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Testing the null of stationarity in the presence of a structural break

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ABSTRACT

A test for stationarity in the presence of a structural break is proposed. An unknown break point is endogenously determined at the value minimizing the test statistic. The break point can be estimated reasonably well under the null hypothesis of stationarity, especially when the magnitude of the break is large.

I . INTRODUCTION

The pioneering work of Perron (1989) illustrates the need to allow for a structural break when testing for a unit root in economic time series. Since Perron's break point is given exogenously, subsequent literature has incorporated an endogenous break point. Allowing for a break point endogenously determined from the data, Zivot and Andrews (1992), and others, consider a minimum unit root statistic for which a break point is determined at the level giving the minimum t-statistic. These tests examine the null hypothesis of a unit root in the presence of a structural break. Lee *et al.* (1997) have shown that tests examining the null of stationarity are also affected by the presence of a structural break. They show that the stationarity tests of Kwiatkowski *et al.* (1992, hereafter denoted KPSS) diverge in the presence of an unaccounted for structural break. This finding illustrates the need for developing stationary tests that allow for a structural break.

This paper proposes a version of the minimum test for the null hypothesis of stationarity in the presence of a structural break. To determine an unknown break point endogenously from the data, a minimum statistic is considered for which the break point is selected at the value minimizing the test statistic. The performance of the test is then investigated, and as to how well it can detect an unknown break point is examined. This task is more meaningful when testing for stationarity than when testing for a unit root, since the null hypothesis of the former implies a stationary process involving a break, while the null distribution of the latter is shown not to depend on the break point. Throughout the paper, '$\rightarrow$' indicates weak convergence as $T \rightarrow \infty$.

II. TEST STATISTICS WITH A FIXED BREAK POINT

As in KPSS, an unobserved components representation is considered in which a time series y_t, $t = 1, 2, \ldots, T$, is decomposed into the sum of a random walk and a stationary error term as follows:

$$y_t = r_t + \varepsilon_t \qquad r_t = r_{t-1} + u_t \tag{1}$$

where ε_t is assumed stationary and u_t is *idd* $(0, \sigma_u^2)$. KPSS test the null hypothesis of stationarity around a level r_0 by examining the null hypothesis $\sigma_u^2 = 0$. When $r_t = \gamma + r_{r_{t-1}+u_t}$ the null hypothesis implies stationarity around the trend function γt.

To allow for a structural break, the following models using different specifications for r_t are considered.

$$(\mathrm{M}_1)^* \; r_t = \delta_1 B_t + r_{t-1} + u_t \tag{2a}$$

$$(\mathrm{M}_2)^* \; r_t = \gamma + \delta_1 B_t + \delta_2 D_t + r_{t-1} + u_t \tag{2b}$$

where $B_t = 1$ for $t = T_B + 1$ and zero otherwise; $D_t = 1$ for $t \geqslant T_B + 1$ and zero otherwise; and T_B stands for the time period when a structural change occurs. The following corresponding models are then considered.

$$(\mathrm{M}_1) \; y_t = \alpha + \delta_1 D_t + e_t \tag{3a}$$

$$(\mathrm{M}_2) \; y_t = \alpha + \gamma t + \delta_1 D_t + \delta_2 DT_t^* + e_t \tag{3b}$$

where $DT_t^* = t - T_B$ for $t \geqslant T_B + 1$ and zero otherwise. Perron (1989) previously considered these models. M_1 describes a stationary process with a one-time shift in the level and M_2 allows for a sudden change in the level followed by a change in the slope of the trend function.

Here stationarity tests that allow for a structural break are proposed. For the time being, it is assumed as in Perron (1989), that the break point is fixed and known *a priori*. Consequently, the DGP and the test regressions are referred to as either M_1 in (3a) for a level stationarity test, or as M_2 in (3b) for a trend stationarity test. The null hypothesis implies stationarity around a break point for M_1, or stationarity around a trend function for M_2. The level stationarity test statistic $\tilde{\eta}_\mu$ allowing for an exogenous break is given by:

$$\tilde{\eta}_\mu = T^{-2} \sum_{t=1}^{T} \tilde{S}_t^2 / \tilde{s}^2(l) \tag{4}$$

