Formulas for the computation of the Tutte polynomial of graphs with parallel classes

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Abstract

We give some reduction formulas for computing the Tutte polynomial of any graph with parallel classes. Several examples are given to illustrate our results.

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1. Introduction

Many polynomials have been extensively researched in relations to certain properties of graphs; graph polynomials play some important roles as they encode various information about graphs. Tutte and chromatic polynomials are two of the most studied graph polynomials. In addition, the Tutte polynomial is applicable in many fields, such as, colorings, flows, network reliability, knot theory, statistical physics, etc.

This paper was motivated by the fact that the chromatic polynomial of a graph is an evaluation of its Tutte polynomial. See [2, 4]. Further, it is easier to find the chromatic polynomial of a given graph than to compute its Tutte polynomial which is often intractable. In fact, multigraphs of same parallel classes share the same chromatic polynomials even though their Tutte polynomials are different. This adds to the level of complexity for finding explicit formulas of Tutte polynomial of graphs in general.
For this reason, many research have been focused on finding more efficient algorithms to reduce the computational steps; see for instance [1, 3], for some reduction formulas for graphs and matroids. Our results are inspired particularly by the work done in [3].

In this paper, we first present a short algorithm describing a sequence of deletion of some edges of a simple graph to get a minor. This process gives rise to three types of minors. Later, we classify multigraphs according to the type of minors obtained from their simplification. Then, for each multigraph of these classes, we give a reduction formula for their Tutte polynomial in terms of the Tutte polynomial of their simplification and the minor of their simplification. Finally, we provide examples to illustrate the reduction formulas obtained for each class of multigraphs.

2. Preliminaries

In this section we review some basic definitions and methods for computing the Tutte polynomial relevant to this paper.

\( G = (V, E) \) denotes an undirected (multi)graph where \( V = V(G) \) and \( E = E(G) \) denote, respectively, the set of vertices and the multiset of unordered pair of vertices called edges or elements of \( G \). An edge \( e \in E(G) \) with ends \( u, v \in V \) is denoted by \( \{u, v\} \) and when \( \{u, v\} \) occurs more than once in \( E \), it is said to be parallel. An edge in a connected graph is an isthmus if its removal leaves a disconnected graph. The special edge \( \{u, u\} \) is called a loop. A graph that admits no parallel edges or loops is said to be simple. The simplification or underlying graph of \( G \) is obtained by removing any loop and repeated edge of \( E \). A graph \( G \) is said to be isomorphic to a graph \( H \) if \( G \) can be obtained by relabeling the vertices of \( H \); and we write \( G \cong H \). Given \( G \), the deletion of an edge \( e \in E \) is denoted by \( G \setminus e = (V, E \setminus e) \). The contraction of \( e \), denoted by \( G/e \), results in identifying the endpoints of \( e \) after its deletion. A minor \( G' \) of \( G \) is a graph obtained from \( G \) through a sequence of edge deletions/contractions. A collection of multigraphs \( G = G_n, G_{n-1}, \ldots, G_1 \) forms a parallel class if their simplifications are isomorphic.

There are several methods for computing the Tutte polynomial of a graph, see [4, 5]. The most widely used technique involves the deletion/contraction operation and it is given by the following:

\( T(I; x, y) = x \) and \( T(L; x, y) = y \) where \( I \) is an isthmus and \( L \) is a loop.

T2. If \( e \) is an edge of the graph \( G \) and \( e \) is neither a loop nor an isthmus, then

\[
T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y).
\]

T3. If \( e \) is a loop or an isthmus of the graph \( G \), then

\[
T(G; x, y) = T(G(e); x, y)T(G/e; x, y).
\]

T4. \( T(G; x, y) = 1 \) if \( G \) is an edgeless or null graph.

