Generalizations of n-Leibniz Algebras and n-Lie Algebras

by
Christian Payne

Honors Thesis

Appalachian State University

Submitted to the Department of Mathematical Sciences
in partial fulfillment of the requirements for the degree of
Bachelor of Science
May 2021

Approved by:

Vicky Klima, Ph.D., Thesis Advisor
Department of Mathematical Sciences

William J. Cook, Ph.D., Co-Advisor and
Honors Director, Department of Mathematical Sciences

Eric Marland, Ph.D., Chair, Department of Mathematical Sciences

Abstract

In general, Lie algebras are vector spaces equipped with an alternating, bilinear product that obeys the Jacobi identity, a sort of product rule. We first explore Leibniz algebras, generalizations of Lie algebras in which the bilinear product is no longer required to be alternating. Then, we will extend our discussion to n-airy operations and see what connections we can make to n-Leibniz and n-Lie algebras.

Contents

1 Background

It would seem negligent to discuss Lie algebras without the consideration of Sophus Lie and his contribution to mathematics. In the late 1800s, Lie studied certain transformation groups that later became known as Lie groups. He began his study of mathematics at the age of 26. According to [Ji], Lie wrote over 7000 pages of mathematical writings. During his study of Lie groups, he discovered Lie algebras. Lie algebras, in general, are vector spaces over a field that are equipped with a binary operation called the bracket satisfying certain properties [EW]. Lie's desire was to understand how Lie algebras and Lie groups provided organization to geometry, differential equations, and other topics in mathematics. In Lie's autobiography entitled *A Sketch of His Life and Work*, Lie wrote: "In particular, it was group theory and its great importance for the differential equations which interested me. But publication in this area went woefully slow. I could not structure it properly, and I was always afraid of making mistakes. Not the small inessential mistakes... No, it was the deep-rooted errors I feared. I am glad that my group theory in its present state does not contain any fundamental errors" [Ji].

2 Preliminary Concepts

We begin our discussion with a review of some algebra concepts. Vector spaces, fields, and bilinear maps are major components of our discussion, and it is important that we describe those concepts here.

Definition 2.1. . A **ring** R is a set with two binary operations, addition (denoted by $a + b$) and multiplication (denoted by ab), such that for all $a, b, c \in \mathbb{R}$:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. There is an additive identity $0 \in R$ such that $a + 0 = a$ for all $a \in R$.
4. There is an element $-a \in R$ such that $a + (-a) = 0$.
5. $a(bc) = (ab)c$.
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

If for every element $a, b \in R, ab = ba$ we have a **commutative ring**. If there exists an element $1 \in R$ such that $1a = a$ for all $a \in R$, then we have a **ring with unity**.

For example, $\mathbb{Z}$ with standard addition and multiplication form a commutative ring with unity. Also, $\mathbb{R}[x]$, the set of polynomials with indeterminate x and real coefficients, form a commutative ring with unity. If we consider the 2×2 matrices with real entries we have a non-commutative ring with unity.

Definition 2.2. A **field** is a non-trivial commutative ring with unity (multiplicative identity) in which every nonzero element is a unit (has a multiplicative inverse).

For example, $\mathbb{R}$ with the standard addition and multiplication form a field.

Definition 2.3. The **characteristic** of a ring R, denoted $\text{char}(R)$, is the smallest positive integer n such that $\underbrace{r + r + \cdots + r}_{n\text{−times}} = 0$ (the additive identity) for all $r \in R$. If there is no such n, then $\text{char}(R) = 0$.

For example, $\text{char}(\mathbb{Z}_n) = n$ and $\text{char}(\mathbb{R}) = 0$.

With the term field defined, we may introduce the notion of a vector space (over a field).

Definition 2.4. A **vector space** over a field $\mathbb{F}$ is a set V equipped with an addition $+ : V \times V \to V$ denoted $\vec{v} + \vec{w}$, and scalar multiplication $\cdot : \mathbb{F} \to V$ denoted $s\vec{v}$, such that the following properties holds:

1. **commutativity:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for all $\vec{u}, \vec{v} \ \in V$.

2. **associativity:** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ and $a(b\vec{v}) = (ab)\vec{v}$ for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $a, b \in \mathbb{F}$.

3. **additive identity (zero vector):** There exists an element $0 \in V$ such that for every element $\vec{v} \in V$, $\vec{v} + \vec{0} = v$.

4. **additive inverse:** For every element $\vec{v} \in V$, we can find an element $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$.

5. **multiplicative identity:** $1\vec{v} = \vec{v}$ for all $\vec{v} \in V$.

6. **distributive property:** $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ and $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

Vector spaces are sets that are equipped with a suitable vector addition and scalar multiplication. Typically, we use $\mathbb{R}$ or $\mathbb{C}$ as our field, but there are other possibilities. The following definition will develop the structure for the Lie bracket.

Definition 2.5. Let V, W be vector spaces over $\mathbb{F}$. A map $T : V \to W$ is a **linear map** if T satisfies the following properties:

1. $T(x + y) = T(x) + T(y)$ for all $x, y, \in V$

2. $T(\lambda x) = \lambda T(x)$ for all $\lambda \in \mathbb{F}$ and $x \in V$.

If we have a map $T : V \to V$ that satifies the properties above, we have **linear operator**.

Definition 2.6. A **bilinear map** on a vector space V over a field $\mathbb{F}$ is a map

$$(-,-) : V \times V \to V$$

such that

$$(\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2, \vec{w}) = \lambda_1(\vec{v}_1, \vec{w}) + \lambda_2(\vec{v}_2, \vec{w}),$$

$$(\vec{w}, \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2) = \lambda_1(\vec{w}, \vec{v}_1) + \lambda_2(\vec{w}, \vec{v}_2)$$

for all $\vec{v}_1, \vec{v}_2, \vec{w} \in V$ and $\lambda_1, \lambda_2 \in \mathbb{F}$

In essence, a bilinear map preserves vector addition and scalar multiplication in both entries. Understanding a bilinear bracket can help us derive important properties of Leibniz and Lie algebras.

Definition 2.7. An **algebra** over a field $\mathbb{F}$ is a vector space L over $\mathbb{F}$ equipped with a bilinear map,

$$\cdot : L \times L \to L \quad \text{denoted} \quad (x, y) \mapsto xy.$$

If multiplication is commutative, then we have a **commutative algebra.** If multiplication is assosicative, we have an **associative algebra**. If we have an algebra with a multiplicative identity, then we have a **unital algebra**. We define an **endomorphism** T on an algebra L if T as a linear operator, T, such that $T(ab) = T(a)T(b)$ for all $a, b \in L$.

Example 2.1. Let $C^\infty(\mathbb{R})$ be the vector space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. For $f, g, \in C^\infty(\mathbb{R})$, we define the product fg by the pointwise multiplication: $(fg)(x) = f(x)g(x)$. With this definition, $C^\infty(\mathbb{R})$ is a commutative, associative unital algebra.

3 Leibniz Algebras

3.1 Basic Definitions

Definition 3.1. A **(left) Leibniz algebra** L is a vector space equipped with a bilinear map (multiplication)

$$[-,-] : L \times L \to L$$

satisfying the (left) Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c, \in L$.

The Leibniz identity has a structure similar to the product rule from calculus. The element a in the Leibniz identity acts as the derivative operator in the product rule.

$$\frac{d}{dx}\left[f(x)g(x)\right] = \frac{d}{dx}\left[f(x)\right]g(x) + f(x)\frac{d}{dx}\left[g(x)\right]$$

Definition 3.2. Let A be an algebra over a field $\mathbb{F}$. A **derivation** of A is an $\mathbb{F}$-linear map $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$.

With this definition, we can see that for Leibniz algebras left multiplication operators, $L_a : L \to L$ given by $L_a(x) = [a, x]$, are in fact derivations.

Consider the following straightforward example of a derivation on the associative algebra introduced in Example 2.1. The usual derivative, $Df = f'$, is a derivation on $C^\infty(\mathbb{R})$ by the product rule [EW]:

$$D(fg) = (fg)' = f'g + fg' = (Df)g + f(Dg).$$

We now introduce the operator ad_x. We define $\mathrm{ad}_x(y) = [x, y]$ for every x and y in our Lie algebra.

Theorem 3.1. Let L be a Leibniz algebra and $x \in L$. The adjoint operator ad_x is a derivation.