The residuals $\tilde{e}_t$ from the regression (3a) are used to construct $\tilde{S}_t = \sum_{j=1}^{t} \tilde{e}_j$. $\tilde{s}^2(l)$ is the estimate of the long-run variance $\sigma^2 = \lim T^{-1}(\sum e_t)^2$. The long-run variance is constructed via non-parametric estimation of $\hat{s}^2(l) = \hat{\Gamma}_0 + 2\sum w_j \hat{\Gamma}_j$ by choosing a truncation lag parameter l and a set of weights w_j, $j = 1, \ldots, l$, where $\hat{\Gamma}_j$ is the jth sample autocovariance of the residuals $\tilde{e}_t$. The corresponding trend stationarity test statistic $\tilde{\eta}_\tau$ is obtained in the same manner from the regression (3b).

The next theorem states the asymptotic null distribution of the $\tilde{\eta}_\mu$ statistic under the strong mixing regularity conditions of Phillips and Perron (1988, p. 336).

Theorem 1. Suppose that the DGP is subject to a structural break as in (3a). Under the null hypothesis of stationarity around the break, $\sigma_u^2 = 0$,

$$\tilde{\eta}_\mu \rightarrow \lambda^2 \int_0^1 V_1(b_1)^2 db_1 + (1-\lambda)^2 \int_0^1 V_2(b_2)^2 db_2 \qquad (5)$$

where V_1 and V_2 are two independent Brownian bridges; $V_i(b_i) = W(b_i) - b_i W(1)$, for $i = 1,\ 2$, with $0 < b_1 = b/\lambda < 1$ and $0 < b_2 = (b-\lambda)/(1-\lambda) < 0$, where W is a Brownian motion.

Proof. See Lee (1996b).

For the $\tilde{\eta}_\tau$ statistic, a partial sum of the residuals is obtained from the regression (3b). The resulting asymptotic distribution is the same as in expression (5), except that the term $V_i(b_i)$ is replaced by the second-level Brownian bridge whose expression appears in KPSS (1992, p. 167). The limiting distribution is expressed as a weighted sum of two independent terms. Each of them can be obtained from the regression using the subsample of before or after the structural break. The symmetry of the distribution around $\lambda = 0.5$ is easily observed, since we can interchange λ and $(1-\lambda)$ in this case. The asymptotic distributions of the test statistics $\tilde{\eta}_\mu$ and $\tilde{\eta}_\tau$ can be simulated accordingly. Critical values are calculated via Monte Carlo simulation using a sample size of 2000 with 50 000 replications. Results are shown in Table 1.

Table 1. *Upper tail critical values for $\bar{\eta}_\mu$ and $\bar{\eta}_\tau$*

λ	10%	5%	2.5%	1%
Upper tail percentiles of the distribution of $\bar{\eta}_\mu$				
0.1	0.2880	0.3809	0.4712	0.6009
0.2	0.2301	0.3028	0.3819	0.4842
0.3	0.1880	0.2452	0.3064	0.3878
0.4	0.1597	0.2004	0.2431	0.3003
0.5	0.1531	0.1891	0.2256	0.2690
Upper tail percentiles of the distribution of $\bar{\eta}_\tau$				
0.1	0.0981	0.1223	0.1467	0.1788
0.2	0.0794	0.0984	0.1171	0.1429
0.3	0.0654	0.0792	0.0933	0.1134
0.4	0.0556	0.0656	0.0761	0.0904
0.5	0.0528	0.0615	0.0701	0.0814

I I I . TEST STATISTICS WITH AN UNKNOWN BREAK POINT

In practical estimation, one rarely knows the break point T_B *a priori*. To determine an unknown break point endogenously from the data, a minimum stationarity test is considered. The estimation scheme for the minimum stationarity test is to choose a break point that gives the *most* favorable result for the null of stationarity around a break. This means that the estimate of λ is obtained at the value that *minimizes* the stationarity statistic . This scheme diOEers in principle from that of the minimum unit root test that adopts the *least* favourable result for the null hypothesis. Both the minimum stationarity test and the minimum unit root test share the notion of taking the minimum of the statistics over a range of λ between 0 and 1. It appears obvious that a supreme test is not appropriate for testing stationarity, because it tends to maximize the error sum of squares. Therefore, we consider minimum level and trend stationarity statistics with an endogenous break as follows:

$$\mathrm{Inf}\ \tilde{\eta}_\mu = \inf_{\lambda \in \Lambda} \tilde{\eta}_\mu(\lambda) \quad \text{(6a)}$$

$$\mathrm{Inf}\ \tilde{\eta}_\tau = \inf_{\lambda \in \Lambda} \tilde{\eta}_\tau(\lambda) \quad \text{(6b)}$$

where Λ is a closed subset of (0, 1) and $\tilde{\lambda} = \tilde{T}_B/T$. $\tilde{T}_B$ is the estimated break point from a sample of size T. Here, $\tilde{\lambda}$ has a well-defined probability distribution over Λ, and the true value of λ is included in the sample space of $\tilde{\lambda}$. The same is true for $\tilde{T}_B$ defined over $(0, 1, \ldots, T)$. Then the asymptotic distribution of the minimum statistic $\mathrm{Inf}\,\tilde{\eta}_\mu$ is given by:

$$\int_0^1 \Phi[\tilde{\eta}_\mu(\tilde{\lambda}) \mid \tilde{\lambda} = \lambda] P(\tilde{\lambda} = \lambda)\, d\lambda \qquad (7)$$

where $\Phi[\tilde{\eta}_\mu(\tilde{\lambda}) \mid \tilde{\lambda} = \lambda]$ is the asymptotic distribution of $\mathrm{Inf}\,\tilde{\eta}_\mu$ conditional on $\tilde{\lambda} = \lambda$, and $P(\tilde{\lambda} = \lambda)$ is the probability that $\tilde{\lambda} = \lambda$. The asymptotic distribution of $\mathrm{Inf}\,\tilde{\eta}_\tau$ is obtained in a similar manner.

As an extreme case, suppose that $P(\tilde{\lambda} = \lambda) = 1$ and a break point is correctly estimated. Then the null distribution of $\mathrm{Inf}\,\tilde{\eta}_\mu$ approaches asymptotically that of $\tilde{\eta}_\mu$, which assumes an exogenous break.

Corollary 1. Suppose that $\tilde{T}_B \to T_B$ and $\tilde{\lambda} \to \lambda$ as $T \to \infty$. Then, the asymptotic null distribution of the $\mathrm{Inf}\,\tilde{\eta}_\mu$ statistic converges to that of the $\tilde{\eta}_\mu$ statistic.

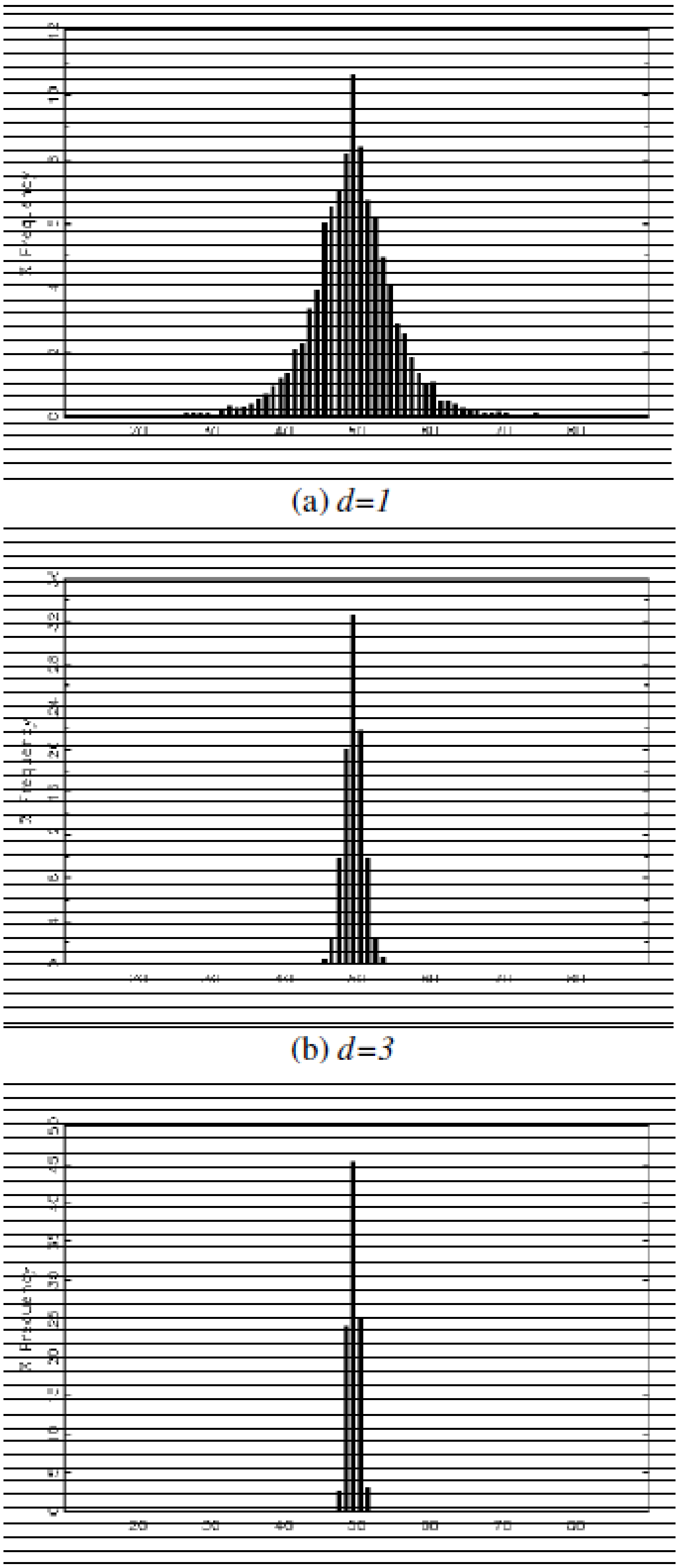
Proof. See Lee (1996b).

