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$\mathscr{F}$-WORM colorings of some 2-trees: partition vectors

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Abstract

Suppose $\mathscr{F} = \{F_1, \dots, F_t\}$ is a collection of distinct subgraphs of a graph $G = (V, E)$. An $\mathscr{F}$-*WORM* coloring of G is the coloring of its vertices such that no copy of each subgraph $F_i \in \mathscr{F}$ is monochrome or rainbow. This generalizes the notion of F-WORM coloring that was introduced recently by W. Goddard, K. Wash, and H. Xu. A (restricted) partition vector $(\zeta_\alpha, \dots, \zeta_\beta)$ is a sequence whose terms ζ_r are the number of $\mathscr{F}$-WORM colorings using exactly r colors, with $\alpha \le r \le \beta$. The partition vectors of complete graphs and those of some 2-trees are discussed. We show that, although 2-trees admit the same partition vector in classic proper vertex colorings which forbid monochrome K_2, their partition vectors in K_3-WORM colorings are different.

Keywords: 2*-tree, maximal outerplanar, partition, Stirling numbers.*

Math. Subj. Class.: 05C15, 05C10

1 Preliminaries

A *partition* σ of a set S is a set of nonempty subsets or *blocks* of S such that each element of S is in exactly one of the subsets of S. The number of blocks of σ is its *rank* and a partition of rank r is simply called an r-*partition*. For instance, the Stirling number of the second kind, $\left\{ {n \atop r} \right\}$ counts the number r-partitions of the set $[n] = \{1, 2, \dots, n\}$.

Consider the mapping $c\colon S \to [x]$ being an x-coloring of the elements of S. A subset $A \subseteq S$ is said to be *monochrome* if all of its elements share the same color and A is *rainbow* if all of its elements have different colors. As such, a coloring $c(S)$ is a partition of the set

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S since all of the elements of S are assigned a color; elements that share the same color belong to the same block (monochrome subsets), and different blocks are used for those with distinct colors (rainbow subsets).

Let $G = (V, E)$ denote a simple graph and suppose $\mathscr{F} = \{F_1, \ldots, F_t\}$ is a collection of some distinct subgraphs $F_i \subseteq G$, $1 \le i \le t$. An *$\mathscr{F}$-WORM* coloring of G is the coloring of the vertices of G such that no copy of each subgraph F_i is monochrome or rainbow. When $\mathscr{F}$ has only one member, say F, we write F-WORM coloring; this special case was first introduced by W. Goddard, K. Wash and H. Xu, and independently studied by Cs. Bujtás and Zs. Tuza [5, 7, 8, 12, 13]. We note that this coloring requirement makes sense only if each $F_i \in \mathscr{F}$ is of order three or more. However, for a generalization purpose if some F_j is of order 2, we allow only rainbow copies of F_j in order to meet the classic proper (vertex) coloring requirement. Suppose $\mathscr{H} = (\mathcal{V}, \mathcal{E})$ is a hypergraph. If $|e| = s$ for each hyperedge $e \in \mathcal{E}$, then $\mathscr{H}$ is said to be *s-uniform*. Given any vertex coloring of $\mathscr{H}$, if no $e \in \mathcal{E}$ is monochrome, $\mathscr{H}$ is called a *$\mathcal{D}$-hypergraph* or simply a hypergraph. When no $e \in \mathcal{E}$ is rainbow, $\mathscr{H}$ is called a *cohypergraph*. In the event no $e \in \mathcal{E}$ is monochrome or rainbow, then $\mathscr{H}$ is called a *bihypergraph*. Moreover, if G is a hypergraph and each subgraph $F_i = E_r$, the null graph on r-vertices, then an $\mathscr{F}$-WORM coloring of G is a proper (vertex) coloring of an r-uniform bihypergraph; Cs. Bujtás and Zs. Tuza [7, 8] also noted this strong relation between $\mathscr{F}$-WORM coloring and mixed hypergraph colorings, a theory that was first introduced by the second author [25, 26]. Thus, the notion of $\mathscr{F}$-WORM colorings generalizes several well known coloring constraints. Given an $\mathscr{F}$-WORM coloring, the sequence $(\zeta_\alpha, \ldots, \zeta_\beta)$ whose terms, ζ_r, are the number of r-partitions is called a *(restricted) partition vector*, with $\alpha \le r \le \beta$. In general, partition vectors have some added benefits in the study of log-concave and unimodal sequences which often arise in algebra, combinatorics, computer science, even in probability and statistics (see for e.g., [2, 4, 11]). A sequence of non-negative terms $(a_0, \ldots, a_n)$ is called *log-concave* if $a_i^2 \ge a_{i-1}a_{i+1}$ for $i = 1, \ldots, n-1$. Such sequence is also said to be *unimodal* if it has no gap (i.e., there is no i with $a_{i-1} \ne 0$, $a_i = 0$ and $a_{i+1} \ne 0$) and there is an index $0 \le j \le n$ such that $a_0 \le \ldots \le a_j \ge \ldots \ge a_n$. Further, partition vectors are closely related to colorings; each ζ_r gives the number of $\mathscr{F}$-WORM colorings using exactly r colors, in which case α and β are the *lower* and *upper chromatic numbers*, respectively. In [7], it is shown that it is NP-hard to determine α and it is NP-complete to decide whether or not a graph G admits a K_3-WORM coloring using $k \ge 2$ colors. Moreover, the integer set $S = \{x : \alpha \le x \le \beta\}$ commonly known as *feasible set*, has been the subject of numerous research publications (see e.g., [6, 15, 18, 27]). We note that the term chromatic spectrum has also been used for feasible set in some of the aforementioned literatures. Further, we call the rank-generating function

