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LM threshold unit root tests

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ABSTRACT

We build on the threshold unit root tests in Enders and Granger (1998) and develop tests based on Lagrange Multiplier (LM) unit root tests. The asymptotic properties are derived and finite sample properties are examined in simulations.

1. INTRODUCTION

Conventional linear unit root tests assume a symmetric adjustment process under the stationary alternative. However, a growing body of research finds evidence of nonlinear or asymmetric adjustments in many economic time series. To address these issues in a pioneering work, Enders and Granger (1998, EG) develop Dickey–Fuller (DF) based threshold tests to test the null of a unit root against an asymmetric stationary alternative. Their first test is based on the threshold autoregressive (TAR) model developed by Tong (1983), where the autoregressive decay process depends on whether the *level* of the demeaned and detrended series is above or below the threshold level. EG additionally develop "momentum threshold autoregressive" (M-TAR) tests, where the speed of adjustment depends on whether the *change* in the demeaned and detrended series is above or below the threshold level. We build on the threshold models in EG and develop new threshold tests based on the Lagrange Multiplier (LM) unit root tests of Schmidt and Phillips, 1992 and Schmidt and Lee, 1991. In the linear framework, it is well known that LM unit root tests can be more powerful than DF unit root tests in many cases.[3] Overall, we find that similar greater power of the LM unit test carries over to the threshold framework.

2. LM THRESHOLD UNIT ROOT TESTS

Consider the data generating process (DGP) based on the unobserved components representation:

(1)

$$y_t = \delta' Z_t + e_t, e_t = \beta e_{t-1} + \varepsilon_t,$$

where Z_t contains deterministic terms. Following the LM (score) principle, the null restriction of $\beta = 1$ is imposed to give the regression in differences:

(2)

$$\Delta yt = \delta' \Delta Zt + ut.$$

Let $\tilde{S}_t$ be the demeaned and detrended series $\tilde{S}_t = y_t - \tilde{\psi} - Z_t \tilde{\delta}$, where $\tilde{\delta}$ is the coefficient estimated in the regression of Δy_t on ΔZ_t and $\tilde{\psi}$ is the restricted MLE ($\tilde{\psi} = y_1 - Z_1 \tilde{\delta}$). Following EG, an LM TAR model can be described by:

(3)

$$\Delta y_t = I_t \phi_1 \left[\tilde{S}_{t-1}\right] + (1 - I_t)\phi_2 \left[\tilde{S}_{t-1}\right] + \sum_{j=1}^{k} d_j \, \Delta \tilde{S}_{t-j} + e_t,$$

where I_t is the "Heaviside" indicator function:

(4)

$$I_t = 1 \text{ if } \tilde{S}_{t-1} \geq \eta \text{ and } I_t = 0 \text{ if } \tilde{S}_{t-1} < \eta$$

and η is the threshold value. An LM M-TAR (momentum) model can be similarly described by:

(5)

$$I_t = 1 \text{ if } \Delta\tilde{S}_{t-1} \geq \eta \text{ and } I_t = 0 \text{ if } \Delta\tilde{S}_{t-1} < \eta.$$

The null and alternative hypotheses can be described by:

(6)

H_0: $\phi_1 = \phi_2 = 0$ and Ha: $\phi_1 < 0$ and/or $\phi_2 < 0$.

Under the null hypothesis the DGP is symmetric and there is a unit root in both regimes, while under the alternative hypothesis there is a stationary process in at least one regime.