The same is true for the $\mathrm{Inf}\,\tilde{\eta}_\tau$ test. It is worth noticing that the asymptotic null distribution of $\mathrm{Inf}\,\tilde{\eta}_\mu$ is not free of λ. An intuitive reason is that the null hypothesis of the stationarity statistic should imply λ. This property differs from that of the minimum unit root test of Zivot and Andrews (1992), since the asymptotic distribution of the minimum unit root statistic is free of λ.[1] The method of the minimum stationarity test can be accordingly viewed as a procedure for identifying a break point under the null of stationarity. The break point needs to be correctly estimated to have correct *size* under the null of stationarity. On the other hand, the minimum unit root test allows us to identify a break point when the alternative hypothesis is stationary. In their test, a break point must be correctly estimated to increase *power*.

When testing for stationarity, a break point cannot be consistently estimated under the alternative hypothesis of a unit root. Nunes *et al.* (1995) provide evidence of the difficulty of obtaining a precise ML estimate of a break point when an integrated process is involved. The same is true for the minimum stationarity test under the alternative unit root hypothesis. However, the difficulty of estimating λ does not negate the validity of the minimum stationarity test, since the test statistic simply diverges under the alternative. Thus, stationarity tests have power to reject the null of stationarity if the DGP implies the alternative hypothesis. Simply put, we can say that it is only the estimated break point that is not reliable when the stationary null hypothesis is rejected.

IV. PERFORMANCE

This section shows Monte Carlo simulation results on the performance of the stationarity statistic. Because the behaviour of the trend stationarity tests $\tilde{\eta}_\tau$ and $\mathrm{Inf}\,\tilde{\eta}_\tau$ is not much different from those of the level stationarity tests $\tilde{\eta}_\mu$ and $\mathrm{Inf}\,\tilde{\eta}_\mu$, we focus on the latter in the simulations. Pseudo-*iid* N(0, 1) random numbers were generated using Gauss procedure RNDNS, and all calculations were conducted using the Gauss software version 3.1.4. The DGP implies (3a). Initial values y_0 and e_0 are taken a random numbers, and assume that e_t are $i.i.d.$ and $\sigma_e^2 = 1$. The Gauss application software COINT version 2.0 (Ouliaris and Phillips, 1994) is used to obtain the estimate of the long-run variance in Equation 4. For the choice of a truncation lag, we employ three fixed values of l: $l_0 = 0$, $l_4 = \mathrm{int}\,[4(T/100)^{1/4}]$, and $l_{12} = \mathrm{int}\,[12/(T/100)^{1/4}]$, where 'int' takes the value of the nearest integer to the expression in parentheses. The optimal bandwidth selection procedure of Andrews (1991) is also employed. The pre-whitening procedure of Andrews and Monahan (1992) is not used, since it makes stationarity tests inconsistent.[2] The choice of lag window does not significantly change the results and the Fejer lag window will be used here. All simulation results are calculated using 5000 replications and the size (rejection frequencies at the 5% level when the null hypothesis is true) and power (rejection frequencies when the alternative hypothesis is true) of the tests are evaluated using the upper 5% critical value.