Proof. We consider the proof given by [KM].Let $y, z \in L$. Then

$$\begin{aligned} \mathrm{ad}_x([y,z]) &= [x,[y,z]] \\ &= [[x,y],z] + [y,[x,z]] \\ &= [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z))] \end{aligned}$$

by (left) Leibniz identity. Hence, ad_x is a derivation. □

A right Leibniz algebra has a multiplication whose right multiplication operators act as derivations on the right. The right Leibniz identity states that

$$[[x,y],z] = [[x,z],y] + [x,[y,z]].$$

Thus we can have left or right Leibniz algebras. In our discussion, we will always refer to left Leibniz algebras as Leibniz algebras.

We will incorporate these various definitions in our discussion of Leibniz algebras.

Example 3.1. The right multiplication operator $R_a : L \to L$ is defined by $R_a(x) = [x, a]$ for all $a, x \in L$. Using the (left) Leibniz identity, we can form the following identities.

$$\begin{cases} R_{[b,c]} = R_c R_b + L_b R_c \\ L_b R_c = R_c L_b + R_{[b,c]} \\ L_c L_b = L_{[c,b]} + L_b L_c \end{cases}$$

To verify these identities, assume a, b and c are elements in the Leibniz algebra. Applying the Lebiniz identity we have

$$R_{[b,c]}(a) = [a,[b,c]] = [[a,b],c] + [b,[a,c]] = R_c R_b(a) + L_b R_c(a),$$

verifying the fist identity. Similarly,

$$L_b R_c(a) = [b,[a,c]] = [[b,a],c] + [a,[b,c]] = R_c L_b(a) + R_{[b,c]}(a), \text{ and}$$

$$L_c L_b(a) = [c,[b,a]] = [[c,b],a] + [b,[c,a]] = L_{[c,b]}(a) + L_b L_c(a).$$

Example 3.2. Let A be an associative $\mathbb{F}$-algebra equipped with a linear operator $T : A \to A$ such that $T(T(a)) = T(a)$ for all $a \in A$. Define the multiplication $[,] : A \times A \to A$ by

$$[a,b] := (Ta)b - b(Ta)$$

for all $a, b \in A$.

We will show that the Leibniz identity holds. Observe,

$$\begin{aligned}
[[a,b],c] &= [(Ta)b - b(Ta), c] \\
&= T((Ta)b - b(Ta))c - cT((Ta)b - b(Ta)) \\
&= (Ta)(Tb)c - (Tb)(Ta)c - c(Ta)(Tb) + c(Tb)(Ta), \\
[a,[b,c]] &= [a,[(Tb)c - c(Tb)] = T(a)((Tb)c - c(Tb)) - (((Tb)c - c(Tb))T(a)) \\
&= (Ta)(Tb)c - (Ta)c(Tb) - (Tb)c(Ta) + c(Tb)(Ta), \text{ and} \\
[b,[a,c]]] &= [b,(Ta)c - c(Ta)] \\
&= (Tb)((Ta)c - c(Ta)) - ((Ta)c - c(Ta))(Tb) \\
&= (Tb)(Ta)c - (Tb)c(Ta) - (Ta)c(Tb) + c(Ta)(Tb).
\end{aligned}$$

Putting it all together yields our desired result.

$$\begin{aligned} [[a,b],c]+[b,[a,c]] &= (Ta)(Tb)c-(Tb)(Ta)c-c(Ta)(Tb)+c(Tb)(Ta) \\ &\quad +(Tb)(Ta)c-(Tb)c(Ta)-(Ta)c(Tb)+c(Ta)(Tb) \\ &= (Ta)(Tb)c+c(Tb)(Ta)-(Tb)c(Ta)-(Ta)c(Tb) \\ &= (Ta)(Tb)c-(Ta)c(Tb)-(Tb)c(Ta)+c(Tb)(Ta) \\ &= [a,[b,c]] \end{aligned}$$

3.2 Cyclic Leibniz Algebras

Before we discuss the topic of cyclic Leibniz algebras, let's consider the powers of x for a Leibniz algebra. We define the powers of a Leibniz algebra to be the following:

$$x^1 = x$$

$$x^2 = [x,x]$$

$$x^3 = [x,[x,x]]$$

$$x^4 = [x,[x,[x,x]]]$$

$$\vdots$$

$$x^n = \underbrace{[x,[x,[x,[x,...[x,x]...]]]]}_{n-\text{times}}$$

We begin with the following theorem highlighting the reasoning behind our definitions for the powers of x. When we bracket in any other way of association, we always obtain 0.

Theorem 3.2. For all $x, y \in L$, $[x^n, y] = 0$ for any integer $n \geq 2$.

Proof. We will proceed by induction.

Base Case: Let $x, y \in L$. By the Leibniz identity, we have that

$$[x,[x,y]] = [[x,x],y] + [x,[x,y]].$$

Then, if we subtract the bracket $[x,[x,y]]$ from both sides, we obtain

$$[x,[x,y]] - [x,[x,y]] = [[x,x],y] + [x,[x,y]] - [x,[x,y]]$$

which means that

$$0 = [[x,x],y].$$

Thus, $[x^2, y] = 0$ and the base case holds.

Inductive Hypothesis: We assume for some positive integer $k \geq 2$ that $[x^k, y] = 0$ for all $x, y \in L$.

Inductive Step: For $x, y \in L$ consider the bracket

$$[x, \underbrace{[x,[x,[x,...[x}_{k-\text{times}},y]...]]]] = [x,[x^k,y]] = [[x,x^k],y] + [x^k,[x,y]],$$

where the last equality follows from the Leibniz identity. By the inductive hypothesis, we have that

$$[x,0] = [x^{k+1},y] + 0.$$

which implies

$$[x^{k+1},y] = 0.$$

Thus, by the Principle of Mathematical Induction, this theorem holds. $\square$

For an associative algebra, we consider the multiplication $x^3 = (xx)x = x(xx)$; however, for a Leibniz algebra, $(xx)x = [x^2, x] = 0$. In other words, the way we group elements in a Leibniz algebra matters, and we have that Leibniz algebras are not power associative. Because $[x^n, y] = 0$ for $n \geq 2$, we define our powers of x^n with all of our brackets associated to the right.

Let's consider another family of examples of Leibniz algebras known as n-dimensional cyclic Leibniz algebras. Let L be an n-dimensional Leibniz algebra over $\mathbb{F}$ generated by a single element x. This means that $L = \text{span}\{x, x^2, x^3, \dots\}$. Since L is n-dimensional, we must have $L = \text{span}\{x, x^2, ..., x^n\}$ and we have $x^{n+1} = [x, x^n] = c_1 x + \cdots + c_n x^n$ for some $c_1, ..., c_n \in \mathbb{F}$.

Theorem 3.3. The linear combination for x^{n+1} does not contain x^1.

Proof. We will follow the proof by [CHKM].

Recall Theorem 3.1 which states $[x^n, y] = 0$ for any $n \geq 2$. If $\dim(L) = 1$, then $x^2 = 0$ and we are done, so suppose that $\dim(L) > 1$. We apply the Leibniz identity once more.

$$
\begin{aligned}
0 = [x, 0] = [x, [x^n, x]] &= [[x, x^n], x] + [x^n, [x, x]] \\
&= [x^{n+1}, x] + [x^n, x^2] \\
&= [x^{n+1}, x] + 0 \\
&= \left[\sum_{i=1}^{n} c_i x^i, x\right] = \sum_{i=1}^{n} c_i [x^i, x] \\
&= c_1 x^2 + \sum_{i=2}^{n} c_i [x^i, x] \\
&= c_1 x^2 + \sum_{i=2}^{n} c_i 0 \\
&= c_1 x^2
\end{aligned}
$$

Since $\dim(L) = n > 1$, we conclude $x^2 \neq 0$, and thus $c_1 = 0$. Therefore, $x^{n+1} = \sum_{i=2}^{n} c_i x^i$, which is a summation that does not involve $i = 1$. Furthermore, any linear combination of $x^2, x^3, x^4, \ldots, x^n$ can be a choice for $[x, x^n]$ (see [CHKM], proposition 3.2). $\square$

3.3 Properties of Leibniz Algebras

In this section, we will explore many definitions and properties of Leibniz algebras. In our discussion, we will illustrate the definitions using the example of a cyclic algebra defined below.