One may consider:

(7)

$$\begin{aligned} I_t = I\left(\tilde{S}_{t-1} \geq \eta\right) &= I\left(\sigma^{-1}\ T^{-1/2}\tilde{S}_{t-1} > \sigma^{-1}T^{-1/2}\eta\right) \\ &= I\left(\sigma^{-1}T^{-1/2}\tilde{S}_{t-1} > \eta^*\right), \end{aligned}$$

where $\eta^* = \sigma^{-1}\ T^{-1/2}\eta$ is the normalized threshold parameter and $\sigma^2 = T^{-1}E(\sum \tilde{S}_{t-1})^2$. Since η can potentially take on any value of $\tilde{S}_{t-1}$, providing critical values for different $\tilde{S}_{t-1}$ at each possible value of η can be impractical. To mitigate this, we transform the threshold variable into the percentile of the sorted threshold variable $\sigma^{-1}\ T^{-1/2}\tilde{S}_{t-1}^{*}$ defined over all values of $\tilde{S}_{t-1}$ (with trimming to eliminate endpoints). Let $\tilde{S}_{t-1}^{*}(\tau) = h^{*}$ be the τ-th percentile value of the empirical distribution of $\tilde{S}_{t-1}^{*}$, such that $P[\sigma^{-1}\ T^{-1/2}\tilde{S}_{t-1}^{*} \leq h^{*}] = P[\sigma^{-1}\ T^{-1/2}\tilde{S}_{t-1}^{*} \leq \tilde{S}_{t-1}^{*}(\tau)] = \tau$. Then, it follows that:

(8)

$$I_t = I\left(\sigma^{-1}\ \mathrm{T}^{-1/2}\,\tilde{S}_{t-1} > \tilde{S}^*_{t-1}(\tau)\right) \rightarrow I(W(r) > h^*) = I(W^*(r) > \tau),$$

where $W(r)$ is the usual Brownian motion defined on $r \in [0, 1]$, and $W^{*}(r)$ maps to $W(r)$ in such a way that the threshold value (h^{*}) matches the threshold percentile parameter (τ) and the value of the indicator function is maintained. A similar percentile procedure can be defined for the M-TAR model by replacing $\tilde{S}_{t-1}$ with $\Delta\tilde{S}_{t-1}$, where $\Delta\tilde{S}_{t-1}^{*}$ is defined as the sorted threshold variable of $\Delta\tilde{S}_{t-1}$ and $\Delta\tilde{S}_{t-1}^{*}(\tau) = h$ is the τ-th percentile of the empirical distribution of $\Delta\tilde{S}_{t-1}^{*}$. Then, it follows that:

(9)

$$I_t = I\left(\Delta\tilde{S}_{t-1} > h\right) = I\left(\sigma_1^{-1}\ \Delta\tilde{S}_{t-1} > \sigma_1^{-1}\Delta\tilde{S}^*_{t-1}(\tau)\right) \rightarrow I(U(r) > \tau),$$

where $U(r)$ is a process defined on [0, 1] and $\sigma_1^2 = T^{-1}E(\sum \Delta \tilde{\tilde{S}}_{t-1})^2$. Thus, the threshold parameter is transformed into a percentile parameter (τ) defined on the interval [0, 1] where the asymptotic distribution of the test statistic depends only on τ.

An F-statistic to test the null hypothesis $\phi_1 = \phi_2 = 0$ can be defined as follows:

(10)

$$F(\tau) = \hat{\rho}(\tau)' V(\hat{\rho}(\tau))^{-1} \hat{\rho}(\tau) / 2,$$

where $\hat{\rho}(\tau) = (\hat{\rho}_1(\tau), \hat{\rho}_2(\tau))'$ is the OLS estimator from regression (3) with variance $V(\hat{\rho}(\tau))$.[4] The coefficient estimate $\hat{\rho}(\tau)$ is obtained by controlling for the effects of the remaining deterministic terms and any significant augmented terms. Since I_t and $(1 - I_t)$ are orthogonal, $F(\tau)$ is the sum of the two quadratic forms:

(11)

$$F(\tau) = 0.5\left[\hat{\rho}_1(\tau)' V(\hat{\rho}_1(\tau))^{-1} \ \hat{\rho}_1(\tau) + \hat{\rho}_2(\tau)' V(\hat{\rho}_2(\tau))^{-1} \ \hat{\rho}_2(\tau)\right],$$

where $\hat{\rho}_i(\tau)$ and $V(\hat{\rho}_i(\tau))$, $i = 1,2$, are the corresponding OLS estimate and error variance, respectively. The asymptotic distribution is described as follows:

Theorem 1. *Let* $V(r) = W(r) - rW(1)$ *be a standard Brownian bridge that is the weak limit of the partial sum residual process* $T^{-1/2}\tilde{\tilde{S}}_{t-1}$, *and let* $\underline{V}(r) = V(r) - \int V(r)dr$ *be a demeaned Brownian Bridge where each term is defined on the interval* $r \in [0, 1]$. *Under the null hypothesis,* $\phi_1 = \phi_2 = 0$ *and* $F(\tau)$ *follows as* $T \to \infty$*:*

(12)

$$F(\tau) \to \frac{1}{8} \ \frac{\sigma_u^2}{\sigma_*^2} \sum_{k=1}^{2} \int (\underline{V}(r) \cdot I_k \cdot \underline{V}(r)')^{-1},$$

where $I_1 = I(W^{?}(r) > \tau)$ *or* $I(U(r) > \tau)$ *and* $I_2 = 1 - I_1$. σ_u^2 *is the usual error variance and* σ^{*2} *is the long-run variance;* $\sigma^{*2} = T^{-1}E(\sum S_t)^2$ *with* $S_t = u_1 + \ldots + u_t$.

Proof. The distributions of $\hat{\rho}(\tau)$ and $V(\hat{\rho}(\tau))$ are given as a function of the demeaned Brownian Bridge as demonstrated in (A.9) and (A.11) in Lee and Strazicich (2003). The distribution of $F(\tau)$ is accordingly obtained from these expressions while interacting with the indicator functions allowing for regime change.

In Table 1, we report critical values of the LM TAR and LM M-TAR F-statistics for the case where the (percentile) threshold parameter is known prior to testing. Critical values were obtained using the DGP in Eq. (1) under the unit root null ($\beta = 1$, implying $\phi_1 = \phi_2 = 0$) and calculated using 50,000 replications. Since the F-statistics are not invariant to the percentile parameter, we report critical values at different threshold parameters, $\tau = 0.1, 0.2, 0.3, 0.4$, and 0.5. The critical values for $\tau = 0.6, 0.7, .0.8$, and 0.9 are symmetric and other critical values can be interpolated.

Table 1
Critical values of the F-statistic in LM threshold unit root tests.

T	%	LM TAR with threshold variable $\tilde{S}_{t-1}$					LM M-TAR with threshold variable $\Delta\tilde{S}_{t-1}$				
		τ					τ				
		0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
50	10	3.701	3.654	3.584	3.508	3.512	3.562	3.610	3.624	3.645	3.696
	5	4.555	4.536	4.434	4.369	4.382	4.458	4.484	4.508	4.567	4.610
	1	6.602	6.538	6.308	6.386	6.484	6.503	6.602	6.582	6.589	6.791
100	10	3.768	3.639	3.585	3.512	3.512	3.541	3.547	3.610	3.591	3.618
	5	4.565	4.446	4.400	4.371	4.326	4.356	4.365	4.433	4.464	4.434
	1	6.483	6.243	6.291	6.312	6.158	6.336	6.318	6.386	6.461	6.330
250	10	3.798	3.670	3.561	3.517	3.493	3.520	3.533	3.515	3.547	3.549
	5	4.591	4.459	4.378	4.295	4.280	4.335	4.326	4.322	4.325	4.351
	1	6.395	6.195	6.218	6.113	6.096	6.115	6.133	6.159	6.143	6.260
1000	10	3.850	3.704	3.585	3.511	3.460	3.516	3.539	3.543	3.547	3.493
	5	4.671	4.488	4.378	4.304	4.271	4.334	4.343	4.338	4.344	4.302
	1	6.563	6.313	6.141	6.041	6.099	6.156	6.169	6.090	6.129	6.060