(a) $d=1$

(b) $d=3$

(c) $d=5$

Fig. 1. *Empirical distribution of estimated break points*

The performance of the statistics under the null hypothesis of stationarity is examined. The sample size is given as $T = 100$. Results are reported in Table 2. Experiment A examines whether $\tilde{\eta}_\mu$ and $\text{Inf}\,\tilde{\eta}_\mu$ are invariant to changes of a structural break under *iid* errors for which using l_0 is most appropriate. Results from both tests show that size is not affected by the location of the break. The size remains similar when $\lambda = 0.5$ or 0.2. In addition, $\tilde{\eta}_\mu$ is mostly invariant to the different magnitudes of the level shift parameter d, where $d = 1$, 2, or 3. This is an important result, indicating that the stationarity test statistic $\tilde{\eta}_\mu$ does not depend on the parameters describing the break. As expected, $\text{Inf}\,\tilde{\eta}_\mu$ is affected by the magnitude of d. The $\text{Inf}\,\tilde{\eta}_\mu$ test rejects the true null hypothesis less often than it should when d is small. This result occurs because $P(\tilde{\lambda} = \lambda)$ departs from one. This phenomenon, though not critical, leads to a mild size distortion. However, as d gets bigger the size of the test gets closer to its nominal size of 5%. Figure 1 provides empirical distributions of the estimated break point $\tilde{\lambda}$ for different values of d. The empirical distribution shows that $\tilde{\lambda}$ centers on the true value of λ (0.5) in the DGP, while its variance gets smaller as d increases. When d is large enough the estimated $\tilde{\lambda}$ converges to a mass point λ, and the distribution of the corresponding $\text{Inf}\,\tilde{\eta}_\mu$ test would be the same as that of the $\tilde{\eta}_\mu$ test, which assumes a given fixed break point. This result indicates that the distribution of $\text{Inf}\,\tilde{\eta}_\mu$ approaches asymptotically that of $\tilde{\eta}_\mu$ as d gets larger. In the finite sample, however, the distribution is somewhat dependent on the degree of accuracy of estimating the true break. In this sense, though not critical it is recommended in the finite sample to use adjusted bootstrapping critical values to avoid size distortion, when the estimated break coefficient is relatively small.

Experiment B investigates the size of the tests in the presence of autocorrelated errors. We consider AR(1) errors of the form $e_t = \rho e_{t-1} + v_t$, where v_t is *i.i.d.* Table 2 presents the results for $\rho = 0, \pm 0.2, +0.5$, and ± 0.8, using various lag selections. Results with l_0 and l_4 lags exhibit non-negligible size distortions for $\rho > 0.5$, but the results using l_{12} and the data driven optimal bandwidth lags are fairly good. Results are comparable to the KPSS test in the absence of a break (KPSS, Table 3). As in KPSS, even the tests using optimal lags exhibit noticeable distortions when errors are strongly and negatively correlated, especially for $\rho = -0.8$.

Experiment C considers the power of the tests. We report simulation results in Table 3 for different values of σ_u^2 (with $\sigma_\varepsilon^2 = 1$) under the assumption of *iid* errors, so that the power comparison would not be affected by potential size distortions. As explained in KPSS, the power of the test increases as T increases.[3] The power of the tests using the optimal bandwidth lags is somewhat lower than the power of the tests using l_0 and l_4 lags. In some cases, it is lower than that of the tests using l_{12} lags. It is expected that using l_0 or l_4 lags produces better power, since the optimal choice of the truncation lag under *iid* errors is zero (l_0), and estimating with too many lags typically results in a loss of power.

V. SUMMARY

In this paper a minimum stationarity test in the presence of a structural break has been proposed. An unknown break is endogenously determined at the point minimizing the stationarity test statistic. It is shown that the minimum stationarity test performs reasonably well in identifying the unknown break point especially when the magnitude of the structural break is large. Under the alternative hypothesis of a unit root, the minimum stationarity test cannot detect the break point precisely, but this difficulty does not negate the validity of the test. The test is shown to have sufficient power to reject the null hypothesis of stationarity, whether or not the break point is precisely estimated.