Example 3.3. Suppose we have a Leibniz Algebra L with basis $\beta = \{x, x^2, x^3\}$ and $x^4 = x^2$. We obtain the following multiplication table.

	x	x^2	x^3
x	x^2	x^3	x^2
x^2	0	0	0
x^3	0	0	0

To prove that the Leibniz identity holds for all elements, we show that the identity holds for all combinations of the basis elements. We now verify the Leibniz identity by checking all of the cases. If $k \geq 2$, $[x^k, [a, b]] = [[x^k, a], b] + [a, [x^k, b]]$ holds since all terms are zero according to Theorem 3.1. Thus we only need to check the Leibniz identities whose left hand sides are the following:

1. $[x, [x, x]] = [[x, x], x] + [x, [x, x]] = [x^2, x] + [x, x^2] = x^3$

2. $[x, [x^2, x^2]] = [[x, x^2], x^2] + [x^2, [x, x^2]] = [x \cdot x^2] + [x^2, x^3] = 0$

3. $[x, [x^2, x^3]] = [[x, x^2], x^3] + [x^2, [x, x^3]] = [x^3, x^3] + [x^2, x^2] = 0$

4. $[x, [x^3, x^2]] = [[x, x^3], x^2] + [x^3, [x, x^2]] = [x^2, x^2] + 0 = 0$

5. $[x, [x^3, x^3] = [[x, x^3], x^3] + [x^3, [x, x^3]] = 0 + 0 = 0$

Therefore, we have verified the Leibniz identity.

In our future discussion, it will be helpful to know the bracket between two arbitrary elements of L Consider the following bracket:

$$\begin{aligned}
&[c_1x + c_2x^2 + c_3x^3, c_4x + c_5x^2 + c_6x^3] \\
&\quad = [c_1x, c_4x] + [c_1x, c_5x^2] + [c_1x, c_6x^3] + [c_2x^2, c_4x] + [c_2x^2, c_5x^2] \\
&\qquad + [c_2x^2, c_6x^3] + [c_3x^3, c_4x] + [c_3x^3, c_5x^2] + [c_3x^2, c_6x^3] \qquad (3.1) \\
&\quad = c_1c_4[x, x] + c_1c_5[x, x^2] + c_1c_6[x, x^3] + c_2c_4[x^2, x] + c_2c_5[x^2, x^2] \\
&\qquad + c_2c_6[x^2, x^3] + c_3c_4[x^3, x] + c_3c_5[x^3, x^2] + c_3c_6[x^2, x^3] \\
&\quad = c_1c_4x^2 + c_1c_5x^3 + c_1c_6x^2 = (c_1c_4 + c_1c_6)x^2 + c_1c_5x^3
\end{aligned}$$

for all $c_1, \ldots, c_6 \in \mathbb{F}$.

Definition 3.3. A **Leibniz subalgebra** of L is a vector space $K \subseteq L$ such that $[x, y] \in K$ for all $x, y \in K$.

An important example of a Leibniz subalgebra is Leib(L), defined below.

Definition 3.4. For any Leibniz algebra L, we denote the $\text{Leib}(L) = \text{span}\{[a, a] \mid a \in L\}$.

Informally, Leib(L) is the set of all linear combinations of the squares of the elements.

As an example, consider finding Leib(L) for the cyclic Lie algebra $L = \text{span}\,\{x, x^2, x^3\}$ defined in Example 3.3. Using the computation of the bracket in (3.1), we let $c_4 = c_1, c_5 = c_2$, and $c_6 = c_3$. Then we have the following bracket:

$$[c_1x + c_2x^2 + c_3x^3, c_1x + c_2x^2 + c_3x^3] = (c_1c_1 + c_1c_3)x^2 + c_1c_2x^3$$

for all $c_1, \ldots, c_6 \in \mathbb{F}$. Notice then that $\text{Leib}(L) = \text{span}\{x^2, x^3\}$.

Definition 3.5. A **(left) ideal** of a Leibniz algebra L is a subspace I of L such that $[x, y] \in I$ for all $x \in L$ and $y \in I$ or briefly $[L, I] \subseteq I$. Likewise a **(right) ideal** of a

Leibniz algebra L is a subspace I of L such that $[x, y] \in I$ for all $x \in I$ and $y \in L$ or briefly $[I, L] \subseteq I$. If I is both a left and right ideal, then I is a **(two-sided) ideal** of L. If I is an ideal, we write $I \triangleleft L$.

We will show that Leib(L) for this algebra is an ideal by direct computation. To show that the Leib(L) is a left ideal, let $z \in L$ and $y \in \text{Leib}(L)$ then $z = c_1x + c_2x^2 + c_3x^3$ and $y = d_2x^2 + d_3x^3$ for some $c_i, d_i \in \mathbb{F}$. Referring to the general computation in (3.1) and taking $c_4 = 0$, $c_5 = d_2$ and $c_6 = d_3$ we have

$$[c_1x + c_2x^2 + c_3x^3, d_2x^2 + d_3x^3] = +c_1d_2x^3 + c_1d_3x^2$$

for all $c_1, \ldots, c_6 \in \mathbb{F}$. Our computation above shows that $[L, \text{Leib}(L)] \subseteq \text{Leib}(L)$.

To show that the Leib(L) is a right ideal, let $z \in \text{Leib}(L)$ and $y \in L$ and then $z = c_2x^2 + c_3x^3$ and $y = d_1x + d_2x^2 + d_3x^3$ for some $c_i, d_i \in \mathbb{F}$. Referring to the general computation in (3.1) and taking $c_1 = 0$, $c_4 = d_1, c_5 = d_2$ and $c_6 = d_3$ we have

$$[c_2x^2 + c_3x^3, d_1x + d_2x^2 + d_3x^3] = 0 + 0 + 0 + 0 + 0 + 0 = 0$$

for all $c_1, \ldots, c_6 \in \mathbb{F}$. Our computation above shows that $[\text{Leib}(L), L] \subseteq \text{Leib}(L)$. We have shown that Leib(L) is both a left and right ideal. Therefore, Leib(L) $\triangleleft$ L. for this Leibniz algebra.

It can be shown that for all Leibniz algebras, the Leib(L) is an ideal of L.

Theorem 3.4. For all Leibniz algebras, Leib(L) is an ideal of L.

Proof. By definition, the Leib(L) is a right ideal since we are bracketing squares on the left and therefore, we will always obtain 0. We only need to show that the Leib(L) is a left ideal.

For any elements $a, b \in L$ we have that

$$\begin{aligned}[a + [b, b], a + [b, b]] &= [a + [b, b], a] + [a + [b, b], [b, b]] \\ &= [a, a] + [[b, b], a] + [a, [b, b]] + [[b, b], [b, b]] \\ &= [a, a] + 0 + [a, [b, b]] + 0\end{aligned}$$

Therefore, by the Leibniz identity $[a, [b, b]] = [a + [b, b], a + [b, b]] - [a, a]$ for any $a, b \in L$. Notice that both brackets on the right hand side are squares of an element, so $[a, [b, b]] \in \text{Leib}(L)$. Since this holds for an arbitrary square $[b, b]$ and we can extend this result linearly, Leib(L) is a left ideal. Thus the Leib$(L) \triangleleft L$ for all Leibniz algebras. $\square$

Definition 3.6. Let L be a Leibniz algebra. L is **simple** if and only if $[L, L] \neq \text{Leib}(L)$ and $\{0\}$, Leib(L), and L are the only ideals of L.

Our previous computation reveals that for the cyclic Lie algebra, L, introduced in Example 3.3, $[L, L] = \text{span}\{x^2, x^3\} = \text{Leib}(L)$ which means this cyclic algebra is not simple.

Definition 3.7. Let L be a Leibniz algebra. Then the series of ideals $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots$ where $L^{(0)} = L$, $L^{(1)} = [L, L]$, and generally $L^{(i+1)} = [L^{(i)}, L^{(i)}]$ for $i \geq 0$ is called the **derived series** of L.

Consider the derived series of the cyclic algebra, L, introduced in Example 3.3. We have already shown that $L^{(1)} = [L, L] = \text{Leib}(L)$. Moreover, we will show that $L \supseteq L^{(1)} = \text{Leib}(L) \supseteq L^{(2)} = \{0\}$. Consider the following bracket.

$$\begin{aligned}&[c_1x^2 + c_2x^3, c_3x^2 + c_4x^3] \\ &\quad = [c_1x^2, c_3x^2] + [c_1x^2, c_4x^3] + [c_2x^3, c_3x^2] + [c_2x^3, c_4x^3] \\ &\quad = c_1c_3[x^2, x^3] = c_1c_3[x^2, x^2] + c_1c_4[x^2, x^3] + c_2c_3[x^3, x^2] + c_2c_4[x^3, x^3] \\ &\quad = 0 + 0 + 0 + 0\end{aligned}$$

for all $c_1, \ldots, c_4 \in \mathbb{F}$. Therefore, $L^{(2)} = [L^{(1)}, L^{(1)}] = \{0\}$.

Definition 3.8. A Leibniz algebra L is **solvable** if $L^{(m)} = \{0\}$ for some integer $m \geq 0$.

Our work above proves that the cyclic algebra introduced in Example 3.3 is solvable.