$$\sigma(G|_{\mathscr{F}}; x) = \sum_{k=\alpha}^{\beta} \zeta_k x^k$$

the *restricted partition polynomial* of G subject to an $\mathscr{F}$-WORM coloring. Note that, when x^i is replaced by the falling factorial power $x^{\underline{i}} = x(x-1)(x-2)\cdots(x-i+1)$, the polynomial

$$\sigma(G|_{\mathscr{F}}; x) = \sum_{k=\alpha}^{\beta} \zeta_k x^{\underline{k}}$$

counts all $\mathscr{F}$-WORM colorings using at most x colors. Some variants of restricted partition polynomials have been well studied. For instance when $G = E_n$, $\sigma(G|_{\emptyset}; x)$ is the

Bell polynomial which is a widely studied tool in combinatorial analysis [9, 24]. Also, $\sigma(G|_{K_2}; x)$ has been recently called Stirling polynomial [11] although it was first introduced by Korfhage as σ-polynomial [17]. In particular, when written in the falling factorial power of x, $\sigma(G|_{K_2}; x) = \chi(G; x)$ is the well known chromatic polynomial [3, 22]. Thus, a restricted partition polynomial extends both the chromatic polynomial and the Stirling polynomial of graphs. In this paper, in Section 2, we determine the partition vectors of some mixed hypergraphs. Later, in Section 3, we investigate K_3-WORM colorings of some 2-trees. We find that, while 2-trees admit the same partition vector given any (classic) proper vertex coloring, it is not true for their K_3-WORM colorings. To support this argument, we present two non-isomorphic members of 2-trees which have different partition vectors. In Section 4, we conclude this paper with $\mathscr{F}$-WORM colorings when $\mathscr{F}$ includes a family of 2 or more graphs such as Path, Star or Cycle.

2 Coloring K_n with forbidden monochrome or rainbow subgraphs

We begin by establishing a connection between K_s-WORM coloring of a complete graph K_n and mixed hypergraph colorings.

Theorem 2.1. *The partition vector in a K_s-WORM coloring of K_n is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $3 \le s < n$.*

Proof. A partition of the vertices of K_n into r blocks, i.e., $\left\{ {n \atop r} \right\}$, guarantees that no subset of $V(K_n)$ with size $r \le s-1$ is rainbow. Moreover, to forbid a monochrome K_s, it suffices to ensure that no subset contains s or more elements, given each r-partition. This implies that $n \le r(s-1)$. Hence, $\lfloor \frac{n}{s-1} \rfloor \le r \le s-1$, giving the result. □

An s-uniform hypergraph whose hyperedges are all the subsets of size $s \ge 3$ of its vertex set is called an *s-uniform complete hypergraph*.