3. Types of graphs with parallel classes

In this section, we present an algorithm called DEL that is used to classify any graph. From a simple graph \( G \), we obtain a minor through a series of edge deletion; \( G \) is classified according to the type of minor we obtain from DEL.
Suppose $G$ is a simple graph with some of its edges labeled, $e_1, e_2, \ldots, e_n$. Let $M_n = \{e_1, e_2, \ldots, e_n\} \subseteq E(G)$. By considering only the elements of $M_n$ in $G$, we obtain a special minor for $G$ based on the deleted elements of $M_n$ using the following algorithm:

**Algorithm DEL**
1. If all the elements of $M_n$ are isthmuses in $G$, STOP, otherwise go to step 2.
2. Choose any element $e_i \in M_n$ which is not an isthmus in $G$.
3. Delete $e_i$ from $M_n$.
4. If $M_n \neq \emptyset$, repeat step 1, otherwise STOP.

**Types of minors**

For any simple graph $G$, this algorithm DEL results in a minor $G'$, of one of the following three groups:

- **Type-i** None of the elements of $M_n$ are deleted. Thus, they are all isthmuses in $G$ and the minor $G' = G$.
- **Type-ii** All the elements of $M_n$ are deleted. In this the case the minor $G' = G \setminus M_n$.
- **Type-iii** Some elements of $M_n$ are deleted and its remaining elements become isthmuses. Hence, the minor $G' = G \setminus M_l$ with $l < n$ and the deleted elements are $e_1, e_2, \ldots, e_l$.

Let $G = G_n, G_{n-1}, \ldots, G_1$ be a collection of all multigraphs whose simplification is $G$. It is clear that $G_n, G_{n-1}, \ldots, G_1$ are isomorphic up to parallel class. Consider a member say, $G_n$ with parallel edges, $e_1, e_2, \ldots, e_n$. Denote by $G$, the graph obtained from $G_n$ by replacing each edge $e_i$ in $G_n$ by a single edge (with the same label) $e_i$ in $G$, for $i = 1, \ldots, n$. Thus, $M_n = \{e_1, e_2, \ldots, e_n\} \subseteq E(G)$ and $G$ a simplification of $G_n$. Since $G$ is simple, we apply DEL to $M_n$ to obtain the minor $G'$ of $G$. Therefore we say that $G_n$ (or $G$) is of **Type-i**, **Type-ii** or **Type-iii** if $G'$ is of **Type-i**, **Type-ii** or **Type-iii**, respectively.

4. Tutte polynomials of parallel classes

In this section we state and prove the main results of this paper. In particular, we give expressions for the Tutte polynomials of the different types of parallel classes defined in Section 3.

In the next two Lemmas, we give a recursion for computing the Tutte polynomial of parallel class whose members have each, one single parallel edge.

**Lemma 4.1.** Let $G$ be a simple graph with an edge $e$ which is an isthmus. Let $G_1$ be a graph whose simplification is $G$ such that $e$ is parallel to $k$ edges in $G_1$ and $k \geq 1$. Then the Tutte polynomial of $G_1$ is

$$T(G_1; x, y) = (x + \left(\sum_{i=1}^{k} y^i\right))T(G/e; x, y).$$
Theorem 4.1. Simplification of graphs generalize the results in Section 3. Any parallel class in terms of the Tutte polynomial of its simplification. Thus, the next theorems

Lemma 4.2. Suppose $G$ is a simple graph with an edge $e$ which is not an isthmus. Let $G_1$ be a graph whose simplification is $G$ such that $e$ is parallel to $k$ edges in $G_1$ and $k \geq 1$. Then the Tutte polynomial of $G_1$ is

$$T(G_1; x, y) = (\sum_{i=0}^{k} y^i)T(G; x, y) - (\sum_{i=1}^{k} y^i)T(G/e; x, y).$$

Proof. Recall that $T(G; x, y) = T(G/e; x, y) + T(G/e; x, y)$ since $e$ is not an isthmus. Hence we write

$$T(G/e; x, y) = T(G; x, y) - T(G/e; x, y).$$

Furthermore, if the edge $e$ is contracted in $G_1$, then all edges parallel to $e$ become loops in the minor $G_1/e$. Induction on $k$ completes the proof.

We now present the main results of this paper that give the Tutte polynomial of a member of any parallel class in terms of the Tutte polynomial of its simplification. Thus, the next theorems generalize the results in Section 3.

Recall that if a graph $G_n$ is of Type-i, then all the elements of $E_n = \{e_1, e_2, \cdots, e_n\}$ in the simplification of $G_n$ are isthmuses.