Table 2. *Size of stationarity tests*

Exp	λ	d	ρ	$\bar{\eta}_\mu$ l_0	l_4	l_{12}	opt	Inf $\bar{\eta}_\mu$ l_0	l_4	l_{12}	opt
A	0.5	1	0	0.050	0.039	0.031	0.046	0.005	0.002	0.001	0.003
		3		0.051	0.039	0.032	0.046	0.011	0.014	0.013	0.011
		5		0.053	0.039	0.031	0.049	0.014	0.006	0.014	0.015
	0.2	1	0	0.052	0.053	0.053	0.047	0.006	0.000	0.000	0.003
		3		0.052	0.051	0.054	0.052	0.012	0.000	0.000	0.001
		5		0.047	0.059	0.046	0.057	0.009	0.000	0.000	0.001
B	0.5	1	0	0.050	0.039	0.031	0.046	0.006	0.003	0.000	0.002
			0.8	0.931	0.318	0.054	0.060	0.593	0.010	0.001	0.000
			0.5	0.509	0.103	0.037	0.079	0.138	0.103	0.000	0.002
			0.2	0.153	0.051	0.032	0.074	0.024	0.003	0.000	0.004
			−0.2	0.009	0.029	0.030	0.028	0.000	0.000	0.001	0.001
			−0.5	0.000	0.015	0.028	0.019	0.000	0.003	0.001	0.000
			−0.8	0.000	0.002	0.022	0.045	0.000	0.000	0.000	0.000
	0.5	3	0	0.051	0.039	0.032	0.046	0.011	0.014	0.000	0.011
			0.8	0.928	0.331	0.054	0.057	0.778	0.037	0.005	0.008
			0.5	0.520	0.095	0.036	0.078	0.245	0.019	0.004	0.000
			0.2	0.162	0.056	0.034	0.073	0.062	0.007	0.004	0.017
			−0.2	0.011	0.029	0.033	0.029	0.001	0.005	0.010	0.005
			−0.5	0.000	0.017	0.029	0.017	0.000	0.004	0.009	0.002
			−0.8	0.000	0.002	0.021	0.042	0.000	0.000	0.002	0.002

Notes: 'opt' refers to the case where the optimal bandwidth procedure is used ($\sigma_u^2 = 0$, AR(1) error; $T = 100$).

Table 3. *Power of stationarity tests*

Exp	λ	d	σ_u^2	$\bar{\eta}_\mu$ l_0	l_4	l_{12}	opt	Inf $\bar{\eta}_\mu$ l_0	l_4	l_{12}	opt
C	0.5	1	0.001	0.048	0.040	0.033	0.044	0.003	0.000	0.000	0.002
			0.1	0.428	0.339	0.176	0.362	0.121	0.059	0.032	0.107
			1	0.995	0.859	0.512	0.388	0.923	0.377	0.151	0.153
			100	0.999	0.871	0.533	0.356	0.972	0.437	0.174	0.149
			10000	0.999	0.867	0.526	0.355	0.973	0.487	0.151	0.128
	0.5	3	0.001	0.045	0.042	0.037	0.045	0.003	0.000	0.000	0.002
			0.1	0.424	0.341	0.184	0.396	0.192	0.079	0.050	0.114
			1	0.994	0.860	0.525	0.386	0.942	0.375	0.177	0.183
			100	0.999	0.871	0.530	0.373	0.973	0.388	0.152	0.130
			10000	0.999	0.867	0.532	0.365	0.975	0.411	0.162	0.147

Notes: $\sigma_u^2 > 0$; *did* errors; $T = 100$

NOTES

1 Recently, the assumption of no structural break under the null of a unit root for the Zivot and Andrews test has been criticized by Lee *et al.* (1998). This assumption might be necessary to have the null distribution free of λ Since the null hypothesis does not include a break, rejection of the null hypothesis does not imply rejection of a unit root, but implies rejection of a unit root *without* break, potentially leading to spurious rejections of the unit root null.

2 See Lee (1996a) for further discussion.

3 Results using other sample sizes are not reported here, but are available upon request.

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