Corollary 2.2. *The partition vector of an s-uniform complete bihypergraph is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $s \ge 3$.*

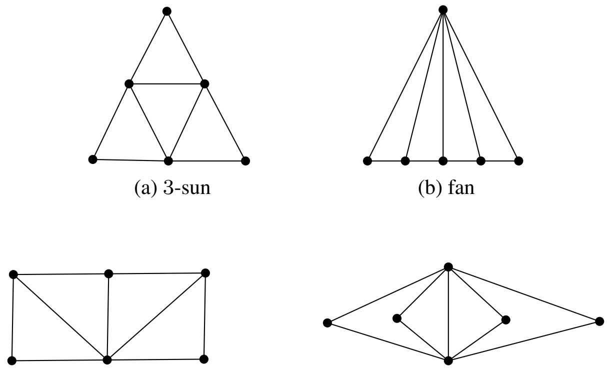
Removing either restriction on r gives each of the next result.

Corollary 2.3. *The partition vector of an s-uniform complete hypergraph is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_s)$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $s \ge 2$.*

Corollary 2.4. *The partition vector of an s-uniform complete cohypergraph is $(\zeta_1, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $s \ge 3$.*

3 Partition vectors of some 2-trees

As a generalization of a tree, a *k-tree* on n vertices (with $1 \le k \le n$) is a graph which arises from a K_k by adding $n - k \ge 1$ new vertices, each joined to a K_k in the old graph; this process generates several non-isomorphic k-trees, $k \ge 1$. Figure 1 depicts four non-isomorphic 2-trees on 6 vertices. K-trees are chordal graphs which are known to admit at least one simplicial elimination ordering ([10]). Recall, a graph is *chordal* if it does not contain an induced cycle of length 4 or more. The characterization of families of graphs by forbidden subgraphs is an old tradition in graph theory and k-trees, despite



(a) 3-sun (b) fan (c) snake (d) $\theta(1,2,2,2,2)$

Figure 1: Some non-isomorphic 2-trees.

being ubiquitous, have yet to be fully classified even in the case when $k = 1$. Adding some additional restrictions on the coloring of certain subgraphs besides K_2 and E_n may help in the analysis of the structure of the graphs that contain them. To help support this claim, we begin with the partition vectors in the coloring of 2-trees when monochrome K_2 are forbidden. These vectors do not characterize any member of k-trees, since non-isomorphic k-trees do share the same partition vector as shown, later, in Corollary 3.2.

Proposition 3.1. *The equality*

$$x^{\underline{k}}(x-k)^{n-k} = \sum_{t=k+1}^{n} \left\{ {n-k \atop t-k} \right\} x^{\underline{t}}$$

holds for all $1 \leq k \leq n$.

Proof. Since $x^n = \sum_{t=1}^{n} \left\{ {n \atop t} \right\} x^{\underline{t}}$, this implies that $(x-k)^{n-k} = \sum_{t=1}^{n-k} \left\{ {n-k \atop t} \right\} (x-k)^{\underline{t}}$ and

$$\begin{aligned} x^{\underline{k}}(x-k)^{n-k} &= \sum_{t=1}^{n-k} \left\{ {n-k \atop t} \right\} x(x-1)\cdots(x-k-1)(x-k)^{\underline{t}} \\ &= \sum_{t=1}^{n-k} \left\{ {n-k \atop t} \right\} x^{\underline{t+k}} \\ &= \sum_{t=k+1}^{n} \left\{ {n-k \atop t-k} \right\} x^{\underline{t}}, \end{aligned}$$

giving the result. □

Corollary 3.2. *The partition vector of any k-tree on $n-k$ simplicial vertices such that no K_2 is monochrome is $(\zeta_{k+1}, \ldots, \zeta_n)$, where $\zeta_r = \left\{ {n-k \atop r-k} \right\}$.*