**Theorem 4.1.** Let $G$ be the simplification of a graph $G_n$ which is of Type-i with $E_n = \{e_1, e_2, \cdots, e_n\}$ being the set of all parallel edges. If the edge $e_j$ is parallel to $j_p$ edges in $G_n$ for any $j \in \{1, 2, \cdots, n\}$, then the Tutte polynomial of $G_n$ is

$$T(G_n; x, y) = \prod_{j=1}^{n}(x + (\sum_{i=1}^{j_p} y^i))T(G/E_n; x, y).$$

Proof. By applying Lemma 4.1 on one parallel edge at a time, starting with $e_n$ and repeating the process for $e_{n-1}, \cdots, e_1$ successively, we get

$$T(G_n; x, y) = \left(x + \sum_{i=1}^{n_p} y^i\right)T(G_{n-1}/e_n; x, y)$$

$$= \left(x + (\sum_{i=1}^{n_p} y^i)\right)\left(x + (\sum_{i=1}^{n-1_p} y^i)\right)T(G_{n-2}/e_1/e_2; x, y)$$

$$\vdots$$

$$= \left(x + (\sum_{i=1}^{n_p} y^i)\right)\cdots(x + (\sum_{i=1}^{1_p} y^i))T(G/e_1/e_2/\cdots/e_n; x, y)$$

$$= \prod_{j=1}^{n}(x + (\sum_{i=1}^{j_p} y^i))T(G/E_n; x, y).$$
Recall that if a graph $G_n$ is of Type-ii, then in the process of obtaining the minor $G'$, all the elements of $E_n = \{e_1, e_2, \cdots, e_n\}$ are deleted in $G$.

**Theorem 4.2.** Let $G_n$ be a graph of Type-ii whose simplification is $G$ and let $E_n = \{e_1, e_2, \cdots, e_n\}$ be the set of all parallel edges in $G_n$. If the edge $e_j$ is parallel to $j$ edges for any $j \in \{1, 2, \cdots, n\}$ and $S \subseteq E_n$ such that $S = \{e_1, e_2, \cdots, e_t\}$ where $t \leq n$ and the complement of $S$, $\overline{S} = \{e_{t+1}, e_{t+2}, \cdots, e_n\}$, then the Tutte polynomial of $G_n$ is

$$T(G_n; x, y) = (-1)^{|S|} \sum_{S \subseteq E_n} \prod_{t=1}^{t} \prod_{i=1}^{l_p} (\sum_{y^i})[\prod_{t=t+1}^{n} \prod_{i=0}^{l_p} (\sum_{y^i})]T(G\setminus S; x, y)$$

*Proof.* Note that when $S = \emptyset$, $\overline{S} = E_n$.

The proof is by induction on $n$ with the case when $n = 1$ being Lemma 4.2. When $n = 2$, $E_2$ has two parallel elements namely, $e_1$ and $e_2$. We apply repeatedly Lemma 4.2 on $e_2$ then on $e_1$ to get

$$T(G_2; x, y) = \left( \sum_{i=0}^{2p} y^i \right)T(G_1; x, y) - \left( \sum_{i=1}^{2p} y^i \right)T(G_1\setminus e_2; x, y)$$

$$= \left( \sum_{i=0}^{2p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G; x, y) - \left( \sum_{i=1}^{1p} y^i \right)T(G\setminus e_1; x, y)]$$

$$- \left( \sum_{i=1}^{2p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G\setminus e_2; x, y) - \left( \sum_{i=0}^{1p} y^i \right)T(G\setminus e_2\setminus e_1; x, y)]$$

$$= \left( \sum_{i=0}^{2p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G; x, y) - \left( \sum_{i=1}^{1p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G\setminus e_1; x, y)]$$

$$- \left( \sum_{i=1}^{2p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G\setminus e_2; x, y) - \left( \sum_{i=1}^{1p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G\setminus e_1\setminus e_2; x, y)$$

$$= \left( \sum_{i=0}^{2p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G; x, y) - \left( \sum_{i=1}^{1p} y^i \right)[\sum_{i=0}^{1p} y^i]T(G\setminus e_1; x, y)]$$

Now assume the result is true for some graph $G_q$ with $q \geq 1$ parallel elements. Consider a graph $G_{q+1} = G_q \cup e_{q+1}$, where $e_{q+1}$ is a parallel element. Thus, the simplification of $G_q$ is equal to the simplification of $G_{q+1}$ which is $G$. If $S$ is a subset of $E_q$ then $S$ is also a subset of $E_{q+1}$ since $E_q \subseteq E_{q+1}$, by definition.

