1, e_2, \cdots, e_n$. Let $M_n = \{e_1, e_2, \cdots, e_n\} \subseteq E(G)$. By considering only the elements of M_n in G, we obtain a special minor for G based on the deleted elements of M_n using the following algorithm:

Algorithm DEL

1. If all the elements of M_n are isthmuses in G, STOP, otherwise go to step 2.
2. Choose any element $e_i \in M_n$ which is not an isthmus in G.
3. Delete e_i from M_n.
4. If $M_n \neq \emptyset$, repeat step 1, otherwise STOP.

Types of minors

For any simple graph G, this algorithm DEL results in a minor G', of one of the following three groups:

- **Type**-i None of the elements of M_n are deleted. Thus, they are all isthmuses in G and the minor $G' = G$.
- **Type**-ii All the elements of M_n are deleted. In this the case the minor $G' = G \backslash M_n$.
- **Type**-iii Some elements of M_n are deleted and its remaining elements become isthmuses. Hence, the minor $G' = G \backslash M_l$ with $l < n$ and the deleted elements are $e_1, e_2, \cdots, e_l$.

Let $\mathcal{G} = G_n, G_{n-1}, \ldots, G_1$ be a collection of all multigraphs whose simplification is G. It is clear that $G_n, G_{n-1}, \ldots, G_1$ are isomorphic up to parallel class. Consider a member say, G_n with parallel edges, $e_1, e_2, \cdots, e_n$. Denote by G, the graph obtained from G_n by replacing each edge e_i in G_n by a single edge (with the same label) e_i in G, for $i = 1, \ldots, n$. Thus, $M_n = \{e_1, e_2, \cdots, e_n\} \subseteq E(G)$ and G a simplification of G_n. Since G is simple, we apply DEL to M_n to obtain the minor G' of **Type**-i, **Type**-ii or **Type**-iii, as previously described. Further, we classify G_n according to the type of its minor G' of G. Therefore we say that G_n (or $\mathcal{G}$) is of **Type**-i, **Type**-ii or **Type**-iii if G' is of **Type**-i, **Type**-ii or **Type**-iii, respectively.

4. Tutte polynomials of parallel classes

In this section we state and prove the main results of this paper. In particular, we give expressions for the Tutte polynomials of the different types of parallel classes defined in Section 3.

In the next two Lemmas, we give a recursion for computing the Tutte polynomial of parallel class whose members have each, one single parallel edge.

Lemma 4.1. *Let G be a simple graph with an edge e which is an isthmus. Let G_1 be a graph whose simplification is G such that e is parallel to k edges in G_1 and $k \geq 1$. Then the Tutte polynomial of G_1 is*

$$T(G_1; x, y) = (x + (\sum_{i=1}^{k} y^i))T(G/e; x, y).$$

Proof. Note that if the edge e of G_1 is contracted, then all the edges parallel to e become loops in the minor G_1/e. Furthermore, since e is an isthmus $T(G;x,y) = xT(G/e;x,y)$. This establishes the base case, when $k = 1$, for a proof by induction. An induction on k completes the proof. □

Lemma 4.2. *Suppose G is a simple graph with an edge e which is not an isthmus. Let G_1 be a graph whose simplification is G such that e is parallel to k edges in G_1 and $k \geq 1$. Then the Tutte polynomial of G_1 is*

$$T(G_1;x,y) = (\sum_{i=0}^{k} y^i)T(G;x,y) - (\sum_{i=1}^{k} y^i)T(G\backslash e;x,y).$$

Proof. Recall that $T(G;x,y) = T(G\backslash e;x,y) + T(G/e;x,y)$ since e is not an isthmus. Hence we write

$$T(G/e;x,y) \quad = \quad T(G;x,y) - T(G\backslash e;x,y). \tag{1}$$

Furthermore, if the edge e is contracted in G_1, then all edges parallel to e become loops in the minor G_1/e. Induction on k completes the proof. □

We now present the main results of this paper that give the Tutte polynomial of a member of any parallel class in terms of the Tutte polynomial of its simplification. Thus, the next theorems generalize the results in Section 3.