Since multiplying on the left by a square yields zero, $(\text{Leib}(L))^{(1)} = [\text{Leib}(L), \text{Leib}(L)] = \{0\}$ for any Leibniz algebra L. Thus $\text{Leib}(L)$ is a solvable ideal.

Definition 3.9. The **radical**, $\text{rad}(L)$, is the maximal solvable ideal of L. If I and J are solvable ideals, then $I + J$ is a solvable ideal. More generally any sum of solvable ideals is itself a solvable ideal. We can take the sum of all solvable ideals and get this maximal solvable ideal that contains all solvable ideal [DE]. This is the radical.

We can think of the radical of a Leibniz algebra being the ideal that contains all of the other solvable ideals. Therefore, for solvable algebras such as the cyclic algebra defined in Example 3.3, the radical of L is L.

Definition 3.10. L is **semisimple** if and only if $\text{rad}(L) = \text{Leib}(L)$.

Based on our previous computation, we have shown that the cyclic algebra, L, introduced in Example 3.3 is not semisimple because the radical, $\text{rad}(L)$, does not equal the $\text{Leib}(L)$.

Definition 3.11. For a Leibniz algebra L the series of ideals $L = L^0 \supseteq L^1 \supseteq L^2 \supseteq \ldots$, where $L^1 = [L, L]$ and generally $L^{i+1} = [L, L^i]$, is called the **lower central series** of L.

Let's now find the lower central series of the cyclic algebra L from Example 3.3. Recall that $L^0 = L$.

$$\begin{aligned}
L^1 &= [L, L] = \text{Leib}(L) \\
L^2 &= [L, L^1] = [L, \text{Leib}(L)] \\
&= \{[c_1x + c_2x^2 + c_3x^3, c_4x^2 + c_5x^3] \mid c_1, \ldots, c_5 \in \mathbb{F}\} \\
&= \{c_1c_4[x, x^2] + c_1c_5[x, x^3] + c_2c_4[x^2, x^2] + c_2c_5[x^2, x^3] + c_3c_4[x^3, x^2] + c_3c_5[x^3, x^3] \mid c_i \in \mathbb{F}\} \\
&= \text{span}\{x^2, x^3\} \\
&= \text{Leib}(L). \\
L^3 &= [L, L^2] = [L, \text{Leib}(L)] = \text{Leib}(L).
\end{aligned}$$

Definition 3.12. A Leibniz algebra is **nilpotent** if $L^m = 0$ for some positive integer m.

When we observe the lower central series for this cyclic Leibniz algebra L from Example 3.3, we notice that when $n \geq 2$, $L^n = \text{Leib}(L)$. For this reason, L is not nilpotent.

Definition 3.13. A Leibniz algebra L is **abelian** is $[a, b] = 0$ for all $a, b \in L$.

Theorem 3.5. If a Leibniz algebra is abelian, then it is nilpotent. If a Lie algebra is nilpotent, then it is solvable.

We first suppose that L is abelian. Then, $L^1 = [L, L] = \{0\}$. Therefore, L is nilpotent. Now, suppose that L is nilpotent. If L is nilpotent, then $L^m = \{0\}$ for some $m \geq 1$. Then $L^{(1)} = L^1, L^{(2)} = [L^{(1)}, L^{(1)}] = [[L, L], [L, L]] \subseteq [L, [L, L]] = [L, L^1] = L^2$. Through induction we could show that $L^{(m)} \subseteq L^m$. Therefore, $L^{(m)} \subseteq L^m = \{0\}$ and we have that L is solvable.

4 Lie Algebras

Historically, Lie algebras were introduced in 1800s by Sophus Lie, while Leibniz algebras were introduced in the late 1900s by Loday. Our goal is to see how Lie algebra properties generalize to Leibniz algebra properties. In this section, we will explore various examples of Lie algebras from [EW, KM, FP] and discuss the similarities and differences between Lie and Leibniz algebras.

4.1 Basic Properties

Definition 4.1. Let $\mathbb{F}$ be a field. A **Lie algebra** over $\mathbb{F}$ is a vector space L, together with a bilinear map, the *Lie bracket* $[-,-] : L \times L \to L$, satisfying the following properties:

$[x, x] = 0$ for all $x \in L$ (alternating)

$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$ (Jacobi identity)

Let x and y be elements of a Lie algebra L. By the alternating property,

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$$

and thus $[x, y] = -[y, x]$ for any $x, y \in L$, making all Lie algebra products *skew-symmetric*. It is worth noting that skew-symmetry is equivalent to alternation as long as our Lie algebra is defined over a field of characteristic not equal to two. This reveals that Lie algebras are almost commutative and that by switching the the components in the bracket, we are only off by a sign.

We will use the skew-symmetry property of Lie algebras to re-write the Jacobi identity as the Leibniz identity. For example, we show that the Jacobi identity implies the left Leibniz identity.

The Jacobi identity states $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z \in L$. Then,

$$\begin{aligned}
[x,[y,z]]+[z,[x,y]] &= -[y,[z,x]] \\
[x,[y,z]]+[z,[x,y]] &= [[z,x],y] \\
[x,[y,z]] &= [[z,x],y]-[z,[x,y]] \\
[x,[y,z]] &= [[z,x],y]+[[x,y],z] \\
[x,[y,z]] &= -[y,[z,x]]+[[x,y],z] \\
[x,[y,z]] &= [y,-[z,x]]+[[x,y],z] \\
[x,[y,z]] &= [y,[x,z]]+[[x,y],z].
\end{aligned}$$

Notice that we have written the Jacobi identity as the Leibniz identity using the skew-symmetry property of Lie algebras. We reach an important conclusion.

Theorem 4.1. Every Lie algebra is in fact a Leibniz algebra.

It is the case that Lie algebras specialize Leibniz algebras. We see that Lie algebras are just Leibniz algebras that also satisfy the alternating property. With the alternating property, we can re-write the Jacobi identity as the Leibniz identity. Previously, we have discussed cyclic Leibniz algebras. Due to alternation, all cyclic Lie algebras, however, are trivial 1-dimensional algebras. Also, the Leib(L) is the trivial ideal, $\{0\}$, for any Lie algebra. This is due to the fact that if we bracket any element in a Lie algebra with itself, we obtain 0.

Theorem 4.2. All one dimensional Lie algebras are abelian.

Proof. Suppose L is a one-dimensional Lie algebra. Then L has a basis $\beta = \{x\}$. Let $\alpha, \beta \in \mathbb{F}$. Then

$$[\alpha x, \beta x] = \alpha\beta[x,x] = 0.$$

Thus L is an Abelian Lie algebra. □

4.2 General Linear Algebra $\mathfrak{gl}(V)$ and $\mathfrak{gl}(n, \mathbb{F})$

In the study of Lie algebras, one of the many versatile algebras is the *general linear algebra*. Suppose that V is a vector space over $\mathbb{F}$. We use the notation $\mathfrak{gl}(V)$ to denote the set of all linear maps from V to V. This becomes a Lie algebra when we define the Lie bracket by

$$[x, y] = x \circ y - y \circ x \text{ for } x, y \in \mathfrak{gl}(V)$$

where $\circ$ denotes the composition of the two maps.

Often times, we aim to work with matrices. We use the notation $\mathfrak{gl}(n, \mathbb{F})$ to denote the set of all $n \times n$ matrices with coefficients from $\mathbb{F}$. We define the Lie bracket of $\mathfrak{gl}(n, \mathbb{F})$ to be

$$[x, y] = xy - yx$$

where xy denotes regular matrix multiplication.

Definition 4.2. If L_1 and L_2 are Lie algebras over a field $\mathbb{F}$, then we say that a map $\varphi : L_1 \to L_2$ is a **homomorphism** if φ is a linear map and $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L_1$. Of course, an invertible (i.e., one-to-one and onto) homomorphism is called a **isomorphism** and if there is an isomorphism between L_1 and L_2 we say they are **isomorphic** and write $L_1 \cong L_2$.

If V is of dimension n, then $\mathfrak{gl}(V)$ and $\mathfrak{gl}(n, \mathbb{F})$ are isomorphic essentially because we can work with the transformations themselves or the matrices for the transformations (once we fix a basis).

An important homomorphism in the study of Lie algebras is the **adjoint homomorphism**. Suppose we have a Lie algebra L. We define $\text{ad} : L \to \mathfrak{gl}(L)$ by $\text{ad}(x) = \text{ad}_x$.