Consider any subset $S'$ of $E_{q+1}$. Note that there are two types of subsets of $E_{q+1}$ : those that contain the element $e_{q+1}$ and those that do not. By applying Lemma 4.2 on the parallel element $e_{q+1}$, we get

$$T(G_{q+1}; x, y) = \left( \sum_{i=0}^{(q+1)p} y^i \right)T(G_q; x, y) - \left( \sum_{i=1}^{(q+1)p} y^i \right)T(G_q\setminus e_{q+1}; x, y).$$

(2)
Since $e_{q+1}$ is not in $G_q$, it is not an element of $E_q$. Thus, $e_{q+1}$ is not in any $S \subseteq E_q$. Therefore, $e_{q+1}$ is in all the minors of $G_q \setminus S$ when computing $T(G_q; x, y)$. Furthermore, if we consider the set $E_{q+1}$ and any subset $S'$ which do not contain $e_{q+1}$, it is clear that $S' = S \subseteq E_q \subseteq E_{q+1}$. Hence, we obtain that

$$
(q+1)_p \sum_{i=0}^{(q+1)_p} y^i T(G_q; x, y) = (-1)^{|S|} \prod_{S \subseteq E_q} \sum_{i=0}^{(q+1)_p} y^i \prod_{i=0}^{l_p} (\sum_{i=0}^{q} y^i) T(G_q \setminus S; x, y)
$$

On the other hand, when $e_{q+1}$ is deleted in $G_q$, it means $e_{q+1}$ is not in any minor of the form $G \setminus e_{q+1} \setminus S$ in the computation of $T(G_q \setminus e_{q+1}; x, y)$. Thus, if we consider the set $E_{q+1}$ and any subset $S''$ containing $e_{q+1}$, then it is clear that $S'' = S' \cup e_{q+1} \subseteq E_q \cup e_{q+1} = E_{q+1}$. Moreover, if $S = \{e_1, e_2, \ldots, e_t\}$ then $S'' = \{e_1, e_2, \ldots, e_t, e_{t+1}\}$ where $e_{t+1} = e_{t+1}$ and $t \leq q$. Furthermore, the complement of $S''$, $\overline{S''} = \{e_{t+2}, e_{t+3}, \ldots, e_n\}$. Hence, we can rewrite

$$
-(q+1)_p \sum_{i=1}^{(q+1)_p} y^i T(G_q \setminus e_{q+1}; x, y)
$$

$$
= (-1)(-1)^{|S|} \prod_{S \subseteq E_q} \sum_{i=1}^{q + 1} y^i \prod_{i=1}^{l_p} (\sum_{i=1}^{q} y^i) T(G_q \setminus e_{q+1}; S; x, y)
$$

$$
= (-1)^{|S''|} \sum_{S'' \subseteq E_q \cup e_{q+1}} \prod_{i=1}^{t + 1} y^i \prod_{i=1}^{q + 1} y^i \prod_{i=0}^{l_p} \prod_{i=0}^{q + 1} y^i T(G \setminus S' \cup e_{q+1}; x, y)
$$