Note: τ is the percentile threshold parameter. Critical values are for the F-statistic to test the joint null hypothesis $\phi_1 = \phi_2 = 0$ when τ is known *a priori*. All regressions include an intercept and trend. The critical values for $\tau = 0.6$, 0.7, 0.8, and 0.9 are symmetric around the reported critical values for $\tau = 0.4$, 0.3, 0.2, and 0.1; other critical values can be interpolated.

If the threshold parameter is unknown prior to testing, we propose to jointly determine the percentile parameter and number of augmented terms (*k*) by minimizing the sum of squared residuals in regression (3). Equivalently, the percentile parameter can be obtained by maximizing the F-statistic testing the null hypothesis $\phi_1 = \phi_2 = 0$ by performing a grid search over all possible values of τ (after trimming) to give:

(13)

$$\hat{\tau} = argmaxF(\hat{\tau}).$$

This same F-statistic is used to test the null hypothesis that $\phi_1 = \phi_2 = 0$ and we denote this as "F-max." The F-max test statistic will be a supreme of the distribution of $F(\hat{\tau})$ given in Eq. (12). The critical values are provided in Table 2. The critical values were calculated using 5,000 replications.

Table 2
Critical values of the F-max statistic in LM threshold unit root tests.

%	LM TAR with threshold variable $\tilde{S}_{t-1}$					LM M-TAR with threshold variable $\Delta\tilde{S}_{t-1}$				
	T=50	100	250	500	1000	T=50	100	250	500	1000
10	5.246	5.170	5.231	5.255	5.293	3.956	3.881	3.817	3.804	3.841
5	6.263	6.164	6.109	6.102	6.171	4.971	4.756	4.663	4.702	4.646
1	8.538	8.127	8.285	8.296	8.306	7.316	6.935	6.621	6.494	6.388

Note: Critical values are for the maximum F-statistic (F-max) that jointly determines the percentile threshold parameter τ and tests the unit root null hypothesis $\phi_1 = \phi_2 = 0$. All regressions include an intercept and trend.

3. FINITE SAMPLE PROPERTIES

In this section, we provide Monte Carlo simulations to investigate the finite sample power properties of the LM TAR and LM M-TAR tests and compare power with similar versions of the DF based tests. We compare power at different persistent parameters (ϕ_1 and ϕ_2) and different percentile threshold parameters (τ = 0.5 and τ = 0.3). The LM TAR and LM M-TAR tests are denoted by TAR_{LM} and $MTAR_{LM}$ and the corresponding DF based tests are denoted by TAR_{EG} and $MTAR_{EG}$, respectively. To perform our simulations, pseudo-iid $N(0,1)$ random numbers were generated using RATS version 7.0, where the initial values of y_0 and ε_0 are assumed to be random and $\sigma_\varepsilon^2 = 1$. The simulations were calculated using 5,000 replications in sample size $T = 100$. The results are displayed in Table 3.

Table 3
Power comparisons of F-statistic ($T = 100$).