We will verify that the adjoint operator is in fact a homomorphism. Let $g, h, x \in L$ then,

$$\begin{aligned} \mathrm{ad}_{[g,h]}(x) &= [[g,h],x] \\ &= [[g,x],h] + [g,[h,x]] \\ &= [g,[h,x]] - [h,[g,x]] \\ &= \mathrm{ad}_g(\mathrm{ad}_h(x)) - \mathrm{ad}_h(\mathrm{ad}_g(x)) \\ &= (\mathrm{ad}_g \circ \mathrm{ad}_h - \mathrm{ad}_h \circ \mathrm{ad}_g)(x) \\ &= [\mathrm{ad}_g, \mathrm{ad}_h](x) \end{aligned}$$

where we get the second line from the first line by calling on the (right) Leibniz identity. Here we showed that $\mathrm{ad}([g,h]) = \mathrm{ad}_{[g,h]}$ and $[\mathrm{ad}(g), \mathrm{ad}(h)] = [\mathrm{ad}_g, \mathrm{ad}_h]$ agree on all $x \in L$. Therefore ad is a homomorphism.

4.3 Subalgebras of $\mathfrak{gl}(V)$ and $\mathfrak{gl}(n, \mathbb{F})$

We now consider looking at subalgebras of $\mathfrak{gl}(V)$. It is important to note that the definitions of subalgebras, nilpotentcy, solvability, and the radicals of an algebra are identical in the Lie and Leibniz algebra setting.

The general linear Lie algebra $\mathfrak{gl}(n, \mathbb{F})$ has commonly known subalgebras. The set of all $n \times n$ matrices with coefficients from $\mathbb{F}$ whose trace is 0 is denoted by $\mathfrak{sl}(n, \mathbb{F})$. We call this the *special linear Lie algebra*. Similarly, we have the subalgebra $\mathfrak{b}(n, \mathbb{F})$ that represents upper triangular matrices and $\mathfrak{n}(n, \mathbb{F})$ which represents the strictly upper triangular matrices. Let's explore important features of these subalgebras.

4.3.1 The Special Linear Lie Algebra: $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$

Let's consider $\mathfrak{sl}(2,\mathbb{C})$. This Lie algebra has a basis formed of matrices we denote as h, e, f where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

With these matrices, we can re-write $\mathfrak{sl}(2,\mathbb{C})$ as span$\{h, e, f\}$ over $\mathbb{F}$. Furthermore, we can compute brackets among the basis elements

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f.$$

Then, by skew-symmetry and linearity, we see that $\mathfrak{sl}(2,\mathbb{C})$ is a subalgebra. We can show that not only is $\mathfrak{sl}(2,\mathbb{C})$ a subalgebra of $\mathfrak{gl}(2,\mathbb{C})$, but also that $\mathfrak{sl}(n,\mathbb{C})$ is a subalgebra of $\mathfrak{gl}(n,\mathbb{C})$.

Theorem 4.3. $\mathfrak{sl}(n,\mathbb{F})$ is a subalgebra of $\mathfrak{gl}(n,\mathbb{F})$.

Proof. Clearly, the set of all $n \times n$ matrices with trace 0 are elements in the vector space $\mathfrak{gl}(n,\mathbb{F})$. Notice that the trace map $\mathrm{tr} : \mathfrak{gl}(n,\mathbb{F}) \to \mathbb{F}$ is linear and that $\mathfrak{sl}(n,\mathbb{F})$ is by definition the kernel (= nullspace) of this map. Thus $\mathfrak{sl}(n,\mathbb{F})$ is a subspace of $\mathfrak{gl}(n,\mathbb{F})$. Let $A, B \in \mathfrak{sl}(n,\mathbb{F})$. Then $A, B \in \mathfrak{gl}(n,\mathbb{F})$ where $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(B) = 0$. We utilize the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ to proceed with the proof. Further, we have that

$$\mathrm{tr}([A, B]) = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$$

since the trace map is linear and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Therefore, for any matrix $A, B \in \mathfrak{sl}(n,\mathbb{F})$, we have that $[A, B] \in \mathfrak{sl}(n,\mathbb{F})$. Therefore, $\mathfrak{sl}(n,\mathbb{F})$ is a subalgebra of $\mathfrak{gl}(n,\mathbb{F})$. □

4.3.2 Upper Triangular Matrices: $\mathfrak{b}(n, \mathbb{F})$

We now consider the set of all $n \times n$ matrices that are upper triangular. These matrices have a 0 in every entry below the main diagonal of the sqaure matrix. In this section, we will explore some features of this particular subalgebra.

It can be shown that $\mathfrak{b}(n, \mathbb{F})$ (upper triangular $n \times n$ matrices over $\mathbb{F}$) is a solvable Lie algebra.

Theorem 4.4. $L = \mathfrak{b}(3, \mathbb{F})$ is solvable.

Proof. Let $A, B \in L$. Then

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \text{ and } B = \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{pmatrix}$$

where $a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{F}$.

Recall that $L^{(1)} = [L, L]$. Furthermore,

$$[A, B] = AB - BA$$

$$= \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{pmatrix} - \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

$$= \begin{pmatrix} 0 & ah - gb + bj - hd & ai + bk - gc + cl - he - if \\ 0 & 0 & dk - je + el - kf \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we have matrices of the form $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ We now consider the bracket $L^{(2)} = [A, B]$ where $A, B \in L^{(1)}$. Then we have that the following bracket

$$= \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & af - cd \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

.

Now, we have bracket

$$L^{(3)} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that $L^{(3)} = \{0\}$ and we have a solvable Lie algebra. □

We consider a nilpotent subalgebra of $\mathfrak{gl}(3, \mathbb{F})$.

Suppose $L = \text{span}\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}(3, F)$.

Theorem 4.5. *L is a subalgebra of $\mathfrak{gl}(3, F)$.*

Proof. Clearly, L is a subspace since it is the span of the those three matrices. We only need to check that we are closed under the bracket.

Let $A, B \in L$. Then, $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & d & 0 \\ 0 & 0 & 0 \\ e & f & 0 \end{pmatrix}$ where $a, b, c, d, e, f \in \mathbb{F}$.

Futhermore, $[A, B] = AB - BA$.

$$AB - BA = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix} \begin{pmatrix} 0 & d & 0 \\ 0 & 0 & 0 \\ e & f & 0 \end{pmatrix} - \begin{pmatrix} 0 & d & 0 \\ 0 & 0 & 0 \\ e & f & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (bd - ea) & 0 \end{pmatrix} \in L$$

Therefore, we have shown that L is a subalgebra of $gl(3, F)$.

□

Theorem 4.6. Each matrix $A \in L$ is nilpotent. That is, $A^n = 0$ for some $n \in \mathbb{Z}_{>0}$.

Proof. Let $A \in L$. Then $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix}$ for some $a, b, c \in \mathbb{F}$. Then

$$A^2 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & ab & 0 \end{pmatrix}.$$

Furthermore,

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & ab & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, when $n = 3$, $A^3 = 0$, and every $A \in L$ is nilpotent. □

As a corollary to Engel's theorem we know if V is finite dimensional and L is a subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent linear transformations then L is a nilpotent Lie algebra [DE]. With this corollary, we know that subalgebras of $\mathfrak{gl}(n, \mathbb{F})$ consisting of nilpotent matrices are nilpotent Lie subalgebras. This implies that L is nilpotent.

4.4 Connections Between Leibniz and Lie algebra Definitions

Next, we present several classical Lie definitions than differ slightly from their Leibniz counterparts. We will see, however, that the Lie and Leibniz definitions are equivalent for Leibniz algebras that are also Lie. This is due to the additional alternating property for Lie algebras which forces $\text{Leib}(L) = \{0\}$ for any Leibniz algebra that is also Lie. Informally, $\text{Leib}(L)$ determines how close a Leibniz algebra is from being Lie.

Definition 4.3. An **ideal** of a Lie algebra L is a subspace I of L such that $[x, y] \in I$ for all $x \in L, y \in I$.

For a Leibniz algebra, we must consider the fact that an ideal is both a left and right ideal. For a Lie algebra, we need to check only one side. The alternating property of Lie algebras implies that if we switch the elements in a bracket, we are only going to change the sign of the overall bracket. Since our bracket is bilinear, the ideal absorbs the negative sign, and the need to check the other sided ideal is no longer required.

Definition 4.4. L is said to be **simple** if it has no ideals other than $\{0\}$ and L and it is not abelian.

The Leib is also important when we discuss the definition of simple and semisimple algebras. Recall that a Leibniz algebra is simple if and only if $[L, L] \neq \text{Leib}(L)$ and $\{0\}$, $\text{Leib}(L)$ and L are the only ideals of L. On the other hand, for a Lie algebra L, if L has no ideals other than $\{0\}$ and L and L is not abelian, then L is simple. Because the $\text{Leib}(L) = \{0\}$, we see that both definitions really coincide with each other.