Recall that if a graph G_n is of **Type**-i, then all the elements of $E_n = \{e_1, e_2, \cdots, e_n\}$ in the simplification of G_n are isthmuses.

Theorem 4.1. *Let G be the simplification of a graph G_n which is of* ***Type**-i with $E_n = \{e_1, e_2, \cdots, e_n\}$ being the set of all parallel edges. If the edge e_j is parallel to j_p edges in G_n for any $j \in \{1, 2, \cdots n\}$, then the Tutte polynomial of G_n is*

$$T(G_n;x,y) \quad = \quad [\prod_{j=1}^{n}(x + (\sum_{i=1}^{j_p} y^i))]T(G/E_n;x,y).$$

Proof. By applying Lemma 4.1 on one parallel edge at a time, starting with e_n and repeating the process for $e_{n-1}, \cdots, e_1$ successively, we get

$$\begin{aligned}
T(G_n;x,y) &= (x + \sum_{i=1}^{n_p} y^i)T(G_{n-1}/e_n;x,y) \\
&= (x + (\sum_{i=1}^{n_p} y^i))(x + (\sum_{i=1}^{n-1_p} y^i))T(G_{n-2}/e_1/e_2;x,y) \\
&\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&= (x + (\sum_{i=1}^{n_p} y^i))(x + (\sum_{i=1}^{n-1_p} y^i)) \cdots (x + (\sum_{i=1}^{1_p} y^i))T(G/e_1/e_2/\cdots/e_n;x,y) \\
&= [\prod_{j=1}^{n}(x + (\sum_{i=1}^{j_p} y^i))]T(G/E_n;x,y).
\end{aligned}$$

□

Recall that if a graph G_n is of **Type**-ii, then in the process of obtaining the minor G', all the elements of $E_n = \{e_1, e_2, \cdots, e_n\}$ are deleted in G.

Theorem 4.2. *Let G_n be a graph of **Type**-ii whose simplification is G and let $E_n = \{e_1, e_2, \cdots, e_n\}$ be the set of all parallel edges in G_n. If the edge e_j is parallel to j_p edges for any $j \in \{1, 2, \cdots n\}$ and $S \subseteq E_n$ such that $S = \{e_1, e_2, \cdots, e_t\}$ where $t \leq n$ and the complement of S, $\overline{S} = \{e_{t+1}, e_{t+2}, \cdots, e_n\}$, then the Tutte polynomial of G_n is*

$$T(G_n; x, y) = (-1)^{|S|} \sum_{S \subseteq E_n} [\prod_{l=1}^{t} (\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{n} (\sum_{i=0}^{l_p} y^i)] T(G \backslash S; x, y)$$

Proof. Note that when $S = \emptyset$, $\overline{S} = E_n$.

The Proof is by induction on n with the case when $n = 1$ being Lemma 4.2. When $n = 2$, E_2 has two parallel elements namely, e_1 and e_2. We apply repeatedly Lemma 4.2 on e_2 then on e_1 to get