Proof. It is easy to see that the left side of the equality of the formula in Proposition 3.1 is that of the chromatic polynomial of any k-tree, $k \geq 1$. The result follows from the right side of that equality. □

We note the quantity $\left\{ {n-k \atop r-k} \right\} = \left\{ {n \atop r} \right\}^{(k)}$ is known for counting the number of k-nonconsecutive r-partitions of n elements (see e.g., [16]); a partition of the set $\{1, \ldots, n\}$ is said to be $k-$*nonconsecutive* whenever x and y are in the same block, $|x-y| \geq k$.

Recall that a graph is called *outerplanar* if it can be embedded in the plane in such a way that every vertex lies on the outer cycle. A *maximal outerplanar (MOP)* graph is an outerplanar graph with a maximum number of edges [21]. Graphs such as 3-sun, fan and snake are some well known MOPs; these graphs are depicted in Figures 1(a), 1(b) and 1(c), respectively. Laskar and Mulder [19, 20] characterized MOPs as the intersection of any two of the following graphs: chordal, path-neighborhood, and triangle graphs $T(G)$ which are trees. Recall, a *path-neighborhood graph* is a graph in which every vertex neighborhood induces a path and the *triangle graph* $T(G)$ of G has the triangles of G as its vertices, and two vertices of $T(G)$ are adjacent whenever their corresponding triangles in G share an edge [1]. Simply put, MOPs are members of 2-trees. Here, by considering K_3-WORM colorings of 2-trees, we have found that their partition vectors are uniquely determined and the process reveals that MOPs are 2-trees with the characteristic that every edge is shared by at most two triangles.

Theorem 3.3. *Suppose G is a 2-tree such that its triangle graph is a path. Then the number of colorings of its vertices such that no triangle is monochrome or rainbow is*

$$P(G) = \sum_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor} \phi(n-1, j) x(x-1)^j,$$

where

$$\phi(n-1, j) = \begin{cases} a_{n-1,j} + a_{n-1, \lfloor \frac{n-1}{2} \rfloor + j} & \text{if } j < \frac{n}{2} \\ a_{n-1, \frac{n}{2}} & \text{otherwise} \end{cases}$$

and the values $a_{i,j}$'s satisfying,

(i)

$$a_{1,1} = 1, a_{i,1} = 2 \text{ for each } 2 \leq i \leq n-1$$

and, for each $k = 1, \ldots, \lfloor \frac{i}{2} \rfloor$,

(ii)

$$a_{2k,j} = \begin{cases} a_{2k-1,j} + a_{2k-1,j+k-1} & \text{if } 2 \leq j \leq k \\ a_{2k-1,j-k} & \text{if } k+1 \leq j \leq i \end{cases}$$

(iii)

$$a_{2k+1,j} = \begin{cases} a_{2k,j} + a_{2k,j+k-1} & \text{if } 2 \leq j \leq k \\ 1 & \text{if } j = k+1 \\ a_{2k,j-k-1} & \text{if } k+2 \leq j \leq i. \end{cases}$$

Proof. Suppose $G = (V, E)$ is a 2-tree on $n \geq 3$ whose triangle graph is a path. Then, there exists a simplicial elimination ordering $\pi = \{u_1, \ldots, u_n\}$, such that u_i is adjacent to the edge with endpoints (u_{i-1}, u_{i-2}). Let $G_1 := u_1$, $G_2 := u_1u_2$, and $G_i := G_{i-1} \cup \{u_i\}$ where u_i is adjacent to the pair (u_{i-1}, u_{i-2}) in G_i for all $i \geq 3$. Suppose c is any coloring of G and denote by $P(G)$ the restricted number of colorings of G. For $n = 3$, we count the colorings when u_1u_2 is rainbow and when u_1u_2 is monochrome, separately. If we denote