Lemma 4.7. $\mathfrak{sl}(2, \mathbb{F})$ is a simple Lie algebra when the characteristic of $\mathbb{F}$ is not 2.

Proof. We will follow Misra's proof in [KM]. Recall that $\{e, f, h\}$ form a basis for $\mathfrak{sl}(2, \mathbb{F})$ and that $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Suppose I is an ideal of $\mathfrak{sl}(2, F)$ and $I \neq \{0\}$. To prove the Lemma, it is enough to show that $I = \mathfrak{sl}(2, F)$. Suppose $0 \neq x = \alpha e + \beta f + \gamma h \in I$ for some $\alpha, \beta, \gamma \in \mathbb{F}$. Then $[e, x] = \beta h - 2\gamma e \in I$ and $[e, \beta h - 2\gamma e] = -2\beta e \in I$. If $\beta \neq 0$, then $e \in I$ since $\text{char}(\mathbb{F}) \neq 2$. So $h = [e, f] \in I$ and $[h, f] = -2f \in I$. Hence $\{e, f, h\} \subseteq I$ and $I = \mathfrak{sl}(2, F)$. If $\beta = 0$, then $x = \alpha e + \gamma h$ and $[e, x] = -2\gamma e \in I$. Now if $\gamma \neq 0$, then $e \in I$ and as before $I = \mathfrak{sl}(2, \mathbb{F})$. If $\gamma = 0$, then $x = \alpha e \neq 0$, hence $\alpha \neq 0$ and $e \in I$. So as before $I = \mathfrak{sl}(2, \mathbb{F})$. □

Similarly, we say a Lie algebra is **semisimple** if it has no non-zero solvable ideals or equivalently if $\text{rad}(L) = \{0\}$. The Leibniz algebra definition says that if $\text{rad}(L) = \text{Leib}(L)$, then L is semisimple. We see here the importance of the Leib. Because the Leib is $\{0\}$ for Lie algebras, these two definitions are inherently the same.

4.5 The Jacobian

Recall from a calculus course the discussion of the Jacobian determinant, which appeared often when changing variables within multiple integrals.

Definition 4.5. Formally, let f be a multivariate function with two independent variables x and y. Let g also be a multivariate function with two independent variables x and y. Then the **Jacobian** is defined as follows:

$$\text{Jac}\{f, g\} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = f_x g_y - g_x f_y.$$

Let's see an example of the Jacobian of two multivariable functions. Let $f = r\cos\theta$ and

$g = r\sin\theta$. We will find the Jacobian for the change of variables.

$$\text{Jac}\{f,g\} = \begin{vmatrix} f_r & f_\theta \\ g_r & g_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

where vertical bars signal we are taking the determinant of our matrix. Now that we have introduced the Jacobian and an example, we will show why the vector space $C^\infty(\mathbb{R}^2)$ with the bracket defined by the Jacobian is in fact a Lie algebra.

Example 4.1. Consider the Jacobian product over $C^\infty(\mathbb{R}^2)$, the vector space of all continuously differential functions in two variables. We need to show that this algebra satisfies the alternating property and the Jacobi identity. Suppose $f, g,$ and h are all differentiable functions with two independent variables x and y. We will define the bracket as

$$[f,g] = \text{Jac}\{f,g\}$$

First, we have that

$$[f,f] = \text{Jac}\{f,f\} = \begin{vmatrix} f_x & f_y \\ f_x & f_y \end{vmatrix} = 0$$

since we are taking the determinant of two rows that are the same. Thus, we have satisfied the alternating property. We demonstrated in earlier work that this alternation property proves anti-symmetry.

Let's verify that the Jacobi identity holds for this problem on an ordered basis. The Jacobi identity states that

$$[h,[f,g]] = [[h,f],g] + [f,[h,g]].$$

We can break this problem into parts and then combine elements at the end.

$$[f,g] = \det \begin{bmatrix} f_x & g_x \\ f_y & g_y \end{bmatrix} = f_x g_y - f_y g_x$$

$$[h,[f,g]] = \det \begin{bmatrix} h_x & (f_x g_y - f_y g_x)_x \\ h_y & (f_x g_y - f_y g_x)_y \end{bmatrix} = \det \begin{bmatrix} h_x & f_{xx} g_y + f_x g_{yx} - f_{yx} g_x - f_y g_{xx} \\ h_y & f_{xy} g_y + f_x g_{yy} - f_{yy} g_x - f_y g_{xy} \end{bmatrix}$$

$$[h,[f,g]] = h_x f_{xy} g_y + h_x f_x g_{yy} - h_x f_{yy} g_x - h_x f_y g_{xy} - h_y f_{xx} g_y - h_y f_x g_{yx} + h_y f_{yx} g_x + h_y f_y g_{xx}$$

$$[[h,f],g] = \det \begin{bmatrix} (h_x f_y - h_y f_x)_x & g_x \\ (h_x f_y - h_y f_x)_y & g_y \end{bmatrix} = \det \begin{bmatrix} h_x f_{yx} + f_y h_{xx} - h_y f_{xx} - f_x h_{yx} & g_x \\ h_x f_{yy} f_y h_{xy} - h_y f_{xy} - f_x h_{yy} & g_y \end{bmatrix}$$

$$[[h,f],g] = g_y h_x f_{yx} + g_y f_y h_{xx} - g_y h_y f_{xx} - g_y f_x h_{yx} - g_x h_x f_{yy} - g_x f_y h_{xy} + g_x h_y f_{xy} + g_x f_x h_{yy}$$

$$[f,[h,g]] = \det \begin{bmatrix} f_x & (h_x g_y - h_y g_x)_x \\ f_y & (h_x g_y - h_y g_x)_y \end{bmatrix} = \det \begin{bmatrix} f_x & h_x g_{yx} + g_y h_{xx} - h_y g_{xx} - g_x h_{yx} \\ f_y & h_x g_{yy} + g_y h_{xy} - h_y g_{xy} - g_x h_{yy} \end{bmatrix}$$

$$[f,[h,g]] = f_x h_x g_{yy} + f_x g_y h_{xy} - f_x h_y g_{xy} - f_x g_x h_{yy} - f_y h_x g_{yx} - f_y g_y h_{xx} + f_y h_y g_{xx} + f_y g_x h_{yx}$$

We now substitute in each element and verify the Jacobi identity.

$[[h,f],g]+[f,[h,g]] = g_y h_x f_{yx} + g_y f_y h_{xx} - g_y h_y f_{xx} - g_y f_x h_{yx} - g_x h_x f_{yy} - g_x f_y h_{xy} + g_x h_y f_{xy} + g_x f_x h_{yy} + f_x h_x g_{yy} + f_x g_y h_{xy} - f_x h_y g_{xy} - f_x g_x h_{yy} - f_y h_x g_{yx} - f_y g_y h_{xx} + f_y h_y g_{xx} + f_y g_x h_{yx} = h_x f_{xy} g_y + h_x f_x g_{yy} - h_x f_{yy} g_x - h_x f_y g_{xy} - h_y f_{xx} g_y - h_y f_x g_{yx} + h_y f_{yx} g_x + h_y f_y g_{xx} = [h,[f,g]]$, as desired.

4.6 Binary Cross Product

The discussion of the cross product often arises in the third course of the calculus sequence. Most often we think of the cross product occurring in $\mathbb{R}^3$. Geometrically, the cross product produces an orthogonal vector to two given vectors. Let's first discuss how the cross product is performed.

Suppose that $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ where $x_i, y_i \in \mathbb{R}$. Then $x \times y$ (the cross product of x and y) is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$
$$= \mathbf{i}(x_2y_3 - y_2x_3) - \mathbf{j}(x_1y_3 - y_1x_3) + \mathbf{k}(x_1y_2 - y_1x_2)$$

The top row of the cross product has an orthonormal set of basis vectors for $\mathbb{R}^3$ that we denote as $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Further, $\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Now that we understand the structure of the cross product, let's explain why $\mathbb{R}^3$ with the cross product is in fact a Lie algebra.

Consider the mapping

$$[-, -] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

where $[x, y] = x \times y$.

Let's consider $[x, x]$ where $x = (x_1, x_2, x_3)$ and $x_i \in \mathbb{R}$.

$$[x, x] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} x_2 & x_3 \\ x_2 & x_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} x_1 & x_3 \\ x_1 & x_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} x_1 & x_2 \\ x_1 & x_2 \end{vmatrix} = \mathbf{0}$$

We have shown that the cross product satisfies anti-symmetry.

We now will show that the Jacobi identity holds for the cross product.