$$\begin{aligned}
T(G_2; x, y) &= (\sum_{i=0}^{2_p} y^i) T(G_1; x, y) - (\sum_{i=1}^{2_p} y^i) T(G_1 \backslash e_2; x, y) \\
&= (\sum_{i=0}^{2_p} y^i)[(\sum_{i=0}^{1_p} y^i) T(G; x, y) - (\sum_{i=1}^{1_p} y^i) T(G \backslash e_1; x, y)] \\
&- (\sum_{i=1}^{2_p} y^i)[(\sum_{i=0}^{1_p} y^i) T(G \backslash e_2; x, y) - (\sum_{i=1}^{1_p} y^i) T(G \backslash e_2 \backslash e_1; x, y)] \\
&= (\sum_{i=0}^{2_p} y^i)(\sum_{i=0}^{1_p} y^i) T(G; x, y) - (\sum_{i=0}^{2_p} y^i)(\sum_{i=1}^{1_p} y^i) T(G \backslash e_1; x, y) \\
&- (\sum_{i=1}^{2_p} y^i)(\sum_{i=0}^{1_p} y^i) T(G \backslash e_2; x, y) - (\sum_{i=1}^{2_p} y^i)(\sum_{i=1}^{1_p} y^i) T(G \backslash e_1 \backslash e_2; x, y) \\
&= (-1)^{|S|} \sum_{S \subseteq E_2} [\prod_{l=1}^{t} (\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{2} (\sum_{i=0}^{l_p} y^i)] T(G \backslash S; x, y).
\end{aligned}$$

Now assume the result is true for some graph G_q with $q \geq 1$ parallel elements. Consider a graph $G_{q+1} = G_q \cup e_{q+1}$, where e_{q+1} is a parallel element. Thus, the simplification of G_q is equal to the simplification of G_{q+1} which is G. If S is a subset of E_q then S is also a subset of E_{q+1} since $E_q \subset E_{q+1}$. by definition.

Consider any subset S' of E_{q+1}. Note that there are two types of subsets of E_{q+1} : those that contain the element e_{q+1} and those that do not. By applying Lemma 4.2 on the parallel element e_{q+1}, we get

$$T(G_{q+1}; x, y) = (\sum_{i=0}^{(q+1)_p} y^i) T(G_q; x, y) - (\sum_{i=1}^{(q+1)_p} y^i) T(G_q \backslash e_{q+1}; x, y). \tag{2}$$

Since e_{q+1} is not in G_q, it is not an element of E_q. Thus, e_{q+1} is not in any $S \subset E_q$. Therefore, e_{q+1} is in all the minors of $G\backslash S$ when computing $T(G_q; x, y)$. Furthermore, if we consider the set E_{q+1} and any subset S' which do not contain e_{q+1}, it is clear that $S' = S \subseteq E_q \subset E_{q+1}$ Hence, we obtain that

$$
\begin{aligned}
(\sum_{i=0}^{(q+1)_p} y^i)T(G_q; x, y) &= (-1)^{|S|}(\sum_{i=0}^{(q+1)_p} y^i) \sum_{S\subseteq E_q} [\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{q}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S; x, y) \\
&= (-1)^{|S|}(\sum_{i=0}^{(q+1)_p} y^i) \sum_{\substack{S=S'\subseteq E_{q+1} \\ e_{q+1}\notin S'}} [\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{q}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S; x, y) \\
&= (-1)^{|S'|} \sum_{S'\subseteq E_{q+1}} [\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{q+1}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S'; x, y).
\end{aligned}
$$

On the other hand, when e_{q+1} is deleted in G_q, it means e_{q+1} is not in any minor of the form $G\backslash e_{q+1}\backslash S$ in the computation of $T(G_q\backslash e_{q+1}; x, y)$. Thus, if we consider the set E_{q+1} and any subset S'' containing e_{q+1}, then it is clear that $S'' = S \cup e_{q+1} \subseteq E_q \cup e_{q+1} = E_{q+1}$. Moreover, if $S = \{e_1, e_2, \cdots, e_t\}$ then $S'' = \{e_1, e_2, \cdots, e_t, e_{t+1}\}$ where $e_{q+1} = e_{t+1}$ and $t \leq q$. Furthermore, the complement of S'', $\overline{S''} = \{e_{t+2}, e_{t+3}, \cdots, e_n\}$. Hence, we can rewrite