$A_1 = x(x-1)$ and $B_1 = x$ then $P(G_3) = A_1+(x-1)B_1+A_1$. Set $A_2 := A_1+(x-1)B_1$ and $B_2 := A_1$ and clearly A_2 and B_2 count the number of colorings where $c(u_3) \neq c(u_2)$ and $c(u_3) = c(u_2)$, respectively. For all $n \geq 3$, at each iteration, we separate the terms that count $c(u_i) \neq c(u_{i-1})$ from those that count $c(u_i) = c(u_{i-1})$, giving the recursion $P(G_n) = A_{n-1} + B_{n-1}$, where $A_{n-1} := A_{n-2} + (x-1)B_{n-2}$ and $A_{n-1} := A_{n-2}$. Now use A_1 and B_1 as basis for the previous recursion and record at each iterative step the coefficients $a_{i,j}$'s of each expression $(x-1)^k$, for $1 \leq i, j \leq n-1$. By letting $a_{1,1} = 1$, it is easy to verify that the coefficients $a_{i,j}$'s satisfy the conditions (i) – (iii). For instance, when $n = 3$, $a_{2,1} = 2$, $a_{2,2} = a_{1,1} = 1$. Now, define an $(n-1) \times (n-1)$ matrix A whose entries are the coefficients $a_{i,j}$'s of $P(G_{i+1})$ with $P(G_2) = x(x-1)$. It follows that $P := xA \cdot Q$, where

$$P = \begin{bmatrix} P(G_2) \\ P(G_3) \\ \vdots \\ P(G_n) \end{bmatrix}, \qquad A = \begin{bmatrix} a_{1,1} & & & \\ a_{2,1} & a_{2,2} & & \\ \vdots & \vdots & \ddots & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix},$$

and

$$Q = \left[Q^1 \mid Q^2\right]^T,$$

with

$$Q^1 = \left[(x-1)^1 \quad \cdots \quad (x-1)^{\lceil \frac{n-1}{2} \rceil}\right] \text{ and}$$
$$Q^2 = \left[(x-1)^1 \quad \cdots \quad (x-1)^{\lfloor \frac{n-1}{2} \rfloor}\right].$$

Thus,

$$\begin{aligned} P(G) = P(G_n) &= \\ x\Big(\sum_{1 \leq k \leq \lceil \frac{n-1}{2} \rceil} a_{n-1,k}(x-1)^k &+ \sum_{1+\lceil \frac{n-1}{2} \rceil \leq k \leq n-1} a_{n-1,k}(x-1)^{k-\lceil \frac{n+1}{2} \rceil}\Big) = \\ &\sum_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor} \phi(n-1,j)x(x-1)^j, \end{aligned} \tag{3.1}$$

where

$$\phi(n-1,j) = \begin{cases} a_{n-1,j} + a_{n-1,\lfloor \frac{n-1}{2} \rfloor + j} & \text{if } j < \frac{n}{2} \\ a_{n-1,\frac{n}{2}} & \text{otherwise,} \end{cases}$$

giving the result. □

Remark 3.4. The argument in Theorem 3.3 requires that, at each iteration, no newly added vertex is joined to a previously used edge of a triangle. Also, this argument obviously applies to the case when one endpoint (but not both) of an edge is reused during the iteration process as in the case of a fan, for example. Moreover, because the recursive process is independent of the choice of the edge in the old graph, the result includes all such 2-trees which have the characteristic that each edge in the graph is shared by at most two triangles; this is a unique characteristic of MOPs, giving the next result.

Corollary 3.5. *If G is a MOP then its partition vector given any K_3-WORM coloring is $(\zeta_2, \ldots, \zeta_{\lfloor \frac{n}{2} \rfloor+1})$, where*

$$\zeta_r = \sum_{j=r-1}^{\lfloor \frac{n}{2} \rfloor} \phi(n-1, j) \left\{ {j \atop r-1} \right\},$$

with

$$\phi(n-1, j) = \begin{cases} a_{n-1,j} + a_{n-1,\lfloor \frac{n-1}{2} \rfloor + j} & \text{if } j < \frac{n}{2} \\ a_{n-1,\frac{n}{2}} & \text{otherwise,} \end{cases}$$

and the values $a_{i,j}$'s satisfying,

(i)

$$a_{1,1} = 1, a_{i,1} = 2 \text{ for each } 2 \leq i \leq n-1$$

and, for each $k = 1, \ldots, \lfloor \frac{i}{2} \rfloor$,

(ii)

$$a_{2k,j} = \begin{cases} a_{2k-1,j} + a_{2k-1,j+k-1} & \text{if } 2 \leq j \leq k \\ a_{2k-1,j-k} & \text{if } k+1 \leq j \leq i \end{cases}$$