The Jacobi Identity states that $[\mathbf{i},[\mathbf{j},\mathbf{k}]]+[\mathbf{j},[\mathbf{k},\mathbf{i}]]+[\mathbf{k},[\mathbf{i},\mathbf{j}]]=\mathbf{0}$.

$$[\mathbf{i},[\mathbf{j},\mathbf{k}]]=[\mathbf{i},\mathbf{i}]=\mathbf{0} \text{ (according to the alternating property)}$$

$$[\mathbf{j},[\mathbf{k},\mathbf{i}]]=[\mathbf{j},\mathbf{j}]=\mathbf{0}$$

$$[\mathbf{k},[\mathbf{i},\mathbf{j}]]=[\mathbf{k},\mathbf{k}]=\mathbf{0}.$$

Notice that we obtain the result $[\mathbf{i},[\mathbf{j},\mathbf{k}]]+[\mathbf{j},[\mathbf{k},\mathbf{i}]]+[\mathbf{k},[\mathbf{i},\mathbf{j}]]=\mathbf{0}+\mathbf{0}+\mathbf{0}=\mathbf{0}$ as desired.

Theorem 4.8. $\mathbb{R}^3$ equipped with the cross product is a simple Lie algebra.

Proof. Suppose there is a non-zero ideal $I\subseteq\mathbb{R}^3$ and let x be a non-zero element of I. Choose $y\in\mathbb{R}^3$ such that y is linearly independent from x. Then $[x,y]\neq 0$ is orthogonal to both x and y and $[x,y]\in I$ since I is an ideal. Furthermore, $\{x,[x,y],[x,[x,y]]\}\subseteq I$ forms an orthogonal basis for $\mathbb{R}^3$. Therefore $I=\mathbb{R}^3$ and the cross product Lie algebra is simple.

□

5 Ternary Lie Algebras

Definition 5.1. Let $[-,-,-]$ be a trilinear map on a $\mathbb{F}$-vector space L. We say that $(L,[-,-,-])$ is a ternary Lie algebra or a 3-Lie algebra if the map $[-,-,-]$ satisfies for all $x_1,..,x_5\in L$ the Filippov identity:

$$[x_1,x_2,[x_3,x_4,x_5]]=[[x_1,x_2,x_3],x_4,x_5]+[x_3,[x_1,x_2,x_4],x_5]+[x_3,x_4,[x_1,x_2,x_5]]$$

and the alternation property:

$$[x_1,x_2,x_3]=0$$

if $x_i = x_j$ for some $i \neq j$. [AEM].

Example 5.1. Recall the previous definition of the cross product. We now extend our discussion of the binary cross product to the ternary cross product. Consider the following bracket:

$$[-,-] : \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$$

We define

$$[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \mathbf{x} \times \mathbf{y} \times \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_3 \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{vmatrix}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3,$ and $\mathbf{e}_4$ mean the following

$$\mathbf{e}_1 = \langle 1, 0, 0, 0 \rangle, \mathbf{e}_2 = \langle 0, 1, 0, 0 \rangle, \mathbf{e}_3 = \langle 0, 0, 1, 0 \rangle \text{ and } \mathbf{e}_4 = \langle 0, 0, 0, 1 \rangle.$$

Because the ternary cross product Lie algebra is built on top of a determinant, we can easily see that the Filippov identity and alternating property are satisfied and thus we have a ternary Lie algebra. To show that the Filippov identity holds, consider the work presented in [FP].

Example 5.2. Consider an algebra over $C^\infty(\mathbb{R}^n)$ the set of all differentiable functions. Suppose all of the following functions are differentiable functions with 3 independent variables. We will define the bracket as:

$$[f, g, h] = \text{Jac}\{f, g, h\} = \begin{vmatrix} f_{x_1} & f_{x_2} & f_{x_3} \\ g_{x_1} & g_{x_2} & g_{x_3} \\ h_{x_1} & h_{x_2} & h_{x_3} \end{vmatrix}$$

Then this is a ternary Lie algebra [AEM].

Definition 5.2. We say that H is a **ternary Lie sub-algebra** of $(L, [-,-,-])$ if it is closed under the bracket. This means that $[H, H, H] \subseteq H$.

Definition 5.3. A subspace I of L is called an **ideal** if $[I, L, L] \subset I$.

For the ternary cross product Lie algebra, there are only two trivial subalgebras. The subalgebras are $\{0\}$ and itself. This algebra has no non-trivial ideals. A proof of the more general result follows in the next section.

Definition 5.4. A ternary Lie algebra is said to be **simple** if it has at least one non-zero bracket and has no proper ideal.

Our discussion above shows that the cross-product ternary Lie algebra is simple.

6 n-airy Leibniz and Lie Algebras

Let's continue our discussion of Lie algebras that are not binary. We will start by looking at definitions that generalize to n-ary Lie algebras. Consider the following bracket:

$$\underbrace{[-,-,\ldots,-]}_{\textbf{n times}} : \underbrace{V \times V \times V \times \cdots \times V}_{\textbf{n times}} \to V$$

such that the following identity holds

$$[x_1, ..., x_{n-1}, [y_1, ..., y_n]] = \sum_{i=1}^{n} [y_1, ..., y_{i-1}, [x_1, ..., x_{n-1}, y_i], y_{i+1}, ..., y_n]$$

If the n-airy algebra only satisfies the bracket above, then we have an n-airy Leibniz algebra. If the following alternating property holds, then we have a n-Lie algebra.

$$[x_1, x_2, \ldots, x_n] = 0$$

if $x_i = x_j$ for some $i \neq j$. This identity states if any elements appears twice in the same bracket, then the bracket will equal 0.

6.1 The Generalized Cross Product

We can extend our discussion of the binary and ternary cross products to an $(n-1)$-airy cross product on $\mathbb{R}^n$ To do this, let's consider the types and number of vectors we need.

Suppose we are working in $\mathbb{R}^n$. In order to proceed with the cross product, we need $n-1$ vectors from $\mathbb{R}^n$. We also need the orthonormal basis vectors along the top row of the determinant. We will denote these vectors as $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ where

$$\mathbf{e}_1 = \langle 1, 0, 0, ..., 0\rangle, \mathbf{e}_2 = \langle 0, 1, 0, 0, 0, ..., 0\rangle, \ldots, \mathbf{e}_n = \langle 0, 0, 0, \ldots, 1\rangle$$

With this information, here is how we can set up the general cross product.

Let $x_i = \langle x_{i1}, x_{i2}, ..., x_{in}\rangle$.

Therefore, we have a mapping from $\underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{(n-1)-\text{times}}$ to $\mathbb{R}^n$ defined by

$$\mathbf{x}_1 \times \mathbf{x}_2 \times ... \times \mathbf{x}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(n-1)1} & x_{(n-1)2} & \cdots & x_{(n-1)n} \end{vmatrix}$$

Theorem 6.1. $\mathbb{R}^n$ equipped with the $(n-1)$-airy cross product is a simple Lie algebra.

Proof. For sake of contradiction, suppose there is an ideal I of dimension $0 < k < n$. Let $\{x_1, \ldots, x_k\}$ be an orthogonal basis for I and extend this basis to an orthogonal basis for $\mathbb{R}^n$, $\{x_1, \ldots, x_k, \ldots, x_n\}$. Then the bracket $[x_1, x_2, \ldots, x_{n-1}] \in I$ since I is an ideal and

$x_1 \in I$. We have that $[x_1, x_2, \ldots, x_{n-1}] \neq 0$ is perpendicular to $x_1, x_2, \cdots, x_{n-1}$. Recall $k \leq n-1$ which means $[x_1, x_2, \ldots, x_{n-1}]$ is in I but linearly independent from each of the basis vectors for I, a contradiction.

□

6.2 Generalized Jacobian n-airy Lie Algebra

Consider the vector space $C^\infty(\mathbb{R}^n)$ which contains all continuously differentiable functions in n variables. Suppose $\{f, g, h, \cdots, j\}$ is a list of n continously diferentiable functions in the variables $x_1, x_2, \cdots, x_n$. We will define the bracket as

$$[f, g, h, \ldots, j] = \text{Jac}\{f, g, h, \cdots, j\} = \begin{vmatrix} f_{x_1} & f_{x_2} & f_{x_3} & \cdots & f_{x_n} \\ g_{x_1} & g_{x_2} & g_{x_3} & \cdots & g_{x_n} \\ h_{x_1} & h_{x_2} & h_{x_3} & \cdots & h_{x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_{x_1} & j_{x_2} & j_{x_3} & \cdots & j_{x_n} \end{vmatrix}$$

Since we are working with determinants, we can see that the generalized Jacobi identity holds. See [FP] for a proof.