$$
\begin{aligned}
&-(\sum_{i=1}^{(q+1)_p} y^i)T(G_q\backslash e_{q+1}; x, y) \\
&= (-1)(-1)^{|S|}(\sum_{i=1}^{(q+1)_p} y^i) \sum_{S\subseteq E_q} [\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{n}(\sum_{i=0}^{l_p} y^i)]T(G\backslash e_{q+1}\backslash S; x, y) \\
&= (-1)^{|S''|} \sum_{S\cup e_{q+1}\subseteq E_q\cup e_{q+1}} [\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)(\sum_{i=1}^{(q+1)_p} y^i)][\prod_{l=t+1}^{q}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S\cup e_{q+1}; x, y) \\
&= (-1)^{|S''|} \sum_{S''\subseteq E_{q+1}} [\prod_{l=1}^{t+1}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+2}^{q+1}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S''; x, y).
\end{aligned}
$$

Now considering the fact that all subsets of E_{q+1} are either the form S' or S'' and returning back to Equation 2, we get

$$\begin{aligned} T(G_{q+1};x,y) &= (\sum_{i=0}^{(q+1)_p} y^i)T(G_q;x,y) - (\sum_{i=1}^{(q+1)_p} y^i)T(G_q\backslash e_{q+1};x,y) \\ &= (-1)^{|S'|} \sum_{S'\subseteq E_{q+1}} [\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{q+1}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S';x,y) \\ &+ (-1)^{|S''|} \sum_{S''\subseteq E_{q+1}} [\prod_{l=1}^{t+1}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+2}^{q+1}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S'';x,y). \\ &= (-1)^{|S|} \sum_{S\subseteq E_{q+1}} [\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{q+1}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S;x,y). \end{aligned}$$

□

Recall that if a graph G_n is of **Type**-iii, then some edges of $\{e_1, e_2, \cdots, e_n\}$ are not in the minor G' and those that are present in G' are isthmuses in G'.

The proof for the next theorem follows directly from Theorem 4.1 and Theorem 4.2.

Theorem 4.3. *Let G_n be a graph of **Type**-iii, such that in G_n the edge e_j is parallel to j_p edges for any $j \in \{1, 2, \cdots n\}$. Suppose $E_r = \{e_1, e_2, \cdots, e_r\}$ is a maximal set of edges which are not isthmuses in G_n for some $r < n$, such that $\overline{E_r} = \{e_{r+1}, e_{r+2}, \cdots, e_n\}$ is a set of isthmuses in the minor $G\backslash E_r$. If $S \subseteq E_r$ then the Tutte polynomial of G_n is*

$$\begin{aligned} T(G_n;x,y) &= (-1)^{|S|} \sum_{S\subseteq E_r} (-1)^{|S|}[\prod_{l=1}^{t}(\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{n}(\sum_{i=0}^{l_p} y^i)]T(G\backslash S;x,y) \\ &+ (-1)^r[\prod_{l=1}^{r}(\sum_{i=1}^{l_p} y^i)][\prod_{l=r+1}^{n}(x + \sum_{i=1}^{l_p} y^i)]T(G\backslash E_r/\overline{E_r};x,y). \end{aligned}$$

5. Examples

We conclude this paper with some examples that illustrate the recurrence formulas presented in the previous sections, for each type of parallel class.

Example 5.1. *Let a multigraph G_2 be defined by the vertex set $V(G_1) = \{1, 2, 3, 4, 5, 6\}$ and the edge set*

$$E(G_2) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{4, 5\}, \{5, 4\}, \{3, 6\}, \{6, 3\}\}.$$

If we let $e_1 = \{4, 5\}$ and $e_2 = \{3, 6\}$, then $E_2 = \{\{4, 5\}, \{3, 6\}\}$ and $1_p = 1$ and $2_p = 1$. The simplification of G_2, is the graph G defined by the vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and the edge set

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{4, 5\}, \{3, 6\}\}.$$

The minor G' obtained after applying algorithm DEL is defined by the vertex set $V(G') = \{1, 2, 3, 4, 5, 6\}$ and the edge set

$$E(G') = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{4, 5\}, \{3, 6\}\}.$$