(iii)

$$a_{2k+1,j} = \begin{cases} a_{2k,j} + a_{2k,j+k-1} & \text{if } 2 \leq j \leq k \\ 1 & \text{if } j = k+1 \\ a_{2k,j-k-1} & \text{if } k+2 \leq j \leq i. \end{cases}$$

Proof. Apply Proposition 3.1 (when $k = 1$) to the factors of the parameter ϕ in (3.1) and combine like expressions, to obtain the restricted partition polynomial

$$\sigma(G|_{K_3}) = \sum_{r=2}^{\lfloor \frac{n}{2} \rfloor+1} \Big(\sum_{j=r-1}^{\lfloor \frac{n}{2} \rfloor} \phi(n-1, j) \left\{ {j \atop r-1} \right\} x^{\underline{r}} \Big),$$

giving the result. □

Observations.

1. When $r = \lfloor \frac{n}{2} \rfloor + 1$, the upper partition number $\zeta_{\lfloor \frac{n}{2} \rfloor+1} = \phi(n-1, \lfloor \frac{n}{2} \rfloor)$, where

$$\phi(n-1, \lfloor \tfrac{n}{2} \rfloor) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 + \frac{n-1}{2} & \text{otherwise.} \end{cases}$$

   These values indicate that MOPs with even number of vertices admit a unique K_3-WORM coloring.

2. Also, it is worth noting that when $r = 2$, we have the lower partition number $\zeta_2 = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \phi(n, j) = \sum_j a_{i,j}$. Further, if we let $b_{i-1} = \sum_j a_{i,j}$ for all $i \geq 2$, the sequence $\{b_n\}$ satisfies the shifted Fibonacci recurrence given by $b_1 = 3$, $b_2 = 5$ and $b_n = b_{n-1} + b_{n-2}$, for $n \geq 3$.

3. If each triangle of G is replaced by a hyperedge (of size 3), the previous result also gives the partition vector of several nonlinear 3-uniform acyclic bihypergraphs, which include the complete 3-uniform interval bihypergraphs [26]; 3-uniform bihypergraphs often appear in communication models for cyber security [14].

Obviously, there are other members of 2-trees who have 3 or more triangles sharing the same edge as subgraphs. Here, we present the other extremal case of 2-trees when all triangles share a single edge, say u_1u_2. This 2-tree, often denoted by $\theta(1,2,\ldots,2)$, is a member of the well known n-bridge graphs. See Figure 1(d) for an example of a 5-bridge. Note that $\theta(1,2,\ldots,2)$ is a maximal planar graph but not a MOP, for all $n \geq 5$.

Corollary 3.6. *Suppose $G = \theta(1,2,\ldots,2)$, an $(n-1)$-bridge graph on $n \geq 3$ vertices. The partition vector of a K_3-WORM coloring of G is $(\zeta_2,\ldots,\zeta_{n-1})$ where*

$$\zeta_r = \begin{cases} 2^{n-2}+1 & \text{if } r = 2 \\ \left\{ {n-2 \atop r-1} \right\} & \text{otherwise.} \end{cases}$$

Proof. Count the number of colorings when the shared edge u_1u_2 is monochrome and when it is rainbow, giving $x(x-1)^{n-2} + 2^{n-2}x(x-1)$ colorings. Now apply Proposition 3.1 (when $k = 1$) to the terms of the expression to obtain the result. ☐

We leave it to the reader to verify that the previous values in the partition vector when $G = \theta(1,2,\ldots,2)$ are different from those of MOPs, for all $n \geq 5$.