6.3 Nilpotent n-Lie algebras

Let's consider the n-Lie algebra introduced by [MW]. Suppose $J = \mathbb{F}[x_1, x_2, ..., x_n]$ and that $f_1, f_2, ..., f_n$ are elements in our algebra. Then we can define the bracket $[f_1, f_2, ..., f_n]$ to be the determinant of the Jacobian matrix of partial derivatives. This Lie algebra is infinite dimensional. In order for this Lie algebra to be nilpotent, it must be finite dimensional. We will truncate the dimension in the following way. Let's consider the monomials of degree 3 or more and the subspace generated by them. We will call this subspace B. Now consider the monomials of degree of degree r where $r \geq 3$ and let I_r be the subspace linearly generated by them. We can easily see that the subspace $I_r \subset B$. Thus, we will define $J_r = {}^{B}\!\big/_{I_r}$.

Theorem 6.2. J_r is a nilpotent n-Lie algebra.

We will proceed with this proof using the work of [MW].

In order to prove this theorem, we need to show that S is a subalgebra, $I_r \triangleleft S$, and that J_r is a nilpotent Lie algebra.

Let's begin the proof by showing that S is a subalgebra. To do this, we use the following lemma:

Lemma 6.3. If $f_j = \prod_{i=1}^{n} x_i^{p_{ij}} \in J$ are monomials for $j = 1, 2, ..., n$ then

$$[f_1, f_2, ..., f_n] = \det([p_{ij}]) \prod_{i=1}^{n} x_i^{q_i}$$

where $q_i = p_{i1} + p_{i2} + ... + p_{in} - 1$. Furthermore, $\deg([f_1, f_2, ..., f_n]) = \left(\sum_{j=1}^{n} x_i^{q_i}\right) - 1$.

Proof. We begin with

$$\frac{\partial f_i}{\partial x_i} = p_{ij} x_i^{-1} f_j$$

Note that if we take the partial of the jth term in the row, we have some constant p_{ij}, subtract 1 from the exponent of our variable x_i, and we keep the orginal term f_j. Let $f_j = \prod_{i=1}^{n} x_i^{p_i j}$ where $\sum_{i=1}^{n} p_{ij} = \deg(f_j)$. Using the properties of determinants, we have that

$$
\begin{aligned}
[f_1, f_2, ...f_n] &= \det\left(\left[\frac{\partial f_j}{\partial x_i}\right]\right) \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\left(\frac{\partial f_{\sigma(1)}}{\partial x_1} \cdot \frac{\partial f_{\sigma(2)}}{\partial x_2} \cdots \frac{\partial f_{\sigma(n)}}{\partial x_n}\right) \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\left(p_{1\sigma(1)} x_1^{-1} f_{\sigma(1)} p_{2\sigma(2)} x_2^{-1} f_{\sigma(2)} \ldots p_{n\sigma(n)} x_n^{-1} f_{\sigma(n)}\right) \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\left(p_{1\sigma(1)} p_{2\sigma(2)} \ldots p_{n\sigma(n)} x_1^{-1} x_2^{-1} \ldots x_n^{-1} \prod_{i,j=1}^{n} x_i^{p_{i\sigma(j)}}\right) \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\left(p_{1\sigma(1)} p_{2\sigma(2)} \ldots p_{n\sigma(n)} \prod_{i=1}^{n} x_i^{p_{i\sigma(1)}+p_{i\sigma(2)}+\ldots+p_{i\sigma(n)}-1}\right)
\end{aligned}
$$

We begin by taking the determinant of the matrix of the partial derivatives of f. We can consider the determinant operator as an operation over the permutation group S_n. With permutations, we can have a positive or negative sign. An even permutation is positive while an odd permutation is negative. We then pick an element in row 1, an element in row 2, and so on. Once we have picked those elements in the row, we multiply each of the entries with the sign of the permutation. Then we compute this for every permutation and add the various permutations. With this method, we find the partial derivatives over the permutations and compute the power rule on each of the permutations. Then we move the common factors of x_i^{-1} and group all of the common powers of x over the permutations and multiply them.

Since $x_i^{p_{i1}+p_{i2}+...+p_{in}-1} = x_i^{p_{i\sigma(1)}+p_{i\sigma(2)}+...+p_{i\sigma(n)}-1}$ we have the following:

$$\begin{aligned}
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \left(p_{1\sigma(1)}p_{2\sigma(2)}\dots p_{n\sigma(n)} \prod_{i=1}^{n} x_i^{p_{i\sigma(1)}+p_{i\sigma(2)}+...+p_{i\sigma(n)}-1} \right) \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \left(p_{1\sigma(1)}p_{2\sigma(2)}\dots p_{n\sigma(n)} \prod_{i=1}^{n} x_i^{p_{i1}+p_{i2}+...+p_{in}-1} \right) \\
&= \prod_{i=1}^{n} x_i^{p_{i1}+p_{i2}+...+p_{in}-1} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \left(p_{1\sigma(1)}p_{2\sigma(2)}\dots p_{n\sigma(n)} \right) \\
&= \det([p_{ij}]) \prod_{i=1}^{n} x_i^{q_i}
\end{aligned}$$

where $q_i = p_{i1} + ... + p_{in} - 1$. We now have proved the first part of the lemma. Let's now look at the second part of the lemma. We have that

$$\begin{aligned}
\deg([f_1, ..., f_n]) &= \sum_{i=1}^{n} q_i \\
&= \sum_{i=1}^{n} (p_{i1} + ... + p_{in} - 1) \\
&= \left(\sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij} \right) - n \\
&= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} \right) - n \\
&= \left(\sum_{j=1}^{n} \deg(f_j) \right) - n
\end{aligned}$$

We can see clearly that the monomials $f_j = \prod_{i=1}^{n} x_i^{p_{ij}}$ form a basis for J. If $h_1, ..., h_n \in S$ and $h_i = \sum_{j=1}^{n} c_j h_{ij}$ where h_j is a monomial whose coefficient is 1, then

$$[h_1, ..., h_n] = \left[\sum_{j_1=1}^{n} c_{j_1} h_{1j_1}, ..., \sum_{j_1=1}^{n} c_{jn} h_{nj_n} \right] = \sum_{j_1=1}^{n} c_{j_1} \dots \sum_{j_n=1}^{n} c_{j_n} \left[h_{1j_1}, ..., h_{nj_n} \right]$$

Then for any fixed $j_1, ..., j_n$

$$\deg([f_{1_{j_1}}, ..., f_{n_{j_n}}]) = \left(\sum_{r=1}^{n} \deg(f_{r_{j_r}})\right) - n \geq \left(\sum_{j=1}^{n} 3\right) - n = 2n \geq (2)(2) = 4.$$

We have now shown that that S is a subalgebra of J. Now suppose that $h_1, ..., \hat{h}_i, ..., h_n \in S$ and $h_i \in I_r$. Then for fixed $h_1, ..., h_n$

$$\deg([f_1, ..., f_n]) \geq \left(\sum_{j \neq i} \deg(f_j)\right) + \deg(f_i) - n$$

$$= 3(n-1) + r - n = 2n + r - 3 \geq (2)(2) + r - 3 = r + 1.$$

This concludes that $I \lhd S$ and that ${}^{S}\!/_{I_r} = J_r$ is an n-Lie algebra. All we need to show is that J_r is nilpotent.

We will now prove the following lemma to complete the proof:

Lemma 6.4. If $f \in J_r^s$ then $\deg(f) \geq 2(s-1)(n) - 3(s-2)$.

We will proceed with the proof through induction on s. If $s = 1$ we get that if $f \in J_r$ then $\deg(f) \geq 2(1-1) - 3(1-2) = 3$. The base case holds and we assume this pattern continues for k and consider $k+1$. Let $h_1 \in J_r^s$ and $h_2, ..., h_n \in J_r$. Using the previous lemma we have that

$$\begin{aligned} &\deg([h_1, h_2, ..., h_n]) \\ &\qquad \geq \quad 2(s-1)n - 3(s-2) + (3(n-1) - n) \\ &\qquad = \quad 2sn - 2n - 3s + 6 + 3n - 3 - n \\ &\qquad = \quad 2sn - 3(s-1). \end{aligned}$$

□

7 Conclusions and Future Work

In our research, we have observed the properties of Lie algebras and the way their definitions generalize to Leibniz algebras and n-Lie algebras. In future research, we can look at the relationship between n-Lie algebras and Leibniz algebras. Another area of interest for future research is the concept of a n-ary cyclic Leibniz algebra. Can we form a non-Lie n-ary cyclic Leibniz algebra? If we can, how would we incorporate the bracket and the operations with the powers? These are questions for future study.

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