*G' is a minor of **Type**-i, that is all the edges of E_2, e_1 and e_2 are isthmuses in G'. Thus G_2 is a graph of **Type**-i. To find the Tutte polynomial of G_2 we apply Theorem 4.1 where $E_2 = \{e_1, e_2\}$ and $G/E_2 \cong C_4$, a cycle on 4 vertices.*

$$\begin{aligned}
T(G_2; x, y) &= [\prod_{j=1}^{2}(x + (\sum_{i=1}^{j_p} y^i))]T(G/S; x, y) \\
&= (x + y)(x + y)T(G/e_1/e_2; x, y) \\
&= (x^2 + 2xy + y^2)T(C_4; x, y) \\
&= (x^2 + 2xy + y^2)(y + x + x^2 + x^3) \\
&= x^3 + x^4 + x^5 + y^3 + 3yx^2 + 3y^2x + 2yx^3 + 2yx^4 + y^2x^2 + y^2x^3.
\end{aligned}$$

Example 5.2. *Let a multigraph H_2 be defined by the vertex set $V(H_2) = \{1, 2, 3, 4\}$ and the edge set*

$$E(H_2) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}, \{3, 2\}, \{1, 4\}\}.$$

If we let $e_1 = \{4, 1\}$ and $e_2 = \{2, 3\}$ then $E_2 = \{\{4, 1\}, \{2, 3\}\}$ and $1_p = 1$ and $2_p = 1$. The simplification of H_2 is a graph H defined by the vertex set $V(H) = \{1, 2, 3, 4\}$ and the edge set

$$E(H) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}\}.$$

*The minor H' obtained after applying algorithm DEL is defined by the vertex set $V(H') = \{1, 2, 3, 4\}$ and the edge set $E(H') = \{\{1, 2\}, \{1, 3\}, \{3, 4\}\}$. H' is a minor of **Type**-ii as the edges of E_2, e_1 and e_2, are not in H'. Hence H_2 is a graph of **Type**-ii. To find the Tutte polynomial of H_2, we apply Theorem 4.2 where $S \subseteq E_2 = \{\{4, 5\}, \{3, 6\}\}$. If $S = \emptyset$ then $H\backslash S = G$ and also if $S = \{e_1\}$ then $G\backslash S$ is isomorphic to $C_3 \cup e$ where e is an isthmus. Similarly, if $S = \{e_2\}$ then $G\backslash S$ is isomorphic to $C_3 \cup e$ where e is an isthmus. Finally, if $S = \{e_1, e_2\}$ this implies*

$E_2 = \{e_1, e_2\}$ and $H\backslash S \cong P_4$, a path on 4 vertices.

$$
\begin{aligned}
T(H_2; x, y) &= (-1)^{|S|} \sum_{S \subseteq E_2} [\prod_{l=1}^{t} (\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{2} (\sum_{i=0}^{l_p} y^i)] T(H\backslash S; x, y) \\
&= (-1)^{|\emptyset|} [\prod_{l=1}^{2} (\sum_{i=0}^{l_p} y^i)] T(H\backslash \emptyset; x, y) \\
&+ (-1)^{|\{e_1\}|} [\prod_{l=1}^{1} (\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{2} (\sum_{i=0}^{l_p} y^i)] T(H\backslash \{e_1\}; x, y) \\
&+ (-1)^{|\{e_2\}|} [\prod_{l=1}^{1} (\sum_{i=1}^{l_p} y^i)][\prod_{l=t+1}^{2} (\sum_{i=0}^{l_p} y^i)] T(H\backslash \{e_2\}; x, y) \\
&+ (-1)^{|\{e_1, e_2\}|} [\prod_{l=1}^{2} (\sum_{i=0}^{l_p} y^i)] T(H\backslash \{e_1, e_2\}; x, y) \\
&= [(1+y)(1+y)] T(H\backslash \emptyset; x, y) - [(1+y)y] T(H\backslash \{e_1\}; x, y) \\
&- [(1+y)y] T(H\backslash \{e_2\}; x, y) + [y^2] T(H\backslash \{e_1\}; x, y) \\
&= [(1 + 2y + y^2] T(H; x, y) - 2y(1+y) x T(C_3; x, y) \\
&+ y^2 T(P_4; x, y) \\
&= [1 + 2y + y^2][y + x + 2x^2 + x^3 + 2yx + y^2] \\
&- [2y^2 + 2y][y + x + x^2] + y^2 x^3 \\
&= x + 2x^2 + x^3 + y + y^2 + 3y^3 + y^4 + 2xy + 3y^2 x.
\end{aligned}
$$