Now considering the fact that all subsets of $E_{q+1}$ are either the form $S'$ or $S''$ and returning back
to Equation 2, we get

\[
T(G_{q+1}; x, y) = (\sum_{i=0}^{(q+1)p} y^i)T(G_q; x, y) - (\sum_{i=1}^{(q+1)p} y^i)T(G_q \setminus e_{q+1}; x, y)
\]

\[
= (-1)^{|S'|} \sum_{S' \subseteq E_{q+1}} \left( \prod_{t=1}^{t+1} (\sum_{i=1}^{l_p} y^i) \right) \left( \prod_{t=1+1}^{t+1} (\sum_{i=0}^{l_p} y^i) \right) T(G \setminus S'; x, y)
\]

\[
+ (-1)^{|S''|} \sum_{S'' \subseteq E_{q+1}} \left( \prod_{t=1}^{t+1} (\sum_{i=1}^{l_p} y^i) \right) \left( \prod_{t=1+1}^{t+1} (\sum_{i=0}^{l_p} y^i) \right) T(G \setminus S''; x, y).
\]

\[
= (-1)^{|S|} \sum_{S \subseteq E_{q+1}} \left( \prod_{t=1}^{t+1} (\sum_{i=1}^{l_p} y^i) \right) \left( \prod_{t=1+1}^{t+1} (\sum_{i=0}^{l_p} y^i) \right) T(G \setminus S; x, y).
\]

Recall that if a graph \( G_n \) is of Type-iii, then some edges of \{e_1, e_2, \ldots, e_n\} are not in the minor \( G' \) and those that are present in \( G' \) are isthmuses in \( G' \).

The proof for the next theorem follows directly from Theorem 4.1 and Theorem 4.2.

**Theorem 4.3.** Let \( G_n \) be a graph of Type-iii, such that in \( G_n \) the edge \( e_j \) is parallel to \( j_p \) edges for any \( j \in \{1, 2, \ldots, n\} \). Suppose \( E_r = \{e_1, e_2, \ldots, e_r\} \) is a maximal set of edges which are not isthmuses in \( G_n \) for some \( r < n \), such that \( E_r = \{e_{r+1}, e_{r+2}, \ldots, e_n\} \) is a set of isthmuses in the minor \( G \setminus E_r \). If \( S \subseteq E_r \), then the Tutte polynomial of \( G_n \) is

\[
T(G_n; x, y) = (-1)^{|S|} \sum_{S \subseteq E_r} (-1)^{|S|} \left( \prod_{t=1}^{t+1} (\sum_{i=1}^{l_p} y^i) \right) \left( \prod_{t=1+1}^{t+1} (\sum_{i=0}^{l_p} y^i) \right) T(G \setminus S; x, y)
\]

\[
+ (-1)^{|S|} \left( \prod_{t=1}^{r+1} (\sum_{i=1}^{l_p} y^i) \right) \left( \prod_{t=r+1}^{r+1} (x + \sum_{i=0}^{l_p} y^i) \right) T(G \setminus E_r / E_r; x, y).
\]

5. Examples

We conclude this paper with some examples that illustrate the recurrence formulas presented in the previous sections, for each type of parallel class.

**Example 5.1.** Let a multigraph \( G_2 \) be defined by the vertex set \( V(G_1) = \{1, 2, 3, 4, 5, 6\} \) and the edge set

\[ E(G_2) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{4, 5\}, \{5, 4\}, \{3, 6\}, \{6, 3\}\}. \]

If we let \( e_1 = \{4, 5\} \) and \( e_2 = \{3, 6\} \), then \( E_2 = \{\{4, 5\}, \{3, 6\}\} \) and \( l_p = 1 \) and \( 2_p = 1 \). The simplification of \( G_2 \) is the graph \( G \) defined by the vertex set \( V(G) = \{1, 2, 3, 4, 5, 6\} \) and the edge set

\[ E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{4, 5\}, \{3, 6\}\}. \]
The minor \( G' \) obtained after applying algorithm DEL is defined by the vertex set \( V(G') = \{1, 2, 3, 4, 5, 6\} \) and the edge set

\[
E(G') = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{4, 5\}, \{3, 6\}\}.
\]

\( G' \) is a minor of Type-i, that is all the edges of \( E_2, e_1 \) and \( e_2 \) are isthmuses in \( G' \). Thus \( G_2 \) is a graph of Type-i. To find the Tutte polynomial of \( G_2 \) we apply Theorem 4.1 where \( E_2 = \{e_1, e_2\} \) and \( G/E_2 \cong C_4 \), a cycle on 4 vertices.