ϕ_1	ϕ_2	$\tau = 0.5$				$\tau = 0.3$			
		TAR_{LM}	TAR_{EG}	$MTAR_{LM}$	$MTAR_{EG}$	TAR_{LM}	TAR_{EG}	$MTAR_{LM}$	$MTAR_{EG}$
−0.025	−0.025	0.064	0.056	0.063	0.070	0.072	0.061	0.061	0.067
	−0.05	0.075	0.063	0.082	0.078	0.085	0.068	0.074	0.074
	−0.10	0.110	0.084	0.168	0.119	0.113	0.092	0.119	0.096
	−0.135	0.131	0.100	0.258	0.179	0.139	0.110	0.172	0.116
	−0.15	0.141	0.106	0.308	0.208	0.151	0.115	0.197	0.129
	−0.20	0.174	0.132	0.475	0.342	0.194	0.143	0.304	0.187
−0.05	−0.025	0.080	0.067	0.080	0.082	0.085	0.072	0.088	0.087
	−0.05	0.099	0.082	0.096	0.088	0.104	0.089	0.093	0.090
	−0.10	0.147	0.115	0.167	0.134	0.153	0.125	0.127	0.111
	−0.135	0.187	0.143	0.248	0.195	0.198	0.153	0.169	0.136
	−0.15	0.201	0.153	0.288	0.224	0.213	0.166	0.191	0.146
	−0.20	0.253	0.192	0.454	0.346	0.272	0.205	0.278	0.204
−0.10	−0.025	0.115	0.094	0.177	0.139	0.117	0.101	0.196	0.155
	−0.05	0.156	0.119	0.173	0.147	0.152	0.129	0.195	0.163
	−0.10	0.248	0.187	0.226	0.197	0.248	0.197	0.222	0.193
	−0.135	0.314	0.240	0.296	0.252	0.315	0.253	0.254	0.218
	−0.15	0.339	0.261	0.333	0.284	0.344	0.275	0.269	0.231
	−0.20	0.423	0.338	0.466	0.406	0.438	0.355	0.338	0.288
−0.135	−0.135	0.397	0.313	0.358	0.316	0.392	0.328	0.361	0.320

Note: τ is the percentile threshold parameter. The F-statistic tests the joint unit root null hypothesis $\phi_1 = \phi_2 = 0$ when τ is known *a priori*. All regressions include an intercept and trend.

In nearly every case that we consider, the LM TAR and LM M-TAR tests are more powerful than the DF based tests. The greater power of the LM based tests holds regardless of whether the underlying model is symmetric ($\phi_1 = \phi_2$) or asymmetric ($\phi_1 \neq \phi_2$), and regardless of the threshold value. For example, when $\phi_1 = -0.025$, $\phi_2 = -0.10$, and $\tau = 0.5$, the power of the LM TAR test is 31% greater than the DF TAR test. In the M-TAR tests the differences in power are somewhat greater. For example, when $\phi_1 = -0.025$, $\phi_2 = -0.10$, and $\tau = 0.5$, the power in the LM M-TAR test is 41% greater than the DF M-TAR test. We next compare the LM TAR test with the LM M-TAR test. We see similar power in each test when close to a unit root. As we move away from a unit root the differences in power are mixed, but the power of the LM M-TAR test is generally greater than the power of the LM TAR test. For the LM TAR test, the power with $\tau = 0.3$ is generally greater than with $\tau = 0.5$, although the differences are small. In the LM M-TAR test, the power with $\tau = 0.5$ is generally greater than with $\tau = 0.3$.[5]

4. CONCLUSION

We build on the threshold unit root tests developed in Enders and Granger (1998) and provide new threshold tests based on Lagrange Multiplier (LM) unit root tests. In addition, by adopting a percentile value the nuisance parameter problem is mitigated and one standard set of critical values can be utilized in each model. When the threshold value is unknown prior to testing, we adopt a supreme type test. Asymptotic properties are derived and finite sample properties are examined in simulations. Overall, we find that the suggested tests have favorably comparable power properties.

NOTES

3. See, e.g., Stock, 1994 and Vougas, 2003 provides simulation results showing that LM tests are more powerful than the corresponding DF tests.

4. If desired, we can allow for a delay parameter *d* and utilize $\tilde{S}\tilde{S}_{t\text{-}d}$ or $\Delta\tilde{S}\tilde{S}_{t\text{-}d}$ in Eqs. (3) and (5), respectively. We consider only $d = 1$ in our simulations.

5. In findings omitted here to conserve space, we examined simulations with a larger sample size of T = 250 and found similar results. These results are available from the authors upon request.

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