Example 5.3. *Let N_2 be a multigraph defined by the vertex set $V(N_2) = \{1, 2, 3, 4\}$ and the edge set*

$$E(N_2) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{2, 1\}, \{4, 3\}\}.$$

*If we let $e_1 = \{1, 2\}$ and $e_2 = \{3, 4\}$ then $E_2 = \{e_1, e_2\}$ and $1_p = 1$ and $2_p = 1$. The simplification of N_2 is a graph N defined by the vertex set $V(N) = \{1, 2, 3, 4\}$ and the edge set $E(N) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. The minor N' obtained after applying the algorithm DEL is a graph defined by the vertex set $V(N') = \{1, 2, 3, 4\}$ and the edge set $E(N') = \{\{1, 4\}, \{3, 4\}, \{2, 3\}\}$. N' is a minor of **Type**-iii, that is e_1 is not in N' and e_2 is an isthmus in N'. Thus, N_2 is a graph of **Type**-iii. To find the Tutte polynomial of N_2 we apply Theorem 4.3 where $S \subseteq E_1 = \{e_1\}$. Therefore if $S = \emptyset$ then $N/S = N$ and if $S = \{e_1\}$ this implies that $N\backslash S$ is isomorphic to P_4. Note that $N\backslash E_1/\overline{E_1} \cong P_3$.*

$$
\begin{aligned}
T(N_2;x,y) &= (-1)^{|S|}\sum_{S\subseteq E_r}(-1)^{|S|}[\prod_{l=1}^{t}(\sum_{i=1}^{l_p}y^i)][\prod_{l=t+1}^{n}(\sum_{i=0}^{l_p}y^i)]T(N\backslash S;x,y)\\
&+ (-1)^r[\prod_{l=1}^{r}(\sum_{i=1}^{l_p}y^i)][\prod_{l=r+1}^{n}(x+\sum_{i=1}^{l_p}y^i)]T(M\backslash E_r/\overline{E_r};x,y)\\
&= (-1)^{|S|}\sum_{S\subseteq E_1}[\prod_{l=1}^{t}(\sum_{i=1}^{l_p}y^i)][\prod_{l=t+1}^{2}(\sum_{i=0}^{l_p}y^i)]T(N\backslash S;x,y)\\
&+ (-1)^1[\prod_{l=1}^{1}(\sum_{i=1}^{l_p}y^i)][\prod_{l=2}^{2}(x+\sum_{i=1}^{l_p}y^i)]T(N\backslash E_1/\overline{E_1};x,y).\\
&= [(1+y)(1+y)]T(N\backslash\emptyset;x,y)-[(1+y)y]T(N\backslash\{e_1\};x,y)\\
&- y[x+y]T(N\backslash e_1/e_2;x,y).\\
&= [1+2y+y^2]T(C_4;x,y)-y(1+y)xT(P_4;x,y)\\
&+ [y(y+x)]T(P_3;x,y)\\
&= [1+2y+y^2][y+x+x^2+x^3]-y(1+y)x^4-[y^2+yx]x^2\\
&= x+x^2+x^3+y+2y^2+y^3+2yx+2yx^2+y^2x.
\end{aligned}
$$

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