\[
T(G_2; x, y) = \prod_{j=1}^{2} (x + \sum_{i=1}^{j_p} y^i))T(G/S; x, y)
\]

\[
= (x + y)(x + y)T(G/e_1/e_2; x, y)
\]

\[
= (x^2 + 2xy + y^2)T(C_4; x, y)
\]

\[
= (x^2 + 2xy + y^2)(y + x + x^2 + x^3)
\]

\[
= x^3 + x^4 + x^5 + y^3 + 3yx^2 + 3y^2x + 2yx^3 + 2yx^4 + y^2x^2 + y^2x^3.
\]

**Example 5.2.** Let a multigraph \( H_2 \) be defined by the vertex set \( V(H_2) = \{1, 2, 3, 4\} \) and the edge set

\[
E(H_2) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}, \{3, 2\}, \{1, 4\}\}.
\]

If we let \( e_1 = \{4, 1\} \) and \( e_2 = \{2, 3\} \) then \( E_2 = \{\{4, 1\}, \{2, 3\}\} \) and \( 1_p = 1 \) and \( 2_p = 1 \). The simplification of \( H_2 \) is a graph \( H \) defined by the vertex set \( V(H) = \{1, 2, 3, 4\} \) and the edge set

\[
E(H) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}\}.
\]

The minor \( H' \) obtained after applying algorithm DEL is defined by the vertex set \( V(H') = \{1, 2, 3, 4\} \) and the edge set \( E(H') = \{\{1, 2\}, \{1, 3\}, \{3, 4\}\} \). \( H' \) is a minor of Type-ii as the edges of \( E_2, e_1 \) and \( e_2 \), are not in \( H' \). Hence \( H_2 \) is a graph of Type-ii. To find the Tutte polynomial of \( H_2 \), we apply Theorem 4.2 where \( S \subseteq E_2 = \{\{4, 5\}, \{3, 6\}\} \). If \( S = \emptyset \) then \( H \setminus S = G \) and also if \( S = \{e_1\} \) then \( G \setminus S \) is isomorphic to \( C_3 \cup e \) where \( e \) is an isthmus. Similarly, if \( S = \{e_2\} \) then \( G \setminus S \) is isomorphic to \( C_3 \cup e \) where \( e \) is an isthmus. Finally, if \( S = \{e_1, e_2\} \) this implies
If we let \( e = \{e_1, e_2\} \), and \( H\setminus S \cong P_4 \), a path on 4 vertices.

\[
T(H_2; x, y) = (-1)^{|S|} \sum_{S \subseteq \mathcal{E}_2} \left( \prod_{t=1}^{l_p} \left( \sum_{i=1}^{l_p} y^i \right) \right) \prod_{l=t+1}^{2} \left( \sum_{i=0}^{l_p} y^i \right) T(H \setminus S; x, y)
\]

\[
= (-1)^{|S|} \prod_{l=1}^{2} \left( \sum_{i=0}^{l_p} y^i \right) T(H \setminus \emptyset; x, y)
\]

\[
+ (-1)^{|\{e_1\}|} \prod_{l=1}^{1} \left( \sum_{i=1}^{l_p} y^i \right) \prod_{l=t+1}^{2} \left( \sum_{i=0}^{l_p} y^i \right) T(H \setminus \{e_1\}; x, y)
\]

\[
+ (-1)^{|\{e_2\}|} \prod_{l=1}^{1} \left( \sum_{i=1}^{l_p} y^i \right) \prod_{l=t+1}^{2} \left( \sum_{i=0}^{l_p} y^i \right) T(H \setminus \{e_2\}; x, y)
\]

\[
+ (-1)^{|\{e_1,e_2\}|} \prod_{l=1}^{2} \left( \sum_{i=0}^{l_p} y^i \right) T(H \setminus \{e_1, e_2\}; x, y)
\]

\[
= [(1 + y)(1 + y)]T(H \setminus \emptyset; x, y) - [(1 + y)y]T(H \setminus \{e_1\}; x, y)
\]

\[
- [(1 + y)y]T(H \setminus \{e_2\}; x, y) + [y^2]T(H \setminus \{e_1\}; x, y)
\]

\[
= [(1 + 2y + y^2)T(H; x, y) - 2y(1 + y)xT(C_3; x, y)
\]

\[
+ y^2T(P_4; x, y)
\]

\[
= [1 + 2y + y^2][y + x + 2x^2 + x^3 + 2yx + y^2]
\]

\[
- [2y^2 + 2y][y + x + x^2] + y^2x^3
\]

\[
= x + 2x^2 + x^3 + y + y^2 + 3y^3 + y^4 + 2xy + 3y^2 x.
\]