4 Conclusion

We've shown that while 2-trees admit the same partition vector given any proper vertex coloring, it is not the case with their K_3-WORM colorings. We hope these results indicate the importance of WORM colorings in general in the analysis of the structures of some well-known graphs which could not be classified with the usual proper vertex colorings. For a potential future research, we introduce some generalizations of $\mathscr{F}$-WORM colorings when $\mathscr{F}$ includes multiple graphs such as Path, Star or Cycle. In the next results, C_n, $K_{1,n-1}$, and P_n^* denote an n-cycle, an n-star, and an n-path that includes a fixed vertex (apex) u_1, respectively.

Corollary 4.1. *Suppose G is a fan on $n \geq 4$ vertices. If G has a K_3-WORM coloring then G admits an $\mathscr{F}$-WORM coloring with $\mathscr{F} = \{P_s^*, K_{1,t}, C_r, \theta(1,n_1,n_2)\}$ where $s \geq 4$, $\lfloor \frac{n-1}{2} \rfloor \leq t \leq n-1$, $r \geq 3$, and $2 \leq n_1 \leq n_2$ such that $n_1 + n_2 \leq n$.*

Proof. Suppose G is a fan on $n \geq 4$ vertices which we can construct as follow: start with a triangle, say (u_1,u_2,u_3), and iteratively add $n-3$ new vertices such that each additional vertex u_i is adjacent to the pair (u_1,u_{i-1}), for $i = 4,\ldots,n$. Assume G admits a K_3-WORM coloring.

(i) Observe that for $s \geq 4$, every path $P_s^* \subseteq G$ contains the subgraph $u_1u_iu_{i+1}$ for some i $(2 \leq i \leq n-2)$. If some 3-path (that includes u_1) is monochrome/rainbow then the triangle (u_1,u_i,u_{i+1}) is monochrome/rainbow, violating the K_3-WORM coloring assumption. Hence G admits a P_s^*-WORM coloring for all $s \geq 4$.

(ii) By letting the vertices of $K_{1,t} \subseteq G$ be all the vertices of G, it follows that $t \leq n-1$. Now, consider the coloring such $c(u_1) = c(u_{2k})$ and $c(u_1) \neq c(u_{2k+1})$ for $k = 1, \ldots, \lceil \frac{n-1}{2} \rceil$. Clearly, such coloring does not violate our assumption of K_3-WORM coloring of G. Hence, the lower bound of t is satisfied by letting the vertices of $K_{1,t}$ be $\{u_1, u_2\} \cup \{u_{2k+1} : k = 1, \ldots, \lceil \frac{n-1}{2} \rceil\}$, which guarantees a $K_{1,t}$-WORM coloring for all $t \geq \lceil \frac{n-1}{2} \rceil$.

(iii) For $r \geq 4$, since every cycle $C_r \subseteq G$ includes the apex u_1, there exists an $s \leq r$ such that $P_s^* \subseteq C_r$, with $4 \leq s \leq r \leq n$. From (i), G admits a C_r-WORM coloring. The case when $r = 3$ is trivial.

(iv) Likewise, since $\theta(1, n_1, n_2)$ contains $C_{1+q} \subseteq G$ with $q \in \{n_1, n_2\}$, the result follows from (iii) that, for all $2 \leq n_1 \leq n_2$ such that $n_1 + n_2 \leq n$, G admits a $\theta(1, n_1, n_2)$-WORM coloring. □

Note that the converse of the statement in Corollary 4.1 is not true. Using a similar argument as in the previous proof establishes the next result; recall, a *snake* (see Figure 1(c)) is a 3-sun-free maximal outerplanar graph with at least four vertices.

Corollary 4.2. *Suppose G is a snake on $n \geq 4$ vertices. If G has a K_3-WORM coloring then G admits an $\mathscr{F}$-WORM coloring, where $\mathscr{F} = \{C_r, \theta(1,2,2)\}$ with $3 \leq r \leq n$.*

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