**Example 5.3.** Let \( N_2 \) be a multigraph defined by the vertex set \( V(N_2) = \{1, 2, 3, 4\} \) and the edge set

\[
E(N_2) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{2, 1\}, \{4, 3\}\}.
\]

If we let \( e_1 = \{1, 2\} \) and \( e_2 = \{3, 4\} \) then \( E_2 = \{e_1, e_2\} \) and \( l_p = 1 \) and \( 2_p = 1 \). The simplification of \( N_2 \) is a graph \( N \) defined by the vertex set \( V(N) = \{1, 2, 3, 4\} \) and the edge set \( E(N) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\} \). The minor \( N' \) obtained after applying the algorithm DEL is a graph defined by the vertex set \( V(N') = \{1, 2, 3, 4\} \) and the edge set \( E(N') = \{\{1, 4\}, \{3, 4\}, \{2, 3\}\} \). \( N' \) is a minor of **Type-iii**, that is \( e_1 \) is not in \( N' \) and \( e_2 \) is an isthmus in \( N' \). Thus, \( N_2 \) is a graph of **Type-iii**. To find the Tutte polynomial of \( N_2 \) we apply Theorem 4.3 where \( S \subseteq E_1 = \{e_1\} \). Therefore if \( S = \emptyset \) then \( N/\emptyset = N \) and if \( S = \{e_1\} \) this implies that \( N\setminus S \) is isomorphic to \( P_4 \). Note that \( N\setminus E_1/E_1 \cong P_3 \).
\[ T(N_2; x, y) = (-1)^{|S|} \sum_{S \subseteq E_r} (-1)^{|S|} \left( \prod_{l=1}^{r} \left( \sum_{i=1}^{l_p} y^i \right) \right) \left( \prod_{l=1}^{t} \left( \sum_{i=0}^{l_p} y^i \right) \right) T(N \setminus S; x, y) \]

\[ + \ (-1)^{|S|} \sum_{S \subseteq E_1} \left( \prod_{l=1}^{t} \left( \sum_{i=0}^{l_p} y^i \right) \right) \left( \prod_{l=1}^{r} \left( \sum_{i=1}^{l_p} y^i \right) \right) T(M \setminus E_r/E_r; x, y) \]

\[ = (-1)^{|S|} \sum_{S \subseteq E_1} \left[ \prod_{l=1}^{t} \left( \sum_{i=1}^{l_p} y^i \right) \right] \left[ \prod_{l=1}^{r+1} \left( \sum_{i=0}^{l_p} y^i \right) \right] T(N \setminus S; x, y) \]

\[ + \ (-1)^{|S|} \sum_{S \subseteq E_1} \left[ \prod_{l=1}^{t} \left( \sum_{i=1}^{l_p} y^i \right) \right] \left[ \prod_{l=1}^{r+1} \left( \sum_{i=1}^{l_p} y^i \right) \right] T(N \setminus E_1/E_1; x, y) \]

\[ = [(1 + y)(1 + y)]T(N \setminus \emptyset; x, y) - [(1 + y)y]T(N \setminus \{e_1\}; x, y) \]

\[ - y[x + y]T(N \setminus \{e_1/e_2\}; x, y) \]

\[ = [1 + 2y + y^2]T(C_4; x, y) - y(1 + y)x T(P_4; x, y) \]

\[ + \ [y(y + x)] T(P_3; x, y) \]

\[ = [1 + 2y + y^2][y + x + x^2 + x^3] - y(1 + y)x^4 - [y^2 + yx]x^2 \]

\[ = x + x^2 + x^3 + y + 2y^2 + y^3 + 2yx + 2yx^2 + y^2 x. \]

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References


